Global Dynamics of SIR Model with Switched Transmission Rate

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Abstract

We propose a new epidemiological model, based on the classical SIR model, taking additionally into account a switching prevention strategy. After a disease begins to spread the general public will begin to take action, which will reduce the transmission rate. However, as the threat of epidemics becomes less severe these preventative measures may be revoked by various reasons and so the transmission rate increases again. We model this phenomenon by introducing a switched system within the classical SIR model. Our results indicate that if preventative measures are introduced or taken away before certain conditions are met the disease will continue to linger or the epidemics can recur after the intervention.

The Classical SIR Model

The population consists of: susceptible individuals (S) who can contract the disease; infected individuals (I); and, recovered individuals (R) who are immune to the disease (Agur et. al. 1993).

$$\frac{dS}{dt} = \mu N - \frac{\beta SI}{N} - \mu S,$$

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I - \mu I, \qquad (1)$$

$$\frac{dR}{dt} = \gamma I - \mu R,$$

N the population size, β the transmission rate, γ the recovery rate and μ the birth/mortality rate. The equality of the birth and mortality rates ensures the conservation of the population size.

$$S + I + R = N \Rightarrow R = N - S - I$$

This assumption reduces the system (1) to the first two equations.

$$\frac{dS}{dt} = -\beta IS - \mu S + \mu,$$

$$\frac{dI}{dt} = \beta IS - (\gamma + \mu)I \qquad (2)$$

The Modified Model

The key parameter of model (2) is called *basic reproduction number* defined as

$$R_0 = \beta/(\gamma + \mu).$$

- If $R_0 < 1$, the infection dies out in the long run;
- If $R_0 > 1$, the infection spreads in the population.

Rescaling the time leads to the equivalent normalized system

$$\frac{dS}{dt} = -R_0 IS - \alpha S + \alpha,$$

$$\frac{dI}{dt} = R_0 IS - I, \qquad (3)$$

with $\alpha = \mu/(\gamma + \mu) < 1$.

Switching Conditions

One problem with the classical system is that it does not account for adaptive behavior of the population. For instance, public action reduces transmission rate and revoking preventative measures as threat subsides increase transmission rate. Hence, we propose a switch system to account for this change in transmission rate.

$$R_0(t) = \begin{cases} R_0^{nat} & \text{if either } I(\tau) < I_{int} \text{ for all } \tau \in [0,t] \\ & \text{or there is } t_1 \in [0,t] \text{ such that } I(t_1) \le I_{nat} \\ & \text{and } I(\tau) < I_{int} \text{ for all } \tau \in (t_1, t]; \\ R_0^{int} & \text{if there is } t_1 \in [0,t] \text{ such that } I(t_1) \ge I_{int} \\ & \text{and } I(\tau) > I_{nat} \text{ for all } \tau \in (t_1, t] \end{cases} \qquad (4)$$

where

$$R_0^{nat} = \frac{\beta_{nat}}{\gamma + \mu}, \qquad R_0^{int} = \frac{\beta_{int}}{\gamma + \mu}.$$

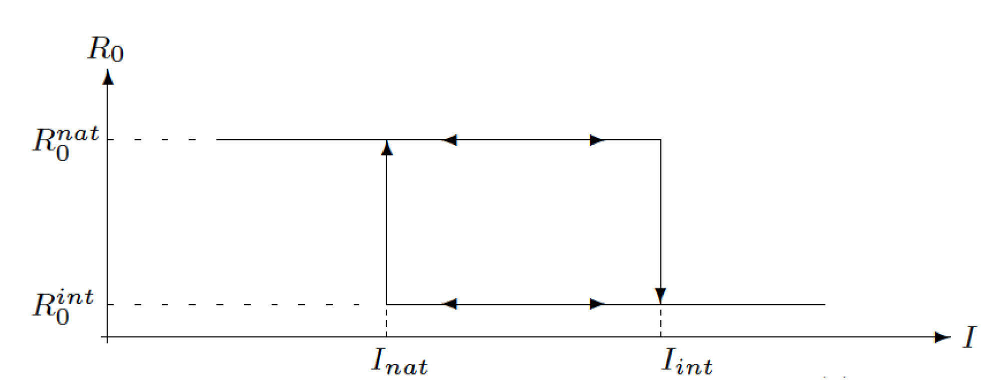

Fig. 1: The non-ideal relay operator

The possible equilibrium for the switch system (3):
- Infection free: $E^* = (0, 1)$
- Endemic: $E = (I_E, S_E)$ where $I_E = \alpha(1 - \frac{1}{R_0^{nat}})$, $S_E = \frac{1}{R_0^{nat}}$

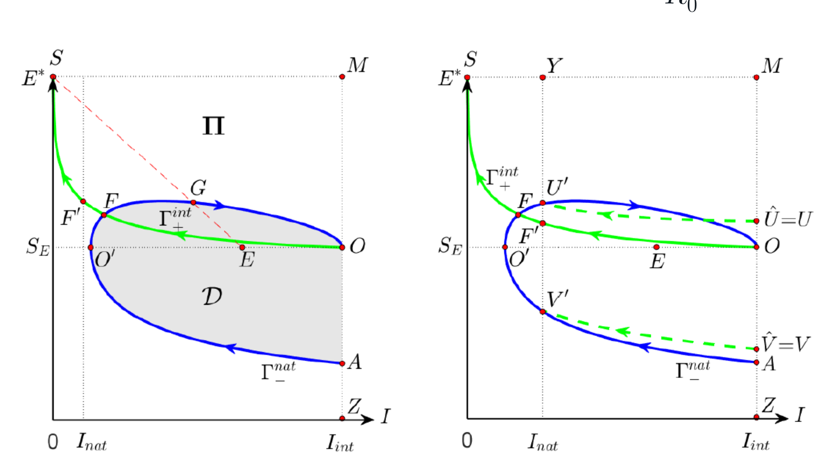

Fig. 2: Trajectories of Γ_+^{int}, and Γ_-^{nat}

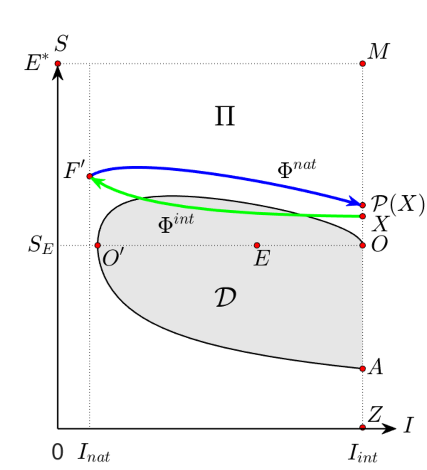

Fig. 3: Poincaré map of Γ_+^{int}, and Γ_-^{nat}

Theorem

Suppose that $S_F > S_E = 1/R_0^{nat}$.Then, the following statements hold:

Case I:

If $I_{nat} < I_F$, then every trajectory of switched system starting outside the domain $\mathcal{D}$ and following initially the vector field Φ^{nat} makes infinitely many switches.

- At least one stable periodic orbit with 2 switches per period
- Poincaré map $\mathcal{P}$ is defined on the whole segment OM

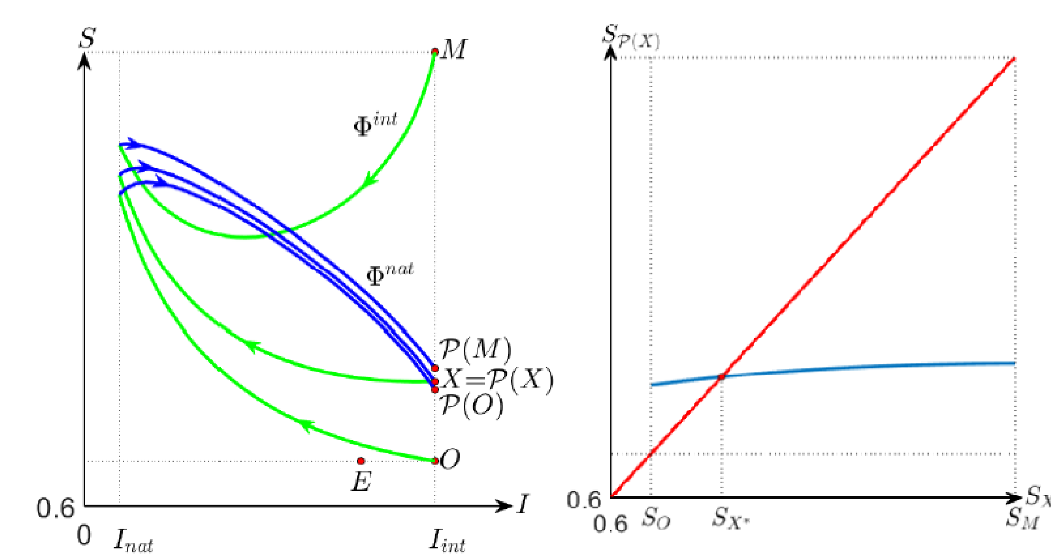

Fig. 4: Periodic Behavior.

Case II:

If $I_{nat} > I_F$, then either all the trajectories of switched system following initially the vector field Φ^{nat} converge to the endemic equilibrium E or there is a periodic trajectory with 2 switches per period.

- Poincaré map $\mathcal{P}$ is defined on the segment $\Omega_{\mathcal{P}} = UM \setminus \{M\}$ and is order preserving on its domain.
- All trajectories of the switched system starting on the segment OU converge to the endemic equilibrium E
- If $I_{nat} > I_G$, then all trajectories of switched system following initially the vector field Φ^{nat} converge to the equilibrium E.

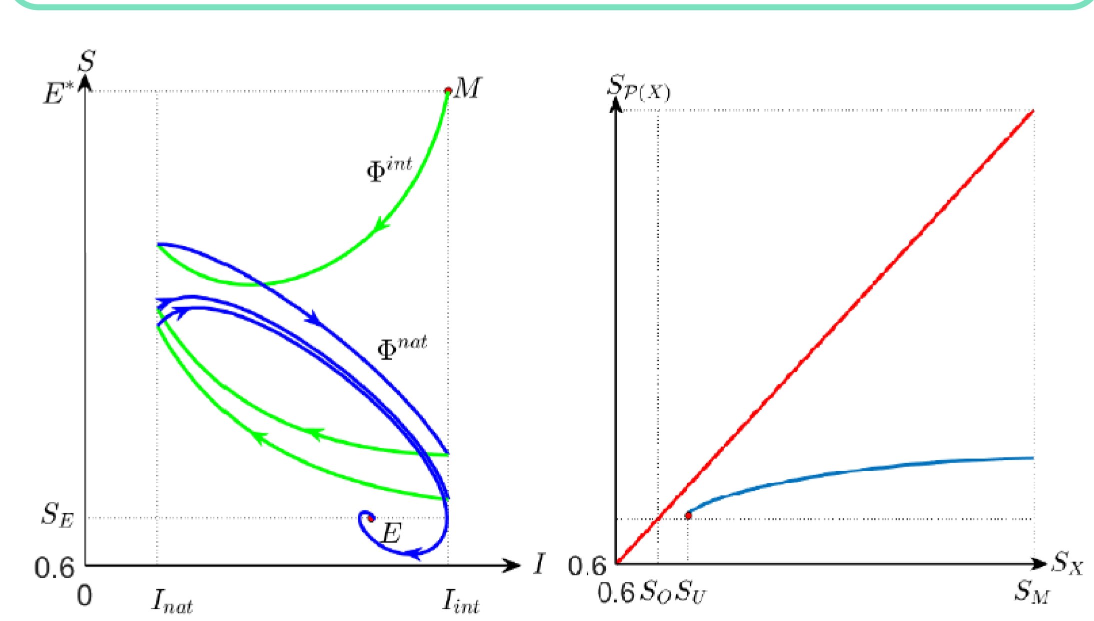

Fig. 5: Convergence to Endemic.

Example: Measles

Parameter	Value	Description
R_0^{nat}	18	Estimates of the basic reproduction number range from 14 to 19 for measles (Bjornstad et al. 2002, Chiew et al. 2014).
α	0.00025	This value for $\alpha = \mu/(\gamma + \mu) \approx \mu/\gamma$ is based on the average life expectancy of $1/\mu = 75$ years and the recovery rate $\gamma = 1$ week^{-1} (Agur et al. 1993).
R_0^{int}	0.3 – 0.8	As a result of the intervention the basic reproduction number drops to a value less than 1. (Chiew et al. 2014)

Fig. 6: Typical Parameters for Measles

For $R_0 \approx 18$ the model predicts the extinction of measles at the immunization rate of about 95%. This suggests that the disease is endemic in low income countries and probably still endemic worldwide.

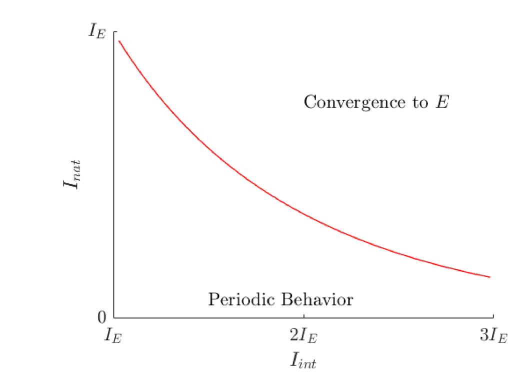

Fig. 7: Relation of pair of thresholds (I_{int}, I_{nat}) for measles

Also, Figure 7 indicates that for a chosen value of I_{int}, the lower threshold I_{nat} should not be set too small else periodic behavior will be witnessed. Hence, optimal conditions would encourage stopping interventions while a small subset of the population is still infected. In this example, the endemic equilibrium corresponds to a population of 0.02% infected individuals. Hence, intervention should be stopped when the infected popultion is within 0.01% to 0.02% i.e. $0.01 < I_{nat} < 0.02$.

Conclusion

We concluded that by choosing appropriate threshold the system can be steered towards the endemic equilibrium. From a public policy stand point this means temporary introduction of preventive measures such as educating the public, school closures, or quarantine via the two threshold method can be effective in confining the scale of the epidemic. However, this method is not effective in eradicating the disease. This is witness by reducing infected population pass I_{nat}, which simultaneously increases the susceptible population. So in the long run this creates favorable conditions for the disease to return.

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