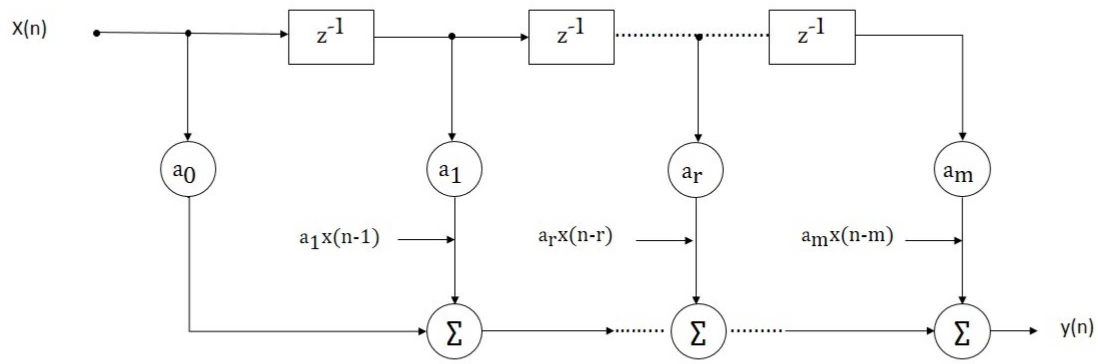
Step By Step Design of Digital Filters

Finite-impulse response (FIR) filters are characterized by the equation,

$$y(x) = a_0x(n) + a_1x(n-1) + a_2x(n-2) + ... + a_mx(n-m)$$

where, $a_0, a_1, ...$ are constants.

The implementation of this equation is as



output $y(n)$ depends only on the present and past values of input $x(n)$. There are no feedback connections from the output to input.

Basic principles of FIR Filter Design

FIR filters are designed by assuming that the magnitude of x-fer function $H(w)$ is unity. i.e,

$$|H(\omega)| = 1$$
$$|H(\omega)| = \left|\frac{y(\omega)}{x(\omega)}\right|$$
$$|y(\omega)| = |x(\omega)|$$

output in frequency domain = input in frequency domain.

i.e, An FIR filters does not introduce any losses to signals that get x-mitted through it.

It must be noted that all types of IIR filters are designed by choosing a suitable and finite value of $absH(w)$ which is less than 1.

$*$ We generally express the magnitude and phase angle of the x-fer function as

$$H(\omega) = Me^{-j\omega n} = Me^{j\theta}$$

where, M (1) is the magnitude and $\theta = \omega n$ represents the phase angle.
Since, ω is a constant, we find that, as time-factor n increases, θ decreases at a constant rate. For this reason FIR filters are called constant-phase filters.

Problem:

Design a Low pass FIR filters for the following specification:
cut-off freq. = 500 Hz
sampling freq. = 2000 Hz
Order of the filter, N = 10
filter length required, L = N+1 = 11

Solution:

Step 1:

Normalization of cut-off frequency: We normalize the cut-off frequency as

$$\omega_c = 2\pi\frac{f_c}{f_s} = 2\pi\frac{500}{2000} = \frac{\pi}{2}$$

Step 2:

Fixing the x-fer function to be used: We fix the transfer function as

$$H(\omega) = \begin{cases} 1, -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2} \\ 0, elsewhere \end{cases}$$

Step 3:

Determine the impulse response of the filter:

From IDFT we get,

$$h(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H(\omega)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi n}\frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j}$$

$$= \frac{1}{\pi n}\sin(\frac{n\pi}{2})$$

We can also express this as,

$$h(n) = \frac{1}{2}\frac{\sin(n\pi/2)}{(n\pi/2)} = \frac{1}{2}\sin c(\frac{n\pi}{2}) \tag{1}$$

Step 4:

Determining the co-efficient the impulse response sequence :

Substitute various values of n and determine the corresponding values of h(n).
Thus, for n=0, we have,

$h(0) = \frac{1}{2}\frac{\sin 0}{0} = \frac{1}{2} = 0.5$; [L's Hospital's rules, $\frac{\sin\theta}{\theta} = \frac{\sin 0}{0}$=1]

for n=1, we have,

$h(1) = \frac{1}{2}\frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi} = 0.3183$

for n=2,

$h(2) = \frac{1}{2}\frac{\sin(\pi)}{\pi} = 0$

for n=3,

$h(3) = \frac{1}{2}\frac{\sin(1.5\pi)}{1.5\pi} = \frac{-1}{1.5\pi} = \frac{-1}{3\pi} = -0.1061$

for n=4,

$h(4) = 0$

for n=5,

$h(5) = \frac{1}{2}\frac{\sin(2.5\pi)}{2.5\pi} = \frac{1}{5\pi} = 0.0637$

We stop our computation at this point, since the required length of the filter L=N+1=11, and we can achieve this length by truncating the number of samples at n=5.

It may be noted that the computation is symmetrical about the origin, and we find

$$h(n) = h(-n)$$

[For example, we have computed the value of $h(1) = \frac{1}{\pi}$, Then we infer that $h(-1) = h(1) = \frac{1}{\pi}$. Similarly, $h(-1) = h(1) = 0$ and so on.]

Thus impulse response,

$h(n) = (\frac{1}{5\pi}, 0, -\frac{1}{3\pi}, 0, \frac{1}{\pi}, \frac{1}{2}, \frac{1}{\pi}, 0, -\frac{1}{3\pi}, 0, \frac{1}{5\pi})$

<u>Step 5:</u>

Determining the x-fer function back from the impulse response sequence:

For this, we write

$$H(z) = \frac{1}{5\pi}.z^5 + 0.z^4 + \frac{-1}{3\pi}.z^3 + 0.z^2 + \frac{1}{\pi}.z^1 + \frac{1}{2} + \frac{1}{\pi}.z^{-1} + 0.z^{-2} + \frac{-1}{3\pi}.z^{-3} + 0.z^{-4} \quad (2)$$

There exist one problem with this computation. The negative value of frequencies (i.e terms containing positive power of z, such as z^1, z^2 etc) indicating that we are going to realize a non-casual filter.

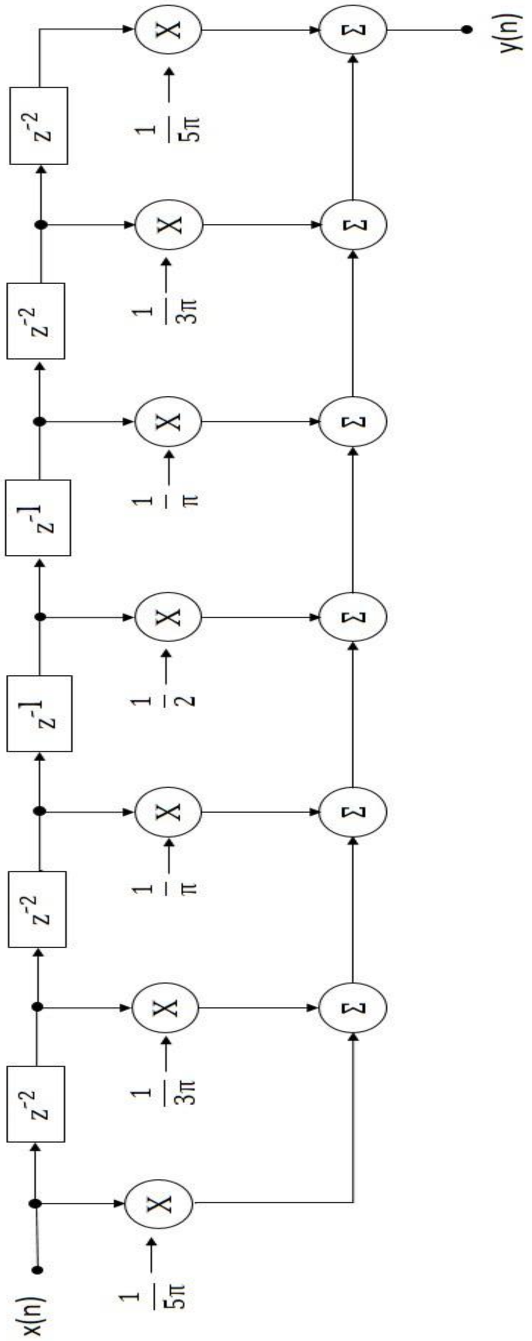
To make the filter casual and physically realizable, we multiply all the co-efficient with an appropriate power of z^{-n} (in this case, with z^{-5}).

$$H(z) = \frac{1}{5\pi} + \frac{-1}{3\pi}.z^{-2} + \frac{1}{\pi}.z^{-4} + \frac{1}{2}.z^{-5} + \frac{1}{\pi}.z^{-6} + \frac{-1}{3\pi}.z^{-8} + \frac{1}{5\pi}.z^{-10} \quad (3)$$

This equation shows the transfer function of a physically realizable low-pass FIR filter.

Step 6:

Implementation of the filter:



Step 7:

Adding more filter co-efficient:

The accuracy of an FIR filter can be increased by increasing the number of filter co-efficient.
It may be noted that this a trian and error procedure.

Assuming that we want to find filter co-efficient upto n, where n is odd, co-efficient in this case follow the pattern $h(\pm n) = (1/n\pi)$.

For example, we have $h(\pm 7) = (1/7\pi), h(\pm 9) = (1/9\pi)$ and so on.

It can also be seen that the even numbered co-efficients are all zeros in this case.

For example, $h(\pm 4) = h(\pm 6) = ... = 0$.

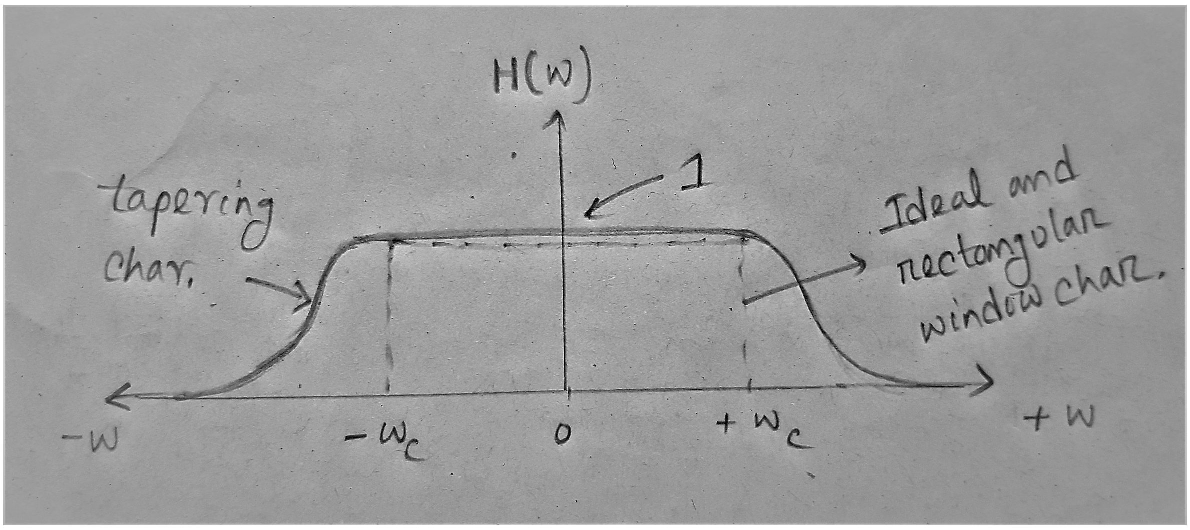
Thus, we find that there will be infinite number of values of filter co-efficients as, n$\longrightarrow \pm\infty$
We can also see that the larger the n, the smaller the value of the filter co-efficients. Because of this, there is no meaning in finding the co-efficients beyond a certain value of n.

The actual number of co-efficients to be computed depends on the accuracy we need in the realization of the filter, as well as on the number of bits the computer used can handle.

Design of FIR filters using window Functions

Finite register length of computers necessitates abrupt termination of FIR filter co-efficients at some finite value of n. In turn, this gives rise to the Gibbs's Phenomenon which is dangerous in many situations, as it gives rise to sharp transients may destroy the hardware used for the construction of the filter. So it must be avoided at all costs.

To prevent the occurrence of the Gibb's Phenomenon, we must avoid abrupt termination of filter co-efficients. The way to avoid abrupt termination is to use a tapering characteristic, as shown in fig.



To get a gradually tapering transfer characteristic, similar to one shown in fig, we make use of window functions.

"window functions are mathematically functions that are designed to have tapering characteristics"

For Example, a half-cosine wave is a window function of this type.

When the impulse response h(n), derived from the transfer function H(w), is multiplied by an appropriate window function w(n), we get a modified impulse h'(n), which shows a set of gradually decreasing filter co-efficients.

These filter co-efficients, in turn, will ensure the absence of the Gibb's Phenomenon from the operating regions of the filter.

Thus, we see that, to prevent the occurrence of the Gibb's Phenomenon, we must use a modified impulse response given by,

$$h'(n) = h(n) \times w(n) \tag{1}$$

Typical Window functions in Use

The rectangular Window:

The rectangular window is defined as

$$W_{REC}(n) = \begin{cases} 1, -M \leq n \leq M \\ 0, elsewhere. \end{cases} \tag{2}$$

The response characteristic of this window is shown in previous fig. Now, we substitute of (2) in equation (1) yields the modified impulse response,

$$h'(n) = \begin{cases} h(n) \times W_{REC}(n) = 1, -M \leq n \leq M \\ 0, elsewhere. \end{cases} \tag{3}$$

Equations (3) says that the modified impulse response h'(n) is the same as the original impulse response, h(n). This means that this window will result in abrupt cut-off filter co-efficients, which will lead to generation of the Gibb's Phenomenon, Hence this window is never used for the practical design of FIR Filters.

The Hann(or Hanning) Window

The Hann or Hanning window (named after J. Von Hann, the originator of the window) is defined by the expression.

$$W_{HAN}(n) = 0.5 + 0.5cos(\frac{2\pi n}{N}), -N/2 \leq n \leq N/2 \qquad (1)$$

Where, N = order of the filter.

Putting $M = \frac{N}{2}$, we get

$$W_{HAN}(n) = 0.5 + 0.5cos(\frac{\pi n}{M}), -M \leq n \leq M \qquad (2)$$

Changing the limits, equation (1) may also be written as,

$$W_{HAN}(n) = 0.5 - 0.5cos(\frac{\pi n}{N}), 0 \leq n \leq N \qquad (3)$$

[But we use equation (2) for easiness]

Let us now use equation (2) and calculate $W_{HAN}(n)$ for a tenth order (i.e, N=10) window, nothing that $W_{HAN}(n) = W_{HAN}(-n)$.

Thus,

$$
\begin{aligned}
W_{HAN}(0) &= 0.5 + 0.5\cos(0) = 1.0000\\
W_{HAN}(1) = W_{HAN}(-1) &= 0.5 + 0.5\cos(\pi/4) = 0.9045\\
W_{HAN}(2) = W_{HAN}(-2) &= 0.5 + 0.5\cos(\pi/4) = 0.6545\\
W_{HAN}(3) = W_{HAN}(-3) &= 0.5 + 0.5\cos(\pi/4) = 0.3455\\
W_{HAN}(4) = W_{HAN}(-4) &= 0.5 + 0.5\cos(\pi/4) = 0.0955\\
W_{HAN}(5) = W_{HAN}(-5) &= 0.5 + 0.5\cos(\pi/4) = 0.0000
\end{aligned}
$$

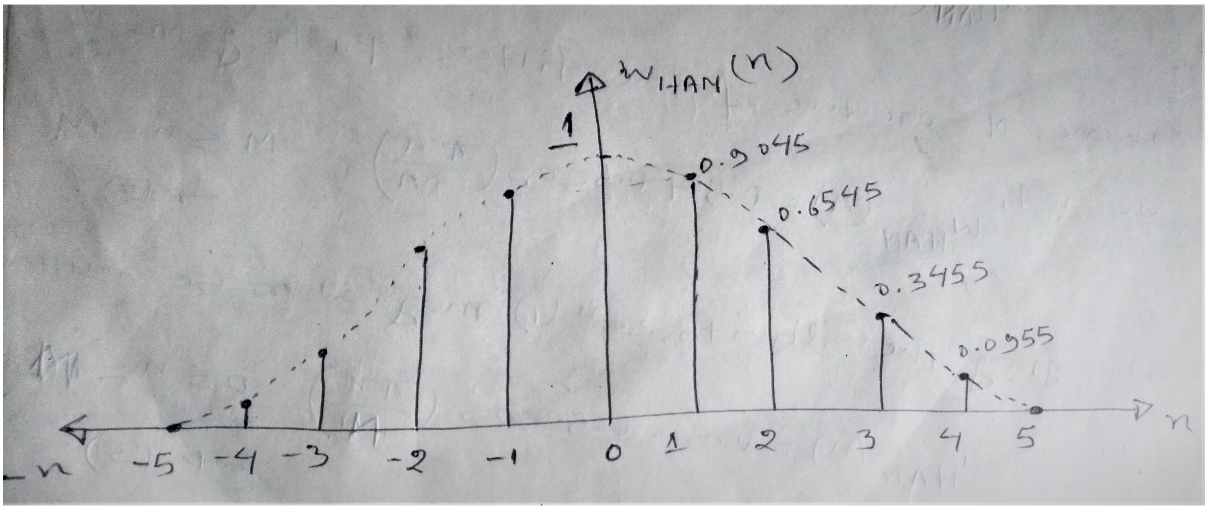


Figure: Plot of the function W$_{HAN}$(n)

This waveform is known as the raised cosine waveform, as it loos like the positive half-cycle of a cosine wave raised in the middle portion, and falling-off sharply.

The Hamming Window

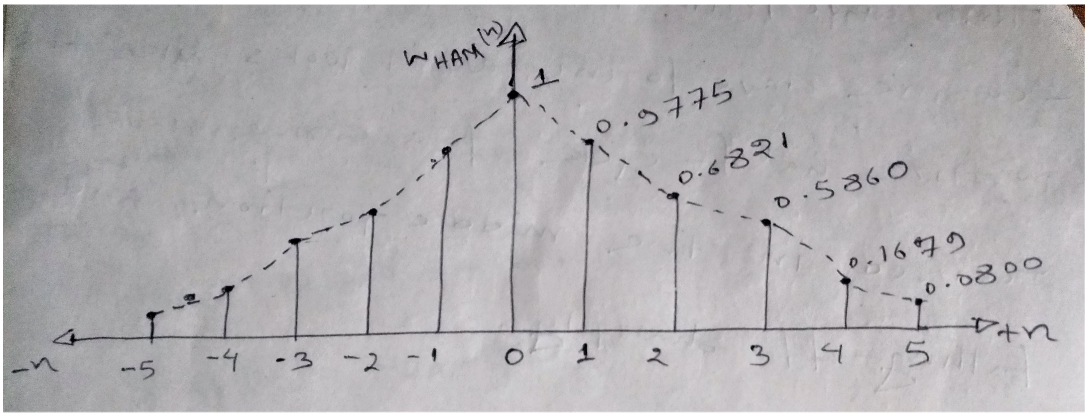
The Hamming window (named after R.W. Hamming, the originator of the window) is defined by the expression.

$$W_{HAM}(n) = 0.54 + 0.464cos(\frac{\pi n}{M}), -M \leq n \leq M$$

and,

$$W_{HAM}(n) = 0.54 - 0.464cos(\frac{\pi n}{N}), 0 \leq n \leq N$$

$*$ The same procedure we follow, as Hann Window and we get the waveform as like,



This curve shown that, this waveform is also a cosine function, but its base is not resting on the x-axis, i.e, zero at, $n = \pm N$. This window is known as the raised-cosine window with platform.

Problem:

Use (a) the Hamming window and (b) the Hann window to the solution given in precious example to determine the modified impulse response.

Solution:

We know, from previous example, the impulse response is,

$$h(n) = \frac{sin(n\pi/2)}{2(n\pi/2)}$$

(a) solution using the Hamming window:

We have the Hamming, defined as,

$$W_{HAM}(n) = 0.54 + 0.46cos(\frac{\pi n}{M}), -M \leq n \leq M$$

then, modified impulse response is obtained with N = 10, is,

$$\begin{aligned} h'(n) &= h(n) \times W_{HAM}(n) \\ &= \frac{sin(n\pi/2)}{2(n\pi/2)}[0.54 + 0.46cos(\frac{\pi n}{5})] \end{aligned}$$

Now, we compute h'(n), for different values of h(n) and $W_{HAM}(n)$ [computed in previous example] for n=0,1,2,3,4,5

$$
\begin{aligned}
h'(0) =& 0.5 \times 1 = 0.5 \\
h'(1) =& h'(-1) = 0.3183 \times 0.9775 = 0.3111 \\
h'(2) =& h'(-2) = 0 \\
h'(3) =& h'(-3) = -0.1061 \times 0.5860 = -0.0622 \\
h'(4) =& h'(-4) = 0 \\
h'(5) =& h'(-5) = 0.0637 \times 0.08 = 0.0051
\end{aligned}
$$

Using the value of modified impulse response, we find the x-fer function fo the desired low pass FIR filter as,

$$H(Z) = 0.0051Z^5+0\times Z^4-0.0622Z^3+0\times Z+0.3111z^1+0.5+0.3111Z^{-1}+0\times$$

This is also a non causal system, it contains negative frequencies Z^n. To make it caused, we multiplying whole equation with Z^{-5}

$$H(Z) = 0.0051-0.0178Z^{-2}+0.3111Z^{-4}+0.5Z^{-5}+0.3111Z^{-6}-030178Z^{-8}+$$

This can be implemented by using an appropriate FIR structure.

(b) solution using the Hann window:

We have the modified impulse response using the Hann window given by,

$$\begin{aligned} h'(n) =& h(n) \times W_{HAN}(n) \\ =& \frac{sin(n\pi/2)}{2(n\pi/2)}[0.5 + 0.5cos(\frac{\pi n}{5})] \end{aligned}$$

Now, we have,

$$\begin{aligned} h'(0) =& 0.5 \times 1 = 0.5 \\ h'(1) =& h'(-1) = 0.3183 \times 0.9045 = 0.2879 \\ h'(2) =& h'(-2) = 0 \\ h'(3) =& h'(-3) = -0.1061 \times 0.3455 = -0.0367 \\ h'(4) =& h'(-4) = 0 \\ h'(5) =& h'(-5) = 0 \end{aligned}$$

The cause filter is given by,

$$H(Z) - -0.0367Z^{-2} + 0.2879Z^{-4} + 0.5Z^{-5} + 0.2879Z^{-6} - 0.0367Z^{-8}$$

The Balckman Window

Defined by,

$$W_{BLA} = 0.42 - 0.5cos(\frac{\pi n}{m}) + 0.08cos(\frac{2\pi n}{m}), M \leq n \leq -M$$

This is the modification of the Hanning window.

The Barlett window

The Barlett's triangular window is given by,

$$W_{BAR} = \begin{cases} n/m, 0 \leq n \leq M \\ 2 - n/M, M \leq n \leq N \end{cases}$$

This represents a triangular function, with sharp cut-off characteristic.