CHAPTER 17

Partial Differential Equations

17.1 INTRODUCTION

The reader has, already been introduced to the notion of partial differential equations. Here, we shall begin by studying the ways in which partial differential equations are formed. Then we shall investigate the solutions of special types of partial differential equations of the first and higher orders.

In what follows x and y will, usually be taken as the independent variables and z, the dependent variable so that $z = f(x, y)$ and we shall employ the following notation :

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

17.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; *the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.* The method is best illustrated by the following examples :

Example 17.1. *Derive a partial differential equation (by eliminating the constants) from the equation*

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \qquad ...(i)$$

Solution. Differentiating (i) partially with respect to x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

and

$$\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (i), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

Example 17.2. *Form the partial differential equations (by eliminating the arbitrary functions) from*
(*a*) $z = (x + y)\,\phi\,(x^2 - y^2)$ (*P.T.U., 2009*)
(*b*) $z = f(x + at) + g(x - at)$ (*V.T.U., 2009*) (*c*) $f(x^2 + y^2, z - xy) = 0$ (*S.V.T.U., 2007*)

Solution. (*a*) We have $z = (x + y)\,\phi\,(x^2 - y^2)$

Differentiating z partially with respect to x and y,

$$p = \frac{\partial z}{\partial x} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,2x + \phi\,(x^2 - y^2), \quad ...(i)$$

$$q = \frac{\partial z}{\partial y} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,(-2y) + \phi\,(x^2 - y^2) \quad ...(ii)$$

From (*i*), $$p - \frac{z}{x + y} = 2x\,(x + y)\,\phi'\,(x^2 - y^2)$$

From (*ii*), $$q - \frac{z}{x + y} = -2y\,(x + y)\,\phi'\,(x^2 - y^2)$$

Division gives $$\frac{p - z/(x + y)}{q - z/(x + y)} = -\frac{x}{y}$$

i.e., $[p(x + y) - z]y + [q\,(x + y) - z]x$

i.e., $(x + y)\,(py + qx) - z(x + y) = 0$

Hence $py + qz = z$ is required equation.

(*b*) We have $z = f(x + at) + g(x - at)$ ...(*i*)

Differentiating z partially with respect to x and t,

$$\frac{\partial z}{\partial x} = f'\,(x + at) + g'(x - at),\ \frac{\partial^2 z}{\partial x^2} = f''\,(x + at) + g''(x - at) \quad ...(ii)$$

$$\frac{\partial z}{\partial t} = af'\,(x + at) - ag'(x - at),\ \frac{\partial^2 z}{\partial t^2} = a^2 f''\,(x + at) + a^2 g''(x - at) = a^2\frac{\partial^2 z}{\partial x^2} \quad [\text{By } (ii)]$$

Thus the desired partial differential equation is $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$

which is an equation of the second order and (*i*) is its solution.

(*c*) Let $x^2 + y^2 = u$ and $z - xy = v$ so that $f(u, v) = 0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

or $$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-y + p) = 0 \quad ...(i)$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-x + q) = 0 \quad ...(ii)$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (*i*) and (*ii*), we get

$$\begin{vmatrix} 2x & -y + p \\ 2y & -x + q \end{vmatrix} = 0 \quad \text{or} \quad xq - yp = x^2 - y^2.$$

Example 17.3. *Find the differential equation of all planes which are at a constant distance a from the origin.* (*V.T.U., 2009 S ; Kurukshetra, 2006*)

Solution. The equation of the plane in 'normal form' is

$$lx + my + nz = a \quad ...(i)$$

where l, m, n are the d.c.s of the normal from the origin to the plane.

Then $$l^2 + m^2 + n^2 = 1 \text{ or } n = \sqrt{(1 - l^2 - m^2)}$$

$\therefore$ (*i*) becomes $$lx + my + \sqrt{(1 - l^2 - m^2)}\ z = a \qquad ...(ii)$$

Differentiating partially w.r.t. x, we get

$$l + \sqrt{(1 - l^2 - m^2)}\ .\ p = 0 \qquad ...(iii)$$

Differentiating partially w.r.t. y, we get

$$m + \sqrt{(1 - l^2 - m^2)}\ .\ q = 0 \qquad ...(iv)$$

Now we have to eliminate l, m from (*ii*), (*iii*) and (*iv*).

From (*iii*), $l = -\sqrt{(1 - l^2 - m^2)}\ .\ p$ and $m = -\sqrt{(1 - l^2 - m^2)}\ .\ q$

Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

or $$(l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2 \text{ or } 1 - l^2 - m^2 = 1 - \frac{p^2 + q^2}{1 + p^2 + q^2} = \frac{1}{1 + p^2 + q^2}$$

Also $$l = -\frac{p}{\sqrt{(1 + p^2 + q^2)}} \text{ and } m = -\frac{q}{\sqrt{(1 + p^2 + q^2)}}$$

Substituting the values of l, m and $1 - l^2 - m^2$ in (*ii*), we obtain

$$\frac{-px}{\sqrt{(1 + p^2 + q^2)}} - \frac{qy}{\sqrt{(1 + p^2 + q^2)}} + \frac{1}{\sqrt{(1 + p^2 + q^2)}}\ z = a$$

or $z = px + qy + a\sqrt{(1 + p^2 + q^2)}$ which is the required partial differential equation.

PROBLEMS 17.1

From the partial differential equation (by eliminating the arbitrary constants from :

1. $z = ax + by + a^2 + b^2$.
2. $(x - a)^2 + (y - b)^2 + z^2 = c^2$. (*Kottayam, 2005*)
3. $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$ (*Anna, 2009*)
4. $z = a \log \left\{\frac{b(y-1)}{1-x}\right\}$ (*J.N.T.U., 2002 S*)
5. Find the differential equation of all spheres of fixed radius having their centres in the xy-plane. (*Madras 2000 S*)
6. Find the differential equation of all spheres whose centres lie on the z-axis. (*Kerala, 2005*)

Form the partial differential equations (by eliminating the arbitrary functions) from :

7. $z = f(x^2 - y^2)$ (*S.V.T.U., 2008*)
8. $z = f(x^2 + y^2) + x + y$ (*Anna, 2009*)
9. $z = yf(x) + xg(y)$. (*V.T.U., 2004*)
10. $z = x^2 f(y) + y^2 g(x)$. (*Anna, 2003*)
11. $z = f(x) + e^y g(x)$.
12. $xyz = \phi(x + y + z)$.
13. $z = f_1(x) f_2(y)$.
14. $z = e^{my} \phi(x - y)$. (*P.T.U., 2002*)
15. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$. (*V.T.U., 2010 ; J.N.T.U., 2010 ; Madras, 2000*)
16. $z = f_1(y + 2x) + f_2(y - 3x)$. (*Kurukshetra, 2005*)
17. $v = \frac{1}{r}[f(r - at) + F(r + at)]$.
18. $z = xf_1(x + t) + f_2(x + t)$.
19. $F(xy + z^2, x + y + z) = 0$. (*V.T.U., 2006*)
20. $F(x + y + z, x^2 + y^2 + z^2) = 0$. (*S.V.T.U., 2007*)
21. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.

17.3 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution $$f(x, y, z, a, b) = 0 \qquad ...(1)$$

of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a particular integral.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ, which is called the *general integral*.

The envelope of the family of surfaces (1), with parameters a and b, if it exists, is called a *singular integral*. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

17.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Example 17.4. *Solve* $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$ *(V.T.U., 2010)*

Solution. Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y).$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \text{ and } \int g(y)dy = v(y).$$

Thus $z = \dfrac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

Example 17.5. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ *and* $\dfrac{\partial z}{\partial x} = 1$.

Solution. If z were function of x alone, the solution would have been $z = A\sin x + B\cos x$, where A and B are constants. Since z is a function of x and y, A and B can be arbitrary functions of y. Hence the solution of the given equation is $z = f(y)\sin x + \phi(y)\cos x$

$$\therefore \quad \frac{\partial z}{\partial x} = f(y)\cos x - \phi(y)\sin x$$

When $x = 0$; $z = e^y$, $\quad \therefore e^y = \phi(y)$. $\quad$ When $x = 0$, $\dfrac{\partial z}{\partial x} = 1$, $\quad \therefore \; 1 = f(y)$.

Hence the desired solution is $z = \sin x + e^y \cos x$.

Example 17.6. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, *for which* $\dfrac{\partial z}{\partial y} = -2\sin y$ *when* $x = 0$ *and* $z = 0$ *when* y *is an odd multiple of* $\pi/2$. *(V.T.U., 2010 S)*

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x, keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \qquad \text{...}(i)$$

When $x = 0$, $\dfrac{\partial z}{\partial y} = -2 \sin y$, $\quad \therefore \quad -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$

$\therefore$ (*i*) becomes $\dfrac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y, keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \qquad ...(ii)$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$$\therefore \qquad 0 = 0 + 0 + g(x) \quad \text{or} \quad g(x) = 0 \qquad [\because \cos(2n+1)\pi/2 = 0]$$

Hence from (*ii*), the complete solution is $z = (1 + \cos x) \cos y$.

PROBLEMS 17.2

Solve the following equations :

1. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a.$
2. $\dfrac{\partial^2 z}{\partial x^2} = xy.$
3. $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x.$
4. $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y).$
5. $\dfrac{\partial^2 z}{\partial y^2} = z$, gives that when $y = 0$, $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$
6. $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$, $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0.$

17.5 LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the first order, commonly known as *Lagrange's Linear equation**, is of the form

$$Pp + Qq = R \qquad ...(1)$$

where P, Q and R are functions of x, y, z. This equation is called a *quasi-linear equation*. When P, Q and R are independent of z it is known as *linear equation*.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u, v) = 0$ ...(2)

where u, v are some functions of x, y, z.

Differentiating (2) partially with respect to x and y.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0 \text{ and } \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0.$$

Eliminating $\dfrac{\partial \phi}{\partial u}$ and $\dfrac{\partial \phi}{\partial v}$, we get $\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z} p & \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z} p \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} q & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial z} q \end{vmatrix} = 0$

which simplifies to $\left(\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}\right) p + \left(\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}\right) q = \left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right)$ ...(3)

This is of the same form as (1).

Now suppose $u = a$ and $v = b$, where a, b are constants, so that

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv = 0.$$

*See footnote p. 142.

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}.$$

or $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$ ...(4) [By virtue of (1) and (3)]

The solutions of these equations are $u = a$ and $v = b$.

$\therefore$ $\phi(u, v) = 0$ is the required solution of (1).

Thus to solve the equation $Pp + Qq = R$.

(*i*) *form the subsidiary equations* $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

(*ii*) *solve these simultaneous equations by the method of § 16.10 giving $u = a$ and $v = b$ as its solutions.*

(*iii*) *write the complete solution as $\phi(u, v) = 0$* or $u = f(v)$.

Example 17.7. *Solve* $\frac{y^2z}{x}p + xzq = y^2$. (*Kottayam, 2005*)

Solution. Rewriting the given equation as

$$y^2zp + x^2zq = y^2x,$$

The subsidiary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$

The first two fractions give $x^2dx = y^2dy$.

Integrating, we get $x^3 - y^3 = a$ ...(*i*)

Again the first and third fractions give $xdx = zdz$

Integrating, we get $x^2 - z^2 = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2).$$

Example 17.8. *Solve* $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$. (*V.T.U., 2010 ; S.V.T.U., 2009*)

Solution. Here the subsidiary equations are $\frac{dx}{mz-ny}=\frac{dy}{mx-lz}=\frac{dz}{ly-mx}$

Using multipliers x, y, and z, we get each fraction $= \frac{xdx+ydy+zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a$ ...(*i*)

Again using multipliers l, m and n, we get each fraction $= \frac{ldx+mdy+ndz}{0}$

$\therefore$ $ldx + mdy + ndx = 0$ which on integration gives $lx + my + nz = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

Example 17.9. *Solve* $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. (*V.T.U., 2010 ; Anna, 2009 ; S.V.T.U., 2008*)

Solution. Here the subsidiary equations are $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$

From the last two fractions, we have $\frac{dy}{y}=\frac{dz}{z}$

which on integration gives $\log y = \log z + \log a$ or $y/z = a$ ...(*i*)

Using multipliers x, y and z, we have

$$\text{each fraction} = \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)} \quad \therefore \quad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}=\frac{dz}{z}$$

which on integration gives $\log(x^2+y^2+z^2) = \log z + \log b$

or $$\frac{x^2+y^2+z^2}{z} = b \qquad ...(ii)$$

Hence from (*i*) and (*ii*), the required solution is $x^2+y^2+z^2 = zf(y/z)$.

Example 17.10. *Solve* $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$. *(P.T.U., 2009 ; Bhopal, 2008 ; S.V.T.U. 2007)*

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\therefore$ $$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ which on integration gives}$$

$$\log x + \log y + \log z = \log a \quad \text{or} \quad xyz = a \qquad ...(i)$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$\therefore$ $$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0, \text{ which on integrating gives}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \qquad ...(ii)$$

Hence from (*i*) and (*ii*), the complete solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Example 17.11. *Solve* $(x^2-yz)p + (y^2-zx)q = z^2-xy$. *(Bhopal, 2008 ; V.T.U., 2006 ; Madras, 2000)*

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \qquad ...(i)$$

$$\text{Each of these equations} = \frac{dx-dy}{x^2-y^2-(y-x)z} = \frac{dy-dz}{y^2-z^2-x(z-y)}$$

i.e., $$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} \quad \text{or} \quad \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, $$\log(x-y) = \log(y-z) + \log c \quad \text{or} \quad \frac{x-y}{y-z} = c \qquad ...(ii)$$

$$\text{Each of the subsidiary equations } (i) = \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} \qquad ...(iii)$$

$$\text{Also each of the subsidiary equations} = \frac{dx+dy+dz}{x^2+y^2+z^2-yz-zx-xy} \qquad ...(iv)$$

Equating (*iii*) and (*iv*) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $$\int (xdx + ydy + zdz) = \int (x + y + z) d\,(x + y + z) + c'$$

or $$x^2 + y^2 + z^2 = (x + y + z)^2 + 2c' \quad \text{or} \quad xy + yz + zx + c' = 0 \qquad \ldots(v)$$

Combining (*ii*) and (*v*), the general solution is

$$\frac{x - y}{y - z} = f\,(xy + yz + zx).$$

PROBLEMS 17.3

Solve the following equations :

1. $xp + yq = 3z$.
2. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.
3. $(z - y)p + (x - z)q = y - x$.
4. $p \cos (x + y) + q \sin (x + y) = z$.
5. $pyz + qzx = xy$.
6. $p \tan x + q \tan y = \tan z$.
7. $p - q = \log (x + y)$.
8. $xp - yq = y^2 - x^2$ (*J.N.T.U., 2002 S*)
9. $(y + z)p - (z + x)\,q = x - y$.
10. $x(y - z)p + y(z - x)q = z(x - y)$. (*Bhopal, 2007*)
11. $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$. (*V.T.U., 2010 ; Anna, 2008*)
12. $y^2p - xyq = x(z - 2y)$. (*S.V.T.U., 2008*)
13. $(y^2 + z^2)p - xyq + zx = 0$. (*P.T.U., 2009 ; V.T.U., 2009*)
14. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. (*Kerala, 2005*)
15. $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$.

17.6 NON-LINEAR EQUATIONS OF THE FIRST ORDER

Those equations in which p and q occur other than in the first degree are called *non-linear partial differential equations of the first order*. The *complete solution* of such an equation contains only two arbitrary constants (*i.e.*, equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.]

Here we shall discuss four standard forms of these equations.

Form I. f(p, q) = 0, *i.e., equations containing p and q only.*

Its complete solution is $z = ax + by + c$ ...(1)

where a and b are connected by the relation $f(a, b) = 0$ ...(2)

[Since from (1), $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$, which when substituted in (2) give $f(p, q) = 0$].

Expressing (2) as $b = \phi(a)$ and substituting this value of b in (1), we get the required solution as $z = ax + \phi(a)y + c$ in which a and c are arbitrary constants.

Example 17.12. *Solve $p - q = 1$.* (*Anna, 2009*)

Solution. The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + a - 1y + c$ is the desired solution.

Example 17.13. *Solve $x^2p^2 + y^2q^2 = z^2$.* (*Anna, 2008 ; Bhopal, 2008 ; Kerala, 2005 ; Kurukshetra, 2005*)

Solution. Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \qquad \ldots(i)$$

and setting $$\frac{dx}{x} = du, \frac{dy}{y} = dv, \frac{dz}{z} = dw \text{ so that } u = \log x, v = \log y, w = \log z.$$

Then (*i*) becomes $$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1$$

i.e., $P^2 + Q^2 = 1$ where $P = \dfrac{\partial w}{\partial u}$ and $Q = \dfrac{\partial w}{\partial v}$.

Its complete solution is $w = au + bv + c$ ...(*ii*)

where $a^2 + b^2 = 1$ or $b = \sqrt{(1-a^2)}$.

$\therefore$ (*ii*) becomes $w = au + \sqrt{(1-a^2)}\, v + c$

or $$\log z = a \log x + \sqrt{(1-a^2)} \log y + c \text{ which is the required solution.}$$

Form II. f(z, p, q) = 0, *i.e., equations not containing x and y.*

As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \qquad q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a \frac{dz}{du}$$

Substituting the values of p and q in $f(z, p, q) = 0$, we get

$$f\left(z, \frac{\partial z}{\partial u}, a \frac{dz}{du}\right) = 0 \text{ which is an ordinary differential equation of the first order.}$$

Rewiting it as $\dfrac{dz}{du} = \phi(z, a)$ it can be easily integrated giving

$$F(z, a) = u + b, \text{ or } x + ay + b = F(z, a) \text{ which is the desired complete solution.}$$

Thus to solve $f(z, p, q) = 0$,

(*i*) *assume* $u = x + ay$ *and substitute* $p = dz/du$, $q = a\, dz/du$ *in the given equation;*

(*ii*) *solve the resulting ordinary differential equation in z and u;*

(*iii*) *replace u by* $x + ay$.

Example 17.14. *Solve* $p(1 + q) = qz$. (*Madras, 2000 S*)

Solution. Let $u = x + ay$, so that $p = dz/du$ and $q = a\, dz/du$.

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \text{ or } a\frac{dz}{du} = az - 1 \quad \text{or} \quad \int \frac{a\, dz}{az - 1} = \int du + b$$

or $$\log(az - 1) = u + b \text{ or } \log(az - 1) = x + ay + b$$

which is the required complete solution.

Example 17.15. *Solve* $q^2 = z^2p^2(1 - p^2)$. (*J.N.T.U., 2005 ; Kerala, 2005*)

Solution. Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = a\frac{dz}{du} \text{ and } q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}$$

$\therefore$ The given equation becomes $\left(\dfrac{dz}{du}\right)^2 = a^2z^2\left(\dfrac{dz}{du}\right)^2 \left\{1 - a^2\left(\dfrac{dz}{du}\right)^2\right\}$ ...(*i*)

or $$a^4z^2\left(\frac{dz}{du}\right)^2 = a^2z^2 - 1 \quad \text{or} \quad \frac{dz}{du} = \frac{\sqrt{(a^2z^2 - 1)}}{a^2z}$$

Integrating, $$\int \frac{a^2z}{\sqrt{(a^2z^2 - 1)}}\, dz = \int du + c \quad \text{or} \quad (a^2z^2 - 1)^{1/2} = u + c$$

i.e., $$a^2z^2 = (y + ax + c)^2 + 1 \qquad [\because u = y + ax]$$

The second factor in (*i*) is $dz/du = 0$. Its solution is $z = c'$.

Example 17.16. *Solve* $z^2(p^2x^2 + q^2) = 1$. (*Bhopal, 2008 S*)

Solution. Given equation can be reduced to the above form by writing it as

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(i)$$

Putting $X = \log x$, so that $x\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial X}$, (i) takes the standard form

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(ii)$$

Let $u = X + ay$ and put $\dfrac{\partial z}{\partial X}=\dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y}=a\dfrac{dz}{du}$ in (ii), so that

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=1 \quad \text{or} \quad \sqrt{(1+a^2)}\,zdz=\pm du$$

Integrating, $\sqrt{(1+a^2)}\,z^2 = \pm 2u + b = \pm 2(X + ay) + b$

or $z^2\sqrt{(1+a^2)} = \pm 2(\log x + ay) + b$

which is the complete solution required.

Form III. f(x, p) = F(y, q), *i.e., equations in which z is absent and the terms containing x and p can be separated from those containing y and q.*

As a trial solution assume that $f(x, p) = F(y, q) = a$, say

Then solving for p, we get $p = \phi(x)$

and solving for q, we get $q = \psi(y)$

Since $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = pdx + qdy$

$\therefore$ $dz = \phi(x)dx + \psi(y)dy$

Integrating, $z = \int\phi(x)dx + \int\psi(y)dy + b$

which is the desired complete solution containing two constants a and b.

Example 17.17. *Solve $p^2 + q^2 = x + y$.* *(Bhopal, 2006 ; Madras, 2003)*

Solution. Given equation is $p^2 - x = y - q^2 = a$, say

$\therefore$ $p^2 - x = a$ gives $p = \sqrt{(a + x)}$

and $y - q^2 = a$ gives $q = \sqrt{(y - a)}$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{(a+x)}\,dx + \sqrt{(y-a)}\,dy$$

$\therefore$ integrating gives, $z = \dfrac{2}{3}(a + x)^{3/2} + \dfrac{2}{3}(y - a)^{3/2} + b$

which is the required complete solution.

Example 17.18. *Solve $z^2(p^2 + q^2) = x^2 + y^2$.* *(Bhopal, 2008)*

Solution. The equation can be reduced to the above form by writing it as

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial y}\right)^2=x^2+y^2 \quad ...(i)$$

and putting $zdz = dZ$, *i.e.*, $Z = \dfrac{1}{2}z^2$

$$\therefore \quad \frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}=z\frac{\partial z}{\partial x}=P$$

and $$\frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial y}=z\frac{\partial z}{\partial y}=Q$$

$\therefore$ (*i*) becomes $\quad P^2 + Q^2 = x^2 + y^2$

or $\quad P^2 - x^2 = y^2 - Q^2 = a$, say.

$$\therefore \quad P = \sqrt{(x^2 + a)} \text{ and } Q = \sqrt{(y^2 - a)}.$$

$$\therefore \quad dZ = Pdx + Qdy \text{ gives}$$

$$dZ = \sqrt{(x^2 + a)}\, dx + \sqrt{(y^2 - a)}\, dy$$

Integrating, we have
$$Z = \frac{1}{2} x \sqrt{(x^2 + a)} + \frac{1}{2} a \log [x + \sqrt{(x^2 + a)}] + \frac{1}{2} y \sqrt{(y^2 - a)} - \frac{1}{2} a \log [y + \sqrt{(y^2 - a)}] + b$$

or
$$z^2 = x \sqrt{(x^2 + a)} + y \sqrt{(y^2 - a} + a \log \frac{x + \sqrt{(x^2 + a)}}{y + \sqrt{(y^2 - a)}} + 2b$$

which is the required complete solution.

Example 17.19. *Solve* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.
(*Bhopal, 2006 ; Rajasthan, 2006 ; V.T.U., 2003*)

Solution. This equation can be reduced to the form $f(x, q) = F(y, q)$ by putting $u = x + y$, $v = x - y$ and taking $z = z(u, v)$.

Then
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = P + Q$$

and
$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = P - Q, \text{ where } P = \frac{\partial z}{\partial u}, Q = \frac{\partial z}{\partial v}$$

Substituting these, the given equation reduces to

$$u(2P)^2 + v(2Q)^2 = 1 \quad \text{or} \quad 4P^2 u = 1 - 4Q^2 v = a \text{ (say)}$$

$$P = \pm \frac{1}{2} \sqrt{\frac{a}{u}}, Q = \pm \frac{1}{2} \sqrt{\frac{1 - a}{v}}$$

$$\therefore \quad dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = Pdu + Qdv$$

$$= \pm \frac{\sqrt{a}}{2} \frac{du}{\sqrt{u}} \pm \frac{\sqrt{1 - a}}{2} \frac{dv}{\sqrt{v}}$$

Integrating, we have
$$z = \pm \sqrt{a} \sqrt{u} \pm \sqrt{1 - a} \sqrt{v} + b$$

or
$$z = \pm \sqrt{a (x + y)} \pm \sqrt{(1 - a)(x - y)} + b$$

which is the required complete solution.

Form IV. $z = px + qy + f(p, q)$: *an equation analogous to the Clairaut's equation (§ 11.14).*

Its complete solution is $z = ax + by + f(a, b)$ which is obtained by writing a for p and b for q in the given equation.

Example 17.20. *Solve* $z = px + qy + \sqrt{(1 + p^2 + q^2)}$. (*Anna, 2009*)

Solution. Given equation is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{(1 + p^2 + q^2)}$

$\therefore$ Its complete solution is $z = ax + by + \sqrt{(1 + a^2 + b^2)}$.

PROBLEMS 17.4

Obtain the complete solution of the following equations :

1. $pq + p + q = 0$.
2. $p^2 + q^2 = 1$. (*Osmania, 2000*)
3. $z = p^2 + q^2$. (*Anna, 2005 S ; J.N.T.U., 2002 S*)
4. $p(1 - q^2) = q(1 - z)$ (*Anna, 2006*)
5. $yp + xq + pq = 0$.
6. $p + q = \sin x + \sin y$.

7. $p^2 - q^2 = x - y$.
8. $\sqrt{p} + \sqrt{q} = x + y$.
9. $p^2 + q^2 = x^2 + y^2$. (*Osmania, 2003*)
10. $z = px + qy + \sin(x + y)$.
11. $\sqrt{p} + \sqrt{q} = 2x$. (*J.N.T.U., 2006*)
12. $z = px + qy - 2\sqrt{(pq)}$.
13. $(x - y)(px - qy) = (p - q)^2$. [**Hint.** Use $x + y = u, xy = v$]

17.7 CHARPIT'S METHOD*

We now explain a general method for finding the complete integral of a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y\, z, p, q) = 0 \qquad \text{...(1)}$$

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy \qquad \text{...(2)}$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ ...(3)
then we can solve (1) and (3) for p and q and substitute in (2). This will give the solution provided (2) is integrable.

To determine ϕ, we differentiate (1) and (3) with respect to x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(4)}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(5)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(6)}$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}q + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(7)}$$

Eliminating $\frac{\partial p}{\partial x}$ between the equations (4) and (5), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad \text{...(8)}$$

Also eliminating $\frac{\partial q}{\partial y}$ between the equations (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \qquad \text{...(9)}$$

Adding (8) and (9) and using $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$,

we find that the last terms in both cancel and the other terms, on rearrangement, give

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} = 0 \qquad \text{...(10)}$$

i.e.,

$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} = 0 \qquad \text{...(11)}$$

This is Lagrange's linear equation (§ 17.5) with x, y, z, p, q as independent variables and ϕ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

*Charpit's memoir containing this method was presented to the Paris Academy of Sciences in 1784.

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\partial \phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (3), which alongwith (1) will give the values of p and q to make (2) integrable. Of course, we should take the simplest of the integrals so that it may be easier to solve for p and q.

Example 17.21. *Solve* $(p^2 + q^2)y = qz$. *(V.T.U., 2007 ; Hissar, 2005)*

Solution. Let $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{-2py} = \frac{dy}{z - 2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating, $p^2 + q^2 = c^2$ ...(*ii*)

Now to solve (*i*) and (*ii*), put $p^2 + q^2 = c^2$ in (*i*), so that $q = c^2y/z$

Substituting this value of q in (*ii*), we get $p = c\sqrt{(z^2 - c^2y^2)/z}$

Hence $$dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2 - c^2y^2)}\,dx + \frac{c^2y}{z}dy$$

or $$zdz - c^2y\,dy = c\sqrt{(z^2 - c^2y^2)}\,dx \quad \text{or} \quad \frac{\frac{1}{2}d(z^2 - c^2y^2)}{\sqrt{(z^2 - c^2y^2)}} = c\,dx$$

Integrating, we get $\sqrt{(z^2 - c^2y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$ which is the required complete integral.

Example 17.22. *Solve* $2xz - px^2 - 2qxy + pq = 0$. *(Rajasthan, 2006)*

Solution. Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\therefore$ $dq = 0$ or $q = a$.

Putting $q = a$ in (*i*), we get $p = \dfrac{2x(z - ay)}{x^2 - a}$

$$\therefore \quad dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a}dx + ady \quad \text{or} \quad \frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}dx$$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

or $z - ay = b(x^2 - a)$ or $z = ay + b(x^2 - a)$

which is the required complete solution.

Example 17.23. *Solve* $2z + p^2 + qy + 2y^2 = 0$. *(J.N.T.U., 2005 ; Kurukshetra, 2005)*

Solution. Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

Charpit's subsidiary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a - x)^2]$$

$\therefore \qquad dz = pdx + qdy = (a - x)dx - \dfrac{1}{y}[2z + 2y^2 + (a - x)^2]dy$

Multiplying both sides by $2y^2$,

$$2y^2dz + 4yz\,dy = 2y^2(a - x)dx - 4y^3dy - 2y(a - x)^2dy$$

Integrating $\quad 2zy^2 = -[y^2(a - x)^2 + y^4] + b$

or $\quad y^2[(x - a)^2 + 2z + y^2] = b$, which is the desired solution.

PROBLEMS 17.5

Solve the following equations :

1. $z = p^2x + q^2x$.
2. $z^2 = pq\,xy$. (Anna, 2009 ; V.T.U., 2004)
3. $1 + p^2 = qz$.
4. $pxy + pq + qy = yz$. (J.N.T.U., 2006 ; Kurukshetra, 2006)
5. $p(p^2 + 1) + (b - z)q = 0$.
6. $q + xp = p^2$. (Osmania, 2003)

17.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \dots(1)$$

in which k's are constants, is called a *homogeneous linear partial differential equation of the nth order with constant coefficients*. It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\dfrac{\partial^r}{\partial x^r} = D^r$ and $\dfrac{\partial^r}{\partial y^r} = D'^r$. (1) becomes $(D^n + k_1D^{n-1}D'^r + D' + \dots + k_nD'^n)z = F(x, y)$

or briefly $\qquad f(D, D')z = F(x, y) \qquad \dots(2)$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely : the *complementary function* and *the particular integral*.

The complementary function is the complete solution of the equation $f(D, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

17.9 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $\dfrac{\partial^2 z}{\partial x^2} + k_1 \dfrac{\partial^2 z}{\partial x \partial y} + k_2 \dfrac{\partial^2 z}{\partial y^2} = 0 \qquad \dots(1)$

which in *symbolic form is* $(D^2 + k_1DD' + k_2D'^2)z = 0 \qquad \dots(2)$

Its symbolic operator equated to zero, *i.e.*, $D^2 + k_1DD' + k_2D'^2 = 0$ is called the *auxiliary equation* (A.E.)

Let its root be $\quad D/D' = m_1, m_2$.

Case I. *If the roots be real and distinct* then (2) is equivalent to

$$(D - m_1D')(D - m_2D')z = 0 \qquad \dots(3)$$

It will be satisfied by the solution of

$$(D - m_2D')z = 0, \text{ i.e., } p - m_2q = 0.$$

This is a Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}, \text{ whence } y + m_2x = a \text{ and } z = b.$$

$\therefore$ its solution is $z = \phi(y + m_2x)$.

Similarly (3) will also be satisfied by the solution of

$$(D - m_1D')z = 0, \quad i.e., \quad \text{by } z = f(y + m_1x)$$

Hence the complete solution of (1) is $z = f(y + m_1x) + \phi(y + m_2x)$.

Case II. *If the roots be equal* (*i.e.*, $m_1 = m_2$), then (2) is equivalent to

$$(D - m_1D')^2z = 0 \quad ...(4)$$

Putting $(D - m_1D')z = u$, it becomes $(D - m_1D')u = 0$ which gives

$$u = \phi(y + m_1x)$$

$\therefore$ (4) takes the form $(D - m_1'D)z = \phi(y + m_1x)$ or $p - m_1q) = \phi(y + m_1x)$

This is again Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(y + m_1x)}$$

giving $y + m_1x = a$ and $dz = \phi(a)\,dx$, *i.e.*, $z = \phi(a)x + b$

Thus the complete solution of (1) is

$$z - x\phi(y + m_1x) = f(y + m_1x). \quad i.e., \quad z = f(y + m_1x) + x\phi(y + m_1x).$$

Example 17.24. *Solve* $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$

Solution. Given equation in *symbolic form* is $(2D^2 + 5DD' + 2D'^2)z = 0$.

Its *auxiliary equation* is $2m^2 + 5m + 2 = 0$, where $m = D/D'$.

which gives $m = -2, -1/2$.

Here the complete solution is $z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

which may be written as $z = f_1(y - 2x) + f_2(2y - x)$.

Example 17.25. *Solve* $4r + 12s + 9t = 0$. *(P.T.U., 2010)*

Solution. Given equation in *symbolic form* is $(4D^2 + 12DD' + 9D'^2)z = 0$

for $r = \frac{\partial^2 z}{\partial x^2} = D^2z,\ s = \frac{\partial^2 z}{\partial x \partial y} = DD'z$ and $t = \frac{\partial^2 z}{\partial y^2} = D'^2z.$

$\therefore$ Its *auxiliary equation* is $4m^2 + 12m + 9 = 0$, whence $m = -3/2, -3/2$

Hence the complete solution is $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$.

17.10 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $(D^2 + k_1DD' + k_2D'^2)z = F(x, y)$ *i.e.*, $f(D, D')z = F(x, y)$.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')}F(x, y)$$

Case I. *When* $F(x, y) = e^{ax + by}$

Since $De^{ax+by} = ae^{ax+by}$; $D'e^{ax+by} = be^{ax+by}$

$\therefore$ $D^2e^{ax+by} = a^2e^{ax+by}$; $DD'e^{ax+by} = abe^{ax+by}$

and $D'^2e^{ax+by} = b^2e^{ax+by}$

$\therefore$ $(D^2 + k_1DD' + k_2D'^2)e^{ax+by} = (a^2 + k_1ab + k_2b^2)\,e^{ax+by}$

i.e., $f(D, D')e^{ax+by} = f(a, b)\,e^{ax+by}$

Operating both sides by $1/f(D, D')$, we get

$$\text{P.I.} = \frac{1}{f(D, D')}e^{ax+by} = \frac{1}{f(a, b)}e^{ax+by}$$

Case II. *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

Since $D^2 \sin(mx + ny) = -m^2 \sin(mx + ny)$

$DD' \sin(mx + ny) = -mn \sin(mx + ny)$

and $D'^2 \sin(mx + ny) = -n^2 \sin(mx + ny)$.

$\therefore$ $f(D^2, DD', D'^2) \sin(mx + ny) = f(-m^2, -mn, -n^2) \sin(mx + ny)$

Operating both sides by $1/f(D^2, DD', D'^2)$, we get

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin(mx+ny) = \frac{1}{f(-m^2, -mn, -n^2)} \sin(mx+ny)$$

Similarly about the P.I. for $\cos(mx + ny)$.

Case III. *When* $F(x, y) = x^m y^n$, m and n being constants.

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n.$$

To evaluate it, we expand $[f(D, D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and then operate on $x^m y^n$ term by term.

Case IV. *When $F(x, y)$ is any function of x and y.*

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

To evaluate it, we resolve $1/f(D, D')$ into partial fractions treating $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

where c is replaced by $y + mx$ after integration.

17.11 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots + k^n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

Its *symbolic form* is $\quad (D^n + k_1 D^{n-1} D' + \ldots + k_n D'^n) z = F(x, y)$

or briefly $\quad f(D, D') z = F(x, y)$

Step I. To find the C.F.

(*i*) *Write the A.E.*

i.e., $\quad m^n + k_1 m^{n-1} + \ldots + k_n = 0$ *and solve it for m.*

(*ii*) *Write the C.F. as follows*

Roots of A.E.	*C.F.*
1. m_1, m_2, m_3 ... (distinct roots)	$f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \ldots$
2. m_1, m_1, m_3 ... (two equal roots)	$f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_3x) + \ldots$
3. m_1, m_1, m_1 ... (three equal roots)	$f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \ldots$

Step II. To find the P.I.

From the *symbolic form*, P.I. $= \dfrac{1}{f(D, D')} F(x, y)$.

(*i*) *When* $F(x, y) = e^{ax+by}$ P.I. $= \dfrac{1}{f(D, D')} e^{ax+by}$ [Put $D = a$ and $D' = b$]

(*ii*) *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx+ny) \qquad [\text{Put } D^2 = -m^2, DD' = -mn, D'^2 = -n^2]$$

(*iii*) *When* $F(x, y) = x^m y^n$, P.I. $= \dfrac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$.

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

(*iv*) *When $F(x, y)$ is any function of x and y* P.I. $= \dfrac{1}{f(D, D')} F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D-mD'}F(x, y) = \int F(x, c-mx)\,dx \text{ where } c \text{ is replaced by } y + mx \text{ after integration.}$$

Example 17.26. *Solve $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$.* (*Madras, 2006*)

Solution. A.E. of the given equation is $m^2 + 4m - 5 = 0$ *i.e.*, $m = 1, -5$

$\therefore$ C.F. $= f_1(y + x) + f_2(y - 5x)$

$$\text{P.I.} = \frac{1}{D^2 + 4DD' - 5D'^2}\sin(2x + 3y) \quad [\text{Put } D^2 = -2^2, DD' = -2\times 3, D'^2 = -3^2$$

$$= \frac{1}{-4 + 4(-6) - 5(-9)}\sin(2x + 3y) = \frac{1}{17}\sin(2x + 3y).$$

Hence the C.S. is $z = f_1(y + x) + f_2(y - 5x) + \frac{1}{17}\sin(2x + 3y)$.

Example 17.27. *Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\partial y} = \cos x \cos 2y$.* (*Bhopal, 2008 S*)

Solution. Given equation in *symbolic form* is $(D^2 - DD')z = \cos x \cos 2y$.

Its A.E. is $m^2 - m = 0$, whence $m = 0, 1$.

$\therefore$ C.F. $= f_1(y) + f_2(y + x)$

$$\text{P.I.} = \frac{1}{D^2 - DD'}\cos x \cos 2y = \frac{1}{2}\frac{1}{D^2 - DD'}[\cos(x + 2y) + \cos(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x + 2y)\right. \qquad [\text{Put } D^2 = -1, DD' = -2]$$

$$\left. + \frac{1}{D^2 - DD'}\cos(x - 2y)\right] \qquad [\text{Put } D^2 = -1, DD' = 2]$$

$$= \frac{1}{2}\left[\frac{1}{-1+2}\cos(x + 2y) + \frac{1}{-1-2}\cos(x - 2y)\right] = \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y)$$

Hence the C.S. is $z = f_1(y) + f_2(y + x) + \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y)$.

Example 17.28. *Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2\partial y} = 2e^{2x} + 3x^2y$.* (*S.V.T.U., 2007*)

Solution. Given equation in *symbolic form* is

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$$

Its A.E. is $m^3 - 2m^2 = 0$, whence $m = 0, 0, 2$.

$\therefore$ C.F. $= f_1(y) + xf_2(y) + f_3(y + 2x)$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D'}(2e^{2x} + 3x^2y) = 2\frac{1}{D^3 - 2D^2D'}e^{2x} + 3\frac{1}{D^3(1 - 2D'/D)}x^2y$$

$$= 2\frac{1}{2^3 - 2\,.\,2^2(0)}e^{2x} + \frac{3}{D^3}(1 - 2D'/D)^{-1}x^2y = \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \ldots\right)x^2y$$

$$= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{D}x^2\,.\,1\right) = \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{3}x^3\right) \qquad \left[\because \frac{1}{D}f(x) = \int f(x)\,dx\right]$$

$$= \frac{1}{4}e^{2x} + 3y\frac{x^5}{3\,.\,4\,.\,5} + 2\,.\,\frac{x^6}{4\,.\,5\,.\,6} \qquad \left[\because \frac{1}{D^3}f(x) = \int\left[\int\left(\int f(x)\,dx\right)dx\right]dx\right]$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Hence the C.S. is $z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5y + x^6)$.

Example 17.29. *Solve* $r - 4s + 4t = e^{2x+y}$.

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+y}$.

i.e., in *symbolic form* $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$.

Its A.E. is $(m - 2)^2 = 0$, whence $m = 2, 2$.

$$\therefore \qquad \text{C.F.} = f_1(y + 2x) + xf_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

The usual rule fails because $(D - 2D')^2 = 0$ for $D = 2$ and $D' = 1$.

$\therefore$ to obtain the P.I., we find from $(D - 2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx)\, dx = \int e^{2x + (c-2x)}\, dx = xe^c = xe^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

and from $(D - 2D')z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x+(c-2x)} dy = \frac{1}{2}x^2 e^c = \frac{1}{2}x^2 e^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

Hence the C.S. is $z = f_1(y + 2x) + xf_2(y + 2x) + \frac{1}{2}x^2 e^{2x+y}$.

Example 17.30. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + y)$. *(P.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Given equation in *symbolic form* is $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$

Its A.E. is $m^2 + m - 6 = 0$ whence $m = -3, 2$.

$$\therefore \qquad \text{C.F.} = f_1(y - 3x) + f_2(y + 2x).$$

Since $D^2 + DD' - 6D'^2 = -2^2 - (2)(1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x + y) = \frac{1}{(D + 3D')(D - 2D')}\cos(2x + y)$$

$$= \frac{1}{D + 3D'}\left[\int \cos(2x + \overline{c - 2x})dx\right]_{c \to y+2x} = \frac{1}{D + 3D'}\left[\int \cos c\, dx\right]_{c \to y+2x}$$

$$[\because\ y = c - mx = c - 2x$$

$$= \frac{1}{D + 3D'} x\cos(y + 2x) = \left[\int x\cos(\overline{c + 3x} + 2x)dx\right]_{c \to y-3x} = \left[\int x\cos(5x + c)dx\right]_{c \to y-3x}$$

$$= \left[\frac{x\sin(5x + c)}{5} + \frac{\cos(5x + c)}{25}\right]_{c \to y-3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{x}{5}\sin(5x + \overline{y - 3x}) + \frac{1}{25}\cos(5x + \overline{y - 3x}) = \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

Hence the C.S. is

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y).$$

Example 17.31. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y \cos x.$ *(Anna, 2005 S ; U.P.T.U., 2003)*

or $r + s - 6t = y \cos x.$ *(Bhopal, 2008 ; S.V.T.U., 2008)*

Solution. Its *symbolic form* is $(D^2 + DD' - 6D'^2)z = y \cos x$
and the A.E. is $m^2 + m - 6 = 0$, whence $m = -3, 2$.

$$\therefore \quad \text{C.F.} = f_1(y - 3x) + f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D + 3D')} y \cos x = \frac{1}{D - 2D'}\left[\int (c + 3x) \cos x \, dx\right]_{c \to y - 3x}$$

$$[\because \ y = c - mx = c + 3x]$$

$$= \frac{1}{D - 2D'}[(c + 3x) \sin x + 3 \cos x]_{c \to y - 3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{D - 2D'}(y \sin x + 3 \cos x) = \left[\int \{(c - 2x) \sin x \; 3 \cos x\} \, dx\right]_{c \to y - 2x}$$

$$= [(c - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x]_{c \to y + 2x}$$

$$= -y \cos x + \sin x$$

Hence the C.S. is $z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x$.

Example 17.32. *Solve* $4\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 16 \log (x + 2y).$

Solution. Its *symbolic form* is $4D^2 - 4DD' + D'^2 = 16 \log (x + 2y)$
and the A.E. is $4m^2 - 4m + 1 = 0$, $m = 1/2, 1/2$.

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$$

$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log (x + 2y) = 4\frac{1}{\left(D - \frac{1}{2}D'\right)}\left\{\frac{1}{D - \frac{1}{2}D'} \log (x + 2y)\right\}$$

$$= 4\frac{1}{D - \frac{1}{2}D'}\left[\int \log\left\{x + 2\left(c - \frac{x}{2}\right)\right\} dx\right]_{c \to y + x/2} \qquad [\because \ y = c - mx = c - x/2]$$

$$= 4\frac{1}{D - \frac{1}{2}D'}\left[\int \log (2c) \, dx\right]_{c \to y + x/2} = 4\frac{1}{D - \frac{1}{2}D'}[x \log (x + 2y)]$$

$$= 4\left[\int \left\{x \log\left[x + 2\left(c - \frac{x}{2}\right)\right]\right\} dx\right]_{c \to y + x/2} = 4\left[\log 2c \int x \, dx\right]_{c \to y + x/2} = 2x^2 \log (x + 2y)$$

Hence the C.S. is $z = f_1\left(y + \dfrac{x}{2}\right) + x f_2\left(y + \dfrac{x}{2}\right) + 2x^2 \log (x + 2y)$.

PROBLEMS 17.6

Solve the following equations :

1. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 x}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 0.$

2. $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x + 2y}.$ *(Burdwan, 2003)*

3. $(D^2 - 2DD' + D'^2)z = e^{x + y}.$ *(Bhopal, 2007)*

4. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 5\dfrac{\partial^3 z}{\partial x \partial y^2} - 2\dfrac{\partial^3 z}{\partial y^3} = e^{2x + y}.$ *(Bhopal, 2008)*

5. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x.$ (*P.T.U., 2009 S*)

6. $\frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = E \sin pt.$

7. $\frac{\partial^3 z}{\partial x^3} - \frac{4\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2 \sin (3x + 2y).$ (*S.V.T.U., 2007*)

8. $(D^3 - 7DD'^2 - 6D'^3)z = \cos (x + 2y) + 4.$ (*Anna, 2008*)

9. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos (x + 2y).$ (*U.P.T.U., 2006*)

10. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y.$ (*U.P.T.U., 2003*)

11. $(D^2 - DD')z = \cos 2y (\sin x + \cos x).$

12. $(D^2 - D'^2)z = e^{x-y} \sin (x + 2y).$ (*Anna, 2009*)

13. $(D^2 + 3DD' + 2D'^2)z = 24xy.$

14. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2.$

15. $(D^2 - DD' - 2D'^2)z = (y - 1)e^x.$ (*Bhopal, 2006*)

16. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y.$

17. $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y.$ (*P.T.U., 2005*)

17.12 NON-HOMOGENEOUS LINEAR EQUATIONS

If in the equation $\quad f(D, D')z = F(x, y) \quad$ ...(1)

the polynomial expression $f(D, D')$ is not homogeneous, then (1) is a non-homogeneous linear partial differential equation. As in the case of homogeneous linear partial differential equations, its complete solution = C.F. + P.I.

The methods to find P.I. are the same as those for homogeneous linear equations.

To find the C.F., we factorize $f(D, D')$ into factors of the form $D - mD' - c$. To find the solution of $(D - mD' - c)z = 0$, we write it as $p - mq = cz$ ...(2)

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

Its integrals are $y + mx = a$ and $z = be^{cx}$.

Taking $b = \phi(a)$, we get $z = e^{cx} \phi(y + mx)$

as the solution of (2). The solution corresponding to various factors added up, give the C.F. of (1).

Example 17.32. *Solve* $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin (x + 2y).$ (*U.P.T.U., 2004*)

Solution. Here $\quad f(D, D') = (D + D')(D + D' - 2)$

Since the solution corresponding to the factor $D - mD' - c$ is known to be

$$z = e^{cx} \phi(y + mx)$$

$$\therefore \quad \text{C.F.} = \phi_1(y - x) + e^{2x} f_2(y - x)$$

$$\therefore \quad \text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin (x + 2y)$$

$$= \frac{1}{-1 + 2(-2) + (-4) - 2D - 2D'} \sin (x + 2y)$$

$$= -\frac{1}{2(D + D') + 9} \sin (x + 2y) = -\frac{2(D + D') - 9}{4(D^2 + 2DD' + D'^2) - 81} \sin (x + 2y)$$

$$= \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)]$$

Hence the complete solution is

$$z = \phi_1(y - x) + e^{2x} \phi_2(y - x) + \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)].$$

PROBLEMS 17.7

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial y}-z=e^{-x}$.
2. $(D-D'-1)(D-D'-2)z=e^{2x-y}$.
3. $(D+D'-1)(D+2D'-3)z=4+3x+6y$.
4. $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial y}=x^2+y^2$. (*Madras, 2000 S*)
5. $(D^2+DD'+D'-1)z=\sin(x+2y)$. (*S.V.T.U., 2009*)
6. $(2DD'+D'^2-3D')z=3\cos(3x-2y)$.

17.13 NON-LINEAR EQUATIONS OF THE SECOND ORDER

We now give a method due to *Monge**, for integrating the equation $Rr + Ss + Tt = V$ ...(1)
in which R, S, T, V are functions of x, y, z, p and q.

Since $$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + tdy, \text{ and } dq = sdx + tdy,$$

we have $$r = (dp - tdy)/dx \text{ and } t = (dq - sdx)/dy.$$

Substituting these values of r and t in (1), and rearranging the terms, we get

$$(Rdpdy + Tdqdx - Vdxdy) - s(Rdy^2 - Sdydx + Tdx^2) = 0 \qquad ...(2)$$

Let us consider the equations

$$Rdy^2 - Sdydx + Tdx^2 = 0 \qquad ...(3)$$

$$Rdpdy + Tdqdx - Vdxdy = 0 \qquad ...(4)$$

which are known as *Monge's equations*.

Since (3) can be factorised, we obtain its integral first. In case the factors are different, we may get two distinct integrals of (3). Either of these together with (4) will give an integral of (4). If need be, we may also use the relation $dz = pdx + qdy$ while solving (3) and (4).

Let $u(x, y, z, p, q) = a$ and $v(x, y, z, p, q) = b$ be the integrals of (3) and (4) respectively. Then $u = a$, $v = b$ evidently constitute a solution of (2) and therefore, of (1) also. Taking $b = \phi(a)$, we find a general solution of (1) to be $v = \phi(u)$, which should be further integrated by methods of first order equations.

Example 17.34. *Solve* $(x-y)(xr - xs - ys + yt) = (x+y)(p-q)$. (*S.V.T.U., 2007*)

Solution. Monge's equations are

$$xdy^2 + (x+y)dy\,dx + ydx^2 = 0 \qquad ...(i)$$

$$xdpdy + ydqdx - \frac{x+y}{x-y}(p-q)\,dydx = 0 \qquad ...(ii)$$

(*i*) may be factorised as $(xdy + ydx)(dx + dy) = 0$ whose integrals are $xy = c$ and $x + y = c$.

Taking $xy = c$ and dividing each term of (*ii*) by xdy or its equivalent $-ydx$, we get

$$dp - dq - \frac{dx-dy}{x-y}(p-q) = 0 \quad \text{or} \quad \frac{d(p-q)}{p-q} - \frac{d(x-y)}{x-y} = 0$$

This gives on integration $(p-q)/(x-y) = c$.

Hence a first integral of the given equation is $p - q = (x-y)\,\phi(xy)$ which is a Lagrange's linear equation. Its subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x-y)\phi(xy)}$$

From the first two equations, we have $x + y = a$

Using this, we have

$$dz = -\phi(ax - x^2)\,.\,(a - 2x)\,dx \quad \text{which gives } z = \phi_1(ax - x^2) + b$$

Writing $b = \phi_2(a)$ and $a = x + y$, we get

$$z = \phi_1(xy) + \phi_2(x+y).$$

* Named after *Gaspard Monge* (1746–1818), Professor at Paris.

Obs. Had we started with the integral $x + y = c$ and divided each term of (*ii*) by dx or $-dy$, we would have arrived at the same solution.

Example 17.35. *Solve* $y^2r - 2ys + t = p + 6y$. (*Osmania, 2002*)

Solution. Monge's equations are $y^2dy^2 + 2ydydx + dx^2 = 0$ ...(*i*)

and $$y^2dpdy + dqdx - (p + 6y)dydx = 0 \quad ...(ii)$$

(*i*) gives $$(ydy + dx)^2 = 0 \text{ i.e. } y^2 + 2x = c \quad ...(iii)$$

Putting $ydy = -dx$ in (*ii*), we get

$$ydp - dq + (p + 6y)dy = 0 \quad \text{or} \quad (ydp + pdy) - dq + 6ydy = 0$$

whose integral is $$py - q + 3y^2 = a$$

Combining this with (*iii*), we get the integral $py - q + 3y^2 = \phi(y^2 + 2x)$

The subsidiary equations for this Lagrange's linear equation are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\phi(y^2 + 2x) - 3y^2}$$

From the first two equations, we have $y^2 + 2x = c$

Using this, we have $dz + [\phi(c) - 3y^2]\, dy = 0$

whose solution is $$z + y\phi(c) - y^3 = b.$$

Hence the required solution is $z = y^3 - y\phi(y^2 + 2x) + \psi(y^2 + 2x)$.

PROBLEMS 17.8

Solve :

1. $(q + 1)s = (p + 1)t$.
2. $r - t\cos^2 x + p \tan x = 0$.
3. $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$. (*J.N.T.U., 2006*)
4. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$.
5. $q^2r - 2pqs + p^2t = pq^2$.
6. $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$.

17.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 17.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. The equation $\dfrac{\partial^2 z}{\partial x^2} + 2xy\left(\dfrac{\partial z}{\partial x}\right)^2 + \dfrac{\partial z}{\partial y} = 5$ is of order .... and degree ....
2. The complementary function of $(D^2 - 4DD' + 4D'^2)z = x + y$ is ......
3. The solution of $\dfrac{\partial^2 z}{\partial y^2} = \sin(xy)$ is ......
4. A solution of $(y - z)p + (z - x)q = x - y$ is ....
5. The particular integral of $(D^2 + DD')z = \sin(x + y)$ is ....
6. The partial differential equation obtained from $z = ax + by + ab$ by eliminating a and b is ......
7. Solution of $\sqrt{p} + \sqrt{q} = 1$ is ......
8. Solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ is ......
9. Solution of $p - q = \log(x + y)$.
10. The order of the partial differential equation obtained by eliminating f from $z = f(x^2 + y^2)$, is ......
11. The solution of $x\, \partial z/\partial x = 2x + y$ is
12. By eliminating a and b from $z = a(x + y) + b$, the p.d.e. formed is ......
13. The solution of $[D^3 - 3D^2D' + 2DD'^2]z = 0$ is ......
14. By eliminating the arbitrary constants from $z = a^2x + ay^2 + b$, the partial differential equation formed is ...... (*Anna, 2008*)
15. A solution of $u_{xy} = 0$ is of the form ....
16. If $u = x^2 + t^2$ is a solution of $c^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, then $c =$ ......

17. The general solution of $u_{xx} = xy$ is ....

18. The complementary function of $r - 7s + 6t = e^{x+y}$ is ....

19. The solution of $xp + yq = z$ is

(i) $f(x^2, y^2) = 0$ (ii) $f(xy, yz)$ (iii) $f(x, y) = 0$ (iv) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

20. The solution of $(y - z)p + (z - x)q = x - y$, is

(i) $f(x^2 + y^2 + z^2) = xyz$ (ii) $f(x + y + z) = xyz$

(iii) $f(x + y + z) = x^2 + y^2 + z^2$ (iv) $f(x^2 + y^2 + z^2, xyz) = 0$.

21. The partial differential equation from $z = (c + x)^2 + y$ is

(i) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (ii) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$ (iii) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (iv) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$.

22. The solution of $p + q = z$ is

(i) $f(xy, y \log z) = 0$ (ii) $f(x + y, y + \log z) = 0$

(iii) $f(x - y, y - \log z) = 0$ (iv) None of these.

23. Particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$ is

(i) $\frac{1}{2}e^{x+2y}$ (ii) $-\frac{x}{2}e^{x+2y}$ (iii) xe^{x+2y} (iv) x^2e^{x+2y}.

24. The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(i) $z = (1 + x + x^2) f(y)$ (ii) $z = (1 + y + y^2) f(x)$

(iii) $z = f_1(x) + y f_2(x) + y^2 f_3(x)$ (iv) $z = f_1(y) + x f_2(y) + x^2 f_3(y)$.

25. Particular integral of $(D^2 - D'^2)z = \cos(x + y)$ is

(i) $x \cos(x + y)$ (ii) $\frac{x}{2}\cos(x + y)$ (iii) $x \sin(x + y)$ (iv) $\frac{x}{2}\sin(x + y)$

26. The solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ is

(i) $z = f_1(y + x) + f_2(y - x)$ (ii) $z = f_1(y + x) + f_1(y - x)$

(iii) $z = f(x^2 - y^2)$ (iv) $z = f(x^2 + y^2)$.

27. $xu_x + yu_y = u^2$ is a non-linear partial differential equation. (True or False)

28. $xu_x + u_{xx} = 0$ is a non-linear partial differential equation. (True or False)

29. $u = x^2 - y^2$ is a solution of $u_{xx} + u_{yy} = 0$. (True or False)

30. $u = e^{-t} \sin x$ is a solution of $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$. (True or False)

31. $x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 2u$ is an ordinary differential equation. (True or False)

CHAPTER 18

Applications of Partial Differential Equations

18.1 INTRODUCTION

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem*.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions. Most of the boundary value problems involving linear partial differential equations can be solved by the following method.

18.2 METHOD OF SEPARATION OF VARIABLES

It involves a solution which breaks up into a product of functions each of which contains only one of the variables. The following example explains this method :

Example 18.1. *Solve (by the method of separation of variables) :*

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0. \qquad \textit{(P.T.U., 2009 S ; Bhopal 2008 ; U.P.T.U., 2005)}$$

Solution. Assume the trial solution $z = X(x)Y(y)$ ...(*i*)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \qquad \text{where } X' = \frac{dX}{dx}, Y' = \frac{dY}{dy} \text{ etc.}$$

Separating the variables, we get $\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}$ ...(*ii*)

Since x and y are independent variables, therefore, (*ii*) can only be true if each side is equal to the same constant, a (say).

$\therefore \qquad \dfrac{X'' - 2X'}{X} = a$, i.e. $X'' - 2X' - aX = 0$ ...(iii)

and $\qquad -Y'/Y = a, \quad i.e., \quad Y' + aY = 0$ ...(iv)

To solve the ordinary linear equation (*iii*), the auxiliary equation is

$$m^2 - 2m - a = 0, \quad \text{whence } m = 1 \sqrt{(1 + a)}.$$

$\therefore$ the solution of (*iii*) is $\quad X = c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}$

and the solution of (*iv*) is $\qquad Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (*i*), we get

$$z = \{c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}\} \cdot c_3 e^{-ay}$$

i.e., $$z = \{k_1 e^{[1 + \sqrt{(1+a)}]x} + k_2 e^{[1 - \sqrt{(1+a)}]x}\} \, e^{-ay}$$

which is the required complete solution.

Obs. In practical problems, the unknown constants a, k_1, k_2 are determined from the given boundary conditions.

Example 18.2. *Using the method of separation of variables, solve* $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ *where* $u(x, 0) = 6e^{-3x}$.

(V.T.U., 2009 ; Kurukshetra, 2006 ; Kerala, 2005)

Solution. Assume the solution $u(x, t) = X(x)T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X' - X)\,T = 2XT'$$

or $$\frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$\therefore \qquad X' - X - 2kX = 0 \quad \text{or} \quad \dfrac{X'}{X} = 1 + 2k \qquad ...(i) \qquad \text{and} \qquad \dfrac{T'}{T} = k$ ...(ii)

Solving (*i*), $\qquad \log X = (1 + 2k)x + \log c \quad \text{or} \quad X = ce^{(1 + 2k)x}$

From (*ii*), $\qquad \log T = kt + \log c' \quad \text{or} \quad T = c'e^{kt}$

Thus $\qquad u(x, t) = XT = cc'\, e^{(1 + 2k)x}\, e^{kt}$ ...(iii)

Now $\qquad 6e^{-3x} = u(x, 0) = cc'\, e^{(1 + 2k)x}$

$\therefore \qquad cc' = 6 \text{ and } 1 + 2k = -3 \quad \text{or} \quad k = -2$

Substituting these values in (*iii*), we get

$u = 6e^{-3x}\, e^{-2t} \quad i.e., \quad u = 6e^{-(3x + 2t)}$ which is the required solution.

PROBLEMS 18.1

Solve the following equations by the method of *separation of variables* :

1. $py^3 + qx^2 = 0$. (*V.T.U., 2011 ; S.V.T.U., 2008*)
2. $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} = 0$. (*V.T.U., 2008*)
3. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$. (*J.N.T.U., 2006*)
4. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x = 0$. (*S.V.T.U., 2008*)
5. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$. (*V.T.U., 2008 S*)
6. Find a solution of the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u$ in the form $u = f(x)\,g(y)$. Solve the equation subject to the conditions $u = 0$ and $\partial u/\partial x = 1 + e^{-3y}$, when $x = 0$ for all values of y. (*Andhra, 2000*)

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

A number of problems in engineering give rise to the following well-known partial differential equations :

(*i*) *Wave equation* : $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

(*ii*) *One dimensional heat flow equation* : $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

(*iii*) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation* : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(*iv*) *Transmission line equations.*

(*v*) *Vibrating membrane.* Two dimensional wave equation.

(*vi*) *Laplace's equation* in three dimensions.

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solution gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING—WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

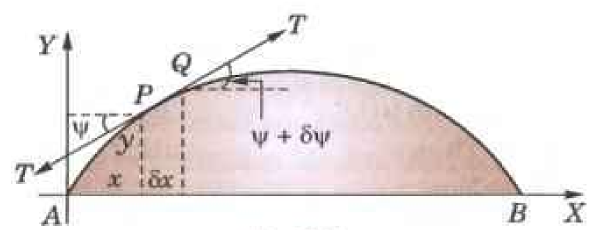


Fig. 18.1

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis ; so that the motion takes place entirely in the xy-plane. Figure 18.1 shows the string in the position APB at time t. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element.

$$= T \sin(\psi + \delta\psi) - T \sin\psi = T[\sin(\psi + \delta\psi) - \sin\psi]$$

$$= T\,[\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \,.\, \frac{\partial^2 y}{\partial t^2} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x\right] \quad i.e., \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x}{\delta x}\right]$$

Taking limits as $Q \to P$ *i.e.*, $dx \to 0$, we have $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, where $c^2 = \dfrac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $$\frac{\partial^2 y}{\partial t^2} = X \, . \, T'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X'' \, . \, T$$

Substituting these in (1), we get $XT'' = c^2 X'' T$ i.e., $\dfrac{X''}{X} = \dfrac{1}{c^2} \dfrac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2 X}{dx^2} - kX = 0 \quad ...(3) \quad \text{and} \quad \frac{d^2 T}{dt^2} - kc^2 T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say $X = c_1 e^{px} + c_2 e^{-px}$; $T = c_3 e^{cpt} + c_4 e^{-cpt}$.

(*ii*) *When k is negative* and $= -p^2$ say $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(*iii*) *When k is zero.* $X = c_9 x + c_{10}$; $T = c_{11} t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}) \quad ...(5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad ...(6)$$

$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.*, of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(8)$$

is the only suitable solution of the *wave equation*. *(Bhopal, 2008)*

Example 18.3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin (\pi x / l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$y(x, t) = a \sin (\pi x / l) \cos (\pi ct / l)$. *(V.T.U., 2010 ; S.V.T.U., 2008 ; Kerala, 2005 ; U.P.T.U., 2004)*

Solution. The vibration of the string is given by $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad ...(ii) \quad \text{and} \quad y(l, t) = 0 \quad ...(iii)$$

Since the initial transverse velocity of any point of the string is zero,

therefore, $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(iv)$$

Also $$y(x, 0) = a \sin (\pi x / l) \quad ...(v)$$

Now we have to solve (*i*) subject to the *boundary conditions* (*ii*) and (*iii*) and *initial conditions* (*iv*) and (*v*). Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(vi)$$

By (*ii*), $$y(0, t) = C_1 (C_3 \cos cpt + C_4 \sin cpt) = 0$$

For this to be true for all time, $C_1 = 0$.

Hence $$y(x, t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt) \quad ...(vii)$$

and $$\frac{\partial y}{\partial t} = C_2 \sin px \, \{C_3(-cp \, . \sin cpt) + C_4(cp \, . \cos cpt)\}$$

$\therefore$ By (*iv*), $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \, . \, (C_4 \, cp) = 0, \text{ whence } C_2 C_4 cp = 0.$$

If $C_2 = 0$, (*vii*) will lead to the trivial solution $y(x, t) = 0$,

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (*vii*) becomes $y(x, t) = C_2 C_3 \sin px \cos cpt$ ...(*viii*)

$\therefore$ By (*iii*), $y(l, t) = C_2C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. $\therefore$ $pl = n\pi$, *i.e.*, $p = n\pi/l$, where n is an integer.

Hence (*i*) reduces to $y(x, t) = C_2C_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$.

[These are the solutions of (*i*) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its **spectrum.**]

Finally, imposing the last condition (*v*), we have $y(x, 0) = C_2C_3 \sin \dfrac{n\pi x}{l} = a \sin \dfrac{\pi x}{l}$

which will be satisfied by taking $C_2C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l}$ ...(*ix*)

Obs. We have from (*ix*) $\dfrac{\partial^2 y}{\partial t^2} = -a\left(\dfrac{\pi c}{l}\right)^2 \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l} = -\left(\dfrac{\pi c}{l}\right)^2 y.$

This shows that the motion of each point $y(x, t)$ of the string is simple harmonic with period $= 2\pi/(\pi c/l)$, *i.e.*, $2l/c$.

Thus we can look upon (*ix*) as a sine wave $y = y_0 \sin (\pi x/l)$ of wave length l, wave-velocity c and amplitude $y_0 = a \cos (\pi ct/l)$ which varies harmonically with time t. Whatever t may be, $y = 0$ when $x = 0, l, 2l, 3l$ etc. and these points called *nodes*, remain undisturbed during wave motion. Thus (*ix*) represents a *stationary sine wave* of varying amplitudes whose frequency is $c/2l$. Such waves often occur in electrical and mechanical vibratory systems.

Example 18.4. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.*

(*Rajasthan, 2006 ; V.T.U., 2003 ; J.N.T.U., 2002*)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = y_0 \sin^3 \left(\dfrac{\pi x}{l}\right)$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \ldots(iv)$$

Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*), $y(l, t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{cn\pi t}{l}\right) \quad \ldots(v)$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l}\right)\frac{cn\pi}{l}\left(-c_3 \sin \frac{cn\pi t}{l} + c_4 \cos \frac{cn\pi t}{l}\right)$$

By (*iv*), $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \dfrac{n\pi x}{l}\right)\dfrac{cn\pi}{l} \cdot c_4 = 0$, *i.e.* $c_4 = 0$.

Thus (*v*) becomes $y(x, t) = c_2c_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l} = b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$

Adding all such solutions the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \ldots(vi)$$

$\therefore$ from (*iii*), $$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or $$y_0 \left\{ \frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right\} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

Comparing both sides, we have

$$b_1 = 3\,y_0/4,\ b_2 = 0,\ b_3 = -\,y_0/4,\ b_4 = b_5 = \ldots = 0.$$

Hence from (*vi*), the desired solution is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

Example 18.5. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.*

(Bhopal, 2008 ; Madras, 2006 ; J.N.T.U., 2005 ; P.T.U., 2005)

Solution. The equation of the string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = \mu x(l - x)$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \text{...(iv)}$$

The solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $$y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $$y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$$

Also by (*ii*) $$y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0 \text{ for all } t.$$

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right) \qquad \text{...(v)}$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \left(-c_3 \sin \frac{n\pi ct}{l} + c_4 \cos \frac{n\pi ct}{l} \right)$$

$\therefore$ by (*iv*) $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \,.\, c_4 = 0$$

Thus (*v*) becomes $$y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Adding all such solutions, the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \text{...(vi)}$$

From (*iii*), $$\mu(lx - x^2) = y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l} \int_0^l \mu(lx - x^2) \sin \frac{n\pi x}{l}\, dx, \text{ by Fourier half-range sine series}$$

$$= \frac{2\mu}{l} \left\{ \left| (lx - x^2)\left(-\frac{\cos n\pi x/l}{n\pi/l} \right) \right|_0^l - \int_0^l (l - 2x) \left(-\frac{\cos n\pi x/l}{n\pi/l} \right) dx \right\}$$

$$= \frac{2\mu}{l} \cdot \frac{1}{n\pi} \left\{ \int_0^l (l - 2x) \frac{\cos n\pi x}{l} dx \right\} = \frac{2\mu}{n\pi} \left\{ \left| (l - 2x) \frac{\sin n\pi x/l}{n\pi/l} \right|_0^l - \int_0^l (-2) \frac{\sin n\pi x/l}{n\pi/l} dx \right\}$$

$$= \frac{2\mu}{n\pi} \cdot \frac{2l}{n\pi} \int_0^l \sin \frac{n\pi x}{l} dx = \frac{4\mu l}{n^2\pi^2} \left| \frac{-\cos n\pi x/l}{n\pi/l} \right|_0^l = \frac{4\mu l^2}{n^3\pi^3} \{1 - (-1)^n\}$$

Hence from (vi), the desired solution is

$$y(x, t) = \frac{4\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$= \frac{8\mu l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi}{l} x \cos \frac{(2m-1)\pi ct}{l}.$$

Example 18.6. *A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement $y(x, t)$.*

(S.V.T.U., 2008 ; V.T.U., 2008 ; U.P.T.U., 2006)

Solution. The equation of the vibrating string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$ ...(iv)

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\, (c_3 \cos cpt + c_4 \sin cpt)$

Also $y(l, t) = c_2 \sin pl\, (c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = \frac{n\pi}{l}$, n being an integer.

Thus $y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t \right)$

By (iii), $0 = c_2 c_3 \sin \frac{n\pi x}{l}$ for all x *i.e.*, $c_2 c_3 = 0$

$\therefore$ $y(x, t) = b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$ where $b_n = c_2 c_4$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l} \qquad ...(v)$$

Now $\frac{\partial y}{\partial t} = \sum b_n \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l} \cos \frac{cn\pi t}{l}$

By (iv), $v_0 \sin^3 \frac{\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$

or $\frac{v_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$ $[\because \ \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta]$

$$= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + ...$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \ldots$$

$$\therefore \quad b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_3 = \ldots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9 \sin \frac{\pi x}{l} \sin \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \sin \frac{3c\pi t}{l}\right).$$

Example 18.7. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.* (Anna, 2009 ; U.P.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l - x) \qquad ...(iv)$$

As in example 18.6, the general solution of (i) satisfying the conditions (ii) and (iii) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . \sin \frac{n\pi ct}{l} \qquad ...(v)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . \cos \frac{n\pi ct}{l} . \left(\frac{n\pi c}{l}\right)$$

By (iv), $$\lambda x(l - x) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{l}$$

$$\therefore \quad \frac{\pi c}{l} nb_n = \frac{2}{l}\int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2\lambda}{l}\left| (lx - x^2)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}[1 - (-1)^n]$$

or $$b_n = \frac{4\lambda l^3}{c\pi^4 n^4}[1 - (-1)^n] = \frac{8\lambda l^3}{c\pi^4 (2m - 1)^4} \text{ taking } n = 2m - 1.$$

Hence, from (v), the desired solution is

$$y = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^4} \sin \frac{(2m - 1)\pi x}{l} \sin \frac{(2m - 1)\pi ct}{l}.$$

Example 18.8. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.* (Kerala, 2005)

Solution. Let B and C be the points of the trisection of the string $OA (= l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$ (say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(i)$$

and the boundary conditions are

$$y(0, t) = 0 \qquad ...(ii)$$

$$y(l, t) = 0 \qquad ...(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad ...(iv)$$

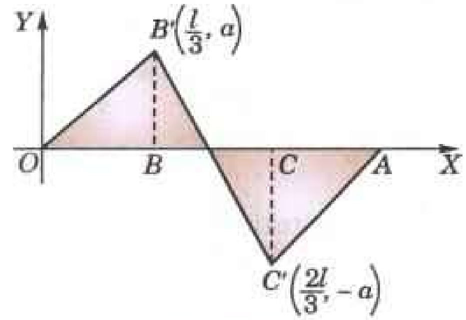


Fig. 18.2

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)\, x$;

the equation of $B'C'$ is $\qquad y - a = \dfrac{-2a}{(l/3)}\left(x - \dfrac{l}{3}\right)$, i.e., $y = \dfrac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $\qquad y = \dfrac{3a}{l}(x - l)$

Hence the fourth boundary condition is

$$\left.\begin{aligned} y(x, 0) &= \frac{3a}{l}x, \; 0 \le x \le \frac{l}{3} \\ &= \frac{3a}{l}(l - 2x), \; \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3a}{l}(x - l), \; \frac{2l}{3} \le x \le l \end{aligned}\right\} \qquad ...(v)$$

As in example 18.6, the solution of (i) satisfying the boundary conditions (ii), (iii) and (iv), is

$$y(x, t) = b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Where } b_n = C_2C_3]$$

Adding all such solutions, the most general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad ...(vi)$$

Putting $t = 0$, we have $y(x, 0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ $\qquad ...(vii)$

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ by (1) of § 10.7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l - 2x)\sin\frac{n\pi x}{l}\,dx + \int_{2l/3}^{l} \frac{3a}{l}(x - l)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left[\left| x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_0^{l/3}\right.$$

$$+ \left|(l - 2x)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{l/3}^{2l/3}$$

$$\left. + \left|(x - l)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1).\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2}\sin\frac{n\pi}{3}[1 + (-1)^n] \qquad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n \sin\frac{n\pi}{3}\right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (vi) gives

$$y(x, t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Take } n = 2m]$$

$$= \frac{9a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\frac{2m\pi}{3}\sin\frac{2m\pi x}{l}\cos\frac{2m\pi ct}{l} \qquad \ldots(vii)$$

Putting $x = l/2$ in (vii), we find that the displacement of the mid-point of the string, *i.e.* $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v.

Then $\dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v}$

and $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly, $$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting in (1), we get $\dfrac{\partial^2 y}{\partial u \partial v} = 0$ ...(2)

Integrating (2) w.r.t. v, we get $\dfrac{\partial y}{\partial u} = f(u)$ ...(3)

where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u, we obtain

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e. $$y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \ldots(4)$$

This is the *general solution of the wave equation* (1).

Now to determine ϕ and ψ, suppose initially $u(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

Differentiating (4) w.r.t. t, we get $\dfrac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

At $t = 0$, $$\phi'(x) = \psi'(x) \qquad \ldots(5)$$

and $$y(x, 0) = \phi(x) + \psi(x) = f(x) \qquad \ldots(6)$$

(5) gives, $\phi(x) = \psi(x) + k$

$\therefore$ (6) becomes $2\psi(x) + k = f(x)$

or $$\psi(x) = \frac{1}{2}[f(x) - k] \text{ and } \phi(x) = \frac{1}{2}[f(x) + k]$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2}[f(x+ct)+k] + \frac{1}{2}[f(x-ct)-k] = f(x+ct)+f(x-ct) \quad ...(7)$$

which is the *d'Alembert's solution* of the wave equation* (1) (*V.T.U., 2011 S*)

Obs. The above solution gives a very useful method of solving partial differential equations by change of variables.

Example 18.9. *Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.* (*V.T.U., 2011*)

Solution. By d'Alembert's method, the solution is

$$y(x, t) = \frac{1}{2}[f(x+ct)+f(x-ct)]$$

$$= \frac{1}{2}[k\{\sin(x+ct) - \sin 2(x+ct)\} + k\{\sin(x-ct) - \sin 2(x-ct)\}]$$

$$= k[\sin x \cos ct - \sin 2x \cos 2ct]$$

Also $\quad y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\quad \partial y(x, 0)/\partial t = k(-c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t=0} = 0$

i.e., the given boundary conditions are satisfied.

PROBLEMS 18.2

1. Solve completely the equation $\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2$, representing the vibrations of a string of length l, fixed at both ends, given that $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$, $0 < x < l$. (*Bhopal, 2007 S ; U.P.T.U., 2005*)
2. Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ under the conditions $u(0, t) = 0$, $u(l, t) = 0$ for all t ; $u(x, 0) = f(x)$ and $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 < x < l$.
3. Find the solution of the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection
$$f(x) = \frac{2k}{l}x \text{ when } 0 < x < \frac{l}{2}, = \frac{2k}{l}(l-x) \text{ when } \frac{l}{2} < x < l,$$
and initial velocity zero. (*Bhopal, 2006 ; Kerala, M.E., 2005*)
4. A tightly stretched string of length l has its ends fastened at $x = 0$, $x = l$. The mid-point of the string is then taken to height h and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release. (*Anna, 2005*)
5. A tightly stretched string with fixed end points at $x = 0$ and $x = 1$, is initially in a position given by
$$f(x) = \begin{cases} x, 0 \le x \le \dfrac{1}{2} \\ 1-x, \dfrac{1}{2} \le x \le 1 \end{cases}$$
If it is released from this position with velocity a, perpendicular to the x-axis, show that the displacement $u(x, t)$ at any point x of the string at any time $t > 0$, is given by
$$u(x,t) = \frac{4\sqrt{2}}{\pi^2}\left[\sum_{n=1}^{\infty}\left\{\frac{\sin[(4pi-3)\pi x]\cos[(4pi-3)\pi at - \pi/4]}{(4n-3)^2} - \frac{\sin[(4pi-1)\pi x]\cos[(4pi-1)\pi at - \pi/4]}{(4n-1)^2}\right\}\right]$$
6. If a string of length l is initially at rest in equilibrium position and each of its points is given a velocity v such that $v = cx$ for $< x < l/2$
$c(l \,.\, x)$ for $l/2 < x < l$, determine the displacement $y(x, t)$ at anytime t. (*Anna, 2008*)
7. Using d'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection :
(*i*) $f(x) = a(x - x^2)$ (*Kerala, M. Tech., 2005*) (*ii*) $f(x) = a \sin^2 \pi x$.

*See footnote of p. 373.

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section α(cm^2). Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of flow as the positive x-axis (Fig. 18.3). Let ρ be the density (gr/cm^3), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

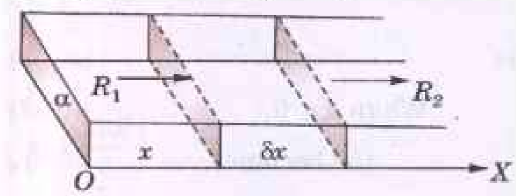


Fig. 18.3

Let $u(x, -t)$ be the temperature at a distance x from O. If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (*ii*) p. 466, the quantity of heat in this slab $= s\rho\alpha\ \delta x \delta u$. Hence the rate of increase of heat in this slab, *i.e.*, $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.

Now by (*A*) of p. 466, $R_1 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x$ and $R_2 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$

the negative sign appearing as a result of (*i*) on p. 466.

Hence $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x + k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$ *i.e.*, $\dfrac{\partial u}{\partial t} = \dfrac{k}{s\rho}\left\{\dfrac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right\}$

Writing $k/s\rho = c^2$, called the *diffusivity* of the substance (cm^2/sec.), and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} \qquad ...(1)$$

This is the *one-dimensional heat-flow equation.* (*V.T.U., 2011*)

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x) \,.\, T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2X''T, \text{ i.e., } X''/X = T'/c^2T \qquad ...(2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3) \qquad \text{and} \qquad \frac{dT}{dt} - kc^2T = 0 \qquad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$X = c_1e^{px} + c_2e^{-px},\ T = c_3\, e^{c^2p^2t} ;$$

(*ii*) *When k is negative* and $= -p^2$, say :

$$X = c_4 \cos px + c_5 \sin px,\ T = c_6\, e^{-c^2p^2t} ;$$

(*iii*) *When k is zero :*

$$X = c_7x + c_8,\ T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1e^{px} + c_2e^{-px})\, c_3e^{c^2p^2t} \qquad ...(5)$$

$$u = (c_4 \cos px + c_5 \sin px)\, c_6e^{-c^2p^2t} \qquad ...(6)$$

$$u = (c_7x + c_8)c_9 \qquad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.*, of the form

$$u = (C_1 \cos px + C_2 \sin px)\, e^{-c^2p^2t} \qquad ...(8)$$

is the only suitable solution of the heat equation.

Example 18.10. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ *and* $u(1, t) = 0$, *where* $0 < x < 1$, $t > 0$.

Solution. The solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

is $$u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-p^2 t} \quad ...(ii)$$

When $x = 0$, $u(0, t) = c_1 e^{-p^2 t} = 0$ *i.e.*, $c_1 = 0$.

$\therefore$ (*ii*) becomes $u(x, t) = c_2 \sin px e^{-p^2 t}$ ...(*iii*)

When $x = 1$, $u(1, t) = c_2 \sin p \,.\, e^{-p^2 t} = 0$ or $\sin p = 0$

i.e., $p = n\pi$.

$\therefore$ (*iii*) reduces to $u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x$ where $b_n = c_2$

Thus the general solution of (*i*) is $u(x, t) = \sum b_n e^{-n^2\pi^2 t} \sin n\pi x$ ...(*iv*)

When $t = 0$, $3 \sin n\pi x = u(0, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x$

Comparing both sides, $b_n = 3$

Hence from (*iv*), the desired solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

Example 18.11. *Solve the differential equation* $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(*i*) *u is not infinite for* $t \to \infty$, (*ii*) $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$,

(*ii*) $u = lx - x^2$ *for* $t = 0$, *between* $x = 0$ *and* $x = l$. (*P.T.U., 2007*)

Solution. Substituting $u = X(x)T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad i.e., \quad X''/X = \frac{T'}{\alpha^2 T} = -k^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2X}{dx^2} + k^2 X = 0 \text{ and } \frac{dT}{dt} + k^2\alpha^2 T = 0 \quad ...(1)$$

Their solutions are $X = c_1 \cos kx + c_2 \sin kx$, $T = c_3 e^{-k^2\alpha^2 t}$ ...(2)

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4 e^{kx} + c_5 e^{-kx},\ T = c_6 e^{k^2\alpha^2 t} \quad ...(3)$$

If $k^2 = 0$, the solutions are $X = c_7 x + c_8$, $T = c_9$ ...(4)

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ *i.e.*, the given condition (*i*) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (*ii*) to (4), we get $c_7 = 0$.

$$\therefore \quad u = XT = c_8 c_9 = a_0 \quad \text{(say)} \quad ...(5)$$

From (2), $$\frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx)\, kc_3 e^{-k^2\alpha^2 t}$$

Applying the condition (*ii*), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e., $c_2 = 0$ and $kl = n\pi$ (n an integer)

$$\therefore \quad u = c_1 \cos kx \,.\, c_3 e^{-k^2\alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right) \frac{e^{-n^2\pi^2\alpha^2 t}}{l^2} \quad ...(6)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma a_n \cos(n\pi x/l)\, e^{-n^2\pi^2 a^2 t/l^2} \quad ...(7)$$

Now using the condition (*iii*), we get

$$lx - x^2 = a_0 + \Sigma a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left|(lx - x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right.$$

$$\left. - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left\{0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right\} = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-4m^2\pi^2 a^2 t/l^2}.$$

Example 18.12. (*a*) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* (*U.P.T.U., 2005*)

(*b*) *Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* (*Madras, 2000 S*)

Solution. (*a*) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (*i*) reduces to $\partial^2 u/\partial x^2 = 0$.

Its general solution is $u = ax + b$ ...(*ii*)

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (*ii*) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \dfrac{100}{l}x$ ...(*iii*)

Also the *boundary conditions* for the subsequent flow are

$$u(0, t) = 0 \text{ for all values of } t \quad ...(iv)$$

and

$$u(l, t) = 0 \text{ for all values of } t \quad ...(v)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (*i*) subject to the initial condition (*iii*) and the boundary conditions (*iv*) and (*v*).

Now the solution of (*i*) is of the form

$$u(x, t) = (C_1 \cos px + C_2 \sin px)\,e^{-c^2p^2t} \quad ...(vi)$$

By (*iv*), $u(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and (*vi*) reduces to $u(x, t) = C_2 \sin px \,.\, e^{-c^2p^2t}$ ...(*vii*)

Applying (*v*), (*vii*) gives $u(l, t) = C_2 \sin pl \,.\, e^{-c^2p^2t} = 0$, for all values of t.

This requires $\sin pl = 0$ *i.e.*, $pl = n\pi$ as $C_2 \neq 0$. $\therefore$ $p = n\pi/l$, where n is any integer.

Hence (*vii*) reduces to $u(x, t) = b_n \sin\dfrac{n\pi x}{l}\cdot e^{-c^2n^2\pi^2t/l^2}$, where $b_n = C_2$.

[These are the solutions of (i) satisfying the boundary conditions (iv) and (v). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (i) satisfying the boundary conditions (iv) and (v) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2} \qquad ...(viii)$$

Putting $t = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad ...(ix)$$

In order that the condition (iii) may be satisfied, (iii) and (ix) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l} \int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \left[x \left\{ - \frac{\cos(n\pi x/l)}{(n\pi/l)} \right\} - (1) \left\{ - \frac{\sin(n\pi x/l)}{(n\pi/l)^2} \right\} \right]_0^l = \frac{200}{l^2} \left(- \frac{l^2}{n\pi} \cos n\pi \right) = \frac{200}{n\pi} (-1)^{n+1}$$

Hence ($viii$) gives $u(x, t) = \dfrac{200}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-(cn\pi/l)^2 t}$

(b) Here the initial condition remains the same as (iii) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \qquad ...(x)$$
$$u(l, t) = 80 \text{ for all values of } t \qquad ...(xi)$$

In part (a), the boundary values ($i.e.$, the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad ...(xii)$$

where $u_s(x)$ is a solution of (i) involving x only and satisfying the boundary conditions (x) and (xi) ; $u_t(x, t)$ is then a function defined by (xii). Thus $u_s(x)$ is a steady state solution of the form (ii) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (ii) we get

$$u_s(x) = 20 + (60/l)x \qquad ...(xiii)$$

Putting $x = 0$ in (xii), we have by (x),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \qquad ...(xiv)$$

Putting $x = l$ in (xii), we have by (xi),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \qquad ...(xv)$$

Also $$u_t(x, 0) = u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20 \right) \qquad \text{[by } (iii) \text{ and } (xiii)$$

$$= \frac{40x}{l} - 20 \qquad ...(xvi)$$

Hence (xiv) and (xv) give the boundary conditions and (xvi) gives the initial condition relative to the transient solution. Since the boundary values given by (xiv) and (xv) are both zero, therefore, as in part (a), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2 p^2 t}$

By (xiv), $u_t(0, t) = C_1 e^{-c^2 p^2 t} = 0$, for all values of t.

Hence $C_1 = 0$ and $$u_t(x, t) = C_2 \sin px \,.\, e^{-c^2 p^2 t} \qquad ...(xvii)$$

Applying (xv), it gives $u_t(l, t) = C_2 \sin pl e^{-c^2 p^2 t} = 0$ for all values of t.

This requires $\sin pl = 0$, $i.e.$ $pl = n\pi$ as $C_2 \neq 0$. $p = n\pi/l$, when n is any integer.

Hence ($xvii$) reduces to $u_t(x, t) = b_n \sin \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$ where $b_n = C_2$.

Adding all such solutions, the most general solution of (*xvii*) satisfying the boundary conditions (*xiv*) and (*xv*) is

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2} \quad ...(xviii)$$

Putting $t = 0$, we have $u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ ...(*xix*)

In order that the condition (*xvi*) may be satisfied, (*xvi*) and (*xix*) must be same. This requires the expansion of $(40/l)\,x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where} \quad b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right)\sin\frac{n\pi x}{l}\,dx = -\frac{40}{nx}(1 + \cos n\pi)$$

i.e., $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

Hence (*xviii*) becomes $u_t(x, t) = \sum_{n=2,4\ldots}^{\infty}\left(\frac{-80}{n\pi}\right)\sin\frac{n\pi x}{l}\cdot e^{-c^2n^2\pi^2t/l^2}$ [Take $n = 2m$]

$$= -\frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\cdot e^{-4c^2m^2\pi^2t/l^2} \quad ...(xx)$$

Finally combining (*xiii*) and (*xx*), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\cdot e^{-4c^2m^2\pi^2t/l^2}.$$

Example 18.13. *The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. Let the heat equation be $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ ...(*i*)

In steady state condition, u is independent of time and depends on x only, (*i*) reduces to

$$\partial^2 u/\partial x^2 = 0. \quad ...(ii)$$

Its solution is $u = a + bx$

Since $u = 30$ for $x = 0$ and $u = 80$ for $x = 20$, therefore $a = 30$, $b = (80 - 30)/20 = 5/2$

Thus the initial conditions are expressed by

$$u(x, 0) = 30 + \frac{5}{2}x \quad ...(iii)$$

The boundary conditions are $u(0, t) = 40$, $u(20, t) = 60$

Using (*ii*), the steady state temperature is

$$u(x, 0) = 40 + \frac{60 - 40}{20}x = 40 + x \quad ...(iv)$$

To find the temperature u in the intermediate period,

$$u(x, t) = u_s(x) + u_t(x, t)$$

where $u_s(x)$ is the steady state temperature distribution of the form (*iv*) and $u_t(x, t)$ is the transient temperature distribution which decreases to zero as t increases.

Since $u_t(x, t)$ satisfies one dimensional heat equation

$$\therefore \quad u(x, t) = 40 + x + \sum_{n=1}^{\infty}(a_n \cos px + b_n \sin px)\,e^{-p^2t} \quad ...(v)$$

$$u(0, t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2t} \quad \text{whence } a_n = 0.$$

$\therefore$ (v) reduces to $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin pxe^{-p^2t} \quad ...(vi)$$

Also $$u(20, t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20\, pe^{-p^2t}$$

or $$\sum_{n=1}^{\infty} b_n \sin 20\, pe^{-p^2t} = 0 \text{ i.e., } \sin 20p = 0 \text{ i.e., } p = n\pi/20$$

Thus (vi) becomes $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-n\pi t/20} \quad ...(vii)$$

Using (iii), $$30 + \frac{5}{2}x = u(0, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

or $$\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where $b_n = \frac{2}{20}\int_0^{20}\left(\frac{3x}{2} - 10\right)\sin\frac{n\pi x}{20}\,dx = -\frac{20}{n\pi}(1 + 2\cos n\pi)$

Hence from (vii), the desired solution is

$$u = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1 + 2\cos n\pi}{n}\sin\frac{n\pi x}{20}e^{-(n\pi/20)^2 t}.$$

Example 18.14. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ (= 100 cm) of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0 \text{ for all } t \quad ...(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2} = 0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore$ $b = 0$ and $a = 1$.

Thus the initial condition is $u(x, 0) = x \quad 0 < x < l.$ ...(iii)

Now the solution of (i) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px)e^{-c^2p^2t}$ ...(iv)

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)e^{-c^2p^2t} \quad ...(v)$$

Putting $x = 0$, $\left(\frac{\partial u}{\partial x}\right)_0 = c_2 pe^{-c^2p^2t} = 0$ for all t. [By (ii)]

$\therefore$ $c_2 = 0$

Putting $x = l$ in (v), $\left(\frac{\partial u}{\partial x}\right)_l = -c_1 p \sin pl e^{-c^2p^2t}$ for all t. [By (ii)]

$\therefore$ $c_1 p \sin pl = 0$ i.e., p being $\neq 0$, either $c_1 = 0$ or $\sin pl = 0$.

When $c_1 = 0$, (iv) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.

or $pl = n\pi$ or $p = n\pi/l$, $n = 0, 1, 2, ...$

Hence (*iv*) becomes $u(x, t) = c_1 \cos \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$.

$\therefore$ the most general solution of (*i*) satisfying the boundary conditions (*ii*) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} \qquad (\text{where } A_n = c_1) \quad ...(vi)$$

Putting $t = 0$, $u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \dfrac{n\pi x}{l} = x$ [by (*iii*)]

This requires the expansion of x into a half range cosine series in $(0, l)$.

Thus $$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x/l \qquad \text{where } a_0 = \frac{2}{l}\int_0^l x\,dx = l$$

and $$a_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l} dx = \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$$

$= 0$, where n is even ; $= -4l/n^2\pi^2$, when n is odd.

$\therefore$ $$A_0 = \frac{a_0}{2} = l/2, \text{ and } A_n = a_n = 0 \text{ for } n \text{ even} ; = -4l/n^2\pi^2 \text{ for } n \text{ odd.}$$

Hence (*vi*) takes the form

$$u(x, t) = \frac{l}{2} + \sum_{n=1,3,...}^{\infty} \frac{4l}{n^2\pi^2} \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2(2n-1)^2\pi^2 t/l^2} \quad ...(vii)$$

This is the required temperature at a point P_1 distant x from end A at any time t.

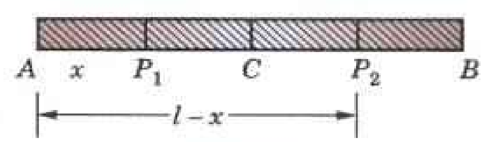


Fig. 18.4

Obs. *The sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.*

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18.4).

If $AP_1 = BP_2 = x$ (say), then $AP_2 = l - x$.

$\therefore$ Replacing x by $l - x$ in (*vii*), we get the temperature at P_2 as

$$u(l - x, t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi(l-x)}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}}$$

$$= \frac{l}{2} + \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}} \quad ...(viii)$$

$$\left\{ \because \cos \frac{(2n-1)\pi(l-x)}{l} = \cos\left[2n\pi - \pi - \frac{(2n-1)\pi x}{l}\right] = -\cos \frac{(2n-1)\pi x}{l} \right.$$

Adding (*vii*) and (*viii*), we get $u(x, t) + u(l - x, t) = l = 100°$C.

PROBLEMS 18.3

1. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is

$$u(x, 0) = x, \qquad 0 \le x \le 50$$
$$= 100 - x, \qquad 50 \le x \le 100.$$

Find the temperature $u(x, t)$ at any time. (*Bhopal, 2007 ; S.V.T.U., 2007 ; Kurukshetra, 2006*)

2. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length l, whose ends are kept at temperature 0°C and whose initial temperature in (°C) is given by $ax (l - x)/l^2$. (*P.T.U., 2009*)
3. A rod 30 cm. long, has its ends A and B kept at 20° and 80°C respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $u(x, t)$ taking $x = 0$ at A. (*Anna, 2008*)
4. A bar of 10 cm long, with insulated sides has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t. (*P.T.U., 2010*)

   Show that the temperature at the middle point of the bar remains unaltered for all time, regardless of the material of the bar.
5. Solve the following boundary value problem :
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \qquad \frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0, \; u(x, 0) = x. \qquad \text{(S.V.T.U., 2008)}$$
6. The temperatures at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.
7. Find the solution of $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$, such that

   (*i*) θ is not infinite when $t \to + \infty$; (*ii*) $\left.\begin{matrix} \frac{\partial \theta}{\partial x} = 0 & \text{when } x = 0 \\ \theta = 0, & \text{when } x = l \end{matrix}\right\}$ for all values of t ;

   (*iii*) $\theta = \theta_0$, when $t = 0$, for all values of x between 0 and l. (*S.V.T.U., 2008*)
8. Find the solution of $\frac{\partial V}{\partial t} = k \frac{\partial^2 V}{\partial x^2}$ having given that $V = V_0 \sin nt$ when $x = 0$ for all values of t and $V = 0$ when x is very large.

18.6 TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate of uniform thickness α (cm), density ρ (gr/cm^3), specific heat s (cal/gr deg) and thermal conductivity k (cal/cm sec deg). Let XOY plane be taken in one face of the plate (Fig. 18.5). If the temperature at any point is independent of the z-coordinate and depends only on x, y and time t, then the flow is said to be two-dimensional. In this case, the heat flow is in the XY-plane only and is zero along the normal to the XY-plane.

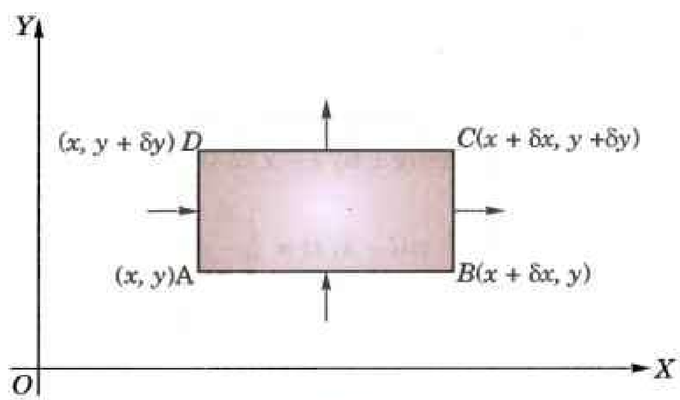


Fig. 18.5

Consider a rectangular element $ABCD$ of the plane with sides δx and δy. By (A) on p. 466, the amount of heat entering the element in 1 sec. from the side AB
$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_y$$
and the amount of heat entering the element in 1 second from the side $AD = -k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_x$

The quantity of heat flowing out through the side CD per sec. $= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$

and the quantity of heat flowing out through the side BC per second $= -k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

Hence the total gain of heat by the rectangular element $ABCD$ per second
$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}\right] + k\alpha\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right]$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\partial x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}}{\partial y}\right] \quad ...(1)$$

Also the rate of gain of heat by the element

$$= \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t} \quad ...(2)$$

Thus equating (1) and (2),

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right] = \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t}.$$

Dividing both sides by $\alpha\delta x\delta y$ and taking limits as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho s\frac{\partial u}{\partial t}$$

i.e.,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2\left(\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right) \text{ where } c^2 = k/\rho s \text{ is the } \textit{diffusivity}. \quad ...(3)$$

Hence the equation (3) gives the temperature distribution of the plane in the *transient state.*

Cor. In the *steady state*, u is independent of t, so that $\partial u/\partial t = 0$ and the above equation reduces to,

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0}$$

which is the well known **Laplace's equation** *in two dimensions.*

Obs. When the stream lines are curves in space, *i.e.*, the heat flow is three dimensional, we shall similarly arrive at the equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

In a *steady state*, it reduces to $\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} + \dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{z}^2} = \mathbf{0}$

which is the *three dimensional* **Laplace's equation.**

18.7 SOLUTION OF LAPLACE'S EQUATION

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0} \quad ...(1)$$

Let $u = X(x)Y(y)$ be a solution of (1).

Substituting it in (1), we get $\dfrac{d^2X}{dx^2}Y + X\dfrac{\partial^2 Y}{\partial y^2} = 0$

or separating the variables, $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -\dfrac{1}{Y}\dfrac{d^2Y}{dy^2}$ ...(2)

Since x and y are independent variables, (2) can hold good only if each side of (2) is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2Y}{dy^2} + kY = 0.$$

Solving these equations, we get

(*i*) *When k is positive* and is equal to p^2, say

$$X = c_1 e^{px} + c_2 e^{-px},\ Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative*, and is equal to $-p^2$, say

$$X = c_5 \cos px + c_6 \sin px,\ Y = c_7 e^{py} + c_8 e^{-py}$$

(*iii*) *When k is zero* ; $X = c_9 x + c_{10}$, $Y = c_{11} y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad ...(3)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad ...(4)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad ...(5)$$

Of these we take that solution which is consistent with the given boundary conditions.

(*V.T.U., 2011 S ; Kerala, 2005*)

Temperature distribution in long plates

Example 18.15. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π ; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.*

(*P.T.U., 2005 ; J.N.T.U., 2002 S*)

Solution. In the steady state (Fig. 18.6), the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(i)$$

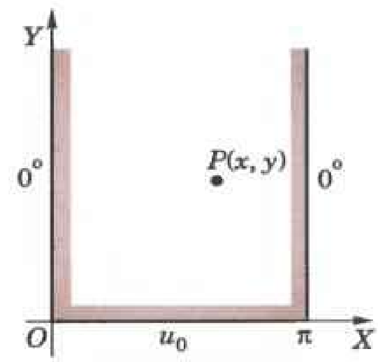


Fig. 18.6

The boundary conditions are $u(0, y) = 0$ for all values of y ...(*ii*)

$u(\pi, y) = 0$ for all values of y ...(*iii*)

$u(x, \infty) = 0$ in $0 < x < \pi$ ...(*iv*)

$u(x, 0) = u_0$ in $0 < x < \pi$ ...(*v*)

Now the three possible solutions of (*i*) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad ...(vi)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad ...(vii)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad ...(viii)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (*vi*) cannot satisfy the condition (*ii*) for $u \neq 0$ for $x = 0$, for all values of y. The solution (*viii*) cannot satisfy the condition (*iv*). Thus the only possible solution is (*vii*), *i.e.* of the form

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad ...(ix)$$

By (*ii*), $u(0, y) = C_1(C_3 e^{py} + C_4 e^{-py}) = 0$ for all y.

Hence $C_1 = 0$ and (*ix*) reduces to

$$u(x, y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py}) \quad ...(x)$$

By (*iii*), $u(\pi, y) = C_2 \sin p\pi(C_3 e^{py} + C_4 e^{-py}) = 0$, for all y.

This requires $\sin p\pi = 0$, *i.e.* $p\pi = n\pi$ as $C_2 \neq 0$. $\therefore$ $p = n$, an integer.

Also to satisfy the condition (*iv*), *i.e.*, $u = 0$ as $y \to \infty$, $C_3 = 0$.

Hence (*x*) takes the form $u(x, y) = b_n \sin nx \,.\, e^{-ny}$, where $b_n = C_2 C_4$.

$\therefore$ the most general solution satisfying (*ii*), (*iii*) and (*iv*) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \,.\, e^{-ny} \quad ...(xi)$$

Putting $y = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx \quad ...(xii)$$

In order that the condition (*v*) may be satisfied, (*v*) and (*xii*) must be same. This requires the expansion of u as a half-range Fourier sine series in $(0, \pi)$. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin nx \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{2u_0}{n\pi}[1-(-1)^n]$$

i.e., $b_n = 0$, if n is even ; $= 4u_0/n\pi$, if n is odd.

Hence (*xi*) becomes $u(x, y) = \dfrac{4u_0}{\pi}\left[e^{-y}\sin x + \dfrac{1}{3}e^{-3y}\sin 3x + \dfrac{1}{5}e^{-5y}\sin 5x + \ldots\right]$.

Temperature distribution in finite plates

Example 18.16. *Solve the Laplace equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin n\pi x/l$. *(V.T.U., 2011 ; J.N.T.U., 2006 ; Kerala M. Tech., 2005, U.P.T.U., 2004)*

Solution. The three possible solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(i)$$

are
$$u = (c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py) \qquad \ldots(ii)$$
$$u = (c_5\cos px + c_6\sin px)(c_7e^{py} + c_8e^{-py}) \qquad \ldots(iii)$$
$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \qquad \ldots(iv)$$

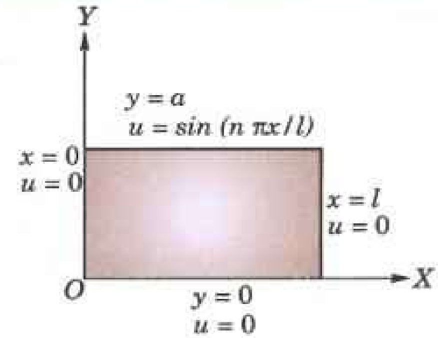


Fig. 18.7

We have to solve (*i*) satisfying the following boundary conditions

$$u(0, y) = 0 \quad \ldots(v) \qquad u(l, y) = 0 \quad \ldots(vi)$$
$$u(x, 0) = 0 \quad \ldots(vii) \qquad u(x, a) = \sin n\pi x/l \quad \ldots(viii)$$

Using (*v*) and (*vi*) in (*ii*), we get

$$c_1 + c_2 = 0, \quad \text{and} \quad c_1e^{pl} + c_2e^{-pl} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$ which lead to trivial solution. Similarly, we get a trivial solution by using (*v*) and (*vi*) in (*iv*). Hence the suitable solution for the present problem is solution (*iii*). Using (*v*) in (*iii*), we have $c_5(c_7e^{py} + c_8e^{-py}) = 0$ *i.e.*, $c_5 = 0$

$\therefore$ (*iii*) becomes $\quad u = c_6 \sin px(c_7e^{py} + c_8e^{-py}) \qquad \ldots(ix)$

Using (*vi*), we have $c_6 \sin pl\,(c_7e^{py} + c_8e^{-py}) = 0$

$\therefore$ either $c_6 = 0$ or $\sin pl = 0$

If we take $c_6 = 0$, we get a trivial solution.

Thus $\sin pl = 0$ whence $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, \ldots\ldots$

$\therefore$ (*ix*) becomes $u = c_6 \sin(n\pi x/l)(c_7e^{n\pi y/l} + c_8e^{-n\pi y/l}) \qquad \ldots(x)$

Using (*vii*), we have $0 = c_6 \sin n\pi x/l \,.\, (c_7 + c_8)$ *i.e.*, $c_8 = -c_7$.

Thus the solution suitable for this problem is

$$u(x, y) = b_n \sin\frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l}) \quad \text{where} \quad b_n = c_6c_7$$

Now using the condition (*viii*), we have

$$u(x, a) = \sin\frac{n\pi x}{l} = b_n \sin\frac{n\pi x}{l}(e^{n\pi a/l} - e^{-n\pi a/l}),$$

we get
$$b_n = \frac{1}{(e^{n\pi a/l} - e^{-n\pi a/l})}$$

Hence the required solution is

$$u(x, y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}}\sin\frac{n\pi x}{l} = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)}\sin\frac{n\pi x}{l}.$$

Example 18.17. *The function $v(x, y)$ satisfies the Laplace's equation in rectangular coordinates (x, y) and for points within the rectangle $x = 0, x = a, y = 0, y = b$, it satisfies the conditions $v(0, y) = v(a, y) = v(x, b) = 0$ and $v(x, 0) = x(a - x)$, $0 < x < a$. Show that $v(x, y)$ is given by*

$$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x/a}{(2n+1)^3}\,\frac{\sinh(2n+1)\pi(b-y)/a}{\sinh(2n+1)\pi b/a}$$

(Madras, 2003)

Solution. The only possible solution of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad ...(i)$$

is of the form $\qquad v(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(ii)$

The boundary conditions are

$$v(0, y) = 0 \; ; \quad v(a, y) = 0 \qquad ...(iii)$$

$$v(x, b) = 0 \qquad ...(iv)$$

$$v(x, 0) = x(a - x), \; 0 < x < a. \qquad ...(v)$$

Using (iii) $\qquad v(0, y) = c_1(c_3 e^{py} + c_4 e^{-py}) = 0 \quad i.e., \quad c_1 = 0.$

$\therefore$ (ii) becomes $\qquad v(x, y) = c_2 \sin px \, (c_3 e^{py} + c_4 e^{-py}) \qquad ...(vi)$

Again using (iii), $\qquad v(a, y) = c_2 \sin pa \, (c_3 e^{py} + c_4 e^{-py}) = 0.$

i.e., $\qquad \sin pa = 0$, *i.e.* $pa = n\pi$ or $p = n\pi/a$

$\therefore$ (vi) becomes $\qquad v(x, y) = c_2 \sin \dfrac{n\pi x}{a} \left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}} \right)$

or $\qquad v(x, y) = \sin \dfrac{n\pi x}{a} (A e^{n\pi y/a} + B e^{-n\pi y/a})$ where $A = c_2 c_3, B = c_2 c_4 \qquad ...(vii)$

Now using (iv), $\qquad v(x, b) = \sin \dfrac{n\pi x}{a} \left(A e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}} \right) = 0$

i.e., $\qquad A e^{n\pi b/a} + B e^{-n\pi b/a} = 0$ or $A e^{n\pi b/a} - B e^{-n\pi b/a} = -\dfrac{1}{2} b_n$ (say)

Thus (vii) becomes $\qquad v(x, y) = \sin \dfrac{n\pi x}{a} \cdot \dfrac{1}{2} b_n \left\{ e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a} \right\}$

$$= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}$$

$\therefore$ the most general solution of (i) is

$$v(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a} \qquad ...(viii)$$

Using the condition (v), we have

$$x(a - x) = v(x, 0) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

where $\qquad b_n \sinh \dfrac{n\pi b}{a} = \dfrac{2}{a} \displaystyle\int_0^a x(a - x) \sin \dfrac{n\pi x}{a} \, dx$

$$= \frac{2}{a} \left| (ax - x^2)\left(\frac{-\cos n\pi x/a}{n\pi/a} \right) - (a - 2x)\left(-\frac{\sin n\pi x/a}{(n\pi/a)^2} \right) + (-2)\left\{ \frac{\cos n\pi x/a}{(n\pi/a)^3} \right\} \right|_0^a$$

$$= 0 - 0 + \frac{4a^2}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \frac{8a^2}{n^3\pi^3} \text{ when } n \text{ is odd, otherwise zero when } n \text{ is even.}$$

Hence from $(viii)$, the required solution is

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}$$

or $\qquad v(x, y) = \dfrac{8a^2}{\pi^3} \displaystyle\sum_{n=0}^{\infty} \dfrac{\sinh (2n+1)\pi(b-y)/a}{(2n+1)^3 \sinh (2n+1)\pi b/a} \sin \dfrac{(2n+1)\pi x}{a}.$

PROBLEMS 18.4

1. A long rectangular plate of width a cm. with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$, $v(x, \infty) = 0$, $v(x, 0) = kx$. Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot e^{-n\pi y/a} \sin \frac{n\pi x}{a}. \qquad (J.N.T.U., 2005)$$

2. A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100 \sin (\pi x/8), \quad 0 < x < 8 ;$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane is given by

$$u(x, y) = 100e^{-\pi y/8} \sin (\pi x/8).$$

3. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by

$$u = 20x \qquad \text{for } 0 \le x \le 5$$

and

$$u = 20(10 - x) \qquad \text{for } 5 \le x \le 10$$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature u at any point (x, y) is given by

$$u = \frac{40}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-(2n-1)\pi y/10} \qquad (Anna, 2009)$$

4. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$, with conditions given : $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

5. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by

$$u(x, 20) = x(20 - x), \text{ when } 0 < x < 20,$$

while other three edges are kept at 0°C. Find the steady state temperature in the plate. *(Madras, 2003)*

6. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm. and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y). Hence show that the temperature at the centre of the plate

$$= \frac{200}{\pi}\left[\frac{1}{\cosh \pi/2} - \frac{1}{3 \cosh 3\pi/2} + \frac{1}{5 \cosh 5\pi/2} - \dots\right].$$

7. A square thin metal plate of side a is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = 0$, $y = a$ are kept at zero temperature, the edge $y = 0$ is insulated and the edge $x = a$ is kept at constant temperature T_0. Show that in the steady state conditions, the temperature $u(x, y)$ at the point (x, y) is given by

$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sinh \dfrac{(2n-1)\pi x}{2a} \cos \dfrac{(2n-1)\pi y}{2a}}{(2n-1) \sinh \dfrac{(2n-1)\pi}{2}}.$$

8. A rectangular plate has sides a and b. Taking the side of length a as OX and that of length b as OY and other sides to be $x = a$ and $y = b$, the sides $x = 0$, $x = a$, $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \dfrac{\pi x}{a}$. Find the temperature $u(x, y)$ in the steady-state.

18.8 (1) LAPLACE'S EQUATION IN POLAR COORDINATES

In the study of steady-state temperature distribution in a rectangular plate, it is usually convenient to employ Cartesian coordinates as hitherto done. Sometimes Polar coordinates (r, θ) are found to be more useful and the Cartesian form of Laplace's equation is replaced by its polar form :

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

(See Ex. 5.24, p. 213-214)

(2) Solution of Laplace's equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(1)$$

Assume that a solution of (1) is of the form $u = R(r) \,.\, \phi(\theta)$ where R is a function of r alone and ϕ is a function of θ only.

Substituting it in (1), we get $r^2R''\phi + rR'\phi + R\phi'' = 0$ or $\phi(r^2R'' + rR') + R\phi'' = 0$.

Separating the variables $\dfrac{r^2R'' + rR'}{R} = -\dfrac{\phi''}{\phi}$ ...(2)

Clearly the left side of (2) is a function of r only and the right side is a function of θ alone. Since r and θ are independent variables, (2) can hold good only if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - kR = 0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2\phi}{d\theta^2} + k\phi = 0 \quad ...(4)$$

Putting $r = e^z$, (3) reduces to $\dfrac{d^2R}{dz^2} - kR = 0$ ...(5)

Solving (5) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p},\ \phi = c_3 \cos p\theta + c_4 \sin p\theta$$

(*ii*) *When k is negative* and $= -p^2$, say

$$R = c_5 \cos pz + c_6 \sin pz = c_5 \cos (p \log r) + c_6 \sin (p \log r),\ \phi = c_7e^{p\theta} + c_8e^{-p\theta}$$

(*iii*) *When k is zero* :

$$R = c_9z + c_{10} = c_9 \log r + c_{10},\ \phi = c_{11}\theta + c_{12}$$

Thus the three possible solutions of (1) are

$$u = (c_1r^p + c_2r^{-p})\,(c_3 \cos p\theta + c_4 \sin p\theta) \qquad ...(6)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)]\,(c_7e^{p\theta} + c_8e^{-p\theta}) \qquad ...(7)$$

$$u = (c_9 \log r + c_{10})\,(c_{11}\theta + c_{12}) \qquad ...(8)$$

Of these solutions, we have to take that solution which is consistent with the physical nature of the problem. The general solution will consist of a sum of terms of type (6), (7) or (8). (*S.V.T.U., 2008*)

Example 18.18. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r, \theta) = \frac{4T}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1} \sin (2n-1)\,\theta. \qquad \textit{(Kerala M. Tech., 2005)}$$

Solution. Take the centre of the circle as the pole and bounding diameter as the initial line as in Fig. 18.8. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(i)$$

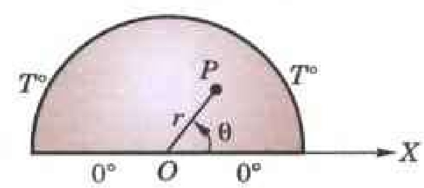


Fig. 18.8

The boundary conditions are :

$$u(r, 0) = 0 \quad \text{in } 0 \le r \le a \qquad ...(ii)$$

$$u(r, \pi) = 0 \quad \text{in } 0 \le r \le a \qquad ...(iii)$$

and $$u(a, \theta) = T \qquad ...(iv)$$

The three possible solutions of (*i*) are

$$u = (c_1r^p + c_2r^{-p})\,(c_3 \cos p\theta + c_4 \sin p\theta) \qquad ...(v)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)]\,(c_7e^{p\theta} + c_8e^{-p\theta}) \qquad ...(vi)$$

$$u = (c_9 \log r + c_{10})\,(c_{11}\theta + c_{12}) \qquad ...(vii)$$

From (*ii*) and (*iii*), $u = 0$ when $r = 0$ *i.e.*, u must be finite at the origin. Thus the solutions (*vi*) and (*vii*) are to be rejected. Hence the only suitable solution is (*v*).

By (*ii*), $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_3 = 0$

Hence $c_3 = 0$ and (*v*) becomes $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\theta$ ...(*viii*)

By (*iii*), $u(r, \pi) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\pi = 0.$

As $c_4 \neq 0$, $\sin p\pi = 0$, *i.e.*, $p = n$, where n is any integer.

Hence (*viii*) reduces to $u(r, \theta) = (c_1 r^n + c_2 r^{-n})\, c_4 \sin n\theta$ ...(*ix*)

Since $u = 0$, when $r = 0$, $\therefore$ $c_2 = 0$ and (*ix*) becomes

$u(r, \theta) = b_n r^n \sin n\theta$, where $b_n = c_1 c_4$.

$\therefore$ the most general solution of (*i*) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad \text{...}(x)$$

Putting $r = a$,
$$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta. \qquad \text{...}(xi)$$

In order that (*iv*) may be satisfied, (*iv*) and (*xi*) must be same. This requires the expansion of T as a half-range Fourier sine series in $(0, \pi)$. Thus

$$T = \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{where} \quad B_n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta \, d\theta = \frac{2T}{n\pi}(1 - \cos n\pi) \quad \text{and } B_n = b_n a^n$$

$\therefore$
$$b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n}(1 - \cos n\pi)$$

i.e.,
$$b_n = 0, \text{ if } n \text{ is even}$$
$$= \frac{4T}{n\pi a^n}, \text{ if } n \text{ is odd.}$$

Hence (*x*) gives
$$u(r, \theta) = \frac{4T}{\pi}\left\{\frac{(r/a)}{1}\sin\theta + \frac{(r/a)^3}{3}\sin 3\theta + \frac{(r/a)^5}{5}\sin 5\theta + \dots\right\}$$

Example 18.19. *The bounding diameter of a semi-circular plate of radius a cm is kept at 0°C and the temperature along the semi-circular boundary is given by*

$$u(a, \theta) = \begin{cases} 50\theta, & \text{when } 0 < \theta \le \pi/2 \\ 50(\pi - \theta) & \text{when } \pi/2 < \theta < \pi \end{cases}$$

Find the steady-state temperature function u (r, θ). (Madras, 2003)

Solution. We know that $u(r, \theta)$ satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial \theta^2} + r\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{...}(i)$$

The boundary conditions are $u(r, \theta) = 0$, $u(r, \pi) = 0$ ...(*ii*)

and $u(a, \theta) = 50\theta$ for $0 \le \theta \le \pi/2$; $u(a, \theta) = 50(\pi - \theta)$ for $\pi/2 \le \theta \le \pi$ ...(*iii*)

As in example 18.18, the most general solution of (*i*) satisfying the boundary conditions (*ii*) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad \text{...}(iv)$$

Putting $r = a$,
$$r(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

In order that the boundary condition (*iii*) is satisfied, we have $r(a, \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$

where
$$b_n a^n = B_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} 50\theta \sin n\theta \, d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta)\sin n\theta \, d\theta\right\} \qquad \text{...}(v)$$

$$= \frac{100}{\pi}\left\{\left| \theta\left(\frac{-\cos n\theta}{\theta}\right) - (1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_0^{\pi/2} + \left|(\pi-\theta)\left(\frac{-\cos n\theta}{n}\right) - (-1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{100}{\pi}\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin n\pi/2}{n^2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin n\pi/2}{n^2}\right\} = \frac{200}{\pi n^2}\sin n\pi/2.$$

When n is even $B_n = 0$, so taking $n = 1, 3, 5$ etc, (*iv*) gives

$$u(r, \theta) = \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{200}{\pi n^2}\sin\frac{n\pi}{2}\right)\frac{1}{a^n}\cdot r^n \sin n\theta$$

$$= \frac{200}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^2}\left(\frac{r}{a}\right)^{2m-1} \sin(2m-1)\theta.$$

[Taking $n = 2m - 1$, $n = 1, 3, 5, \ldots$; gives $m = 1, 2, 3$, $\sin n\pi/2 = \sin(2m-1)\pi/2 = (-1)^{m-1}$. This gives the required temperature function.

PROBLEMS 18.5

1. A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta), = k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r, \theta)$ of the plate assuming the lateral surfaces of the plate to be insulated.
2. A semi-circular plate of radius 10 cm has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and on the circumference the temperature distribution maintained is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$. Determine the temperature distribution $u(r, \theta)$ at any point on the plate.
3. A plate in the shape of truncated quadrant of a circle, is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. It has its faces insulated and heat flows in plane curves. It is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the temperature distribution.
4. Determine the steady state temperature at the points on the sector $0 \le \theta \le \pi/4$, $0 \le r \le a$ of a circular plate, if the temperature is maintained at 0°C along the side edges and at a constant temperature k°C along the curved edges.
5. Find the steady-state temperature in a circular plate of radius a which has one-half of its circumference at 0°C and the other half at 60°C.
6. If the radii of the inner and outer boundaries of a circular annulus area 10 cm and 20 cm and
$$u(10, \theta) = 15\cos\theta,\ u(20, \theta) = 30\sin\theta,$$
find the value of $u(r, \theta)$ in the annulus. [$u(r, \theta)$ satisfies Laplace equation in the interior of the annulus.]
7. A plate in the form of a ring is bounded by the lines $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature along the boundaries are
$$u(2, \theta) = 10\sin\theta + 6\cos\theta,\ u(4, \theta) = 17\sin\theta + 15\cos\theta$$
Find the steady-state temperature $u(r, \theta)$ in the ring.

18.9 (1) VIBRATING MEMBRANE—TWO DIMENSIONAL WAVE EQUATION

We shall now derive the equation for the vibrations of a tightly stretched membrane, such as the membrane of a drum. We shall assume that the membrane is uniform and the tension T in it per unit length is the same in all directions at every point.

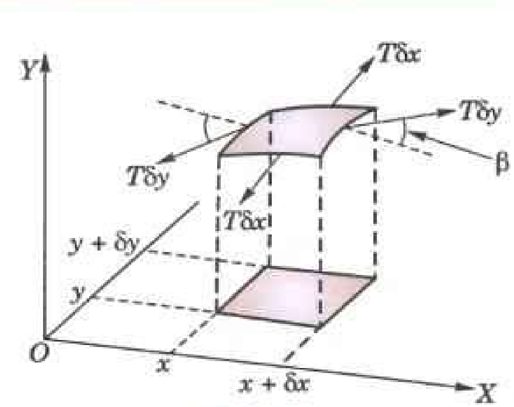


Fig. 18.9

Consider the forces acting on an element $\delta x \delta y$ of the membrane (Fig. 18.9). Forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let u be its small displacement perpendicular to the xy-plane, so that the forces $T\delta y$ on its opposite edges of length δy make angles α and β to the horizontal. So their vertical component

$= T\delta y \sin\beta - T\delta y \sin\alpha$

$= T\delta y(\tan\beta - \tan\alpha)$ approximately, since α and β are small

$$= T\delta y\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\} = T\delta y\,\frac{\partial^2 u}{\partial x^2}\,\delta x, \text{ up to a first order of approximation.}$$

Similarly, the vertical component of the force $T\delta x$ acting on the edges of length δx

$$= T\delta x\,\frac{\partial^2 u}{\partial y^2}\,\delta y$$

If m be the mass per unit area of the membrane, then the equation of motion of the element is

$$m\delta x\delta y\,\frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]\delta x\delta y \quad \text{or} \quad \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} = \mathbf{c}^2\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right] \quad \text{where} \quad c^2 = T/m \qquad ...(1)$$

This is the wave equation in two dimensions.

(2) Solution of the two-dimensional wave equation - Rectangular membrane. Assume that a solution of (1) is of the form $u = X(x)Y(y)T(t)$

Substituting this in (1) and dividing by XYT, we get

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

This can hold good if each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2 X}{dx^2} + k^2 X = 0, \quad \frac{d^2 Y}{dy^2} + l^2 Y = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} + (k^2 + l^2)\,c^2 T = 0$$

Hence a solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)\,(c_3 \cos ly + c_4 \sin ly) \times [c_5 \cos\sqrt{(k^2 + l^2)}ct + c_6 \sin\sqrt{(k^2 + l^2)}ct] \qquad ...(2)$$

Now suppose the membrane is rectangular and is stretched between the lines $x = 0$, $x = a$, $y = 0$, $y = b$. Then the condition $u = 0$ when $x = 0$ gives

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos\sqrt{(k^2 + l^2)ct} + c_6 \sin\sqrt{(k^2 + l^2)ct}] \quad i.e., \quad c_1 = 0.$$

Then putting $c_1 = 0$ in (2) and applying the condition $u = 0$ when $x = a$, we get $\sin ka = 0$ or $k = m\pi/a$. (m being an integer)

Similarly, applying the conditions $u = 0$, when $y = 0$ and $y = b$, we obtain

$$c_3 = 0 \quad \text{and} \quad l = n\pi/b \qquad (n \text{ being an integer})$$

Thus the solution (2) becomes

$$u(x, y, t) = c_2 c_4 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,(c_5 \cos p_{mn}t + c_6 \sin p_{mn}t)$$

where $p_{mn} = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$ ...(3)

[These are the solutions of the wave equation (1) which are zero on the boundary of the rectangular membrane. These functions are called **eigen functions** and the numbers p_{mn} are the **eigen values** of the vibrating membrane.]

Choosing the constants c_2 and c_4 so that $c_2 c_4 = 1$, we can write the general solution of the equation (1) as

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,(A_{mn}\cos pt + B_{mn}\sin pt) \qquad ...(4)$$

If the membrane starts from rest from the initial position $u = f(x, y)$, *i.e.*, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then (3) gives $B_{mn} = 0$.

Also using the condition $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

This is *double Fourier series*. Multiplying both sides by $\sin(m\pi x/a)\sin(n\pi y/b)$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one, becomes zero. Hence we obtain

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dydx = \frac{ab}{4} A_{mn} \qquad ...(5)$$

which gives the coefficients in the solution and is called the **generalised Euler's formula.**

Rectangular Membranes

Example 18.20. *Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2\pi y$.*

Solution. Taking $a = b = 1$ and $f(x, y) = A \sin \pi x \sin 2\pi y$, in (5), we get

$$A_{mn} = 4 \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y \, dydx$$

$$= 4A \int_0^1 \sin \pi x \sin m\pi x \, dx \left(\int_0^1 \sin 2\pi y \sin n\pi y \, dy \right) = 0, \quad \text{for } m \neq 1$$

$$= 4A \left(\frac{1}{2}\right) \int_0^1 \sin 2\pi y \sin n\pi y \, dy, \text{ for } m = 1 \qquad \left[\because \int_0^1 \sin \pi x \sin \pi x \, dx = \tfrac{1}{2}\right]$$

i.e.,
$$A_{mn} = 2A \int_0^1 \sin 2\pi y \sin n\pi y \, dy = 0, \text{ for } n \neq 2$$

$$= 2A \left(\frac{1}{2}\right), \text{ for } n = 2.$$

$\therefore$ $A_{12} = A$. Also from (3), $p_{mn} = \pi\sqrt{(m^2 + n^2)}$ $\qquad [\because a = b = 1 = c]$

$\therefore$ $p_{12} = \pi\sqrt{(1^2 + 2^2)} = \sqrt{5}\,\pi$.

Hence from (4), the required solution is $u(x, y, t) = A \sin \pi x \sin 2\pi y \cos (\sqrt{5}\,\pi t)$.

Example 18.21. *Find the vibration $u(x, y, t)$ of a rectangular membrane $(0 < x < a, 0 < y < b)$ whose boundary is fixed given that it starts from rest and $u(x, y, 0) = hxy(a - x)(b - y)$.*

Solution. Proceeding as in § 18.9 (**2**), we have from (4),

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \text{ where } p = \pi c\sqrt{[(m/a)^2 + (n/b)^2}$$

Since the membrane starts from rest $\partial u/\partial t = 0$ when $t = 0$,

$$\therefore \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (-A_{mn}\, p \sin pt + pB_{mn} \cos pt) = 0 \text{ when } t = 0$$

This gives $B_{mn} = 0$

$$\therefore \quad u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \qquad ...(i)$$

Then
$$hxy(a - x)(b - y) = u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where
$$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b hxy(a-x)(b-y) \,.\, \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dydx$$

$$= \frac{4h}{ab} \left\{ \int_0^a x(a-x) \sin \frac{m\pi x}{a} dx \right\} \left\{ \int_0^b y(b-y) \sin \frac{n\pi y}{b} dy \right\}$$

$$= \frac{4h}{ab} \left| (ax - x^2)\left(\frac{-\cos m\pi x/a}{m\pi/a}\right) - (a - 2x)\left\{\frac{-\sin \frac{m\pi x}{a}}{(m\pi/a)^2}\right\} + (-2)\frac{\cos m\pi x/a}{(m\pi/a)^3} \right|_0^a$$

$$\times \left| (by - y^2)\left(\frac{-\cos n\pi y/b}{n\pi/b}\right) - (b - 2y)\left\{\frac{-\sin n\pi/b}{(n\pi/b)^2}\right\} + (-2)\frac{\cos n\pi y/b}{(n\pi/b)^3} \right|_0^b$$

$$= \frac{4h}{ab} \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} (1 - \cos m\pi)(1 - \cos n\pi)$$

Hence from (i), we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $A_{mn} = \dfrac{16\,ha^2b^2}{m^3n^3\pi^6} (1 - \cos m\pi)(1 - \cos n\pi)$ and $p = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$

Circular Membranes*

Example 18.22. *A circular membrane of unit radius fixed along its boundary starts vibrating from rest and has initial deflection $u(r, 0) = f(r)$. Show that the deflection $u(r, t)$ of the membrane at any instant is given by*

$$u(r, t) = \sum_{m=1}^{\infty} A_m \cos(c\alpha_m t) \,.\, J_o(\alpha_m r) \text{ where } A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r)\, dr,$$

and $\alpha_m (m = 1, 2, ...)$ are the positive roots of the Bessel function $J_0(k) = 0$.

Solution. The vibrations of a plane circular membrane are governed by 2-dimensional wave equation in polar coordinates *i.e.*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

For a radially symmetric membrane (in which u does not depend on θ) the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) \qquad ...(i)$$

For the given membrane fixed along its boundary, the boundary condition is

$$u(1, t) = 0 \quad \text{for all } t \geq 0 \qquad ...(ii)$$

For solutions not depending on θ,

initial deflection $u(r, 0) = f(r)$ ...(iii)

and initial velocity $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = 0$ ...(iv)

which are the initial conditions. We find the solutions $u(r, t) = R(r)T(t)$ ...(v)

satisfying the boundary condition (ii).

Differentiating and substituting (v) in (i), we get

$$\frac{\partial^2 T/\partial t^2}{c^2 T} = \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} \right) = -k^2 \text{ (say)}$$

This leads to $\dfrac{\partial^2 T}{\partial t^2} + p^2 T = 0$ where $p = ck$ ...(vi)

and $$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + k^2 R = 0 \qquad ...(vii)$$

Now putting $s = kr$, (vii) transforms to $\dfrac{d^2R}{ds^2} + \dfrac{1}{s}\dfrac{dR}{ds} + R = 0$ which is Bessel's equation. Its general solution $R = aJ_0(s) + bY_0(s)$ where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, we must have $b = 0$. Then taking $a = 1$, we get

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the circular membrane, we must have $J_0(k) = 0$, which is satisfied for

$$k = \alpha_m,\ m = 1, 2, ...$$

*Drums, telephones and microphones provide examples of circular membrane and as such are quite useful in engineering.

Thus the solutions of (*vii*) are $R(r) = J_0(\alpha_m r)$, $m = 1, 2, \ldots$ and the corresponding solutions of (*vi*) are $T(t) = A_m \cos p_m t + B_m \sin p_m t$, where $p_m = ck_m = c\alpha_m$.

Hence the general solution of (*i*) satisfying (*ii*) are

$$u(r, t) = (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

which are the *eigen functions* of the problem and the corresponding *eigen values* are p_m.

To find that solution which also satisfies the initial conditions (*iii*) and (*iv*), consider the series

$$u(r, t) = \sum_{m=1}^{\infty} (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

Putting $t = 0$ and using (*iii*), we get $u(r, 0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) = f(r)$

Here, the constants A_m must be the coefficients of Fourier-Bessel series [(8) page 560] with $m = 0$, *i.e.*,

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r\, f(r)\, J_0(\alpha_m r)\, dr$$

Using (*iv*), we get $B_m = 0$. Hence the result.

PROBLEMS 18.6

1. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\,\pi ct)$.
2. Find the deflection $u(r, t)$ of the circular membrane of unit radius if $c = 1$, the initial velocity is zero and the initial deflection is $0.25\,(1 - r^2)$.

18.10 TRANSMISSION LINE

Consider a cable l km in length, carrying an electric current with resistance R ohms/km, inductance L henries/km ; capacitance C farads/km and leakance G mhos/km (Fig. 18.10).

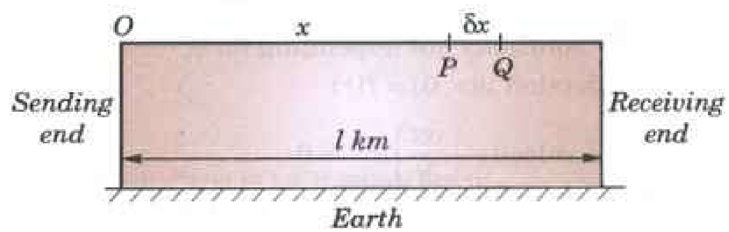


Fig. 18.10

Let the instantaneous voltage and current at any point P, distant x km from the sending end O, and at time t sec be $v(x, t)$ and $i(x, t)$ respectively. Consider a small length $PQ (= \delta x)$ of the cable.

Now since the voltage drop across the segment δx

= voltage drop due to resistance + voltage drop due to inductance

$$\therefore \qquad -\delta v = iR\delta x + L\delta x \,.\, \frac{\partial i}{\partial t}$$

and dividing by δx and taking limits as $\delta x \to 0$, we get

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \qquad \ldots(1)$$

Similarly the current loss between P and Q

= current lost due to capacitance and leakance

$\therefore \qquad -\delta i = C\dfrac{\partial v}{\partial t}\delta x + Gv\delta x$ from which as before, we get

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \qquad \ldots(2)$$

Rewriting the simultaneous partial differential equations (1) and (2) as

$$\left(R + L\frac{\partial}{\partial t}\right) i + \frac{\partial v}{\partial x} = 0 \qquad \ldots(3)$$

and $$\frac{\partial i}{\partial x} + \left(C\frac{\partial}{\partial t} + G\right)v = 0, \quad ...(4)$$

we shall eliminate i and v in turn.

$\therefore$ operating (3) by $\frac{\partial}{\partial x}$ and (4) by $\left(R + L\frac{\partial}{\partial t}\right)$ and subtracting

$$\frac{\partial^2 v}{\partial x^2} - \left(R + L\frac{\partial}{\partial t}\right)\left(C\frac{\partial}{\partial t} + G\right)v = 0$$

or $$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (LG + RC)\frac{\partial v}{\partial t} + RGv \quad ...(5)$$

Again operating (3) by $\left(C\frac{\partial}{\partial t} + G\right)$ and (4) by $\frac{\partial}{\partial x}$ and subtracting

$$\left(C\frac{\partial}{\partial t} + G\right)\left(R + L\frac{\partial}{\partial t}\right)i - \frac{\partial^2 i}{\partial x^2} = 0$$

or $$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (LG + RC)\frac{\partial i}{\partial t} + RGi \quad ...(6)$$

which is (5) with v replaced by i. The equations (5) and (6) are called the *telephone equations*.

Cor. 1. *If* $L = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad ...(7) \qquad \frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t} \quad ...(8)$$

which are known as the *telegraph equations.*

Rewriting (7) as $\frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2}$, we observe that it is similar to the heat equation [(1) p. 611].

Cor. 2. *If* $R = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \quad ...(9) \qquad \frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \quad ...(10)$$

which are called the *radio equations.*

Rewriting (9) as $\frac{\partial^2 v}{\partial t^2} = k^2\frac{\partial^2 v}{\partial x^2}$ where $k^2 = \frac{1}{LC}$,

its general solution is $v(x, t) = f_1(x + kt) + f_2(x - kt)$. [See (4) p. 609]

Similarly from (10), $i(x, t) = F_1(x + kt) + F_2(x - kt)$.

Thus the voltage $v(x, t)$ for the current $i(x, t)$ at any point along the lossless transmission line can be obtained by the superposition of a progressive wave and a receding wave travelling with equal velocities (k). This is the case of oscillations of $v(x, t)$ and $i(x, t)$ at high frequencies.

Cor. 3. *If* $L = C = 0$, *e.g.*, in the case of a submarine cable, then (5) gives

$$\frac{\partial^2 v}{\partial x^2} = GRv, \text{ i.e. } (D^2 - GR)v = 0$$

$$\therefore \quad v(x) = A\cosh(\sqrt{GR}\,.\,x) + B\sinh(\sqrt{GR}\,.\,x) \quad ...(11)$$

Since by (1), $$Ri = -\frac{\partial v}{\partial x} = -\sqrt{GR}\,[A\sinh(\sqrt{GR}\,.\,x) + B\cosh(\sqrt{GR}\,.\,x)]$$

$$\therefore \quad i(x) = -\sqrt{G/R}\,[A\sinh(\sqrt{GR}\,.\,x) + B\cosh(\sqrt{GR}\,.\,x)] \quad ...(12)$$

If $v(0) = v_0$ and $i(0) = i_0$, then $v_0 = A$ and $i_0 = -\sqrt{(G/R)}B$.

Hence writing $\sqrt{(GR)} = \gamma$ and $\sqrt{(R/G)} = z_0$, (11) and (12) give

$$v(x) = v_0\cosh\gamma x - i_0 z_0\sinh\gamma x \quad ...(13)$$

and $$i(x) = i_0\cosh\gamma x - \frac{v_0}{z_0}\sinh\gamma x. \quad ...(14)$$

Obs. Steady-state solutions. We have so far considered the transient state solutions only. The steady-state solutions of transmission lines are however, obtained by assuming $v = Ve^{i\omega t}$ and $i = Ie^{i\omega t}$, where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear differential equations of the second order which can be solved at once.

Example 18.23. *Neglecting R and G, find the e.m.f. $v(x, t)$ in a line of length l, t seconds after the ends were suddenly grounded, given that $i(x, 0) = i_0$ and $v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l}$.* (S.V.T.U., 2008)

Solution. Since R and G are negligible, we use the *Radio equation* $\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$ ...(*i*)

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, \ v(l, t) = 0 \qquad ...(ii)$$

Also the initial conditions are $i(x, 0) = i_0$

and $$v(x, 0) = e_1 \sin \pi x/l + e_5 \sin 5\pi x/l \qquad ...(iii)$$

$$\therefore \quad \frac{\partial i}{\partial x} = -c\frac{\partial v}{\partial t} \quad \text{gives} \quad \frac{\partial v}{\partial t}(x, 0) = 0 \qquad ...(iv)$$

Let $v = X(x)T(t)$ be the solution of (*i*).

$\therefore$ (*i*) gives $$X''T = LCXT''$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -k^2 \text{ (say)}$$

$$\therefore \quad X'' + k^2X = 0 \quad \text{and} \quad T'' + (k^2/LC)T = 0$$

Solving these equations, we get

$$v = (c_1 \cos kx + c_2 \sin kx)\left(c_3 \cos \frac{k}{\sqrt{LC}} . t + c_4 \sin \frac{k}{\sqrt{LC}} t\right)$$

Using the boundary conditions (*ii*), we get

$$c_1 = 0 \quad \text{and} \quad k = n\pi/l.$$

$$\therefore \quad v = \sin \frac{n\pi x}{l}\left(a_n \cos \frac{n\pi}{l\sqrt{LC}} t + b_n \sin \frac{n\pi}{l\sqrt{LC}} t\right)$$

Using the initial condition (*iv*), we get $b_n = 0$

$$\therefore \quad v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}} t$$

Thus the most general solution of (*i*) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Finally by the initial condition (*iii*), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum a_n \sin \frac{n\pi x}{l}$$

$\therefore$ $a_1 = e_1$ and $a_5 = e_5$ while all other a's are zero.

Hence $$v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

Example 18.24. *A telephone line 3000 km. long has a resistance of 4 ohms/km. and a capacitance of 5×10^{-7} farad/km. Initially both the ends are grounded so that the line is uncharged. At time $t = 0$, a constant e.m.f. E is applied to one end, while the other end is left grounded. Assuming the inductance and leakance to be negligible, show that the steady state current of the grounded end at the end of 1 sec. is 5.3%.*

Solution. Since $L = 0$, $G = 0$, we use the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$$

Let $v = X(x)T(t)$ be its solution so that

$$TX'' = RCXT' \quad \text{or} \quad \frac{X''}{X} = RC\frac{T'}{T} = -k^2 \text{ (say)}$$

$\therefore$ $$X'' + k^2X = 0 \text{ and } T' + (k^2/RC)T = 0$$

Solving these equations, we get

$$X = c_1 \cos kx + c_2 \sin kx,\ T = c_3e^{-k^2t/RC}$$

giving $$v = (c_1 \cos kx + c_2 \sin kx)\,c_3e^{-k^2t/RC} \quad ...(i)$$

When $t = 0$; $v = 0$ at $x = 0$ and $v = 0$ at $x = l$

$\therefore$ $$0 = c_1c_3 \,;\, 0 = (c_1 \cos kl + c_2 \sin kl)c_3$$

i.e., $$c_1c_3 = 0 \text{ and } kl = n\pi \ (n \text{ an integer})$$

Putting these in (i) and adding a linear term, we have

$$v = a_0x + b_0 + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2} \quad ...(ii)$$

The end conditions of the problem are

$$v = 0 \text{ at } x = 0 \text{ and } v = E \text{ at } x = l$$

Using these, (ii) gives $b_0 = 0$ and $a_0 = E/l$

Then (ii) becomes $$v = \frac{E}{l}x + \sum b_n \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2}$$

Also $v = 0$ when $t = 0$, we get $-Ex/l = \Sigma b_n \sin n\pi x/l$

This requires the expansion of $(-Ex/l)$ as a half-range sine series in $(0, l)$.

$\therefore$ $$b_n = \frac{2}{l}\int_0^l \left(\frac{-Ex}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(\frac{-Ex}{l}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-E}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l = \frac{2}{l}\left(\frac{El}{n\pi}\cos n\pi\right) = \frac{2E}{n\pi}(-1)^n.$$

Thus $$v = \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2} \quad ...(iii)$$

Also when $L = 0$, $\dfrac{-\partial v}{\partial x} = Ri$

i.e., $$i = -\frac{1}{R}\frac{\partial v}{\partial x} = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2}$$

At the grounded end $(x = 0)$, the current is

$$i = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n\, e^{-n^2\pi^2t/RCl^2}$$

When $t = 1$ sec, $$i = -\frac{E}{lR}\left(1 - 2e^{\pi^2/RCl^2} + 2e^{-4\pi^2/RCl^2} - ...\right) \quad ...(iv)$$

Since $$\frac{\pi^2}{RCl^2} = \frac{(3.14)^2}{4(5\times10^{-7})(3000)^2} = 0.548$$

$\therefore$ $$e^{-\pi^2/RCl^2} = e^{-0.548} = 0.578$$

When $t \to \infty$, $i \to -E/lR$

Hence from (*iv*), we have

$$i = -\frac{E}{lR}\{1 - 2(0.578) + 2(0.578)^4 - 2(0.578)^9 + ...\}$$
$$= -\frac{E}{lR}\{1 - 1.156 + 0.223 - 0.014 + ...\}$$
$$= i_\infty(0.053) = 5.3\% \text{ of } i_\infty.$$

Example 18.25. *A transmission line 1000 kilometers long is initially under steady-state conditions with potential 1300 volts at the sending end (x = 0) and 1200 volts at the receiving end (x = 1000). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential v(x, t).* (Andhra, 2000)

Solution. The equation of the telegraph line is

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad \text{or} \quad \frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2} \qquad ...(i)$$

v_s = initial steady voltage satisfying $\frac{\partial^2 v}{\partial x^2} = 0 = 1300 - x/10 = v(x, 0)$ ...(*ii*)

v_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached = $1300 - 1.3x$

$\therefore$ $v(x, t) = v_s' + v_t(x, t)$ where $v_t(x, t)$ is the transient part

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{l} \qquad \text{[By } (viii)\text{, p. 614]} \qquad ...(iii)$$

where l = 1000 kilometers.

Putting $t = 0$, we have from (*ii*) and (*iii*)

$$1300 - 0.1x = v(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e.
$$1.2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^1 1.2\sin\frac{n\pi x}{l}\,dx = \frac{2400}{\pi}\cdot\frac{(-1)^{n+1}}{n}$$

Hence
$$v(x, t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{1000}.$$

PROBLEMS 18.7

1. Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin(\pi x/l)$.
   Also R and G are negligible.
2. Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity equal to $l/\sqrt{(LC)}$, where L is the self-inductance and C is the capacitance.
3. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.
4. Obtain the solution of the *radio equation*
$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$
   appropriate to the case when a periodic e.m.f. $V_0 \cos pt$ is applied at the end $x = 0$ of the line.

18.11 LAPLACE'S EQUATION IN THREE DIMENSIONS

We have seen that the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

which is the *Laplace's equation in three dimensions.*

Laplace's equation also arises in the study of gravitational potential at (x, y, z) of a particle of mass m situated at (ξ, η, ζ) given by

$$\frac{Gm}{r} \text{ where } r = \sqrt{[(x-\zeta)^2 + (y-\eta)^2 + (z-\zeta)^2]}$$

This function is called the *potential of the gravitational field* and satisfies the Laplace's equation.

If a mass of density ρ at (ξ, η, ζ) is distributed throughout a region R, then the gravitational potential u at an external point (x, y, z) is given by

$$u(x, y, z) = G \iiint_R \frac{\rho}{r} \, d\xi \, d\eta \, d\zeta \qquad ...(2)$$

Since $\nabla^2(1/r) = 0$ and ρ is independent of x, y and z, we get

$$\nabla^2 u = \iiint_R \rho \, \nabla^2 (1/r) \, d\xi \, d\eta \, d\zeta = 0.$$

This shows that the gravitational potential defined by (2) also obeys Laplace's equation.

Thus Laplace's equation (1) is one of the most important equations arising in connection with numerous problems of physics and engineering. *The theory of its solutions is called the* **potential theory** *and its solutions are called the* **harmonic functions.**

In most of the problems leading to Laplace's equation, it is required to solve the equation subject to certain boundary conditions. A proper choice of coordinate system makes the solution of the problem simpler. Now we proceed to take up the solutions of (1) in its other forms.

18.12 SOLUTIONS OF THREE DIMENSIONAL LAPLACE'S EQUATION

(1) **Cartesian form of $\nabla^2 \mathbf{u} = 0$** is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$

Let $\qquad u = X(x)Y(y)Z(z) \qquad ...(2)$

be a solution of (1). Substituting (2) in (1) and dividing by XYZ, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = 0 \qquad ...(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

As x, y, z are independent, this will hold good only if F_1, F_2, F_3 are constants. Assuming these constants to be $k^2, l^2, -(k^2 + l^2)$ respectively, (3) leads to the equations

$$\frac{d^2X}{dx^2} - k^2X = 0, \quad \frac{d^2Y}{dy^2} - l^2Y = 0, \quad \frac{d^2Z}{dz^2} + (k^2 + l^2)Z = 0$$

Their solutions are $\qquad X = c_1e^{kx} + c_2e^{-kx}, \; Y = c_3e^{ly} + c_4e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z$$

Thus a possible solution of (1) is

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly})\{c_5 \cos \sqrt{(k^2+l^2)}z + c_6 \sin \sqrt{(k^2+l^2)}z\}.$$

Since the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, an alternative solution of (1) will be

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\{c_5e^{\sqrt{(k^2+l^2)}z} + c_6e^{-\sqrt{(k^2+l^2)}z}\}.$$

(2) **Cylindrical form of $\nabla^2 \mathbf{u} = 0$** is $\dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$

Let $u = R(\rho)\, H(\phi)\, Z(z)$ [(*iv*) p. 359]

be a solution of (1). Substituting it in (1), and dividing by RHZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 H}\frac{d^2H}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \qquad ...(2)$$

Assuming that $\dfrac{d^2H}{d\phi^2} = -n^2H$ and $\dfrac{d^2Z}{dz^2} = k^2Z,$ ..(3)

(2) reduces to $\dfrac{1}{R}\left(\dfrac{d^2R}{d\rho^2} + \dfrac{1}{\rho}\dfrac{dR}{d\rho}\right) - \dfrac{n^2}{\rho^2} + k^2 = 0$

or $$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (k^2\rho^2 - n^2)R = 0.$$

This is Bessel's equation [§ 16.10 (1)] and its solution is $R = c_1J_n(k\rho) + c_2Y_n(k\rho)$.

Also the solutions of equations (3) are

$$H = c_3 \cos n\phi + c_4 \sin n\phi,\ Z = c_5e^{kz} + c_6e^{-kz}$$

Thus a solution of (1) is

$$u = [c_1J_n(k\rho) + c_2Y_n(k\rho)]\,[c_3 \cos n\phi + c_4 \sin n\phi]\,[c_5\, e^{kz} + c_6e^{-kz}]$$

which is known as a *cylindrical harmonic.* (*Assam, 1999*)

(3) Spherical form of ∇^2u = 0 is

$$\frac{\partial^2u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2} = 0 \qquad ...(1) \quad [(iv)\text{ p. } 361]$$

Let $u = R(r)\, G(\theta)\, H(\phi)$ be a solution of (1).

Then $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right) + \frac{1}{G}\left(\frac{d^2G}{d\theta^2} + \cot\theta\frac{dG}{d\theta}\right) + \frac{1}{H\sin^2\theta}\frac{d^2H}{d\phi^2} = 0$$

Putting $\dfrac{1}{R}\left(r^2\dfrac{d^2R}{dr^2} + 2r\dfrac{dR}{dr}\right) = n(n+1)$ ...(2) and $\dfrac{1}{H}\dfrac{d^2H}{d\phi^2} = -m^2,$ ...(3)

the above equation takes the form

$$\frac{d^2G}{d\theta^2} + \cot\theta\frac{dG}{d\theta} + [n(n+1) - m^2\operatorname{cosec}^2\theta]G = 0 \qquad ...(4)$$

Now differentiating the *Legendre's equation* (§ 16.13)

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

m times with respect to x and writing $u = d^my/dx^m$, we get

$$(1-x^2)u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0 \qquad ...(5)$$

Now putting $G = (1-x^2)^{m/2}\, u$ in (5), we get

$$(1-x^2)\frac{d^2G}{dx^2} - 2x\frac{dG}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]G = 0 \qquad ...(6)$$

Now putting $x = \cos\theta$ in (6), it reduces to (4) and its solution is

$$G = c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)$$

The solution of (3) is $H = c_3\cos m\phi + c_4\sin m\phi$

To solve (2), write $R = r^k$, so that $k(k-1) + 2k = n(n+1)$ which gives $k = n$ or $-(n+1)$

Thus $R = c_5r^n + c_6r^{-n-1}$

Hence the general solution of (1) is

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)\}(c_3\cos m\phi + c_4\sin m\phi) \times (c_5r^n + c_6r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic.*

Example 18.26. *Find the potential in the interior of a sphere of unit radius when the potential on the surface is $f(\theta) = \cos^2 \theta$.*

Solution. Take the origin at the centre of the given sphere S. Since the potential is independent of ϕ on S, so also is the potential at any point. Therefore, the Laplace's equation in spherical co-ordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad ...(i)$$

Putting $u(r, \theta) = R(r)\, G(\theta)$ in (i) and proceeding as in § 18.12 (3), we obtain the equations

$$\frac{\partial^2 G}{d\theta^2} + \cot\theta \frac{dG}{d\theta} + n(n+1)G = 0 \qquad ...(ii)$$

and

$$\frac{1}{R}\left(r^2 \frac{d^2 R}{dr^2} + 2r\frac{dR}{dr}\right) = n(n+1) \qquad ...(iii)$$

Putting $\cot\theta = v$, (ii) takes the form

$$(1 - v^2)\frac{d^2 G}{dv^2} - 2v\frac{dG}{dv} + n(n+1)G = 0$$

which is Legendre's equation. Its solutions are

$$G = P_n(v) = P_n(\cos\theta) \text{ for } n = 0, 1, 2, ...$$

The solutions of (iii) are $R_n(r) = r^n$, $\overline{R_n}(r) = 1/r^{n+1}$.

Hence the equation (i) has the following two sets of solutions

$$u_n(r, \theta) = c_n r^n P_n(\cos\theta) \text{ and } \bar{u}_n(r, \theta) = c_n P_n(\cos\theta)/r^{n+1}, \text{ where } n = 0, 1, 2, ...$$

For points inside S, we have the general equation $u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta) \qquad ...(iv)$

On the boundary of S, $u(1, \theta) = f(\theta) \quad \therefore \quad f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$

which is Fourier-Legendre expansion of $f(\theta)$. Hence by (5) p. 560, we have

$$c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(\theta)\, P_n(x)\, dx \text{ where } x = \cos\theta.$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1} x^2 P_n(x) dx \qquad [\because \ f(\theta) = \cos^2\theta]$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1}\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right]P_n(x)\, dx \qquad [\because \ P_2(x) = \tfrac{1}{2}(3x^2 - 1)]$$

Using the orthogonality of Legendre polynomials, we get

$c_n = 0$, except for $n = 0, 2$. Hence

$$c_0 = \frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1} P_0^2(x)\, dx = \frac{1}{3},\ c_2 = \frac{5}{2}\cdot\frac{2}{3}\int_{-1}^{0} P_2^2(x)\, dx = \frac{2}{3}.$$

Substituting in (iv), we get $u(r, \theta) = \frac{1}{3} + \frac{2}{3} r^2 P_2(\cos\theta)$ or $u(r, \theta) = \frac{1}{3} + r^2(\cos^2\theta - \frac{1}{3})$.

PROBLEMS 18.8

1. Show that a solution of Laplace's equation in cylindrical co-ordinates, which remains finite at $r = 0$, may be expressed in the form

$$u = \sum_{n=0}^{\infty} J_n(kr)\, \{e^{kz}(A_n \cos n\theta + B_n \sin n\theta) + e^{-kz}(C_n \cos n\theta + D_n \sin n\theta)\}.$$

2. The potential on the surface of a unit sphere is $f(\theta) = \cos 2\theta$. Show that the potential at all points of space is given by

$$u(r, \theta) = 2r^2(\cos^2\theta - 1/3) - \frac{1}{3} \text{ for } r < 1,$$

and
$$u(r, \theta) = 2r^{-3}(\cos^2\theta - 1/3) - r^{-1}/3 \text{ for } r > 1.$$

3. Show that in spherical polar coordinates (r, θ, ϕ), Laplace's equation possesses solutions of the form

$$(Ar^n + B/r^{n+1})P_n(\mu)e^{\pm im\phi},$$

where $\mu = \cos\theta$, A, B, m, n are constants and $P_n(\mu)$ satisfies Legendre's equation

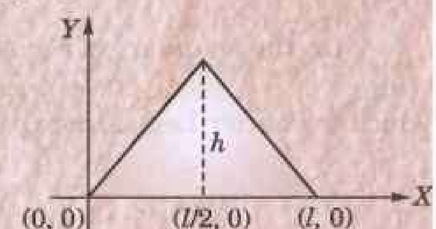
$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}P_n = 0.$$

18.13 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 18.9

Fill up the blanks in each of the following questions :

1. The radio equations for the potential and current are ....
2. The partial differential equation representing variable heat flow in three dimensions, is ....
3. Temperature gradient is defined as ....
4. The differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is classified as ....
5. The partial differential equation of the transverse vibrations of a string is ....
6. The steady state temperature of a rod of length l whose ends are kept at 30° and 40° is ....
7. The equation $u_t = c^2u_{xx}$ is classified as ....
8. The two dimensional steady state heat flow equation in polar coordinates is ....
9. The mathematical function of the initial form of the string given by the following graph is ....



10. When a vibrating string fastened to two points l apart, has an initial velocity u_0, its initial conditions are ....
11. In two dimensional heat flow, the temperature along the normal to the xy-plane is ....
12. If a square plate has its faces and the edge $y = 0$ insulated, its edges $x = 0$ and $x = a$ are kept at zero temperature and the fourth edge is kept at temperature u, then the boundary conditions for this problem are ....
13. If the ends $x = 0$ and $x = l$ are insulated in one dimensional heat flow problems, then the boundary conditions are ....
14. D'Alembert's solution of the wave equation is ....
15. The partial differential equation of 2-dimensional heat flow in ....
16. A rod 50 cm long with insulated sides has its end A and B kept at 20° and 70°C respectively. The steady state temperature distribution of the rod is .... *(Anna, 2008)*
17. The three possible solutions of Laplace equation in polar coordinates are ....
18. Solution of $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$, is ....
19. Solution of $\dfrac{\partial z}{\partial x} + 4z = \dfrac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is ....
20. In the equation $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial x^2}$, α^2 represents ....
21. The telegraph equations for potential and current are ......
22. The general solution of one-dimensional heat flow equation when both ends of the bar are kept at zero temperature, is of the form ....
23. The three possible solutions of Laplace equation $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ are ....

CHAPTER 24

Empirical Laws and Curve-fitting

24.1 INTRODUCTION

In many branches of applied mathematics, it is required to express a given data, obtained from observations, in the form of a *Law* connecting the two variables involved. Such a *Law* inferred by some scheme is known as *Empirical Law.* For example, it may be desired to obtain the law connecting the length and the temperature of a metal bar. At various temperatures, the length of the bar is measured. Then, by one of the methods explained below, a law is obtained that represents the relationship existing between temperature and length for the observed values. This relation can then be used to predict the length at an arbitrary temperature.

(2) Scatter diagram. To find a relationship between the set of paired observations x and y (say), we plot their corresponding values on the graph taking one of the variables along the x-axis and other along the y-axis *i.e.* (x_1, y_1), (x_2, y_2), (x_n, y_n). The resulting diagram showing a collection of dots is called a *scatter diagram*. A smooth curve that approximates the above set of points is known as the *approximating curve.*

(3) Curve fitting. Several equations of different types can be obtained to express the given data approximately. But the problem is to find the equation of the curve of '*best fit*' which may be most suitable for predicting the unknown values. The process of finding such an equation of 'best fit' is known as *curve-fitting.*

If there are n pairs of observed values then it is possible to fit the given data to an equation that contains n arbitrary constants for we can solve n simultaneous equations for n unknowns. If it were desired to obtain an equation representing these data but having less than n arbitrary constants, then we can have recourse to any of the four methods : *Graphical method, Method of Least squares, Method of Group averages* and *Method of Moments*. The graphical method fails to give the values of the unknowns uniquely and accurately while the other methods do. *The method of Least squares is, probably, the best to fit a unique curve to a given data.* It is widely used in applications and can be easily implemented on a computer.

24.2 GRAPHICAL METHOD

When the curve representing the given data is a **linear law** $y = mx + c$ *; we proceed as follows :*

(*i*) *Plot the given points on the graph paper* taking a suitable *scale.*

(*ii*) *Draw the straight line of best fit* such that the points are evenly distributed about the line.

(*iii*) *Taking two suitable points* (x_1, y_1) *and* (x_2, y_2) *on the line, calculate* m, the slope of the line and c, its intercept on y-axis.

When the points representing the observed values do not approximate to a straight line, a smooth curve is drawn through them. From the shape of the graph, we try to infer the law of the curve and then reduce it to the form $y = mx + c$.

24.3 LAWS REDUCIBLE TO THE LINEAR LAW

We give below some of the laws in common use, indicating the way these can be reduced to the linear form by suitable substitutions :

(1) *When the law is* $y = mx^n + c$.

Taking $x^n = X$ and $y = Y$ the above law becomes $Y = mX + c$

(2) *When the law is* $y = ax^n$.

Taking logarithms of both sides, it becomes $\log_{10} y = \log_{10} a + n \log_{10} x$

Putting $\log_{10} x = X$ and $\log_{10} y = Y$, it reduces to the form $Y = nX + c$, where $c = \log_{10} a$.

(3) *When the law is* $y = ax^n + b \log x$.

Writing it as $\dfrac{y}{\log x} = a\dfrac{x^n}{\log x} + b$ and taking $x^n/\log x = X$ and $y/\log x = Y$,

the given law becomes, $Y = aX + b$.

(4) *When the law is* $y = ae^{bx}$

Taking logarithms, it becomes $\log_{10} y = (b \log_{10} e)\, x + \log_{10} a$

Putting $x = X$ and $\log_{10} y = Y$, it takes the form $Y = mX + c$ where $m = b \log_{10} e$ and $c = \log_{10} a$.

(5) *When the law is* $xy = ax + by$.

Dividing by x, we have $y = b\dfrac{y}{x} + a$.

Putting $y/x = X$ and $y = Y$, it reduces to the form $Y = bX + a$.

Example 24.1. *R is the resistance to maintain a train at speed V; find a law of the type* $R = a + bV^2$ *connecting R and V, using the following data :*

V (miles/hour) :	10	20	30	40	50
R (lb/ton) :	8	10	15	21	30

Solution. Given law is $R = a + bV^2$ ...(*i*)

Taking $V^2 = x$ and $R = y$, (*i*) becomes

$$y = a + bx \quad ...(ii)$$

which is a *linear law*.

Table for the values of x and y is as follows :

x	100	400	900	1600	2500
y	8	10	15	21	30

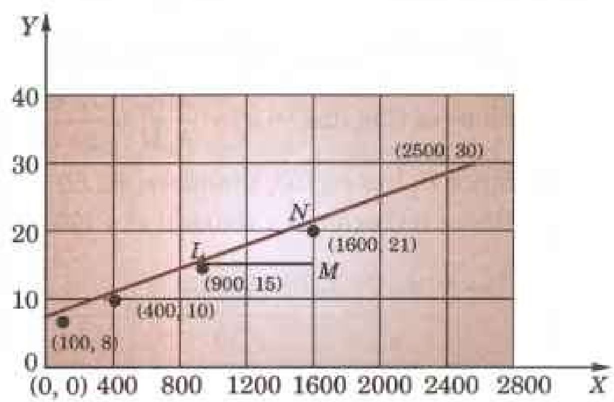


Fig. 24.1

Plot these points. Draw the straight line of best fit through these points (Fig. 24.1)

Slope of this line (= b)

$$= \frac{MN}{LM} = \frac{21 - 15}{1600 - 900} = \frac{6}{700} = 0.0085 \text{ nearly.}$$

Since L (900, 15) lies on (*ii*),

$\therefore$ $$15 = a + 0.0085 \times 900,$$

whence $$a = 15 - 7.65 = 7.35 \text{ nearly.}$$

Example 24.2. *The following values of x and y are supposed to follow the law* $y = ax^2 + b \log_{10} x$. *Find graphically the most probable values of the constants a and b.*

x	2.85	3.88	4.66	5.69	6.65	7.77	8.67
y	16.7	26.4	35.1	47.5	60.6	77.5	93.4

Solution. Given law is $y = ax^2 + b \log_{10} x$

i.e.
$$\frac{y}{\log_{10} x} = a\frac{x^2}{\log_{10} x} + b \quad ...(i)$$

Taking $x^2/\log_{10} x = X$ and $y/\log_{10} x = Y$

(*i*) becomes $Y = aX + b$ ...(*ii*)

This is a *linear law*. Table for the values of X and Y is as follows :

$X = x^2/\log_{10} x$	17.93	25.56	32.49	42.87	53.75	67.80	80.83
$Y = y/\log_{10} x$	35.59	44.83	52.50	62.90	73.65	87.04	99.56
Points	P_1	P_2	P_3	P_4	P_5	P_6	P_7

Plot these points and draw the straight line of best fit through these points (Fig. 24.2).

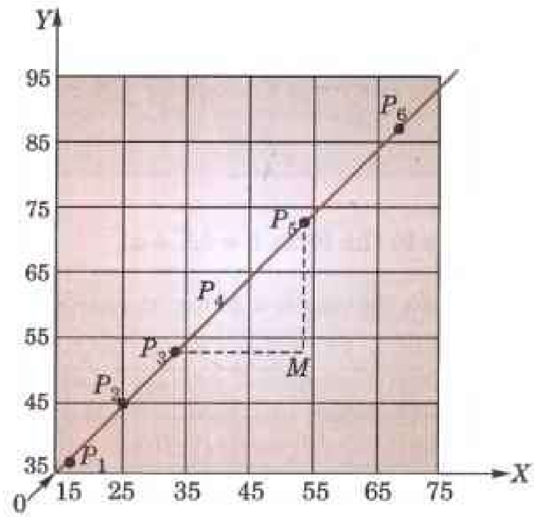


Fig. 24.2

$$\text{Slope of this line } (= a) = \frac{MP_5}{P_3M} = \frac{73.65 - 52.50}{53.75 - 32.49} = \frac{21.15}{21.26} = 0.99$$

Since P_3 lies on (*ii*), therefore, $52.50 = 0.99 \times 32.49 + b$ whence $b = 20.2$

Hence (*i*) becomes $y = (0.99)\, x^2 + (20.2) \log_{10} x$.

Example 24.3. *The values of x and y obtained in an experiment are as follows :*

x	2.30	3.10	4.00	4.92	5.91	7.20
y	33.0	39.1	50.3	67.2	85.6	125.0

Th probable law is $y = ae^{bx}$. Test graphically the accuracy of this law and if the law holds good, find the best values of the constants.

Solution. Given law is $y = ae^{bx}$ ...(*i*)

Taking logarithms to base 10, we have $\log_{10} y = \log_{10} a + (b \log_{10} e)\, x$

Putting $x = X$ and $\log_{10} y = Y$, it becomes $y = (b \log_{10} e)\, X + \log_{10} a$ ...(*ii*)

Table for the values of X and Y is as under :

$X = x$	2.30	3.10	4.00	4.92	5.91	7.20
$Y = \log_{10} y$	1.52	1.59	1.70	1.83	1.93	2.1
Points	P_1	P_2	P_3	P_4	P_5	P_6

Scale : 1 small division along x–axis = 0.1

10 small divisions along y–axis = 0.1.

Plot these points and draw the line of best fit. As these points are lying almost along a straight line, the given law is nearly accurate (Fig. 24.3).

Now slope of this line ($= b \log_{10} e$)

$$= \frac{MN}{NM} = 0.12$$

whence $$b = \frac{0.12}{\log_{10} e} = 0.12 \times 2.303 = 0.276$$

Since the point L (4, 1.71) lies on (*ii*), therefore, $1.71 = 0.12 \times 4 + \log_{10} a$ whence $a = 17$ nearly.

Hence the curve of best fit is $y = 17\, e^{0.276x}$.

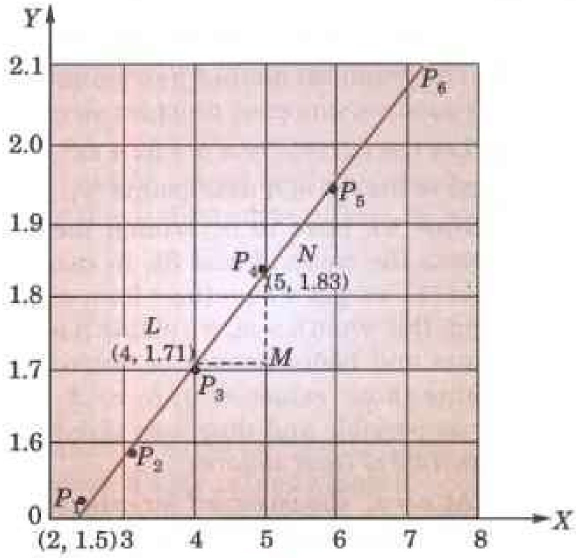


Fig. 24.3

PROBLEMS 24.1

1. If p is the pull required to lift the weight by means of a pulley block, find a linear law of the form $p = a + bw$, connecting p and w, using the following data :

w (lb) :	50	70	100	120
p (lb) :	12	15	21	25

Compute p, when $w = 150$ lb.

2. The resistance R of a carbon filament lamp was measured at various values of the voltage V and the following observations were made :

Voltage V...	62	70	78	84	92
Resistance R...	73	70.7	69.2	67.8	66.3

Assuming a law of the form $R = \frac{a}{V} + b$, find by graphical method the best value of a and b.

3. Verify if the values of x and y, related as shown in the following table, obey the law $y = a + b\sqrt{x}$. If so, find graphically the values of a and b.

x :	500	1,000	2,000	4,000	6,000
y :	0.20	0.33	0.38	0.45	0.51

4. The following values of T and l follow the law $T = al^n$. Test if this is so and find the best values of a and n.

$T = 1.0$	1.5	2.0	2.5
$l = 25$	56.2	100	1.56

5. Find the best value of a and b if $y = ax + b \log_{10} x$ is the curve which represents most closely the observed values given below :

x :	2	3	4	5	6
y :	3.74	5.99	7.47	8.92	9.86

6. Fit the curve $y = ae^{bx}$ to the following data :

x :	0	2	4
y :	5.1	10	31.1

(*Coimbatore, 1997*)

7. The following are the results of an experiment on friction of bearings. The speed being constant, corresponding values of the coefficient of friction and the temperature are shown in the table :

t :	120	110	100	90	80	70	60
μ :	0.0051	0.0059	0.0071	0.0085	0.00102	0.00124	0.00148

If μ and t are given by the law $\mu = ae^{bt}$, find the values of a and b by plotting the graph for μ and t.

24.4 PRINCIPLE OF LEAST SQUARES

The graphical method has the obvious drawback of being unable to give a unique curve of fit. *The principle of least squares, however, provides an elegant procedure for fitting a unique curve to a given data.*

Let the curve, $y = a + bx + cx^2 + ... + kx^{m-1}$ ...(1)

be fitted to the set of n data points $(x_1, y_1), (x_2, y_2)...(x_n, y_n)$.

Now we have to determine the constants $a, b, c,...k$ such that it represents the curve of best fit. In case $n = m$, on substituting the values (x_i, y_i) in (1), we get n equations from which a unique set of n constants can be found. But when $n > m$, we obtain n equations which are more than the m constants and hence cannot be solved for these constants. So we try to determine those values of $a, b, c,...k$ which satisfy all the equations as nearly as possible and thus may give the best fit. In such cases, we apply the *principle of least squares*.

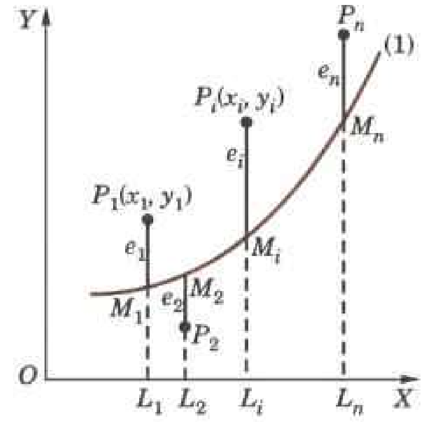


Fig. 24.4

At $x = x_i$, the *observed* (or *experimental*) *value* of the ordinate is $y_i = P_iL_i$ and the corresponding value on the fitting curve (1) is $a + bx_i + cx_i^2 + ... + kx_i^m = M_iL_i$ ($= \eta_i$, say) which is the *expected* (or *calculated*) *value* (Fig. 24.4). The difference of the observed and the expected values *i.e.* $y_i - \eta_i (= e_i)$ is called the *error* (or *residual*) at $x = x_i$. Clearly some of the errors $e_1, e_2, ... e_n$ will be positive and others negative. Thus to give equal weightage to each error, we square each of these and form their sum *i.e.* $E = e_1^2 + e_2^2 + ... e_n^2$.

The curve of best fit is that for which e's are as small as possible i.e., E, the sum of the squares of the errors is a minimum. This is known as the *principle of least squares* and was suggested by *Legendre** in 1806.

Obs. The principle of least squares does not help us to determine the form of the appropriate curve which can fit a given data. It only determines the best possible values of the constants in the equation when the form of the curve is known before hand. The selection of the curve is a matter of experience and practical considerations.

24.5 (1) METHOD OF LEAST SQUARES

For clarity, suppose it is required to fit the curve

$$y = a + bx + cx^2 \quad ...(1)$$

to a given set of observations $(x_1, y_1), (x_2, y_2), ..., (x_5, y_5)$. For any x_i, the observed value is y_i and the expected value is $\eta_i = a + bx_i + cx_i^2$ so that the error $e_i = y_i - \eta_i$.

$\therefore$ the sum of the squares of these errors is

$$E = e_1^2 + e_2^2 + ... + e_5^2$$
$$= [y_1 - (a + bx_1 + cx_1^2)]^2 + [y_2 - (a + bx_2 + cx_2^2)]^2 + ... + [y_5 - (a + bx_5 + cx_5^2)]^2$$ [See § 5.12 (3)]

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0 = 2[y_1 - (a + bx_1 + cx_1^2)] - 2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2[y_5 - (a + bx_5 + cx_5^2)] \quad ...(2)$$

$$\frac{\partial E}{\partial b} = 0 = -2x_1[y_1 - (a + bx_1 + cx_1^2)] - 2x_2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2x_5[y_5 - (a + bx_5 + cx_5^2)] \quad ...(3)$$

$$\frac{\partial E}{\partial c} = 0 = -2x_1^2[y_1 - (a + bx_1 + cx_1^2)] - 2x_2^2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2x_5^2[y_5 - (a + bx_5 + cx_5^2)] \quad ...(4)$$

Equation (2) simplifies to

$$y_1 + y_2 + ... + y_5 = 5a + b(x_1 + x_2 + ... + x_5) + c(x_1^2 + x_2^2 + ... + x_5^2)$$

i.e., $$\Sigma y_i = 5a + b\Sigma x_i + c\,\Sigma x_i^2 \quad ...(5)$$

* See footnote on p. 311.

Equation (3) becomes

$$x_1y_1 + x_2y_2 + ... + x_5y_5 = a(x_1 + x_2 + ... + x_5) + b(x_1^2 + x_2^2 + ... + x_5^2) + c(x_1^3 + x_2^3 + ... + x_5^3)$$

i.e., $$\Sigma x_iy_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3 \quad ...(6)$$

Similarly (4) simplifies to $\Sigma x_i^2y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4$ ...(7)

The equations (5), (6) and (7) are known as *Normal equations* and can be solved as simultaneous equations in a, b, c. The values of these constants when substituted in (1) give the desired curve of best fit.

(2) Working procedure

(a) To fit the straight line $y = a + bx$

(i) Substitute the observed set of n values in this equation.

(ii) Form normal equations for each constant

i.e., $$\Sigma y = na + b\Sigma x, \ \Sigma xy = a\Sigma x + b\Sigma x^2$$

[The normal equation for the unknown a is obtained by multiplying the equations by the coefficient of a and adding. The normal equation for b is obtained by multiplying the equations by the coefficient of b (*i.e.*, x) and adding.]

(iii) Solve these normal equations as simultaneous equations for a and b.

(iv) Substitute the values of a and b in $y = a + bx$, which is the required line of best fit.

(b) To fit the parabola : $y = a + bx + cx^2$

(i) Form the normal equations $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

and $$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

[The normal equation for c has been obtained by multiplying the equations by the coefficient of c (*i.e.*, x^2) and adding.]

(ii) Solve these as simultaneous equations for a, b, c.

(iii) Substitute the values of a, b, c in $y = a + bx + cx^2$, which is the required parabola of best fit.

(c) In general, the curve $y = a + bx + cx^2 + ... + kx^{m-1}$ can be fitted to a given data by writing m normal equations.

Example 24.4. *If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form $P = mW + c$ connecting P and W, using the following data :*

$P = 12$	15	21	25
$W = 50$	70	100	120

where P and W are taken in kg-wt. Compute P when W = 150 kg. wt. (U.P.T.U., 2007 ; V.T.U., 2002)

Solution. The corresponding normal equations are

$$\left.\begin{aligned}\Sigma P &= 4c + m\Sigma W\\ \Sigma WP &= c\Sigma W + m\Sigma W^2\end{aligned}\right\} \quad ...(i)$$

The values of ΣW etc. are calculated by means of the following table :

W	P	W^2	WP
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	14400	3000
Total = 340	73	31800	6750

$\therefore$ The equations (i) becomes $73 = 4c + 340m$ and $6750 = 340c + 31800m$

i.e., $$2c + 170m = 365 \quad ...(ii)$$

and $$34c + 3180m = 675 \quad ...(iii)$$

Multiplying (ii) by 17 and substracting from (iii), we get

$$m = 0.1879 \quad \therefore \text{ from } (ii), \quad c = 2.2785$$

Hence the line of best fit is

$$P = 2.2759 + 0.1879\ W$$

When $W = 150$ kg., $P = 2.2785 + 0.1879 \times 150 = 30.4635$ kg.

Obs. The calculations get simplified when the central values of x is zero. It is therefore, advisable to make the central value zero, if it be not so. This is illustrated by the next example.

Example 24.5. *Fit a second degree parabola to the following data :*

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

(P.T.U., 2006)

Solution. Let $u = x - 2$ and $v = y$ so that the parabola of fit $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \quad ...(i)$$

The normal equations are

$$\Sigma v = 5A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 12.9 = 5A + 10C$$

$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^2 \quad \text{or} \quad 11.3 = 10B$$

$$\Sigma u^2 v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad 33.5 = 10A + 34C$$

Solving these as simultaneous equations, we get

$$A = 1.48, \qquad B = 1.13, \qquad C = 0.55.$$

$\therefore$ (*i*) becomes, $\quad v = 1.48 + 1.13u + 0.55u^2$

or $\quad y = 1.48 + 1.13\,(x - 2) + 0.55\,(x - 2)^2$

Hence $y = 1.42 - 1.07x + 0.55x^2$.

Example 24.6. *Fit a second degree parabola to the following data :*

$x = 1.0$	1.5	2.0	2.5	3.0	3.5	4.0
$y = 1.1$	1.3	1.6	2.0	2.7	3.4	4.1

(V.T.U., 2009 ; Bhopal, 2008)

Solution. We shift the origin to (2.5, 0) and take 0.5 as the new unit. This amounts to changing the variable x to X, by the relation $X = 2x - 5$.

Let the parabola of fit be $y = a + bX + cX^2$. The values of ΣX etc., are calculated as below :

x	X	y	Xy	X^2	X^2y	X^3	X^4
1.0	–3	1.1	–3.3	9	9.9	–27	81
1.5	–2	1.3	–2.6	4	5.2	–8	16
2.0	–1	1.6	–1.6	1	1.6	–1	1
2.5	0	2.0	0.0	0	0.0	0	0
3.0	1	2.7	2.7	1	2.7	1	1
3.5	2	3.4	6.8	4	13.6	8	16
4.0	3	4.1	12.3	9	36.9	27	81
Total	0	16.2	14.3	28	69.9	0	196

The normal equations are

$$7a + 28c = 16.2\ ; \qquad 28b = 14.3\ ; \qquad 28a + 196c = 69.9$$

Solving these as simultaneous equations, we get

$$a = 2.07,\ b = 0.511,\ c = 0.061$$

$\therefore \quad y = 2.07 + 0.511X + 0.061X^2$

Replacing X by $2x - 5$ in the above equation, we get

$$y = 2.07 + 0.511\,(2x - 5) + 0.061\,(2x - 5)^2$$

which simplifies to $y = 1.04 - 0.198x + 0.244x^2$. This is the required parabola of best fit.

Example 24.7. *Fit a second degree parabola to the following data :*

x	1989	1990	1991	1992	1993	1994	1995	1996	1997
y	352	356	357	358	360	361	361	360	359

(U.P.T.U., 2009)

Solution. Taking $u = x - 1993$ and $v = y - 357$, the equation $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \qquad ...(i)$$

x	$u = x - 1993$	y	$v = y - 357$	uv	u^2	u^2v	u^3	u^4
1989	−4	352	−5	20	16	−80	−64	256
1990	−3	360	−1	3	9	−9	−27	81
1991	−2	357	0	0	4	0	−8	16
1992	−1	358	1	−1	1	1	−1	1
1993	0	360	3	0	0	0	0	0
1994	1	361	4	4	1	4	1	1
1995	2	361	4	8	4	16	8	16
1996	3	360	3	9	9	27	27	81
1997	4	359	2	8	16	32	64	256
Total	$\Sigma u = 0$		$\Sigma v = 11$	$\Sigma uv = 51$	$\Sigma u^2 = 60$	$\Sigma u^2v = -9$	$\Sigma u^3 = 0$	$\Sigma u^4 = 708$

The normal equations are

$$\Sigma v = 9A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 11 = 9A + 60C$$

$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^3 \quad \text{or} \quad 51 = 60B \quad \text{or} \quad B = \frac{17}{20}$$

$$\Sigma u^2v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad -9 = 60A + 708C$$

On solving these equations, we get $A = \dfrac{694}{231}, B = \dfrac{17}{20}, C = -\dfrac{247}{924}$

$\therefore$ (*i*) becomes $\quad v = \dfrac{694}{231} + \dfrac{17}{20}u - \dfrac{247}{924}u^2$

or $$y - 357 = \frac{694}{231} + \frac{17}{20}(x - 1933) - \frac{247}{924}(x - 1933)^2$$

or $$y = \frac{694}{231} - \frac{32861}{20} - \frac{247}{924}(1933)^2 + \frac{17}{20}x + \frac{247 \times 3866}{924}x - \frac{247}{924}x^2$$

or $$y = 3 - 1643.05 - 998823.36 + 357 + 0.85x + 1033.44x - 0.267x^2$$

Hence $$y = -1000106.41 + 1034.29x - 0.267x^2.$$

PROBLEMS 24.2

1. By the method of least squares, find the straight line that best fits the following data :

x :	1	2	3	4	5
y :	14	27	40	55	68

(U.P.T.U., 2008)

2. Fit a straight line to the following data :

Year x :	1961	1971	1981	1991	2001
Production y :	8	10	12	10	16

(in thousand tons)

and find the expected production in 2006.

3. A simply supported beam carries a concentrated load P (lb) at its mid-point. Corresponding to various values of P, the maximum deflection Y (in) is measured. The data are given below :

P:	100	120	140	160	180	200
Y:	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$.

4. The results of measurement of electric resistance R of a copper bar at various temperatures t°C are listed below :

t :	19	25	30	36	40	45	50
R :	76	77	79	80	82	83	85

Find a relation $R = a + bt$ where a and b are constants to be determined by you.

5. Find the best possible curve of the form $y = a + bx$, using method of least squares for the data :

x :	1	3	4	6	8	9	11	14
y :	1	2	4	4	5	7	8	9

(*V.T.U., 2011*)

6. Fit a straight line to the following data

(*a*)

x :	1	2	3	4	5	6	7	8	9
y :	9	8	10	12	11	13	14	16	5

(*Bhopal, 2008*)

(*b*)

x :	6	7	7	8	8	8	9	9	10
y :	5	5	4	5	4	3	4	3	3

(*J.N.T.U., 2008*)

7. Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the observations :

x :	− 3	− 2	− 1	0	1	2	3
y :	4.63	2.11	0.67	0.09	0.63	2.15	4.58

(*V.T.U., 2006 ; J.N.T.U., 2000 S*)

8. Fit a parabola $y = a + bx + cx^2$ to the following data :

x :	2	4	6	8	10
y :	3.07	12.85	31.47	57.38	91.29

(*V.T.U., 2003 S*)

9. Fit a second degree parabola to the following data :

x :	1	2	3	4	5	6	7	8	9	10
y :	124	129	140	159	228	289	315	302	263	210

(*U.P.T.U., 2009*)

10. The following table gives the results of the measurements of train resistances ; V is the velocity in miles per hour. R is the resistance in pounds per ton :

V:	20	40	60	80	100	120
R:	5.5	9.1	14.9	22.8	33.3	46.0

If R is related to V by the relation $R = a + bV + cV^2$, find a, b, and c. (*U.P.T.U., 2002*)

11. The velocity V of a liquid is known to vary with temperature according to a quadratic law $V = a + bT + cT^2$. Find the best values of a, b and c for the following table :

T:	1	2	3	4	5	6	7
V:	2.31	2.01	3.80	1.66	1.55	1.47	1.41

(*U.P.T.U., MCA, 2010*)

24.6 FITTING OF OTHER CURVES

(1) $y = ax^b$

Taking logarithms, $\log_{10} y = \log_{10} a + b \log_{10} x$

i.e., $Y = A + bX$ where $X = \log_{10} x$, $Y = \log_{10} y$ and $A = \log_{10} a$. (*i*)

$\therefore$ The normal equations for (*i*) are : $\Sigma Y = nA + b\Sigma X$, $\Sigma XY = A\Sigma X + b\Sigma X^2$

from which A and b can be determined. Then a can be calculated from $A = \log_{10} a$.

(2) $y = ae^{bx}$ (*Exponential curve*)

Taking logarithms, $\log_{10} y = \log_{10} a + bx \log_{10} e$

i.e., $Y = A + Bx$ where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b \log_{10} e$

Here the normal equations are : $\Sigma Y = nA + B\Sigma x$, $\Sigma xY = A\Sigma x + B\Sigma x^2$

from which A, B can be found and consequently a, b can be calculated.

(3) $xy^a = b$ (*or $pv^\gamma = k$*) (*Gas equation*)

Taking logarithms, $\log_{10} x + a \log_{10} y = \log_{10} b$ or $\log_{10} y = \dfrac{1}{a} \log_{10} b - \dfrac{1}{a} \log_{10} x$.

This is of the form $Y = A + BX$

where $X = \log_{10} x$, $Y = \log_{10} y$, $A = \frac{1}{a} \log_{10} b$, $B = -\frac{1}{a}$.

Here also the problem reduces to finding a straight line of best fit through the given data.

Example 24.8. *Find the least squares fit of the form $y = a_0 + a_1x^2$ to the following data :*

x :	-1	0	1	2
y :	2	5	3	0

(U.P.T.U., 2008)

Solution. Putting $x^2 = X$, we have $y = a_0 + a_1X$

$\therefore$ the normal equations are : $\Sigma y = 4a + a_1\Sigma X$; $\Sigma Y = a_0\Sigma X + a_1\Sigma X^2$.

The values of ΣX, ΣX^2 etc. are calculated below :

x	y	X	X^2	XY
-1	2	1	1	2
0	5	0	0	0
1	3	1	1	3
2	0	4	16	0
	$\Sigma y = 10$	$\Sigma X = 10$	$\Sigma X^2 = 18$	$\Sigma XY = 5$

$\therefore$ the normal equations become $10 = 400 + 6a_1$; $5 = 600 + 18a_1$

Solving these equations we get, $a_0 = 4.167$, $a_1 = -1.111$.

Hence the curve of best fit is $y = 4.167 - 1.111X$ *i.e.*, $y = 4.167 - 1.111x^2$.

Example 24.9. *An experiment gave the following values :*

v (ft/min) :	350	400	500	600
t (min) :	61	26	7	26

It is known that v and t are connected by the relation $v = at^b$. Find the best possible values of a and b.

Solution. We have $\log_{10} v = \log_b a + b \log_{10} t$

or $y = A + 6X$, where $X = \log_{10} t$, $y = \log_{10} v$, $A = \log_{10} a$

$\therefore$ the normal equations are

$$\Sigma Y = 4A + b\Sigma X \quad ...(i)$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \quad ...(ii)$$

Now ΣX etc. are calculated as in the following table :

v	t	$X = \log_{10} t$	$y = \log_{10} v$	XY	X^2
350	61	1.7853	2.5441	4.542	3.187
400	26	1.4150	2.6021	3.682	2.002
500	7	0.8451	2.6990	2.281	0.714
600	2.6	0.4150	2.7782	1.153	0.172
Total		4.4604	10.6234	11.658	6.075

$\therefore$ Equations (i) and (ii) become

$$4A + 4.46b = 10.623 \text{ ; } 4.46A + 6.075b = 11.658$$

Solving these, $A = 2.845$, $b = -0.1697$

$\therefore$ $a = \text{antilog } A = \text{antilog } 2.845 = 699.8$.

Example 24.10. *Predict the mean radiation dose at an altitude of 3000 feet by fitting an exponential curve to the given data :*

Altitude (x) :	50	450	780	1200	4400	4800	5300
Dose of radiation (y) :	28	30	32	36	51	58	69

(S.V.T.U., 2007 ; J.N.T.U., 2003)

Solution. Let $y = ab^x$ be the exponential curve.

Then $\log_{10} y = \log_{10} a + x \log_{10} b$

or $Y = A + Bx$ where $Y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$

$\therefore$ the normal equations are

$$\Sigma Y = 7A + B\,\Sigma x \quad ...(i)$$

$$\Sigma x\,Y = A\,\Sigma x + B\,\Sigma x^2 \quad ...(ii)$$

Now Σx etc. are calculated as follows :

x	y	$Y = \log_{10} y$	xY	x^2
50	28	1.447158	72.3579	2500
450	30	1.477121	664.7044	202500
780	32	1.505150	1174.0170	608400
1200	36	1.556303	1867.5636	1440000
4400	51	1.707570	7513.3080	19360000
4800	58	1.763428	8464.4544	23040000
5300	69	1.838849	9745.8997	28090000
$\Sigma = 16980$		11.295579	29502.305	72743400

$\therefore$ equations (i) and (ii) become

$$11.295579 = 7A + 16980B$$

$$29502.305 = 16980A + 72743400B$$

Solving these equations, we get $A = 1.4521015$, $B = 0.0000666289$

$\therefore$ $\log_{10} y = Y = 1.4521015 + 0.0000666289x$

Hence $y\ (at\ x = 3000) = 44.874$ *i.e.* 44.9 approx.

Example 24.11. *The pressure and volume of a gas are related by the equation $pv^\gamma = k$, γ and k being constants. Fit this equation to the following set of observations :*

p (kg/cm²)	:	*0.5*	*1.0*	*1.5*	*2.0*	*2.5*	*3.0*
v (litres)	:	*1.62*	*1.00*	*0.75*	*0.62*	*0.52*	*0.46*

(V.T.U., 2011)

Solution. We have $\log_{10} p + \gamma \log_{10} v = \log_{10} k$

or $$\log_{10} v = \frac{1}{\gamma} \log_{10} k - \frac{1}{\gamma} \log_{10} p \quad \text{or} \quad Y = A + BX$$

where $X = \log_{10} p, Y = \log_{10} v, A = \frac{1}{\gamma} \log_{10} k, B = -\frac{1}{\gamma}$.

$\therefore$ the normal equations are

$$\Sigma Y = 6A + B\Sigma X \quad ...(i)$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad ...(ii)$$

Now ΣX etc. are calculated as follows :

p	v	$X = \log_{10} p$	$Y = \log_{10} v$	XY	X^2
.5	1.62	−0.3010	0.2095	−0.0630	0.0906
1.0	1.00	0.0000	0.0000	−0.0000	0.0000
1.5	0.75	0.1761	−0.1249	−0.0220	0.0310
2.0	0.62	0.3010	−0.2076	−0.0625	0.0906
2.5	0.52	0.3979	−0.2840	−0.1130	0.1583
3.0	0.46	0.4771	−0.3372	−0.1609	0.2276
Total		1.0511	−0.7442	−0.4214	0.5981

$\therefore$ equations (i) and (ii) become

$$6A + 1.0511B = -0.7442$$

$$1.0511A + 0.5981B = -0.4214$$

Solving these, we get $A = 0.0132$, $B = -0.7836$.

$\therefore \quad \gamma = -1/B = 1.1276$ and $k = \text{antilog}(A\gamma) = \text{antilog}(0.0168) = 1.039$.

Hence the equation of best fit is $pv^{1.276} = 1.039$.

PROBLEMS 24.3

1. If V (km/hr) and R (kg/ton) are related by a relation of the type $R = a + bV^2$, find by the method of least squares a and b with the help of the following table :

V :	10	20	30	40	50
R :	8	10	15	21	30

(Indore, 2008)

2. Using the method of least squares fit the curve $y = ax + bx^2$ to following observations :

x :	1	2	3	4	5
y :	1.8	5.1	8.9	14.1	19.8

3. Fit the curve $y = ax + b/x$ to the following data :

x :	1	2	3	4	5	6	7	8
y :	5.4	6.3	8.2	10.3	12.6	14.9	17.3	19.5

(U.P.T.U., 2010)

4. Estimate y at $x = 2.25$ by fitting the *indifference curve* of the form $xy = Ax + B$ to the following data :

x :	1	2	3	4
y :	3	1.5	6	7.5

(J.N.T.U., 2003)

5. Find the least square curve $y = ax + b/x$ for the following data :

x :	1	2	3	4
y :	-1.5	0.99	3.88	7.66

(Madras, 2003)

6. Predict y at $x = 3.75$, by fitting a *power curve* $y = ax^b$ to the given data :

x :	1	2	3	4	5	6
y :	298	4.26	5.21	6.10	6.80	7.50

(J.N.T.U., 2003)

7. Fit the curve of the form $y = ae^{bx}$ to the following data :

x :	77	100	185	239	285
y :	2.4	3.4	7.0	11.1	19.6

(V.T.U., 2011 S ; J.N.T.U., 2006)

8. Obtain the least squares fit of the form $f(t) = ae^{-3t} + be^{-2t}$ for the data :

x :	0.1	0.2	0.3	0.4
$f(t)$:	0.76	0.58	0.44	0.35

(U.P.T.U., 2008)

9. The voltage v across a capacitor at time t seconds is given by the following table :

t :	0	2	4	6	8
v :	150	63	28	12	5.6

Use the method of least squares to fit a curve of the form $v = ae^{kt}$ to this data.

10. Using method of least squares, fit a relation of the form $y = ab^x$ to the following data :

x :	2	3	4	5	6
y :	144	172.8	207.4	248.8	298.5

(Tiruchirapalli, 2001)

24.7 METHOD OF GROUP AVERAGES

Let the straight line, $\quad y = a + bx \quad$ ...(1)

fit the set of n observations

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ quite closely. (Fig. 24.5)

When $x = x_1$, the *observed (or experimental) value of* $y = y_1 = L_1P_1$ and from (1),

$$y = a + bx_1 = L_1M_1,$$

which is known as the *expected* (or *calculated*) *value* of y at L_1.

Then e_1 = observed value at L_1 – expected value at L_1

$$= y_1 - (a + bx_1) = M_1P_1,$$

which is called the *error* (or *residual*) at x_1. Similarly the errors for the other observations are

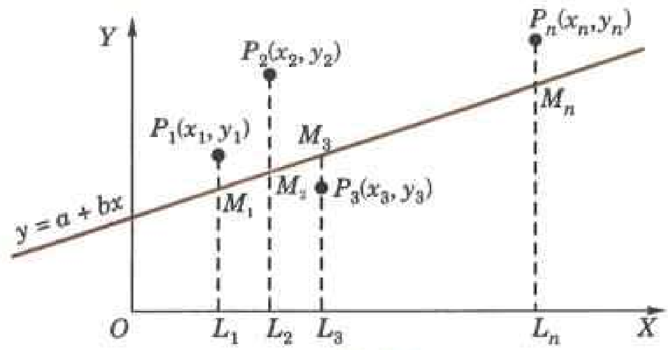


Fig. 24.5

$$e_2 = y_2 - (a + bx_2) = M_2P_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$e_n = y_n - (a + bx_n) = M_nP_n$$

Some of these errors may be positive and others negative.

The method of group averages is based on the assumption that the sum of the residuals is zero. To find the constants a and b is (1), we require two equations. As such we divide the data into two groups : the first containing k observations

$$(x_1, y_1), (x_2, y_2) \dots (x_k, y_k) ;$$

and the second group having the remaining $n - k$ observations

$$(x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2}), \dots, (x_n, y_n).$$

Assuming that the sum of the errors in each group is zero, we get

$$\{y_1 - (a + bx_1)\} + \{y_2 - (a + bx_2)\} + \dots \{y_k - (a + bx_k)\} = 0$$

$$\{y_{k+1} - (a + bx_{k+1})\} + \{y_{k+2} - (a + bx_{k+2})\} + \dots + \{y_n - (a + bx_n)\} = 0$$

On simplification, we obtain

$$\frac{y_1 + y_2 + \dots + y_k}{k} = a + b\,\frac{x_1 + x_2 + \dots + x_k}{k} \qquad \dots(2)$$

$$\frac{y_{k+1} + y_{k+2} + \dots + y_n}{n-k} = a + b\,\frac{x_{k+1} + x_{k+2} + \dots + x_n}{n-k} \qquad \dots(3)$$

In (2), $\frac{1}{k}(x_1 + x_2 + \dots + x_k)$ and $\frac{1}{k}(y_1 + y_2 + \dots + y_k)$ are simply the average values of x's and y's of the first group. Hence the equations (2) and (3) are obtained from (1) by replacing x and y by their respective averages of the two groups. Solving (2) and (3), we get a and b.

Obs. The main drawback of this method is that a different grouping of the observations will give different values of a and b. In practice, we divide the data in such a way that each group contains almost an equal number of observations.

Example 24.12. *The latent heat of vaporisation of steam r, is given in the following table at different temperatures t :*

t :	*40*	*50*	*60*	*70*	*80*	*90*	*100*	*110*
r :	*1069.1*	*1063.6*	*1058.2*	*1052.7*	*1049.3*	*1041.8*	*1036.3*	*1030.8*

For this range of temperature, a relation of the form r = a + bt is known to fit the data. Find the values of a and b by the method of group averages. (*Madras, 2003*)

Solution. Let us divide the data into two groups each containing four readings. Then we have

t	r	t	r
40	1069.1	80	1049.3
50	1063.6	90	1041.8
60	1058.2	100	1036.3
70	1052.7	110	1030.8
$\Sigma t = 220$	$\Sigma r = 4243.6$	$\Sigma t = 380$	$\Sigma r = 4158.2$

Substituting the averages of t's and r's of the two groups in the given relation, we get

$$\frac{4243.6}{4} = a + b\,\frac{220}{4} \quad i.e.,\ 1060.9 = a + 55b \qquad ...(i)$$

$$\frac{4158.2}{4} = a + b\,\frac{380}{4} \quad i.e.,\ 1039.55 = a + 95b \qquad ...(ii)$$

Solving (*i*) and (*ii*), we obtain

$$a = 1090.26,\ b = -\,0.534.$$

24.8 FITTING A PARABOLA

We have applied the method of averages to *linear law* involving two constants only. To fit the parabola

$$y = a + bx + cx^2 \qquad ...(1)$$

which contains three constants, to a set of observations, we proceed as follows :

Let (x_1, y_1) be a point on (1) satisfying the given data so that

$$y_1 = a + bx_1 + cx_1^2$$

Then $$y - y_1 = b\,(x - x_1) + c\,(x^2 - x_1^2)$$

or $$\frac{y - y_1}{x - x_1} = b + c\,(x + x_1)$$

Putting $x + x_1 = X$ and $(y - y_1)/(x - x_1) = Y$, it takes the linear form

$$Y = b + cX.$$

Now b and c can be found as before.

Example 24.13. *The corresponding values of x and y are given by the following table :*

x :	*87.5*	*84.0*	*77.8*	*63.7*	*46.7*	*36.9*
y :	*292*	*283*	*270*	*235*	*197*	*181*

Solution. Taking $x = 84$, $y = 283$ as a particular point on $y = a + bx + cx^2$,

we get $$283 = a + b\,(84) + c\,(84)^2 \qquad ...(i)$$

$\therefore$ $$y - 283 = b\,(x - 84) + c\,[x^2 - (84)^2]$$

or $$\frac{y - 283}{x - 84} = b + c\,(x + 84)$$

i.e., $$Y = b + cX \qquad ...(ii)$$

where $X = x + 84$, $Y = (y - 283)/(x - 84)$.

Now we have the following table of values :

x	y	$X = x + 84$	$Y = (y - 283)/(x - 84)$
87.5	292	171.5	2.571
84.0	283	—	—
77.8	270	161.8	2.097
		$\Sigma X = 333.3$	$\Sigma Y = 4.668$
63.7	235	147.7	2.364
46.7	197	130.7	2.306
36.9	181	120.9	2.166
		$\Sigma X = 399.3$	$\Sigma Y = 6.836$

Substituting the averages of X and Y in (*ii*), we get

$$\frac{4.668}{2} = b + c\,\frac{333.3}{2} \quad i.e.,\ 2.33 = b + 166.65\,c \qquad ...(iii)$$

$$\frac{6.836}{3} = b + c\,\frac{399.3}{3} \quad i.e.,\ 2.28 = b + 131.1\,c \qquad ...(iv)$$

(iv)–(iii) gives $c = 0.0014$

and (iii) gives $b = 2.0967$ i.e., 2.1 nearly

From (i), we get $a = 96.9988$ i.e., 97 nearly.

Hence the parabola of fit is

$$y = 97 + 2.1x + .0014x^2.$$

Example 24.14. *The train resistance R (lbs/ton) is measured for the following values of its velocity V (km/hr) :*

V :	20	40	60	80	100
R :	5	9	14	25	36

If R is related to V by the formula $R = a + bV^n$, find a, b, and n.

Solution. To find a, we take the following three values of v which are in G.P. :

$$v_1 = 20, \quad v_2 = 40, \quad v_3 = 80$$

Then $R_1 = 5, \quad R_2 = 9, \quad R_3 = 25$

$\therefore \quad (R_1 - a)(R_3 - a) = (R_2 - a)^2$

whence
$$a = \frac{R_1R_3 - R_2^2}{R_1 + R_3 - 2R_2} = 3.67$$

Thus $R - 3.67 = bV^n$ or $\log_{10}(R - 3.67) = \log_{10} b + n \log_{10} V$

i.e., $$Y = k + nX \qquad ...(i)$$

where $X = \log_{10} V$, $Y = \log_{10}(R - 3.67)$, $k = \log_{10} b$.

Now we have the following table of values :

V	R	$X = \log_{10} V$	$Y = \log_{10}(R - 3.67)$
20	5	1.3010	0.1238
40	9	1.6021	0.7267
60	14	1.7782	1.0141
		$\Sigma X = 4.6813$	$\Sigma Y = 1.8646$
80	25	1.9031	1.3290
100	36	2.0000	1.5096
		$\Sigma X = 3.9031$	$\Sigma X = 2.8396$

Substituting the averages of X's and Y's in (i), we obtain

$$\frac{1.8646}{2} = k + n\frac{4.6813}{2} \qquad i.e., \; 0.6215 = k + 1.5604\,n \qquad ...(ii)$$

$$\frac{2.8386}{2} = k + n\frac{3.9031}{2} \qquad i.e., \; 1.4193 = k + 1.9516\,n \qquad ...(iii)$$

Solving (ii) and (iii), we get $n = 2.04$, $k = -2.56$ approx.

$$b = \text{antilog } k = \text{antilog}(-2.56) = 0.0028.$$

PROBLEMS 24.4

1. Fit a straight line of the form $y = a + bx$ to the following data by the method of group averages :

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30

(*Tiruchirapalli, 2001*)

2. The weights of a calf taken at weekly intervals are given below :

Age :	1	2	3	4	5	6	7	8	9	10
Weight :	52.5	58.7	65.0	70.2	75.4	81.1	87.2	95.5	102.2	108.4

Find a straight line of best fit.

3. Using the method of averages, fit a parabola $y = ax^2 + bx + c$ to the following data :

x:	20	40	60	80	100	120
y:	5.5	9.1	14.9	22.8	33.3	46.0

4. While testing a centrifugal pump, the following data is obtained. It is assumed to fit the equation $y = a + bx + cx^2$, where x is the discharge in litre/sec and y, head in metres of water. Find the values of the constants a, b, c by the method of group averages.

x:	2	2.5	3	3.5	4	4.5	5	5.5	6
y:	18	17.8	17.5	17	15.8	14.8	13.3	11.7	9

5. By the method of averages, fit a curve of the form $y = ae^{bx}$ to the following data :

x:	5	15	20	30	35	40
y:	10	14	25	40	50	62

(*Madras, 2002*)

24.9 METHOD OF MOMENTS

Let $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ be the set of n observations such that

$$x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h \text{ (say)}$$

We define the *moments of the observed values* of y as follows :

m_1, the 1st moment $= h\Sigma y$

m_2, the 2nd moment $= h\Sigma xy$

m_3, the 3rd moment $= h\Sigma x^2 y$ and so on.

Let the curve fitting the given data be $y = f(x)$. Then the *moments of the calculated values* of y are

μ_1, the 1st moment $= \int y\,dx$

μ_2, the 2nd moment $= \int xy\,dx$

μ_3, the 3rd moment $= \int x^2 y\,dx$ and so on.

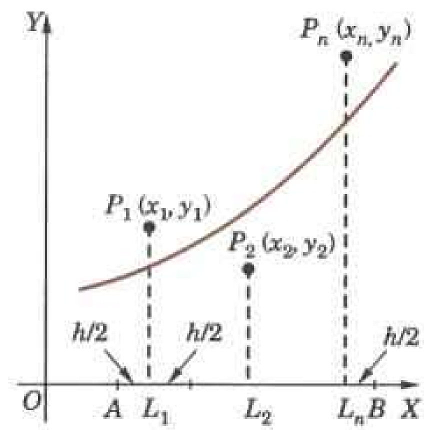


Fig. 24.6

This method is based on the assumption that the moment of the observed values of y are respectively equal to the moments of the calculated values of y i.e., $m_1 = \mu_1, m_2 = \mu_2, m_3 = \mu_3$ *etc. These equations (known as observation equations)* are used to determine the constants in $f(x)$.

m's are calculated from the tabulated values of x and y while μ's are computed as follows :

In Fig. 24.6, y_1 the ordinate of P_1 $(x = x_1)$, can be taken as the value of y at the mid-point of the interval $(x_1 - h/2, x_1 + h/2)$. Similarly, y_n, the ordinate of P_n $(x = x_n)$, can be taken as the value of y at the mid-point of the interval $(x_n - h/2, x_n + h/2)$. If A and B be the points such that

$$OA = x_1 - h/2 \text{ and } OB = x_n + h/2,$$

then
$$\mu_1 = \int y\,dx = \int_{x_1 - h/2}^{x_n + h/2} f(x)\,dx$$

$$\mu_2 = \int_{x_1 - h/2}^{x_n + h/2} xf(x)\,dx$$

and
$$\mu_3 = \int_{x_1 - h/2}^{x_n + h/2} x^2 f(x)\,dx.$$

Example 24.15. *Fit a straight line $y = a + bx$ to the following data by the method of moments :*

x:	1	2	3	4
y:	16	19	23	26

(*Madras, 2001 S*)

Solution. Since only two constants a and b are to be found, it is sufficient to calculate the first two moments in each case. Here $h = 1$.

$$m_1 = h\Sigma y = 1\,(16 + 19 + 23 + 26) = 84$$

$$m_2 = h\Sigma xy = 1\,(1 \times 16 + 2 \times 19 + 3 \times 22 + 4 \times 26) = 227$$

To compute the moments of calculated values of $y = a + bx$, the limits of integration will be $1 - h/2$ and $4 + h/2$ *i.e.*, 0.5 to 4.5

$$\therefore \qquad \mu_1 = 2\int_{0.5}^{4.5}(a + bx)\,dx = \left| ax + b\frac{x^2}{2} \right|_{0.5}^{4.5} = 4a + 10b$$

$$\mu_2 = \int_{0.5}^{4.5} x(a + bx)\,dx = 10a + \frac{91}{3}b.$$

Thus, the observation equations $m_r = \eta_r\ (r = 1, 2)$ are $4a + 10b = 84$; $10a + \frac{91}{3}b = 227$

Solving these, $a = 13.02$ and $b = 3.19$.

Hence the required equation is $y = 13.02 + 3.19x$.

Example 24.16. *Given the following data :*

x :	*0*	*1*	*2*	*3*	*4*
y :	*1*	*5*	*10*	*22*	*38*

find the parabola of best fit by the method of moments.

Solution. Let the parabola of best fit be $y = a + bx + cx^2$ ...(*i*)

Since three constants are to be found, we calculate the first three moments in each case. Here $h = 1$.

$$m_1 = h\Sigma y = 1\,(1 + 5 + 10 + 22 + 38) = 76$$
$$m_2 = h\Sigma xy = 1\,(0 + 5 + 20 + 66 + 152) = 243$$
$$m_3 = h\Sigma x^2y = 1\,(0 + 5 + 40 + 198 + 608) = 851$$

For computing the moments of calculated values of (*i*), the limits of integration will be $0 - h/2$ and $4 + h/2$ *i.e.*, -0.5 and 4.5.

$$\therefore \qquad \mu_1 = \int_{-0.5}^{4.5}(a + bx + cx^2)dx = 5a + 10b + 30.4c$$

$$\mu_2 = \int_{-0.5}^{4.5} x\,(a + bx + cx^2)\,dx = 10a + 30.4b + 102.5c$$

$$\mu_3 = \int_{-0.5}^{4.5} x^2\,(a + bx + cx^2)\,dx = 30.4a + 102.5b + 369.1c$$

Thus the observation equations $m_r = \mu_r\ (r = 1, 2, 3)$ are

$$5a + 10b + 30.4c = 76\ ;\ 10a + 30.4 + 102.5c = 243\ ;\ 30.4a + 102.5b + 369.1c = 851$$

Solving these equations, we get $a = 0.4$, $b = 3.15$, $c = 1.4$.

Hence the parabola of best fit is $y = 0.4 + 3.15x + 1.4x^2$.

PROBLEMS 24.5

1. Use the method of moments to fit the straight line $y = a + bx$ to the data :

x :	1	2	3	4
y :	0.17	0.18	0.23	0.32

2. Fit a straight line to the following data, using the method of moments :

x :	1	3	5	7	9
y :	1.5	2.8	4.0	4.7	6.0

(*Madras, 2001*)

3. Fit a parabola of the form $y = a + bx + cx^2$ to the data :

x :	1	2	3	4
y :	1.7	1.8	2.3	3.2

by the method of moments.

4. By using the method of moments, fit a parabola to the following data :

x :	1	2	3	4
y :	0.30	0.64	1.32	5.40

(*Madras, 2000 S*)

24.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 24.6

Fill up the blanks or choose the correct answer in the following problems :

1. The law $y = ax^2 + bx$ converted to linear form is ....
2. The gas equation $pv^r = k$ can be reduced to $y = a + bx$ where $a =$ .... and $b =$ ....
3. The principle of *'least squares'* states that ....
4. $y = ax^b + c$ in linear form is ....
5. To fit the straight line $y = mx + c$ to n observations, the normal equations are
   (*i*) $\Sigma y = n\,\Sigma x + \Sigma cm$, $\Sigma xy = c\,\Sigma x^2 + c\,\Sigma n$.
   (*ii*) $\Sigma y = m\,\Sigma x + nc$, $\Sigma xy = m\,\Sigma x^2 + c\,\Sigma x$.
   (*iii*) $\Sigma y = c\,\Sigma x + m\,\Sigma n$, $\Sigma xy = c\,\Sigma x^2 + m\,\Sigma x$.
6. To fit $y = ab^x$ by least square method, normal equations are ....
7. The observation equations for fitting a straight line by *method of moments* are ....
8. The *method of group averages* is based on the assumption that the sum of the residuals is ....
9. $y = ax^2 + b\log_{10} x$ reduced to linear law takes the form ....
10. Given $\begin{bmatrix} x: & 0 & 1 & 2 \\ y: & 0 & 1.1 & 2.1 \end{bmatrix}$ then the straight line of best fit is ....
11. The *method of moments* is based on the assumption that ....
12. In $y = a + bx$, $\Sigma x = 50$, $\Sigma y = 80$, $\Sigma xy = 1030$, $\Sigma x^2 = 750$ and $n = 10$, then $a =$ ...., $b =$ ....
13. $y = x/(ax + b)$ in linear form is ....
14. If $y = a + bx + cx^2$ and

x :	0	1	2	3	4
y :	1	1.8	1.3	2.5	7.3

   then the first normal equation is :
   (α) $15 = 5a + 10b + 29c$, (β) $15 = 5a + 10b + 31c$
   (γ) $12.9 = 5a + 10b + 30c$ (δ) $34 = 5a + 10b + 27c$.
15. If $y = 2x + 5$ is the best fit for 8 pairs of values (x, y) by the method of least squares and $\Sigma y = 120$, then $\Sigma X =$
   (*a*) 35 (*b*) 40 (*c*) 45 (*d*) 30.

CHAPTER 25

Statistical Methods

25.1 INTRODUCTION

Statistics deals with the methods for collection, classification and analysis of numerial data for drawing valid conclusions and making reasonable decisions. It has meaningful applications in production engineering, in the analysis of experimental data, etc. The importance of statistical methods in engineering is on the increase. As such we shall now introduce the student to this interesting field.

25.2 (1) COLLECTION OF DATA

The collection of data constitutes the starting point of any statistical investigation. Data may be collected for each and every unit of the whole lot (*population*), for it would ensure greater accuracy. But complete enumeration is prohibitively expensive and time consuming. As such out of a very large number of items, a few of them (a *sample*) are selected and conclusions drawn on the basis of this sample are taken to hold for the population.

(2) **Classification of data.** The data collected in the course of an inquiry is not in an easily assimilable form. As such, its proper classification is necessary for making intelligent inferences. The classification is done by dividing the raw data into a convenient number of groups according to the values of the variable and finding the frequency of the variable in each group.

Let us, for example, consider the raw data relating to marks obtained in Mechanics by a group of 64 students :

79	88	75	60	93	71	59	85
84	75	82	68	90	62	88	76
65	75	87	74	62	95	78	63
78	82	75	91	77	69	74	68
67	73	81	72	63	76	75	85
80	73	57	88	78	62	76	53
62	67	97	78	85	76	65	71
78	89	61	75	95	60	79	83

This data can conveniently be grouped and shown in a tabular form as follows :

Class	*Frequency*	*Cumulative frequency*
50—54	1	1
55—59	2	3
60—64	9	12
65—69	7	19
70—74	8	27
75—79	17	44
80—84	6	50
85—89	8	58
90—94	3	61
95—99	3	64
	Total = 64	

It would be seen from the above table that there is one student getting marks between 50—54, two students getting marks between 55—59, nine students getting marks between 60—64 and so on. Thus the 64 figure have been put into only 10 groups, called the **classes**. The width of the class is called the **class interval** and the number in that interval is called the **frequency.** The mid-point or the mid-value of the class is called the **class mark.** The above table showing the classes and the corresponding frequencies is called a *frequency table. Thus a set of raw data summarised by distributing it into a number of classes alongwith their frequencies is known as a* **frequency distribution**.

While forming a frequency distribution, the number of classes should not ordinarily exceed 20, and should not, in general, be less than 10. As far as possible, the class intervals should be of equal width.

(3) **Cumulative frequency.** In some investigations, we require the number of items less than a certain value. We add up the frequencies of the classes upto that value and call this number as the *cumulative frequency*. In the above table, the third column shows the cumulative frequencies, *i.e.*, the number of students, getting less than 54 marks, less than 59 marks and so on.

25.3 GRAPHICAL REPRESENTATION

A convenient way of representing a sample frequency distribution is by means of graphs. It gives to the eyes the general run of the observations and at the same time makes the raw data readily intelligible. We give below the important types of graphs in use :

(1) **Histogram.** A histogram is drawn by erecting rectangles over the class intervals, such that the areas of the rectangles are proportional to the class frequencies. If the class intervals are of equal size, the height of the rectangles will be proportional to the class frequencies themselves (Fig. 25.1).

(2) **Frequency polygon.** A frequency polygon for an ungrouped data can be obtained by joining points plotted with the variable values as the abscissae and the frequencies as the ordinates. For a grouped distribution, the abscissae of the points will be the mid-values of the class intervals. In case the intervals are equal, the frequency polygon can be obtained by joining the middle points of the upper sides of the rectangles of the histogram by straight lines (shown by dotted lines in Fig. 25.1). If the class intervals become very very small, the frequency polygon takes the form of a smooth curve called the *frequency curve*.

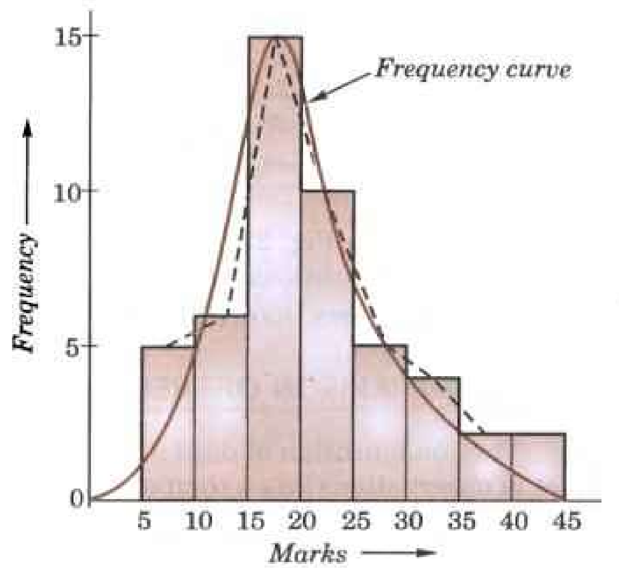


Fig. 25.1

(3) **Cumulative frequency curve-Ogive.** Very often, it is desired to show in a diagrammatic form, not the relative frequencies in the various intervals, but the cumulative frequencies above or below a given value. For example, we may wish to read off from a diagram the number or proportions of people whose income is not less than any given amount, or proportion of people whose height does not exceed any stated value. Diagrams of

this type are known as *cumulative frequency curves* or *ogives*. These are of two kinds '*more than*' or '*less than*' and typically they look somewhat like a long drawn S (Fig. 25.2).

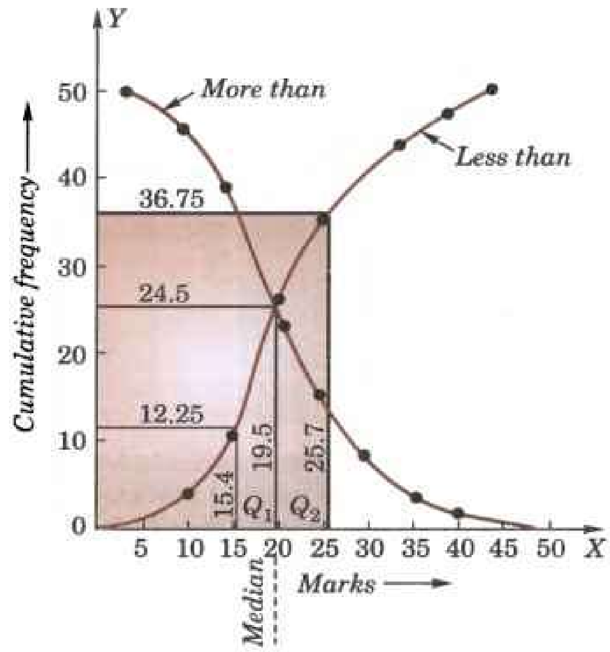


Fig. 25.2

Example 25.1. *Draw the histogram, frequency polygon, frequency curve and the ogive 'less than' and 'more than' from the following distribution of marks obtained by 49 students :*

Class	*Frequency*	*Cumulative frequency*	
(Marks group)	*(No. of students)*	*(Less than)*	*(More than)*
5—10	5	5	49
10—15	6	11	44
15—20	15	26	38
20—25	10	36	23
25—30	5	41	13
30—35	4	45	8
35—40	2	47	4
40—45	2	49	2

Solution. In Fig. 25.1, the rectangles show the *histogram*; the dotted polygon represents the *frequency polygon* and the smooth curve is the *frequency curve*.

The *ogives* 'less than' and 'more than' are shown in Fig. 25.2.

25.4 COMPARISON OF FREQUENCY DISTRIBUTIONS

The condensation of data in the form of a frequency distribution is very useful as far as it brings a long series of observations into a compact form. But in practice, we are generally interested in comparing two or more series. The inherent inability of the human mind to grasp in its entirety even the data in the form of a frequency distribution compels us to seek for certain constants which could concisely give an insight into the important characteristics of the series. The chief constants which summarise the fundamental characteristics of the frequency distributions are (*i*) *Measures of central tendency*, (*ii*) *Measures of dispersion and* (*ii*) *Measures of skewness*.

25.5 MEASURES OF CENTRAL TENDENCY

A frequency distribution in general, shows clustering of the data around some central value. Finding of this central value or the average is of importance, as it gives a most representative value of the whole group.

Different methods give different averages which are known as the *measures of central tendency*. The commonly used measures of central value are *Mean, Median, Mode, Geometric mean and Harmonic mean.*

(1) Mean. If $x_1, x_2, x_3, ..., x_n$ are a set of n values of a variate, then the *arithmetic mean* (or simply *mean*) is given by

$$\bar{\mathbf{x}} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}, \text{ i.e. } \frac{\Sigma \mathbf{x}_i}{\mathbf{n}} \quad ...(1)$$

In a *frequency distribution*, if $x_1, x_2, ..., x_n$ be the mid-values of the class-intervals having frequencies $f_1, f_2, ..., f_n$ respectively, we have

$$\bar{\mathbf{x}} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{\Sigma \mathbf{f}_i \mathbf{x}_i}{\Sigma \mathbf{f}_i} \quad ...(2)$$

Calculation of mean. Direct method of computing especially when applied to grouped data involves heavy calculations and in order to avoid these, the following formulae are generally used :

I. Short-cut method $\quad \bar{\mathbf{x}} = \mathbf{A} + \dfrac{\Sigma \mathbf{f}_i d_i}{\Sigma \mathbf{f}_i}$ ...(3)

II. Step-deviation method $\quad \bar{\mathbf{x}} = \mathbf{A} + \mathbf{h}\dfrac{\Sigma \mathbf{f}_i \mathbf{u}_i}{\Sigma \mathbf{f}_i}$ ...(4)

where $d = x - A$ and $u = (x - A)/h$, A being an arbitrary origin and h the equal class interval.

Proof. If $x_1, x_2, ..., x_n$ are the mid-values of the classes with frequencies $f_1, f_2, ..., f_n$, we have

$$\Sigma f_i x_i = \Sigma f_i (A + d_i) = A\Sigma f_i + \Sigma f_i d_i$$

$$\therefore \quad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

Further $u_i = d_i/h$ or $d_i = hu_i$. Substituting this value in (3), we get (4).

Obs. *The algebraic sum of the deviations of all the variables from their mean is zero, for*

$$\Sigma f_i (x_i - \bar{x}) = \Sigma f_i x_i - \bar{x}\,\Sigma f_i = \Sigma f_i x_i - \frac{\Sigma f_i x_i}{\Sigma f_i} \cdot \Sigma f_i = 0.$$

Cor. *If $\bar{x}_1, \bar{x}_2$ be the means of two samples of size n_1 and n_2, then the mean $\bar{x}$ of the combined sample of size $n_1 + n_2$ is given by*

$$\bar{\mathbf{x}} = \frac{\mathbf{n}_1\bar{\mathbf{x}}_1 + \mathbf{n}_2\bar{\mathbf{x}}_2}{\mathbf{n}_1 + \mathbf{n}_2}$$

For $\quad n_1\bar{x}_1$ = sum of all observations of the first sample,

and $\quad n_2\bar{x}_2$ = sum of all observations of the second sample.

$\therefore$ sum of the observations of the combined sample $= n_1\bar{x}_1 + n_2\bar{x}_2$.

Also number of the observations in the combined sample $= n_1 + n_2$.

$\therefore$ mean of the combined sample $= \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$.

Example 25.2. *The following is the frequency distribution of a random sample of weekly earnings of 509 employees :*

Weekly earnings :	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
No. of employees :	3	6	10	15	24	42	75	90	79	55	36	26	19	13	9	7

Calculate the average weekly earnings.

Solution. The calculations are arranged in the following table. The arbitrary origin is generally taken as the value corresponding to the maximum frequency.

By direct method, we have

$$\text{Mean } \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{13,315}{509} = 26.16$$

By step-deviation method, we have

$$\bar{x} = A + h\frac{\Sigma fu}{\Sigma f} = 25 + 2 \times \frac{295}{509}$$

$$= 25 + 1.16 = 26.16, \text{ which is same as found above.}$$

Weekly earnings	Mid value x	No. of employees f	Step deviations		
			$f \times x$	$u = (x - 25)/2$	$f \times u$
10—12	11	3	33	−7	−21
12—14	13	6	78	−6	−36
14—16	15	10	150	−5	−50
16—18	17	15	255	−4	−60
18—20	19	24	456	−3	−72
20—22	21	42	882	−2	−84
22—24	23	75	1725	−1	−75
24—26	25	90	2250	0	−398
26—28	27	79	2133	1	79
28—30	29	55	1595	2	110
30—32	31	36	1116	3	108
32—34	33	26	858	4	104
34—36	35	19	665	5	95
36—38	37	13	481	6	78
38—40	39	9	351	7	63
40—42	41	7	287	8	56
					+ 693
		$\Sigma f = 509$	$\Sigma fx = 13{,}315$		$\Sigma fu = 295$

(2) Median. *If the values of a variable are arranged in the ascending order of magnitude, the median is the middle item if the number is odd and is the mean of the two middle items if the number is even.* Thus the median is equal to the mid-value, *i.e.*, the value which divides the total frequency into two equal parts.

For the grouped data,

$$\textbf{Median} = \mathbf{L} + \frac{(\frac{1}{2}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

where L = lower limit of the median class, N = total frequency,

f = frequency of the median class, h = width of the median class,

and C = cumulative frequency upto the class preceding the median class.

Quartiles. Quartiles are those values which divide the frequency into four equal parts, when the values are arranged in the ascending order of magnitude. The **lower quartile** (Q_1) is mid-way between the lower extreme and the median. The **upper quartile** (Q_3) is midway between the median the upper extreme.

For the grouped data, these are calculated by the formulae :

$$\mathbf{Q_1} = \mathbf{L} + \frac{(\frac{1}{4}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

and

$$\mathbf{Q_3} = \mathbf{L} + \frac{(\frac{3}{4}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

where L = lower limit of the class in which Q_1 or Q_3 lies, f = frequency of this class, h = width of the class

and C = cumulative frequency upto the class preceding the class in which Q_1 or Q_3 lies.

The difference between the upper and lower quartiles, i.e., Q_3—Q_1 is called the **inter-quartile range.**

Obs. The ogives give a ready method of marking on the curve the values of the median and the quartiles. The two ogives 'less than' and 'more than' cut each other at the median (Fig. 25.2).

(3) Mode. *The mode is defined as that value of the variable which occurs most frequently, i.e., the value of the maximum frequency.*

For a grouped distribution, it is given by the formula

$$\textbf{Mode} = \mathbf{L} + \frac{\Delta_1}{\Delta_1 + \Delta_2}\,\mathbf{h}$$

where L = lower limit of the class containing the mode,

Δ_1 = excess of modal frequency over frequency of preceding class,

Δ_2 = excess of modal frequency over following class,

and h = size of modal class.

For a frequency curve (Fig. 25.1), the abscissa of the highest ordinate determines the value of the mode. There may be one or more modes in a frequency curve. Curves having a single mode are termed as *unimodal*, those having two modes as *bi-modal* and those having more than two modes as *multi-modal*.

Obs. In a symmetrical distribution, the mean, median and mode coincide. For other distributions, however, they are different and are known to be connected by the empirical relationship :

Mean – Mode = 3(Mean – Median).

Example 25.3. *Calculate median and the lower and upper quartiles from the distribution of marks obtained by 49 students of example 25.1. Find also the semi-interquartile range and the mode.*

Solution. *Median* (or 49/2) falls in the class (15—20) and is given by

$$15 + \frac{(49/2) - 11}{15} \times 5 = 15 + \frac{13.5}{3} = 19.5 \text{ marks.}$$

Lower quartile Q_1 (or 49/4) = 12.25) also falls in the class 15—20.

$$\therefore \qquad Q_1 = 15 + \frac{(49/4) - 11}{15} \times 5 = 15 + \frac{12.5}{3} = 15.4 \text{ marks}$$

Upper quartile (or $\frac{3}{4} \times 49 = 36.75$) falls in the class 25—30.

$$\therefore \qquad Q_3 = 25 + \frac{36.75 - 36}{5} \times 5 = 25.75 \text{ marks.}$$

Semi-interquartile range = $\frac{1}{2}(Q_3 - Q_1) = \dfrac{25.75 - 15.4}{2} = \dfrac{10.35}{2} = 5.175.$

Mode. It is seen that the mode value falls in the class 15—20. Employing the formula for the grouped distribution, we have

$$Mode = 15 + \frac{15 - 6}{(15 - 6) + (15 - 10)} \times 5 = 18.2 \text{ marks.}$$

Obs. In Fig. 25.2, the ogives meet at a point whose abscissa is 19.5 which is the *median* of the distribution. The values for the lower and upper quartiles are similarly seen to be 15.4 (for frequency 12.25) and 25.7 (for frequency 36.75).

Example 25.4. *Given below are the marks obtained by a batch of 20 students in a certain class test in Physics and Chemistry.*

Roll No. of students	*Marks in Physics*	*Marks in Chemistry*	*Roll No. of students*	*Marks in Physics*	*Marks in Chemistry*
1	53	58	11	25	10
2	54	55	12	42	42
3	52	25	13	33	15
4	32	32	14	48	46
5	30	26	15	72	50
6	60	85	16	51	64
7	47	44	17	45	39
8	46	80	18	33	38
9	35	33	19	65	30
10	28	72	20	29	36

In which subject is the level of knowledge of the students higher ?

Solution. The subject for which the value of the median is higher will be the subject in which the level of knowledge of the students is higher. To find the median in each case, we arrange the marks in ascending order of magnitude :

Sr. No.	Marks in Physics	Marks in Chemistry	Sr. No.	Marks in Physics	Marks in Chemistry
1	25	10	11	46	42
2	28	15	12	47	44
3	29	25	13	48	46
4	30	26	14	51	50
5	32	30	15	52	55
6	33	32	16	53	58
7	33	33	17	54	64
8	35	36	18	60	72
9	42	38	19	65	80
10	45	39	20	72	85

Median marks in Physics = A.M. of marks of 10th and 11th terms

$$= \frac{45+46}{2} = 45.5$$

Median marks in Chemistry = A.M. of marks of 10th and 11th items.

$$= \frac{39+42}{2} = 40.5$$

Since the median marks in Physics is greater than the median marks in Chemistry; the level of knowledge in Physics is higher.

Example 25.5. *An incomplete frequency distribution is given as below :*

Variable :	*10—20*	*20—30*	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
Frequency :	*12*	*30*	*?*	*65*	*?*	*25*	*18*

Given that the total frequency is 229 and median is 46, find the missing frequencies.

Solution. Let f_1, f_2 be the missing frequencies of the classes 30—40 and 50—60 respectively.

Since the median lies in the class 40—50,

$$\therefore \qquad 46 = 40 + \frac{229/2 - (12 + 30 + f_1)}{65} \times 10$$

which gives $f_1 = 33.5$ which can be taken as 34.

$$\therefore \qquad f_2 = 229 - (12 + 30 + 34 + 65 + 25 + 18) = 45.$$

(4) Geometric mean. If $x_1, x_2, ..., x_n$ are a set of n observations, then the *geometric mean* is given by

$$\text{G.M.} = (x_1 x_2 \ldots x_n)^{1/n}$$

or
$$\log \text{G.M.} = \frac{1}{n}(\log x_1 + \log x_2 + \ldots + \log x_n) \qquad \ldots(1)$$

In a frequency distribution, let $x_1, x_2, ..., x_n$ be the central values with corresponding frequencies $f_1, f_2, ..., f_n$, we have

$$\text{G.M.} = \left[(x_1)^{f_1} \cdot (x_2)^{f_2} \ldots (x_n)^{f_n}\right]^{1/n} \quad \text{where } n = \Sigma f_i.$$

or
$$\log \text{G.M.} = \frac{1}{n}[f_1 \log x_1 + f_2 \log x_2 + \ldots + f_n \log x_n] \qquad \ldots(2)$$

Hence (1) and (2) show that logarithm of G.M. = A.M. of logarithms of the values.

(5) Harmonic mean. If $x_1, x_2, \ldots x_n$ be a set of n observations, then the *harmonic mean is defined as the reciprocal of the* (*arithmetic*) *mean of the reciprocals of the quantities.* Thus

$$\text{H.M.} = \frac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

In a frequency distribution, $\text{H.M.} = \dfrac{1}{\frac{1}{n}\left(\frac{f_1}{x_1} + \frac{f_2}{x_2} + \ldots + \frac{f_n}{x_n}\right)}$ where $n = \Sigma f_i$.

Example 25.6. *Three cities A, B, C are equidistant from each other. A motorist travels from A to B at 30 km/hr, from B to C at 40 km/hr, from C to A at 50 km/hr. Determine the average speed.*

Solution. Let $AB = BC = CA = s$ km

Time taken to travel from A to $B = s/30$

Time taken to travel from B to $C = s/40$

Time taken to travel from C to $A = s/50$

$\therefore$ average time taken $= \frac{1}{3}\left(\frac{s}{30} + \frac{s}{40} + \frac{s}{50}\right)$

Thus the average speed $= \dfrac{s}{\frac{1}{3}\left(\frac{s}{30} + \frac{s}{40} + \frac{s}{50}\right)}$

In other words, the average speed is the harmonic mean of 30, 40, 50 km/hr.

Hence the average speed $= \dfrac{1}{\frac{1}{3}\left(\frac{1}{30} + \frac{1}{40} + \frac{1}{50}\right)} = 38.3$ km/hr.

Obs. Of the various measures of central tendency, *the mean* is the most important for it can be computed easily. *The median*, though more easily calculable, cannot be applied with ease to theoretical analysis. Median is of advantage when there are exceptionally large and small values at the ends of the distribution.

The mode, though most easily calculated, has the least significance. It is particularly misleading in distributions which are small in numbers or highly unsymmetrical.

The geometrical mean though difficult to compute, finds application in cases like populations where we are concerned with a quantity whose changes tend to be directly proportional to the quantity itself.

The harmonic mean is useful in limited situations where time and rate or prices are involved.

PROBLEMS 25.1

1. Draw the histogram and frequency polygon for the following distribution. Also calculate the arithmetic mean :

Class interval	0—99	100—199	200—299	300—399	400—499	500—599	600—699	700—799
Frequency	10	54	184	264	246	40	1	1

2. The following marks were given to a batch of candidates :

66	62	45	79	32	51	56	60	51	49
25	42	54	54	58	70	43	58	50	52
38	67	50	59	48	65	71	30	46	55
82	51	63	45	53	40	35	56	70	52
67	55	57	30	63	42	74	58	44	55

Draw a cumulative frequency curve.

Hence find the proportion of candidates securing more than 50 marks. Also mark off the median, the first and third quartiles.

3. Find the mean, median and mode for the following :

Mid Value :	15	20	25	30	35	40	45	50	55
Frequency :	2	22	19	14	3	4	6	1	1

(Kerala, 1990)

4. Calculate mean, median and mode of the following data relating to weight of 120 articles :

Weight (in gm) :	0—10	10—20	20—30	30—40	40—50	50—60
No. of articles :	14	17	22	26	23	18

5. The population of a country was 300 million in 1971. It became 520 million in 1989. Calculate the percentage compound rate of growth per annum.

[**Hint.** Use $P_n = P_0(1 + r)^n$, r being the growth rate.]

6. The number of divorces per 1000 marriages in the United States increased from 84 in 1970 to 108 in 1990. Find the annual increase of the divorce rate for the period 1970 to 1990.

7. An aeroplane flies along the four sides of a square at speeds of 100, 200, 300 and 400 km/hr. respectively. What is the average speed of the plane in its flight around the square.

8. A man having to drive 90 km. wishes to achieve an average speed of 30 km/hr. For the first half of the journey, he averages only 20 km/hr. What must be his average speed for the second half of the journey if his overall average is to be 30 km/hr.

9. Following table gives the cumulative frequency of the age of a group of 199 teachers. Find the mean and median age of the group.

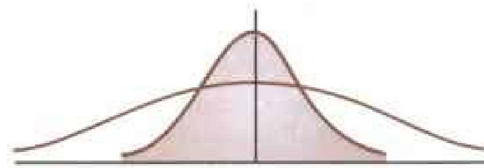
Age in years :	20—25	25—30	30—35	35—40	40—45	45—50	50—55	55—60	60—65	65—70
Cum. frequ. :	21	40	90	130	146	166	176	186	195	199

10. Recast the following cumulative table in the form of an ordinary frequency distribution and determine the median and the mode :

No. of days absent	*No. of students*	*No. of days absent*	*No. of students*
Less than 5	29	Less than 30	644
Less than 10	224	Less than 35	650
Less than 15	465	Less than 40	653
Less than 20	582	Less than 45	655
Less than 25	634		

25.6 MEASURES OF DISPERSION

Although measures of central tendency do exhibit one of the important characteristics of a distribution, yet they fail to give any idea as to how the individual values differ from the central value, *i.e.*, whether they are closely packed around the central value or widely scattered away from it. Two distributions may have the same mean and the same total frequency, yet they may differ in the extent to which the individual values may be spread about the average (*See* Fig. 25.3). The magnitude of such a variation is called *dispersion*. The important measures of dispersion are given below :



Same mean different dispersion

Fig. 25.3

(1) Range. *This is the simplest measure of dispersion and is given by the difference between the greatest and the least values in the distribution.* If the weekly wages of a group of labourers are

₹ 21 23 28 25 35 42 39 48

then *range* = Max. value – Min. value = 48 – 21 = ₹ 27.

(2) Quartile deviation or semi-interquartile range. One half of the interquartile range is called *quartile deviation*, or *semi-interquartile range*. If Q_1 and Q_3 are the first and third quartiles, the semi-interquartile range

$$Q = \tfrac{1}{2}(Q_3 - Q_1).$$

(3) Mean deviation. The mean deviation is the mean of the absolute differences of the values from the mean, median or mode. Thus *mean deviation (M.D.)*

$$= \frac{1}{n} \Sigma f_i \, |x_i - A|$$

where A is either the mean or the median or the mode. As the positive and negative differences have equal effects, only the absolute value of differences is taken into account.

(4) Standard deviation. The most important and the most powerful measure of dispersion is the *standard deviation (S.D.)* : generally denoted by σ. It is computed as the square root of the mean of the squares of the differences of the *variate* values from their mean.

Thus *standard deviation (S.D.)*

$$\sigma = \sqrt{\left[\frac{\Sigma \mathbf{f}_i(\mathbf{x}_i - \overline{\mathbf{x}})^2}{\mathbf{N}}\right]} \qquad ...(1)$$

where N is the total frequency Σf_i.

If however, the deviations are measured from any other value, say A, instead of $\overline{x}$, it is called the *root-mean-square deviation*.

The square of the standard deviation is known as the **variance.**

Calculation of S.D. The change of origin and the change of scale considerably reduces the labour in the calculation of standard deviation. The formulae for the computation of σ are as follows :

I. Short-cut method

$$\sigma = \sqrt{\left[\frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2\right]} \qquad ...(2)$$

II. Step-deviation method

$$\sigma = h\sqrt{\left\{\frac{\Sigma f_i d_i'^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i'}{\Sigma f_i}\right)^2\right\}} \qquad ...(3)$$

where $d_i = x_i - A$ and $d'_i = (x_i - A)/h$, being the assumed mean and h the equal class interval.

Proof. We know that $x_i - \bar{x} = (x_i - A) - (\bar{x} - A)$

$$\therefore \quad \Sigma f_i(x_i - \bar{x})^2 = \Sigma f_i[d_i - (\bar{x} - A)]^2 = \Sigma f_i d_i^2 + (\bar{x} - A)^2 \Sigma f_i - 2(\bar{x} - A)\Sigma f_i d_i$$

$$= \Sigma f_i d_i^2 - \frac{(\Sigma f_i d_i)^2}{\Sigma f_i} \qquad \left[\because \quad \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}\right]$$

Hence $$\sigma^2 = \frac{\Sigma f_i(x - \bar{x})^2}{\Sigma f_i} = \frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2$$

Further $d'_i = (x_i - A)/h = d_i/h$ or $d_i = hd'_i$, then substituting this value in (2), we get (3).

Obs. *The root mean square deviation is least when measured from the mean.*

The root mean square deviation is given by

$$s^2 = \frac{\Sigma f_i d_i^2}{\Sigma f_i} \quad \text{and} \quad \frac{\Sigma f_i d_i}{\Sigma f_i} = \left[A + \frac{\Sigma f_i d_i}{\Sigma f_i}\right] - A = \bar{x} - A$$

$\therefore$ from (2), we have $\quad s^2 = \sigma^2 + (\bar{x} - A)^2 \qquad ...(4)$

This shows that s^2 is always $> \sigma^2$ and the least value of $s^2 = \sigma^2$. This occurs when $A = \bar{x}$.

25.7 (1) COEFFICIENT OF VARIATION

The ratio of the standard deviation to the mean, is known as the *coefficient of variation*. As this is a ratio having no dimension, it is used for comparing the variations between the two groups with different means. It is often expressed as a percentage.

$\therefore$ *Coefficient of variation* $= \frac{\sigma}{\bar{x}} \times 100$

(2) Relations between measures of dispersion

(*i*) *Quartile deviation* = 2/3 *(standard deviation)*

(*ii*) *Mean deviation* = 4/5 *(standard deviation)*

25.8 STANDARD DEVIATION OF THE COMBINATION OF TWO GROUPS

If m_1, σ_1 *be the mean and S.D. of a sample of size* n_1 and m_2, σ_2 *be those for a sample of size* n_2*, then the S.D.* σ *of the combined sample of size* $n_1 + n_2$ *is given by*

$$(\mathbf{n_1 + n_2})\boldsymbol{\sigma}^2 = \mathbf{n_1}\boldsymbol{\sigma}_1^2 + \mathbf{n_2}\boldsymbol{\sigma}_2^2 + \mathbf{n_1 D_1^2 + n_2 D_2^2}$$

where $D_i = m_i - m$, m *being the mean of combined sample.*

From (4), we have $ns^2 = n\sigma^2 + n(\bar{x} - A)^2$ where n is the size of the sample.

i.e. sum of the squares of the deviations from $A = n\sigma^2 + n(\bar{x} - A)^2$.

Now let us apply this result to the first given sample taking A at m. Then, sum of the squares of the deviations of n_1 items from $m = n_1\sigma_1^2 + n_1(m_1 - m)^2$ ...(5)

Similarly for the second given sample taking A at m, sum of the squares of the deviations of n_2 items from $m = n_2\sigma_2^2 + n_2(m_2 - m)^2$ ...(6)

Adding (5) and (6), sum of the squares of the deviations of $n_1 + n_2$ items from m

$$= n_1\sigma_1^2 + n_2\sigma_2^2 + n_1(m_1 - m)^2 + n_2(m_2 - m)^2$$

$$\therefore \quad (n_1 + n_2)\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 D_1^2 + n_2 D_2^2$$

This result can be extended to the combination of any number of samples, giving a result of the form

$$(\Sigma n_i)\sigma^2 = \Sigma(n_i\sigma_i^2) + \Sigma(n_i D_i^2).$$

Example. 25.7. *Calculate the mean and standard deviation for the following :*

Size of item :	6	7	8	9	10	11	12
Frequency :	3	6	9	13	8	5	4

(V.T.U., 2001)

Solution. The calculations are arranged as follows :

Size of item x	*Frequency f*	*Deviation* $d = x - 9$	$f \times d$	$f \times d^2$
6	3	− 3	− 9	27
7	6	− 2	− 12	24
8	9	− 1	− 9	9
9	13	0	0	0
10	8	1	8	8
11	5	2	10	20
12	4	3	12	36
	$\Sigma f = 48$		$\Sigma fd = 0$	$\Sigma fd^2 = 124$

$$\therefore \qquad mean = 9 + \frac{\Sigma fd}{\Sigma f} = 9$$

$$Standard\ deviation = \sqrt{\left[\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}\right]} = \sqrt{\left(\frac{\Sigma fd^2}{\Sigma f}\right)} = \sqrt{\left(\frac{124}{48}\right)} = 1.607.$$

Example 24.8. *Calculate the mean and standard deviation of the following frequency distribution :*

Weekly wages is ₹	*No. of men*
4.5—12.5	4
12.5—20.5	24
20.5—28.5	21
28.5—36.5	18
36.5—44.5	5
44.5—52.5	3
52.5—60.5	5
60.5—68.5	8
68.5—76.5	2

Solution. The calculations are arranged in the table below :

Wages class ₹	*Mid value x*	*No. of men f*	*Step deviation* $d' = \frac{x - 32.5}{8}$	fd'	fd'^2
4.5—12.5	8.5	4	− 3	− 12	36
12.5—20.5	16.5	24	− 2	− 48	96
20.5—28.5	24.5	21	− 1	− 21	21
28.5—36.5	32.5	18	0	0	0
36.5—44.5	40.5	5	1	5	5
44.5—52.5	48.5	3	2	6	12
52.5—60.5	56.5	5	3	15	45
60.5—68.5	64.5	8	4	32	128
68.5—76.5	72.5	2	5	10	50
		$\Sigma f = 90$		$\Sigma fd' = -13$	$\Sigma fd'^2 = 393$

$$\therefore \qquad mean\ wage = 32.5 + 8 \times \frac{\Sigma fd'}{\Sigma f} = 32.5 + 8\left(\frac{-13}{90}\right) = ₹\ 31.35$$

$$Standard\ deviation = 8\sqrt{\frac{\Sigma fd'^2}{\Sigma f} - \left(\frac{\Sigma fd'}{\Sigma f}\right)^2} = 8\sqrt{\frac{393}{90} - \left(\frac{-13}{90}\right)^2} = ₹\ 16.64.$$

Example 25.9. *The following are scores of two batsmen A and B in a series of innings :*

A :	12	115	6	73	7	19	119	36	84	29
B :	47	12	16	42	4	51	37	48	13	0

Who is the better score getter and who is more consistent ? (V.T.U., 2004)

Solution. Let x denote score of A and y that of B.

Taking 51 as the origin, we prepare the following table :

x	$d(= x - 51)$	d^2	y	$\delta(= y - 51)$	δ^2
12	– 39	1521	47	– 4	16
115	64	4096	12	– 39	1521
6	– 45	2025	16	– 35	1225
73	22	484	42	– 9	81
7	– 44	1936	4	– 47	2209
19	– 32	1024	51	0	0
119	68	4624	37	– 14	196
36	– 15	225	48	– 3	9
84	33	1089	13	– 38	1444
29	– 22	484	0	– 51	2601
Total	– 10	17508		– 240	9302

For A, A.M. $\bar{x} = 51 + \dfrac{\Sigma d}{n} = 51 - \dfrac{10}{10} = 50$

$$\text{S.D. } \sigma_1 = \sqrt{\left\{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2\right\}} = \sqrt{\{1750.8 - (-1)^2\}} = 41.8$$

$\therefore$ coefficient of variation $= \dfrac{\sigma_1}{\bar{x}} \times 100 = \dfrac{41.8}{50} \times 100 = 83.6\%$

For B, A.M. $\bar{y} = 51 + \dfrac{\Sigma \delta}{n} = 51 - \dfrac{240}{10} = 27$

$$\text{S.D. } \sigma_2 = \sqrt{\left\{\frac{\Sigma \delta^2}{n} - \left(\frac{\Sigma \delta}{n}\right)^2\right\}} = \sqrt{\{930.2 - (-24)^2\}} = 18.8$$

$\therefore$ coefficient of variation $= \dfrac{\sigma^2}{\bar{y}} \times 100 = \dfrac{18.8}{27} \times 100 = 69.6\%$

Since the A.M. of $A >$ A.M. of B, it follows that A is a better score getter (*i.e.*, more efficient) than B.

Since the coefficient of variation of $B <$ the coefficient of variation of A, it means that B is more consistent than A. Thus even though A is a better player, he is less consistent.

Example 25.10. *The numbers examined, the mean weight and S.D. in each group of examination by three medical examiners are given below. Find the mean weight and S.D. of the entire data when grouped together.*

Med. Exam.	*No. Examined*	*Mean Wt. (lbs.)*	*S.D. (lbs.)*
A	50	113	6
B	60	120	7
C	90	115	8

Solution. We have $n_1 = 50,\ \bar{x}_1 = 113,\ \sigma_1 = 6$

$n_2 = 60,\ \bar{x}_2 = 120,\ \sigma_2 = 7$

$n_3 = 90,\ \bar{x}_3 = 115,\ \sigma_3 = 8.$

If $\bar{x}$ is the mean of the entire data,

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3} = \frac{50 \times 113 + 60 \times 120 + 90 \times 115}{50 + 60 + 90} = \frac{23200}{200} = 116 \text{ lb.}$$

If σ is the S.D. of the entire data,

$$N\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1D_1^2 + n_2D_2^2 + n_3D_3^2$$

where $N = n_1 + n_2 + n_3 = 200$, $D_1 = \bar{x}_1 - \bar{x} = -3$, $D_2 = \bar{x}_2 - \bar{x} = 4$ and $D_3 = \bar{x}_3 - \bar{x} = -1$.

$$\therefore \quad 200\sigma^2 = 50 \times 36 + 60 \times 49 + 90 \times 64 + 50 \times 9 + 60 \times 16 + 90 \times 1$$

$$= 1800 + 2940 + 5760 + 450 + 960 + 90$$

$$\sigma^2 = \frac{12000}{200} = 60. \quad \text{Hence} \quad \sigma = \sqrt{60} = 7.746 \text{ lb.}$$

PROBLEMS 25.2

1. The crushing strength of 8 cement concrete experimental blocks, in metric tonnes per sq. cm., was 4.8, 4.2, 5.1, 3.8, 4.4, 4.7, 4.1 and 4.5. Find the mean crushing strength and the standard deviation.
2. Show that the variance of the first n positive integers is $\frac{1}{12}(n^2 - 1)$. (*V.T.U., 2003*)
3. The mean of five items of an observation is 4 and the variance is 5.2. If three of the items are 1, 2 and 6, then find the other two. (*V.T.U., 2002*)
4. For the distribution

x :	5	6	7	8	9	10	11	12	13	14	15
f :	18	15	34	47	68	90	80	62	35	27	11

find the mean, median and lower and upper quartiles, variance and the standard deviation.

5. The following table shows the marks obtained by 100 candidates in an examination. Calculate the mean, median and standard deviation :

Marks obtained :	1—10	11—20	21—30	31—40	41—50	51—60
No. of candidates :	3	16	26	31	16	8

(*Osmania, 2003 S ; V.T.U., 2003 S*)

6. Compute the quartile deviation and standard deviation for the following :

x :	100—109	110—119	120—129	130—139	140—149	150—159	160—169	170—179
f :	15	44	133	150	125	82	35	16

7. Calculate (*i*) mean deviation about the mean, (*ii*) mean deviation about the median for the following distribution :

Class :	3—4.9	5—6.9	7—8.9	9—10.9	11—12.9	13—14.9	15—16.9
f :	5	8	30	82	45	24	6

(*Madras, 2002*)

8. Two observers bring the following two sets of data which represent measurements of the same quantity :

I.	105.1	103.4	104.2	104.7	104.8	105.0	104.9
II.	105.3	105.1	104.8	105.2	106.7	102.9	103.1

Calculate the standard deviation in each case. Which set of data is more reliable ? Can the same conclusion be reached by calculating the mean deviation ?

Obs. The smaller the coefficient of variation, the greater is the *reliability* or *consistency* in the data.

9. The heights and weights of the 10 armymen are given below. In which characteristics are they more variable ?

Height in cm.	170	172	168	177	179	171	173	178	173	179
Weight in kg.	75	74	75	76	77	73	76	75	74	75

10. The index number of prices of two articles A and B for six consecutive weeks are given below :

A :	314	326	336	368	404	412
B :	330	331	320	318	321	330

Find which has a more variable price ?

11. The scores of two golfers A and B in 12 rounds are given below. Who is the better player and who is the more consistent player ?

A :	74	75	78	72	78	77	79	81	79	76	72	71
B :	87	84	80	88	89	85	86	82	82	79	86	80

12. The scores obtained by two batsmen A and B in 10 matches are given below :

A :	30	44	66	62	60	34	80	46	20	38
B :	34	46	70	38	55	48	60	34	45	30

Calculating mean, S.D. and coefficient of variation for each batsman, determine who is more efficient and who is more consistent.

13. Find the mean and standard deviation of the following two samples put together :

Sample No.	*Size*	*Mean*	*S.D.*
1	50	158	5.1
2	60	164	4.6

14. A distribution consists of three components with frequencies 200, 250 and 300 having means 25, 10 and 15 and S.Ds. 3, 4 and 5 respectively. Show that the mean of the combined distribution is 16 and its S.D. is 7.2 approximately.

25.9 (1) MOMENTS

The rth moment about the mean $\bar{x}$ of a distribution is denoted by μ_r and is given by

$$\mu_r = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^r \quad ...(1)$$

The corresponding moment about any point a is defined as

$$\mu_r' = \frac{1}{N}\Sigma f_i(x_i - a)^r \quad ...(2)$$

In particular, we have $\mu_0 = \mu_0' = 1$ ...(3)

$$\mu_1 = \frac{1}{N}\Sigma f_i(x_i - \bar{x}) = 0;\ \mu_1' = \frac{1}{N}\Sigma f_i(x_i - a) = \bar{x} - a = d, \text{ say} \quad ...(4)$$

$$\mu_2 = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^2 = \sigma^2. \quad ...(5)$$

(2) Moments about the mean in terms of moments about any point.

We have

$$\mu_r = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^r = \frac{1}{N}\Sigma f[(x_i - a) - (\bar{x} - a)]^r$$

$$= \frac{1}{N}\Sigma f_i(X_i - d)^r \quad \text{where} \quad X_i = x_i - a,\ d = \bar{x} - a.$$

$$= \frac{1}{N}[\Sigma f_i X_i^r - {}^rC_1 d\Sigma f_i X_i^{r-1} + {}^rC_2 d^2 \Sigma f_i X_i^{r-2} - ...]$$

$$= \mu_r' - {}^rC_1\, d\mu_{r-1}' + {}^rC_2 d^2 \mu_{r-2}' - ...... \quad ...(6)$$

In particular,

$$\mu_2 = \mu'_2 - \mu'^2_1 \quad ...(7)$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 \quad ...(8)$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1 \quad ...(9)$$

These three results should be committed to memory. It should be noted that in each of these relations, the sum of the coefficients of the various terms on the right side is zero. Also each term on the right side is of the same dimension as the term on the left.

25.10 SKEWNESS

Skewness measures the degree of asymmetry or the departure from symmetry. If the frequency curve has a longer 'tail' to the right, *i.e.*, the mean is to the right of the mode [as in Fig. 25.4 (*a*)], then the distribution is said to have *positive skewness*. If the curve is more elongated to the left, then it is said to have *negative skewness* [Fig. 25.4 (*b*)].

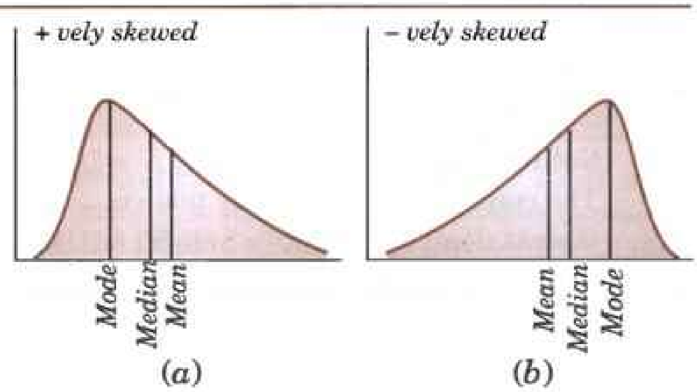


Fig. 25.4

The following three measures of skewness deserve mention :

(*i*) *Pearson's* coefficient of skewness* $= \dfrac{\text{mean} - \text{mode}}{\sigma}$

* After the English statistician and biologist *Karl Pearson* (1857–1936) who did pioneering work and found the English school of statistics.

(*ii*) *Quartile coefficient of skewness* $= \dfrac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$

Its value always lies between – 1 and + 1.

(*iii*) *Coefficient of skewness based on third moment* $\gamma_1 = \sqrt{\beta_1}$.

where $\beta_1 = \mu_3^2/\mu_2^3$

Thus $\gamma_1 = \sqrt{\beta_1}$ gives the simplest measure of *skewness*.

25.11 KURTOSIS

Kurtosis measures the degree of peakedness of a distribution and is given by $\beta_2 = \mu_4/\mu_2^2$.

(*a*) *Leptokurtic* (*b*) *Platykurtic* (*c*) *Mesokurtic*

Fig. 25.5

$\gamma_2 = \beta_2 - 3$ gives the excess of *Kurtosis*. The curves with $\beta_2 > 3$ are called *Leptokurtic* and those with $\beta_2 < 3$ as *Platykurtic*. The normal curve for which $\beta_2 = 3$, is called *Mesokurtic* [Fig. 25.5].

Example 25.11. *The first four moments about the working mean 28.5 of a distribution are 0.294, 7.144, 42.409 and 454.98. Calculate the moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.* (V.T.U., 2[illegible] S)

Solution. The first four moments about the arbitrary origin 28.5 are $\mu'_1 = 0.294$, $\mu'_2 = 7.144$, $\mu'_3 = 42.409$, $\mu'_4 = 454.98$.

$$\therefore \quad \mu'_1 = \frac{1}{N}\sum f_i(x_i - 28.5) = \frac{1}{N}\sum f_i x_i - 28.5 = \bar{x} - 28.5 = 0.294 \text{ or } \bar{x} = 28.794$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = 7.144 - (0.294)^2 = 7.058$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 = 42.409 - 3(7.144)(0.294) + 2(0.294)^3 = 36.151.$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$$
$$= 454.98 - 4(42.409) \times (0.294) + 6(7.144)(0.294)^2 - 3(0.294)^4 = 408.738$$

Now
$$\beta_1 = \mu_3^2/\mu_2^3 = (36.151)^2/(7.058)^3 = 3.717$$
$$\beta_2 = \mu_4/\mu_2^2 = 408.738/(7.058)^2 = 8.205.$$

$\therefore \quad \gamma_1 = \sqrt{\beta_1} = 1.928$, which indicates considerable skewness of the distribution.

$\gamma_2 = \beta_2 - 3 = 5.205$ which shows that the distribution is *leptokurtic*.

Example 25.12. *Calculate the median, quartiles and the quartile coefficient of skewness from the following data :*

Weight (lbs) :	*70–80*	*80–90*	*90–100*	*100–110*	*110–120*	*120–130*	*130–140*	*140–150*
No. of persons :	12	18	35	42	50	45	20	8

Solution. Here total frequency $N = \Sigma f_i = 230$.

The cummulative frequency table is

Weight (lbs) :	70–80	80–90	90–100	100–110	110–120	120–130	130–140	140–150
f :	12	18	35	42	50	45	20	8
cum. f. :	12	30	65	107	157	202	222	230

Now $N/2 = 230/2 = 115$th item which lies in 110–120 group.

$$\therefore \quad \text{median or } Q_2 = L + \frac{N/2 - C}{f} \times h = 110 + \frac{115 - 107}{50} \times 10 = 111.6$$

Also $N/4 = 230/4 = 57.5$ *i.e.* Q_1 is 57.5th or 58th item which lies in 90–100 group.

$$\therefore \qquad Q_1 = L + \frac{N/4 - C}{f} \times h = 90 + \frac{57.5 - 30}{35} \times 10 = 97.85$$

Similarly, $3N/4 = 172.5$ *i.e.* Q_3 is 173rd item which lies in 120–130 group.

$$\therefore \qquad Q_3 = L + \frac{3N/4 - C}{f} \times h = 120 + \frac{172.5 - 157}{45} \times 10 = 123.44$$

Hence quartile coefficient of skewness $= \dfrac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}$

$$= \frac{97.85 + 123.44 - 2 \times 111.6}{123.44 - 97.85} = -0.07 \text{ (approx.)}.$$

PROBLEMS 25.3

1. Calculate the first four moments of the following distribution about the mean :

x :	0	1	2	3	4	5	6	7	8
f :	1	8	28	56	70	56	28	8	1

Also evaluate β_1 and β_2. (*V.T.U., 2004 ; Madras, 2003*)

2. The following table gives the monthly wages of 72 workers in a factory. Compute the standard deviation, quartile deviation, coefficients of variation and skewness. (*V.T.U., 2001*)

Monthly wages (in ₹)	*No. of workers*	*Monthly wages (in ₹)*	*No. of workers*
12.5—17.5	2	37.5—42.5	4
17.5—22.5	22	42.5—47.5	6
22.5—27.5	19	47.5—52.5	1
27.5—32.5	14	52.5—57.5	1
32.5—37.5	3		

3. Find Pearson's coefficient of skewness for the following data :

Class :	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Frequency :	5	9	14	20	25	15	8	4

(*V.T.U., 2000 S*)

4. Compute the quartile coefficient of skewness for the following distribution :

x :	3–7	8–12	13–17	18–22	23–27	28–32	33–37	38–42
f :	2	108	580	175	80	32	18	5

(*Madras, 2002 ; V.T.U., 2000*)

Also compute the measure of skewness based on the third moment.

5. The first three moments of a distribution about the value 2 of the variable are 1, 16 and − 40. Show that the mean = 3, the variance = 15 and $\mu_3 = -86$. (*V.T.U., 2003 S*)
6. Compute skewness and kurtosis, if the first four moments of a frequency distribution $f(x)$ about the value $x = 4$ are respectively 1, 4, 10 and 45. (*Coimbatore, 1999*)
7. In a certain distribution, the first four moments about a point are − 1.5, 17, − 30 and 108. Calculate the moments about the mean, β_1 and β_2 ; and state whether the distribution is leptokurtic or platykurtic ?

25.12 CORRELATION

So far we have confined our attention to the analysis of observations on a single variable. There are, however, many phenomenae where the changes in one variable are related to the changes in the other variable. For instance, the yield of a crop varies with the amount of rainfall, the price of a commodity increases with the reduction in its supply and so on. Such a simultaneous variation, *i.e.* when the changes in one variable are associated or followed by changes in the other, is called *correlation*. Such a data connecting two variables is called *bivariate population*.

If an increase (or decrease) in the values of one variable corresponds to an increase (or decrease) in the other, the correlation is said to be *positive*. If the increase (or decrease) in one corresponds to the decrease (or increase) in the other, the correlation is said to be *negative*. If there is no relationship indicated between the variables, they are said to be *independent or uncorrelated*.

To obtain a measure of relationship between the two variables, we plot their corresponding values on the graph, taking one of the variables along the x-axis and the other along the y-axis. (Fig. 25.6).

Let the origin be shifted to $(\bar{x}, \bar{y})$, where $\bar{x}, \bar{y}$ are the means of x's and y's that the new co-ordinates are given by

$$X = x - \bar{x}, \qquad Y = y - \bar{y}.$$

Now the points (X, Y) are so distributed over the four quadrants of XY-plane that the product XY is positive in the first and third quadrants but negative in the second and fourth quadrants. The algebraic sum of the products can be taken as describing the trend of the dots in all the quadrants.

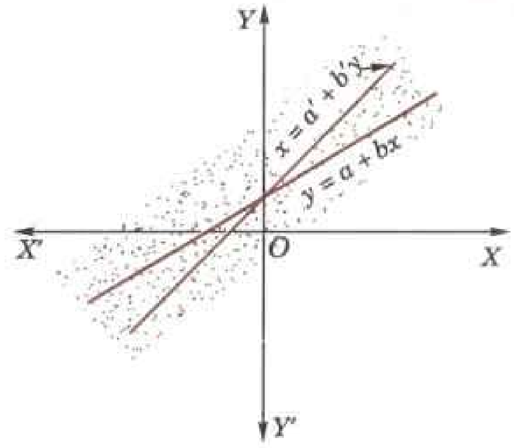


Fig. 25.6

$\therefore$ (*i*) If ΣXY is positive, the trend of the dots is through the first and third quadrants,

(*ii*) if ΣXY is negative the trend of the dots is in the second and fourth quadrants, and

(*iii*) if ΣXY is zero, the points indicate no trend *i.e.* the points are evenly distributed over the four quadrants.

The ΣXY or better still $\frac{1}{n}\Sigma XY$, *i.e.*, the average of n products may be taken as a measure of correlation. If we put X and Y in their units, *i.e.*, taking σ_x as the unit for x and σ_y for y, then

$$\frac{1}{n}\Sigma \frac{X}{\sigma_x} \cdot \frac{Y}{\sigma_y}, \text{ i.e., } \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

is the *measure of correlation*.

25.13 COEFFICIENT OF CORRELATION

The numerical measure of correlation is called the *coefficient of correlation* and is defined by the relation

$$r = \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

where X = deviation from the mean $\bar{x} = x - \bar{x}$, Y = deviation from the mean $\bar{y} = y - \bar{y}$,

σ_x = S.D. of x-series, σ_y = S.D. of y-series and n = number of values of the two variables.

Methods of calculation :

(*a*) *Direct method.* Substituting the value of σ_x and σ_y in the above formula, we get

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2 \Sigma Y^2)}} \qquad \text{...(1)}$$

Another form of the formula (1) which is quite handy for calculation is

$$r = \frac{n\Sigma xy - \Sigma x \Sigma y}{\sqrt{[\{n\Sigma x^2 - (\Sigma x)^2\} \times \{n\Sigma y^2 - (\Sigma y)^2\}]}} \qquad \text{...(2)}$$

(*b*) *Step-deviation method.* The direct method becomes very lengthy and tedious if the means of the two series are not integers. In such cases, use is made of assumed means. If d_x and d_y are step-deviations from the assumed means, then

$$r = \frac{n\Sigma d_x d_y - \Sigma d_x \Sigma d_y}{\sqrt{[\{n\Sigma d_x^2 - (\Sigma d_x)^2\} \times \{n\Sigma d_y^2 - (\Sigma d_y)^2\}]}} \qquad \text{...(3)}$$

where $d_x = (x - a)/h$ and $d_y = (y - b)/k$.

Obs. The change of origin and units do not alter the value of the correlation coefficient since r is a pure number.

(*c*) *Co-efficient of correlation for grouped data.* When x and y series are both given as frequency distributions, these can be represented by a two-way table known as the *correlation-table*. It is double-entry table with one series along the horizontal and the other along the vertical as shown on page 848. The co-efficient of correlation for such a *bivariate frequency distribution* is calculated by the formula.

$$r = \frac{n(\Sigma f d_x d_y) - (\Sigma f d_x)(\Sigma f d_y)}{\sqrt{[\{n\Sigma f d_x^2 - (\Sigma f d_x)^2\} \times \{n\Sigma f d_y^2 - (\Sigma f d_y)^2\}]}} \quad ...(4)$$

where d_x = deviation of the central values from the assumed mean of x-series,
d_y = deviation of the central values from the assumed mean of y-series,
f is the frequency corresponding to the pair (x, y)
and $n(= \Sigma f)$ is the total number of frequencies.

Example 25.13. *Psychological tests of intelligence and of engineering ability were applied to 10 students. Here is a record of ungrouped data showing intelligence ratio (I.R.) and engineering ratio (E.R.). Calculate the co-efficient of correlation.*

Student	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
I.R.	105	104	102	101	100	99	98	96	93	92
E.R.	101	103	100	98	95	96	104	92	97	94

(Andhra, 2000)

Solution. We construct the following table :

Student	*Intelligence ratio*		*Engineering ratio*				
	x	$x - \bar{x} = X$	y	$y - \bar{y} = Y$	X^2	Y^2	XY
A	105	6	101	3	36	9	18
B	104	5	103	5	25	25	25
C	102	3	100	2	9	4	6
D	101	2	98	0	4	0	0
E	100	1	95	−3	1	9	−3
F	99	0	96	−2	0	4	0
G	98	−1	104	6	1	36	−6
H	96	−3	92	−6	9	36	18
I	93	−6	97	−1	36	1	6
J	92	−7	94	−4	49	16	28
Total	990	0	980	0	170	140	92

From this table, mean of x, *i.e.*, $\bar{x} = 990/10 = 99$ and mean of y, *i.e.* $\bar{y} = 980/10 = 98$.

$$\Sigma X^2 = 170, \Sigma Y^2 = 140 \text{ and } \Sigma XY = 92.$$

Substituting these values in the formula (1) p. 744, we have

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2 \Sigma Y^2)}} = \frac{92}{\sqrt{(170 \times 140)}} = 92/154.3 = 0.59.$$

Example 25.14. *The correlation table given below shows that the ages of husband and wife of 53 married couples living together on the census night of 1991. Calculate the coefficient of correlation between the age of the husband and that of the wife.* *(J.N.T.U., 2003)*

Age of husband	*Age of wife*						*Total*
	15–25	*25–35*	*35–45*	*45–55*	*55–65*	*65–75*	
15–25	1	1	–	–	–	–	2
25–35	2	12	1	–	–	–	15
35–45	–	4	10	1	–	–	15
45–55	–	–	3	6	1	–	10
55–65	–	–	–	2	4	2	8
65–75	–	–	–	–	1	2	3
Total	3	17	14	9	6	4	53

Solution.

Age of husband				*Age of wife x-series*							*Suppose* $d_x = \frac{x-40}{10}$, $d_y = \frac{y-40}{10}$		
				15–25	25–35	35–45	45–55	55–65	65–75				
Years			*Mid pt. x*	20	30	40	50	60	70	*Total f*			
				– 20	– 10	0	10	20	30				
Age group	*Mid pt. y*		d_x / d_y	–2	–1	0	1	2	3		fd_y	fd_y^2	fd_xd_y
15–25	20	–20	–2	**4** 1	**2** 1					2	–4	8	**6**
25–35	30	–10	–1	**4** 2	**12** 12	**0** 1				15	–15	15	**16**
35–45	40	0	0		**0** 4	**0** 10	**0** 1			15	0	0	**0**
45–55	50	10	1			**0** 3	**6** 6	**2** 1		10	10	10	**8**
55–65	60	20	2				**4** 2	**16** 4	**12** 2	8	16	32	**32**
65–75	70	30	3					**6** 1	**18** 2	3	9	27	**24**
Total *f*				3	17	14	9	6	4	53 = *n*	16	92	**86**
fd_x				– 6	–17	0	9	12	12	10	Thick figures in small sqs. stand for fd_xd_y		
fd_x^2				12	17	0	9	24	36	98			
fd_xd_y				**8**	**14**	**0**	**10**	**24**	**30**	**86**	**Check :** $\Sigma fd_xd_y = 86$ from both sides		

With the help of the above correlation table, we have

$$r = \frac{n(\Sigma fd_xd_y) - (\Sigma fd_x)(\Sigma fd_y)}{\sqrt{[\{n\Sigma fd_x^2 - (\Sigma fd_x)^2\} \times [\{n\Sigma fd_y^2 - (\Sigma fd_y)^2\}]}}$$

$$= \frac{53 \times 86 - 10 \times 16}{\sqrt{[(53 \times 98 - 100) \times (53 \times 92 - 256)]}} = \frac{4398}{\sqrt{(5094 \times 4620)}} = \frac{4398}{4850} = 0.91 \text{ (approx.)}.$$

25.14 LINES OF REGRESSION

It frequently happens that the dots of the scatter diagram generally, tend to cluster along a well defined direction which suggests a linear relationship between the variables x and y. Such a line of best-fit for the given distribution of dots is called the *line of regression* (Fig. 25.6). In fact there are two such lines, one giving the best possible mean values of y for each specified value of x and the other giving the best possible mean values of x for given values of y. The former is known as the *line of regression of y on x* and the latter as the *line of regression of x on y*.

Consider first the line of regression of y on x. Let the straight line satisfying the general trend of n dots in a scatter diagram be

$$y = a + bx \qquad ...(1)$$

We have to determine the constants a and b so that (1) gives for each value of x, the best estimate for the average value of y in accordance with the *principle of least squares* (page 816), therefore, the normal equations for a and b are

$$\Sigma y = na + b\Sigma x \qquad ...(2)$$

and
$$\Sigma xy = a\Sigma x + b\Sigma x^2 \qquad ...(3)$$

(2) gives $\quad \frac{1}{n}\Sigma y = a + b \cdot \frac{1}{n}\Sigma x$ *i.e.*, $\bar{y} = a + b\bar{x}$.

This shows that $(\bar{x}, \bar{y})$, *i.e.*, the means of x and y, lie on (1).

Shifting the origin to $(\bar{x}, \bar{y})$, (3) takes the form

$$\Sigma(x-\bar{x})(y-\bar{y}) = a\Sigma(x-\bar{x}) + b\Sigma(x-\bar{x})^2, \text{ but } a\Sigma(x-\bar{x}) = 0,$$

$$\therefore \quad b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{\Sigma XY}{\Sigma X^2} = \frac{\Sigma XY}{n\sigma_x^2} = r\frac{\sigma_y}{\sigma_x} \qquad \left[\because r = \frac{\Sigma XY}{n\sigma_x\sigma_y}\right]$$

Thus the line of best fit becomes $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$ ...(4)

which is the equation of *the line of regression of y on x*. Its slope is called the *regression coefficient of y on x*.

Interchanging x and y, we find that the line of regression of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \qquad ...(5)$$

Thus the *regression coefficient of y on x* $= r\sigma_y/\sigma_x$ ...(6)

and the *regression coefficient of x on y* $= r\sigma_x/\sigma_y$ ...(7)

Cor. *The correlation coefficient r is the geometric mean between the two regression co-efficients.*

For $\quad r\frac{\sigma_y}{\sigma_x} \times r\frac{\sigma_x}{\sigma_y} = r^2.$

Example 25.15. *The two regression equations of the variables x and y are $x = 19.13 - 0.87y$ and $y = 11.64 - 0.50x$. Find (i) mean of x's, (ii) mean of y's and (iii) the correlation coefficient between x and y.*

(V.T.U., 2004 ; Anna, 2003 ; Burdwan, 2003)

Solution. Since the mean of x's and the mean of y's lie on the two regression lines, we have

$$\bar{x} = 19.13 - 0.87\,\bar{y} \qquad ...(i)$$
$$\bar{y} = 11.64 - 0.50\,\bar{x} \qquad ...(ii)$$

Multiplying (*ii*) by 0.87 and subtracting from (*i*), we have

$$[1 - (0.87)(0.50)]\ \bar{x} = 19.13 - (11.64)(0.87) \text{ or } 0.57\ \bar{x} = 9.00 \text{ or } \bar{x} = 15.79$$

$$\therefore \quad \bar{y} = 11.64 - (0.50)(15.79) = 3.74$$

$\therefore$ regression coefficient of y on x is -0.50 and that of x on y is -0.87.

Now since the coefficient of correlation is the geometric mean between the two regression coefficients.

$$\therefore \quad r = \sqrt{[(-0.50)(-0.87)]} = \sqrt{(0.43)} = -0.66.$$

[– ve sign is taken since both the regression coefficients are – ve]

Example 25.16. *In the following table are recorded data showing the test scores made by salesmen on an intelligence test and their weekly sales :*

Salesmen	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Test scores	*40*	*70*	*50*	*60*	*80*	*50*	*90*	*40*	*60*	*60*
Sales (000)	*2.5*	*6.0*	*4.5*	*5.0*	*4.5*	*2.0*	*5.5*	*3.0*	*4.5*	*3.0*

Calculate the regression line of sales on test scores and estimate the most probable weekly sales volume if a salesman makes a score of 70.

Solution. With the help of the table below, we have

$$\bar{x} = \text{mean of } x \text{ (test scores)} = 60 + 0/10 = 60$$

$$\bar{y} = \text{mean of } y \text{ (sales)} = 4.5 + (-4.5)/10 = 4.05.$$

Regression line of sales (y) on scores (x) is given by

$$y - \bar{y} = r(\sigma_y / \sigma_x)(x - \bar{x})$$

where
$$r\frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{\sigma_x \sigma_y} \times \frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{(\sigma_x)^2} = \left[\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{n}\right] \Big/ \left[\Sigma d_x^2 - (\Sigma d_x)^2 / n\right]$$

$$= \frac{140 - \dfrac{0 \times (-4.5)}{10}}{2400 - 0^2/10} = \frac{140}{2400} = 0.06$$

$\therefore$ the required regression line is

$$y - 4.05 = 0.06(x - 60) \quad \text{or} \quad y = 0.06x + 0.45.$$

For $x = 70$, $y = 0.06 \times 70 + 0.45 = 4.65$.

Thus the most probable weekly sales volume for a score of 70 is 4.65.

Test scores x	*Sales* y	*Deviation of x from assumed mean (= 60)* d_x	*Deviation of y from assumed average (= 4.5)* d_y	$d_x \times d_y$	d_x^2	d_y^2
40	2.5	−20	−2	40	400	4
70	6.0	10	1.5	15	100	2.25
50	4.5	−10	0	0	100	0
60	5.0	0	0.5	0	0	2.25
80	4.5	20	0	0	400	0
50	2.0	−10	−2.5	25	100	6.25
90	5.5	30	1	30	900	1.00
40	3.0	−20	−1.5	30	400	2.25
60	4.5	0	0	0	0	0
60	3.0	0	−1.5	0	0	2.25
		$\Sigma d_x = 0$	$\Sigma d_y = -4.5$	$\Sigma d_x d_y = 140$	$\Sigma d_x^2 = 2400$	$\Sigma d_y^2 = 18.25$

Example 25.17. *If θ is the angle between the two regression lines, show that*

$$\tan\theta = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

Explain the significance when $r = 0$ and $r = \pm 1$. (U.P.T.U., 2007 ; V.T.U., 2007)

Solution. The equations to the line of regression of y on x and x on y are

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \text{ and } x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$\therefore$ their slopes are $m_1 = r\sigma_y/\sigma_x$ and $m_2 = \sigma_y/r\sigma_x$

Thus
$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sigma_y / r\sigma_x - r\sigma_y / \sigma_x}{1 + \sigma_y^2/\sigma_x^2} = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

When $r = 0$, $\tan\theta \to \infty$ or *$\theta = \pi/2$ i.e. when the variables are independent, the two lines of regression are perpendicular to each other.*

When $r = \pm 1$, $\tan\theta = 0$ i.e., $\theta = 0$ or π. Thus the lines of regression coincide i.e., there is perfect correlation between the two variables.

Example 25.18. *In a partially destroyed laboratory record, only the lines of regression of y on x and x on y are available as $4x - 5y + 33 = 0$ and $20x - 9y = 107$ respectively. Calculate $\bar{x}, \bar{y}$ and the coefficient of co-relation between x and y.* (S.V.T.U., 2009 ; U.P.T.U., 2009 ; V.T.U., 2005)

Solution. Since the regression lines pass through $(\bar{x}, \bar{y})$, therefore,

$$4\bar{x} - 5\bar{y} + 33 = 0, \quad 20\bar{x} - 9\bar{y} = 107.$$

Solving these equations, we get $\bar{x} = 13$, $\bar{y} = 17$.

Rewriting the line of regression of y on x as $y = \frac{4}{5}x + \frac{33}{5}$, we get

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{4}{5} \qquad ...(i)$$

Rewriting the line of regression of x on y as $x = \frac{9}{20}y + \frac{107}{9}$, we get

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{9}{20} \qquad ...(ii)$$

Multiplying (i) and (ii), we get

$$r^2 = \frac{4}{5} \times \frac{9}{20} = 0.36 \qquad \therefore \qquad r = 0.6$$

Hence $r = 0.6$, the positive sign being taken as b_{yx} and b_{xy} both are positive.

Example 25.19. *Establish the formula* $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$

Hence calculate r from the following data :

x :	21	23	30	54	57	58	72	78	87	90
y :	60	71	72	83	110	84	100	92	113	135

(U.P.T.U., 2002)

Solution. (a) Let $z = x - y$ so that $\bar{z} = \bar{x} - \bar{y}$.

$$\therefore \qquad z - \bar{z} = (x - \bar{x}) - (y - \bar{y})$$

or $$(z - \bar{z})^2 = (x - \bar{x})^2 + (y - \bar{y})^2 - 2(x - \bar{x})(y - \bar{y})$$

Summing up for n terms, we have

$$\Sigma(z - \bar{z})^2 = \Sigma(x - \bar{x})^2 + \Sigma(y - \bar{y})^2 - 2\Sigma(x - \bar{x})(y - \bar{y})$$

or $$\frac{\Sigma(z - \bar{z})^2}{n} = \frac{\Sigma(x - \bar{x})^2}{n} + \frac{\Sigma(y - \bar{y})^2}{n} - 2\frac{\Sigma(x - \bar{x})(y - \bar{y})}{n}$$

i.e., $$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y \qquad \left[\because r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}\right]$$

which is the required result.

(b) To fine r, we have to calculate σ_x, σ_y and σ_{x-y}. We make the following table :

x	$X = x - 54$	X^2	y	$Y = y - 100$	Y^2	$y - x$	$(x - y)^2$
21	– 33	1089	60	– 40	1600	39	1521
23	– 31	961	71	– 29	841	48	2304
30	– 24	576	72	– 28	784	42	1764
54	0	0	83	– 17	289	29	841
57	3	9	110	10	100	53	2809
58	4	16	84	– 16	256	26	676
72	18	324	100	0	0	28	784
78	24	576	92	– 8	64	14	196
87	33	1089	113	13	169	26	676
90	36	1296	135	35	1225	45	2025
Total	30	5936		– 80	5328	350	13596

$$\therefore \qquad \sigma_x^2 = \frac{\Sigma X^2}{N} - \left(\frac{\Sigma X}{N}\right)^2 = \frac{5636}{10} - \left(\frac{30}{10}\right)^2 = 593.6 - 9 = 584.6$$

$$\sigma_y^2 = \frac{\Sigma Y^2}{N} - \left(\frac{\Sigma Y}{N}\right)^2 = \frac{5328}{10} - \left(\frac{-80}{10}\right)^2 = 532.8 - 64 = 468.8$$

$$\sigma^2_{x-y} = \frac{\Sigma(x-y)^2}{N} - \left\{\frac{\Sigma(x-y)}{N}\right\}^2 = 1359.6 - 1225 = 134.6$$

From the above formula,

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y} = \frac{584.6 + 468.8 - 134.6}{2 \times 24.18 \times 23.85} = 0.876.$$

Example 25.20. *While calculating correlation coefficient between two variables x and y from 25 pairs of observations, the following results were obtained : $n = 25$, $\Sigma x = 125$, $\Sigma x^2 = 650$, $\Sigma y = 100$, $\Sigma y^2 = 460$, $\Sigma xy = 508$.*

Later it was discovered at the time of checking that the pairs of values $\begin{array}{c|c} x & y \\ \hline 8 & 12 \\ 6 & 8 \end{array}$ *were copied down as* $\begin{array}{c|c} x & y \\ \hline 6 & 14 \\ 8 & 6 \end{array}$.

Obtain the correct value of correlation coefficient. (V.T.U., 2011 S ; S.V.T.U., 2009)

Solution. To get the correct results, we subtract the incorrect values and add the corresponding correct values.

$\therefore$ The correct results would be

$$\Sigma n = 25,\ \Sigma x = 125 - 6 - 8 + 8 + 6 = 125,\ \Sigma x^2 = 650 - 6^2 - 8^2 + 8^2 + 6^2 = 650$$
$$\Sigma y = 100 - 14 - 6 + 12 + 8 = 100,\ \Sigma y^2 = 460 - 14^2 - 6^2 + 12^2 + 8^2 = 436$$
$$\Sigma xy = 508 - 6 \times 14 - 8 \times 6 + 8 \times 12 + 6 \times 8 = 520$$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{[\{n\Sigma x^2 - (\Sigma x)^2\}\{n\Sigma y^2 - (\Sigma y^2)\}]}} = \frac{25 \times 520 - 125 \times 100}{\sqrt{[\{25 \times 650 - (125)^2\}\{25 \times 436 - (100)^2\}]}}$$

$$= \frac{20}{\sqrt{(25 \times 36)}} = \frac{2}{3}.$$

25.15 STANDARD ERROR OF ESTIMATE

The sum of the squares of the deviations of the points from the line of regression of y on x is

$\Sigma(y - a - bx)^2 = \Sigma(Y - bX)^2$, where $X = x - \bar{x}$, $Y = y - \bar{y}$

$$= \sum\left(Y - r\frac{\sigma_y}{\sigma_x}X\right)^2 = \Sigma Y^2 - 2r(\sigma_y/\sigma_x)\,\Sigma XY + r^2(\sigma_y^2/\sigma_x^2)\,\Sigma X^2$$

$$= n\sigma_y^2 - 2r(\sigma_y/\sigma_x)\,r\,.\,n\sigma_x\sigma_y + r^2(\sigma_y^2/\sigma_x^2)\,.\,n\sigma_x^2 = n\sigma_y^2(1 - r^2).$$

Denoting this sum of squares by nS_y^2, we have $S_y = \sigma_y\sqrt{(1 - r^2)}$ ...(1)

Since S_y is the root mean square deviation of the points from the regression line of y on x, it is called the *standard error of estimate* of y. Similarly the standard error of estimate of x is given by

$$S_x = \sigma_x\sqrt{(1 - r^2)} \qquad ...(2)$$

Since the sum of the squares of deviations cannot be negative, it follows that

$$r^2 \le 1 \quad \text{or} \quad -1 \le r \le 1.$$

i.e., correlation coefficient lies between – 1 and 1. (J.N.T.U., 2006)

If $r = 1$ or -1, the sum of the squares of deviations from either line of regression is zero. Consequently each deviation is zero and all the points lie on both the lines of regression. These two lines coincide and we say that the correlation between the variables is *perfect*. The nearer r^2 is to unity the closer are the points to the lines of regression. Thus *the departure of r^2 from unity is a measure of departure from linearity of the relationship between the variables.*

25.16 RANK CORRELATION

A group of n individuals may be arranged in order to merit with respect to some characteristic. The same group would give different orders for different characteristics. Considering the orders corresponding to two characteristics A and B, the correlation between these n pairs of ranks is called the *rank correlation* in the characteristics A and B for that group of individuals.

Let x_i, y_i be the ranks of the ith individuals in A and B respectively. Assuming that no two individuals are bracketed equal in either case, each of the variables taking the values 1, 2, 3, ..., n, we have

$$\bar{x} = \bar{y} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

If X, Y be the deviations of x, y from their means, then

$$\Sigma X_i^2 = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 + n(\bar{x})^2 - 2\bar{x}\,\Sigma x_i = \Sigma n^2 + \frac{n(n+1)^2}{4} - 2\frac{n+1}{2} \cdot \Sigma n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)^2}{4} - \frac{n(n+1)^2}{2} = \frac{1}{12}(n^3 - n)$$

Similarly $\quad \Sigma Y_i^2 = \dfrac{1}{12}(n^3 - n)$

Now let $\quad d_i = x_i - y_i$ so that $d_i = (x_i - \bar{x}) - (y_i - \bar{y}) = X_i - Y_i$

$\therefore \quad \Sigma d_i^2 = \Sigma X_i^2 + \Sigma Y_i^2 - 2\Sigma X_i Y_i$

or $\quad \Sigma X_i Y_i = \frac{1}{2}(\Sigma X_i^2 + \Sigma Y_i^2 - \Sigma d_i^2) = \dfrac{1}{12}(n^3 - n) - \dfrac{1}{2}\Sigma d_i^2.$

Hence the correlation coefficient between these variables is

$$r = \frac{\Sigma X_i Y_i}{\sqrt{(\Sigma X_i^2 \Sigma Y_i^2)}} = \frac{\frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2}{\frac{1}{12}(n^3 - n)} = 1 - \frac{6\,\Sigma d_i^2}{n^3 - n}$$

This is called the *rank correlation coefficient* and is denoted by ρ.

Example 25.21. *Ten participants in a contest are ranked by two judges as follows :*

x :	1	6	5	10	3	2	4	9	7	8
y :	6	4	9	8	1	2	3	10	5	7

Calculate the rank correlation coefficient ρ. (V.T.U., 2002)

Solution. If $\quad d_i = x_i - y_i$, then $d_i = -5, 2, -4, 2, 2, 0, 1, -1, 2, 1$

$\therefore \quad \Sigma d_i^2 = 25 + 4 + 16 + 4 + 4 + 0 + 1 + 1 + 4 + 1 = 60$

Hence $\quad \rho = 1 - \dfrac{6\Sigma d_i^2}{n^3 - n} = 1 - \dfrac{6 \times 60}{990} = 0.6$ nearly.

Example 25.22. *Three judges, A, B, C, give the following ranks. Find which pair of judges has common approach*

A :	1	6	5	10	3	2	4	9	7	8
B :	3	5	8	4	7	10	2	1	6	9
C :	6	4	9	8	1	2	3	10	5	7

(J.N.T.U., 2003)

Solution. Here $n = 10$.

	Ranks by		d_1	d_2	d_3	d_1^2	d_2^2	d_3^2
$A (= x)$	$B (= y)$	$C (= z)$	$x - y$	$y - z$	$z - x$			
1	3	6	– 2	– 3	5	4	9	25
6	5	4	1	1	– 2	1	1	4
5	8	9	– 3	– 1	4	9	1	16
10	4	8	6	– 4	– 2	36	16	4
3	7	1	– 4	6	– 2	16	36	4
2	10	2	– 8	8	0	64	64	0
4	2	3	2	– 1	– 1	4	1	1
9	1	10	8	– 9	1	64	81	1
7	6	5	1	1	– 2	1	1	4
8	9	7	– 1	2	– 1	1	4	1
Total			0	0	0	200	214	60

$$\therefore \qquad \rho(x, y) = 1 - \frac{6\Sigma d_1^2}{n(n^2 - 1)} = 1 - \frac{6 \times 200}{10 \times 99} = -0.2$$

$$\rho(y, z) = 1 - \frac{6\Sigma d_2^2}{n(n^2 - 1)} = 1 - \frac{6 \times 214}{10 \times 99} = -0.3$$

$$\rho(z, x) = 1 - \frac{6\Sigma d_3^2}{n(n - 1)} = 1 - \frac{6 \times 60}{10 \times 99} = 0.6$$

Since $\rho(z, x)$ is maximum, the pair of judges A and C have the nearest common approach.

PROBLEMS 25.4

1. Find the correlation co-efficient and the regression lines of y and x and x on y for the following data :

x :	1	2	3	4	5	
y :	2	5	3	8	7	*(V.T.U., 2010)*

2. Find the correlation coefficient between x and y from the given data :

x :	78	89	97	69	59	79	68	57	
y :	125	137	156	112	107	138	123	108	*(J.N.T.U., 2005)*

3. Find the co-efficient of correlation between industrial production and export using the following data and comment on the result.

Production (in crore tons) :	55	56	58	59	60	60	62	
Exports (in crore tons) :	35	38	38	39	44	43	45	*(Madras, 2000)*

4. Ten people of various heights as under, were requested to read the letters on a car at 25 yards distance. The number of letters correctly read is given below :

Height (in feet) :	5.1	5.3	5.6	5.7	5.8	5.9	5.10	5.11	6.0	6.1
No. of letters :	11	17	19	14	8	15	20	6	8	12

Is there any correlation between heights and visual power ?

5. Using the formula $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$, find r from the following data :

x :	92	89	87	86	83	77	71	63	53	50
y :	86	88	91	77	68	85	52	82	37	57

6. Find the correlation between x (marks in Mathematics) and y (marks in Engineering Drawing) given in the following data :

y \ x	10—40	40—70	70—100	Total
0—30	5	20	—	25
30—60	—	28	2	30
60—90	—	32	13	45
Total	5	80	15	100

7. Find two lines of regression and coefficient of correlation for the data given below :

$n = 18, \Sigma x = 12, \Sigma y = 18, \Sigma x^2 = 60, \Sigma y^2 = 96, \Sigma xy = 48.$ (*U.P.T.U., MCA, 2009*)

8. If the coefficient of correlation between two variables x and y is 0.5 and the acute angle between their lines of regression is $\tan^{-1}(3/8)$, show that $\sigma_x = \frac{1}{2}\sigma_y$. (*V.T.U., 2004*)

9. For two random variables x and y with the same mean, the two regression lines are $y = ax + b$ and $x = \alpha y + \beta$. Show that $\frac{b}{\beta} = \frac{1-a}{1-\alpha}$. Find also the common mean. (*U.P.T.U., 2010*)

10. Two random variables have the regression lines with equations $3x + 2y = 26$ and $6x + y = 31$. Find the mean values and the correlation coefficient between x and y. (*Madras, 2002*)

11. The regression equations of two variables x and y are $x = 0.7y + 5.2$, $y = 0.3x + 2.8$. Find the means of the variables and the coefficient of correlation between them. (*Osmania, 2002*)

12. In a partially destroyed laboratory data, only the equations giving the two lines of regression of y on x and x on y are available and are respectively, $7x - 16y + 9 = 0$, $5y - 4x - 3 = 0$. Calculate the co-efficient of correlation, $\bar{x}$ and $\bar{y}$.

13. The following results were obtained from records of age (x) and blood pressure (y) of a group of 10 men :

$$\left.\begin{array}{lcc} & x & y \\ \textit{Mean} & 53 & 142 \\ \textit{Variance} & 130 & 165 \end{array}\right\} \quad \text{and} \quad \Sigma(x-\bar{x})(y-\bar{y}) = 1220.$$

Find the appropriate regression equation and use it to estimate the blood pressure of a man whose age is 45.

14. Compute the standard error of estimate S_x for the respective heights of the following 12 couples :

Height x of husband (inches) :	68	66	68	65	69	66	68	65	71	67	68	70
Height y of wife (inches) :	65	63	67	64	68	62	70	66	68	67	69	71

15. Calculate the rank correlation coefficient from the following data showing ranks of 10 students in two subjects :

Maths :	3	8	9	2	7	10	4	6	1	5
Physics :	5	9	10	1	8	7	3	4	2	6

16. Find the rank correlation for the following data :

x :	56	42	72	36	63	47	55	49	38	42	68	60
y :	147	125	160	118	149	128	150	145	115	140	152	155

(*S.V.T.U., 2009 ; J.N.T.U., 2003*)

25.17 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 25.5

Select the correct answer or fill up the blanks in each of the following questions :

1. The median of the numbers 11, 10, 12, 13, 9 is

(*a*) 12.5 (*b*) 12 (*c*) 10.5 (*d*) 11.

2. The mode of the numbers 7, 7, 7, 9, 10, 11, 11, 11, 12 is

(*a*) 11 (*b*) 12 (*c*) 7 (*d*) 7 and 11.

3. S.D. is defined as

(a) $\sqrt{\dfrac{\Sigma f\,(x-\bar{x})^2}{\Sigma f}}$ (b) $\dfrac{\Sigma f\,(x-\bar{x})}{\Sigma f}$ (c) $\dfrac{\Sigma f\,(x-\bar{x})^2}{\Sigma f}$.

4. Coefficient of variation is

(a) $\dfrac{\sigma}{\bar{x}} \times 100$ (b) $\dfrac{\sigma}{x}$ (c) $\sqrt{\dfrac{\sigma^2}{x}} \times 100$.

5. Average scores of three batsman A, B, C are respectively 40, 45 and 55 and their S.D.s are respectively 9, 11, 16. Which batsman is more consistent ?

(a) A (b) B (c) C.

6. The equations of regression lines are $y = 0.5x + a$ and $x = 0.4y + b$. The correlation coefficient is

(a) $\sqrt{0.2}$ (b) 0.45 (c) $-\sqrt{0.2}$.

7. If the correlation coefficient is 0, the two regression lines are

(a) parallel (b) perpendicular (c) coincident (d) inclined at 45° to each other.

8. If r_1 and r_2 are two regression coefficients, then signs of r_1 and r_2 depend on ....
9. Regression coefficient of y on x is 0.7 and that of x on y is 3.2. Is the correlation coefficient r consistent ?
10. The standard deviation of the numbers 24, 48, 64, 36, 53 is ....
11. If $y = x + 1$ and $x = 3y - 7$ are the two lines of regression then $\bar{x}$ = ..., $\bar{y}$ = ... and r = ...
12. If the two regression lines are perpendicular to each other, then their coefficient of correlation is ....
13. Quartile deviation is defined as ....
14. The minimum value of correlation coefficient is ....
15. Prediction error of Y is defined as ....
16. If X and Y are independent, then the correlation coefficient between X and Y is ....
17. The point of intersection of the two regression lines is ....
18. The smaller the coefficient of variation, the greater is the .... in the data.
19. The moment coefficient of skewness is given by ....
20. Kurtosis measures the .... of a distribution.
21. The equation of the line of regression of y on x is ....
22. Coefficient of variation = ...
23. The angle between two regression lines is given by ....
24. A frequency curve is said to be Mesokurtic when β_2 is ....
25. Correlation coefficient is the geometrical mean between ....
26. When the variables are independent, the two lines of regression are ....
27. Arithmetic mean of the coefficients of regression is .... than the coefficient of correlation.
28. If two regression lines coincide then the coefficient of correlation is ....
29. The rank coefficient is given by ....
30. The ratio of the standard deviation to the mean is known as ....
31. The value of $\Sigma f(x - \bar{x})$ = ...
32. The value of coefficient of correlation lies between .... and ....
33. If the two regression coefficients are -0.4 and -0.9, then the correlation coefficient is ....
34. A distribution with the following constants is positively skew : $Q_1 = 25.8$, median = 49.0, $Q_3 = 64.2$. (True or False)
35. Quartile coefficient of skewness is $\dfrac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$. (True or False)
36. Skewness indicates peakedness of the frequency distribution. (True or False)

CHAPTER 26

Probability and Distributions

1. Introduction, Principle of counting, Permutations and Combinations. 2. Basic terminology, Definition of probability. 3. Probability and Set notations. 4. Addition law of probability. 5. Independent events — Multiplication law of probability. 6. Baye's theorem. 7. Random variable. 8. Discrete probability distribution. 9. Continuous probability distribution. 10. Expectation, Variance, Moments. 11. Moment generating function. 12. Probability generating function. 13. Repeated trials. 14. Binomial distribution. 15. Poisson distribution. 16. Normal distribution. 17. Probable error. 18. Normal approximation to Binomial distribution. 19. Some other distributions. 20. Objective Type of Questions.

26.1 (1) INTRODUCTION

We often hear such statements : 'It is likely to rain today', 'I have a fair chance of getting admission', and 'There is an even chance that in tossing a coin the head may come up'. In each case, we are not certain of the outcome, but we wish to assess the chances of our predictions coming true. The study of probability provides a mathematical framework for such assertions and is essential in every decision making process. Before defining probability, let us explain a few terms :

(2) Principle of counting. If an event can happen in n_1 ways and thereafter for each of these events a second event can happen in n_2 ways, and for each of these first and second events a third event can happen for n_3 ways and so on, then the number of ways these m event can happen is given by the product $n_1 \cdot n_2 \cdot n_3 \ldots n_m$.

(3) Permutations. A permutation of a number of objects is their arrangement in some definite order. Given three letters a, b, c, we can permute them two at a time as "bc, cb ; ca, ac; ab, ba" yielding 6 permutations. The combinations or groupings are only 3, *i.e.*, bc, ca, ab. Here the order is immetrial.

The number of permutations of n different thing taken r at a time is

$n(n-1)(n-2) \ldots (n-r+1)$, which is denoted by ${}^nP_r$.

Thus $$ {}^nP_r = n(n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with repetitions. The number of permutations of n objects of which n_1 are alike, n_2 are alike and n_3 are alike is $\dfrac{n!}{n_1!\, n_2!\, n_3!}$.

(4) Combinations. *The number of combinations of n different objects taken r at a time is denoted by* ${}^nC_r$. If we take any one of the combinations, its r objects can be arranged in $r!$ ways. So the total number of arrangements which can be obtained from all the combinations is ${}^nP_r = {}^nC_r \cdot r!$.

Thus $$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

Also $$ {}^nC_{n-r} = {}^nC_r$$

e.g., $$ {}^{25}P_4 = 25 \times 24 \times 23 \times 22;\ {}^{25}C_{21} = {}^{25}C_4 = \frac{25 \times 24 \times 23 \times 22}{4 \times 3 \times 2 \times 1}.$$

Example 26.1. *In how many ways can one make a first, second, third and fourth choice among 12 firms leasing construction equipment.* (J.N.T.U., 2003)

Solution. First choice can be made from any of the 12 firms. Thereafter the second choice can be made from among the remaining 11 firms. Then the third choice can be made from the remaining 10 firms and the fourth choice can be made from the 9 firms.

Thus from the *principle of counting*, the number of ways in which first, second, third and fourth choice can be affected $= 12 \times 11 \times 10 \times 9 = 11880$.

Example 26.2. *Find the number of permutations of all the letters of the word (i) Committee (ii) Engineering.*

Solution. (*i*) $n = 9, n_1\,(m, m) = 2, n_2\,(t, t) = 2, n_3\,(e, e) = 2$

$$\therefore \quad \text{no. of permutations} = \frac{n!}{n_1!\,.\,n_2!\,.\,n_3!} = \frac{9!}{2!\,.\,2!\,.\,2!} = 45360.$$

(*ii*) $n = 11, n_1(e\text{'s}) = 3, n_2\,(g, g) = 2, n_3\,(i, i) = 2, n_4\,(n\text{'s}) = 3$

$$\therefore \quad \text{no. of permutations} = \frac{11!}{3!\,2!\,2!\,3!} = 277200.$$

Example 26.3. *From six engineers and five architects a committee is to be formed having three engineers and two architects. How many different committees can be formed if (i) there is no restriction. (ii) two particular engineers must be included. (iii) one particular architect must be excluded.*

Solution. (*i*) Number of committees $^6C_3 \times {}^5C_2 = \dfrac{6\,.\,5\,.\,4}{3\,.\,2\,.\,1} \times \dfrac{5\,.\,4}{2\,.\,1} = 200.$

(*ii*) Here we have to choose one engineer from the remaining four engineers.

$$\therefore \quad \text{no. of committees} = {}^4C_1 \times {}^5C_2 = 4 \times \frac{5\,.\,4}{2\,.\,1} = 40$$

(*iii*) Here we have to choose two architects from the remaining four architects.

$$\therefore \quad \text{no. of committees} = {}^6C_3 \times {}^4C_2 = \frac{6\,.\,5\,.\,4}{3\,.\,2\,.\,1} \times \frac{4\,.\,3}{2\,.\,1} = 120.$$

PROBLEMS 26.1

1. If a test consists of 12 true-false questions, in how many different ways can a student make the test paper with one answer to each question. (J.N.T.U., 2003)
2. How many 4-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9, without repetition ? How many of these are less than 500 ?
3. A student has to answer 9 out of 12 questions. How many choices has he (*i*) if he must answer first two questions (*ii*) if he must answer at least four of the first five questions.
4. How many car number plates can be made if each plate contains two different letters followed by three different digits ? Solve the problem (*a*) with repetitions and (*b*) without repetitions.

26.2 (I) BASIC TERMINOLOGY

(*i*) **Exhaustive events.** A set of events is said to be *exhaustive*, if it includes all the possible events. For example, in tossing a coin there are two exhaustive cases either head or tail and there is no third possibility.

(*ii*) **Mutually exclusive events.** If the occurrence of one of the events procludes the occurrence of all other, then such a set of events is said to be *mutually exclusive*. Just as tossing a coin, either head comes up or the tail and both can't happen at the same time, *i.e.*, these are two mutually exclusive cases.

(*iii*) **Equally likely events.** If one of the events cannot be expected to happen in preference to another then such events are said to be *equally likely*. For instance, in tossing a coin, the coming of the head or the tail is equally likely.

Thus when a *die** is thrown, the turning up of the six different faces of the die are exhaustive, mutually exclusive and equally likely.

(*iv*) **Odds in favour of an event.** If the number of ways favourable to an event A is m and the number of ways not favourable to A is n then *odds in favour of* $A = m/n$ and *odds against* $A = n/m$.

(2) Definition of probability. *If there are n exhaustive, mutually exclusive and equally likely cases of which m are favourable to an event A, then probability (p) of the happening of A is*

$$\mathbf{P(A) = m/n.}$$

As there are $n - m$ cases in which A will not happen (denoted by A'), *the chance of A not happening is q or P (A') so that*

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

i.e., $P(A') = 1 - P(A)$ so that $\mathbf{P(A) + P(A') = 1}$,

i.e., if an event is certain to happen then its probability is unity, while if it is certain not to happen, its probability is zero.

Obs. This definitions of probability fails when
(*i*) number of outcomes is infinite (not exhaustive) and (*ii*) outcomes are not equally likely.

(3) Statistical (or Empirical) definition of probability. *If in n trials, an event A happens m times, then the probability (p) of happening of A is given by*

$$p = P(A) = \underset{n \to \infty}{Lt} \frac{m}{n}$$

Example 26.4. *Find the chance of throwing (a) four, (b) an even number with an ordinary six faced die.*

Solution. (*a*) There are six possible ways in which the die can fall and of these there is only one way of throwing 4. Thus the required chance $= \frac{1}{6}$.

(*b*) There are six possible ways in which the die can fall. Of these there are only 3 ways of getting 2, 4 or 6. Thus the required chance $= 3/6 = \frac{1}{2}$.

Example 26.5. *What is the chance that a leap year selected at random will contain 53 Sundays ?* (*Madras, 2003*)

Solution. A leap year consists of 366 days, so that there are 52 full weeks (and hence 52 Sundays) and two extra days. These two days can be (*i*) Monday, Tuesday (*ii*) Tuesday, Wednesday, (*iii*) Wednesday, Thursday (*iv*) Thursday, Friday (*v*) Friday, Saturday (*vi*) Saturday, Sunday (*vii*) Sunday, Monday.

Of these 7 cases, the last two are favourable and hence the required probability $= \frac{2}{7}$.

Example 26.6. *A five figure number is formed by the digits 0, 1, 2, 3, 4 without repetition. Find the probability that the number formed is divisible by 4.*

Solution. The five digits can be arranged in 5! ways, out of which 4! will begin with zero.

$\therefore$ total number of 5-figure numbers formed $= 5! - 4! = 96$.

Those numbers formed will be divisible by 4 which will have two extreme right digits divisible by 4, *i.e.*, numbers ending in 04, 12, 20, 24, 32, 40.

Now numbers ending in $04 = 3! = 6$, numbers ending in $12 = 3! - 2! = 4$,
numbers ending in $20 = 3! = 6$, numbers ending in $24 = 3! - 2! = 4$,
numbers ending in $32 = 3! - 2! = 4$, and numbers ending in $40 = 3! = 6$.

[The numbers having 12, 24, 32 in the extreme right are $(3! - 2!)$ since the numbers having zero on the extreme left are to excluded.]

* *Die* is a small cube. Dots 1, 2, 3, 4, 5, 6 are marked on its six faces. The outcome of throwing a die is the number of dots on its upper face.

$\therefore$ total number of favourable ways = 6 + 4 + 6 + 4 + 4 + 6 = 30.

Hence the required probability $= \frac{30}{96} = \frac{5}{16}$.

Example 26.7. *A bag contains 40 tickets numbered 1, 2, 3, ... 40, of which four are drawn at random and arranged in ascending order ($t_1 < t_2 < t_3 < t_4$). Find the probability of t_3 being 25?*

Solution. Here exhaustive number of cases $= {}^{40}C_4$

If $t_3 = 25$, then the tickets t_1 and t_2 must come out of 24 tickets numbered 1 to 24. This can be done in ${}^{24}C_2$ ways.

Then t_4 must come out of the 15 tickets (numbering 25 to 40) which can be done in ${}^{15}C_1$ ways.

$\therefore$ favourable number of cases $= {}^{24}C_2 \times {}^{15}C_1$

Hence the probability of t_3 being 25 $= \frac{{}^{24}C_2 \times {}^{15}C_1}{{}^{40}C_4} = \frac{414}{9139}$.

Example 26.8. *An urn contains 5 red and 10 black balls. Eight of them are placed in another urn. What is the chance that the latter then contains 2 red and 6 black balls ?*

Solution. The number of ways in which 8 balls can be drawn out of 15 is ${}^{15}C_8$.

The number of ways of drawing 2 red balls is ${}^5C_2$ and corresponding to each of these ${}^5C_2$ ways of drawing a red ball, there are ${}^{10}C_6$ ways of drawing 6 black balls.

$\therefore$ the total number of ways in which 2 red and 6 black balls can be drawn is ${}^5C_2 \times {}^{10}C_6$.

$\therefore$ the required probability $= \frac{{}^5C_2 \times {}^{10}C_6}{{}^{15}C_8} = \frac{140}{429}$.

Example 26.9. *A committee consists of 9 students two of which are from 1st year, three from 2nd year and four from 3rd year. Three students are to be removed at random. What is the chance that (i) the three students belong to different classes, (ii) two belong to the same class and third to the different class, (iii) the three belong to the same class ?* (V.T.U., 2002 S)

Solution. (*i*) The total number of ways of choosing 3 students out of 9 is ${}^9C_3$, *i.e.*, 84.

A student can be removed from 1st year students in 2 ways, from 2nd year in 3 ways and from 3rd year in 4 ways, so that the total number of ways of removing three students, one from each group is $2 \times 3 \times 4$.

Hence the required chance $= \frac{2 \times 3 \times 4}{{}^9C_3} = \frac{24}{84} = \frac{2}{7}$.

(*ii*) The number of ways of removing two from 1st year students and one from others

$$= {}^2C_2 \times {}^7C_1.$$

The number of ways of removing two from 2nd year students and one from others

$$= {}^3C_2 \times {}^6C_1.$$

The number of ways of removing 2 from 3rd year students and one from others

$$= {}^4C_2 \times {}^5C_1.$$

$\therefore$ the total number of ways in which two students of the same class and third from the others may be removed

$$= {}^2C_2 \times {}^7C_1 + {}^3C_2 \times {}^6C_1 + {}^4C_2 \times {}^5C_1 = 7 + 18 + 30 = 55.$$

Hence, the required chance $= \frac{55}{84}$.

(*iii*) Three students can be removed from 2nd year group in ${}^3C_3$, *i.e.* 1 way and from 3rd year group in ${}^4C_3$, *i.e.*, 4 ways.

$\therefore$ the total number of ways in which three students belong to the same class = 1 + 4 = 5.

Hence the required chance $= \frac{5}{84}$.

Example 26.10. *A has one share in a lottery in which there is 1 prize and 2 blanks ; B has three shares in a lottery in which there are 3 prizes and 6 blanks ; compare the probability of A's success to that of B's success.*

Solution. A can draw a ticket in ${}^3C_1 = 3$ ways.

The number of cases in which A can get a prize is clearly 1.

$\therefore$ the probability of A's success $= \dfrac{1}{3}\cdot$

Again B can draw a ticket in ${}^9C_3 = \dfrac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ ways.

The number of ways in which B gets all blanks $= {}^6C_3 = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

$\therefore$ the number of ways of getting a prize $= 84 - 20 = 64$.

Thus the probability of B's success $= 64/84 = 16/21$.

Hence A's probability of success : B's probability of success $= \dfrac{1}{3} : \dfrac{16}{21} = 7 : 16$.

26.3 PROBABILITY AND SET NOTATIONS

(1) Random experiment. Experiments which are performed essentially under the same conditions and whose results cannot be predicted are known as *random experiments. e.g.,* Tossing a coin or rolling a die are random experiments.

(2) Sample space. The set of all possible outcomes of a random experiment is called *sample space* for that experiment and is denoted by S.

The elements of the sample space S are called the *sample points.*

e.g., On tossing a coin, the possible outcomes are the head (H) and the tail (T). Thus $S = \{H, T\}$.

(3) Event. The outcome of a random experiment is called an *event*. Thus every subset of a sample space S is an *event*.

The null set ϕ is also an event and is called an *impossible event. Probability of an impossible event* is zero *i.e.*, $P(\phi) = 0$.

(4) Axioms

(*i*) *The numerical value of probability lies between 0 and 1.*

i.e., for any event A of S, $0 \le P(A) \le 1$.

(*ii*) *The sum of probabilities of all sample events is unity i.e.*, $P(S) = 1$.

(*iii*) *Probability of an event made of two or more sample events is the sum of their probabilities.*

(5) Notations

(*i*) *Probability of happening of events* **A** *or* **B** is written as $P(A + B)$ or $P(A \cup B)$.

(*ii*) *Probability of happening of* **both the events A and B** is written as $P(AB)$ or $P(A \cap B)$.

(*iii*) '*Event A implies* ($\Rightarrow$) *event B*' is expressed as $A \subset B$.

(*iv*) '*Events A and B are mutually exclusive*' is expressed as $A \cap B = \phi$.

(6) *For any two events A and B,*

$$\boldsymbol{P(A \cap B') = P(A) - P(A \cap B)}$$

Proof. From Fig. 26.1,

$$(A \cap B') \cup (A \cap B) = A$$

$\therefore$ $$P[(A \cap B') \cup (A \cap B)] = P(A)$$

or $$P(A \cap B') + P(A \cap B) = P(A)$$

or $$P(A \cap B') = P(A) - P(A \cap B)$$

Similarly, $P(A' \cap B) = P(B) - P(A \cap B)$

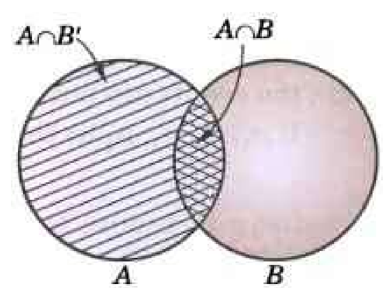


Fig. 26.1

26.4 ADDITION LAW OF PROBABILITY *or* THEOREM OF TOTAL PROBABILITY

(1) *If the probability of an event A happening as a result of a trial is P(A) and the probability of a* **mutually exclusive** *event B happening is P(B), then the probability of* **either** *of the events happening as a result of the trial is* $\boldsymbol{P(A + B)}$ **or** $\boldsymbol{P(A \cup B) = P(A) + P(B)}$.

Proof. Let n be the total number of equally likely cases and let m_1 be favourable to the event A and m_2 be favourable to the event B. Then the number of cases favourable to A or B is $m_1 + m_2$. Hence the probability of A or B happening as a result of the trial

$$= \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

(2) *If A, B, are any two events* **(not mutually exclusive),** *then*

$$\boldsymbol{P(A + B) = P(A) + P(B) - P(AB)}$$

or

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

If the events A and B are any two events then, there are some outcomes which favour both A and B. If m_3 be their number, then these are included in both m_1 and m_2. Hence the total number of outcomes favouring either A or B or both is

$$m_1 + m_2 - m_3.$$

Thus the probability of occurrence of A or B or both

$$= \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

Hence $\qquad P(A + B) = P(A) + P(B) - P(AB)$

or $\qquad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Obs. *When A and B are mutually exclusive* $P(AB)$ or $P(A \cap B) = 0$ and we get

$$P(A + B) \text{ or } P(A \cup B) = P(A) + P(B).$$

In general, for a number of **mutually exclusive** events $A_1, A_2, \ldots A_n$, we have

$$P(A_1 + A_2 + \ldots + A_n) \text{ or } P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

(3) *If A, B, C are any three events, then*

$$\boldsymbol{P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC).}$$

or

$$\boldsymbol{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)}$$

Proof. Using the above result for any two events, we have

$$\begin{aligned}
P(A \cup B \cup C) &= P[(A \cup B) \cup C] \\
&= P(A \cup B) + P(C) - P[(A \cup B) \cap C] \\
&= [P(A) + P(B) - P(A \cap B)] + P(C) - P[(A \cap C) \cup (B \cap C)] && \text{(Distributive Law)} \\
&= P(A) + P(B) + P(C) - P(A \cap B) - \{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)\} \\
& && [\because (A \cap C) \cap (B \cap C) = A \cap B \cap C] \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) && [\because A \cap C = C \cap A.]
\end{aligned}$$

Example 26.11. *In a race, the odds in favour of the four horses H_1, H_2, H_3, H_4 are 1 : 4, 1 : 5, 1 : 6, 1 : 7 respectively. Assuming that a dead heat is not possible, find the chance that one of them wins the race.*

Solution. Since it is not possible for all the horses to cover the same distance in the same time (a dead heat), the events are mutually exclusive.

If p_1, p_2, p_3, p_4 be the probabilities of winning of the horses H_1, H_2, H_3, H_4 respectively, then

$$p_1 = \frac{1}{1+4} = \frac{1}{5} \qquad [\because \text{ Odds in favour of } H_1 \text{ are } 1 : 4]$$

and

$$p_2 = \frac{1}{6}, p_3 = \frac{1}{7}, p_4 = \frac{1}{8}.$$

Hence the chance that one of them wins $= p_1 + p_2 + p_3 + p_4$

$$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840}.$$

Example 26.12. *A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same colour.*

Solution. Two balls out of 14 can be drawn in ${}^{14}C_2$ ways which is the total number of outcomes.

Two white balls out of 8 can be drawn in ${}^{8}C_2$ ways. Thus the probability of drawing 2 white balls

$$= \frac{{}^{8}C_2}{{}^{14}C_2} = \frac{28}{91}$$

Similarly 2 red balls out of 6 can be drawn in ${}^{6}C_2$ ways. Thus the probability of drawing 2 red balls

$$= \frac{{}^{6}C_2}{{}^{14}C_2} = \frac{15}{91}.$$

Hence the probability of drawing 2 balls of the same colour (either both white or both red)

$$= \frac{28}{91} + \frac{15}{91} = \frac{43}{91}.$$

Example 26.13. *Find the probability of drawing an ace or a spade or both from a deck of cards* ?*

Solution. The probability of drawing an ace from a deck of 52 cards = 4/52.

Similarly the probability of drawing a card of spades = 13/52, and the probability of drawing an ace of spades = 1/52.

Since the two events (*i.e.*, a card being an ace and a card being of spades) are not mutually exclusive, therefore, the probability of drawing an ace or a spade

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}.$$

26.5 (1) INDEPENDENT EVENTS

Two events are said to be *independent*, if happening or failure of one does not affect the happening or failure of the other. Otherwise the events are said to be *dependent*.

For two dependent events A and B, the symbol $P(B/A)$ denotes the probability of occurrence of B, when A has already occurred. It is known as the **conditional probability** and is read as a '*probability of B given A*'.

(2) Multiplication law of probability *or* **Theorem of compound probability.** *If the probability of an event A happening as a result of trial is P(A) and after A has happened the probability of an event B happening as a result of another trial (i.e.,* **conditional probability of B given A**) *is $P(B/A)$, then the probability of* **both** *the events A and B happening as a result of two trials is* $\mathbf{P(AB)}$ **or** $\mathbf{P(A \cap B) = P(A) . P(B/A)}$.

Proof. Let n be the total number of outcomes in the first trial and m be favourable to the event A so that $P(A) = m/n$.

Let n_1 be the total number of outcomes in the second trial of which m_1 are favourable to the event B so that $P(B/A) = m_1/n_1$.

Now each of the n outcomes can be associated with each of the n_1 outcomes. So the total number of outcomes in the combined trial is nn_1. Of these mm_1 are favourable to both the events A and B. Hence

$$P(AB) \text{ or } P(A \cap B) = \frac{mm_1}{nn_1} = P(A) \,.\, P(B/A).$$

Similarly, the *conditional probability of A given B is $P(A/B)$.*

$\therefore \quad P(AB)$ or $P(A \cap B) = P(B) \,.\, P(A/B)$

Thus $\quad P(A \cap B) = P(A) \,.\, P(B/A) = P(B) \,.\, P(A/B).$

(3) *If the events A and B are* **independent,** *i.e.*, if the happening of B does not depend on whether A has happened or not, then $P(B/A) = P(B)$ and $P(A/B) = P(A)$.

$\therefore \quad \mathbf{P(AB)}$ **or** $\mathbf{P(A \cap B) = P(A) . P(B).}$

In general, $\quad P(A_1A_2...A_n)$ or $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \,.\, P(A_2) .... P(A_n)$.

* *Cards* : A pack of cards consists of four suits *i.e.*, Hearts, Diamonds, Spades and Clubs. Each suit has 13 cards : an Ace, a King, a Queen, a Jack and nine cards numbered 2, 3, 4, ..., 10. Hearts and Diamonds are *red* while Spades and Clubs are *black*.

Cor. If p_1, p_2 be the probabilities of happening of two independent events, then

(*i*) the probability that the first event happens and the second fails is $p_1(1-p_2)$.

(*ii*) the probability that both events fail to happen is $(1-p_1)(1-p_2)$.

(*iii*) the probability that at least one of the events happens is

$1-(1-p_1)(1-p_2)$. This is commonly known as their **cumulative probability.**

In general, if $p_1, p_2, p_3, \ldots p_n$ be the chances of happening of n independent events, then their cumulative probability (i.e., the chance that at least one of the events will happen) is

$$1-(1-p_1)(1-p_2)(1-p_3)\ldots(1-p_n).$$

Example 26.14. *Two cards are drawn in succession from a pack of 52 cards. Find the chance that the first is a king and the second a queen if the first card is (i) replaced, (ii) not replaced.*

Solution. (*i*) The probability of drawing a king $= \frac{4}{52} = \frac{1}{13}$.

If the card is replaced, the pack will again have 52 cards so that the probability of drawing a queen is 1/13.

The two events being independent, the probability of drawing both cards in succession $= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

(*ii*) The probability of drawing a king $= \frac{1}{13}$.

If the card is not replaced, the pack will have 51 cards only so that the chance of drawing a queen is 4/51.

Hence the probability of drawing both cards $= \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$.

Example 26.15. *A pair of dice is tossed twice. Find the probability of scoring 7 points (a) once, (b) at least once (c) twice.* *(Kurukshetra, 2009 S ; V.T.U., 2004)*

Solution. In a single toss of two dice, the sum 7 can be obtained as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) *i.e.*, in 6 ways, so that the probability of getting 7 = 6/36 = 1/6.

Also the probability of not getting 7 = 1 – 1/6 = 5/6.

(*a*) The probability of getting 7 in the first toss and not getting 7 in the second toss = 1/6 × 5/6 = 5/36.

Similarly, the probability of not getting 7 in the first toss and getting 7 in the second toss = 5/6 × 1/6 = 5/36.

Since these are mutually exclusive events, addition law of probability applies.

$\therefore$ required probability $= \frac{5}{36} + \frac{5}{36} = \frac{5}{18}$.

(*b*) The probability of not getting 7 in either toss $= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$\therefore$ the probability of getting 7 at least once $= 1 - \frac{25}{36} = \frac{11}{36}$.

(*c*) The probability of getting 7 twice $= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Example 26.16. *There are two groups of objects : one of which consists of 5 science and 3 engineering subjects, and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If the number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately*

Solution. Prob. of turning up 3 or 5 $= \frac{2}{6} = \frac{1}{3}$.

Prob. of selecting an engg. subject from first group $= \frac{3}{8}$

$\therefore$ Prob of selecting an engg. subject from first group on turning up 3 or 5

$$= \frac{1}{3} \times \frac{3}{8} = \frac{1}{8} \qquad \ldots(i)$$

Now prob. of not turning 3 or 5 $= 1 - \frac{1}{3} = \frac{2}{3}$.

Prob. of selecting an engg. subject from second group $= \frac{5}{8}$

$\therefore$ prob. of selecting an engg. subject from second group on turning up 3 or 5

$$= \frac{2}{3} \times \frac{5}{8} = \frac{5}{12} \qquad ...(ii)$$

Thus the prob. of selecting an engg. subject

$$= \frac{1}{8} + \frac{5}{12} = \frac{13}{24}. \qquad \text{[From } (i) \text{ and } (ii)\text{]}$$

Example 26.17. *A box A contains 2 white and 4 black balls. Another box B contains 5 white and 7 black balls. A ball is transferred from the box A to the box B. Then a ball is drawn from the box B. Find the probability that it is white.* (V.T.U., 2004)

Solution. The probability of drawing a white ball from box B will depend on whether the transferred ball is black or white.

If a black ball is transferred, its probability is 4/6. There are now 5 white and 8 black balls in the box B. Then the probability of drawing white ball from box B is $\frac{5}{13}$.

Thus the probability of drawing a white ball from urn B, if the transferred ball is black

$$= \frac{4}{6} \times \frac{5}{13} = \frac{10}{39}.$$

Similarly the probability of drawing a white ball from urn B, if the transferred ball is white

$$= \frac{2}{6} \times \frac{6}{13} = \frac{2}{13}.$$

Hence required probability $= \frac{10}{39} + \frac{2}{13} = \frac{16}{39}$.

Example 26.18. *(a) A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd.* (Mumbai, 2006)

(b) Two persons A and B toss an unbiased coin alternately on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning. (Madras, 2000 S)

Solution. (*a*) Let p be the probability of getting a head and q the probability of getting a tail in a single toss, so that $p + q = 1$.

Then probability of getting head on an odd toss

= Probability of getting head in the 1st toss

+ Probability of getting head in the 3rd toss

+ Probability of getting head in the 5th toss + ... ∞

$$= p + qqp + qqqqp + \dots \infty$$

$$= p(1 + q^2 + q^4 + \dots) = p \cdot \frac{1}{1-q^2} \quad (q < 1)$$

$$= p \cdot \frac{1}{(1-q)(1+q)} = p \cdot \frac{1}{p(1+q)} = \frac{1}{1+q}.$$

(*b*) Probability of getting a head = 1/2. Then A can win in 1st, 3rd, 5th, ... throws.

$\therefore$ the chances of A's winning $= \frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{1}{2}\right)^4 \frac{1}{2} + \left(\frac{1}{2}\right)^6 \frac{1}{2} + \cdots$

$$= \frac{1/2}{1-(1/2)^2} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

Hence the chance of B's winning $= 1 - 2/3 = 1/3$.

Example 26.19. *Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability p that the sum is odd, if*

(i) the two cards are drawn together.

(ii) the two cards are drawn one after the other without replacement.

(iii) the two cards are drawn one after the other with replacement. (J.N.T.U., 2003)

Solution. (*i*) Two cards out of 10 can be selected in ${}^{10}C_2 = 45$ ways. The sum is odd if one number is odd and the other number is even. There being 5 odd numbers (1, 3, 5, , , 9) and 5 even numbers (2, 4, 6, 8, 10), an odd and an even number is chosen in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25}{45} = \frac{5}{9}.$$

(*ii*) Two cards out of 10 can be selected one after the other *without replacement* in $10 \times 9 = 90$ ways.

An odd number is selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways

Thus $$p = \frac{25 + 25}{90} = \frac{5}{9}.$$

(*iii*) Two cards can be selected one after the other *with replacement* in $10 \times 10 = 100$ ways.

An odd number in selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25 + 25}{100} = \frac{1}{2}.$$

Example 26.20. *Given* $P(A) = 1/4$, $P(B) = 1/3$ *and* $P(A \cup B) = 1/2$, *evaluate* $P(A/B)$, $P(B/A)$, $P(A \cap B')$ and $P(A/B')$.

Solution. (*i*) Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore \quad \frac{1}{2} = \frac{1}{4} + \frac{1}{3} - P(A \cap B) \text{ or } P(A \cap B) = \frac{1}{12}$$

Thus $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4}.$$

(*ii*) $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/4} = \frac{1}{3}.$$

(*iii*) $$P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}.$$

(*iv*) $$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{1/6}{1 - P(B)} = \frac{1/6}{1 - 1/3} = \frac{1}{4}.$$

Example 26.21. *The odds that a book will be reviewed favourably by three independent critics are 5 to*

Finally, prob. that all the three are favourable $= \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{60}{343}$

Since they are mutually exclusive events, the required prob.

$$= \frac{80}{343} + \frac{45}{343} + \frac{24}{343} + \frac{60}{343} = \frac{209}{343}.$$

Example 26.22. *I can hit a target 3 times in 5 shots, B 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that (i) two shots hit, (ii) atleast two shots hit ?*

(A.M.I.E.T.E., 2003 ; Madras, 2000 S)

Solution. Prob. of A hitting the target = 3/5, prob. of B hitting the target = 2/5

Prob. of C hitting the target = 3/4.

(*i*) In order that two shots may hit the target, the following cases must be considered :

$$p_1 = \text{Chance that } A \text{ and } B \text{ hit and } C \text{ fails to hit} = \frac{3}{5} \times \frac{2}{3} \times \left(1 - \frac{3}{4}\right) = \frac{6}{100}$$

$$p_2 = \text{Chance that } B \text{ and } C \text{ hit and } A \text{ fails to hit} = \frac{2}{5} \times \frac{3}{4} \times \left(1 - \frac{3}{5}\right) = \frac{12}{100}$$

$$p_3 = \text{Chance that } C \text{ and } A \text{ hit and } B \text{ fails to hit} = \frac{3}{4} \times \frac{3}{5} \times \left(1 - \frac{2}{5}\right) = \frac{27}{100}$$

Since these are mutually exclusive events, the probability that any 2 shots hit

$$= p_1 + p_2 + p_3 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = 0.45.$$

(*ii*) In order that at least two shots may hit the target, we must also consider the case of all A, B, C hitting the target [in addition to the three cases of (*i*)] for which

$$p_4 = \text{chance that } A, B, C \text{ all hit} = \frac{3}{5} \times \frac{2}{5} \times \frac{3}{4} = \frac{18}{100}$$

Since all these are mutually exclusive events, the probability of *atleast* two shots hit

$$= p_1 + p_2 + p_3 + p_4 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} + \frac{18}{100} = 0.63.$$

Example 26.23. *A problem in mechanics is given to three students A, B, and C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved.* *(V.T.U., 2004)*

Solution. The probability that A can solve the problem is 1/2.

The probability that A cannot solve the problem is $1 - \frac{1}{2}$.

Similarly the probabilities that B and C cannot solve the problem are $1 - \frac{1}{3}$ and $1 - \frac{1}{4}$.

$\therefore$ the probability that A, B and C cannot solve the problem is $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)$.

Hence the probability that the problem will be solved, *i.e.*, at least one student will solve it

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{4}.$$

Example 26.24. *The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if*

(*i*) *the class consists of 4 boys and 3 girls.*

(*ii*) *the class consists of 3 boys and 3 girls.* *(J.N.T.U., 2003)*

Solution. (*i*) As there are 7 students in the class, the first examined must be a boy.

$\therefore$ prob. that first is a boy $= \dfrac{4}{7}$

Then the prob. that the second is a girl $= \dfrac{3}{6}\cdot$

$\therefore$ prob. of the next boy $= \dfrac{3}{5}$

Similarly the prob. that the fourth is a girl $= \dfrac{2}{4}$,

the prob. that the fifth is a boy $= \dfrac{2}{3}$,

the prob. that the sixth is a girl $= \dfrac{1}{2}$

and the last is a boy $= \dfrac{1}{1}\cdot$

Thus $$p = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{35}.$$

(*ii*) The first student is a boy and the first student is a girl are two mutually exclusive cases. If the first student is a boy, then the probability p_1 that the students alternate is

$$p_1 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}.$$

If the first student is a girl, then the probability p_2 that the students alternate is

$$p_2 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}.$$

Thus the required prob. $p = p_1 + p_2 = \dfrac{1}{20} + \dfrac{1}{20} = \dfrac{1}{10}\cdot$

Example 26.25. (*Huyghen's problem*) *A and B throw alternately with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find his chance of winning.*

(*Madras, 2006 ; J.N.T.U., 2003*)

Solution. The sum 6 can be obtained as follows : (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), *i.e.*, in 5 ways.

The probability of A's throwing 6 with 2 dice is $\dfrac{5}{36}\cdot$

$\therefore$ the probability of A's not throwing 6 is 31/36.

Similarly the probability of B's throwing 7 is 6/36, *i.e.*, $\dfrac{1}{6}\cdot$

$\therefore$ the probability of B's not throwing 7 is 5/6.

Now A can win if he throws 6 in the first, third, fifth, seventh etc. throws.

$\therefore$ the chance of A's wining

$$= \frac{5}{36} + \frac{31}{36}\times\frac{5}{6}\times\frac{5}{36} + \frac{31}{36}\times\frac{5}{6}\times\frac{31}{36}\times\frac{5}{6}\times\frac{5}{36} + \cdots$$

$$= \frac{5}{36}\left[1 + \left(\frac{31}{36}\times\frac{5}{6}\right) + \left(\frac{31}{36}\times\frac{5}{6}\right)^2 + \left(\frac{31}{36}\times\frac{5}{6}\right)^3 + \cdots\right]$$

$$= \frac{5}{36}\cdot\frac{1}{1-(31/36)\times(5/6)} = \frac{5}{36}\times\frac{36\times 6}{61} = \frac{30}{61}.$$

PROBLEMS 26.2

1. (*i*) Given $P(A) = 1/2$, $P(B) = 1/3$ and $P(AB) = 1/4$, find the value $P(A + B)$. (*Burdwan, 2003*)

   (*ii*) Let A and B be two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(A/B)$, $P(A \cup B)$, $P(A'/B')$.

   (*Kurukshetra, 2009 ; V.T.U., 2003 S*)

2. In a single throw with two dice, what is the chance of throwing

(*a*) two aces ? (*b*) 7 ? Is this probability the same as that for getting 7 in two throws of a single die ?

3. Compare the chances of throwing 4 with one dice, 8 with two dice and 12 with three dice.

4. Find the probability that a non-leap year should have 53 Saturdays ? (*Madras, 2003*)

5. When a coin is tossed four times, find the probability of getting (*i*) exactly one head, (*ii*) at most three heads and (*iii*) at least two heads ? (*V.T.U., 2000 S*)

6. Ten coins are thrown simultaneously. Find the probability of getting at least seven heads. (*P.T.U., 2003*)

7. If all the letters of word 'ENGINEER' be written at random, what is the probability that all the latters *E* are found together.

8. A ten digit number is formed using the digits from zero to nine, every digit being used only once. Find the probability that the number is divisible by 4.

9. Four cards are drawn from a pack of 52 cards. What is the chance that

(*i*) no two cards are of equal value ? (*ii*) each belongs to a different suit ?

10. Suppose 5 cards are drawn at random from a pack of 52 cards. If all cards are red what is the probability that all of them are hearts? (*Mumbai, 2005*)

11. Out of 50 rare books, 3 of which are especially valuable, 5 are stolen at random by a thief. What is the probability that

(*a*) none of the 3 is included ? (*b*) 2 of the 3 are included ?

12. Five men in a company of twenty are graduates. If 3 men are picked out of 20 at random, what is the probability that

(*a*) they are all graduates ? (*b*) at least one is graduate ?

13. From 20 tickets marked from 1 to 20, one ticket is drawn at random. Find the probability that it is marked with a multiple of 3 or 5.

14. Five balls are drawn from a bag containing 6 white and 4 black balls. What is the chance that 3 white and 2 black balls are drawn ?

15. The probability of n independent events are $p_1, p_2, p_3, \ldots p_n$. Find the probability that at least one of the events will happen. Use this result to find the chance of getting at least one six in a throw of 4 dice.

16. Find the probability of drawing 4 white balls and 2 black balls without replacement from a bag containing 1 red, 4 black and 6 white balls.

17. A bag contains 10 white and 15 black balls. Two balls are drawn in succession. What is the probability that one of them is black and the other white ?

18. A purse contains 2 silver and 4 copper coins and a second purse contains 4 silver and 4 copper coins. If a coin is selected at random from one of the two purses, what is the probability that it is a silver coin ? (*Osmania, 2002*)

19. A box I contains 5 white balls and 6 black balls. Another box II contains 6 white balls and 4 black balls. A box is selected at random and then a ball is drawn from it : (*i*) what is the probability that the ball drawn will be white ? (*ii*) Given that the ball drawn is white, what is the probability that it came from box I. (*Mumbai, 2006*)

20. A party of n persons take their seats at random at a round table ; find the probability that two specified persons do not sit together.

21. *A* speaks the truth in 75% cases, and *B* in 80% of the cases. In what percentage of cases, are they likely to contradict each other in stating the same fact ? (*V.T.U., 2002 S*)

22. The probability that Sushil will solve a problem is 1/4 and the probability that Ram will solve it is 2/3. If Sushil and Ram work independently, what is the probability that the problem will be solved by (*a*) both of them, (*b*) at least one of them ?

23. A student takes his examination in four subjects, *P*, *Q*, *R*, *S*. He estimates his chances of passing in *P* as 4/5, in *Q* as 3/4, in *R* as 5/6 and in *S* as 2/3. To qualify, he must pass in *P* and at least two other subjects. What is the probability that he qualifies ? (*Madras, 2000 S*)

24. The probability that a 50 year old man will be alive at 60 is 0.83 and the probability that a 45 year old women will be alive at 55 is 0.87. What is the probability that a man who is 50 and his wife who is 45 will both be alive 10 years hence ?

25. If on an average one birth in 80 is a case of twins, what is the probability that there will be at least one case of twins in a maternity hospital on a day when 20 births occur ?

26. Two persons *A* and *B* fire at a target independently and have a probability 0.6 and 0.7 respectively of hitting the target. Find the probability that the target is destroyed.

27. *A* and *B* throw alternately with a pair of dice. The one who throws 9 first wins. Show that the chances of their winning are 9 : 8.

26.6 BAYE'S THEOREM

An event A corresponds to a number of exhaustive events $B_1, B_2, ..., B_n$. If $P(B_i)$ and $P(A/B_i)$ are given, then

$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\Sigma P(B_i)P(A/B_i)}.$$

Proof. By the multiplication law of probability,

$$P(AB_i) = P(A)\,P(B_i/A) = P(B_i)\,P(A/B_i) \qquad ...(1)$$

$$\therefore \qquad P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{P(A)} \qquad ...(2)$$

Since the event A corresponds to $B_1, B_2, ..., B_n$, we have by the addition law of probability,

$$P(A) = P(AB_1) + P(AB_2) + ... + P(AB_n) = \Sigma P(AB_i) = \Sigma P(B_i)\,P(A/B_i) \qquad \text{[By (1)]}$$

Hence from (2), we have $P(B_i/A) = \dfrac{P(B_i)\,P(A/B_i)}{\Sigma P(B_i)\,P(A/B_i)}$

which is known as the *theorem of inverse probability.*

Obs. The probabilities $P(B_i)$, $i = 1, 2, ..., n$ are called *apriori probabilities* because these exist before we get any information from the experiment.

The probabilities $P(A/B_i)$, $i = 1, 2, ..., n$ are called *posteriori probabilities*, because these are found after the experiment results are known.

Example 26.26. *Three machines M_1, M_2 and M_3 produce identical items. Of their respective output 5%, 4% and 3% of items are faulty. On a certain day, M_1 has produced 25% of the total output, M_2 has produced 30% and M_3 the remainder. An item selected at random is found to be faulty. What are the chances that it was produced by the machine with the highest output?*

Solution. Let the event of drawing a faulty item from any of the machines be A, and the event that an item drawn at random was produced by M_i be B_i. We have to find $P(B_i/A)$ for which we proceed as follows :

	M_1	M_2	M_3	*Remarks*
$P(B_i)$	0.25	0.30	0.45	$\because$ sum = 1
$P(A/B_i)$	0.05	0.04	0.03	
$P(B_i)\,P(A/B_i)$	0.0125	0.012	0.0135	sum = 0.38
$P(B_i/A)$	$\frac{0.0125}{0.038}$	$\frac{0.012}{0.038}$	$\frac{0.0135}{0.038}$	by Baye's theorem

The highest output being from M_3, the required probability = 0.0135/0.038 = 0.355.

Example 26.27. *There are three bags : first containing 1 white, 2 red, 3 green balls ; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.* (*J.N.T.U., 2003*)

Solution. Let B_1, B_2, B_3 pertain to the first, second, third bags chosen and A : the two balls are white and red.

Now $\qquad P(B_1) = P(B_2) = P(B_3) = \dfrac{1}{3}$

$P(A/B_1) = P$ (a white and a red ball are drawn from first bag)

$$= ({}^1C_1 \times {}^2C_1)/{}^6C_2 = \frac{2}{15}$$

Similarly $\qquad P(A/B_2) = ({}^2C_1 \times {}^3C_1)/{}^6C_2 = \dfrac{2}{5}$, $P(A/B_3) = ({}^3C_1 \times {}^1C_1)/{}^6C_2 = \dfrac{1}{5}$

By Baye's theorem, $\quad P(B_2/A) = \dfrac{P(B_2)\,P(A/B_2)}{P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\,P(A/B_3)}$

$$= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{15} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{6}{11}.$$

PROBLEMS 26.3

1. In a certain college, 4% of the boys and 1% of girls are taller than 1.8 m. Further more 60% of the students are girls. If a student is selected at random and is found to be taller than 1.8 m., what is the probability that the student is a girl ?
2. In a bolt factory, machines A, B and C manufacture 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B or C ? *(V.T.U., 2006 ; Rohtak, 2005 ; Madras, 2000 S)*
3. In a bolt factory, there are four machines A, B, C, D manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2% in the same order are defective bolts. A bolt is chosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine A or machine D ? *(Hissar, 2007 ; J.N.T.U., 2003)*
4. The contents of three urns are : 1 white, 2 red, 3 green balls ; 2 white, 1 red, 1 green balls and 4 white, 5 red, 3 green balls. Two balls are drawn from an urn chosen at random. These are found to be one white and one green. Find the probability that the balls so drawn came from the third urn. *(Kurukshetra, 2007)*

26.7 RANDOM VARIABLE

If a real variable X be associated with the outcome of a random experiment, then since the values which X takes depend on chance, it is called a *random variable* or a *stochastic variable* or simply a *variate*. For instance, if a random experiment E consists of tossing a pair of dice, the sum X of the two numbers which turn up have the value 2, 3, 4, ..., 12 depending on chance. Then X is the random variable. It is a function whose values are real numbers and depend on chance.

If in a random experiment, the event corresponding to a number a occurs, then the corresponding random variable X is said to assume the value a and the probability of the event is denoted by $P(X = a)$. Similarly the probability of the event X assuming any value in the interval $a < X < b$ is denoted by $P(a < X < b)$. The probability of the event $X \le c$ is written as $P(X \le c)$.

If a random variable takes a finite set of values, it is called a *discrete variate*. On the other hand, if it assumes an infinite number of uncountable values, it is called a *continuous variate.*

26.8 (1) DISCRETE PROBABILITY DISTRIBUTION

Suppose a discrete variate X is the outcome of some experiment. If the probability that X takes the values x_i, is p_i, then

$$P\,(X = x_i) = p_i \text{ or } p(x_i) \text{ for } i = 1, 2,...$$

where (*i*) $p(x_i) \ge 0$ for all values of i, (*ii*) $\Sigma p(x_i) = 1$

The set of values x_i with their probabilities p_i constitute a **discrete probability distribution** *of the discrete variate X.*

For example, the discrete probability distribution for X, the sum of the numbers which turn on tossing a pair of dice is given by the following table :

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

[$\because$ There are $6 \times 6 = 36$ equally likely outcomes and therefore, each has the probability 1/36. We have $X = 2$ for one outcome, *i.e.* (1, 1) ; $X = 3$ for two outcomes (1, 2) and (2, 1); $X = 4$ for three outcomes (1, 3), (2, 2) and (3, 1) and so on.]

(2) Distribution function. *The distribution function $F(x)$ of the discrete variate X is defined by*

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$$ where x is any integer. The graph of $F(x)$ will be stair step form (Fig. 26.2). The distribution function is also sometimes called *cumulative distribution function.*

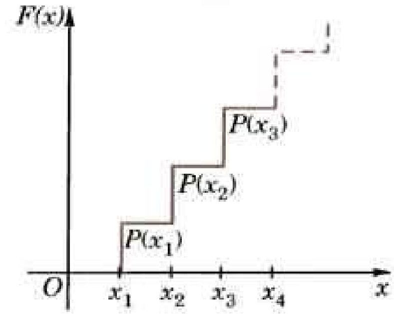


Fig. 26.2

Example 26.28. *A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and variance of the number of successes.* (V.T.U., 2011 S ; Rohtak, 2004)

Solution. Probability of a success $= \frac{2}{6} = \frac{1}{3}$, Probability of failures $= 1 - \frac{1}{3} = \frac{2}{3}$.

$\therefore$ prob. of no success = Prob. of all 3 failures $= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

Probability of one successes and 2 failures $= 3c_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

Probability of two successes and one failure $= 3c_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

Probability of three successes $= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

Now				
	$x_i = 0$	1	2	3
	$p_i = 8/27$	4/9	2/9	1/27

$\therefore$ mean $\qquad \mu = \Sigma p_i x_i = 0 + \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1.$

Also $\qquad \Sigma p_i x_i^2 = 0 + \frac{4}{9} + \frac{8}{9} + \frac{9}{27} = \frac{5}{3}$

$\therefore$ variance $\qquad \sigma^2 = \Sigma p_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}.$

Example 26.29. *The probability density function of a variate X is*

X :	0	1	2	3	4	5	6
$p(X)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(i) Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$. (V.T.U., 2010)
(ii) What will be the minimum value of k so that $P(X \leq 2) > 3$.

Solution. (*i*) If X is a random variable, then

$$\sum_{i=0}^{6} p(x_i) = 1 \text{ i.e., } k + 3k + 5k + 7k + 9k + 11k + 13k = 1 \text{ or } k = 1/49.$$

$\therefore \qquad P(X < 4) = k + 3k + 5k + 7k = 16k = 16/49.$

$P(X \geq 5) = 11k + 13k = 24k = 24/49.$

$P(3 < X \leq 6) = 9k + 11k + 13k = 33k = 33/49.$

(*ii*) $\qquad P(X \leq 2) = k + 3k + 5k = 9k > 0.3$ or $k > 1/30$

Thus minimum value of $k = 1/30$.

Example 26.30. *A random variable X has the following probability function :*

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find the value of the k *(ii) Evaluate $P(X < 6)$, $P(X \geq 6)$*
(iii) $P(0 < X < 5)$. (W.B.T.U., 2005 ; J.N.T.U., 2003)

Solution. (*i*) If X is a random variable, then

$$\sum_{i=0}^{7} p(x_i) = 1, \text{ i.e., } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

i.e., $\qquad 7k^2 + 9k - 1 = 0$ *i.e.* $(10 - k)(k + 1) = 0$ *i.e.,* $k = \frac{1}{10}$

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= 0 + k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7) = 2k^2 + 7k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

(*ii*) $P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= k + 2k + 2k + 3k = 8k = \frac{8}{10} = \frac{4}{5}.$$

26.9 (1) CONTINUOUS PROBABILITY DISTRIBUTION

When a variate X takes every value in an interval, it gives rise to *continuous distribution* of X. The distributions defined by the variates like heights or weights are continuous distributions.

A major conceptual difference, however, exists between discrete and continuous probabilities. When thinking in discrete terms, the probability associated with an event is meaningful. With continuous events, however, where the number of events is infinitely large, the probability that a specific event will occur is practically zero. For this reason, continuous probability statements must be worded somewhat differently from discrete ones. Instead of finding the probability that x equals some value, we find the probability of x falling in a small interval.

Thus the probability distribution of a continuous variate x is defined by a function $f(x)$ such that the probability of the variate x falling in the small interval $x - \frac{1}{2}dx$ to $x + \frac{1}{2}dx$ is $f(x)\,dx$. Symbolically it can be expressed as $P\left(x - \frac{1}{2}dx \leq x \leq x + \frac{1}{2}dx\right) = f(x)\,dx$. Then $f(x)$ is called the *probability density function* and the continuous curve $y = f(x)$ is called the *probability curve*.

The range of the variable may be finite or infinite. But even when the range is finite, it is convenient to consider it as infinite by supposing the density function to be zero outside the given range. Thus if $f(x) = \phi(x)$ be the density function denoted for the variate x in the interval (a, b), then it can be written as

$$\begin{aligned} f(x) &= 0, & x &< a \\ &= \phi(x), & a &\leq x \leq b \\ &= 0, & x &> b. \end{aligned}$$

The density function $f(x)$ is always positive and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (*i.e.*, the total area under the probability curve and the x-axis is unity which corresponds to the requirements that the total probability of happening of an event is unity).

(2) Distribution function

If $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx,$

then $F(x)$ is defined as the **cumulative distribution function** *or simply the* **distribution function** *of the continuous variate X.* It is the probability that the value of the variate X will be $\leq x$. The graph of $F(x)$ in this case is as shown in Fig. 26.3(*b*).

The distribution function $F(x)$ has the following properties :

(*i*) $F'(x) = f(x) \geq 0$, so that $F(x)$ is a non-decreasing function.

(*i*) $F(-\infty) = 0$; (*iii*) $F(\infty) = 1$

(*iv*) $P(a \leq x \leq b) = \int_a^b f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx = F(b) - F(a).$

Example 26.31. (*i*) *Is the function defined as follows a density function?*

$$\begin{aligned} f(x) &= e^{-x}, & x &\geq 0 \\ &= 0, & x &< 0, \end{aligned}$$

(*ii*) *If so, determine the probability that the variate having this density will fall in the interval (1, 2) ?*

(*iii*) *Also find the cumulative probability function F(2) ?*

Solution. (*i*) $f(x)$ is clearly ≥ 0 for every x in (1, 2) and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_0^{\infty} e^{-x}\,dx = 1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

(*ii*) Required probability $= P(1 \le x \le 2) = \int_1^2 e^{-x}\,dx = e^{-1} - e^{-2} = 0.368 - 0.135 = 0.233.$

This probability is equal to the shaded area in Fig. 26.3 (*a*).

(*iii*) Cumulative probability function $F(2)$

$$\int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_0^2 e^{-x}\,dx = 1 - e^{-2} = 1 - 0.135 = 0.865$$

which is shown in Fig. 26.3 (*b*).

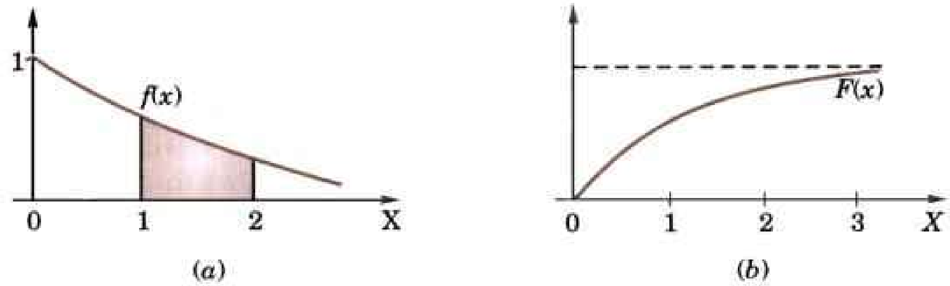


Fig. 26.3

26.10 (1) EXPECTATION

The mean value (μ) *of the probability distribution of a variate X is commonly known as its* **expectation** *and is denoted by E(X).* If $f(x)$ is the probability density function of the variate X, then

$$\sum_i x_i f(x_i) \qquad \text{(discrete distribution)}$$

or

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \qquad \text{(continuous distribution)}$$

In general, expectation of any function $\phi(x)$ is given by

$$E[\phi(x)] = \sum_i \phi(x_i) f(x_i) \qquad \text{(discrete distribution)}$$

or

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x)\,dx \qquad \text{(continuous distribution)}$$

(2) **Variance** *of a distribution is given by*

$$\sigma^2 = \sum_i (x_i - \mu)^2 f(x_i) \qquad \text{(discrete distribution)}$$

or

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx \qquad \text{(continuous distribution)}$$

where σ is the *standard deviation* of the distribution.

(3) *The* **rth moment** *about the mean* (denoted by μ_r) *is defined by*

$$\mu_r = \Sigma (x_i - \mu)^r f(x_i) \qquad \text{(discrete distribution)}$$

or

$$\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r f(x)\,dx \qquad \text{(continuous distribution)}$$

(4) **Mean deviation from the mean** *is given by*

$$\Sigma \,|\, x_i - \mu \,|\, f(x_i) \qquad \text{(discrete distribution)}$$

or by

$$\int_{-\infty}^{\infty} |\, x - \mu \,|\, f(x)\,dx \qquad \text{(continuous distribution)}$$

Example 26.32. *In a lottery, m tickets are drawn at a time out of n tickets numbered from 1 to n. Find the expected value of the sum of the numbers on the tickets drawn.*

Solution. Let $x_1, x_2, ..., x_n$ be the variables representing the numbers on the first, second, ..., nth ticket. The probability of drawing a ticket out of n tickets being in each case $1/n$, we have

$$E(x_i) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + ... + n \cdot \frac{1}{n} = \frac{1}{2}(n+1)$$

$\therefore$ expected value of the sum of the numbers on the tickets drawn

$$= E(x_1 + x_2 + ... + x_m) = E(x_1) + E(x_2) + ... + E(x_m)$$

$$= mE(x_i) = \frac{1}{2} m (n+1).$$

Example 26.33. *X is a continuous random variable with probability density function given by*

$$f(x) = kx \ (0 \le x < 2)$$
$$= 2k \ (2 \le x < 4)$$
$$= -kx + 6k \ (4 \le x < 6)$$

Find k and mean value of X. *(J.N.T.U., 2003)*

Solution. Since the total probability is unity

$\therefore$ $$\int_0^6 f(x)\, dx = 1$$

i.e., $$\int_0^2 kx\, dx + \int_2^4 2k dx + \int_4^6 (-kx + 6k)\, dx = 1$$

or $$k \left| x^2/2 \right|_0^2 + 2k \left| x \right|_2^4 + \left(-kx^2/2 + 6kx \right)_4^6 = 1$$

or $$2k + 4k + (-10k + 12k) = 1 \text{ i.e., } k = 1/8.$$

$$\text{Mean of } X = \int_0^6 x\, f(x)\, dx$$

$$= \int_0^2 kx^2\, dx + \int_2^4 2kx\, dx + \int_4^6 x(-kx + 6k) dx$$

$$= k \left| x^3/3 \right|_0^2 + 2k \left| x^2/2 \right|_2^4 + \left(-k \left| x^3/3 \right|_4^6 + 6k \left| x^2/2 \right|_4^6 \right)$$

$$= k\,(8/3) + k\,(12) - k\,(152/3) + 3k\,(20) = \frac{1}{8}(24) = 3.$$

Example 26.34. *A variate X has the probability distribution*

x	:	-3	6	9
$P(X = x)$	:	$1/6$	$1/2$	$1/3$

Find E (X) and E (X²). Hence evaluate E (2X + 1)².

Solution. $$E(X) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = 11/2.$$

$$E(X)^2 = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = 93/2$$

$\therefore$ $$E(2X+1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1$$
$$= 4\,(93/2) + 4\,(11/2) + 1 = 209.$$

Example 26.35. *The frequency distribution of a measurable characteristic varying between 0 and 2 is as under*

$$f(x) = x^3, \quad 0 \le x \le 1$$
$$= (2-x)^3, \quad 1 \le x \le 2.$$

Calculate the standard deviation and also the mean deviation about the mean.

Solution. Total frequency $N = \int_0^1 x^3\,dx + \int_1^2 (2-x)^3\,dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\therefore$ $$\mu_1' \text{ (about the origin)} = \frac{1}{N}\left[\int_0^1 x \cdot x^3\,dx + \int_1^2 x(2-x)^3\,dx\right]$$

$$= 2\left\{\left|\frac{x^5}{5}\right|_0^1 + \left|-x \cdot \frac{(2-x)^4}{4}\right|_1^2 - \left|\frac{(2-x)^5}{20}\right|_1^2\right\} = 2\left(\frac{1}{5} + \frac{1}{4} + \frac{1}{20}\right) = 1$$

$$\mu_2' \text{ (about the origin)} = \frac{1}{N}\left[\int_0^1 x^2 \cdot x^3 dx + \int_1^2 x^2 (2-x)^3\,dx\right]$$

$$= 2\left\{\left|\frac{x^6}{6}\right|_0^1 + \left|-x^2 \frac{(2-x)^4}{4}\right|_1^2 + \frac{1}{2}\int_1^2 x(2-x)^4\,dx\right\}$$

$$= 2\left\{\frac{1}{6} + \frac{1}{4} + \frac{1}{2}\left[\frac{1}{5} + \frac{1}{30}\right]\right\} = \frac{16}{15}$$

Hence $$\sigma^2 = \mu_2 = \mu_2' - (\mu_1')^2 = \frac{1}{15}$$

i.e., standard deviation $\sigma = \frac{1}{\sqrt{15}}$.

Mean deviation about the mean

$$= \frac{1}{N}\left\{\int_0^1 |x-1|\,x^3\,dx + \int_1^2 |x-1|(2-x)^3\,dx\right\}$$

$$= 2\left\{\int_0^1 (1-x)x^3\,dx + \int_1^2 (x-1)(2-x)^3\,dx\right\}$$

$$= 2\left\{\left(\frac{1}{4} - \frac{1}{5}\right) + \left(0 + \frac{1}{20}\right)\right\} = \frac{1}{5}.$$

26.11 MOMENT GENERATING FUNCTION

(1) *The moment generating function (m.g.f.) of the discrete probability distribution of the variate X about the value $x = a$ is defined as the expected value of $e^{t(x-a)}$* and is denoted by $M_a(t)$. Thus

$$M_a(t) = \Sigma p_i e^{t(x_i - a)} \qquad \text{...(1)}$$

which is a function of the parameter t only.

Expanding the exponential in (1), we get

$$M_a(t) = \Sigma p_i + t\Sigma p_i (x_i - a) + \frac{t^2}{2!}\Sigma p_i (x_i - a)^2 + \ldots + \frac{t^r}{r!}\Sigma x_i (x_i - a)^r + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu'_2 + \ldots + \frac{t^r}{r!}\mu'_r + \ldots \qquad \text{...(2)}$$

where μ'_r is the moment of order r about a. Thus $M_a(t)$ generates moments and that is why it is called the moment generating function. From (2), we find

$$\mu'_r = \text{coefficient of } t^r/r! \text{ in the expansion of } M_a(t).$$

Otherwise differentiating (2) r times with respect to t and then putting $t = 0$, we get

$$\mu'_r = \left[\frac{d^r}{dt^r} M_a(t)\right]_{t=0} \qquad \text{...(3)}$$

Thus the moment about any point $x = a$ can be found from (2) or more conveniently from the formula (3).

Rewriting (1) as

$$M_a(t) = e^{-at}\,\Sigma p_i e^{tx_i} \quad \text{or} \quad M_a(t) = e^{-at} M_0(t) \qquad \text{...(4)}$$

Thus the m.g.f. about the point $a = e^{-at}$ (m.g.f. about the origin).

Obs. *The m.g.f. of the sum of two independent variables is the product of their m.g.fs.* ...(5)

(2) *If $f(x)$ is the density function of a continuous variate X, then the moment generating function of this continuous probability distribution about $x = a$ is given by*

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x)\, dx.$$

Example 26.36. *Find the moment generating function of the exponential distribution*

$f(x) = \frac{1}{c} e^{-x/c}$, $0 \le x \le \infty$, $c > 0$. Hence find its mean and S.D. (Kurukshetra, 2009)

Solution. The moment generating function about the origin is

$$M_0(t) = \int_0^{\infty} e^{tx} \cdot \frac{1}{c} e^{-x/c}\, dx = \frac{1}{c}\int_0^{\infty} e^{(t-1/c)x}\, dx \qquad \left[\because\ |t| < \frac{1}{c}\right]$$

$$= \frac{1}{c}\frac{\left| e^{(t-1/c)x} \right|_0^{\infty}}{|(t-1/c)|} = (1-ct)^{-1} = 1 + ct + c^2t^2 + c^3t^3 + \ldots$$

$$\therefore \quad \mu'_1 = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = (c + 2c^2t + 3c^3t^2 + \ldots)_{t=0} = c$$

$$\mu'_2 = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = 2c^2, \text{ and } \mu_2 = \mu'_2 - (\mu'_1)^2 = 2c^2 - c^2 = c^2.$$

Hence the mean is c and S.D. is also c.

26.12 PROBABILITY GENERATING FUNCTION

The probability generating function (p.g.f.) $P_x(t)$ for a random variable x which takes integral values 0, 1, 2, 3, ... only, is defined by

$$P_x(t) = p_0 + p_1t + p_2t^2 + \ldots = \sum_{n=0}^{\infty} p_n t^n = E(t^x)$$

The coefficient of t^n in the expansion of $P(t)$ in powers of t gives $P(t)_{x=n}$.

$$\frac{\partial P}{\partial t} = \sum_{n=0}^{\infty} np_n t^{n-1} \quad \text{or} \quad \left(\frac{\partial P}{\partial t}\right)_{t=1} = \Sigma np_n = \mu_1'$$

$$\frac{\partial^2 P}{\partial t^2} = \sum_{n=0}^{\infty} n(n-1)\, p_n t^{n-2} \quad \text{or} \quad \left(\frac{\partial^2 P}{\partial t^2}\right)_{t=1} = \Sigma n(n-1)\, p_n = \mu_2' - \mu_1'$$

$$= \mu_2 + \mu_1'^2 - \mu_1' \text{ and so on}$$

Also $\quad \left(\frac{\partial^k P}{\partial t^n}\right)_{t=0} = n!\, p_n,\ k = 1, 2, \ldots n.$

For integral valued variates, we have

$$P_x(e^t) = E(e^{tx}) = m.g.f. \text{ for } x.$$

Obs. The p.g.f. of the sum of two independent random variables is the product of their p.g.f.'s.

Example 26.37. *If x be a random variable with probability generating function $P_x(t)$, find the probability generating function of*

(i) $x + 2$ (ii) $2x$.

Solution. We have $P_x(t) = \sum_{k=0}^{\infty} p_k\, t^k$

(*i*) Probability generating function of $x + 2 = \sum_{k=0}^{\infty} p_k\, t^{k+2} = t^2 \sum_{k=0}^{\infty} p_k\, t^k = t^2 P_x(t)$.

(*ii*) Probability generating function of $2x = \sum_{k=0}^{\infty} p_k\, t^{2k} = \sum_{k=0}^{\infty} p_k (t^2)^k = P(t^2)$.

PROBLEMS 26.4

1. A random variable x has the following probability function :

Values of x :	-2	-1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	k

Find the value of k and calculate mean and variance. *(S.V.T.U., 2007 ; V.T.U., 2004 ; Madras, 2003)*

2. Find the standard deviation for the following discrete distribution :

x :	8	12	16	20	24
$p(x)$:	1/8	1/6	3/8	1/4	1/12

3. Obtain the distribution function of the total number of heads occurring in three tosses of an unbiased coin.
4. Show that for any discrete distribution $\beta_2 \geq 1$.
5. From an urn containing 3 red and 2 white balls, a man is to draw 2 balls at random without replacement, being promised Rs. 20 for each red ball he draws and Rs. 10 for each white one. Find his expectation.
6. Four coins are tossed. What is the expectation of the number of heads ?
7. The diameter of an electric cable is assumed to be a continuous variate with p.d.f. $f(x) = 6x(1-x)$, $0 \leq x \leq 1$. Verify that the above is a p.d.f. Also find the mean and variance.
8. A random variable gives measurements X between 0 and 1 with a probability function

$$f(x) = 12x^3 - 21x^2 + 10x, \quad 0 \leq x \leq 1$$
$$= 0$$

(*i*) Find $P\left(X \leq \frac{1}{2}\right)$ and $P\left(X > \frac{1}{2}\right)$

(*ii*) Find a number k such that $P(X \leq k) = \frac{1}{2}$ *(J.N.T.U., 2003)*

9. The power reflected by an aircraft that is received by a radar can be described by an exponential random variable X.

The probability density of X is given by $f(x) = \begin{cases} \dfrac{1}{x_0} e^{-x/x_0}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

where x_0 is the average power received by the radar.

(*i*) What is the probability that the radar will receive power larger than the power received on the average ? (*ii*) What is the probability that the radar will receive power less than the power received on the average ? *(Mumbai, 2006)*

10. A function is defined as follows :

$$f(x) = 0, \quad x < 2$$
$$= \frac{1}{18}(2x + 3), \quad 2 \leq x \leq 4$$
$$= 0, \quad x > 4.$$

Show that it is a density function. Find the probability that a variate having this density will fall in the interval $2 \leq x \leq 3$?

11. A continuous distribution of a variable x in the range $(-3, 3)$ is defined as

$$f(x) = \frac{1}{16}(3 + x)^2, \quad -3 \leq x < -1$$
$$= \frac{1}{16}(2 - 6x^2), \quad -1 \leq x < 1$$
$$= \frac{1}{16}(3 - x)^2, \quad 1 \leq x \leq 3.$$

Verify that the area under the curve is unity. Show that the mean is zero. *(Kurukshetra, 2005)*

12. The frequency function of a continuous random variable is given by

$$f(x) = y_0 x(2 - x),\ 0 \le x \le 2.$$

Find the value of y_0, mean and variance of x. *(Kerala, 2005 ; J.N.T.U., 2003)*

13. The probability density $p(x)$ of a continuous random variable is given by

$$p(x) = y_0 e^{-|x|},\ -\infty < x < \infty.$$

Prove that $y_0 = 1/2$. Find the mean and variance of the distribution.

(S.V.T.U., 2008 ; Kurukshetra, 2007 ; V.T.U., 2004)

14. If $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$

represents the density of a random variable X, find $E(X)$ and Var (X).

15. A function is defined as under :

$$f(x) = 1/k,\ x_1 \le x \le x_2$$
$$= 0, \text{ elsewhere.}$$

Find the cumulative distribution of the variate x when k satisfies the requirements for $f(x)$ to be a density function.

26.13 REPEATED TRIALS

We know that the probability of getting a head or a tail on tossing a coin is $\frac{1}{2}$. If the coin is tossed thrice, the probability of getting one head and two tails can be combined as $H—T—T$, $T—H—T$, $T—T—H$. The probability of each one of these being $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, *i.e.*, $\left(\frac{1}{2}\right)^3$, their total probability shall be $3(1/2)^3$.

Similarly if a trial is repeated n times and if p is the probability of a success and q that of a failure, then the probability of r successes and $n - r$ failures is given by $p^r q^{n-r}$.

But these r successes and $n - r$ failures can occur in any of the ${}^nC_r$ ways in each of which the probability is same.

Thus the probability of r successes is ${}^nC_r\, p^r q^{n-r}$.

Cor. *The probabilities of at least r successes in n trials*

$= $ the sum of the probabilities of $r, r+1, \ldots, n$ successes

$$= {}^nC_r\, p^r q^{n-r} + {}^nC_{r+1}\, p^{r+1} q^{n-r-1} + \ldots + {}^nC_n\, p^n.$$

26.14 (1) BINOMIAL DISTRIBUTION*

It is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection, yes or no of a particular event is of interest.

If we perform a series of independent trials such that for each trial p is the probability of a success and q that of a failure, then the probability of r successes in a series of n trials is given by ${}^nC_r\, p^r q^{n-r}$, where r takes any integral value from 0 to n. The probabilities of 0, 1, 2, ... r, ..., n successes are, therefore, given by

$$q^n,\ {}^nC_1 pq^{n-1},\ {}^nC_2 p^2 q^{n-2},\ \ldots,\ {}^nC_r\, p^r q^{n-r},\ \ldots,\ p^n.$$

The probability of the number of successes so obtained is called *the* **binomial distribution** for the simple reason that the probabilities are the successive terms in the expansion of the binomial $(q + p)^n$.

$\therefore$ the sum of the probabilities

$$= q^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + p^n = (q+p)^n = 1.$$

(2) Constants of the binomial distribution. The moment generating function about the origin is

$$M_0(t) = E(e^{tx}) = \Sigma\, {}^nC_x\, p^x q^{n-x} e^{tx} \qquad \text{[By (1) § 26.11]}$$
$$= \Sigma\, {}^nC_x\, (pe^t)^x q^{n-x} = (q + pe^t)^n$$

---

* It was discovered by a Swiss mathematician *Jacob Bernoulli* and was published posthumously in 1713.

Differentiating with respect to t and putting $t = 0$ and using (3) § 26.11, we get the mean

$$\mu'_1 = np.$$

Since $M_a(t) = e^{-at} M_0(t)$, the m.g.f. of the binomial distribution about its mean $(m) = np$, is given by

$$M_m(t) = e^{-npt} (q + pe^t)^n = (qe^{-pt} + pe^{qt})^n$$

$$= \left(1 + pq \frac{t^2}{2!} + pq(q^2 - p^2) \frac{t^3}{3!} + pq(q^3 + p^3) \frac{t^4}{4!} + \cdots \right)^n$$

or $$1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \mu_4 \frac{t^4}{4!} + \cdots$$

$$= 1 + npq \frac{t^2}{2!} + npq(q - p) \frac{t^3}{3!} + npq\,[1 + 3(n-2)pq] \frac{t^4}{4!} + \ldots$$

Equating the coefficients of like powers of t on either side, we have

$$\mu_2 = npq, \mu_3 = npq(q - p), \mu_4 = npq\,[1 + 3(n - 2)pq].$$

Also $$\beta_1 = \frac{{\mu_3}^2}{{\mu_2}^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{{\mu_2}^2} = 3 + \frac{1-6pq}{npq}$$

Thus *mean* $= np$, *standard deviation* $= \sqrt{(npq)}$.

skewness $= (1 - 2p)/\sqrt{(npq)}$, *kurtosis* $= \beta_2$.

Obs. The skewness is positive for $p < \frac{1}{2}$ and negative for $p > \frac{1}{2}$. When $p = \frac{1}{2}$, the skewness is zero, *i.e.*, the probability curve of the binomial distribution will be symmetrical (bell-shaped).

As n the number of trials increase indefinitely, $\beta_1 \to 0$, and $\beta_2 \to 3$.

(3) Binomial frequency distribution. If n independent trials constitute one experiment and this experiment be repeated N times, then the frequency of r successes is $N\,{}^nC_r\,p^r\,q^{n-r}$. The possible number of successes together with these expected frequencies constitute the *binomial frequency distribution.*

(4) Applications of Binomial distribution. This distribution is applied to problems concerning : (*i*) Number of defectives in a sample from production line,

(*ii*) Estimation of reliability of systems,

(*iii*) Number of rounds fired from a gun hitting a target,

(*iii*) Radar detection.

Example 26.38. *The probability that a pen manufactured by a company will be defective is 1/10. If 12 such pens are manufactured, find the probability that*

(*a*) *exactly two will be defective.* (*b*) *at least two will be defective.*

(*c*) *none will be defective.* (*V.T.U., 2004 ; Burdwan, 2003*)

Solution. The probability of a defective pen is 1/10 = 0.1

$\therefore$ The probability of a non-defective pen is $1 - 0.1 = 0.9$

(*a*) *The probability that exactly two will be defective*

$$= {}^{12}C_2\,(0.1)^2\,(0.9)^{10} = 0.2301$$

(*b*) *The probability that at least two will be defective*

$$= 1 - (\text{prob. that either none or one is non-defective})$$

$$= 1 - [{}^{12}C_0(0.9)^{12} + {}^{12}C_1(0.1)\,(0.9)^{11}] = 0.3412$$

(*c*) *The probability that none will be defective*

$$= {}^{12}C_{12}(0.9)^{12} = 0.2833.$$

Example 26.39. *In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails.* (*J.N.T.U., 2003*)

Solution. $P(\text{head}) = \frac{1}{2}$ and $P\,(\text{tail}) = \frac{1}{2}$

By binomial distribution, probability of 8 heads and 4 tails in 12 trials is

$$P(X=8) = {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4 = \frac{12!}{8!\,4!} \cdot \frac{1}{2^{12}} = \frac{495}{4096}$$

$\therefore$ the expected number of such cases in 256 sets

$$= 256 \times P(X=8) = 256 \ \frac{495}{4096} = 30.9 = 31 \text{ (say).}$$

Example 26.40. *In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.* (*V.T.U., 2004*)

Solution. Mean number of defectives $= 2 = np = 20p$.

$\therefore$ The probability of a defective part is $p = 2/20 = 0.1$.

and the probability of a non-defective part = 0.9

$\therefore$ The probability of at least three defectives in a sample of 20.

= 1 – (prob. that either none, or one, or two are non-defective parts)

$= 1 - [{}^{20}C_0(0.9)^{20} + {}^{20}C_1(0.1)(0.9)^{19} + {}^{20}C_2(0.1)^2(0.9)^{18}]$

$= 1 - (0.9)^{18} \times 4.51 = 0.323.$

Thus *the number of samples having at least three defective parts* out of 1000 samples

$= 1000 \times 0.323 = 323.$

Example 26.41. *The following data are the number of seeds germinating out of 10 on damp filter paper for 80 sets of seeds. Fit a binomial distribution to these data :*

x :	0	1	2	3	4	5	6	7	8	9	10
f :	6	20	28	12	8	6	0	0	0	0	0

Solution. Here $n = 10$ and $N = \Sigma f_i = 80$

$$\therefore \qquad \text{mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{20+56+36+32+30}{80} = \frac{174}{80} = 2.175$$

Now the mean of a binomial distribution $= np$

i.e., $np = 10p = 2.175 \quad \therefore \quad p = 0.2175, q = 1 - p = 0.7825$

Hence the binomial distribution to be fitted is

$$N(q+p)^n = 80\,(0.7825 + 0.2175)^{10}$$
$$= 80.{}^{10}C_0\,(0.7825)^{10} + 80.{}^{10}C_1\,(0.7825)^9\,(0.2175)^1 + {}^{10}C_2\,(0.7825)^8\,(0.2175)^2 + \ldots + {}^{80}C_9\,(0.7825)^1\,(0.2175)^9 + {}^{80}C_{10}\,(0.2175)^{10}$$
$$= 6.885 + 19.13 + 23.94 + \ldots + 0.0007 + 0.00002$$

$\therefore$ the successive terms in the expansion give the expected or theoretical frequencies which are

x :	0	1	2	3	4	5	6	7	8	9	10
f :	6.9	19.1	24.0	17.8	8.6	2.9	0.7	0.1	0	0	0

PROBLEMS 26.5

1. Determine the binomial distribution for which mean = 2 (variance) and mean + variance = 3. Also find $P(X \le 3)$. (*Kerala, 2005*)
2. An ordinary six-faced die is thrown four times. What are the probabilities of obtaining 4, 3, 2, 1 and 0 faces ?
3. If the chance that one of the ten telephone lines is busy at an instant is 0.2.
   (*a*) What is the chance that 5 of the lines are busy ?
   (*b*) What is the most probable number of busy lines and what is the probability of this number ?
   (*c*) What is the probability that all the lines are busy ? (*V.T.U., 2002 S*)
4. If the probability that a new-born child is a male is 0.6, find the probability that in a family of 5 children there are exactly 3 boys. (*Kuruhshetra, 2005*)

5. If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely. *(P.T.U., 2005)*
6. The probability that a bomb dropped from a plane will strike the target is 1/5. If six bombs are dropped, find the probability that (*i*) exactly two will strike the target, (*ii*) at least two will strike the target.
7. A sortie of 20 aeroplanes is sent on an operational flight. The chances that an aeroplane fails to return is 5%. Find the probability that (*i*) one plane does not return (*ii*) at the most 5 planes do not return, and (*iii*) what is the most probable number of returns ? *(Hissar, 2007)*
8. The probability that an entering student will graduate is 0.4. Determine the probability that out of 5 students (*a*) none (*b*) one and (*c*) at least one will graduate.
9. Out of 800 families with 5 children each, how many would you expect to have (*a*) 3 boys, (*b*) 5 girls, (*c*) either 2 or 3 boys ? Assume equal probabilities for boys and girls. *(V.T.U., 2004)*
10. If 10 per cent of the rivets produced by a machine are defective, find the probability that out of 5 rivets chosen at random (*i*) none will be defective, (*ii*) one will be defective, and (*iii*) at least two will be defective.
11. In a bombing action there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give a 99% chance or better of completely destroying the target. *(V.T.U., 2003 S)*
12. A product is 0.5% defective and is packed in cartons of 100. What percentage contains not more than 3 defectives ?
13. If in a lot of 500 solenoids 25 are defective, find the probability of 0, 1, 2, 3 defective solenoids in a random sample of 20 solenoids.
14. 500 articles were selected at random out of a batch containing 10,000 articles, and 30 were found to be defective. How many defectives articles would you reasonably expect to have in the whole batch ? *(J.N.T.U., 2003)*
15. Fit a binomial distribution for the following data and compare the theoretical frequencies with the actual ones :

x :	0	1	2	3	4	5
f :	2	14	20	34	22	8

(Bhopal, 2006)

16. Fit a binomial distribution to the following frequency distribution :

x :	0	1	2	3	4	5	6
f :	13	25	52	58	32	16	4

(Kurukshetra, 2009 ; S.V.T.U., 2007)

26.15 (1) POISSON DISTRIBUTION*

It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence. The number of persons born blind per year in a large city and the number of deaths by horse kick in an army corps are some of the phenomena, in which this law is followed.

This distribution can be derived as a limiting case of the binomial distribution by making n very large and p very small, keeping np fixed (= m, say).

The probability of r successes in a binomial-distribution is

$$P(r) = {}^nC_r\, p^r q^{n-r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\, p^r q^{n-r}$$

$$= \frac{np(np-p)(np-2p)\cdots(np-\overline{r-1}p)}{r!}\,(1-p)^{n-r}$$

As $n \to \infty$, $p \to 0$ $(np = m)$, we have

$$P(r) = \frac{m^r}{r!} \underset{n\to\infty}{\text{Lt}} \frac{(1-m/n)^n}{(1-m/n)^r} = \frac{m^r}{r!} e^{-m}$$

so that the probabilities of 0, 1, 2..., r,... successes in a Poisson distribution are given by

$$e^{-m},\ me^{-m},\ \frac{m^2e^{-m}}{2!},\ldots,\frac{m^re^{-m}}{r!},\cdots$$

The sum of these probabilities is unity as it should be.

* It was discovered by a French mathematician S.D. Poisson in 1837.

(2) Constants of the Poisson distribution. These constants can easily be derived from the corresponding constants of the binomial distribution simply by making $n \to \infty$, $p \to 0$, $(q \to 1)$ and noting that $np = m$

$$Mean = \text{Lt}\,(np) = m$$
$$\mu_2 = \text{Lt}\,(npq) = m\ \text{Lt}\,(q) = m$$

$\therefore$ *Standard deviation* $= \sqrt{m}$

Also $\mu_3 = m$, $\mu_4 = m + 3m^2$

$\therefore$ *Skewness* $(= \sqrt{\beta_1}\,) = 1/m$, *Kurtosis* $(= \beta_2) = 3 + 1/m$.

Since μ_3 is positive, Poisson distribution is positively skewed and since $\beta_2 > 3$, it is *Leptokurtic.*

(3) Applications of Poisson distribution. This distribution is applied to problems concerning : (*i*) Arrival pattern of 'defective vehicles in a workshop', 'patients in a hospital' or 'telephone calls'.

(*ii*) Demand pattern for certain spare parts.

(*iii*) Number of fragments from a shell hitting a target.

(*iv*) Spatial distribution of bomb hits.

Example 26.42. *If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2,000 individuals more than two will get a bad reaction.* (V.T.U., 2008 ; Kottayam, 2005)

Solution. It follows a Poisson distribution as the probability of occurrence is very small.

Mean $m = np = 2000(0.001) = 2$

Probability that more than 2 will get a bad reaction

= 1 – [prob. that no one gets a bad reaction + prob. that one gets a bad reaction + prob. that two get bad reaction]

$$= 1 - \left[e^{-m} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right] = 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right] \qquad [\because \quad m = 2]$$

$$= 1 - \frac{5}{e^2} = 0.32. \qquad [\because \quad e = 2.718\,]$$

Example 26.43. *In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10, use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets.* (Kurukshetra, 2009 S; Madras, 2006 ; V.T.U., 2004)

Solution. We know that $m = np = 10 \times 0.002 = 0.02$

$$e^{-0.02} = 1 - 0.02 + \frac{(0.02)^2}{2!} - \ldots = 0.9802 \text{ approximately}$$

Probability of no defective blade is $e^{-m} = e^{-0.02} = 0.9802$

$\therefore$ no. of packets containing no defective blade is

$$10{,}000 \times 0.9802 = 9802$$

Similarly the number of packets containing one defective blade $= 10{,}000 \times me^{-m}$

$$= 10{,}000 \times (0.02) \times 0.9802 = 196$$

Finally the number of packets containing two defective blades

$$= 10{,}000 \times \frac{m^2 e^{-m}}{2!} = 10{,}000 \times \frac{(0.02)^2}{2!} \times 0.9802 = 2 \text{ approximately.}$$

Example 26.44. *Fit a Poisson distribution to the set of observations :*

x :	0	1	2	3	4
f :	122	60	15	2	1

(Bhopal, 2007 S ; V.T.U., 2004 ; U.P.T.U., 2003)

Solution. Mean $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{60 + 36 + 6 + 4}{200} = 0.5.$

$\therefore$ mean of Poisson distribution *i.e.*, $m = 0.5$.

Hence the theoretical frequency for r successes is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{200e^{-0.5}(.5)^r}{r!} \text{ where } r = 0, 1, 2, 3, 4$$

$\therefore$ the theoretical frequencies are

x :	0	1	2	3	4
f :	121	61	15	2	0

$(\because \ e^{-.5} = 0.61)$

PROBLEMS 26.6

1. If a random variable has a Poisson distribution such that $P(1) = P(2)$, find
(*i*) mean of the distribution. (*ii*) $P(4)$. (*V.T.U., 2003*)
2. X is a Poisson variable and it is found that the probability that $X = 2$ is two-thirds of the probability that $X = 1$. Find the probability that $X = 0$ and the probability that $X = 3$. What is the probability that X exceeds 3 ?
3. For Poisson distribution, prove that $m\ \mu_2\ \gamma_1\ \gamma_2 = 1$, where symbols have their usual meanings. (*S.V.T.U., 2008*)
4. A certain screw making machine produces on average of 2 defective screws out of 100, and packs them in boxes of 500. Find the probability that a box contains 15 defective screws. (*Kurukshetra, 2006*)
5. A manufacturer knows that the condensers he makes contain on the average 1 % defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers ?
6. A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (*i*) on which there is no demand, (*ii*) on which demand is refused. ($e^{-1.5} = 0.2231$). (*Bhopal, 2008 S ; J.N.T.U., 2003*)
7. The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is probability that in a group of 7, five or more will suffer from it ?
8. The frequency of accidents per shift in a factory is as shown in the following table :

Accidents per shift :	0	1	2	3	4
Frequency :	180	92	24	3	1

Calculate the mean number of accidents per shift and the corresponding Poisson distribution and compare with actual observations.

9. A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimetre equal to 3. Ten 1 c.c., test-tubes are filled with the liquid. Assuming that Poisson distribution is applicable, calculate the probability that all the test-tubes will show growth *i.e.*, contain atleast 1 bacterium each.
10. Find the expectation of the function $\phi(x) = xe^{-x}$ in a Poisson distribution. (*V.T.U., 2003*)
[**Hint** : If m be the mean of the Poisson distribution, then expectation of

$$\phi(x) = \sum_{x=0}^{\infty} \frac{\phi(x) \cdot m^x e^{-m}}{x!} = m \exp. m\,(e^{-1} - m - 1)$$

11. Fit a Poisson distribution to the following :

x :	0	1	2	3	4
f :	46	38	22	9	1

(*Kurukshetra, 2009 ; Bhopal, 2008 ; V.T.U., 2003 S*)

12. Fit a Poisson distribution to the following data given the number of yeast cells per square for 400 squares :

No. of cells per sq. :	0	1	2	3	4	5	6	7	8	9	10
No. of squares :	103	143	98	42	8	4	2	0	0	0	0

(*S.V.T.U., 2007*)

26.16 (1) NORMAL DISTRIBUTION*

Now we consider a continuous distribution of fundamental importance, namely the normal distribution. *Any quantity whose variation depends on random causes is distributed according to the normal law.* Its importance lies in the fact that a large number of distributions approximate to the normal distribution.

* In 1924, *Karl Pearson* found this distribution which *Abraham De Moivre* had discovered as early as 1733. See footnote p. 843 and 647.

Let us define a variate $z = \dfrac{x - np}{\sqrt{(npq)}}$ ...(1)

where x is a binomial variate with mean np and S.D. $\sqrt{(npq)}$ so that z is a variate with mean zero and variance unity. In the limit as n tends to infinity, the distribution of z becomes a continuous distribution extending from $-\infty$ to ∞.

It can be shown that *the limiting form of the binomial distribution (1) for large values of n when neither p nor q is very small, is the normal distribution.* The normal curve is of the form

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \qquad ...(2)$$

where μ and σ are the mean and standard deviation respectively.

(2) Properties of the normal distribution

I. The normal curve (2) is bell-shaped and is symmetrical about its mean. It is unimodal with ordinates decreasing rapidly on both sides of the mean (Fig. 26.3). The maximum ordinate is $1/\sigma\sqrt{(2\pi)}$, found by putting $x = \mu$ in (2).

As it is symmetrical, its mean, median and mode are the same. Its points of inflexion (found by putting $d^2y/dx^2 = 0$ and verifying that at these points $d^3y/dx^3 \neq 0$) are given by $x = \mu \pm \sigma$, *i.e.*, these points are equidistant from the mean on either side.

II. Mean deviation from the mean μ

$$= \int_{-\infty}^{\infty} |x-\mu| \cdot \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} dx \qquad [\text{Put } z = (x-\mu)/\sigma]$$

$$= \frac{\sigma}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} |z| e^{-z^2/2} dz$$

$$= \frac{\sigma}{\sqrt{(2\pi)}} \left[\int_{-\infty}^{0} - ze^{-z^2/2} dz + \int_{0}^{\infty} ze^{-z^2/2} dz \right] = \frac{2\sigma}{\sqrt{(2\pi)}} \int_{0}^{\infty} ze^{-z^2/2} dz$$

$$= \frac{2\sigma}{\sqrt{(2\pi)}} \left| -e^{-z^2/2} \right|_{0}^{\infty} = -\sqrt{\left(\frac{2}{\pi}\right)} \sigma(0-1) = 0.7979\,\sigma \approx (4/5)\,\sigma$$

III. Moments about the mean

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sigma^{2n+1}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} z^{2n+1} e^{-z^2/2} dz \text{ where } z = (x-\mu)/\sigma$$

$= 0$, since the integral is an odd function.

Thus all odd order moments about the mean vanish.

$$\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} z^{2n-1} e^{-z^2/2} \,.\, zdz \qquad [\text{Integrate by parts}]$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} \left[\left| -z^{2n-1} e^{-z^2/2} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2n-1) z^{2n-2} e^{-z^2/2} dz \right]$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} (0-0) + (2n-1)\sigma^2 \mu_{2n-2}$$

Repeated application of this reduction formula, gives

$$\mu_{2n} = (2n-1)(2n-3) \ldots 3 \,.\, 1\sigma^{2n}$$

In particular, $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$.

Hence $\qquad \beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = 0$ and $\beta_2 = \dfrac{\mu_4}{\mu_2^{\,2}} = 3$

i.e., the coefficient of skewness is zero (*i.e.* the curve is symmetrical) and the Kurtosis is 3. This is the basis for the choice of the value 3 in the definitions of platykurtic and leptokurtic (page 844).

IV. The probability of x *lying between* x_1 *and* x_2 is given by the area under the normal curve from x_1 to x_2, *i.e.*, $P(x_1 \le x \le x_2)$

$$= \frac{1}{\sigma\sqrt{(2\pi)}} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{z_1}^{z_2} e^{-z^2/2}\, dz \text{ where } z = (x-\mu)/\sigma,\ dz = dx/\sigma \text{ and } z_1 = (x_1-\mu)/\sigma,\ z_2 = (x_2-\mu)/\sigma.$$

$$= \frac{1}{\sqrt{(2\pi)}} \left\{ \int_0^{z_2} e^{-z^2/2}\, dz - \int_0^{z_1} e^{-z^2/2}\, dz \right\} = P_2(z) - P_1(z)$$

The values of each of the above integrals can be found from the table III–Appendix 2, which gives the values of

$$P(z) = \frac{1}{\sqrt{(2\pi)}} \int_0^{z} e^{-z^2/2}\, dz$$

for various values of z. This integral is called the *probability integral* or the *error function* due to its use in the theory of sampling and the theory of errors.

Using this table, we see that the area under the normal curve from $z = 0$ to $z = 1$, *i.e.* from $x = \mu$ to $\mu + \sigma$ is 0.3413.

$\therefore$ (*i*) The area under the normal curve between the ordinates $x = \mu - \sigma$ and $x = \mu + \sigma$ is 0.6826, ~ *68% nearly. Thus approximately 2/3 of the values lie within these limits.*

(*ii*) The area under the normal curve between $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$ is 0.9544 ~ 95.5%, which implies that *about* $4\frac{1}{2}$ *% of the values lie outside these limits.*

(*ii*) *99.73% of the values lie between* $x = \mu - 3\sigma$ *and* $x = \mu + 3\sigma$ *i.e., only a quarter % of the whole lies outside these limits.*

(*iv*) *95% of the values lie between* $x = \mu - 1.96\sigma$ *and* $x = \mu + 1.96\,\sigma$ *i.e., only 5% of the values lie outside these limits.*

(*v*) *99% of the values lie between* $x = \mu - 2.58\sigma$ *and* $x = \mu + 2.58\sigma$ *i.e., only 1% of the values lie outside these limits.*

(*vi*) 99.9% of the values lie between $x = \mu - 3.29\sigma$ and $x = \mu + 3.29\sigma$.

In other words, a value that deviates more than σ from μ occurs about once in 3 trials. A value that deviates more than 2σ or 3σ from μ occurs about once in 20 or 400 trials. Almost all values lie within 3σ of the mean.

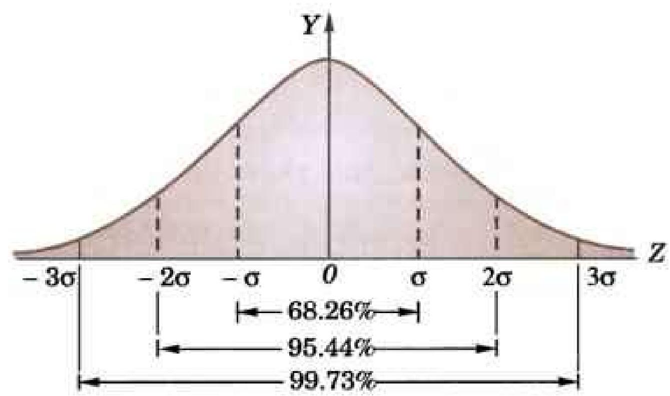


Fig. 26.4

The shape of the standardised normal curve is

$$y = \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} \text{ where } z = (x-\mu)/\sigma \qquad \text{...(3)}$$

and the respective areas are shown in Fig. 26.4. 'z' is called a *normal variate*.

(3) Normal frequency distribution. We can fit a normal curve to any distribution. If N be the total frequency, μ the mean and σ the standard deviation of the given distribution then the curve

$$y = \frac{N}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{...(4)}$$

will fit the given distribution as best as the data will permit. The frequency of the variate between x_1 and x_2 as given by the fitted curve, will be the area under (1) from x_1 to x_2.

(4) Applications of normal distribution. This distribution is applied to problems concerning :

(*i*) Calculation of errors made by chance in experimental measurements.

(*ii*) Computation of hit probability of a shot.

(*iii*) Statistical inference in almost every branch of science.

26.17 PROBABLE ERROR

Any lot of articles manufactured to certain specifications is subject to small errors. In fact, measurement of any physical quantity shows slight error. In general, these errors of manufacture or experiment are of random nature and therefore, follow a normal distribution. While quoting a specification of an experimental result, we usually mention the *probable error* (λ). *It is such that the probability of an error falling within the limits* $\mu - \lambda$ *and* $\mu + \lambda$ *is exactly equal to the chance of an error falling outside these limits, i.e.* the chance of an error lying within $\mu - \lambda$ and $\mu + \lambda$ is $\frac{1}{2}$.

$$\therefore \qquad \frac{1}{\sigma\sqrt{(2\pi)}}\int_{\mu-\lambda}^{\mu+\lambda} e^{-(x-\mu)^2/2\sigma^2}\,dx = \frac{1}{2}$$

or
$$\frac{1}{\sqrt{(2\pi)}}\int_{0}^{\lambda/\sigma} e^{-z^2/2}dz = \frac{1}{4} \qquad \left[z = \frac{x-\mu}{\sigma}\right]$$

The table V, (Appendix 2) gives $\lambda/\sigma = 0.6745$

Hence the probable error $\lambda = 0.6745\sigma \sim \frac{2}{3}\sigma$.

Obs. Quartile deviation $= \frac{1}{2}(Q_3 - Q_1) \sim \frac{2}{3}\sigma$; Mean deviation $= \frac{4}{5}\sigma$ [p. 839]

$\therefore$ *Q.D.* : *M.D.* : *S.D.* = 10 : 12 : 15. *(Madras, 2003)*

Example 26.45. *X is a normal variate with mean 30 and S.D. 5, find the probabilities that (i)* $26 \le X \le 40$, *(ii)* $X \ge 45$ *and (iii)* $|X - 30| > 5$. *(J.N.T.U., 2005)*

Solution. We have $\mu = 30$ and $\sigma = 5$

$$\therefore \qquad z = \frac{X-\mu}{\sigma} = \frac{X-30}{5}$$

(*i*) When $X = 26, z = -0.8$; when $X = 40, z = 2$

$$\begin{aligned} \therefore \quad P(26 \le X \le 40) &= P(-0.8 \le z \le 2) \\ &= P(-0.8 \le z \le 0) + P(0 \le z \le 2) \qquad \text{[Using Table } III\text{]} \\ &= P(0 \le z \le 0.8) + 0.4772 \\ &= 0.2881 + 0.4772 = 0.7653 \end{aligned}$$

(*ii*) When $X = 45, z = 3$

$$\begin{aligned} \therefore \quad P(X \ge 45) &= P(z \ge 3) = 0.5 - P(0 \le z \le 3) \\ &= 0.5 - 0.4986 = 0.0014 \end{aligned}$$

$$\begin{aligned} (iii) \quad P[|X - 30| \le 5] &= P[25 \le X \le 35] \\ &= P(-1 \le z \le 1) = 2P(0 \le z \le 1) \\ &= 2 \times 0.3413 = 0.6826 \\ \therefore \quad P[|X - 30| > 5] &= 1 - P[|X - 30| \le 5] \\ &= 1 - 0.6826 = 0.3174. \end{aligned}$$

Example 26.46. *A certain number of articles manufactured in one batch were classified into three categories according to a particular characteristic, being less than 50, between 50 and 60 and greater than 60. If this characteristic is known to be normally distributed, determine the mean and standard deviation for this batch if 60%, 35% and 5% were found in these categories.*

Solution. Let μ be the mean (at $z = 0$) and σ the standard deviation of the normal curve (Fig. 26.5).

Now 60% of the articles have the characteristic below 50, 35% between 50 and 60 and only 5% greater than 60.

Let the area to the left of the ordinate PQ be 60% and that between the ordinates PQ and ST be 35% so that the areas to the left of PQ ($z = z_1$) and ST ($z = z_2$) are 0.6 and 0.95 respectively, *i.e.*, the area $OPQR = 0.6 - 0.5 = 0.1$ and the area $OSTR = 0.45$.

$\therefore$ area corresponding to $z_1 \left(= \dfrac{50-\mu}{\sigma}\right) = 0.1$

and that corresponding to $z_2 \left(= \dfrac{60-\mu}{\sigma}\right) = 0.45$

From the table III, we have

$(50 - \mu)/\sigma = 0.2533$ and $(60 - \mu)/\sigma = 1.645$

whence $\sigma = 7.543$ and $\mu = 48.092$.

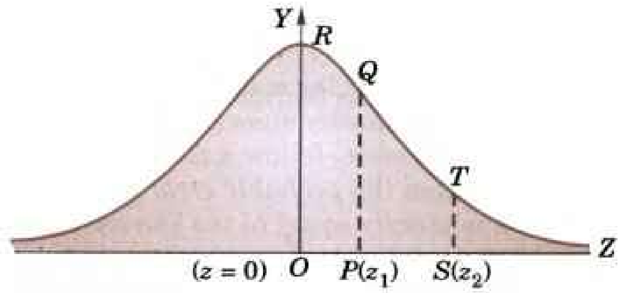


Fig. 26.5

Example 26.47. *In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution.* (V.T.U., 2009 ; S.V.T.U., 2008 ; Kurukshetra, 2007 S)

Solution. Let $\bar{x}$ be the mean and σ the S.D. 31% of the items are under 45 means area to the left of the ordinate $x = 45$. (Fig. 26.6)

When $x = 45$, let $z = z_1$ so that $z_1 = \dfrac{45 - \bar{x}}{\sigma}$ ...(*i*)

$$\therefore \quad \int_{-\infty}^{z_1} \phi(z)\, dz = 0.31 \quad \text{or} \quad \int_{-\infty}^{0} \phi(z)\, dz - \int_{z_1}^{0} \phi(z)\, dz = 0.31$$

$$\text{Hence} \quad \int_{z_1}^{0} \phi(z)\, dz = \int_{-\infty}^{0} \phi(z)\, dz - 0.31 = 0.5 - 0.31 = 0.19$$

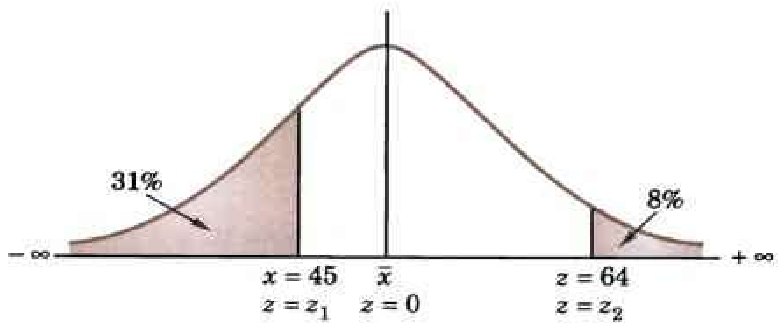


Fig. 26.6

From table III, $z_1 = -0.5$ ...(*ii*)

When $x = 64$, let $z = z_2$ so that $z_2 = (64 - \bar{x})/\sigma$ ...(*iii*)

$$\therefore \quad \int_{z_2}^{\infty} \phi(z)\, dz = 0.08 \quad \text{or} \quad \int_{0}^{\infty} \phi(z)\, dz - \int_{0}^{z_2} \phi(z)\, dz = 0.08$$

$$\text{Hence} \quad \int_{0}^{z_2} \phi(z)\, dz = \int_{0}^{\infty} \phi(z)\, dz - 0.08 = 0.5 - 0.08 = 0.42$$

From table III, $z_2 = 1.4$ ...(*iv*)

From (*i*) and (*ii*), $45 - \bar{x} = -0.5\sigma$

From (*iii*) and (*iv*), $64 - \bar{x} = 1.4\sigma$

Solving these equations, we get $\bar{x} = 50$ and $\sigma = 10$.

Example 26.48. *In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for*

(*a*) *more than 2150 hours,* (*b*) *less than 1950 hours and*

(*c*) *more than 1920 hours and but less than 2160 hours.* (Bhopal, 2008 S ; U.P.T.U., 2008)

Solution. Here $\mu = 2040$ hours and $\sigma = 60$ hours.

(*a*) For $x = 2150$, $z = \dfrac{x - \mu}{\sigma} = 1.833$.

$\therefore$ area against $z = 1.83$ in the table III $= 0.4664$.

We, however, require the area to the right of the ordinate at $z = 1.83$. This area $= 0.5 - 0.4664 = 0.0336$.

Thus the number of bulbs expected to burn for more than 2150 hours

$$= 0.0336 \times 2000 = 67 \text{ approximately.}$$

(*b*) For $x = 1950$, $z = \dfrac{x - \mu}{\sigma} = -1.5$

The area required in this case is to the left of $z = -1.33$

i.e., $= 0.5 - 0.4082$ (table value for $z = 1.33$)

$= 0.0918$.

$\therefore$ the number of bulbs expected to burn for less than 1950 hours

$$= 0.0918 \times 2000 = 184 \text{ approximately.}$$

(*c*) When $x = 1920$, $z = \dfrac{1920 - 2040}{60} = -2$

When $x = 2160$, $z = \dfrac{2160 - 2040}{60} = 2$.

The number of bulbs expected to burn for more than 1920 hours but less than 2160 hours will be represented by the area between $z = -2$ and $z = 2$. This is twice the area from the table for $z = 2$, *i.e.*, $= 2 \times 0.4772 = 0.9544$.

Thus the required number of bulbs $= 0.9544 \times 2000 = 1909$ nearly.

Example 26.49. *If the probability of committing an error of magnitude x is given by*

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2};$$

compute the probable error from the following data :

$m_1 = 1.305$; $m_2 = 1.301$; $m_3 = 1.295$; $m_4 = 1.286$;
$m_5 = 1.318$; $m_6 = 1.321$; $m_7 = 1.283$; $m_8 = 1.289$;
$m_9 = 1.300$; $m_{10} = 1.286$. (*Kurukshetra, 2005*)

Solution. From the given data which is normally distributed, we have

$$\text{mean} = \frac{1}{10}\Sigma m_i = \frac{12.984}{10} = 1.2984$$

and

$$\sigma^2 = \frac{1}{10}\ \Sigma(m_i - \text{mean})^2$$

$$= \frac{1}{10}\ [(0.007)^2 + (0.003)^2 + (0.003)^2 + (0.012)^2 + (0.02)^2 + (0.023)^2 + (0.015)^2 + (0.009)^2 + (0.002)^2 + (0.012)^2\,]$$

$$= 0.0001594 \text{ whence } \sigma = 0.0126.$$

$\therefore$ *probable error* $= \dfrac{2}{3}\sigma = 0.0084$ approx.

Example 26.50. *Fit a normal curve to the following distribution.*

x :	*2*	*4*	*6*	*8*	*10*
f :	*1*	*4*	*6*	*4*	*1*

(*V.T.U., 2001*)

Solution.

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{2 + 16 + 36 + 32 + 10}{16} = 6$$

$$\text{S.D.} = \sqrt{\left[\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2\right]} = \sqrt{(40 - 36)} = 2$$

Taking $\mu = 6$, $\sigma = 2$ and $N = 16$, the equation of the normal curve is

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \text{ or } y = \frac{1}{2\sqrt{2\pi}} e^{-(x-6)^2/8} \qquad ...(i)$$

Area under (i) in (x_1, x_2) or (z_1, z_2)

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_2} e^{-z^2/2} dz - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} dz \qquad \text{where } z = \frac{x-6}{2}$$

To evaluate these integrals, we refer to table III.

Calculations :

Mid x	(x_1, x_2)	(z_1, z_2)	*Area under (i) in* (z_1, z_2)	*Expected frequency*
2	(1, 3)	(– 2.5, – 1.5)	0.4938 – 0.4332	16 × 0.606 = 0.97
4	(3, 5)	(– 1.5, – 0.5)	0.4332 – 0.1915	16 × 0.2417 = 3.9
6	(5, 7)	(– 0.5, 0.5)	0.1915 + 0.1915	16 × 0.383 = 6.1
8	(7, 9)	(0.5, 1.5)	0.4332 – 0.1915	16 × 0.2417 = 3.9
10	(9, 11)	(1.5, 2.5)	0.4938 – 0.4332	16 × 0.606 = 0.97

Hence the expected (theoretical) frequencies corrected to nearest integer are 1, 4, 6, 4, 1 which agree with the observed frequencies. This shows that the normal curve (i) is a proper fit to the given distribution.

PROBLEMS 26.7

1. Show that the standard deviation for a normal distribution is approximately 25% more than the mean deviation.
2. For a normally distributed variate with mean 1 and S.D. 3, find the probabilities that
   (i) $3.43 \le x \le 6.19$ (ii) $-1.43 \le x \le 6.19$.
3. If z is normally distributed with mean 0 and variance 1, find
   (i) $P_r\{z \le -1.64\}$; (ii) z_1 if $P_r\{z \ge z_1\} = 0.84$.
4. In a certain examination, the percentage of candidates passing and getting distinctions were 45 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. (Assume the distribution of marks to be normal). *(Kottayam, 2005)*
5. A manufacturer of air-mail envelopes knows from experience that the weight of the envelopes is normally distributed with mean 1.95 gm and standard deviation 0.05 gm. About how many envelopes weighing (i) 2 gm or more ; (ii) 2.05 gm or more can be expected in a given packet of 100 envelopes.
6. The mean height of 500 students is 151 cm. and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students' heights lie between 120 and 155 cm. *(Burdwan, 2003)*
7. The mean and standard deviation of the marks obtained by 1000 students in an examination are respectively 34.4 and 16.5. Assuming the normality of the distribution, find the approximate number of students expected to obtain marks between 30 and 60.
8. In an examination taken by 500 candidates, the average and the standard deviation of marks obtained (normally distributed) are 40% and 10%. Find approximately
   (i) how many will pass, if 50% is fixed as a minimum ?
   (ii) what should be the minimum if 350 candidates are to pass ?
   (iii) how many have scored marks above 60% ?
9. The mean inside diameter of a sample of 200 washers produced by a machine is 5.02 mm and the standard deviation is 0.05 mm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 4.96 to 5.08 mm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.
   [**Hint.** 4.96 in standard units = (4.96 – 5.02)/0.05 = – 1.2
   5.08 in standard units = (5.08 – 5.02)/0.05 = 1.2
   Proportion of non-defective washers = 2 (area between $z = 0$ and $z = 1.2$)
   = 0.7698 or 77% nearly.
   ∴ percentage of defective washers = 100 – 77 = 23%.]
10. Assuming that the diameters of 1000 brass plugs taken consecutively from a machine, form a normal distribution with mean 0.7515 cm. and standard deviation 0.0020 cm., how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm. ? *(Bhopal, 2002)*

11. It is given that the age of thermostats of a particular make follow the normal law with mean 5 years and S.D. 2 years. 1000 units are sold out every month. How many of them will have to be replaced at the end of the second year.
12. The income of a group of 10,000 persons was found to be normally distributed with mean Rs. 750 p.m., and standard deviation of Rs. 50. Show that, of this group, about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100. *(U.P.T.U., 2004 S)*
13. Find the equation of the best fitting normal curve to the following distribution :

x :	0	1	2	3	4	5
y :	13	23	34	15	11	4

14. Obtain the equation of the normal probability curve that may be fitted to the following data :

Variable :	4	6	8	10	12	14	16	18	20	22	24
Frequency:	1	7	15	22	35	43	38	20	13	5	1

15. A factory turns out an article by mass production and it is found that 10% of the product is rejected. Find the S.D. of the number of rejects and the equation to the normal curve to represent the number of rejects.
[**Hint.** $p = 0.1, q = 0.9, n = 100.$
$\therefore$ binomial distribution of rejects gives mean $= np = 10$, S.D. $= \sqrt{(npq)} = 3$
If this binomial distribution is approximated by a normal distribution, then the equation to the normal curve is
$$y = \frac{100}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where } \mu = 10, \sigma = 3.]$$
16. Given that the probability of committing an error of magnitude x is
$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \text{ show that the probable error is } 0.4769/h.$$

26.18 NORMAL APPROXIMATION TO BINOMIAL DISTRIBUTION

If the number of successes in a Binomial distribution range from x_1 to x_2, then the probability of getting these successes
$$= \sum_{r=x_1}^{x_2} {}^nC_r\, p^r\, q^{n-r}$$
As the number of trials increases, the Binomial distribution becomes approximated to the Normal distribution. The mean np and the variance npq of the binomial distribution will be quite close to the mean and standard deviation of the approximated normal distribution. Thus *for n sufficiently large (≥ 30), the binomial distribution with probability of success p, is approximated by the normal distribution with* $\mu = np$, $\sigma = \sqrt{npq}$.

We must however, be careful to get the correct values of z. For any success x, real class interval is $(x - 1/2, x + 1/2)$. Hence
$$z_1 = \frac{x_1 - \frac{1}{2} - \mu}{\sigma} = \frac{x_1 - \frac{1}{2} - np}{\sqrt{npq}}\,;\; z_2 = \frac{x_2 + \frac{1}{2} - np}{\sqrt{npq}}$$
so that $P(x_1 < x < x_2) = P(z_1 < z < z_2) = \int_{z_1}^{z_2} \phi(z)\,dz$ which can be calculated by using table III–Appendix 2.

Example 26.51. *In a referendum 60% of voters voted in favour. A random sample of 200 voters was selected. What is the probability that in the sample*
(a) more than 130 voted in favour ?
(b) between 105 and 130 inclusive voted in favour ?
(c) 120 voted in favour ?

Solution. Here $n = 200, p = 0.6, q = 0.4$
$$\therefore \quad \mu = np = 200 \times 0.6 = 120;\ \sigma = \sqrt{npq} = \sqrt{48} = 6.928$$
(a) $P(x > 130) = P(x > 130.5) = P\left(x > \frac{130.5 - 120}{\sqrt{48}}\right) = P(z > 1.516) = 0.0648$

(b) $P(105 < x < 130) = P(105.5 < x < 129.5)$

$$= P\left(\frac{105.5 - 120}{\sqrt{48}} < z < \frac{129.5 - 120}{\sqrt{48}}\right) = P(-2.09 < z < 1.37) = 0.8964$$

(c) $P(x = 120) = P(119.5 < x < 120.5)$

$$= P(-0.072 < z < 0.072) = 0.0575.$$

PROBLEMS 26.8

1. A pair of unbiased dice are rolled 180 times and their score recorded. Find
(a) $P(x \le 20)$, (b) $P(20 \le x \le 40)$, (c) $P(20 < x \le 30)$.
2. A marksman has a probability of 0.9 of hitting a target on a single shot. If the marksman has 40 shots, what is the probability that he hits the target (a) at least 35 times; (b) between 34 and 36 times; (c) 37 times.
3. A certain drug is effective in 72% of cases. Given 2000 people are treated with the drug, what is the probability that it will be effective for (a) at least 1400 patients, (b) less than 1390 patients, (c) 1420 patients.

26.19 SOME OTHER DISTRIBUTIONS

Discrete distributions

(1) Geometric distribution. If p be the probability of success and k be the numbers of failures preceding the first success then this distribution is

$$P(k) = q^k p, \qquad k = 0, 1, 2, ..., q = 1 - p.$$

Obviously $$\sum_{k=0}^{\infty} P(k) = p \sum_{k=0}^{\infty} q^k = p \cdot \frac{1}{1-q} = 1.$$

It can easily be shown that mean $= q/p$, and variance $= q/p^2$.

(2) Negative binomial distribution. This distribution gives the probability that the event occurs for the kth time on the rth trial $(r \ge k)$. If p be the probability of occurrence of an event then

$$P(k, r) = {}^{r-1}C_{k-1} p^k q^{r-k}.$$

It contains two parameters p and k. If $k = 1$, the Negative binomial distribution reduces to the geometric distribution.

(3) Hypergeometric distribution. Suppose a bag contains m white and n black balls. If r balls are drawn one at a time (*with replacement*), then the probability that k of them will be white is

$$P(k) = {}^mC_k\,{}^nC_{r-k}/{}^{m+n}C_r,\ k = 0, 1, ...\ r, r \le m, r \le n.$$

This distribution is known as *Hypergeometric distribution.*

For $$\sum_{k=0}^{r} P(k) = 1, \text{ since } \sum_{k=0}^{r} {}^mC_k\,{}^nC_{r-k} = {}^{m+n}C_r$$

This can be proved by equating the coefficient of t^r in

$$(1 + t)^m (t + 1)^n = (1 + t)^{m+n}$$

Continuous distributions

(4) Uniform (or Rectangular) distribution. A random variable X is said to be uniformly distributed over the interval $-\infty < a < b < \infty$, if its density is given by

$$f(x) = \frac{1}{b-a}, a < x < b \qquad ...(i)$$

The distribution given by (i) is called a *uniform distribution*. In this distribution, X takes the values with the same probability.

Its *mean* $$\mu = \int_a^b x \cdot f(x)\, dx = \frac{1}{b-a} \left| \frac{x^2}{2} \right|_a^b = \frac{a+b}{2}.$$

and *variance* $$\sigma^2 = \mu_2' - (\mu)^2 = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2 = \frac{1}{12}(b-a)^2.$$

(5) Gamma distribution. This continuous distribution is given by $f(x) = \dfrac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x}$ for all $x \geq 0$, where r and λ (both > 0) are called the parameters of the *gamma distribution*. Its mean $= r/\lambda$ and variance $= r/\lambda^2$.

Gamma distribution tends to normal distribution as the parameter r tends to infinity.

(6) Exponential distribution. This distribution is a special case of gamma distribution when $r = 1$ so that $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, where λ is a parameter.

It can be seen that mean $= 1/\lambda$, standard deviation $= 1/\lambda$.

This distribution plays an important role in the reliability and queuing theory.

(7) Weibull distribution*. This distribution is given by

$$f(x) = \frac{\alpha}{c} x^{\alpha-1} e^{-x^{\alpha}/c}, x > 0, c > 0$$

where c is a scale parameter and α a shape parameter.

Initially this distribution was used to describe experimentally observed variation in the fatigue resistance of steel and its elastic limits. But it has also been employed to study the variation of length of service of radio service equipment.

Example 26.52. *A die is cast until 6 appears. What is the probability that it must be cast more than 5 times ?*

Solution. Here probability of getting 6 is $p = \dfrac{1}{6}$. Then $q = \dfrac{5}{6}$.

If X is the number of tosses required for the first success, then

$$P(X = x) = q^{x-1} p \text{ for } x = 1, 2, 3, \ldots$$

$\therefore$ required probability $= P(X > 5) = 1 - P(X \leq 5)$

$$= 1 - \sum_{x=1}^{5} \left(\frac{5}{6}\right)^{x-1} \cdot \left(\frac{1}{6}\right) = 1 - \frac{1}{6}\left\{1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^4\right\} = \left(\frac{5}{6}\right)^5.$$

Example 26.53. *A random variable X has a uniform distribution over (– 3, 3), find k for which*

$$P(X > k) = \frac{1}{3}.$$

Also evaluate $P(X < 2)$ *and* $P[|X - 2| < 2]$.

Solution. (*i*) Density of $X = f(x) = \dfrac{1}{b-a} = \dfrac{1}{3-(-3)} = \dfrac{1}{6}$

$$\therefore \quad P(X > k) = 1 - P(X \leq k) = 1 - \int_{-3}^{k} f(x)\, dx$$

$$= 1 - \frac{1}{6}\int_{-3}^{k} dx = 1 - \frac{1}{6}(k+3) = \frac{1}{3} \quad \text{(given)}$$

This gives $k = 1$.

(*ii*) $$P(X < 2) = \int_{-3}^{2} f(x)\, dx = \frac{1}{6}\int_{-3}^{2} dx = \frac{5}{6}.$$

(*iii*) $$P[|X - 2| < 2] = P[2 - 2 < X < 2 + 2] = P[0 < x < 4] = \int_0^3 f(x)\, dx = \frac{1}{6}\int_0^3 dx = \frac{1}{2}.$$

PROBLEMS 26.9

1. Show that the mode of the *geometric distribution* $P(x) = \left(\dfrac{1}{2}\right)^x$, $x = 1, 2, 3 \ldots$, is unity.
2. Show that for the *rectangular distribution* $f(x) = 1$, $0 \leq x \leq 1$,

$$\text{mean} = \frac{1}{2}, \text{ variance} = \frac{1}{12} \text{ and mean deviation} = \frac{1}{4}.$$

* It was first used by Swedish scientist *Weibull* in 1951.

3. Find the mean and variance of the *uniform distribution* given by $f(x) = 1/n$, $x = 1, 2, ..., n$.
4. Show that for the *exponential distribution*
$dP = y_0 e^{-x/\sigma}$, $0 \le x \le \infty$,
the mean and S.D. are both equal to σ.
5. Find the mean and variance of the *exponential distribution* $f(x) = \frac{1}{b} e^{-(x-a)/b}$ $x > a$. (*Mumbai, 2005*)
6. Find the moment generating function for the *triangular distribution* given by
$$f(x) = x, \quad 0 \le x \le 1$$
$$= 2 - x, \quad 1 \le x \le 2.$$
7. Show that for the *Gamma distribution* $f(x) = \frac{e^{-x} x^{l-1}}{\Gamma(l)}$, $0 < x < \infty$, the mean and variance are both equal to l.
8. Find the moment generating function of the *Gamma distribution* $f(x) = \frac{1}{\Gamma\left(\frac{1}{4}\right)} e^{-x} x^{-3/4}$, $x \ge 0$, at the origin.
(*J.N.T.U., 2006 ; Madras, 2000 S*)

[**Chebyshev's inequality***. If x is a continuous random variable with mean μ and variance σ^2, then for any positive real parameter t,
$$P(|x-2| \ge t) \le \sigma^2/t^2 \text{ or } P(|x-\mu| \le t) \ge 1 - \sigma^2/t^2.$$
This result is known as *Chebyshev's inequality*. It gives limits to the probability that the value of the variate chosen at random will differ from mean by more than t.]

9. For the points on a symmetrical die, prove that *Chebyshev's inequality* gives
$$P(|x - \bar{x}| > 2.5) < 0.478,$$
while the actual probability is zero.
10. For the *Geometrical distribution* $P(x) = 2^{-x}$, $x = 1, 2, 3, ...$, prove that *Chebyshev's inequality* gives
$$P(|x-2| < 2) > \frac{1}{2},$$
while the actual probability is 15/16.

26.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 26.10

Select the correct answer or fill up the blanks in each of the following problems :

1. The probability that A happens is 1/3. The odds against happening of A are
(*a*) 2 : 1 (*b*) 2 : 3 (*c*) 3 : 2 (*d*) 5 : 2.
2. The odds in favour of an event A are 5 to 4. The probability of success of A is
(*a*) 4/5 (*b*) 5/9 (*c*) 4/9.
3. The probability that A passes a test is 2/3 and the probability that B passes the same test is 3/5. The probability that only one of them passes is
(*a*) 2/5 (*b*) 4/15 (*c*) 2/15 (*d*) 7/15.
4. A buys a lottery ticket in which the chance of winning is 1/10 ; B has a ticket in which his chance of winning is 1/20. The chance that atleast one of them wins is
(*a*) 1/200 (*b*) 29/200 (*c*) 30/200 (*d*) 170/200.
5. The probability that a non-leap year should have 53 Tuesdays is ...
6. The probability of getting 2 or 3 or 4 from a throw of single dice is ...
7. The mean of the Binomial distribution with n observations and probability of success p, is
(*a*) pq (*b*) np (*c*) $\sqrt{np}$ (*d*) $\sqrt{pq}$.
8. If the mean of a Poisson distribution is m, then S.D. of this distribution is
(*a*) m^2 (*b*) $\sqrt{m}$ (*c*) m (*d*) none of these.

* See footnote on page 571.

9. The S.D. of the Binomial distribution is
(a) $\sqrt{npq}$ (b) $\sqrt{np}$ (c) npq (d) pq.
10. In a Poisson distribution if $2P(x = 1) = P(x = 2)$, then the variance is
(a) 0 (b) -1 (c) 4 (d) 2.
11. If the probability of hitting a target by one shot be $p = 0.8$, then the probability that out of ten shots, seven will hit the target is ...
12. For a Poisson variate $x : P(x = 1) = P(x = 2)$, then the mean of x is ...
13. If $P(A) = 0.35$, $P(B) = 0.73$ and $P(A \cap B) = 0.14$, then $P(A \cap B') = ...$
14. If A and B are independent, $P(B) = 0.14$ and $P(A/B) = 0.24$, then $P(A) = ...$
15. The probability distribution of the number of heads, when two coins are tossed, is ...
16. The multiplication law of probability states that ...
17. The area under the standard normal curve which lies between $z = 0.90$ and $z = -1.85$ is ...
[Given $P(0 < z < 1.85) = 0.4678$, $P(0 < z < 0.9) = 0.3159$]
18. The mean, median and mode of a normal distribution are ...
19. The mean and variance of a Poisson distribution are ...
20. If A and B are two mutually exclusive events, then $P(A \cup B) = ...$
21. For a normal distribution $\beta_1 = ...$ and $\beta_2 = ...$
22. The number of ways in which five people can be lined up to get on a bus are ...
23. A shipment of 10 television sets contains 3 defective sets. The number of ways in which one can purchase 4 of these sets and receive 2 defective sets are ...
24. The probability of getting a total of 5 when a pair of dice is tossed is ...
25. If $P(B) = 0.81$ and $P(A \cap B) = 0.18$, then $P(A/B) = ...$
26. If two unbiased dice are thrown simultaneously, the probability that the sum of the numbers on them is at least 10, is ....
27. If X is a Poisson variate such that $P(X = 2) = P(X = 3)$, then $P(X = 0) = ...$
28. An unbiased die is tossed twice, then the probability of obtaining the sum 6, is ...
29. The variance of Poisson distribution with parameter $\lambda = 2$ is ...
30. The distribution in which mean, median, mode are equal is ...
31. For the Poisson variate, probability of getting at least one success is ...
32. Total number of events in rolling of an ideal die is ...
33. If X be normal with mean 10 and variance 4, then $P(X < 11) = ...$
34. If X is a binomial variate with parameters n and p, then its m.g.f. about the origin is ...
35. In a normal distribution, mean deviation : standard deviation = ...
36. If A and B are independent and $P(A) = 1/2$, $P(B) = 1/3$ then $P(A \cap B) = ...$
37. If X is the random variable representing the outcome of the roll of an ideal die, then $E(X) = ...$
38. If X is a binomial variate with $p = 1/5$ for the experiment of 50 trials, then the standard deviation is ...
39. The area under the whole normal curve is ...
40. Given $X = B(n, p)$, then the conditions under which X tends to a Poisson distribution, are ...
41. If A and B are mutually exclusive events then $P(A \cup B) = ...$
42. The probability of selecting x white balls from a bag containing y white and z red balls is ...
43. The mean of the binomial distribution is ...
44. If A and B are mutually exclusive events, $P(A) = 0.29$, $P(B) = 0.43$, then $P(A \cup B) = ...$ and $P(A \cap B') = ...$
45. If the mean and variance of a binomial variate are 12 and 4, then the distribution is ...
46. If x is a Poisson variable such that $P(x = 2) = 9P(x = 4) + 90P(x = 6)$, then the mean = ...
47. μ_r' the rth moment about the origin in terms of the m.g.f. is ...
48. The chance of throwing 7 in a single throw with two dice is ...
49. If A and B are any two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$, then $P(A/B) = ...$
50. In the roll of an ideal die, the probability of getting a prime number is ...
51. If A and B are mutually exclusive events, $P(A \cup B) = 0.6$, $P(B) = 0.4$, then $P(A) = ...$
52. The probability that a leap year should have 53 Sundays is ...
53. The probability density function of a binomial distribution is ...
54. The probable error is ... times S.D. approximately.
55. To fit a normal distribution, the parameters required are ...

56. A card is drawn from a well-shuffled pack of 52 cards, then the probability of this card being a red coloured ace is ...
57. If $P(1) = P(2)$, then the mean of the Poisson distribution is ...
58. Baye's theorem states that ...
59. If x is a Poisson variate such that $P(x = 1) = 0.3$ and $P(x = 2) = 0.2$, then $P(x = 0) = ...$
60. If A and B are events such that $P(A \cup B) = 3/4$, $P(A \cap B) = 1/4$, and $P(A) = 2/3$, then $P(A) = ...$
61. The chance of throwing 7 in a single throw with two dice is the same as that of getting 7 in two throws of a single die. (True or False)
62. If the mean of a Poisson distribution is 5, then its variance is 10. (True or False)
63. If X is normal with mean 3 and variance 1, then $X - 3$ is a variate with mean 0 and variance 1. (True or False)
64. If X is a binomial variable with parameters $n = 10$, $p = 1/4$, then its standard deviation is 2.25. (True or False)
65. The mean of a binomial distribution is 5 and S.D. is 3. (True or False)
66. The mean and variance of Poisson distribution are equal. (True or False)
67. The graph of the normal distribution is symmetric with respect to the line $y = x$. (True or False)
68. The standard deviation of a binomial distribution is np. (True or False)
69. $f(x) = kx$ in $0 < x < 1$ is a valid probability density function, if $k = ...$
70. If $V(x) = 2$, then $V(2x + 3) = ...$
71. The p.d.f. of an exponential distribution is ...
72. If X is uniformly distributed in $(-2, 3)$, then its variance is ...
73. The variance of Poisson distribution with parameter $\lambda = 2$ is ...
74. In Gamma distribution with parameter l, the variance is ...
75. If $f(x) = kx^3$, $0 < x < 1$ and 0 elsewhere, is a p.d.f., then $k = ...$
76. The m.g.f. of a random variable X is $(1 - 2t)^{-4}$, then $E(x)$ is ...
77. A random variable X has F-distribution with (m, n) degrees of freedom, then $1/X$ has the same distribution with ... degrees of freedom.
78. If X is a continuous random variable having the p.d.f. $f(x)$, then the m.g.f. about the origin is given by ...
79. If $f(X) = X + 2/k$, $X = 1, 2, 3, 4, 5$ is the probability distribution of a discrete random variable, then $k = ...$
80. The p.d.f. of Gamma variate is ...
81. If X is uniformly distributed in $[a, b]$, then $E(X) = ...$
82. The marks obtained by students were found normally distributed with mean 75 and variance 100. The percentage of students who scored more than 75 marks is ...
83. When four unbiased coins are tossed, the probability of getting two heads is ...
84. If $f(x) = \begin{cases} kxe^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$, is the p.d.f. of x, then $k = ...$
85. If x is a uniform distribution defined in the interval $(4, 7)$, then its variance is ...
86. The p.d.f. of a continuous random variable is $f(x) = k/x^3$, $5 \le x \le 10$; 0 elsewhere, then the value of k is
    (a) 1 (b) 50 (c) 200/3 (d) 200.
87. The relation between probability density function and cumulative density function of a random variable is ...
88. If X has Poisson distribution with parameter λ, then P (X is even) $= ...$
89. Range of t- distribution is ...
90. If the p.d.f. of x is $f(x) = kx(1 - x)$, $0 < x < 1$, then $k = ...$
91. The function $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$, is a probability function, then $k = ...$
92. If the random variable x is uniformly distributed in $[0, 3]$, then its p.d.f. is $f(x) = 3$, $0 < x < 3$; 0, elsewhere. (True or False)
93. Exponential distribution $f(x)$ is defined by $f(x) = ae^{-2x}$, $0 < x < \infty$, then $a = ...$
94. The p.d.f. of Beta distribution with $\alpha = 1$, $\beta = 4$ is $f(x) = ...$
95. For a standard normal variate z, $P(-0.72 \le z \le 0) = ...$

CHAPTER 27

Sampling & Inference

27.1 (1) INTRODUCTION

We know that a small section selected from the population is called a *sample* and the process of drawing a sample is called *sampling*. It is essential that a sample must be a *random* selection so that each member of the population has the same chance of being included in the sample. Thus the fundamental assumption underlying theory of sampling is *Random sampling*.

A special case of random sampling in which each event has the same probability p of success and the chance of success of different events are independent whether previous trials have been made or not, is known as *simple sampling*.

The statistical constants of the population such as mean (μ), standard deviation (σ) etc. are called the *parameters*. Similarly, constants for the *sample* drawn from the given population *i.e.*, mean ($\bar{x}$), standard deviation (S) etc. are called the *statistic*. The population parameters are in general, not known and their estimates given by the corresponding sample statistic are used. We use the Greek letters to denote the population parameters and Roman letters for sample statistic.

(2) Objectives of sampling. Sampling aims at gathering the maximum information about the population with the minimum effort, cost and time. The object of sampling studies is to obtain the best possible values of the parameters under specific conditions. Sampling determines the reliability of these estimates. The logic of the sampling theory is the logic of induction in which we pass from a particular (sample) to general (population). Such a generalisation from sample to population is called **Statistical Inference.**

27.2 SAMPLING DISTRIBUTION

Consider all possible samples of size n which can be drawn from a given population at random. For each sample, we can compute the mean. The means of the samples will not be identical. If we group these different means according to their frequencies, the frequency distribution so formed is known as *sampling distribution of the mean*. Similarly we can have *sampling distribution of the standard deviation* etc.

While drawing each sample, we put back the previous sample so that the parent population remains the same. This is called *sampling with replacement* and all the subsequent formulae will pertain to sampling with replacement.

(2) Standard error. The standard deviation of the sampling distribution is called the *standard error* (*S.E.*). Thus the standard error of the sampling distribution of means is called standard error of means. The standard error is used to assess the difference between the expected and observed values. The reciprocal of the standard error is called *precision*.

If $n \geq 30$, a sample is called *large* otherwise *small*. The sampling distribution of large samples is assumed to be normal.

27.3 (1) TESTING A HYPOTHESIS*

To reach decisions about populations on the basis of sample information, we make certain assumptions about the populations involved. Such assumptions, which may or may not be true, are called *statistical hypothesis*. By testing a hypothesis is meant a process for deciding whether to accept or reject the hypothesis. The method consists in assuming the hypothesis as correct and then computing the probability of getting the observed sample. If this probability is less than a certain preassigned value the hypothesis is rejected.

(2) Errors. If a hypothesis is rejected while it should have been accepted, we say that a *Type I error* has been committed. On the other hand, if a hypothesis is accepted while it should have been rejected, we say that a *Type II error* has been made. The statistical testing of hypothesis aims at limiting the Type I error to a preassigned value (say : 5% or 1%) and to minimize the Type II error. The only way to reduce both types of errors is to increase the sample size, if possible.

(3) Null hypothesis. The hypothesis formulated for the sake of rejecting it, under the assumption that it is true, is called the *null hypothesis* and is denoted by H_0. To test whether one procedure is better than another, we assume that there is no difference between the procedures. Similarly to test whether there is a relationship between two variates, we take H_0 that there is no relationship. By accepting a null hypothesis, we mean that on the basis of the statistic calculated from the sample, we do not reject the hypothesis. It however, does not imply that the hypothesis is proved to be true. Nor its rejection implies that it is disproved.

27.4 (1) LEVEL OF SIGNIFICANCE

The probability level below which we reject the hypothesis is known as the *level of significance*. The region in which a sample value falling is rejected, is known as the *critical region*. We generally take two critical regions which cover 5% and 1% areas of the normal curve. The shaded portion in the figure corresponds to 5% level of significance. Thus the *probability of the value of the variate falling in the critical region is the level of significance*.

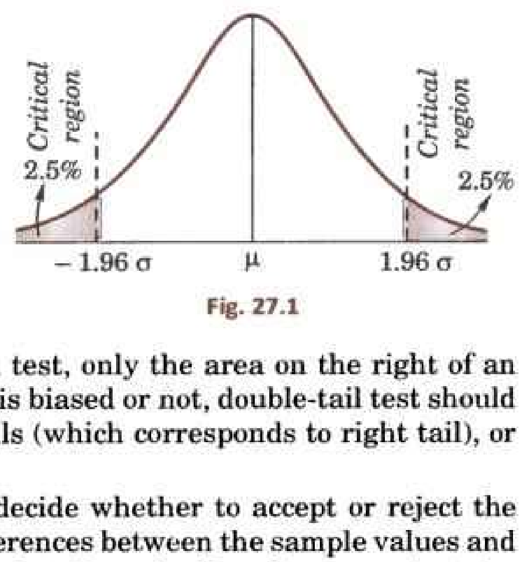


Fig. 27.1

Depending on the nature of the problem, we use a *single-tail test* or *double-tail test* to estimate the significance of a result. In a double-tail test, the areas of both the tails of the curve representing the sampling distribution are taken into account whereas in the single tail test, only the area on the right of an ordinate is taken into consideration. For instance, to test whether a coin is biased or not, double-tail test should be used, since a biased coin gives either more number of heads than tails (which corresponds to right tail), or more number of tails than heads (which corresponds to left tail only).

(2) Tests of significance. The procedure which enables us to decide whether to accept or reject the hypothesis is called the *test of significance*. Here we test whether the differences between the sample values and the population values (or the values given by two samples) are so large that they signify evidence against the hypothesis or these differences are so small as to account for fluctuations of sampling.

27.5 CONFIDENCE LIMITS**

Suppose that the sampling distribution of a statistic S is normal with mean μ and standard deviation σ. As in the Fig. 27.1 the sample statistic S can be expected to lie in the interval $(\mu - 1.96\sigma, \mu + 1.96\sigma)$ for 95% times *i.e.*, we can be confident of finding μ in the interval $(S - 1.96\sigma, S + 1.96\sigma)$ in 95% cases. Because of this, we call

*The American statistician *J. Neyman* (1894—1981) and the English statistician *E.S. Pearson* (1895—1980)–son of *Karl Pearson* (See footnote p. 843), developed a systematic theory of tests around 1930.

***J. Neyman* developed the modern theory and terminology of confidence limits.

$(S - 1.96\sigma, S + 1.96\sigma)$ the 95% confidence interval for estimation of μ. The ends of this interval (*i.e.* $S \pm 1.96\sigma$) are called 95% *confidence limits* (or *fiducial limits*) for S. Similarly $S \pm 2.58\sigma$ are 99% confidence limits. The numbers 1.96, 2.58 etc. are called *confidence coefficients.* The values of confidence coefficients corresponding to various levels of significance can be found from the normal curve area table VI – Appendix 2.

27.6 SIMPLE SAMPLING OF ATTRIBUTES

The sampling of attributes may be regarded as the selection of samples from a population whose members possess the attribute K or not K. The presence of K may be called a success and its absence a failure.

Suppose we draw a simple sample of n items. Clearly it is same as a series of n independent trials with the same probability p of success. The probabilities of 0, 1, 2, ..., n successes are the terms in the binomial expansion of $(q + p)^n$ where $q = 1 - p$.

We know that the mean of this distribution is np and standard deviation is $\sqrt{(npq)}$ *i.e.*, the expected value of success in a sample of size n is np and the standard error is $\sqrt{(npq)}$.

If we consider the proportion of successes, then

(*i*) mean proportion of successes $= np/n = p$.

(*ii*) standard error of the proportion of successes

$$= \sqrt{\left(n \cdot \frac{p}{n} \cdot \frac{q}{n}\right)} = \sqrt{\left(\frac{pq}{n}\right)}$$

and (*iii*) precision of the proportion of successes $= \sqrt{(n/pq)}$, which varies as $\sqrt{n}$, since p and q are constants.

27.7 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

We know that the binomial distribution tends to normal for large n. Suppose we wish to test the hypothesis that the probability of success in such trial is p. Assuming it to be true, the mean μ and the standard deviation σ of the sampling distribution of number of successes are np and $\sqrt{(npq)}$ respectively.

For a normal distribution, only 5% of the members lie outside $\mu \pm 1.96\sigma$ while only 1% of the members lie outside $\mu \pm 2.58\sigma$.

If x be the observed number of successes in the sample and z is the standard normal variate then $z = (x - \mu)/\sigma$.

Thus we have the following test of significance :

(*i*) *If* $|z| < 1.96$, *difference between the observed and expected number of successes is not significant.*

(*ii*) *If* $|z| > 1.96$, *difference is significant at 5% level of significance.*

(*iii*) *If* $|z| > 2.58$, *difference is significant at 1% level of significance.*

Example 27.1. *A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased at 5% level of significance.* (V.T.U., 2007)

Solution. Suppose the coin is unbiased.

Then the probability of getting the head in a toss $= \frac{1}{2}$

$\therefore$ expected number of successes $= \frac{1}{2} \times 400 = 200$

and the observed value of successes = 216

Thus the excess of observed value over expected value = 216 – 200 = 16

Also S.D. of simple sampling $= \sqrt{npq} = \sqrt{\left(400 \times \frac{1}{2} \times \frac{1}{2}\right)} = 10$

Hence $z = \dfrac{x - np}{\sqrt{(npq)}} = \dfrac{16}{10} = 1.6$

As $z < 1.96$, the hypothesis is accepted at 5% level of significance *i.e.*, we conclude that the coin is unbiased at 5% level of significance.

Example 27.2. *A die was thrown 9000 times and a throw of 5 or 6 was obtained 3240 times. On the assumption of random throwing, do the data indicate an unbiased die ?* (V.T.U., 2010)

Solution. Suppose the die is unbiased.

Then the probability of throwing 5 or 6 with one die $= \frac{1}{3}$

The expected number of successes $= \frac{1}{3} \times 9000 = 3000$

and the observed value of successes = 3240

Thus the excess of observed value over expected value $3240 - 3000 = 240$

Also S.D. of simple sampling $= \sqrt{npq} = \sqrt{\left(9000 \times \frac{1}{3} \times \frac{2}{3}\right)} = 44.72$

Hence $z = \frac{x - np}{\sqrt{(npq)}} = \frac{240}{44.72} = 5.4$ nearly.

As $z > 2.58$, the hypothesis has to be rejected at 1% level of significance and we conclude that the die is biased.

Example 27.3. *In a locality containing 18000 families, a sample of 840 families was selected at random. Of these 840 families, 206 families were found to have a monthly income of ₹ 250 or less. It is desired to estimate how many out of 18,000 families have a monthly income of ₹ 250 or less. Within what limits would you place your estimate ?*

Solution. Here $p = \frac{206}{840} = \frac{103}{420}$ and $q = \frac{317}{420}$

∴ standard error of the population of families having a monthly income of ₹ 250 or less

$$= \sqrt{\left(\frac{pq}{n}\right)} = \sqrt{\left(\frac{103}{420} \times \frac{317}{420} \times \frac{1}{840}\right)} = .015 = 1.5\%$$

Hence taking $\frac{103}{420}$ (or 24.5%) to be the estimate of families having a monthly income of ₹ 250 or less in the locality, the limits are $(24.5 \pm 3 \times 1.5)\%$ *i.e.*, 20% and 29% approximately.

27.8 COMPARISON OF LARGE SAMPLES

Two large samples of sizes n_1, n_2 are taken from two populations giving proportions of attributes A's as p_1, p_2 respectively.

(*a*) On the hypothesis that the populations are similar as regards the attribute A, we combine the two samples to find an estimate of the common value of proportion of A's in the populations which is given by

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

If e_1, e_2 be the standard errors in the two samples then

$$e_1^2 = \frac{pq}{n_1} \text{ and } e_2^2 = \frac{pq}{n_2}$$

If e be the standard error of the difference between p_1 and p_2, then

$$e^2 = e_1^2 + e_2^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \quad \therefore \quad z = \frac{p_1 \sim p_2}{e}$$

If $z > 3$, the difference between p_1 and p_2 is real one.

If $z < 2$, the difference may be due to fluctuations of simple sampling.

But if z lies between 2 and 3, then the difference is significant at 5% level of significance.

(*b*) If the proportions of A's are not the same in the two populations from which the samples are drawn, but p_1 and p_2 are the true values of proportions then S.E. e of the difference $p_1 \sim p_2$ is given by

$$e^2 = \frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2}$$

If $z = \frac{p_1 - p_2}{e} < 3$, the difference could have arisen due to fluctuations of simple sampling.

Example 27.4. *In a city A 20% of a random sample of 900 school boys had a certain slight physical defect. In another city B, 18.5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant ?* (V.T.U., 2003 S)

Solution. We have $n_1 = 900, n_2 = 1600$

and $$p_1 = \frac{20}{100} = \frac{1}{5}, p_2 = \frac{18.5}{100}$$

$\therefore$ $$p = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{180 + 296}{900 + 1600} = 0.19$$

and $$q = 1 - 0.19 = 0.81$$

Thus $$e^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right) = 0.19 \times 0.81\left(\frac{1}{900} + \frac{1}{1600}\right) = 0.0017$$

giving $$e = 0.04 \text{ nearly.}$$

Also $$p_1 - p_2 = \frac{1.5}{100} = 0.015 \quad \therefore \quad z = \frac{p_1 - p_2}{e} = \frac{.015}{.04} = 0.37$$

As $z < 1$, the difference between the proportions is not significant.

Example 27.5. *In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations ?* (Coimbatore, 2001)

Solution. Here $p_1 = 0.3, p_2 = 0.25$ so that $p_1 - p_2 = 0.05$.

$\therefore$ $$e^2 = \frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2} = \frac{0.3 \times 0.7}{1200} + \frac{0.25 \times 0.75}{900}$$

so that $$e = 0.0195$$

$\therefore$ $$z = \frac{p_1 - p_2}{e} = \frac{0.05}{0.0195} = 2.5 \text{ nearly}$$

Hence it is unlikely that the real difference will be hidden.

PROBLEMS 27.1

1. A die is tossed 960 times and it falls with 5 upwards 184 times. Is the die biased ? (V.T.U., 2006)
2. 12 dice are thrown 3086 times and a throw of 2, 3, 4 is reckoned as a success. Suppose that 19142 throws of 2, 3, 4 have been made out. Do you think that this observed value deviates from the expected value ? If so, can the deviation from the expected value be due to fluctuations of simple sampling ?
3. Balls are drawn from a bag containing equal number of black and white balls, each ball being replaced before drawing another. In 2250 drawings 1018 black and 1232 white balls have been drawn. Do you suspect some bias on the part of the drawer ?
4. A sample of 1000 days is taken from meteorological records of a certain district and 120 of them are found to be foggy. What are the probable limits to the percentage of foggy days in the district ?
5. In a group of 50 first cousines there were found to be 27 males and 23 females. Ascertain if the observed proportions are inconsistent with the hypothesis that the sexes should be in equal proportion.
6. A random sample of 500 apples was taken from a large consignment and 65 were found to be bad. Estimate the proportion of the bad apples in the consignment as well as the standard error of the estimate. Deduce that the percentage of bad apples in the consignment almost certainly lies between 8.5 and 17.5.
7. 400 children are chosen in an industrial town and 150 are found to be under weight. Assuming the conditions of simple sampling, estimate the percentage of children who are under weight in the industrial town and assign limits within which the percentage probably lies ?

8. A machine produces 16 imperfect articles in a sample of 500. After machine is overhauled, it produces 3 imperfect articles in a batch of 100. Has the machine been improved ? *(Rohtak, 2005 ; Madras, 2003)*
9. One type of aircraft is found to develop engine trouble in 5 flights out of a total of 100 and another type in 7 flights out of a total of 200 flights. Is there a significant difference in the two types of aircrafts so far as engine defects are concerned ?
10. In a sample of 600 men from a certain city, 450 are found smokers. In another sample of 900 men from another city, 450 are smokers. Do the data indicate that the cities are significantly different with respect to the habit of smoking among men ? *(J.N.T.U., 2003)*
11. In a sample of 500 people from a state 280 take tea and rest take coffee. Can we assume that tea and coffee are equally popular in the state at 5% level of significance ?

27.9 (1) SAMPLING OF VARIABLES

We now consider sampling of a variable such as weight, height, etc. Each member of the population gives a value of the variable and the population is a frequency distribution of the variable. Thus a random sample of size n from the population is same as selecting n values of the variables from those of the distribution.

(2) Sampling distribution of the mean. *If a population is distributed normally with mean μ and standard deviation σ, then the means of all positive random samples of size n, are also distributed normally with mean μ and standard error $\sigma/\sqrt{n}$.* This result shows how the precision of a sample mean increases as the sample size increases.

27.10 CENTRAL LIMIT THEOREM

This is a very important theorem regarding the distribution of the mean of a sample *if the parent population is non-normal and the sample size is large.*

If the variable X has a non-normal distribution with mean μ and standard deviation σ, then the limiting distribution of

$$z = \frac{\bar{x} - \mu}{\sigma\sqrt{n}}$$ as $n \to \infty$, *is the standard normal distribution* (*i.e.*, with mean 0 and unit S.D.)

There is no restriction upon the distribution of X except that it has a finite mean and variance. This theorem holds well for a sample of 25 or more which is regarded as large.

Thus *if the population is normal, the sampling distribution of the mean is also normal with mean μ and S.E. $\sigma/\sqrt{n}$, while for large samples the same result holds even if the distribution of the population is non-normal.* This property is of universal use and throws light on the importance of normal distribution in statistical theory.

27.11 CONFIDENCE LIMITS FOR UNKNOWN MEAN

Let the population from which a random sample of size n is drawn, have mean μ and S.D. σ. If μ is not known, there will be a range of values of μ for which observed mean $\bar{x}$ of the sample is not significant at any assigned level of probability. The relative deviation of $\bar{x}$ from μ is $(\bar{x} - \mu)/\sqrt{\sigma}$.

If $\bar{x}$ is not significant at 5% level of probability, then

$$|(\bar{x} - \mu)\sqrt{n}/\sigma| < 1.96 \quad i.e. \quad \bar{x} - 1.96\sigma/\sqrt{n} < \mu < \bar{x} + 1.96\sigma/\sqrt{n}$$

Thus 95% confidence or fiducial limits for the mean of the population corresponding to given sample are $\bar{x} \pm 1.96\sigma/\sqrt{n}$.

Similarly *99% confidence limits for μ are $\bar{x} \pm 2.58\sigma/\sqrt{n}$.*

Example 27.6. *A sample of 900 members is found to have a mean of 3.4 cm. Can it be reasonably regarded as a truly random sample from a large population with mean 3.25 cm and S.D. 1.61 cm.*

Solution. Here $\bar{x} = 3.4$ cm, $n = 900$, $\mu = 3.25$ and $\sigma = 1.61$ cm

$$\therefore \qquad z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{3.4 - 3.25}{1.61/30} = 2.8$$

As $z > 1.96$, the deviation of the sample mean from the mean of the population is significant at 5% level of significance. Hence it cannot be regarded as a random sample.

Example 27.7. *The mean of a certain normal population is equal to the standard error of the mean of the samples of 100 from that distribution. Find the probability that the mean of the sample of 25 from the distribution will be negative ?*

Solution. If μ be the mean and σ the S.D. of the distribution, then

$$\mu = \text{S.E. of the sample means} = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}$$

Also for a sample of size 25, we have $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{25}} = \dfrac{\bar{x} - \sigma/10}{6/5} = \dfrac{5\bar{x}}{\sigma} - \dfrac{1}{2}$

Since $\bar{x}$ is negative, $z < -\frac{1}{2}$.

$\therefore$ the probability that a normal variate $z < -\frac{1}{2}$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{1/2}^{\infty} e^{-\frac{1}{2}z^2} dz$$

$= 0.5 - 0.915 = 0.3085$, from the tables.

Example 27.8. *An unbiased coin is thrown n times. It is desired that the relative frequency of the appearance of heads should lie between 0.49 and 0.51. Find the smallest value of n that will ensure this result with 90% confidence.*

Solution. S.E. of the proportion of heads $= \sqrt{\dfrac{pq}{n}} = \sqrt{\dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{n}} = \dfrac{1}{2\sqrt{n}}$

90% of confidence = 45% or .45 of the total area under the normal curve on each side of the mean.

$\therefore$ the corresponding value of $z = 1.645$, from the tables.

Thus $\quad p \mp 1.645\sigma = 0.49 \quad \text{or} \quad 0.51.$

i.e., $\quad 0.5 - 1.645 \cdot \dfrac{1}{2\sqrt{n}} = 0.49 \quad \text{and} \quad 0.5 + 1.645 \cdot \dfrac{1}{2\sqrt{n}} = 0.51$

whence $\quad \dfrac{1.645}{2\sqrt{n}} = 0.01 \quad \text{or} \quad \sqrt{n} = \dfrac{329}{4} \quad \text{or} \quad n = 6765$ approximately.

27.12 TEST OF SIGNIFICANCE FOR MEANS OF TWO LARGE SAMPLES

(*a*) Suppose two random samples of sizes n_1 and n_2 have been *drawn from the same population* with S.D. σ. We wish to test whether the difference between the sample means $\bar{x}_1$ and $\bar{x}_2$ is significant or is merely due to fluctuations of sampling.

If the samples are independent, then the standard error e of the difference of their means is given by

$$e^2 = e_1^2 + e_2^2$$

where $e_1 = \sigma/\sqrt{n_1}$, $e_2 = \sigma/\sqrt{n_2}$ are the S.E.s of the means of the two samples.

$$\therefore \quad e = \sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}. \quad \text{Hence} \quad z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{(1/n_1 + 1/n_2)}}$$

is normally distributed with mean zero and S.D. 1.

Test of significance (n_1, n_2 being large) :

If $z > 1.96$, then the difference is significant at 5% level of significance.

If $z > 3$, it is highly probable that either the samples have not been drawn from the same population or the sampling is not simple.

(b) *If the samples are known to be drawn from different populations* with means μ_1, μ_2 and standard deviations σ_1 and σ_2. Then the standard error e of their means is given by

$$e = \sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}$$

Assuming that the two populations have the same mean (*i.e.*, $\mu_1 = \mu_2$), the difference of the means of the samples will be normally distributed with mean zero and S.D. e. Now the same procedure of test of significance is applied.

Example 27.9. *The means of simple samples of sizes 1000 and 2000 are 67.5 and 68.0 cm respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 cm.* *(Madras, 2002)*

Solution. We have $\bar{x}_1 = 67.5,\ \bar{x}_2 = 68.0$

$n_1 = 1000,\ n_2 = 2000.$

On the hypothesis, that the samples are drawn from the same population of S.D. $\sigma = 2.5$, we get

$$z = \frac{\bar{x}_1 \sim \bar{x}_2}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{67.5 \sim 68.0}{2.5\sqrt{\left(\frac{1}{1000} + \frac{1}{2000}\right)}}$$

$$= \frac{0.5}{2.5 \times 0.0387} = \frac{0.5}{0.09675} = 5.1$$

Hence the difference between the sample means *i.e.*, 5.1 is very much greater than 1.96 and is therefore significant. Thus, the samples cannot be regarded as drawn from the same population.

Example 27.10. *A sample of height of 6400 soldiers has a mean of 67.85 inches and a standard deviation of 2.56 inches while a simple sample of heights of 1600 sailors has a mean of 68.55 inches and a standard deviation of 2.52 inches. Do the data indicate that the sailors are on the average taller than soldiers?*

Solution. Here $\bar{x}_1 = 67.85,\ \sigma_1 = 2.56,\ n_1 = 6400$

$\bar{x}_2 = 68.55,\ \sigma_2 = 2.52,\ n_2 = 1600.$

$\therefore$ S.E. of the difference of the mean heights is

$$e = \sqrt{\left[\frac{{\sigma_1}^2}{n_1} + \frac{{\sigma_2}^2}{n_2}\right]} = \sqrt{\left[\frac{(2.56)^2}{6400} + \frac{(2.52)^2}{1600}\right]}$$

$$= \sqrt{[.001024 + .003969]} = 0.005 \text{ nearly.}$$

Also difference between the means $= \bar{x}_2 - \bar{x}_1 = 0.7$, which $> 10e$. This is highly significant. Hence the data indicates that the sailors are on the average taller than the soldiers.

PROBLEMS 27.2

1. A sample of 400 items is taken from a normal population whose mean is 4 and variance 4. If the sample mean is 4.45, can the samples be regarded as a simple sample ?
2. To know the mean weights of all 10-year old boys in Delhi, a sample of 225 is taken. The mean weight of the sample is found to be 67 pounds with a S.D. of 12 pounds. Can you draw any inference from it about the mean weight of the population ?
3. A normal population has a mean 0.1 and a S.D. of 2.1. Find the probability that the mean of simple sample of 900 members will be negative.
4. If the mean breaking strength of copper wire is 575 lbs. with a standard deviation of 8.3 lbs., how large a sample must be used in order that there be one chance in 100 that the mean breaking strength of the sample is less than 572 lbs. ?

[**Hint.** $|z| = \left|\frac{\bar{x} - \mu}{\sigma}\sqrt{n}\right| = \frac{3}{8.3}\sqrt{n}$

Also from table IV, $z = 2.33$. Hence $n = 42$ nearly.]

5. A research worker wishes to estimate mean of a population by using sufficiently large sample. The probability is 95% that sample mean will not differ from the true mean by more than 25% of the S.D. How large a sample should be taken ?
6. The density function of a random variable x is $f(x) = ke^{-2x^2+10x}$. Find the upper 5% point of the distribution of the means of the random sample of size 25 from the above population.
7. The means of two large samples of 1000 and 2000 members are 168.75 cms. and 170 cms. respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms.
8. If 60 new entrants in a given university are found to have a mean height of 68.60 inches and 50 seniors a mean height of 69.51 inches ; is the evidence conclusive that the mean height of the seniors is greater than that of the new entrants ? Assume the standard deviation of height to be 2.48 inches.
9. A sample of 100 electric bulbs produced by manufacturer A showed a mean life time of 1190 hours and a standard deviation of 90 hours. A sample of 75 bulbs produced by manufacturer B showed a mean life time of 1230 hours, with a standard deviation of 120 hours. Is there a difference between the mean life time of two brands at a significance level of (*i*) 0.05 (*ii*) 0.01.
10. A random sample of 1000 men from North India shows that their mean wage is ₹ 5 per day with a S.D. of ₹ 1.50. A sample of 1500 men from South India gives a mean wage of ₹ 4.50 per day with a standard deviation of ₹ 2. Does the mean rate of wages varies as between the two regions ?

27.13 SAMPLING OF VARIABLES—SMALL SAMPLES

In case of large samples, sampling distribution approaches a normal distribution and values of sample statistic are considered best estimates of the parameters in a population. It will no longer be possible to assume that statistics computed from small samples are normally distributed. As such, *a new technique has been devised for small samples which involves the concept of 'degrees of freedom' which we explain below.*

Number of degrees of freedom *is the number of values in a set which may be assigned arbitrarily.* For instance, if $x_1 + x_2 + x_3 = 15$ and we assign any values of two of the variables (say : x_1, x_2), then the values of x_3 will be known. The two variables are therefore, free and independent choices for finding the third. Hence these are the degrees of freedom. *If there are n observations, the degrees of freedom (d.f.) are (n – 1).* In other words, while finding the mean of a small sample, one degree of freedom is used up and $(n - 1)$ *d.f.* are left to estimate the population variance.

27.14 (1) STUDENT'S t-DISTRIBUTION

Consider a small sample of size n, drawn from a normal population with mean μ and s.d. σ. If $\bar{x}$ and σ_s be the sample mean and s.d., then the statistic, 't' is defined as

$$t = \frac{\bar{x} - \mu}{\sigma}\sqrt{n} \quad \text{or} \quad t = \frac{\bar{x} - \mu}{\sigma_s}\sqrt{(n-1)},$$

where $\nu = n - 1$ denotes the *df.* of t. If we calculate t for each sample, we obtain the sampling distribution for t. This distribution known as *Student's t-distribution**, is given by

$$y = \frac{y_0}{(1 + t^2/\nu)^{(\nu+1)/2}} \qquad ...(1)$$

where y_0 is constant such that the area under the curve is unity.

(2) Properties of t-distribution.

I. This curve is symmetrical about the line $t = 0$, like the normal curve, since only even powers of t appear in (1). But it is more peaked than the normal curve with the same S.D. The t-curve approaches the horizontal axis less rapidly than the normal curve. Also t-curve attains its maximum value at $t = 0$ so that its mode coincides with the mean. (Fig. 27.2)

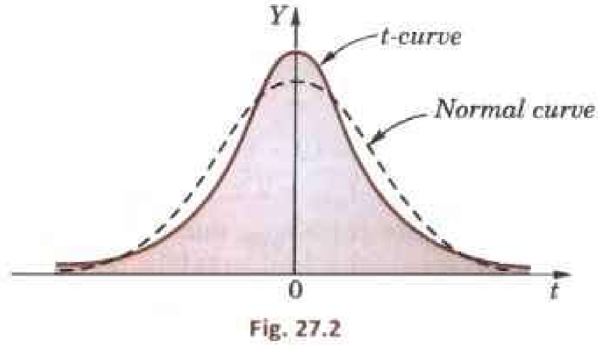


Fig. 27.2

*This distribution was first found by the English statistician *W.S. Gosset* in 1908 who wrote under the pen-name of '*Student*'. *R.A. Fisher* defined t correctly and found its distribution in 1926.

II. The limiting form of t-distribution when $\nu \to \infty$ is given by $y = y_0 e^{-\frac{1}{2}t^2}$ which is a normal curve. This shows that *t is normally distributed for large samples.*

III. The probability P that the value of t will exceed t_0 is given by

$$P = \int_{t_0}^{\infty} y\,dx$$

The values of t_0 have been tabulated for various values of P for various values of ν from 1 to 30 (Table IV – Appendix 2).

IV. Moments about the mean

All the moments of odd order about the mean are zero, due to its symmetry about the line $t = 0$.

Even order moments about the mean are

$$\mu_2 = \frac{\nu}{\nu - 2},\ \mu_4 = \frac{3\nu^2}{(\nu-2)(\nu-4)},\ldots$$

The t-distribution is often used in tests of hypothesis about the mean when the population standard deviation σ is unknown.

27.15 SIGNIFICANCE TEST OF A SAMPLE MEAN

Given a random small sample $x_1, x_2, x_3 \ldots x_n$ from a normal population, we have to test the hypothesis that mean of the population is μ. For this, we first calculate $t = (\bar{x} - \mu)\sqrt{n}/\sigma_s$

where $$\bar{x} = \frac{1}{n}\sum_1^n x_i,\ \sigma_s^2 = \frac{1}{n-1}\sum_1^n (x_i - \bar{x})^2$$

Then find the value of P for the given df from the table.

If the calculated value of $t > t_{0.05}$, the difference between $\bar{x}$ and μ is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1% level of significance.

If $t < t_{0.05}$, the data is said to be consistent with the hypothesis that μ is the mean of the population.

Example 27.11. *A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressure : 5, 2, 8, – 1, 3, 0, – 2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure.* (V.T.U., 2007)

Solution. Let us assume that the stimulus administered to all the 12 patients will increase the B.P. Taking the population to be normal with mean $\mu = 0$ and S.D. σ,

$$\bar{d} = \frac{5+2+8-1+3+0-2+1+5+0+4+6}{12} = 2.583$$

$$\sigma^2 = \frac{\Sigma d^2}{n} - \bar{d}^2 = \frac{1}{12}\,[5^2 + 2^2 + 8^2 + (-1^2) + 3^2 + 0^2 + (-2)^2 + 1^2 + 5^2 + 0^2 + 4^2 + 6^2] - (2.583)^2$$

$$= 8.744. \quad \therefore \quad \sigma = 2.9571$$

Now $$t = \frac{\bar{d} - \mu}{\sigma_s}\sqrt{(n-1)} = \frac{2.583 - 0}{2.9571}\sqrt{(12-1)} = 2.897$$

Here *d.f.* $\gamma = 12 - 1 = 11$.

For $\gamma = 11$, $t_{0.05} = 2.2$, from table IV.

Since the $|t| > t_{0.05}$, our assumption is rejected *i.e.*, the stimulus does not increase the B.P.

Example 27.12. *The nine items of a sample have the following values : 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these differ significantly from the assumed mean of 47.5 ?* (V.T.U., 2010)

Solution. We find the mean and standard deviation of the sample as follows :

x	$d = x - 48$	d^2
45	−3	9
47	−1	1
50	2	4
52	2	4
48	0	0
47	−1	1
49	1	1
53	5	25
51	3	9
Total	10	66

$$\therefore \qquad \bar{x} = \text{mean} = 48 + \frac{\Sigma d}{9} = 48 + \frac{10}{9} = 49.1$$

$$\sigma_s^{\,2} = \frac{\Sigma d^2}{9} - \left(\frac{\Sigma d}{9}\right)^2 = \frac{66}{9} - \frac{100}{81} = \frac{494}{81}$$

$$\therefore \qquad \sigma_s = 2.47$$

Hence $$t = \frac{\bar{x} - \mu}{\sigma_s}\sqrt{(n-1)} = \frac{49.1 - 47.5}{2047}\sqrt{8} = 1.83$$

Here $d.f.\ \nu = 9 - 1 = 8$

For $\nu = 8$, we get from table IV, $t_{0.05} = 2.31$.

As calculated value of $t < t_{0.05}$, the value of t is not significant at 5% level of significance which implies that there is no significant difference between $\bar{x}$ and μ. Thus the test provides no evidence against the population mean being 47.5.

Example 27.13. *A machinist is making engine parts with axle diameter of 0.7 inch. A random sample of 10 parts shows mean diameter 0.742 inch with a standard deviation of 0.04 inch. On the basis of this sample, would you say that the work is inferior ?* (V.T.U., 2009)

Solution. Here we have $\mu = 0.700$, $\bar{x} = 0.742$, $\sigma_x = 0.040$, $n = 10$.

Taking the hypothesis that the product is not inferior *i.e.*, there is no significant difference between $\bar{x}$ and μ.

$$\therefore \qquad t = \frac{\bar{x} - \mu}{\sigma_x}\sqrt{(n-1)} = \frac{0.742 - 0.700}{0.040}\sqrt{(10-1)} = \frac{0.126}{0.040} = 3.16$$

Degrees of freedom $\rho = 10 - 1 = 9$.

For $\rho = 9$, we get from table IV, $t_{0.05} = 2.262$.

As the calculated value of $t > t_{0.05}$, the value of t is significant at 5% level of significance. This implies that $\bar{x}$ differs significantly from μ and the hypothesis is rejected. Hence the work is inferior. In fact, the work is inferior even at 2% level of significance.

Example 27.14. *Show that 95% confidence limits for the mean μ of the population are $\bar{x} \pm \frac{\sigma_s}{\sqrt{n}} t_{0.05}$. Deduce that for a random sample of 16 values with mean 41.5 inches and the sum of the squares of the deviations from the mean 135 inches² and drawn from a normal population, 95% confidence limits for the mean of the population are 39.9 and 43.1 inches.*

Solution. (*a*) The probability P that $t \le t_{0.05}$ is 0.95. Hence the 95% confidence limits for μ are given by

$$\left| \frac{\bar{x} - \mu}{\sigma_s}\sqrt{n} \right| \le t_{0.05}$$

or $$|\bar{x} - \mu| \le \frac{\sigma_s}{\sqrt{n}} t_{0.05} \quad \text{or} \quad \bar{x} - \frac{\sigma_s}{\sqrt{n}} t_{0.05} \le \mu \le \bar{x} + \frac{\sigma_s}{\sqrt{n}} t_{0.05}$$

We can, therefore, say with a confidence coefficient 0.95 that the confidence interval $\bar{x} \pm \frac{\sigma_s}{\sqrt{n}} t_{0.05}$ contains the population mean μ.

(*b*) Here, $n = 16$, $\nu = n - 1 = 15$, $\sigma_s = \sqrt{\frac{135}{15}} = 3$.

Also from table IV, $t_{0.05}$ (for $\nu = 15$) $= 2.13$

$$\therefore \qquad \frac{\sigma_s}{\sqrt{n}} t_{0.05} = \frac{3}{4} \times 2.13 = 1.6 \text{ approx.}$$

Hence the required confidence limits are 41.5 ± 1.6 *i.e.*, 39.9 and 43.1 inches.

27.16 SIGNIFICANCE TEST OF DIFFERENCE BETWEEN SAMPLE MEANS

Given two independent samples $x_1, x_2, x_3, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ with means $\bar{x}$ and $\bar{y}$ and standard deviations σ_x and σ_y from a normal population with the same variance, we have to test the hypothesis that the population means μ_1 and μ_2 are the same.

For this, we calculate
$$t = \frac{\bar{x} \sim \bar{y}}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad \ldots(1)$$

where
$$\bar{x} = \frac{1}{n_1}\sum_1^{n_1} x_i, \quad \bar{y} = \frac{1}{n_2}\sum_1^{n_2} y_i$$

and
$$\sigma_s^2 = \frac{1}{n_1 + n_2 - 2}\left[(n_1 - 1)\sigma_x^2 + (n_2 - 1)\sigma_y^2\right] = \frac{1}{n_1 + n_2 - 2}\left\{\sum_1^{n_1} (x_i - \bar{x})^2 + \sum_1^{n_2} (y_j - \bar{y})^2\right\}$$

It can be shown that the variate t defined by (1) follows the t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

If the calculated value of $t > t_{0.05}$, the difference between the sample means is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1% level of significance.

If $t < t_{0.05}$, the data is said to be consistent with the hypothesis, that $\mu_1 = \mu_2$.

Cor. If the two samples are of the same size and the data are paired, then t is defined by

$$t = \frac{\bar{d} - 0}{(\sigma/\sqrt{n})} \quad \text{where} \quad \sigma^2 = \frac{1}{n-1}\sum_1^{n} (d_i - \bar{d})^2$$

d_i = difference of the ith members of the samples ;

$\bar{d}$ = mean of the differences $= \Sigma d_i/n$; and the number of $d.f. = n - 1$.

Example 27.15. *Eleven students were given a test in statistics. They were given a month's further tuition and a second test of equal difficulty was held at the end of it. Do the marks give evidence that the students have benefitted by extra coaching ?*

Boys :	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
Marks I test :	*23*	*20*	*19*	*21*	*18*	*20*	*18*	*17*	*23*	*16*	*19*
Marks II test :	*24*	*19*	*22*	*18*	*20*	*22*	*20*	*20*	*23*	*20*	*17*

(*V.T.U., 2011 S*)

Solution. We compute the mean and the S.D. of the difference between the marks of the two tests as under :

$$\bar{d} = \text{mean of } d\text{'s} = \frac{11}{11} = 1 \; ; \; \sigma_s^{-2} = \frac{\Sigma(d - \bar{d})^2}{n-1} = \frac{50}{10} = 5 \quad i.e., \quad \sigma_s = 2.24$$

Assuming that the students have not been benefited by extra coaching, it implies that the mean of the difference between the marks of the two tests is zero *i.e.*, $\mu = 0$.

Then $$t = \frac{\bar{d} - \mu}{\sigma_s}\sqrt{n} = \frac{1 - 0}{2.24}\sqrt{11} = 1.48 \text{ nearly and } df\ \nu = 11 - 1 = 10.$$

Students	x_1	x_2	$d = x_2 - x_1$	$d - \bar{d}$	$(d - \bar{d})^2$
1	23	24	1	0	0
2	20	19	–1	–2	4
3	19	22	3	2	4
4	21	18	–3	–4	16
5	18	20	2	1	1
6	20	22	2	1	1
7	18	20	2	1	1
8	17	20	3	2	4
9	23	23	—	–1	1
10	16	20	4	3	9
11	19	17	–2	–3	9
			$\Sigma d = 11$		$\Sigma(d - \bar{d})^2 = 50$

From table IV, we find that $t_{0.05}$ (for $\nu = 10$) = 2.228. As the calculated value of $t < t_{0.05}$, the value of t is not significant at 5% level of significance *i.e.*, the test provides no evidence that the students have benefited by extra coaching.

Example 27.16. *From a random sample of 10 pigs fed on diet A, the increases in weight in a certain period were 10, 6, 16, 17, 13, 12, 8, 14, 15, 9 lbs. For another random sample of 12 pigs fed on diet B, the increases in the same period were 7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 lbs. Test whether diets A and B differ significantly as regards their effect on increases in weight ?*

Solution. We calculate the means and standard deviations of the samples as follows :

	Diet A			*Diet B*	
x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	y_i	$y_i - \bar{y}$	$(y_i - \bar{y})^2$
10	–2	4	7	–8	64
6	–6	36	13	–2	4
16	4	16	22	7	49
17	5	25	15	0	0
13	1	1	12	–3	9
12	0	0	14	–1	1
8	–4	16	18	3	9
14	2	4	8	–7	49
15	3	9	21	6	36
9	–3	9	23	8	64
			10	–5	25
			17	2	4
120	0	120	180	0	314

$$\bar{x} = \frac{120}{10} = 12 \text{ lbs.}, \quad \bar{y} = \frac{180}{12} = 15 \text{ lbs.}$$

$$\sigma_s^2 = [\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2]/(n_1 + n_2 - 2)$$

$$= (120 + 314)/(10 + 12 - 2) = (434/20) = 21.1$$

$\therefore$ $$\sigma_s = 4.65$$

Assuming that the samples do not differ in weight so far as the two diets are concerned *i.e.*, $\mu_1 - \mu_2 = 0$.

Hence $$t = \frac{(\bar{y} - \bar{x}) - 0}{\sigma_s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{15 - 12}{4.65\sqrt{\left(\frac{1}{10} + \frac{1}{12}\right)}} = \frac{3}{4.65}\sqrt{\frac{120}{22}} = 1.6 \text{ nearly}$$

Here $d.f.\ \nu = n_1 + n_2 - 2 = 10 + 12 - 2 = 20.$

For $\nu = 20$, we find $t_{0.05} = 2.09$ [From table IV

$\therefore$ the calculated value of $t < t_{0.05}$.

Hence the difference between the sample means is not significant *i.e.*, the two diets do not differ significantly as regards their effect on increase in weight.

PROBLEMS 27.3

1. Find the student's t for the following variable values in a sample of eight : $-4, -2, -2, 0, 2, 2, 3, 3$; taking the mean of the universe to be zero.

2. A random sample of 10 boys had the following I.Q. :

   70, 120, 110, 101, 88, 83, 95, 98, 107, 100.

   Do these data support the assumption of a population mean I.Q. of 100 (at 5% level of significance) ? (*V.T.U., 2006 ; Coimbatore, 2001*)

3. A sample of 10 measurements of the diameter of a sphere gave a mean of 12 cm and a standard deviation 0.15 cm. Find 95% confidence limits for the actual diameter.

4. A random sample of size 25 from a normal population has the mean $\bar{x} = 47.5$ and s.d. $S = 8.4$. Does this information refute the claim that the mean of the population is $\mu = 42.1$. (*J.N.T.U., 2003*)

5. A process for making certain bearings is under control if the diameter of the bearings have the mean 0.5 cm. What can we say about this process if a sample of 10 of these bearings has a mean diameter of 0.506 cm. and s.d. of 0.004 cm ?

6. A machine is supposed to produce washers of mean thickness 0.12 cm. A sample of 10 washers was found to have a mean thickness of 0.128 cm and standard deviation 0.008. Test whether the machine is working in proper order at 5% level of significance.

7. Find out the reliability of the sample mean of the following data : *Breaking strength of 10 specimens of 1.04 cms diameter hard-drawn copper wire :*

Specimen	1	2	3	4	5	6	7	8	9	10
Breaking Strength (kgs)	578	572	570	568	572	570	570	572	526	584

8. Test runs with 6 models of an experimental engine showed that they operated for 24, 28, 21, 23, 32 and 22 minutes with a gallon of fuel. If the probability of a Type I error is at the most 0.01, is this evidence against a hypothesis that on the average this kind of engine will operate for atleast 29 minutes per gallon of the same fuel. Assume normality. (*J.N.T.U., 2003*)

9. Two horses A and B were tested according to the time (in seconds) to run a particular race with the following results :

Horse A :	28	30	32	33	33	29	and 34
Horse B :	29	30	30	24	27	and 29	

   Test whether you can discriminate between two horses ? (*Rohtak, 2005 ; Coimbatore, 2001*)

10. A group of 10 rats fed on a diet A and another group of 8 rats fed on a different diet B, recorded the following increase in weights :

Diet A :	5	6	8	1	12	4	3	9	6	10	gm
Diet B :	2	3	6	8	10	1	2	8	gm.		

   Does it show that superiority of diet A over that of B ? (*Madras, 2003*)

11. A group of boys and girls were given an intelligence test. The mean score, S.D.s and numbers in each group are as follows :

	Boys	*Girls*
Mean	124	121
S.D.	12	10
n	18	14

   Is the mean score of boys significantly different from that of girls ?

12. The means of two random samples of sizes 9 and 7 are 196.42 and 198.82 respectively. The sum of the squares of the deviations from the mean are 26.94 and 18.73 respectively. Can the sample be considered to have been drawn from the same normal population ? (*Mumbai, 2004*)

27.17 (1) CHI-SQUARE (χ^2) TEST

When a fair coin is tossed 80 times, we expect from theoretical considerations that heads will appear 40 times and tail 40 times. But this never happens in practice *i.e.*, the results obtained in an experiment do not agree exactly with the theoretical results. *The magnitude of discrepancy between observation and theory is given by the quantity* χ^2 (pronounced as *chi-square*). If $\chi^2 = 0$, the observed and theoretical frequencies completely agree. As the value of χ^2 increases, the discrepancy between the observed and theoretical frequencies increases.

(1) Definition. *If $O_1, O_2, ..., O_n$ be a set of observed (experimental) frequencies and $E_1, E_2, ..., E_n$ be the corresponding set of expected (theoretical) frequencies, then χ^2 is defined by the relation*

$$\chi^2 = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + ... + \frac{(O_n - E_n)^2}{E_n}$$

$$= \sum \frac{(O_i - E_i)^2}{E_i} \qquad ...(1)$$

with $n - 1$ degrees of freedom. $[\Sigma O_i = \Sigma E_i = n$ the total frequency$]$

(2) Chi-square distribution*

If $x_1, x_2, ..., x_n$ be n independent normal variates with mean zero and s.d. unity, then it can be shown that $x_1^2 + x_2^2 + ... + x_n^2$, is a random variate having χ^2-distribution with *ndf*.

The equation of the χ^2-curve is

$$y = y_0 e^{-\chi^2/2} (\chi^2)^{(\nu - 1)/2} \qquad ...(2)$$

where $\nu = n - 1$ (Fig. 27.3).

(3) Properties of χ^2-distribution

I. If $\nu = 1$, the χ^2-curve (2) reduces to $y = y_0 e^{-\chi^2/2}$, which is the exponential distribution.

II. If $\nu > 1$, this curve is tangential to x-axis at the origin and is positively skewed as the mean is at ν and mode at $\nu - 2$.

III. The probability P that the value of χ^2 from a random sample will exceed χ_0^2 is given by

$$P = \int_{\chi_0^2}^{\infty} y\, dx.$$

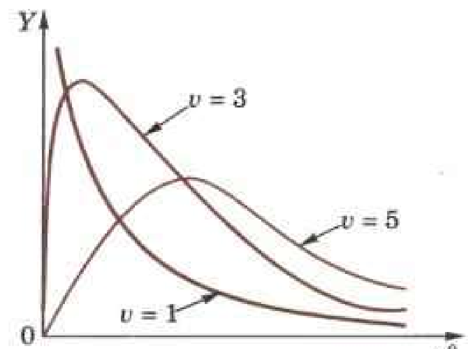


Fig. 27.3

The values of χ_0^2 have been tabulated for various values of P and for values of ν from 1 to 30. (Table-V-Appendix 2)

For $\nu > 30$, the χ^2-curve approximates to the normal curve and we should refer to normal distribution tables for significant values of χ^2.

IV. Since the equation of χ^2-curve does not involve any parameters of the population, *this distribution does not depend on the form of the population* and is therefore, very useful in a large number of problems.

V. Mean $= \gamma$ and *variance* $= 2\gamma$.

27.18 GOODNESS OF FIT

The value of χ^2 is used to test whether the deviations of the observed frequencies from the expected frequencies are significant or not. It is also used to test how will a set of observations fit a given distribution, χ^2 therefore, provides a test of goodness of fit and may be used to examine the validity of some hypothesis about an observed frequency distribution. As a test of goodness of fit, it can be used to study the correspondence between theory and fact.

This is a non-parametric distribution-free test since in this we make no assumption about the distribution of the parent population.

**Hamlet* discovered this distribution in 1875. *Karl Pearson* rediscovered it independently in 1900 and applied it to test 'goodness of fit'.

Procedure to test significance and goodness of fit.

(*i*) Set up a 'null hypothesis' and calculate

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

(*ii*) Find the *df* and read the corresponding values of χ^2 at a prescribed significance level from Table V.

(*iii*) From χ^2-table, we can also find the probability P corresponding to the calculated values of χ^2 for the given *d.f.*

(*iv*) If $P < 0.05$, the observed value of χ^2 is significant at 5% level of significance.

If $P < 0.01$, the value is significant at 1% level.

If $P > 0.05$, it is a good fit and the value is not significant.

Example 27.17. *In experiments on pea breeding, the following frequencies of seeds were obtained :*

Round and yellow	*Wrinkled and yellow*	*Round and green*	*Wrinkled and green*	*Total*
315	*101*	*108*	*32*	*556*

Theory predicts that the frequencies should be in proportions 9 : 3 : 3 : 1. Examine the correspondence between theory and experiment.

Solution. The corresponding frequencies are

$$\frac{9}{16} \times 556, \frac{3}{16} \times 556, \frac{3}{16} \times 556, \frac{1}{16} \times 556 \text{ i.e., } 313, 104, 104, 35.$$

Hence
$$\chi^2 = \frac{(315-313)^2}{313} + \frac{(101-104)^2}{104} + \frac{(108-104)^2}{104} + \frac{(32-35)^2}{35}$$

$$= \frac{4}{313} + \frac{9}{104} + \frac{16}{104} + \frac{9}{35} = 0.51 \quad \text{and} \quad df\, \nu = 4 - 1 = 3.$$

For $\nu = 3$, we have $\chi^2_{0.05} = 7.815$ [From Table V

Since the calculated value of χ^2 is much less than $\chi^2_{0.05}$, there is a very high degree of agreement between theory and experiment.

Example 27.18. *A set of five similar coins is tossed 320 times and the result is*

No. of heads :	*0*	*1*	*2*	*3*	*4*	*5*
Frequency :	*6*	*27*	*72*	*112*	*71*	*32*

Test the hypothesis that the data follow a binomial distribution.

(*Kottayam, 2005 ; P.T.U., 2005 ; V.T.U., 2004*)

Solution. For $\nu = 5$, we have $\chi^2_{0.05} = 11.07$.

p, probability of getting a head $= \frac{1}{2}$; q, probability of getting a tail $= \frac{1}{2}$.

Hence the theoretical frequencies of getting 0, 1, 2, 3, 4, 5 heads are the successive terms of the binomial expansion $320(p+q)^5$

$$= 320[p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5]$$

$$= 320\left[\frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32}\right] = 10 + 50 + 100 + 100 + 50 + 10$$

Thus the theoretical frequencies are 10, 50, 100, 100, 50, 10.

Hence

$$\chi^2 = \frac{(6-10)^2}{10} + \frac{(27-50)^2}{50} + \frac{(72-100)^2}{100} + \frac{(112-100)^2}{100} + \frac{(71-50)^2}{50} + \frac{(32-10)^2}{10}$$

$$= \frac{1}{100}(160 + 1058 + 784 + 144 + 882 + 4840) = \frac{7868}{100} = 78.68$$

and $\quad df\, \nu = 6 - 1 = 5.$

Since the calculated value of χ^2 is much greater than $\chi^2_{0.05}$, the hypothesis that the data follow the binomial law is rejected.

Example 27.19. *Fit a Poisson distribution to the following data and test for its goodness of fit at level of significance 0.05.*

x :	0	1	2	3	4
f :	419	352	154	56	19

$$\therefore \qquad \chi^2 = \frac{(5-4.13)^2}{4.13} + \frac{(18-20.68)^2}{20.68} + \frac{(42-38.92)^2}{38.92} + \frac{(27-27.71)^2}{27.71} + \frac{(8-7.43)^2}{7.43}$$
$$= 0.1833 + 0.3473 + 0.2437 + 0.0182 + 0.0437 = 0.8362$$

As regards the number of degrees of freedom (γ), there are three constraints (*i*) discrepancy between total observed and total estimated frequencies (*ii*) and (*iii*) mean (m) and standard deviation (σ) have been estimated from the sample data. $\therefore \quad r = 5 - 3 = 2.$

For $\gamma = 2$, $\chi^2_{0.05} = 0.103$ from table V.

Since $\chi^2 = 0.8362 > 0.103$. Hence the fit is not good.

PROBLEMS 27.4

1. Five dice were thrown 96 times and the number of times 4, 5 or 6 were thrown were :

No. of dice showing 4, 5 or 6 :	5	4	3	2	1	0
Frequency :	8	18	35	24	10	1

Find the probability of getting this result by chance ?

2. Genetic theory states that children having one parent of blood type M and other blood type N will always be one of the three types M, MN, N and that the proportions of these types will on average be 1 : 2 : 1. A report states that out of 300 children having one M parent and one N parent, 30% were found to be of type M, 45% of type MN and remainder of type N. Test the hypothesis by χ^2 test.

3. A die was thrown 60 times and the following frequency distribution was observed :

Faces :	1	2	3	4	5	6
f_0 :	15	6	4	7	11	17

Test whether the die is unbiased ?

4. The following table gives the number of aircraft accidents that occurred during the various days of the week. Find whether the accidents are uniformly distributed over the week ?

Days :	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
No. of accidents :	14	16	8	12	11	9	14	84

(*Hissar, 2005*)

5. Fit a binomial distribution to the data :

x :	0	1	2	3	4	5
f :	38	144	342	287	164	25

and test for goodness of fit, at the level of significance 0.05.

6. In 1000 extensive sets of trials for an event of small probability, the frequencies f_0 of the number x of successes proved to be :

x :	0	1	2	3	4	5	6	7
f_0 :	305	366	210	80	28	9	2	1

Fit a Poisson distribution to the data and test the goodness of fit.

7. The frequencies of localities according to the number of deaths due to cholera during eight years in 1000 localities is as follows :

No. of deaths :	0	1	2	3	4	5	6	7
No. of localities :	314	355	204	86	29	9	3	0

Fit a suitable distribution to the data and test the goodness of fit.

8. Obtain the equation of the normal curve that may be fitted to the data and test the goodness of fit.

x :	4	6	8	10	12	14	16	18	20	22	24	Total
$f(x)$:	1	7	15	22	35	43	38	20	13	5	1	200

27.19 (1) F-DISTRIBUTION*

Let $x_1, x_2, \ldots x_{n_1}$ and $y_1, y_2, \ldots y_{n_2}$ be the values of two independent random samples drawn from the normal populations σ^2 having equal variances.

* This distribution was introduced by the English statistician *Prof. R.A. Fisher* (1890–1962) who had greatly influenced the development of modern statistics.

Let $\bar{x}_1$ and $\bar{x}_2$ be the sample means and $s_1^2 = \frac{1}{n_1 - 1}\sum_1^{n_1}(x_i - \bar{x})^2$, $s_2^2 = \frac{1}{n_2 - 1}\sum_1^{n_2}(y_i - \bar{y})^2$ be the sample variances.

Then we define F by the relation

$$F = \frac{s_1^2}{s_2^2} \qquad (s_1^2 > s_2^2)$$

This gives F-distribution (also known as variance ratio distribution) with $\gamma_1 = n_1 - 1$ and $\gamma_2 = n - 2$ degrees of freedom. *The larger of the variances is placed in the numerator.*

(2) Properties. I. The F-distribution curve lies entirely in the first quadrant and is *unimodal*.

II. The F-distribution is independent of the population variance σ^2 and depends on γ_1 and γ_2 only.

III. $F_\alpha(\gamma_1, \gamma_2)$ is the value of F for γ_1 and γ_2 of such that the area to the right of F_α is α.

IV. It can be shown that the mode of F-distribution is less than unity.

(3) Significance test. Snedecor's F-tables give 5% and 1% points of significance for F. (Table VI – Appendix 2). 5% points of F mean that area under the F-curve to the right of the ordinate at a value of F, is 0.05. Clearly value of F at 5% significance is lower than that at 1%. F-distribution is very useful for testing the equality of population means by comparing sample variances. As such it forms the basis of *analysis of variance.*

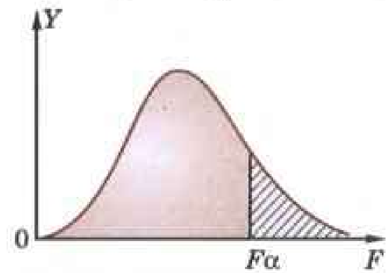


Fig. 27.4

Example 27.21. *Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 inches² and 91 inches² respectively. Can these be regarded as drawn from the same normal population ?* *(V.T.U., 2002)*

Solution. We have $\Sigma(x - \bar{x})^2 = 160$ and $\Sigma(y - \bar{y})^2 = 91$

$\therefore$
$$s_1^2 = \frac{160}{8} = 20$$

and
$$s_2^2 = \frac{91}{7} = 13.$$

Hence
$$F = \frac{s_1^2}{s_2^2} = \frac{20}{13} = 1.54 \text{ nearly.}$$

For $\gamma_1 = 8, \gamma_2 = 7$, we have $F_{0.05} = 3.73$. [From Table VI]

Since the calculated value of $F < F_{0.05}$, the population variances are not significantly different. Thus the two samples can be regarded as drawn from two normal populations with the same variance. If the two populations are to be same, their means should also be the same which can be verified by applying t-test provided the sample means are known.

Example 27.22. *Measurements on the length of a copper wire were taken in 2 experiments A and B as under :*

A's measurements (mm) :	12.29,	12.25,	11.86,	12.13,	12.44,	12.78,	12.77,	11.90,	12.47.
B's measurements (mm) :	12.39,	12.46,	12.34,	12.22,	11.98,	12.46,	12.23,	12.06.	

Test whether B's measurements are more accurate than A's. (The readings taken in both cases being unbiased)

Solution. Readings in both cases being unbiased, B's measurements will be taken more accurate if its population variance is less than that of A's measurements.

Under the hypothesis that the two populations have the same variance (*i.e.* $\sigma_1^2 = \sigma_2^2$), we have

$$F = \frac{s_1^2}{s_2^2}$$

with $\gamma_1 = n_1 - 1 = 8$ and $\gamma_2 = n_2 - 1 = 7$.

We calculate the s.d.'s of the two series as follows :

A's measurements			*B's measurements*		
x	$u = 100(x - 12)$	u^2	y	$v = 100(y - 12)$	v^2
12.29	29	841	12.39	39	1521
12.25	25	625	12.46	46	2116
11.86	– 14	196	12.34	34	1156
12.13	13	169	12.22	22	484
12.44	44	1936	11.98	– 2	4
12.78	78	6084	12.46	46	2116
12.77	77	5929	12.23	23	529
11.90	– 10	100	12.06	6	36
12.47	47	2209			
	289	18089		214	7962

$$\therefore \qquad s_1^2 = \frac{1}{n_1 - 1}\left[18089 - \frac{(289)^2}{n_1}\right] = \frac{1}{8}(18089 - 9280) = 1101.1$$

$$s_2^2 = \frac{1}{n_2 - 1}\left[7962 - \frac{(214)^2}{n_2}\right] = \frac{1}{7}(7962 - 5724) = 319.7$$

$$\therefore \qquad F = \frac{s_1^2}{s_2^2} = \frac{1101.1}{319.7} = 3.44$$

For $\gamma_1 = 8$ and $\gamma_2 = 7$, from table VI, $F_{0.05} = 3.73$ and $F_{0.01} = 6.84$.

Since the calculated value of F is less than both $F_{0.05}$ and $F_{0.01}$, the result is insignificant at both 5% and 1% level.

Hence there is no reason to say that B's measurements are more accurate than those of A's.

27.20 (1) FISHER'S z-DISTRIBUTION

Changing the variable F to z by the substitution $z = \frac{1}{2}\log_e F$ or $F = e^{2z}$ in the F-distribution, we get the *Fisher's z-distribution.*

It is more nearly symmetrical than F-distribution. Table showing the values of z that will be exceeded in simple sampling with probabilities 0.05 and 0.01 have been prepared for various values of v_1 and v_2.

(2) Significance test. As z-table give only critical values corresponding to right hand tail areas, therefore 5% (or 1%) points of z imply that the area to the right of the ordinate at z is 0.05 (or 0.01). In other words, *5% and 1% points of z correspond to 10% and 2% levels of significance respectively.*

Example 27.23. *Two gauge operations are tested for precision in making measurements. One operator completes a set of 26 readings with a standard deviations of 1.34 and the other does 34 readings with a standard deviations of 0.98. What is the level of significance of this difference.*

(Given that for $\nu_1 = 25$ and $\nu_2 = 33$, $z_{0.05} = 0.305$, $z_{0.01} = 0.432$)

Solution. We have $n_1 = 26$, $\sigma_x = 1.34$; $n_2 = 34$, $\sigma_y = 0.98$

$$\therefore \qquad s_1^2 = \frac{n_1}{n_1 - 1}\cdot\sigma_x^2 = \frac{26}{25}(1.34)^2 \approx (1.34)^2 \quad \text{and} \quad s_2^2 = \frac{n_2}{n_2 - 1}\cdot\sigma_y^2 = \frac{34}{33}(0.98)^2 \approx (0.98)^2$$

Hence $\quad F = \left(\frac{1.34}{0.98}\right)^2 = 1.8696 \quad$ and $\quad z = \frac{1}{2}\log_e F = 1.1513 \log_{10} 1.8696 = 0.3129$

Since the calculated value of z is just greater than $z_{0.05}$ and less than $z_{0.01}$, the difference between the standard deviation is just significant at 5% level and insignificant at 1% level.

PROBLEMS 27.5

1. Two samples of 9 and 7 individuals have variances 4.8 and 9.6 respectively. Is the variance 9.6 significantly greater than the variance 4.6 ?
2. Test for breaking strength were carried out on two lots of 5 and 9 steel wires. The variance of one lot was 230 and that of other was 492. Is there a significant difference in their variability ?
3. Show how you would use Fisher's z-test to decide whether the two sets of observations 17, 27, 18, 25, 27, 29, 27, 23, 17 and 16, 16, 20, 16, 20, 17, 15, 21, indicate samples from the same universe.
4. In two groups of ten children each, the increase in weight due to different diets during the same period, were in pounds

   3, 7, 5, 6, 5, 4, 4, 5, 3, 6
   8, 5, 7, 8, 3, 2, 7, 6, 5, 7.

   Is there a significant difference in their variability ?
5. The mean diameter of rivets produced by two firms A and B are practically the same but their standard deviations are different. For 16 rivets manufactured by firm A, the S.D. is 3.8 mm while for 22 rivets manufactured by firm B is 2.9 mm. Do you think products of firm B are of better quality than those of firm A.
6. The I.Q.'s of 25 students from one college showed a variance of 16 and those of an equal number from the other college had a variance of 8. Discuss whether there is any significant difference in variability of intelligence.
7. Two random samples from two normal populations are given below :

Sample I :	16	26	27	23	24	22
Sample II :	33	42	35	32	28	31

   Do the estimates of population variances differ significantly ?

Degrees of freedom :	(5, 5)	(5, 6)	(6, 5)
5% value of F :	5.05	4.39	4.95.

8. Two independent samples of sizes 7 and 6 have the following values :

Sample A :	28	30	32	33	33	29	34
Sample B :	29	30	30	24	27	29	

   Examine whether the samples have been drawn from normal populations having the same variance ?
   [Given that the values of F at 5% level for (6, 5) d.f. is 4.95 and for (5, 6) d.f. is 4.39].

27.21 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 27.6

Select the correct answer or fill up the blanks in each of the following questions :

1. The 'null hypothesis' implies that ....
2. The uses of t-distribution are ....
3. Type I and type II errors are such that ....
4. A single-tailed test is used when ....
5. Control limit theorem states that ....
6. A hypothesis is true, but is rejected. Then this is an error of type ....
7. If the standard deviation of a χ^2 distribution is 10, then its degree of freedom is ....
8. Range of F-distribution is ....
9. A hypothesis is false but accepted, then there is an error of type ....
10. The mean and variance of a χ^2 distribution with 8 degrees of freedom are .... and .... respectively.
11. In a t-distribution of sample size n, the degrees of freedom are ....
12. The test statistic $F = \dfrac{s_1^2}{s_2^2}$ is used when

    (*i*) $s_2^2 > s_1^2$ (*ii*) $s_2^2 < s_1^2$ (*iii*) $s_1^2 = s_2^2$ (*iv*) none of these.
13. The t-test is applicable to samples for which n is ....
14. The two main uses of χ^2-test are ....
15. Range of t-distribution is ....
16. If two samples are taken from two populations of unequal variances, we can apply t-test to test the difference of means. (True or False)
17. The Chi-square distribution is continuous. (True or False)