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Chapter 6
Mrinmoy Sir

1920–1999

Walter R. Evans was born in 1920. He was the recipient of the 1987 American Society of Mechanical Engineers Rufus Oldenburger Medal and the 1988 AACC Richard E. Bellman Control Heritage Award. He passed away at the age of 79 on July 10, 1999 in Whittier, CA.

Walter Evans' principal contribution to the field of automatic control was his invention of the Evans Root Locus Method in 1948 and his subsequent invention of the Spirule, a tool used in conjunction with the root-locus method. Because it codifies very useful frequency information about a feedback system in such an intuitive and appealing graphical form, Evan's root-locus method has enjoyed widespread use in the design of control systems and is now a standard chapter in texts on feedback control systems.

Evans received his B.S. in Electrical Engineering from Washington University in St. Louis, Missouri in 1941 and the M.S. degree in Electrical Engineering from the University of California at Los Angeles in 1951. During his lifetime, he worked as an engineer at several companies, including General Electric, Autonetics, and Ford Aeronautic Company. He also served as an instructor at Washington University for few years.

Root-Locus Technique

OBJECTIVE

The root-locus technique is a graphical method for drawing the locus of roots in the s-plane as a system parameter is varied and hence it is of paramount importance in the design of control systems. Since the root-locus method provides graphical information, an approximate sketch can be used to obtain qualitative information about the stability and performance of the system. The beauty of the root-locus technique is that it gives knowledge about the roots of the closed-loop system from the open-loop pole-zero configuration when one system parameter is varied in the positive range, and all this without solving the roots of the characteristic polynomial of the closed-loop system.

CHAPTER OUTLINE

6.1 Introduction

The roots of the characteristic equation of the closed-loop system determine the dynamic behaviour of the overall system. Thus, for the analysis of problems of control engineering, it is important to study the location of poles and zeros of the closed-loop system in the s-plane, with the variation of a system parameter.

For a higher-order polynomial it is a tedious job to find the roots. The classical technique for factoring the polynomial is not convenient because the computations must be repeated as a single parameter of the open-loop transfer function is varied. Using the root-locus technique, the roots of the closed-loop system (corresponding to a particular value of the variable parameter) can be located on the open-loop pole-zero configuration. It may be noted that the variable parameter is usually the gain K in the present text unless otherwise stated, but any other variable of the open-loop transfer function may also be used.

6.2 Root-Loci for Second-Order System

Consider the second-order system shown in Figure 6.1 which represents a typical position control system. For the negative feedback control system having forward transfer function

$$G(s) = \frac{K}{s(s+4)} \tag{6.1}$$

where K is a gain parameter and feedback transfer function $H(s) = 1$ for a unity-feedback system, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s^2+4s+K} \tag{6.2}$$

Since the dynamic behaviour of a system is controlled by the roots of the characteristic equation, it is important to investigate the variation of the roots of the closed-loop system with the variation of the parameter K. The characteristic equation of the system is

$$s^2 + 4s + K = 0 \tag{6.3}$$

which is analogous to

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

where ζ is the damping coefficient and ω_n is the undamped natural frequency of the closed-loop system. The roots of the closed-loop system are

$$s_1 = -2 + \sqrt{4-K} \quad \text{and} \quad s_2 = -2 - \sqrt{4-K} \tag{6.4}$$

The movement of the roots s_1, s_2 when K is varied in the range $0 \le K < +\infty$ is shown in Figure 6.2.

(i) For $K = 0$: The two roots are at $s_1 = 0$ and $s_2 = -4$. The two roots are the same as the poles of the open-loop transfer function $G(s)H(s) = K/s(s + 4)$ at points A and B of Figure 6.2.

(ii) For $0 < K < 4$: Both the roots s_1 and s_2 are negative real.

(iii) For $K = 4$: Both the roots are repetitive and $s_1 = s_2 = -2$ as at point C.

(iv) For $4 < K < +\infty$: The two roots are complex conjugate pairs with the negative real part equal to -2. $s_1 = -2 + j\sqrt{K-4}$ and $s_2 = -2 - j\sqrt{K-4}$. The two roots move towards points D and E respectively as gain K increases from 4.

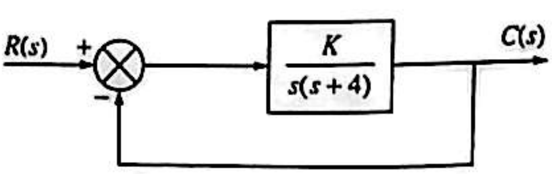


FIGURE 6.1 Typical position control system.

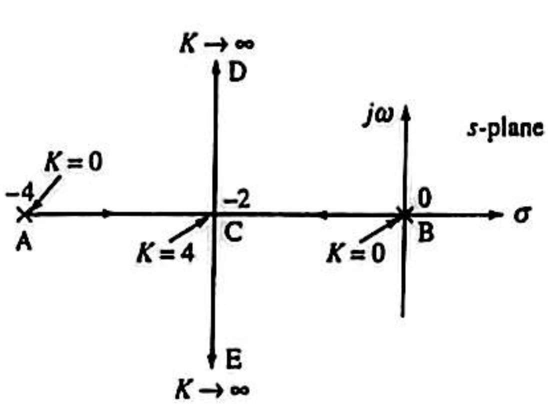


FIGURE 6.2 Root loci for the system shown in Figure 6.1.

The root loci therefore consist of straight lines AB and DE intersecting at C. The loci start from two open-loop poles at $s = 0$ (i.e. at B where $K = 0$) and at $s = -4$ (i.e. at A where $K = 0$). The root-locus starting from A moves towards the right and that starting from B moves towards the left for $0 < K < 4$ till they intersect at C for $K = 4$. For values of $K > 4$, the loci move along the lines CD and CE as shown in Figure 6.2.

From the root loci, the following information about the system behaviour may be inferred.

(i) *Stability:* The closed-loop system is stable for all positive values of gain $0 \le K < +\infty$.

(ii) *Transient response:* For all values of gain K between 0 and 4, the system is overdamped (the damping coefficient $\zeta > 1$); for $4 < K < +\infty$, the system is underdamped ($\zeta < 1$); for $K = 4$, the system is critically damped ($\zeta = 1$) having repetitive roots.

For $\zeta < 1$, the damping coefficient is given by

$$\zeta = 2/\sqrt{K} \tag{6.5}$$

The undamped natural frequency ω_n increases with the increase in K. For all values of $K > 4$, the settling time of the step response is constant since the real parts of the two roots are fixed and are equal to -2.

We have considered an example of a second-order system where the root-locus can be obtained from analytical considerations. Every time with the new parameter value K, we have to determine the roots of the closed-loop system analytically which is a formidable task especially for higher-order systems. The root-locus technique is a systematic graphical method by which we can draw the root-locus diagram of the closed-loop system from open-loop pole-zero information when one variable parameter (say, gain K) is varied in the range $0 \le K < +\infty$.

The root-locus technique was introduced by W.R. Evans in 1948 and the art has since been developed and extensively applied to the analysis and design of control systems. The beauty of the technique is that it gives the complete information of the closed-loop poles from the open-loop pole-zero information without solving for the roots of the closed-loop system when one of the parameters (say, gain K) of the open-loop transfer function is varied.

In control system problems, the complete root-locus diagram is a plot of the loci of the poles of the closed-loop transfer function (or roots of the characteristic equation) when one parameter of the open-loop transfer function is varied from $-\infty$ to $+\infty$. We have in our definition, root loci, inverse root loci and complete root loci plots.

(i) Usually when only one parameter, say gain K, is varied between 0 to $+\infty$ the plot is called the root-locus diagram.

(ii) When K is varied between 0 to $-\infty$, the plot is called the inverse root-locus diagram.

(iii) The plot is called the complete root loci when K is varied in the range $-\infty < K < \infty$, that is, the complete root loci is made up of root loci and inverse root loci plots.

Further, when more than one parameter is considered to be variable, the plot is referred to as a root-contour diagram. Our point of interest is the root loci for the variable parameter K in the range $0 \le K < \infty$.

Drill Problem 6.1

Determine the poles and zeros including those at infinity (if any) of each of the following rational transfer functions $G(s)$. Mark each finite pole by a small cross (×) and each finite zero by a small circle (o) in the s-plane. Graphically, measure the magnitude and phase at $\omega = 2$ rad/s for each of the transfer functions.

(i) $\dfrac{10(s+2)}{s^2(s+1)(s+10)}$ (ii) $\dfrac{10s(s+1)}{(s+2)(s^2+3s+2)}$ (iii) $\dfrac{10(s+2)}{s(s^2+2s+2)}$

Drill Problem 6.2

A unity-feedback system has forward transfer function. $G(s) = K/s(s+2)$. Determine the value of K for the closed-loop poles having damping ratio of 0.5.

Ans. 4

6.3 Basic Conditions for Root Loci

Let us consider the negative feedback system of Figure 6.3 where $G(s)$ is the forward transfer function and $H(s)$ is the feedback ratio. The closed-loop transfer function can be written as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{6.6}$$

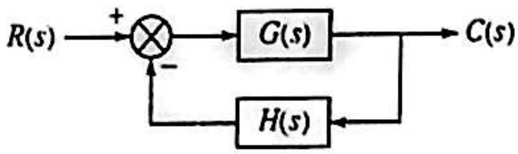


FIGURE 6.3 Negative feedback system.

In a feedback control system without transportation lag, the open-loop transfer function $G(s)H(s)$ is a rational algebraic function and can be written as

$$G(s)H(s) = \frac{K(s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = K\frac{P(s)}{Q(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = KG_1(s)H_1(s); \quad m \le n \tag{6.7}$$

where K is the variable parameter, say, gain, a real quantity varying in the range $0 \le K < +\infty$, and $G_1(s)H_1(s)$ is a complex quantity. $Q(s)$ is an nth order polynomial of s as

$$Q(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

and $P(s)$ is an mth order polynomial of s as

$$P(s) = s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0$$

and n and m are positive integers and $m \le n$.

The characteristic equation is

$$1 + G(s)H(s) = 0 \quad \text{or} \quad G(s)H(s) = -1 \tag{6.8}$$

i.e.

$$KG_1(s)H_1(s) = -1 \quad \text{or} \quad \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = -1 \tag{6.9}$$

This leads to the following two conditions:

(i) The magnitude condition for root loci becomes

$$|G(s)H(s)| = 1 \tag{6.10}$$

i.e.

$$|G_1(s)H_1(s)| = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = \frac{1}{|K|};\ 0 \le K < \infty \tag{6.11}$$

(ii) The phase condition or angle condition for root loci becomes

$$\begin{aligned}\angle G(s)H(s) &= \angle K + \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j)\\ &= (2k+1)\pi;\ k = 0, 1, 2, \cdots\end{aligned} \tag{6.12}$$

or

$$\begin{aligned}\angle G_1(s)H_1(s) &= \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j)\\ &= (2k+1)\pi;\ k = 0, 1, 2, \cdots\end{aligned} \tag{6.13}$$

as K is a real quantity, $\angle K = 0$.

It may be noted that for any given value of K, say K_1, between 0 and $+\infty$, any point P (i.e. $s = s_1$) in the s-plane which satisfies the magnitude and phase conditions of Eqs. (6.11) and (6.13), respectively, is a point on the root loci. Alternatively, the loci of the roots corresponding to all values of K in the range plotted in Figure 6.2 must satisfy both the magnitude and phase conditions, i.e. for any point on the root loci, one can immediately determine the value of the variable parameter K from the magnitude condition that will yield a closed-loop pole at a desired point on the root loci. Similarly, it can be shown that any desired point on the root loci must satisfy the angle condition given by Eq. (6.13).

As an illustration, let

$$G(s)H(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)} \tag{6.14}$$

The poles and zeros are located in the s-plane as shown in Figure 6.4. Let $s = s_1$ be any desired point P on the root loci, then the magnitude condition

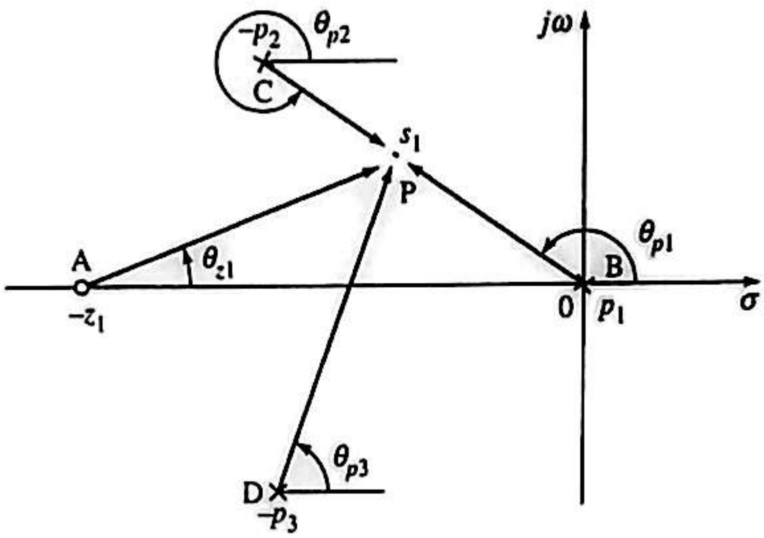


FIGURE 6.4 Pole-zero configuration of $G(s)H(s) = \dfrac{K(s+z_1)}{s(s+p_2)(s+p_3)} \equiv \dfrac{K(s+z_1)}{(s-0)(s+p_2)(s+p_3)}$.

$$\left|G(s)H(s)\right|_{s=s_1} = \left|KG_1(s)H_1(s)\right|_{s=s_1} = 1$$

or

$$\left|G_1(s_1)H_1(s_1)\right| = \frac{1}{|K|} \tag{6.15}$$

i.e.

$$\frac{\left|(s_1+z_1)\right|}{\left|(s_1-0)\right|\left|(s_1+p_2)\right|\left|(s_1+p_3)\right|} = \frac{\text{AP}}{\text{BP}\cdot\text{CP}\cdot\text{DP}} = \frac{1}{|K|} \tag{6.16}$$

The factor $|(s_1 + z_1)|$ is recognized as the length of the vector drawn from the zero at $-z_1$ to the point s_1 and the vector length is represented by AP. The factor $\left|(s_1 + p_2)\right|$ is the length of the vector drawn from the pole at $-p_2$ to s_1 and the vector length is represented by CP. Similarly, the vector length for $\left|(s_1 + p_3)\right|$ is represented by DP and the vector length for the pole at the origin, i.e. $\left|s_1 - 0\right|$ is represented by the vector length BP. Similarly, any point P at $s = s_1$ on the root loci must satisfy the phase condition, i.e.

$$\angle G_1(s_1)H(s_1) = (2k+1)\pi;\ k = 0, 1, 2, \cdots \tag{6.17}$$

or

$$\angle\frac{s_1+z_1}{s_1(s_1+p_2)(s_1+p_3)} = (2k+1)\pi \tag{6.18}$$

or

$$\angle(s_1+z_1) - (\angle(s_1-0) + \angle(s_1+p_2) + \angle(s_1+p_3)) = (2k+1)\pi \tag{6.19}$$

or

$$\theta_{z_1} - (\theta_{p_1} + \theta_{p_2} + \theta_{p_3}) = (2k+1)\pi \tag{6.20}$$

where the angles $\theta_{z_1}, \theta_{p_1}, \theta_{p_2}$, and θ_{p_3} are the arguments of the vectors measured with the positive real axis as the zero reference. The angle in the clockwise direction is negative and the angle in the counterclockwise direction is positive with respect to the positive real axis taken as the reference. It is to be noted that K is a real quantity and its angle contribution is zero.

Consequently, given the pole-zero configuration of an open-loop transfer function, the construction of the root-locus diagram of the closed-loop system involves a search for the s_1 point which will satisfy both the magnitude and angle conditions specified by Eqs. (6.15) and (6.17).

Although searching for all the points s_1 is apparently an almost impossible task, the actual procedure is not so complex. Normally the root loci can be sketched following certain rules of construction as described in the following.

6.4 Rules for the Construction of Root Loci $(0 \le K \le \infty)$

The rules for the construction of root loci are enumerated below:

Rule 1. *The $K = 0$ or the starting points on the root loci are at the poles of $G(s)H(s)$.*

Proof: Rewrite Eq. (6.11) as

$$|G_1(s)H_1(s)| = \frac{\prod_{i=1}^{m} |(s+z_i)|}{\prod_{j=1}^{n} |(s+p_j)|} = \frac{1}{|K|} \tag{6.21}$$

As K approaches zero, the value of $G_1(s)H_1(s)$ in Eq. (6.21) approaches infinity. Again the value of $G_1(s)H_1(s)$ become infinity at poles $s = -p_j$. It means that the starting point (i.e. $K = 0$) of each root loci is a pole (i.e. $s = -p_j$) of $G(s)H(s)$, the open-loop transfer function.

Rule 2. *The $K = +\infty$ or the terminating points on the root loci are the zeros of $G(s)H(s)$.*

Proof: With reference to Eq. (6.21) as K approaches $+\infty$, the value of Eq. (6.21) approaches zero. Again the value of Eq. (6.21) becomes zero at zeros of $G_1(s)H_1(s)$. It means that the terminating point of each root loci (i.e. $K = \infty$) is a zero of $G_1(s)H_1(s)$, which is the same as a zero of the open-loop transfer function $G(s)H(s)$. It may be noted that for any given $G(s)H(s)$, if the number of finite zeros Z is less than the number of finite poles P, then $(P - Z)$ zeros lie at infinity. This is because, for a rational function the total number of poles and zeros must be equal if the poles and zeros at infinity are counted as well.

To illustrate the preceding rules for the construction of root loci, let us consider the open-loop transfer function of the system as

$$G(s)H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)} \tag{6.22}$$

where the gain parameter K varies in the range $0 < K < \infty$. Now let

$$G_1(s)H_1(s) = \frac{s+3}{s(s+5)(s+6)(s^2+2s+2)} = \frac{1}{K}$$

It may be mentioned that the poles and zeros of both $G_1(s)H_1(s)$ and $G(s)H(s)$ are the same. The pole-zero configuration is shown in Figure 6.5. The finite poles are located at $s = 0$,

$s = -5$, $s = -6$ and $s = -1 \pm j1$. The finite zero is located at $s = -3$. Therefore, these five poles are the starting points ($K = 0$ points) of the five root loci. As $G(s)H(s)$ is a rational function, the total number of poles must be equal to the total number of zeros, if the poles and zeros at infinity are counted. Hence, obviously four zeros are located at infinity. One of five root loci terminates at the finite zero of $G(s)H(s)$ at $s = -3$ and the other four root loci terminate at zeros located at infinity.

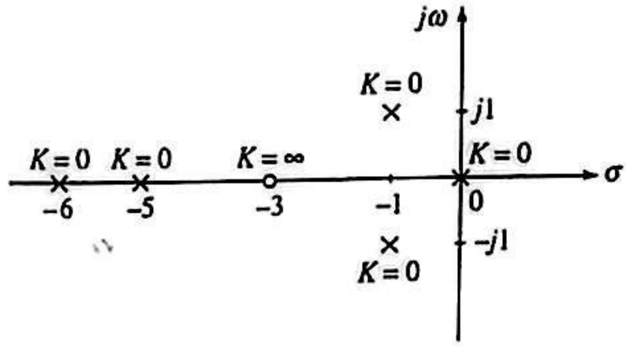


FIGURE 6.5 Pole-zero configuration.

Rule 3. *The number of separate root loci N is given by*

$$N = P \text{ if } P > Z \tag{6.23}$$

$$N = Z \text{ if } Z > P \tag{6.24}$$

Since a complete root loci is formed between a pair of a pole and a zero of $G(s)H(s)$, the total number of root loci of a given system must be equal to P or Z, whichever is greater. In fact, $N = P$ because for a physically realizable system $P \geq Z$. For example, in the open-loop transfer function of Eq. (6.22), the finite number of poles, $P = 5$ and the finite number of zeros, $Z = 1$. Hence the number of separate root loci $N = P = 5$ as $P > Z$.

Rule 4. *Symmetry of the root loci:* The root loci are symmetrical about the real axis in the s-plane. The proof is self evident as the complex roots occur in conjugate pairs and hence root loci are symmetrical about the real axis in s-plane.

Rule 5. *Angles of asymptotes and the intersection of asymptotes on the real axis:* If the number of finite zeros Z is less than the number of finite poles P, then $(P - Z)$ is the number of root loci that must terminate at zeros at infinity. The asymptotes to these $(P - Z)$ number of root loci intersect at a common point (called centroid) σ_A on the real axis, given by

$$\sigma_A = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} \tag{6.25}$$

and are inclined to the real axis at angles ϕ_k given by

$$\phi_k = \frac{(2k+1)\pi}{P-Z}; \quad k = 0, 1, 2, \ldots, (P - Z - 1) \tag{6.26}$$

Equations (6.25) and (6.26) can be proved by rewriting Eq. (6.9) as

$$-K = \frac{\prod_{j=1}^{n}(s+p_j)}{\prod_{i=1}^{m}(s+z_i)} = \frac{s^n + a_{n-1}s^{n-1} + \ldots}{s^m + b_{m-1}s^{m-1} + \ldots} = \frac{s^n - \left(\sum_{j=1}^{n} p_j\right)s^{n-1}}{s^m - \left(\sum_{i=1}^{m} z_i\right)s^{m-1}} \tag{6.27}$$

As lim $s \to \infty$ is the condition of asymptote for large s, we can neglect the lower-order terms of s and Eq. (6.27) is approximated as

$$-K = s^{n-m} - \left(\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i \right) s^{n-m-1} + \ldots$$

by long division and can be approximated to

$$-K \approx s^{n-m} \left(1 - \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{s} \right)$$

or

$$s \left(1 - \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{s} \right)^{\frac{1}{n-m}} = (-K)^{\frac{1}{n-m}} \tag{6.28}$$

Binomial expansion leads to

$$s \left(1 - \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{(n-m)s} + \ldots \right) = (-K)^{\frac{1}{n-m}} \tag{6.29}$$

Again, if the terms higher than the second are neglected for large value of s, we get

$$s - \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m} = (-K)^{\frac{1}{n-m}} \tag{6.30}$$

Substituting $s = \sigma + j\omega$ into Eq. (6.30) and using Demoivre's algebraic theorem, yields

$$(\sigma + j\omega) - \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m} = \left| K^{\frac{1}{n-m}} \right| \left(\cos \frac{(2k+1)\pi}{n-m} + j \sin \frac{(2k+1)\pi}{n-m} \right) \tag{6.31}$$

for $k = 0, 1, 2, \ldots, n - m - 1$

Equating real and imaginary parts, we get

$$\sigma - \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m} = \left| K^{\frac{1}{n-m}} \right| \cos \frac{(2k+1)\pi}{n-m} \tag{6.32}$$

and

$$\omega = \left| K^{\frac{1}{n-m}} \right| \sin \frac{(2k+1)\pi}{n-m} \tag{6.33}$$

Solving for $K^{\frac{1}{n-m}}$ from Eqs. (6.32) and (6.33), we get

$$\left|K^{\frac{1}{n-m}}\right| = \frac{\omega}{\sin\frac{(2k+1)\pi}{n-m}} = \frac{\sigma - \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m}}{\cos\frac{(2k-1)\pi}{n-m}}$$

and solving for ω, we get

$$\omega = \tan\frac{(2k+1)\pi}{n-m}\left[\sigma - \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m}\right] \qquad (6.34)$$

Equation (6.34) represents an equation of straight line in the s-plane which is of the form

$$\omega \approx m(\sigma - \sigma_A) \qquad (6.35)$$

where m is the slope and σ_A is the intersection on the σ-axis. Thus,

$$m = \tan\frac{(2k+1)\pi}{n-m} = \frac{(2k+1)\pi}{P-Z}; \qquad k = 0, 1, \ldots, P-Z-1$$

and

$$\sigma_A = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{n} z_i}{n-m} \qquad (6.36)$$

The intersection of the asymptotes on the real axis is therefore given by

$$\sigma_A = \frac{\Sigma \text{finite poles of } G(s)H(s) - \Sigma \text{finite zeros of } G(s)H(s)}{P-Z} \qquad (6.37)$$

$$= \frac{\Sigma \text{real part of finite poles of } G(s)H(s) - \Sigma \text{real part of finite zeros of } G(s)H(s)}{P-Z}$$

The angles of asymptotes are given by

$$\phi_k = \tan^{-1} m = \frac{(2k+1)\pi}{P-Z} \qquad \text{for} \qquad k = 0, 1, 2, \ldots, (P-Z-1) \qquad (6.38)$$

Considering again the $G(s)H(s)$ of Eq. (6.22), the intersection of asymptotes on the real axis as shown in Figure 6.6 is given by

$$\sigma_A = \frac{\Sigma \text{real part of the finite poles of } G(s)H(s) - \Sigma \text{real part of the finite zeros of } G(s)H(s)}{P-Z}$$

$$= \frac{(0-5-6-1-1)-(-3)}{5-1} = -2.5$$

The angles of asymptotes shown in Figure 6.6 are given by

$$\phi_k = \frac{(2k+1)\pi}{P-Z}; \qquad k = 0, 1, 2, \ldots, P-Z-1$$

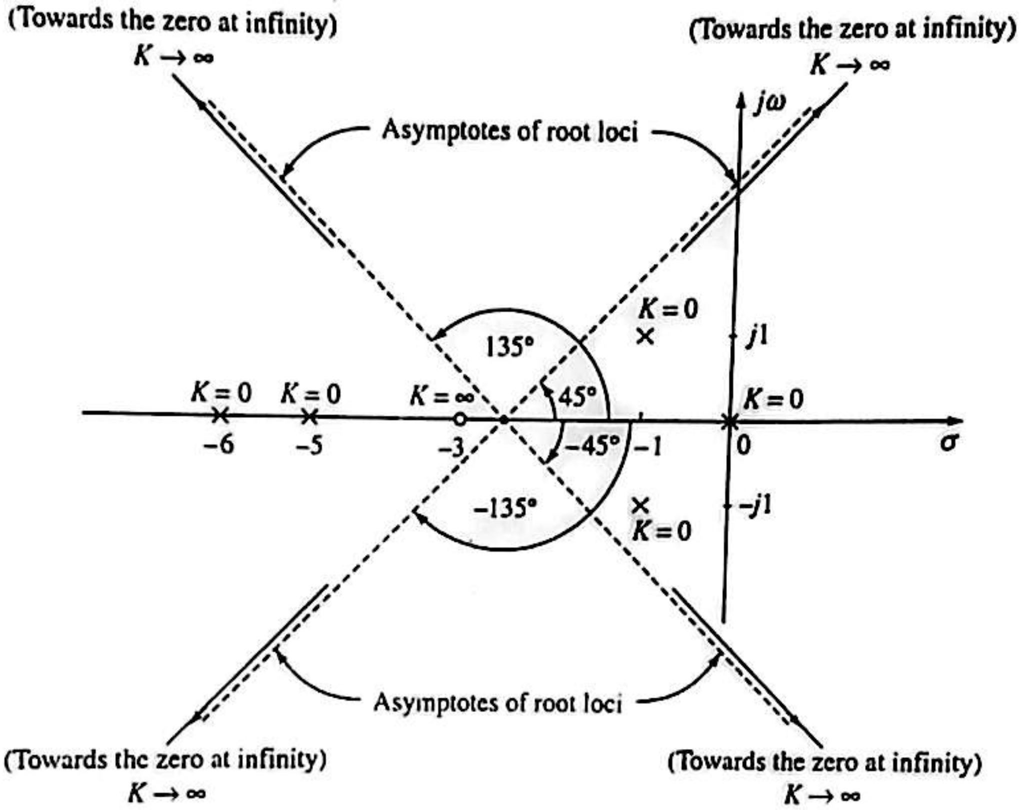


FIGURE 6.6 Angles of asymptotes and intersection of asymptotes on the real axis for $GH(s) = \dfrac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$.

i.e. for $k = 0$, $\phi_0 = \pi/4 = 45°$; $k = 1$; $\phi_1 = 3\pi/4 = 135°$;

$k = 2$, $\phi_2 = 5\pi/4 = -3\pi/4 = -135°$; $k = 3$; $\phi_3 = 7\pi/4 = -\pi/4 = -45°$.

Drill Problem 6.3

Given the open-loop transfer function as $K/(s^2 + 4s + 5)s$, the root loci has one break-in point for $K = 1.852$, justify. Is there any break-away point? Determine the angles of asymptotes and the point of intersection of asymptotes on the real axis. Determine also the angle of departure from the open-loop complex pole in the upper-half of the s-plane.

Ans. 60°, 180°, −60°; −1.33°; −63.43°

Rule 6. *Existence of root loci on the real axis:* Any point on the real axis is a part of the root locus if and only if the number of finite poles and zeros of $G(s)H(s)$ on the real axis to the right of the point is odd.

Proof: This property follows by applying the angle condition at any point on the real axis of the s-plane. At any point s_1 on the real axis, the angles of the vectors from each pair of complex poles or zeros of $G(s)H(s)$ occuring in conjugate pairs will add up to zero (see Figure 6.7(a)). We need consider only the poles and zeros on the real axis. Further, the angle subtended at any point s_1 on the real axis by a real pole or zero of $G(s)H(s)$ to the left of the point is always zero (see Figure 6.7(b)). Hence we need to consider the poles and zeros of $G(s)H(s)$ on the real axis that are to the right of that point. Since each of these subtends an angle π (see Figure 6.7(c)), we must have an odd number of finite poles and zeros to the right of the point on the root loci on the real axis of s-plane to satisfy the angle condition.

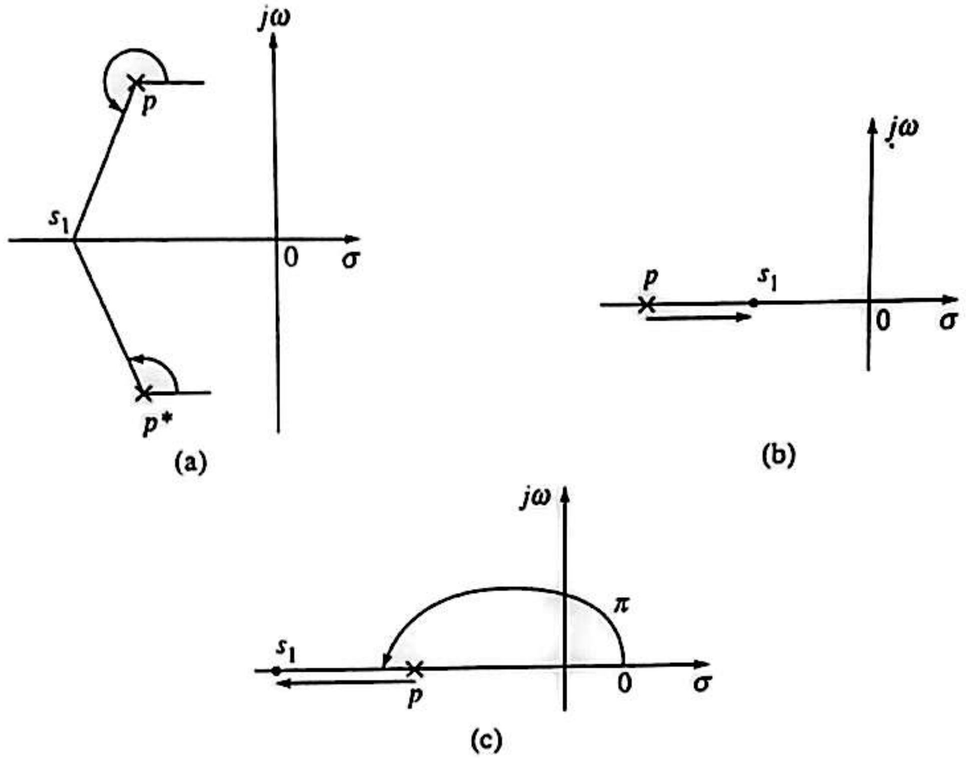


FIGURE 6.7 (a) Sum of the angles subtended at s_1 by complex conjugate poles/zeros is zero, (b) the angle subtended at s_1 by a real pole/zero to the left of the point is 0°, and (c) the angle subtended at s_1 by a real pole/zero to the right of the point is π.

Considering again the open-loop transfer function of Eq. (6.22), the existence of the root loci is shown in Figure 6.8. It is apparent that the occurrence of the loci on the real axis is not affected by the complex poles and zeros of $G(s)H(s)$. There are root loci, i.e. for $0 < K < \infty$ on the real axis between (a) $s = 0$ and $s = -3$ and between (b) $s = -5$ and $s = -6$.

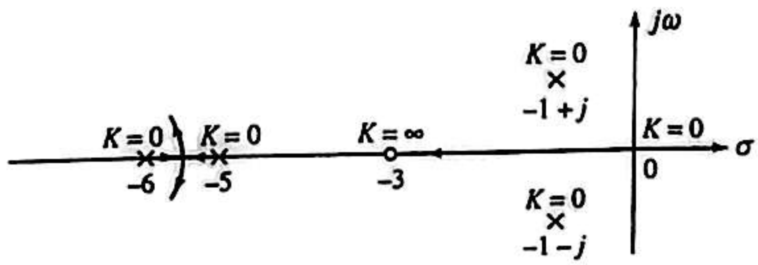


FIGURE 6.8 Existence of root loci on the real axis.

Rule 7. *Intersection of the root loci on the imaginary axis:* If the root-locus crosses the $j\omega$-axis for some values of K, this can be readily determined through the Routh–Hurwitz criterion. From the Routh table we can determine the value of K that makes the system just unstable, as well as the locations of the resulting roots on the $j\omega$-axis. This is illustrated again with the $G(s)H(s)$ of Eq. (6.22), whose characteristic equation is

$$s^5 + 13s^4 + 54s^3 + 82s^2 + (60 + K)s + 3K = 0 \tag{6.39}$$

The Routh tabulation is

$$\begin{array}{llll}
s^5 & 1 & 54 & 60+K \\
s^4 & 13 & 82 & 3K \\
s^3 & 47.7 & 60+0.769K & \\
s^2 & 65.6-0.212K & 3K & \\
s^1 & \dfrac{3940-105K-0.163K^2}{65.6-0.212K} & & \\
s^0 & 3K & &
\end{array}$$

In order to have the closed-loop poles in the left-half of the s-plane, i.e. for the closed-loop system to be stable, the quantities in the first column of the Routh tabulation array should be of the same sign, i.e. positive. Therefore, the following inequalities must be satisfied:

$$\begin{array}{llll}
\text{(i)} & (65.6 - 0.212K) > 0 & \text{or} & K < 309 \\
\text{(ii)} & (3940 - 105K - 0.163K^2) > 0 & \text{or} & K < 35 \\
\text{(iii)} & K \geq 0 & &
\end{array} \tag{6.40}$$

Hence all the roots of the characteristic Eq. (6.39) will stay in the left-half of the s-plane if K lies in the range $0 < K < 35$. This means that the root loci just crosses the imaginary axis when $K \geq 35$. The coordinate at the crossover point on the imaginary axis that corresponds to $K = 35$ can be determined from the auxiliary equation to be formed from Routh's array as

$$A(s) = (65.6 - 0.212K)s^2 + 3K = 0$$

Substituting $K = 35$, we get

$$58.2s^2 + 105 = 0$$

which yields $s = \pm j1.34$, i.e. $\omega = \pm 1.34$

Hence the intersection of the root loci on the imaginary axis will take place at points $\omega = 1.34$ rad/s and $\omega = -1.34$ rad/s for gain $K = 35$.

Drill Problem 6.4

Determine the point of intersection of the root loci with the imaginary axis for a unity-feedback system with forward transfer function $G(s) = K/s(s + 4)(s + 16)$. Find the value of K at this crossover frequency.

Ans. $s = +j8$; $K = 1280$

Rule 8. *Angles of departure (from poles) and angles of arrival (at zeros) of the root loci:* The angle of departure (arrival) of root loci at a pole (zero) of $G(s)H(s)$ denotes its behaviour near the pole (zero). For the root loci ($0 \le K \le \infty$) these angles can be determined by use of Eq. (6.13). This is illustrated with the open-loop transfer function of Eq. (6.22). Suppose it is desired to determine the angle at which the root locus leaves the pole at $s = -1 + j1$ (see Figure 6.9). Note that the unknown angle θ is measured with respect to the real axis with anticlockwise movement as positive and clockwise movement as negative angle. Let us assume that s_1 is a point on the root locus leaving the pole at $s = -1 + j1$ and s_1 is very close to the pole at $s = -1 + j1$. Then the point s_1 must satisfy Eq. (6.13). Thus,

$$\angle G_1(s_1)H_1(s_1) = \angle(s_1 + 3) - [\angle s_1 + \angle(s_1 + 1 + j1) + \angle(s_1 + 5) + \angle(s_1 + 6) + \angle(s_1 + 1 - j1)]$$
$$= (2k + 1)\pi \qquad \text{for } k = 0, 1, 2 \tag{6.41}$$

Since s_1 is very close to the pole at $-1 + j1$, the angles of the vectors are drawn from the finite poles and zero to the point s_1 in Figure 6.9 and then Eq. (6.41) becomes

$$26.6° - (135° + 90° + 14° + 11.4° + \theta) = (2k + 1)180°; \qquad k = 0, 1, 2$$

Therefore,

$$\theta = -43.8° = 316.2°$$

Hence the angle of departure of the root locus from the pole at $(-1 + j1)$ is 316.2°. It may be noted that the angle of departure of the root locus from the complex conjugate pole at $(-1 - j1)$ is also $-316.2°$ as the root loci about the real axis is symmetric.

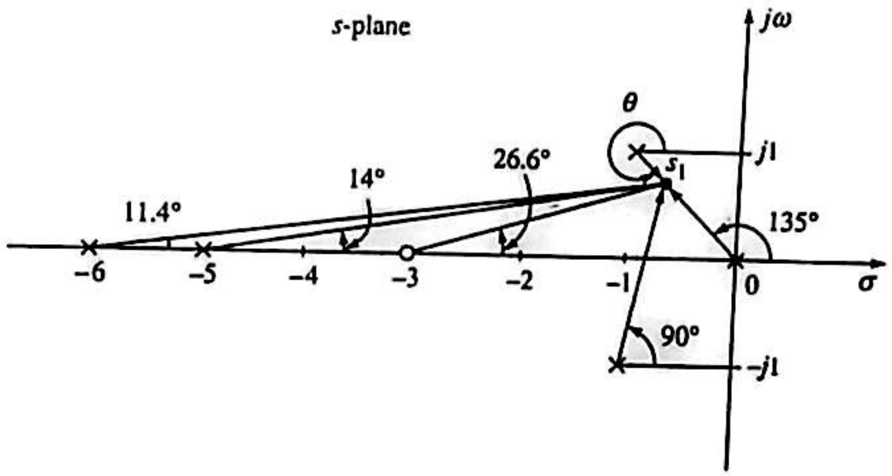


FIGURE 6.9 Determination of angle at which the root locus leaves the pole at $s = -1 + j1$.

Rule 9. *Break-away or break-in points:* Break-away points on the root loci of an equation correspond to multiple-order roots of the equation. A root-locus diagram can, of course, have more than one break-away point. Moreover, the break-away points need not always be on the real axis. However, because of the conjugate symmetry of the root loci, the break-away points must either be real or in complex conjugate pairs. If r number of root loci branches meet at a point, they break away at an angle of $\pm 180°/r$.

Two branches of the root loci on the real axis meet at a point and depart at right angles to it as depicted in Figure 6.10(a). This is called the *break-away* point and is characterized by the fact that the characteristic polynomial has multiple roots at the break-away point. The dual situation occurs when the branches of the root loci enter the real axis. This may be called the *break-in* or *re-entry* point as depicted in Figure 6.10(b). Normally, a break-away point occurs on a segment of the root locus on the real axis commencing from two poles as in Figure 6.10(a), and a re-entry point occurs between two zeros as in Figure 6.10(b). Further, there may be points on the real axis as in Figure 6.10(c) where more than two branches of the root loci meet. The angles at which the branches will leave the real axis are given by $\pm 180°/r$, where r is the number of root loci branches approaching the break-away point. Again, the root-locus may have more than one break-away point as in Figure 6.13. Moreover all the break-away points need not be always on the real axis (see Example 6.1, Figure 6.13). Obviously, if the break-away points are in the complex plane, they must be in complex conjugate pairs because of the property of symmetry of root loci about the real axis.

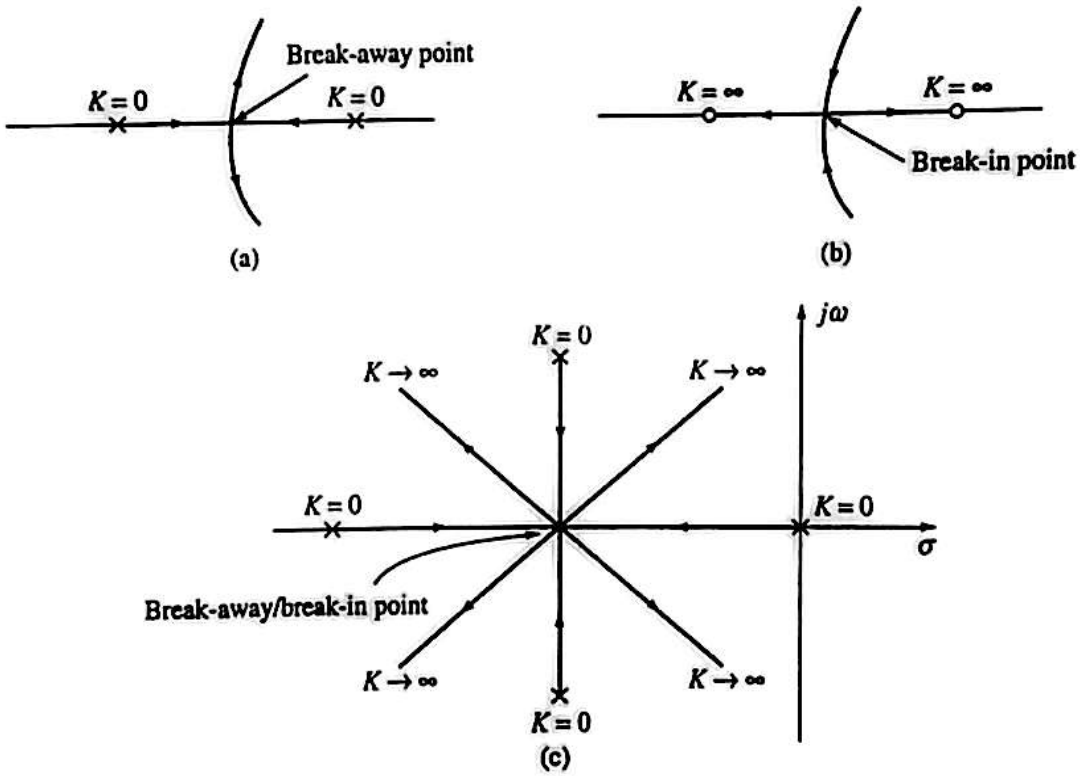


FIGURE 6.10 (a) Break-away point, (b) break-in point, and (c) break-away/break-in point.

Method I

Note that $F(s) = 1 + G(s)H(s) = 0$ has multiple roots at points where

$$\frac{d}{ds}F(s) = 0$$

The break-away or break-in points represent occurrence of multiple roots. Then the break-away/break-in points on the root loci of the characteristic equation

$$F(s) = 1 + G(s)H(s) = 1 + KG_1(s)H_1(s) = 0$$

must satisfy

$$\frac{d}{ds}[G(s)H(s)] = \frac{d}{ds}[G_1(s)H_1(s)] = 0 \tag{6.42}$$

It is important to point out that the condition for the break-away/break-in point given by the above equation is a necessary one but not sufficient. In other words, all break-away/break-in points must satisfy Eq. (6.42), but all solutions of this equation are not necessarily the break-away/break-in points. To be a break-away/break-in point, the solution of Eq. (6.42) must also satisfy the characteristic equation or must be a factor of the characteristic equation for some real value of the parameter K in the range $0 \le K < \infty$.

Break-away/break-in points are thus determined by solving the following equation:

$$\frac{d}{ds}[G(s)H(s)] = 0$$

For higher-order systems the method is laborious, sometimes a formidable task.

Method II

Let us take the characteristic equation as

$$F(s) = 1 + G(s)H(s) = 1 + KG_1(s)H_1(s) = 1 + K\frac{P(s)}{Q(s)} = 0$$

or

$$F(s) = Q(s) + KP(s) = 0 \tag{6.43}$$

where $Q(s)$ and $P(s)$ do not contain K. Note that $F(s) = 0$ has multiple roots at points where

$$\frac{dF(s)}{ds} = 0 \tag{6.44}$$

From Eq. (6.43), we obtain

$$\frac{dF(s)}{ds} = Q'(s) + KP'(s) = 0 \tag{6.45}$$

where $$Q'(s) = \frac{d}{ds}Q(s) \quad \text{and} \quad P'(s) = \frac{d}{ds}P(s)$$

The particular value of K which will yield multiple roots of the characteristic equation as obtained from Eq. (6.45) is

$$K = -\frac{Q'(s)}{P'(s)} \tag{6.46}$$

Substituting the value of K in Eq. (6.43), we get

$$F(s) = Q(s) - \frac{Q'(s)}{P'(s)} \times P(s) = 0$$

or

$$Q(s)P'(s) - Q'(s)P(s) = 0 \tag{6.47}$$

Again from Eq. (6.43),

$$K = -\frac{Q(s)}{P(s)}$$

Then

$$\frac{dK}{ds} = -\frac{Q'(s)P(s) - Q(s)P'(s)}{P^2(s)} \tag{6.48}$$

If dK/ds is set equal to zero, we get the equation which is same as Eq. (6.47), which is the Eq. (6.45) itself and is the condition for occurrence of multiple roots of the characteristic equation $F(s) = 0$ of the closed-loop system. Therefore, the break-away/break-in points can be simply determined from the roots of

$$\frac{dK}{ds} = 0 \tag{6.49}$$

Further, it may be noted that to be the break-away/break-in points, the solution of Eq. (6.49) should satisfy the characteristic equation. In fact, for a break-away point

$$\frac{d^2K}{ds^2} < 0 \tag{6.50}$$

and for a break-in point

$$\frac{d^2K}{ds^2} > 0 \tag{6.51}$$

The value of K at break-away points

As we have the value of s at the break-away point $s = s_1$ from either of the methods I and II, there is a necessity for knowing the value of gain K at the break-away point as

$$K = \left.\frac{1}{G_1(s)H_1(s)}\right|_{s=s_1} \tag{6.52}$$

This will give us important information concerning the design of the system.

As an example, consider $G(s)H(s)$ of Eq. (6.22), having the characteristic equation as

$$\frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)} = -1$$

The break-away points can be obtained by solving $dK/ds = 0$, i.e.

$$s^5 + 13.5s^4 + 66s^3 + 142s^2 + 123s + 45 = 0 \tag{6.53}$$

To determine the break-away points, we have to determine the roots of Eq. (6.53). From common sense and from the structure of root loci, a break-away point should lie between $-6 < s < -5$.

After a few trial-and-error calculations using the bisection theorem, the root of Eq. (6.53) that corresponds to the break-away point is found to be $s = -5.53$. As a check, the point $s = -5.53$ lies on the admissible range on the real axis where root loci can exist (Rule 6). Hence the break-away point is $s = -5.53$ as shown in Figure 6.8. The other four roots of Eq. (6.53) are $s = -0.656 \pm j0.468$ and $s = -3.33 \pm j1.204$. These four points are not the break-away points as they do not satisfy the characteristic equation. After going through all the rules for the construction of root loci, the root loci for the feedback system having the open-loop transfer function as

$$G(s)H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$$

can be drawn as shown in Figure 6.11.

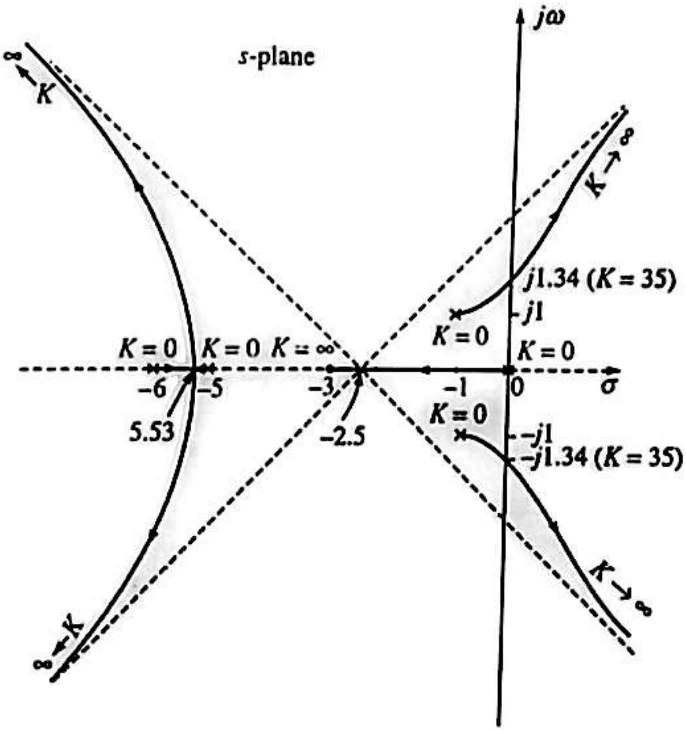


FIGURE 6.11 Root loci of $\dfrac{K(s+3)}{s(s+5)(s+6)(s^2+2s+3)}$.

Drill Problem 6.5

Determine the break-away and break-in points of the root loci plot of a unity-feedback system having the open-loop transfer function as

$$G(s)H(s) = \frac{K(s+3)}{s(s+2)}$$

Ans. -1.268; -4.732

Let us now summarize the steps utilized in evaluating the loci of roots of characteristic equation:

1. Write the characteristic equation in the pole-zero form so that the parameter of interest K appears as $1 + KG_1(s)H_1(s) = 0$.
2. Locate the open-loop poles and zeros of $G_1(s)H_1(s) = 0$.
3. Locate the segments of the real axis where root loci exist.
4. Determine the number of separate root loci.
5. Locate the angles of the asymptotes and the intersection of the asymptotes.
6. Determine the break-away point on the real axis (if any).
7. By utilizing the Routh–Hurwitz criterion, determine the point at which the locus crosses the imaginary axis (if it does so).
8. Estimate the angle of departure of root loci from complex poles and the angle of arrival of root loci at complex zeros.

Note that for solving any particular problem of root loci, we do not need all the rules of construction, but we have to be familiar with all the rules in order to construct the root loci of all systems.

We have studied in detail the construction procedures of root loci. Let us now test our knowledge. Draw the root-locus plot for the transfer function $G(s)H(s) = K/s$. Here the number of root locus is 1. The angle of asymptote is π. The existence of the root locus on the real axis is along the negative real axis where the angle condition is satisfied. See Figure 6.12(a).

Similarly, the root loci for transfer function $G(s)H(s) = K/s^2$, is as shown in Figure 6.12(b). Here the number of root loci is two. The angles of asymptotes are $\pi/2$ and $-\pi/2$. The existence of root loci on the real axis is nil, i.e. no root loci exists on the real axis because of the concerned property, that no root loci exists on the real axis if the sum of finite poles and zeros to the right of the point in s-plane is even.

The angle of departure condition is satisfied only on the imaginary axis.

Now we should be able to draw the rough sketch of root loci at a glance for the following open-loop transfer functions for $0 < K < \infty$.

$GH(s) = K/s$ as in Figure 6.12(a)

$GH(s) = K/s^2$ as in Figure 6.12(b)

$GH(s) = K/(s + 1)$ as in Figure 6.12(c)

$GH(s) = K/(s - 1)$ as in Figure 6.12(d)

$$GH(s) = K/(s+2)^2 \text{ as in Figure 6.12(e)}$$

$$GH(s) = K(s+1)/(s+2)^2 \text{ as in Figure 6.12(f)}$$

$$GH(s) = \frac{K(s+2)}{(s+1)^2} \text{ as in Figure 6.12(g)}$$

$$GH(s) = \frac{K}{s^2+2s+2} \text{ as in Figure 6.12(h)}$$

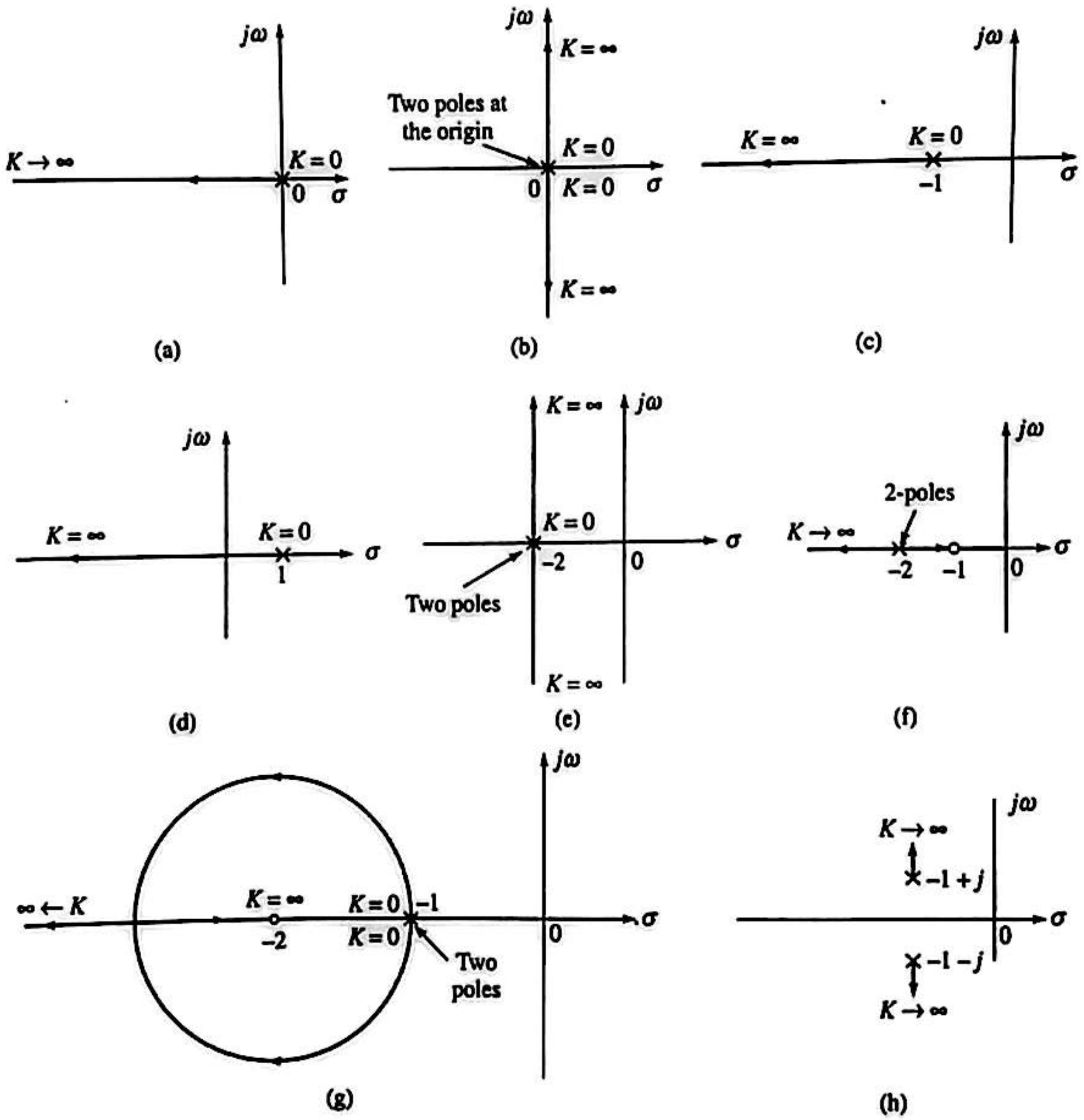


FIGURE 6.12 Root loci: (a) K/s, (b) K/s^2, (c) $K/(s+1)$, (d) $K/(s-1)$, (e) $K/(s+2)^2$, (f) $K(s+1)/(s+2)^2$, (g) $K(s+2)/(s+1)^2$, and (h) $K/(s^2+2s+2)$.

Drill Problem 6.6

Several pole-zero locations of open-loop transfer functions are shown in Figure 6.13. Draw the rough sketch of root loci plot.

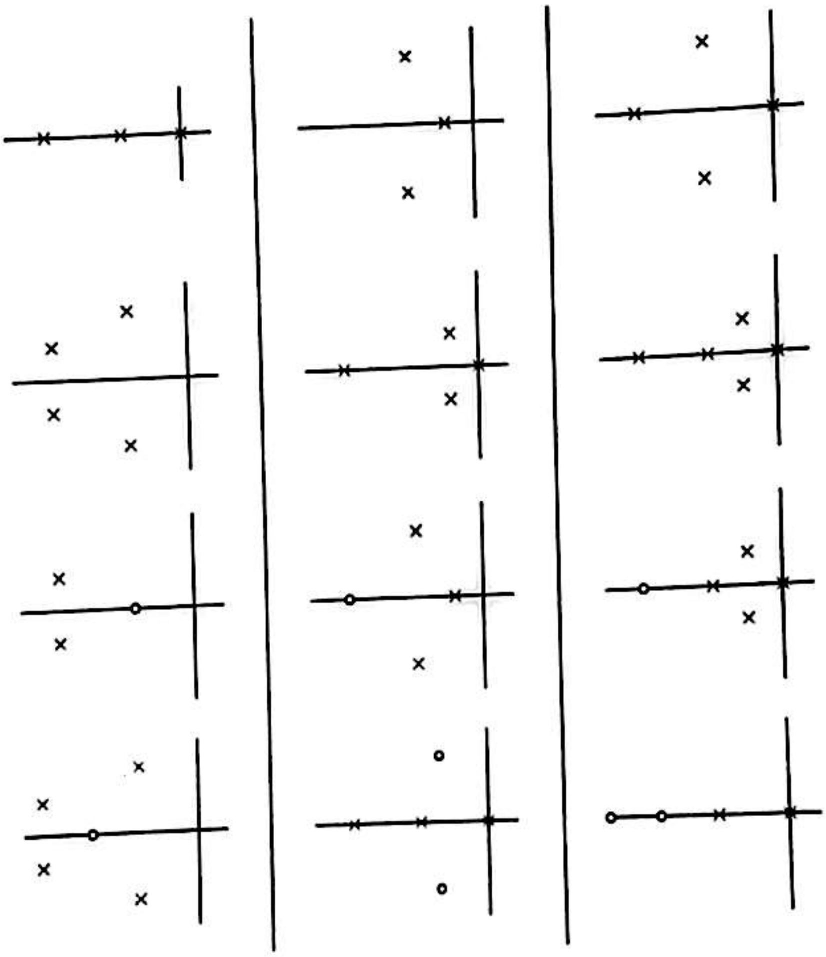


FIGURE 6.13 Drill Problem 6.6.

Rule 10. *Alternative method for obtaining break-away points:* For very high-order systems, solving $dK/ds = 0$ for obtaining the break-away points may be a tedious job, sometimes impossible. An algorithm for finding the break-away points on the root loci ($0 < K < \infty$) was introduced by Remec which is explained as follows:

Let the characteristic equation be

$$F(s) = A_n s^n + A_{n-1}s^{n-1} + \cdots + A_1 s + A_0 = 0 \tag{6.54}$$

Differentiating Eq. (6.54) and equating the result to zero,

$$F'(s) = B_n s^{n-1} + B_{n-1}s^{n-2} + \cdots + B_2 s + B_1 = 0 \tag{6.55}$$

Arranging the coefficients of $F(s)$ and $F'(s)$ in two rows,

$$\begin{matrix} A_n & A_{n-1} & A_{n-2} & \cdots & A_1 & A_0 \\ B_n & B_{n-1} & B_{n-2} & \cdots & B_1 & 0 \end{matrix} \tag{6.56}$$

As an example, consider a third-order characteristic equation,

$$F(s) = A_3 s^3 + A_2 s^2 + A_1 s + A_0$$

Then

$$F'(s) = B_3 s_2 + B_2 s + B_1$$

The tabulation of array is (note that it involves cross-multiplication as in Routh tabulation but not exactly as in Routh's array)

$$\begin{array}{lllll}
s^3 & A_3 & A_2 & A_1 & A_0 \\
s^2 & B_3 & B_2 & B_1 & 0 \\
s^2 & C_3 = \dfrac{B_3A_2 - B_2A_3}{B_3} & C_2 = \dfrac{B_3A_1 - B_1A_3}{B_3} & C_1 = \dfrac{B_3A_0 - 0\times A_3}{B_3} & \\
s^2 & B_3 & B_2 & B_1 &
\end{array}$$

$$\begin{array}{lll}
s^1 & D_3 = \dfrac{B_3C_2 - B_2C_3}{B_3} & D_2 = \dfrac{B_3C_1 - B_1C_3}{B_3} \\
s^1 & E_3 = \dfrac{D_3B_2 - D_2B_3}{D_3} & E_2 = \dfrac{D_3B_1 - 0\times B_3}{D_3} \\
s^1 & D_3 & D_2
\end{array}$$

$$s^0 \quad F_3 = \frac{D_3E_2 - D_2E_3}{D_3}$$

If $F(s)$ has multiple-order roots, which means that the root loci will have break-away points, a row of the tabulation shown above will contain all zero elements, that is, premature termination will occur. The multiple-order roots, which are the break-away points are obtained by solving the equation formed by using the row of coefficients just preceding the row of zeros.

The above alternative method for obtaining the break-away point is illustrated here by taking the open-loop transfer function of the feedback control system as

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

The characteristic equation is

$$F(s) = 1 + G(s)H(s) = 0$$

or
$$F(s) = s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

Then
$$F'(s) = s^3 + 6s^2 + 18s + 20 = 0 \tag{6.57}$$

The following tabulation is made:

$$\begin{array}{llllll} s^4 & 1 & 8 & 36 & 80 & K \end{array}$$

$$\left[\begin{array}{lllll} s^3 & 1 & 6 & 18 & 20 \\ s^3 & 2 & 18 & 60 & K \\ s^3 & 1 & 6 & 18 & 20 \end{array}\right.$$

$$\left[\begin{array}{llll} s^2 & 6 & 24 & K-40 \\ s^2 & 2 & \left(18-\dfrac{K-40}{6}\right) & 20 \\ s^2 & 6 & 24 & K-40 \end{array}\right. \tag{6.58}$$

$$\left[\begin{array}{lll} s^1 & \left(10-\dfrac{K-40}{6}\right) & \left(20-\dfrac{K-40}{3}\right) \\ s^1 & 12 & K-40 \\ s^1 & \left(10-\dfrac{K-40}{6}\right) & \left(20-\dfrac{K-40}{3}\right) \end{array}\right.$$

$$\begin{array}{ll} s^0 & K-64 \end{array}$$

All the elements of the first two of s^1-group would be zero for $K = 100$. Putting this value of $K = 100$ in the equation of the preceding row, i.e.

$$6s^2 + 24s + (K - 40) = 0 \tag{6.59}$$

or
$$s^2 + 4s + 10 = 0 \tag{6.60}$$

Therefore, the break-away points occur at

$$s = -2 \pm j2.45 \tag{6.61}$$

which are the solutions of Eq. (6.60).

For the other multiple roots (i.e. for the break-away points) any other value of $K \neq 100$, i.e. for $K \doteq 64$ all the elements of the s^0-row will be zero. Similarly, putting the value $K = 64$ in the equation formed from the preceding row, we get

$$\left(10 - \frac{K-40}{6}\right)s + \left(20 - \frac{K-40}{3}\right) = 0 \qquad \text{which gives} \quad s = -2$$

Hence $s = -2$ is the other break-away point which, in fact, lies on the admissible range of existence of root loci on the real axis.

EXAMPLE 6.1 Draw the root loci of the characteristic equation of the closed-loop system given as

$$s(s + 4)(s^2 + 4s + 20) + K = 0$$

Solution: Dividing both sides by the terms that do not contain K, we get

$$G(s)H(s) = 1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$$

The poles of $G(s)H(s)$ are at $s = 0$, -4, and $-2 \pm j4$ which are the starting points, i.e. $K = 0$.

The number of finite poles, $P = 4$ and finite zeros $Z = 0$; so the number of root loci, $N = 4$. The root loci are symmetrical about the real axis.

The angles of asymptotes are 45°, 135°, 225°, and 315°. The point of intersection of the asymptotes on the real axis is at $s = -2$.

The existence of asymptotes is on the real axis between $s = 0$ to $s = -4$.

The limiting value for the gain parameter K for stability is obtained as $K = 260$ by Routh–Hurwitz criterion from the characteristic equation $s^4 + 8s^3 + 36s^2 + 80s + K = 0$.

The points of intersection of root loci on the imaginary axis are obtained as $\omega = \pm 3.16$ by formulating the auxiliary equation and putting the limiting value of K in it.

The angle of departure of the root locus from the complex pole $-2 + j4$ is $-90°$.

For the break-away point, with

$$K = -s(s + 4)(s^2 + 4s + 20)$$

$$\frac{dK}{ds} = \frac{d}{ds}(s^4 + 8s^3 + 36s^2 + 80s) = 0$$

or $$s^3 + 6s^2 + 18s + 20 = (s + 2)(s + 2 + j2.45)(s + 2 - j2.45) = 0$$

The break-away points are at $s = -2$ and at $s = -2 \pm j2.45$. The root loci is shown in Figure 6.14.

The break-away points obtained here by solving $dK/ds = 0$, with $H(s) = 1$ and $K = -s(s + 4)(s^2 + 4s + 20)$ of the said open-loop transfer function are the same as those obtained by the tabular method.

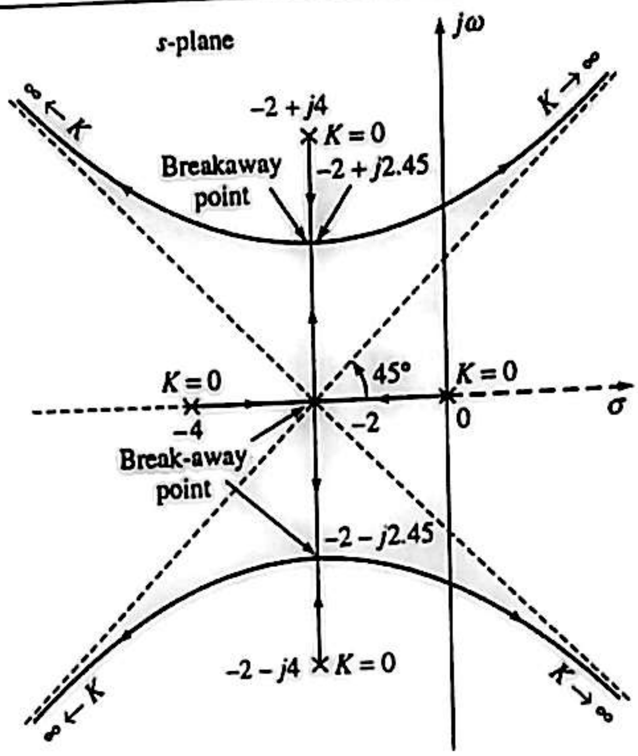


FIGURE 6.14 Example 6.1: root loci.

EXAMPLE 6.2 Draw the root loci of a unity-feedback system having open-loop transfer function $G(s)H(s) = \dfrac{K(s+2)}{s^2+2s+3}$. Determine:

(a) The value of K for which repetitive roots occur.
(b) The range of K for which the closed-loop system becomes underdamped.
(c) The value of K for which the system will have damping ratio of 0.7.

Solution: $G(s)H(s) = \dfrac{K(s+2)}{s^2+2s+3} = \dfrac{K(s+2)}{(s+1-j\sqrt{2})(s+1+j\sqrt{2})}$

In order to draw the root loci, let us follow the following rules:

1. $K = 0$ are the starting points, i.e. the open-loop poles are at, $s = -1 \pm j\sqrt{2}$
2. $K = \infty$, are the terminating points, i.e. one finite zero is at $s = -2$ and the other zero is at $s = \infty$.
3. The number of finite poles, $P = 2$ and the number of finite zeros, $Z = 1$. Hence the number of root loci, $N = 2$. One root locus will terminate at $s = \infty$. Hence only one angle of asymptote is required.
4. The root loci will be symmetric about the real axis.

5. Angles of asymptote are

$$\theta_k = \frac{(2k+1)\pi}{P-Z}; \qquad k = 0, 1, \cdots, \overline{P-Z-1} \Rightarrow \theta_0 = \pi$$

6. Existence of root loci on the real axis is between $s = -2$ and $s = -\infty$.
7. Intersection of root loci with the imaginary axis does not occur.
8. Angle of departure: See Figure 6.15(a) where we choose a test point s_1 in the vicinity of the complex open-loop pole at $(-1 + j\sqrt{2})$. Then if the test point lies on the root loci, the angle condition has to be satisfied and from Figure 6.15(a), we get

$$\theta_z - (\theta_p + \theta_p^*) = 180°$$

or

$$\theta_p = \theta_z - 180° - \theta_p^* = 55° - 180° - 90° = 145°$$

9. The break-in point is obtained from

$$K = \frac{s^2 + 2s + 3}{s + 2}$$

or

$$\frac{dK}{ds} = -\frac{(2s+2)(s+2) - (s^2 + 2s + 3)}{(s+2)^2} = 0$$

which gives $s^2 + 4s + 1 = 0$ or $s = -3.732$ and $s = -0.268$

The point $s = -3.732$ is the break-in point and the other point at $s = -0.268$ is not because it is not within the permissible range of root loci on the real axis. The root loci is drawn in Figure 6.15(b).

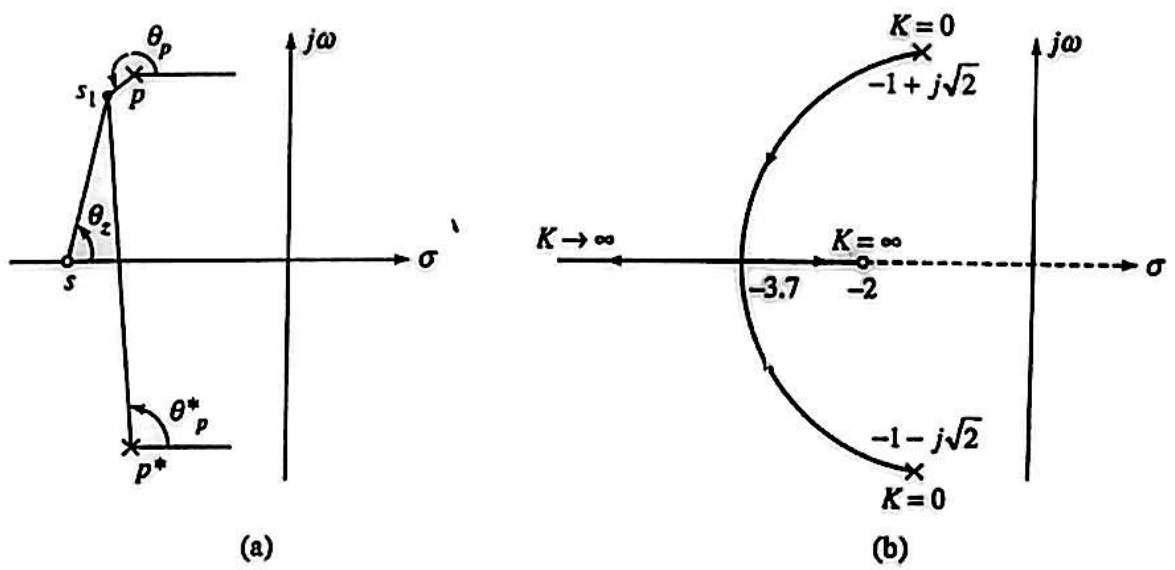


FIGURE 6.15 Example 6.2: (a) angle of departure from $p = -1 + j\sqrt{2}$ to s_1 on the root loci of $G(s)H(s) = K(s + 2)/(s^2 + 2s + 3)$ and (b) root loci of $G(s)H(s) = K(s + 2)/(s^2 + 2s + 3)$.

To show the occurrence of a circular root loci in the present system, we need to derive the equation for the root locus. From the angle condition,

$$(\angle s+2)-(\angle s+1-j\sqrt{2})-(\angle s+1+j\sqrt{2}) = \pm(2k+1)\pi$$

Putting $s = \sigma + j\omega$, we get

$$\tan^{-1}\left(\frac{\omega-\sqrt{2}}{\sigma+1}\right)+\tan^{-1}\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right) = \tan^{-1}\left(\frac{\omega}{\sigma+2}\right) \pm (2k+1)\pi$$

Using the relation, $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$, we get

$$\tan\left[\tan^{-1}\left(\frac{\omega-\sqrt{2}}{\sigma+1}\right)+\tan^{-1}\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right)\right] = \tan\left[\tan^{-1}\left(\frac{\omega}{\sigma+2}\right)\pm(2k+1)\pi\right]$$

or
$$\frac{\dfrac{\omega-\sqrt{2}}{\sigma+1}+\dfrac{\omega+\sqrt{2}}{\sigma+1}}{1-\dfrac{\left(\omega-\sqrt{2}\right)\left(\omega+\sqrt{2}\right)}{(\sigma+1)^2}} = \frac{\dfrac{\omega}{\sigma+2}\pm 0}{1\mp\left(\dfrac{\omega}{\sigma+2}\right)\times 0}$$

or
$$\omega[(\sigma+2)^2+\omega^2-3] = 0$$

or
$$\omega = 0, \quad (\sigma+2)^2+\omega^2 = (\sqrt{3})^2$$

The first equation $\omega = 0$ is the real axis equation from $s = -2$ to $s = -\infty$ and the second one is the equation of a circle with centre at ($\sigma = -2$, $\omega = 0$) and radius equal to $\sqrt{3}$.

It is to be noted that for complicated systems having many poles and zeros, any attempt to derive the equation for root loci is discouraged. Let us now find the answers to the given problem.

(a) The value of gain K at the break-in point (where the repetitive root occurs, that is, the damping factor is just unity) is obtained as

$$\left|\frac{K(s+2)}{s^2+2s+3}\right|_{s=-3.732} = 1 \quad \text{or} \quad K = 5.4641$$

(b) The range of K for which the system becomes underdamped (i.e. complex conjugate poles) is

$$0 < K < 5.4641$$

(c) Here the damping ratio = 0.7, i.e. $\cos\theta = 0.7$, $\theta = 45.57°$

$\therefore \quad \tan\theta = 1.0202 \quad$ or $\quad \omega = 1.0202\sigma$

Putting this value of ω in the equation of the circle,

$$(\sigma + 2)^2 + \omega^2 = (\sqrt{3})^2, \text{ we get } \sigma = -1.6659$$

Again, $\sigma = \zeta\omega_n = 0.7\omega_n$ and $\omega = \omega_n\sqrt{1-\zeta^2} = 1.6995$

Hence the value of K at $s = -1.6659 + j1.6995$ can be obtained as

$$K = \left|\frac{(s+1-j\sqrt{2})(s+1+j\sqrt{2})}{s+2}\right|_{s=-1.6659+j1.6995} = 1.3318$$

Therefore, $K = 1.3318$ is the value of gain where the complex conjugate closed-loop poles will have the damping ratio $\zeta = 0.7$.

EXAMPLE 6.3 Draw the root loci of the closed-loop system for the given open-loop transfer function

$$G(s)H(s) = \frac{K}{s(s+4)(s+4+j4)(s+4-j4)}$$

as K varies from zero to infinity.

Solution:

1. The poles (i.e. $K = 0$ are the starting points of root loci) are located at $s = 0, -4, -4 + j4, -4 - j4$.
2. There is no finite zero.
3. The number of finite poles, $P = 4$ and the number of finite zeros, $Z = 0$, so the number of separate root loci, $N = 4$.
4. The root loci are symmetric about the real axis.
5. Angles of asymptotes are:

$$\theta_k = \frac{(2k+1)\pi}{P-Z}; \qquad k = 0, 1, \ldots, \overline{P-Z-1}$$

$$= \frac{(2k+1)\pi}{4}, \qquad k = 0, 1, \ldots, 3$$

$$= 45°, 135°, 225°, 315°$$

The intersection of asymptotes is

$$\sigma_A = \frac{-4-4-4}{4} = -3$$

The asymptotes are drawn as shown in Figure 6.16(a).

6. Existence of root loci on the real axis is between $s = 0$ and $s = -4$.

7. The characteristic equation is rewritten as

$$s(s+4)(s^2+8s+32)+K = s^4+12s^3+64s^2+128s+K = 0$$

Therefore, the Routh array is:

$$\begin{array}{llll} s^4 & 1 & 64 & K \\ s^3 & 12 & 128 & \\ s^2 & b_1 & K & \\ s^1 & c_1 & & \\ s^0 & K & & \end{array}$$

where $\quad b_1 = \dfrac{12\times 64-128}{12} = 53.33 \qquad$ and $\qquad c_1 = \dfrac{53.33\times 128-12K}{53.33}$

Hence the limiting value of gain for stability is $K = 568.85$ and the roots of the auxiliary equation are

$$53.33s^2+570 = 53.33(s+j3.25)(s-j3.25)$$

The points where the root loci cross the imaginary axis are obtained from

$$53.33s^2+K=0 \qquad \text{which leads to} \qquad \omega = \pm 3.266$$

The limiting value $K = 568.85$ signifies that the closed-loop system is stable for $0 < K < 568.85$. If the gain K is increased beyond $K = 568.85$, the closed-loop system will become unstable. Now if you are asked, what would be the gain margin for the value of gain $K_1 = 100$. The question means that the margin of gain at $K_1 = 100$ would be $20\log_{10}(K/K_1)$ in dB before the system goes to instability.

An alternative approach is to let $s = j\omega$ in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for ω and K. For the present system, the characteristic equation with $s = j\omega$, is

$$(j\omega)^4+12(j\omega)^3+64(j\omega)^2+128(j\omega)+K = 0$$

or

$$(\omega^4-64\omega^2+K)+j4\omega(32-3\omega^2) = 0$$

Equating both the real and imaginary parts of this last equation to zero, we obtain

$$32-3\omega^2 = 0 \qquad \text{and} \qquad \omega^4-64\omega^2+K = 0$$

from which, we obtain $\omega = 3.2666$ and $K = 568.85$.

8. The angle of departure at the complex pole $p_1 = -4+j4$ can be estimated by utilizing the angle criterion as

$$\theta_{p_1} = -135^\circ = +225^\circ$$

9. The break-away point is obtained from $dK/ds = 0$ which gives

$$s^3+9s^2+32s+32 = 0$$

To solve this equation, i.e. to get the roots is a tedious job. The break-away point is therefore estimated by evaluating

$$K = P(s) = -s(s+4)(s+4+j4)(s+4-j4)$$

between $s = -4$ and $s = 0$. We expect the break-away point to lie between $s = -3$ and $s = -1$ and therefore we search for a maximum value of $P(s)$ in that region. The maximum of $P(s)$ is found to lie at approximately $s = -1.5$. A more accurate estimate of the break-away point is normally not necessary or worthwhile. A closer estimate is in fact -1.57 as indicated in Figure 6.16(a). The root loci are shown in Figure 6.16(b).

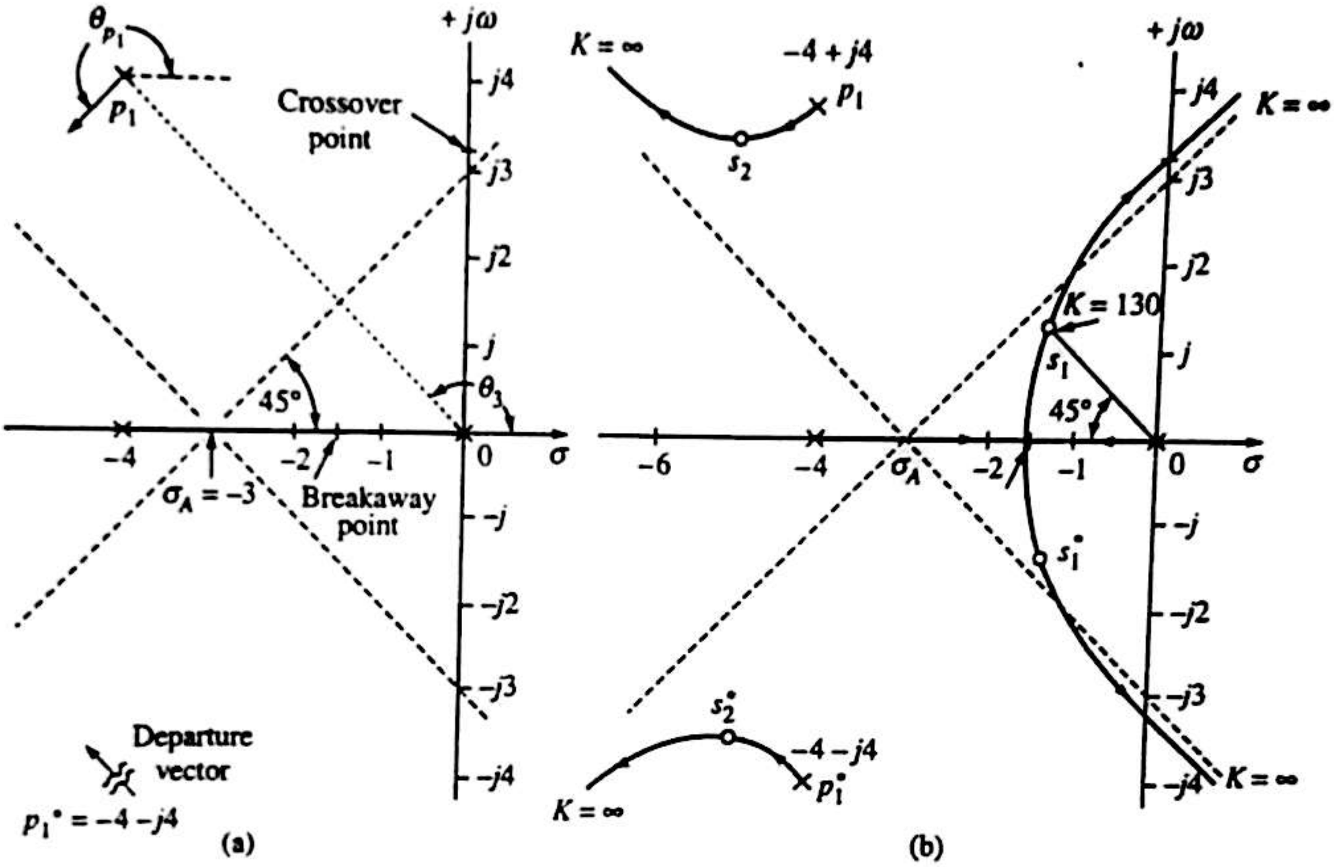


FIGURE 6.16 Example 6.3: (a) asymptotes and (b) root loci.

Suppose you are now asked to find the value of K at $\zeta = \sqrt{2}$. The gain K can be determined graphically as shown in Figure 6.16(b). Draw a line $\theta = 45°$ (as $\zeta = \sqrt{2} = \cos\theta$ gives $\theta = 45°$). The intersection of root loci and $\theta = 45°$ line gives s_1 and s_1^* points graphically. The vector lengths to the root location s_1 from the open-loop poles are evaluated and result in a gain at s_1 of

$$K = |s_1||s_1+4||s_1-p_1||s_1-p_1^*| = (1.9)(3)(3.8)(6) \approx 130$$

The remaining pair of complex conjugate roots of closed-loop system occurs at s_2 and s_2^* when $K = 130$. It is obvious that at $K = 130$, s_1, s_1^* are the dominant pole pair compared to s_2 and s_2^* pole pair.

EXAMPLE 6.4 Draw the root loci of the open-loop transfer function of the feedback control system

$$G(s)H(s) = \frac{K}{s(s+4)(s+5)}$$

Solution: We have finite open-loop poles at $s = 0, -4, -5$ and no finite zero. The number of root loci is $N = 3$, as $P = 3$, $Z = 0$. The root loci on the real axis will exist between 0 and -4, and between -5 and ∞. All the three root loci will terminate at zeros at ∞. The angles of asymptotes are: $\theta_0 = \pi/3$, $\theta_1 = \pi$ and $\theta_2 = 5\pi/4 = -\pi/3$.

The intersection of the asymptotes on the real axis is: $\sigma_A = -3$

Now, $$K = -s(s+4)(s+5) = -s^3 - 9s^2 - 20s$$

The break-away point is obtained from $dK/ds = 0$ which gives

$$-3s^2 - 18s - 20 = 0$$

that is, $s = -1.4725$ and -4.5275 of which the only one at $s = -1.4725$ is the admissible break-away point as it lies on the segment between $s = 0$ and -4. The point -4.5275 does not exist on the admissible segment of the root loci on the real axis and hence need not be considered.

The value of K at break-away point $s = -1.4725$ is obtained as

$$K = -s(s+4)(s+5)\Big|_{s=-1.4725} = 13.128$$

The intersection of root loci with the imaginary axis is determined from Routh's table, with the characteristic equation $s^3 + 9s^2 + 20s + K = 0$. The Routh's table is

s^3	1	20
s^2	9	K
s^1	$\frac{180-K}{9}$	0
s^0	K	

Hence the critical gain before the closed-loop system goes to instability is $K_c = 180$, and the auxiliary equation is $9s^2 + K = 0$. Putting $K = 180$, we get

$$s = \pm j2\sqrt{5} \quad \text{i.e.} \quad \omega = \pm 2\sqrt{5}$$

A sketch of the complete root loci is shown in Figure 6.17.

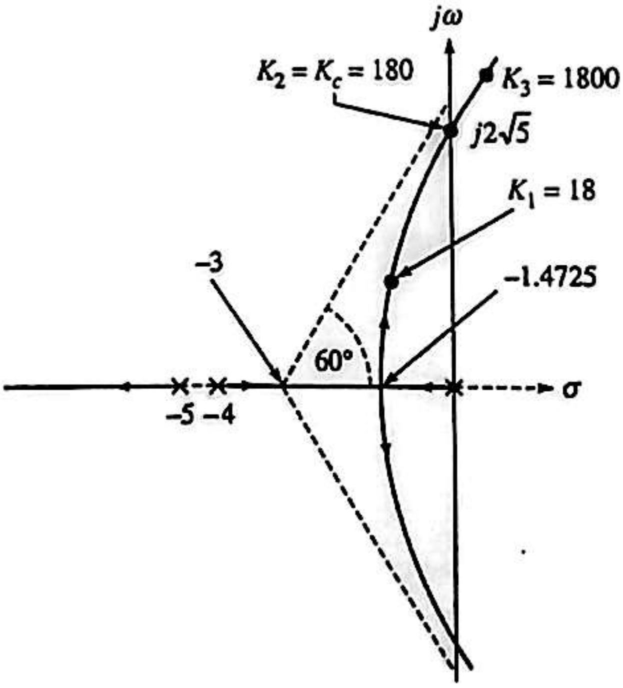


FIGURE 6.17 Example 6.4: root loci.

Now suppose you are asked to find the gain margin for the following gains:

(i) For $K_1 = 18$, the margin of gain is $20\log(K_c/K_1) = 20$ dB, the closed-loop system is stable and margin of gain is 20 dB before the system goes to instability.

(ii) For $K_2 = 180$, the margin of gain is $20\log(K_c/K_2) = 0$ dB; the closed-loop system is criticality (just) stable, no margin of gain is left to play with before the system goes to instability.

(iii) For $K_3 = 1800$, the gain margin is -20 dB, the closed-loop is unstable.

These observations are evident from Figure 6.17. This concept is useful for understanding the design parameters.

EXAMPLE 6.5 Sketch the root loci of the open-loop transfer function of the feedback control system

$$G(s)H(s) = \frac{K}{s(s^2+6s+25)}; \qquad 0 < K < \infty$$

Determine the gain margin at $K = K_1 = 15$ and at $K = K_2 = 1500$.

Solution: The poles are at $s = 0$ and $s = -3 \pm j4$. Here $P = 3$, $Z = 0$. The number of root loci, $N = 3$, each terminating on a zero at infinity. The root loci on the real axis exist on the entire negative real axis. The angles of asymptotes are

$$\theta_0 = \pi/3, \qquad \theta_1 = \pi \qquad \text{and} \qquad \theta_2 = 5\pi/3 = -\pi/3$$

The intersection of asymptotes on the real axis is

$$\sigma_A = \frac{0-3+j4-3-j4}{3} = -2$$

The characteristic polynomial is

$$s^3 + 6s^2 + 25s + K$$

Routh's array becomes:

$$\begin{array}{lcc} s^3 & 1 & 25 \\ s^2 & 6 & K \\ s^1 & \dfrac{150-K}{6} & 0 \\ s^0 & K & \end{array}$$

Hence $K = 150$ when root loci just cross the imaginary axis before going to the positive-half s-plane. The auxiliary equation is $6s^2 + K = 0$.

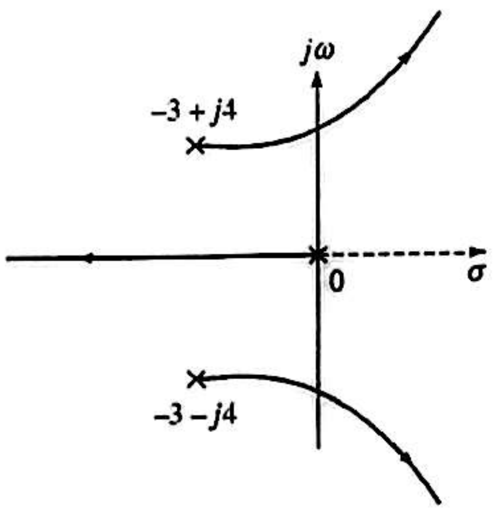


FIGURE 6.18 Example 6.5: root loci.

Putting $K = 150$ in the auxiliary equation, we get $s = \pm j5$, i.e. $\omega = \pm 5$ as the cross-over point on the imaginary axis.

The angle of departure from the upper complex pole $(-3 + j4)$ is calculated as

$$\theta = 180° - (90° + 126.87°) = -36.87°$$

The sketch of the root loci is shown in Figure 6.18. Note that the determination of break-away point is not necessary here, though it may give the value of $s = 2 \pm j2.0817$ from $dK/ds = 0$, which does not satisfy the characteristic equation. We do not have any break-away point on real axis.

Now the gain margin at $K_1 = 15$, is +20 dB, the closed-loop system is therefore stable. For $K_2 = 1500$ the gain margin is –20 dB; the closed-loop system is therefore unstable.

It may be noted that all the rules of construction may not be necessary for drawing the root-locus plot, as can be in this case where the break-away point is not necessary.

EXAMPLE 6.6 Draw the root loci of the open-loop transfer function of the feedback control system

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

Solution: Here $P = 4$, $Z = 0$, $N = 4$. Poles are at $s = 0, -2$ and $-3 \pm j4$. The root loci on the real axis exist only between $s = 0$ and -2. All the four root loci will terminate on zeros at infinity. The intersection of asymptotes on the real axis is $\sigma_A = -2$.

The angles of asymptotes are

$$\theta_0 = \pi/4,\ \theta_1 = 3\pi/4,\ \theta_2 = 5\pi/4 = -3\pi/4, \text{ and } \theta_3 = 7\pi/4 = -\pi/4$$

Here, $$K = -s(s+2)(s^2+65+25)$$

Therefore, $$\frac{dK}{ds} = -(4s^3+24s^2+74s+50) = 0$$

The break-away point is searched between $s = 0$ and $s = -2$ and is found to be at $s = -0.8981$ and the other two are in the non-admissible range on the real axis of root loci and hence ignored. The value of K at the break-away point $s = -0.8981$ is obtained as

$$K = -s(s+2)(s^2+6s+25)\Big|_{s=-0.8981} = 20.206$$

The characteristic polynomial is

$$s^4 + 8s^3 + 37s^2 + 50s + K$$

Routh's array is:

s^4	1	37	K
s^3	8	50	
s^2	30.75	K	
s^1	$\dfrac{1537.5-8K}{30.75}$		
s^0	K		

Hence $K = 1537.5/8 = 192.1875$.

The auxiliary equation is

$$30.75s^2 + K = 0$$

Putting $K = 192.1875$ in the auxiliary equation, we get $s = \pm j2.5$, i.e. $\omega = \pm 2.5$. The angle of departure from the upper complex pole $(-3 + j4)$ is given by

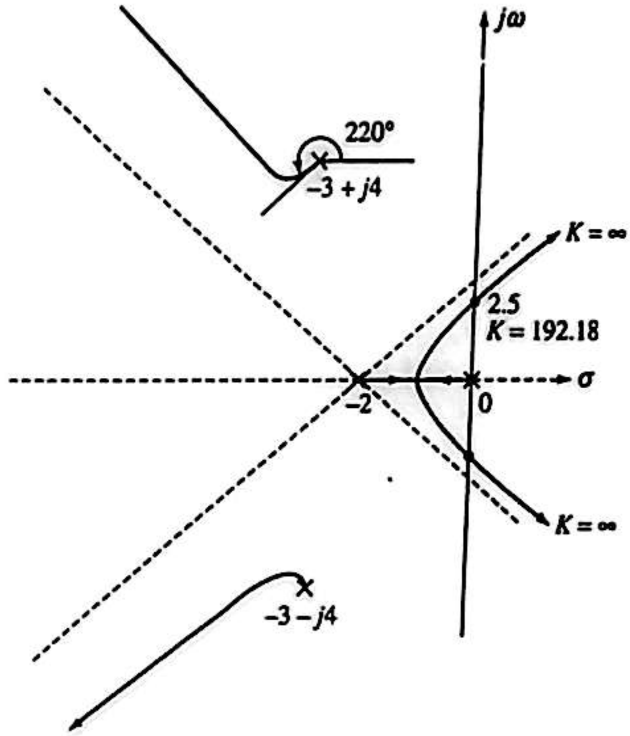


FIGURE 6.19 Example 6.6: root loci.

$$\theta = 180° - (90° + 126.87° + 104.04°) = -140.91°$$

The root loci are shown in Figure 6.19.

EXAMPLE 6.7 Draw the root loci of open-loop transfer function of the feedback control system given as

$$G(s)H(s) = \frac{K(s+3)}{s(s+2)}; \qquad 0 < K < \infty$$

Solution: The poles are at $s = 0$ and -2. The zero is at $s = -3$. The number of poles $P = 2$, and the number of finite zeros $Z = 1$. The number of root loci, $N = 2$. The root loci on the real axis lie between $s = 0$ and -2, and between $s = -3$ to infinity. The angles of asymptotes are: $\theta_0 = \pi$, $\theta_1 = 2\pi = 0°$.

From the characteristic equation the magnitude condition $|GH| = 1/K$ leads to

$$K = -\frac{s(s+2)}{s+3}$$

Then,
$$\frac{dK}{ds} = -(s^2 + 6s + 6) = 0$$

gives $s_1 = -4.1732$ and $s_2 = -1.268$ as the break-away/break-in points. In fact, $s_1 = 1.268$ is the break-away point as $\left.\dfrac{d^2K}{ds^2}\right|_{s=s_1} > 0$, and $s_2 = -4.1732$ is the break-in point as $\left.\dfrac{d^2K}{ds^2}\right|_{s=s_2} < 0$.

Clearly, for finding the break-away/break-in point, dK/ds is the preferred option than dG/ds because for all physically realizable systems the degree of denominator polynomial of open-loop transfer function is higher or equal to that of the numerator polynomial.

The root loci will be circular in the complex s-plane. This may be proved as follows: On the root loci, the angle condition becomes

$$\angle G(s)H(s) = \angle s+3 - \angle s - \angle s+2 = 180^\circ \tag{i}$$

Putting $s = \sigma + j\omega$ in Eq. (i), we have

$$\angle \sigma + j\omega + 3 - \angle \sigma + j\omega - \angle \sigma + j\omega + 2 = 180^\circ \text{ which can be rewritten as}$$

$$\tan^{-1}\left(\frac{\omega}{\sigma+3}\right) - \tan^{-1}\left(\frac{\omega}{\sigma}\right) = 180^\circ + \tan^{-1}\left(\frac{\omega}{\sigma+2}\right)$$

or
$$\tan\left[\tan^{-1}\frac{\omega}{\sigma+3} - \tan^{-1}\frac{\omega}{\sigma}\right] = \tan\left[180^\circ + \tan^{-1}\frac{\omega}{\sigma+2}\right]$$

or
$$\frac{-3\omega}{\sigma(\sigma+3)+\omega^2} = \frac{\omega}{\sigma+2} \qquad \left(\text{Using } \tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \text{ and simplifying}\right)$$

or
$$(\sigma+3)^2 + \omega^2 = (\sqrt{3})^2$$

The above equation represents a circle with centre at ($\sigma = -3$, $\omega = 0$) and radius $\sqrt{3}$. These can be rechecked from the break-away and break-in points. The root loci are shown in Figure 6.20. The gain K at the break-away point -1.268 is obtained as $K_1 = 0.5359$ and at the break-in point -4.1732 the gain is obtained as $K_2 = 7.73$.

Hence the closed-loop system is overdamped for gain K in the ranges, $0 < K < 0.5359$ and $7.73 < K < \infty$ and the closed-loop system is underdamped in the range $0.05359 < K < 7.73$.

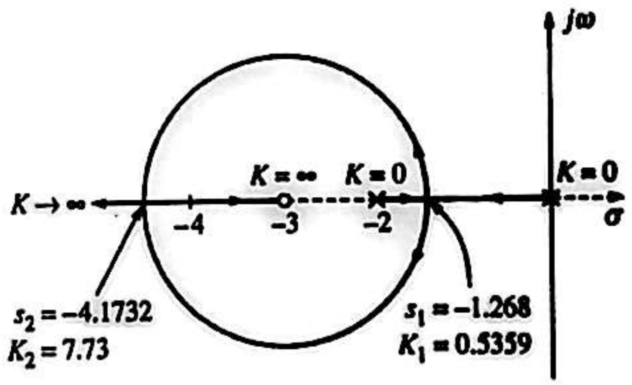


FIGURE 6.20 Example 6.7: root loci.

EXAMPLE 6.8 For the open-loop transfer function $20(1 + ks)/s(s + 1)(s + 4)$, (a) draw the root loci and (b) determine the value of K such that the damping ratio is one of the closed-loop poles, i.e. 0.4.

Solution: (a) $G(s)H(s) = \dfrac{20(1+ks)}{s(s+1)(s+4)}$

The characteristic equation is

$$1 + G(s)H(s) = 0$$

or

$$s^3 + 5s^2 + 4s + 20 + 20ks = 0$$

Assuming $20k = K$, we can rewrite

$$1+\frac{Ks}{s^3+5s^2+4s+20} = 0$$

or

$$1+\frac{Ks}{(s+j2)(s-j2)(s+5)} = 0$$

where $\qquad G(s)H(s) = \dfrac{Ks}{(s+5)(s+j2)(s-j2)}; \qquad$ for $\qquad 0 < (K = 20k) < \infty$

For the construction of the root loci, we follow the following procedure:

(i) Starting points ($K = 0$) are at finite poles, $s = -5$, $s = -j2$, $s = +j2$.

(ii) Terminating points ($K = \infty$) are at $s = 0$, $s = \infty$, $s = \infty$.

(iii) Number of root loci, $N = 3$, the number of finite poles, $P = 3$ and that of finite zeros, $Z = 1$.

(iv) Symmetry of root loci about the real axis holds.

(v) Angles of asymptotes are

$$\theta_0 = \pi/2, \; \theta_1 = -\pi/2$$

The intersection of the asymptotes on the real axis is

$$\frac{\Sigma\text{poles} - \Sigma\text{zeros}}{P-Z} = -\frac{5}{2}$$

(vi) Existence of root loci on real axis is between $s = 0$ to $s = -5$.

(vii) Intersection of root loci with the imaginary axis does not occur in this example.

(viii) Angle of departure (angle θ) from the pole at $s = j2$ is obtained as

$$\theta = 180° - 90° - 21.8° + 90° = 158.2°$$

(ix) Break-away/break-in point does not occur in this example.

The root loci diagram for the system is shown in Figure 6.21.

(b). The damping ratio $\zeta = 0.4$ means that $\zeta = \cos\,\theta = 0.4$, i.e. $\theta = \pm 66.42°$ with the negative-real axis. There are two intersections of the root loci branch in the upper-half of

s-plane with the straight line of angle 66.42°. Thus, these two values of K will give the damping ratio of the closed-loop poles equal to 0.4. At point P the value of K is

$$K = \frac{(s+j2)(s-j2)(s+5)}{s}\bigg|_{s=-1.05+j2.4} = 8.98$$

or

$$k = K/20 = 0.449 \text{ at point P.}$$

At point Q, the value of K is

$$K = \frac{(s+j2)(s-j2)(s+5)}{s}\bigg|_{s=-2.15+j4.95} = 28.26$$

or

$$k = K/20 = 1.413 \text{ at point Q.}$$

Hence for $k = 0.449$, the three closed-loop poles are

$$s = -1.05 \pm j2.4, \quad s = -2.902$$

The closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{20}{s^3 + 5s^2 + 12.98s + 20}$$

The unit-step response becomes

$$c(t) = 1 - 0.747e^{-2.902t} - 0.253e^{-1.05t}\cos(2.4t) - 1.0113e^{-1.05t}\sin(2.4t)$$

For $k = 1.413$, the three closed-loop poles are at $s = -2.15 \pm j4.95$ and $s = -0.6823$.

The closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{20}{s^3 + 5s^2 + 32.26s + 20}$$

The unit-step response is

$$c(t) = 1 - 1.0924e^{-0.6823t} + 0.0924e^{-2.15t}\cos(4.95t) - 0.1102e^{-2.15t}\sin(4.95t)$$

The response is shown in Figure 6.22. Clearly, the oscillatory terms damp out much faster than the purely exponential term. The system with $k = 0.449$ which exhibits faster response with small overshoot has much better characteristic than the system with $k = 1.413$ which exhibits a slow overdamped response. Therefore, we should choose $k = 0.4490$ for the present system.

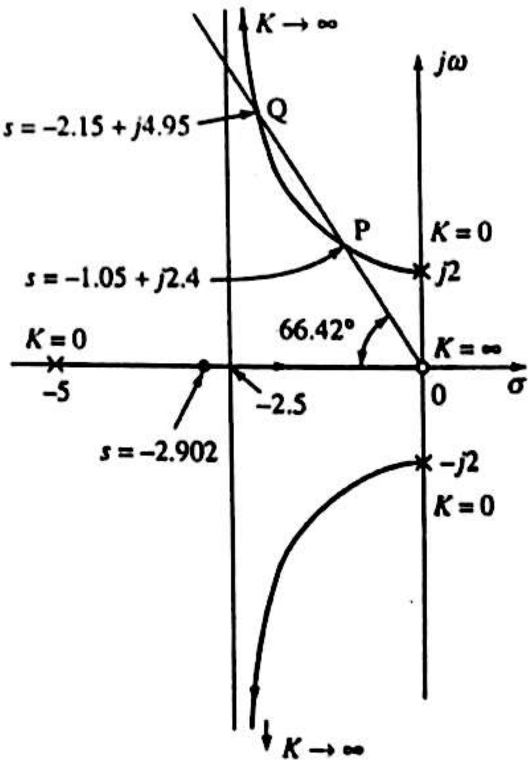


FIGURE 6.21 Example 6.8: root loci.

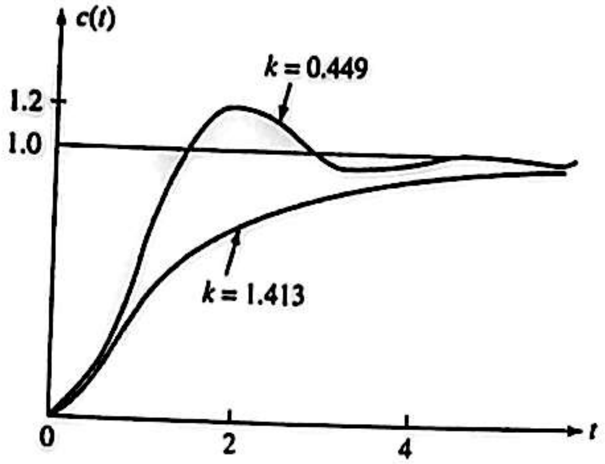


FIGURE 6.22 Example 6.8: system response.

EXAMPLE 6.9 Consider the open-loop system transfer function

$$G(s)H(s) = \frac{K(s^2+1.5s+1.5625)}{(s-0.75)(s+0.25)(s+1.25)(s+2.25)};\quad K>0$$

Draw the root loci diagram. Determine the values of gain for a stable closed-loop system.

Solution: By inspection, the denominator and the numerator polynomials are of proper form. The coefficient of s^2 in the numerator polynomial and that of s^4 in the denominator polynomial is unity.

(i) The poles are at $s = 0.75, -0.25, -1.25$ and -2.25 which are the starting points of root loci, i.e. $K = 0$ points of the root loci.

(ii) Finite zeros at $s = -0.75 \pm j1$ are the terminating points of the two root loci, i.e. $K = \infty$ points and the other two root loci terminate at infinity.

(iii) The number of root loci is 4 as $P = 4$ and $Z = 2$.

(iv) The root loci are symmetrical about the real axis.

(v) The angles of asymptotes are given by

$$\theta_k = \frac{(2k+1)\pi}{P-Z};\qquad k = 0, 1\qquad \text{i.e.}\qquad \theta_0 = \pi/2;\qquad \theta_1 = 3\pi/2$$

The intersection of the asymptotes with the real axis is

$$= \frac{(0.75-0.25-1.25-2.25)-(-0.75-0.75)}{4-2} = -0.75$$

(vi) The existence of root loci on the real axis is in between $s = 0.75$ and $s = -0.25$ and also between $s = -1.25$ and $s = -2.25$.

(vii) The point of intersection of the root loci with the imaginary axis is obtained from Routh–Hurwitz criterion. The characteristic equation of the closed-loop system is obtained from

$$1 + G(s)H(s) = 0$$

i.e. $$s^4 + 3s^3 + (0.875 + K)s^2 + (1.5K - 2.063)s + (1.563K - 0.527) = 0$$

The Routh array for this equation is formulated as

$$\begin{array}{llll}
s^4 & 1 & 0.875 + K & 1.563K - 0.527 \\
s^3 & 3 & 1.5K - 2.063 & \\
s^2 & \dfrac{3(0.875 + K) - (1.5K - 2.063)}{3} & 1.563K - 0.527 & \\
s^1 & A & & \\
s^0 & 1.563K - 0.527 & &
\end{array}$$

To be on the imaginary axis. the s^1 row must be an all-zero row, i.e.

$$A = \frac{3(0.875 + K) - (1.5K - 2.063)}{3} \times (1.5K - 2.063) - 3(1.563K - 0.527) = 0$$

which may be rearranged to give

$$0.75K^2 - 3.377K - 1.643 = 0 \qquad \text{or} \qquad K = 2.251 \pm 2.694$$

Since only positive values of K are required, then $K = 4.945$.
Note that the stable range of the value of K is $\infty > K > 4.945$ for the closed-loop system to be stable.

Putting this value of $K = 4.945$ in the equation obtained from the second row of Routh array, i.e. $3s^2 + (1.5K - 2.063) = 0$, we get

$$\omega = \sqrt{\left(\frac{1.5 \times 4.945 - 2.063}{3}\right)} = \pm 1.34$$

(viii) The angle of arrival at the complex zero $s = -0.75 \pm j1$ is obtained as in Figure 6.23. From the angle criterion, we get

$$(90° + \phi) - \left[(180° - 34°) + 34° + (180° - 63°) + 69°\right] = (2k + 1)\pi; \text{ where } k \text{ is an integer}$$

i.e. $\phi = 90°$

(ix) The break-away/break-in points are obtained from $dK/ds = 0$, as

$$s = 0.26, -1.76 \qquad \text{and} \qquad -0.75 \pm j1.74$$

From the above rules, the root loci may now be drawn as shown in Figure 6.24. It may be noted that it would be a good exercise for the reader to find the value of gain K at the break-away and break-in points. The root loci obtained by MATLAB is also shown (see MATLAB Program 6.1).

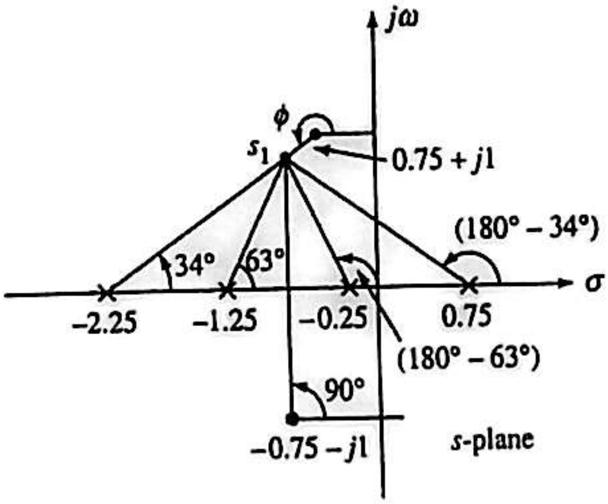


FIGURE 6.23 Example 6.9: application of angle criterion.

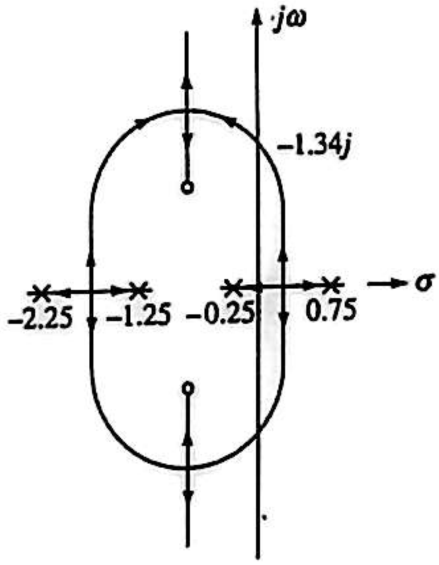
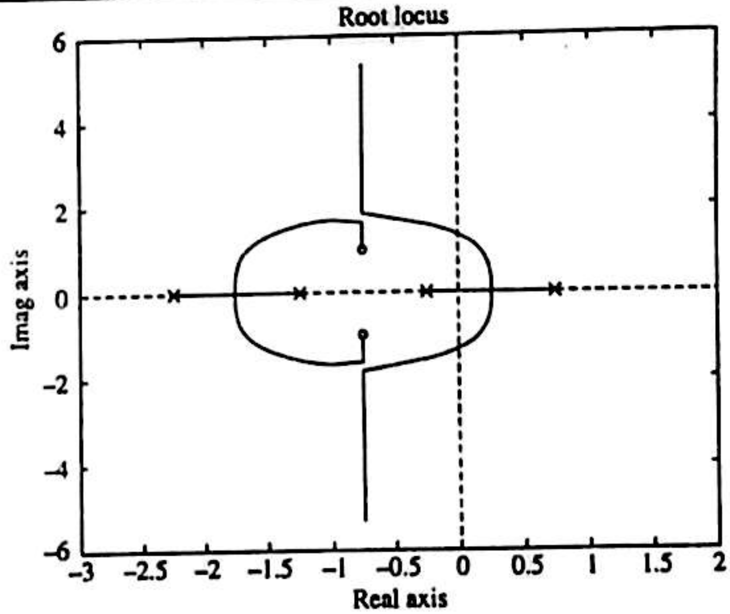


FIGURE 6.24 Example 6.9: root loci.

MATLAB Program 6.1

```
% Root-locus of transfer function of
% Example 6.9
echo off;
clear;
clf;
clc;
num=[1 1.5 1.5625];
a=[1 -0.75];
b=[1 0.25];
c=[1 1.25];
d=[1 2.25];
den1=conv(a,b);
den2=conv(c,d)
den=conv(den1,den2);
t=tf(num,den)
rlocus(t);
title('Root-locus');
```



EXAMPLE 6.10 Draw the root loci of the closed-loop system having open-loop transfer function $G(s)H(s)$ as

$$G(s)H(s) = \frac{Ks}{(s^2+4)(s^2+16)}; \qquad 0 < K < \infty$$

Solution: The gain $K = 0$ points are at $s = j2$, $s = -j2$, $s = j4$, and $s = -j4$. The gain $K = \infty$ points are at $s = 0$ and three other at $s = \infty$. The number of root loci are 4, as $P = 4$ and $Z = 1$. Symmetry of root loci exists about the real axis. Root loci exist between $s = 0$ to $-\infty$ on the negative real axis.

The angles of asymptotes are

$$\theta_k = \frac{(2k+1)\pi}{P-Z}; \qquad k = 0, 1, 2, \text{ i.e. } 60^\circ, 180^\circ \text{ and } -60^\circ$$

The intersection of asymptotes is at $s = 0$. The break-in point occurs at $s = -\infty$ on the negative real axis.

The angle of departure from pole at $s = j4$ is calculated as 0° and that from pole at $s = j2$ is calculated as 180°.

The root loci is drawn in Figure 6.25 for $0 < K < \infty$.

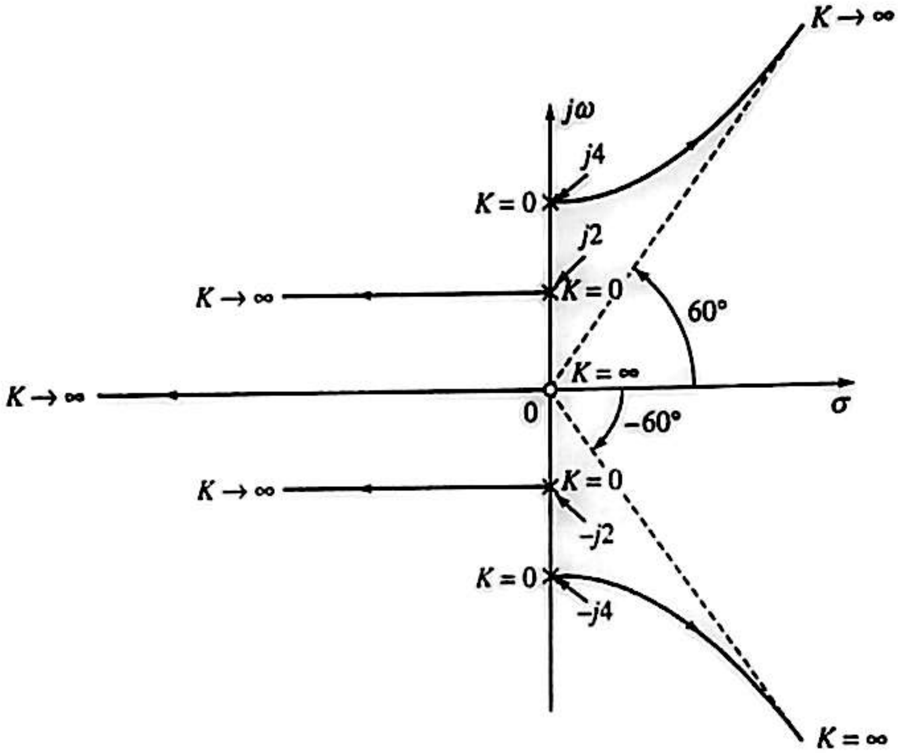


FIGURE 6.25 Example 6.10: root loci.

EXAMPLE 6.11 The block diagram of a feedback control system is shown in Figure 6.26(a). Sketch the root loci for $K \geq 0$ when the switch S is open. Determine the stability of the system as a function of K. Close the switch S so that the minor feedback loop is in effect. Set $K = 1$ and show by the root-locus plot how the system is stabilized when K_t varies. What is

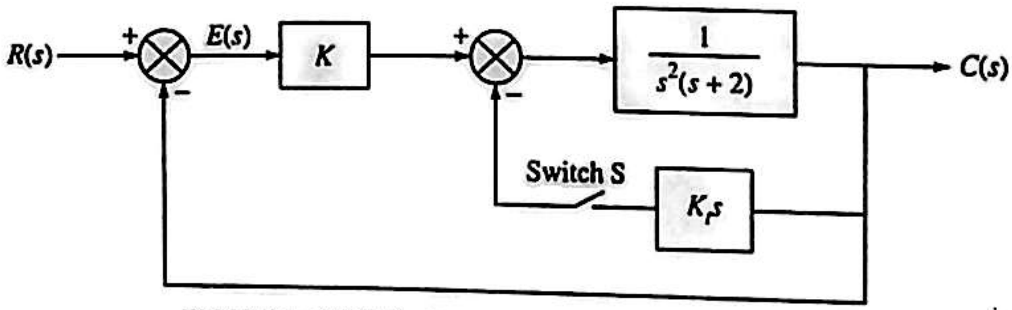


FIGURE 6.26(a) Example 6.11: control system.

the minimum value of K_t which stabilizes the system? What is the system's damped natural frequency corresponding to this value of K_t?

Solution: When switch S is open, the open-loop transfer function becomes

$$GH(s) = \frac{K}{s^2(s+2)}$$

The root-loci plot for $0 < K < \infty$ is shown in Figure 6.26(b).

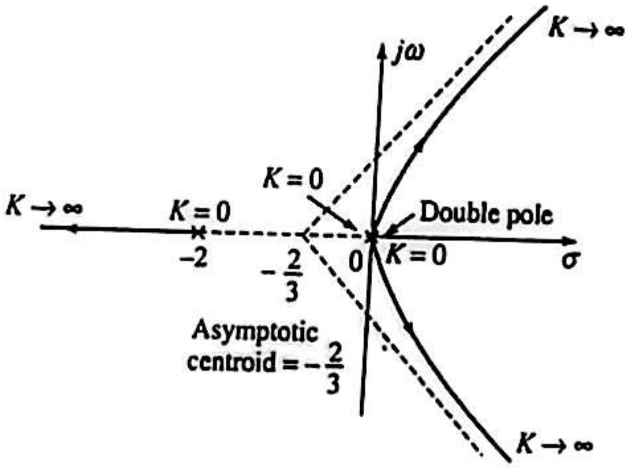


FIGURE 6.26(b) Example 6.11: root-loci plot with switch S open.

When the switch S is closed, the open-loop transfer function becomes:

$$G(s)H(s) = \frac{K}{s^2(s+2) + K_t s}$$

The characteristic equation $1 + G(s)H(s) = 0$ becomes

$$s^2(s + 2) + K_t s + K = 0$$

Now for $K = 1$, we can rearrange the system equation in terms of K_t as variable in the range $0 < K_t < \infty$, then the characteristic equation becomes

$$1 + \frac{K_t s}{s^2(s+2) + 1}$$

Then the pole-zero of the open-loop transfer function $\dfrac{K_t s}{s^2(s+2)+1}$ becomes $s_1 = -2.2$ and $s_{2,3} = 0.1 \pm j0.656$. Now we proceed to draw the root-loci plot for $0 < K_t < \infty$ while $K = 1$. The characteristic equation becomes

$$s^3 + 2s^2 + K_t s + 1 = 0$$

from which by Routh–Hurwitz criterion we get the $j\omega$-axis crossover points as $\pm j0.707$ and K_t at that point is equal to 0.5. In the usual way, the root loci with switch S closed is shown in Figure 6.26(c).

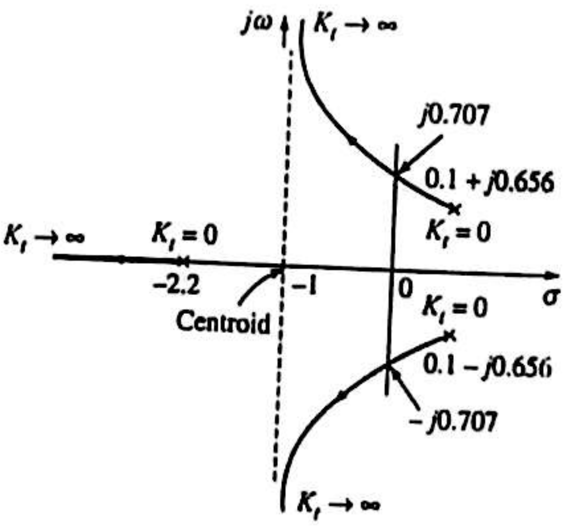


FIGURE 6.26(c) Example 6.11: root-loci plot with switch S closed.

EXAMPLE 6.12 The open-loop transfer function of the given system is

$$G(s)H(s) = \frac{100K(s+5)(s+40)}{s^3(s+100)(s+200)}$$

Draw the root loci for K in the range $0 < K < \infty$.

The same question can be written in a different form as: for the given characteristic equation

$$s^3(s + 100)(s + 200) + 100K(s + 5)(s + 40)$$

draw the root loci for $0 < K < \infty$.

Solution: The starting points of root loci are at 0, 0, 0, –100, and –200. The terminating point of root loci are at –5, – 40, ∞, ∞, and ∞. The number of root loci is 5. The root loci on real axis exist from origin to –5, and from –100 to – 40, and from –200 to infinity. The root loci are symmetrical about the real axis. The angles of asymptotes are $\pi/3$, π, and $-\pi/3$. The intersection of asymptotes on real axis is at –85. The intersection of root loci on $j\omega$-axis is obtained from Routh-tabulation of the characteristic equation as

$$\omega = \pm 25.8 \text{ for } K = 2818 \text{ and } \omega = \pm 77.7 \text{ for } K = 18837$$

The root loci is drawn in Figure 6.27.

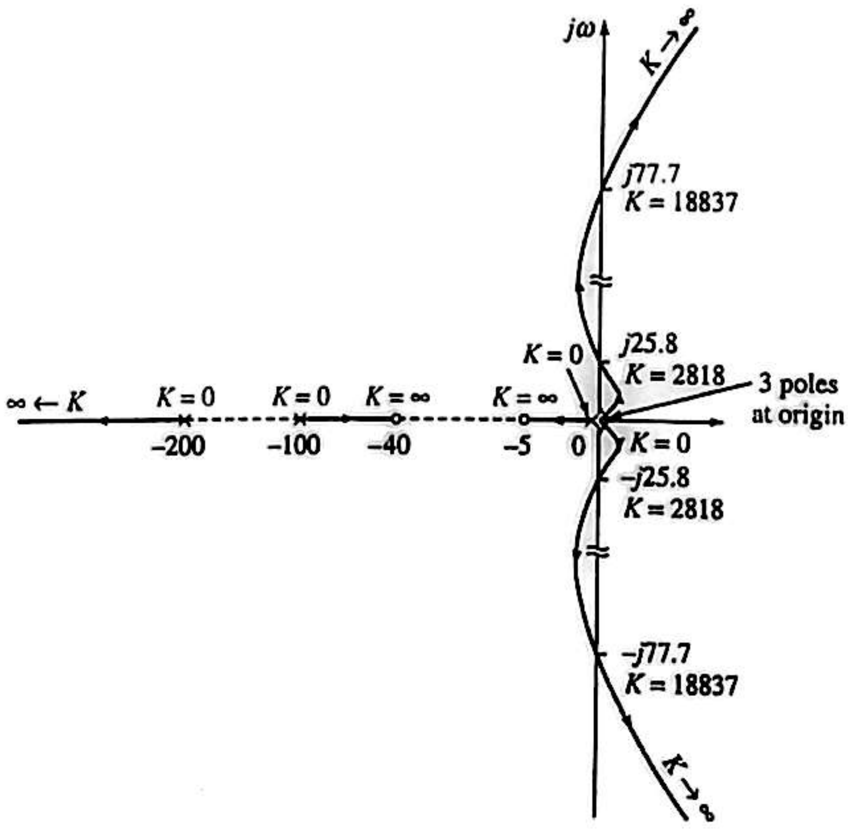


FIGURE 6.27 Example 6.12: root loci.

The concluding comment is that the closed-loop system is conditionally stable for gain K in the positive range. That is, for the range $0 < K < 2818$, the closed-loop system is unstable. Again for the range of gain $77.7 < K < \infty$, the closed-loop system is unstable. For the range of gain $2818 < K < 18837$, the closed-loop system is stable. Hence the closed-loop system is conditionally stable for variable gain K in the range $0 < K < \infty$.

EXAMPLE 6.13 For the system shown in Figure 6.28, draw the root loci and then determine its stability.

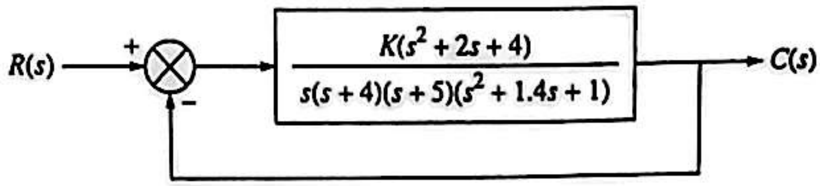


FIGURE 6.28 Example 6.13: control system.

Solution: The characteristic equation of the closed-loop system is

$$1 + G(s)H(s) = 1 + \frac{K(s^2+2s+4)}{s(s+4)(s+6)(s^2+1.4s+1)} = 0$$

or
$$s(s+4)(s+6)(s^2+1.4s+1) + K(s^2+2s+4) = 0$$

Students are recommended to formulate the Routh table from the characteristic equation and verify the following information on closed-loop stability:

(i) The system is stable for $0 < K < 15.6$ and $67.5 < K < 163.6$.
(ii) The system is unstable for $15.6 < K < 67.5$ and $163.6 < K < \infty$.

The system is hence conditionally stable. The root loci is shown in Figure 6.29.

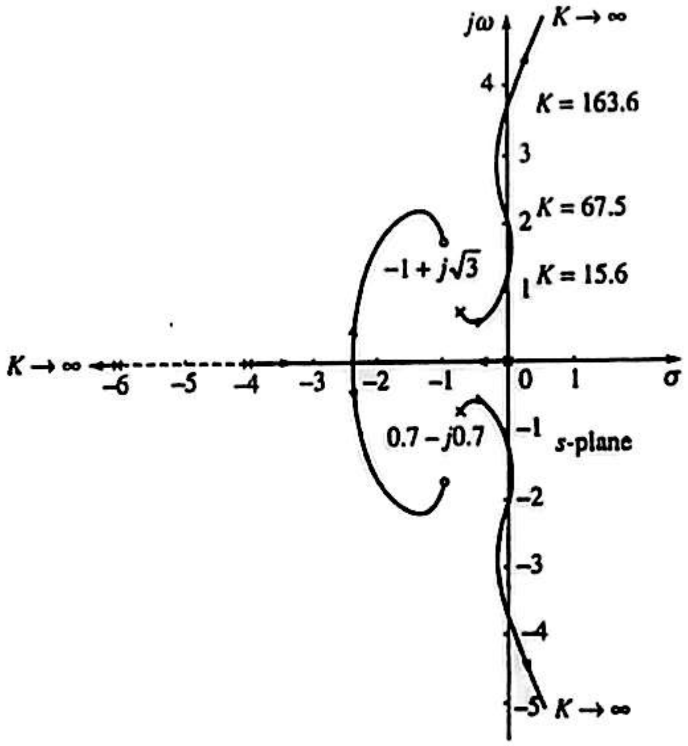


FIGURE 6.29 Example 6.13: root loci.

The same stability information for the conditionally stable system can be obtained from the root-loci plot as that obtained from Routh–Hurwitz criterion. Problems could arise if the designer stipulated a gain between $67.5 < K < 163.6$ as this gain would dampen the transient response but in this case reduction of gain would make the closed-loop system unstable.

The problem can be overcome by using a suitable compensating network $G_c(s)$ in the forward path in cascade as shown in Figure 6.30. With $G_c(s) = (s + 3)/(s + 5)$, the root-loci plot of the compensated system is shown in Figure 6.31. The closed-loop system is now stable for all values of gain K, i.e. $0 < K < \infty$.

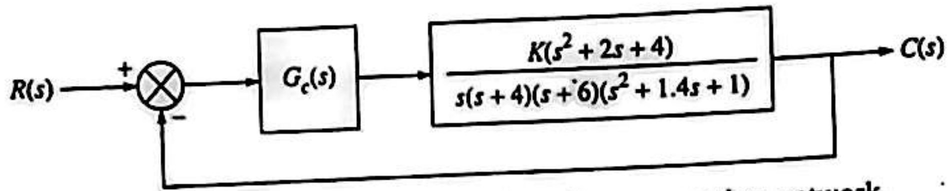


FIGURE 6.30 Example 6.13: addition of compensating network.

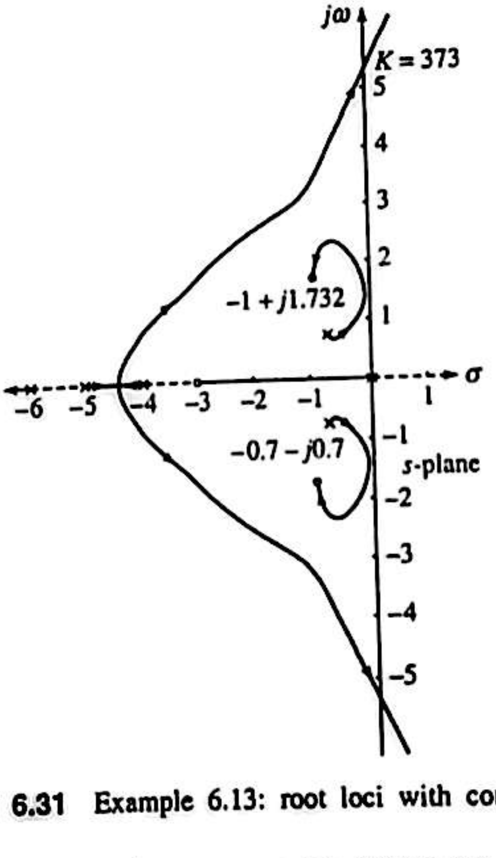
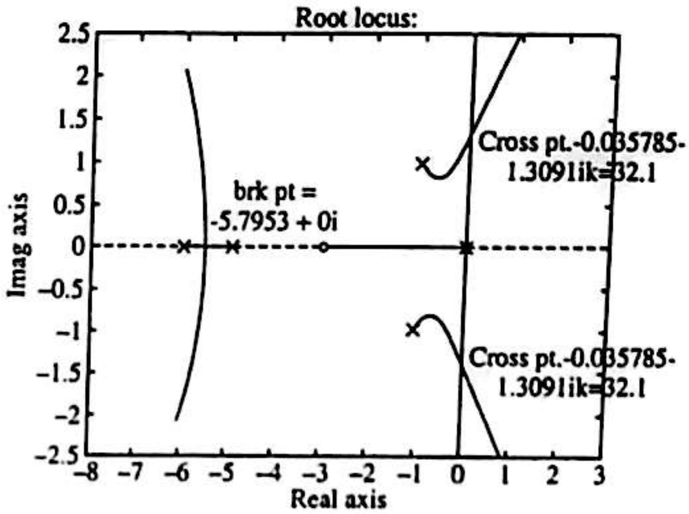


FIGURE 6.31 Example 6.13: root loci with compensator.

We have seen that root loci may be constructed in MATLAB platform. But we are yet to know how to evaluate the salient features (such as break-away/break-in points, intersection point with imaginary axis, uppe. limit of gain before the system goes to instability, etc.) of root-loci plot which are required for design of control systems. Obviously, the professional version of MATLAB is equipped with all the salient features of root-loci plot. The program for determining all the salient features of root loci required for design purposes is incorporated as an executable file with root-loci plot in MATLAB platform. For a better perspective and perception a few relevant programs in C language in MATLAB platform have been given (see MATLAB Program 6.2).

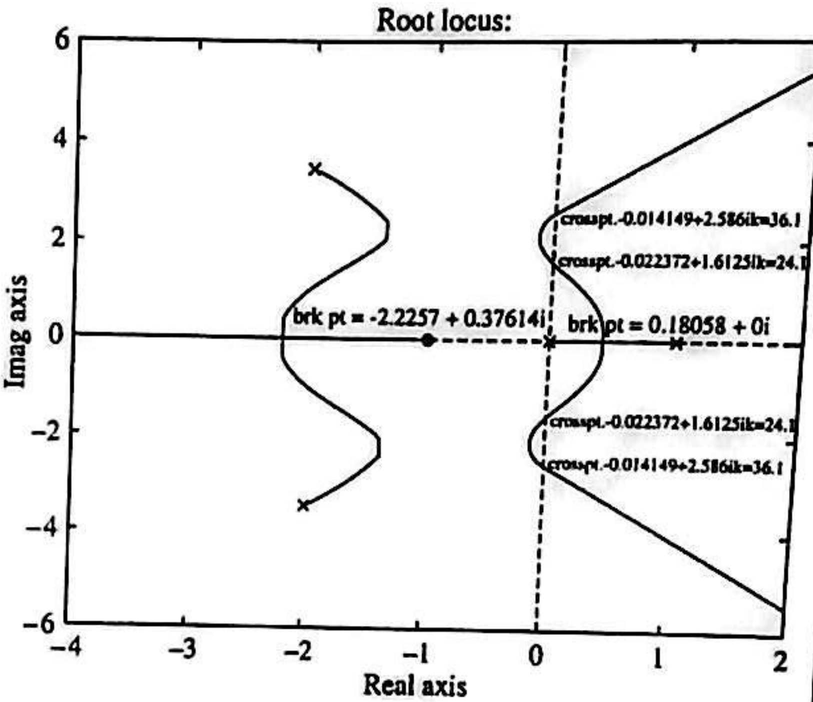
MATLAB Program 6.2

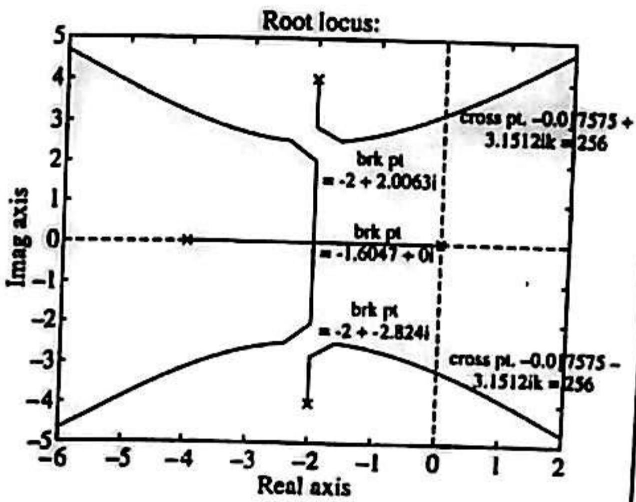
```
% Root loci.m
clear;
pack;
clc;
ch=menu('Root-locus','Demo1','Demo2','Demo3','Enter Tr. Fn.','Quit');
if ch==1,
% Root-loci plot of k(s+3)/s(s+5) (s+6) (s^2+2s+1)
   num=[1 3];
   a = [1 0];
   b = [1 5];
   c = [1 6];
   d = [1 2 2];
   den 1 = conv (a, b);
   den 2 = conv (c, d);
   den = conv (den 1, den 2);
   stp=.8;
end
if ch==2,
   num=[1 1];
   den=[1 3 12 -16 0];
   stp=.8;
end
if ch==3,
% Root-loci plot of G(s)H(s)=k/s(s+4) (s^2+4s+20)
   num=[1];
   a=[1 0];
   b=[1 4];
   den 1=conv [a, b];
   den 2=conv [1 4 20];
   den=conv [den1 den2];
   den=[1 8 36 80 0];
   stp=0.9;
end
if ch==4,
   stp=.8;
   coe=input('input the degree of numerator');
c=1;
coe1=coe+1;
while c<=coe1,
 num(c)=input('enter the value');
  c=c+1;
end
  doe=input('input the degree of numerator');
c=1;
doe1=doe+1;
```



```
while c<=doe1,
   den(c)=input('enter the value');
   c=c+1;
end
 end
 if ch==5,
   quit;
end
[r,k]=rlocus(num,den);
r
k
rlocus(num,den);
title('Root-locus');
len=length(den)-1;
%No of Cols in r
len1=len-1;
l=1;
while l<=len1,
   j=l+1;
   while j<=len,
      z=1;
            while z<=length(r),
%Difference between points
         dif=r(z,l)-r(z,j)
         x=real(dif);
         y=imag(dif);

% Testing brk. Points.
if abs(x)<=stp & abs(y)<=stp
            x1=real(r(z,l));
            y1=imag(r(z,l));
            br1=num2str(x1);
            br2=num2str(y1);
            br=['brk pt= ',br1];
            bq=['  +   ',br2,'i'];
            bf=[br,bq];
          text(x1,y1,bf);
            while  abs(x)<=stp & abs(y)<=stp,
               dif=r(z,l)-r(z,j);
               x=real(dif);
              y=imag(dif);
              z=z+1;
            end
end
        z=z+1;
        end
                     j=j+1;
              end
               l=l+1;
             end
         a=1;
```





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```
while a<=len,
  b=1;
     while b<=length(r),
     m=r(b,a);
 % Testing the crossover points.
if abs(real(r(b,a)))<0.07 & imag(r(b,a))~=0,
m11=num2str(r(b,a));
           m33=num2str(k(b));
           m44=['k=',m33];
           s=['cross pt.',m11,m44];
           text(real(r(b,a)),imag(r(b,a)),s);
          while abs(real(r(b,a)))<0.07 & imag(r(b,a))~=0,
          b=b+1;
          end
             end
      b=b+1;
   end
   a=a+1;
end

%The brk points of the algorithm are calculated using
%the logic that at brk points diff of the roots is
%nearly zero(Threshold value is taken as 0.8 or 0.9).
%So the value calculated manually and using MATLAB
%may not come out to be exactly equal.
```

6.5 Rules for the Construction of Inverse Root Loci

The rules for constructing inverse root loci (IRL) for one parameter, say, gain K varying in the range $-\infty < K \leq 0$ are listed in the following:

(i) *$K = 0$ points:* The $K = 0$ points on the IRL are the poles of the open-loop transfer function $G(s)H(s)$. These are the terminating points of IRL.

(ii) *$K = -\infty$ points:* The $K = -\infty$ points on the IRL are the zeros of $G(s)H(s)$ (including zeros at infinity). These are the starting points of IRL.

(iii) *Number of separate IRL:* The total number of IRL is equal to the number of finite poles P or the number of finite zeros Z, whichever is greater.

(iv) *Symmetry of IRL:* IRL are symmetrical about the real axis.

(v) *Asymptotes of IRL:* For large values of s, the angles of asymptotes are

$$\theta_k = \frac{2k\pi}{P-Z}; \qquad k = 0, 1, 2, \ldots, P - Z - 1 \tag{6.62}$$

The point of intersection of the asymptotes on the real axis is given by

$$\sigma_1 = \frac{\Sigma\text{real parts of poles of } G(s)H(s) - \Sigma\text{real parts of zeros of } G(s)H(s)}{P-Z}$$

(vi) *IRL on real axis:* On a given section on the real axis in the s-plane, the existence of IRL is found for $-\infty < K \leq 0$ in the section only if the total number of finite poles and zeros to the right of the section is even.

(vii) *Intersection of IRL on the imaginary axis:* The values of ω and K at the crossing points of the IRL on the imaginary axis of the s-plane may be obtained by Routh–Hurwitz criterion, similar to the root loci case.

(viii) *Angle of departure and arrival:* The angle of departure of inverse root locus for $-\infty < K \leq 0$ from a zero or the angle of arrival at a pole of $G(s)H(s)$ can be determined by assuming a point s_1 on the inverse root locus that is associated with the zero or pole and which is very close to the zero or pole. The angle of departure or arrival of an inverse root locus is determined from

$$G(s_1)H(s_1) = \sum_{i=1}^{m}(s_1 - z_i) - \sum_{j=1}^{n}(s_1 - p_j) = 2k\pi, \text{ where } k = 0, \pm 1, \pm 2, \ldots \quad (6.63)$$

(ix) *Break-away and break-in points:* The break-away or break-in points on the IRL are determined by finding the roots of

$$\frac{dK}{ds} = 0 \quad \text{or} \quad \frac{d}{ds}[G(s)H(s)] = 0 \quad (6.64)$$

Otherwise, we can use the alternative method as discussed in the context of the construction of root loci.

(x) *Calculation of values of K on the IRL:* The absolute value of K at any point s_1 on the inverse root loci can be determined from the equation

$$|K| = \frac{1}{G(s_1)H(s_1)}$$

$$= \frac{\text{products of lengths of vectors drawn from the poles of } G(s)H(s) \text{ to } s_1}{\text{products of lengths of vectors drawn from zeros of } G(s)H(s) \text{ to } s_1} \quad (6.65)$$

EXAMPLE 6.14 Consider the open-loop transfer function

$$G(s)H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$$

Draw the inverse root loci for $-\infty < K < 0$.

Solution: The pole-zero configuration of the open-loop transfer function $G(s)H(s)$ is shown in Figure 6.32. The construction rules for inverse root loci are:

(i) The starting points ($K = -\infty$) are at $s = -3$, $s = -\infty$, $s = -\infty$, $s = -\infty$ and $s = -\infty$

(ii) The terminating points ($K = 0$) are at $s = 0$, $s = -1 \pm j1$, $s = -5$ and $s = -6$.

(iii) The number of separate IRL = 5 as $P = 5$, $Z = 1$ and $P > Z$.

(iv) There is symmetry of IRL about the real axis.

(v) The angles of asymptotes of IRL for large values of s are given by

$$\theta_k = \frac{2k\pi}{P-Z}; \qquad k = 0, 1, \ldots, P-Z-1$$

i.e. $$\theta_0 = 0°, \ \theta_1 = \pi/2, \ \theta_2 = \pi, \ \theta_3 = 3\pi/2$$

The asymptotes intersect the real axis at –2.5.

(vi) IRL will exist on the real axis if the number of poles and zeros to the right is even. Hence IRL exists between $+\infty$ to 0 and between –3 to –5 and between –6 to $-\infty$.

(vii) There is no intersection of IRL with the imaginary axis for $-\infty < K < 0$ as found out by Routh–Hurwitz criterion.

(viii) The angle of departure or arrival as determined from Eq. (6.65) is $\theta = 180° - 43.8° = 136.2°$.

(ix) The break-away points are obtained by solving $dK/ds = 0$, which gives

$$s^5 + 13.5s^4 + 66s^3 + 142s^2 + 123s + 45 = 0$$

i.e. $$s = -5.53; -0.656 \pm j0.468; \ -3.33 \pm j1.204$$

None of these points are on IRL and hence IRL has no break-away points.

Now the inverse root loci is drawn as shown in Figure 6.32 with the dotted lines for the variable parameter K in the range $-\infty \le K \le 0$. Further, for variable parameter K in the range $0 \le K < \infty$, the root loci is already drawn in Figure 6.11. Now for complete root loci for

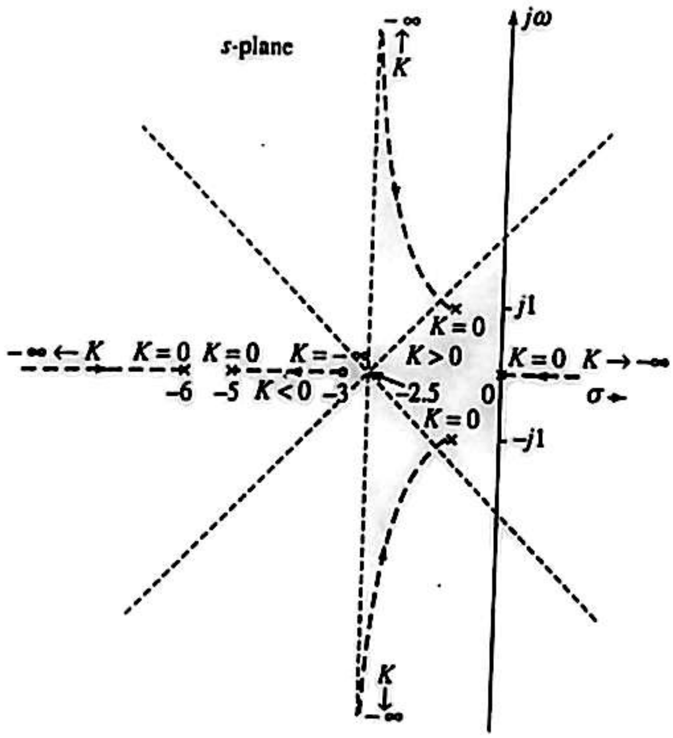


FIGURE 6.32 Example 6.14: inverse root loci for $-\infty \le K \le 0$.

$-\infty < K \leq \infty$, the inverse root loci and root loci have to be added together and are shown in Figure 6.33.

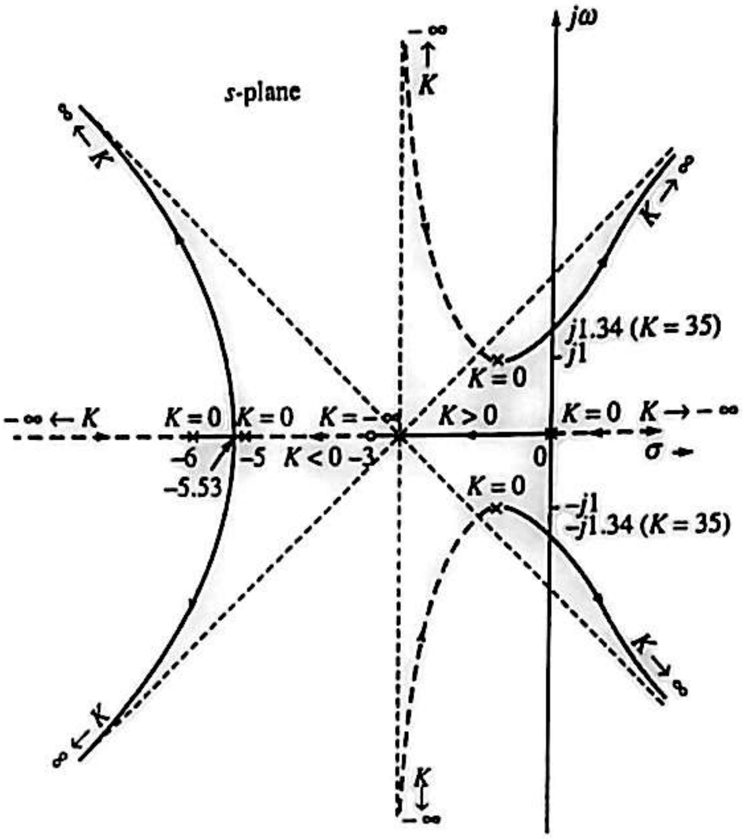


FIGURE 6.33 Example 6.14: complete root loci.

6.6 Effect of Adding Poles and Zeros

The root-locus technique is used to design a control system. In this connection, some of the salient features of the root-locus technique are now highlighted.

6.6.1 Addition of poles

Consider
$$G(s)H(s) = \frac{K}{s(s+a)}; \quad a > 0 \tag{6.66}$$

The root-locus diagram is shown in Figure 6.34(a).

Now a pole at $-b$ is added so that the new transfer function becomes

$$G(s)H(s) = \frac{K}{s(s+a)(s+b)}; \quad |b| > |a| \tag{6.67}$$

Scanned by CamScanner

and the corresponding root loci is shown in Figure 6.34(b). The complex closed-loop poles are going towards the unstable (right-half of s-plane) region for $K > K_1$. The stable range is $0 < K < K_1$.

Next another pole at $-c$ is added so that the new transfer function becomes

$$G(s)H(s) = \frac{K}{s(s+a)(s+b)(s+c)}; \qquad |c| > |b| > |a| \tag{6.68}$$

and the corresponding root loci is shown in Figure 6.34(c). The stable range becomes

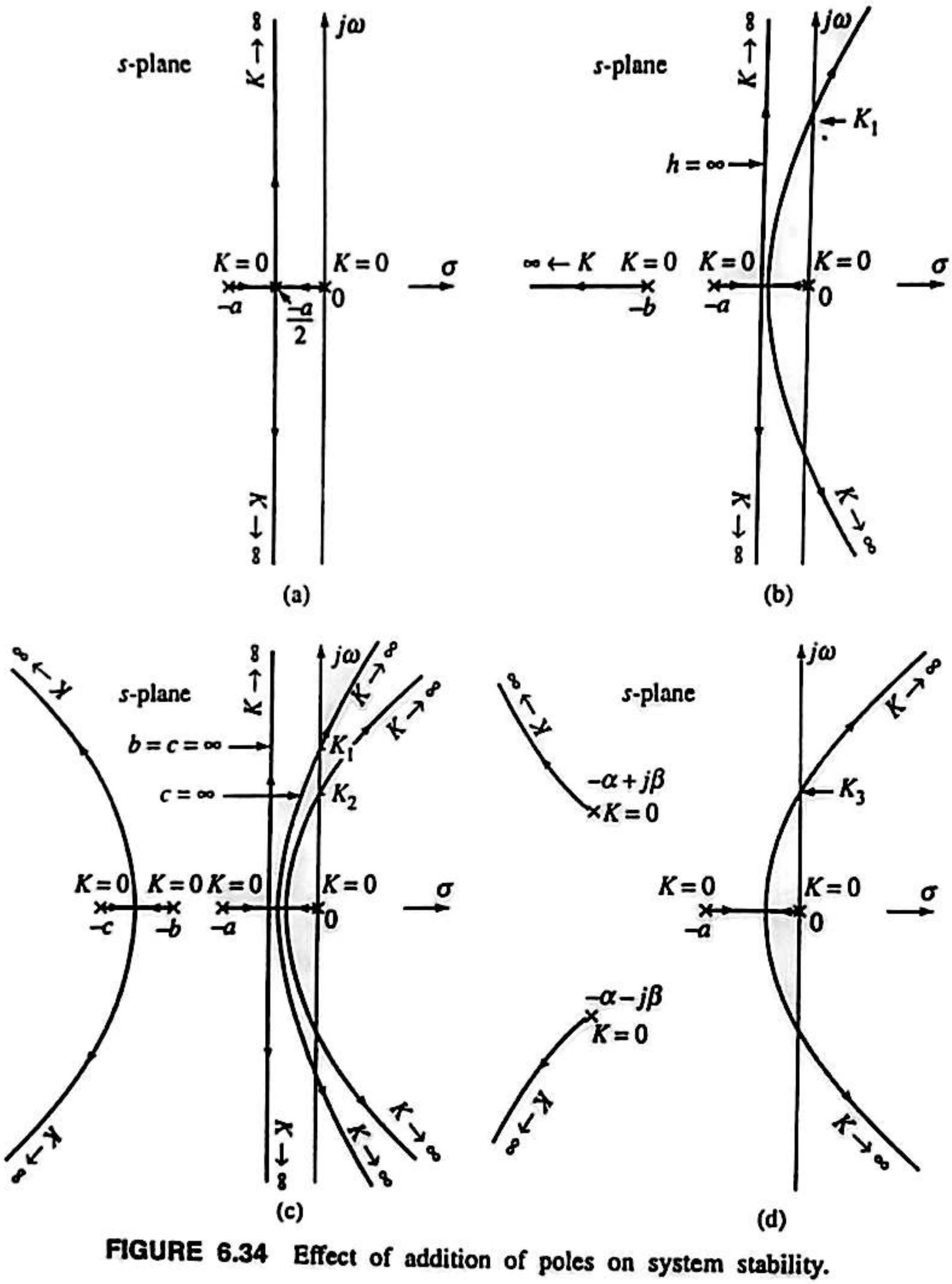


FIGURE 6.34 Effect of addition of poles on system stability.

$0 < K < K_2 < K_1$, i.e. less than the previous case. The closed-loop system is stable up to gain K_2 which is obviously less than K_1.

Next we add two complex conjugate poles to the original system of Figure 6.34(a) so that the new open-loop transfer function of the system becomes

$$G(s)H(s) = \frac{K}{s(s+a)(s+\alpha+j\beta)(s+\alpha-j\beta)}; \qquad |\alpha| > |a| \tag{6.69}$$

The pole-zero configuration is shown in Figure 6.34(d). The root-locus plot is drawn. The system is now stable up to the value of gain $K_3 < K_2$ and hence less stable compared to the original system. If we move the complex poles further, the system becomes even more restricted. Hence we can conclude from the root-locus technique that the system becomes less stable as we add the poles. Addition of poles to $G(s)H(s)$ is the same as the increase in the order of the system. Addition of poles to the function $G(s)H(s)$ has the effect of moving the root loci towards the right-half of s-plane.

6.6.2 Addition of zeros

In a similar fashion, adding zeros to $G(s)H(s) = K/s(s+2)$ [see Figure 6.35(a)] has the effect of moving the root loci towards the left making the system more stable.

Adding a zero at $-b$ where $|b| > |a|$, the root-locus plot is shown in Figure 6.35(b). The root-locus plot has moved towards the left. The closed-loop system becomes more stable compared to the previous case. Move zero at $-b$ towards right so that $|b| < |a|$ as in Figure 6.34(b). The closed-loop poles become real, the system becomes overdamped. Stability improves further. Add two complex zeros at $-b \pm j\beta$. The root-locus plot is shown in Figure 6.35(c). The system stability improves compared to that of Figure 6.35(b).

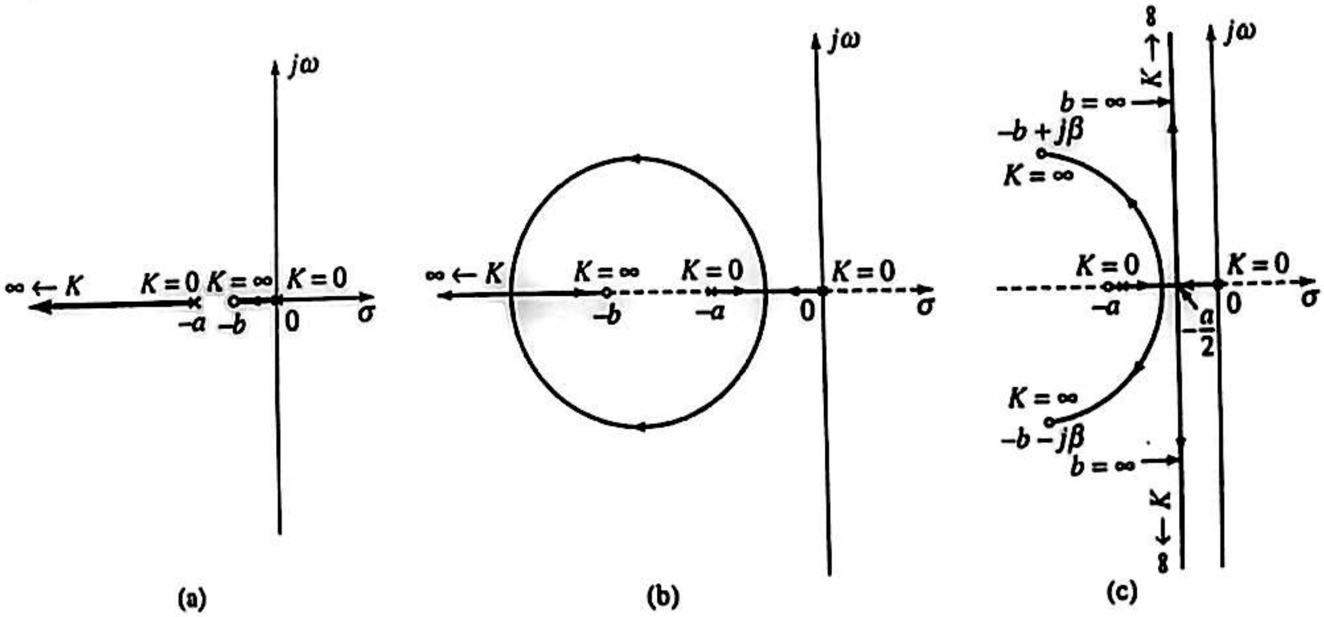


FIGURE 6.35 Effect of addition of zeros on system stability.

6.6.3 Effect of varying the pole position

Consider
$$G(s)H(s) = \frac{K(s+1)}{s^2(s+a)} \tag{6.70}$$

Therefore,
$$K = -\frac{s^2(s+a)}{(s+1)} \tag{6.71}$$

Now $dK/ds = 0$ gives the break-away points as

$$\frac{-(a+3)}{4} \pm \frac{1}{4}\sqrt{a^2 - 10a + 9} \tag{6.72}$$

(i) Let $a = 10$. Then,
$$G(s)H(s) = \frac{K(s+1)}{s^2(s+10)} \tag{6.73}$$

Now $dK/ds = 0$ gives the break-in point at $s = -2.5$ and the break-away point at -4.0.

The root loci on the real axis exist between -1 and ∞ on the negative real axis.

The angles of asymptotes are $\pi/2$, $-\pi/2$. The intersection of asymptotes is at -4.5.

The root-loci plot is shown in Figure 6.36(a).

(ii) Let the pole at $-a$ be shifted from -10 to -9. Then

$$G(s)H(s) = \frac{K(s+1)}{s^2(s+9)} \tag{6.74}$$

Now $dK/ds = 0$ gives the breakaway point at -3. The angles of asymptotes are $\pi/2$ and $-\pi/2$. The intersection of asymptotes is at -4. The root-locus plot is shown in Figure 6.36(b).

(iii) Putting $a = 8$,

$$G(s)H(s) = \frac{K(s+1)}{s^2(s+8)} \tag{6.75}$$

The value of s in equation $dK/ds = 0$ does not satisfy the characteristic equation $1 + G(s)H(s) = 0$. Hence there is no finite non-zero break-away point. The angles of asymptotes are $\pi/2$ and $-\pi/2$. The intersection of asymptotes on the real axis is -3.5. The root-locus plot is shown in Figure 6.36(c).

(iv) For $a = 1$, the pole-zero cancellation will occur. Then

$$G(s)H(s) = \frac{K}{s^2}$$

The root loci lie only on the imaginary axis and are symmetrical about the real axis. The root-locus plot is shown in Figure 6.36(d).

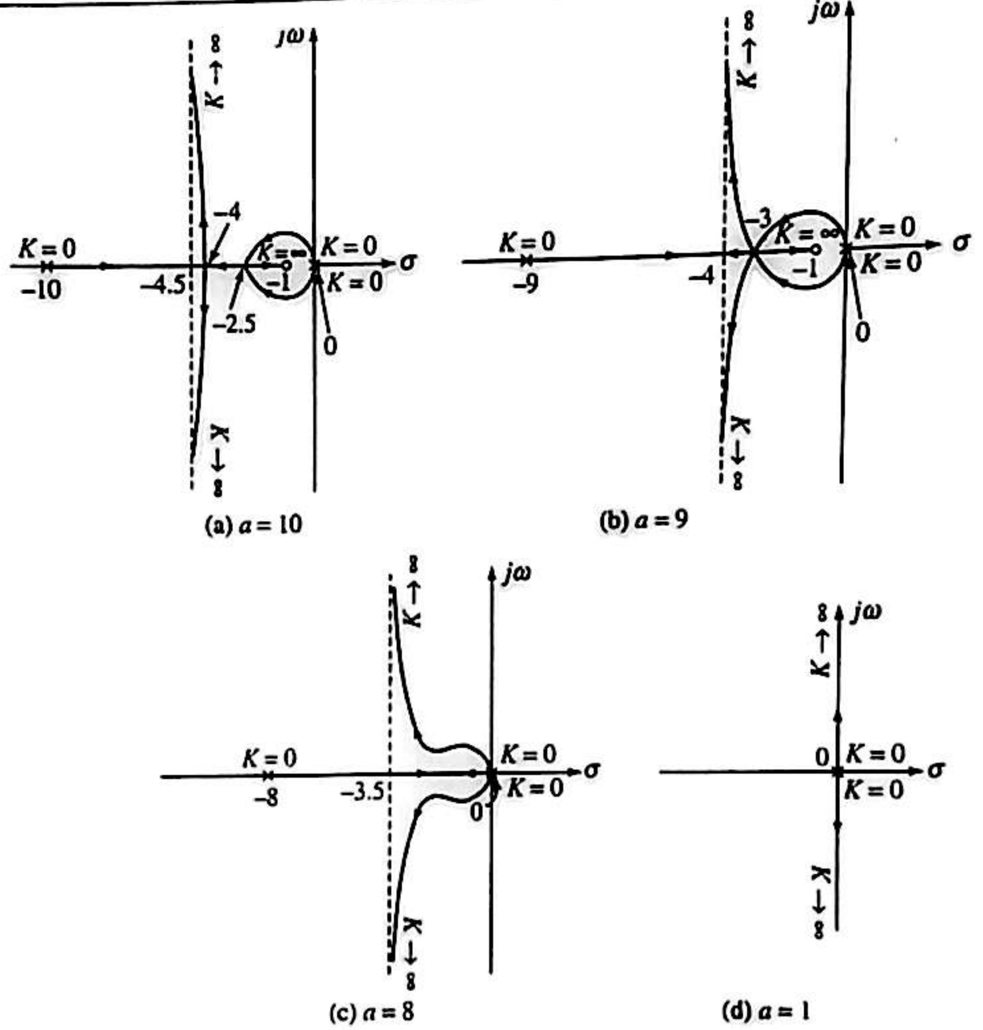


FIGURE 6.36 Root loci for the variable pole of $G(s)H(s) = K(s+1)/s^2(s+a)$; $1 \le a \le 10$, $0 < K < \infty$.

6.6.4 Cancellation of poles and zeros

It is important to note that if the denominator and numerator of $G(s)H(s)$ involve common factors, then the open-loop poles and zeros will cancel each other and the degree of the characteristic equation is reduced by one or more. To illustrate this, consider a negative feedback system having forward transfer function

$$G(s) = \frac{K}{s(s+2)(s+3)}$$

and feedback ratio $H(s) = (s+2)$. Then the characteristic equation from the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s(s+2)(s+3)+K(s+2)}$$

can be written as

$$s(s+2)(s+3)+K(s+2)=0$$

But because of the cancellation of the common term $(s + 2)$, we have the characteristic equation

$$1+G(s)H(s) = 1+\frac{K}{s(s+2)(s+3)} \times (s+2) = 1+\frac{K}{s(s+3)} = \frac{s(s+3)+K}{s(s+3)}$$

The reduced characteristic equation is

$$s(s + 3) + K = 0$$

The root-locus plot does not show all the roots of the characteristic equation, only the roots of the reduced equation. In order to obtain the complete set of closed-loop poles, the cancelled pole of $G(s)H(s)$ must be added to those of the closed-loop poles obtained from the root-locus plot of $G(s)H(s)$. It is to be remembered that the cancelled pole of $G(s)H(s)$ is a closed-loop pole of the original system.

6.7 Root Contour

Thus far we have discussed the root-locus technique for the study of closed-loop systems from the open-loop pole-zero information when one parameter, say, gain K is varied in the range $0 < K < \infty$. The extension of the root-locus technique to the study of the closed-loop system from the open-loop pole-zero information when more than one parameter is varying in the range from 0 to ∞, is known as the root contour. The root contour can be constructed following the same rules that we earlier explained in the context of construction of root loci. This is being illustrated with examples for better understanding.

Consider the system shown in Figure 6.37 where K and α are varying in the range 0 to infinity. The characteristic equation is

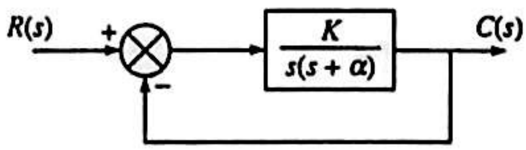


FIGURE 6.37

$$s^2 + \alpha s + K = 0$$

The root-contours are to be plotted by allowing both α and K to vary in the range 0 to ∞. Rewriting the characteristic equation as

$$1+\alpha\left(\frac{s}{s^2+K}\right) = 0 \tag{6.76}$$

such that α appears as a variable (say, gain) parameter and is varying in the range 0 to ∞ for different, particular values of K, i.e. as if the open-loop transfer function is

$$\frac{\alpha s}{s^2 + K} \tag{6.77}$$

For various values of K by allowing α to vary in the range 0 to ∞, the root contour can be constructed. The root contour of Eq. (6.77) originates from poles at $s = \pm j\sqrt{K}$ and terminates on zeros at $s = 0$ and at $s = -\infty$.

Now,

$$\frac{d\alpha}{ds} = -\frac{-(s^2 - K)}{s^2} = 0 \tag{6.78}$$

gives the break-away point at $s = \pm\sqrt{K}$.

It is evident from Figure 6.37 that the break-away point has to lie on the real axis between $s = 0$ to $s = -\infty$. Therefore, only $s = -\sqrt{K}$ corresponds to the break-away point. The root contours for various values of K with α varying from 0 to ∞ are shown in Figure 6.38.

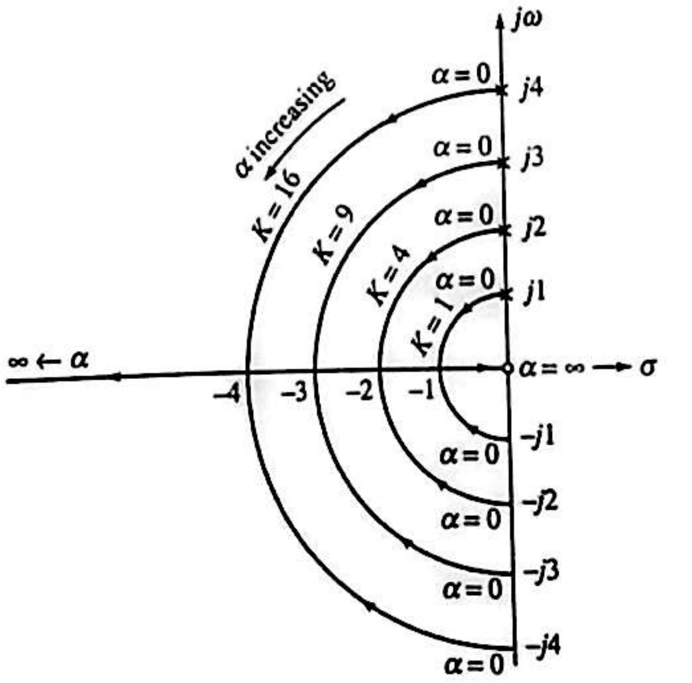
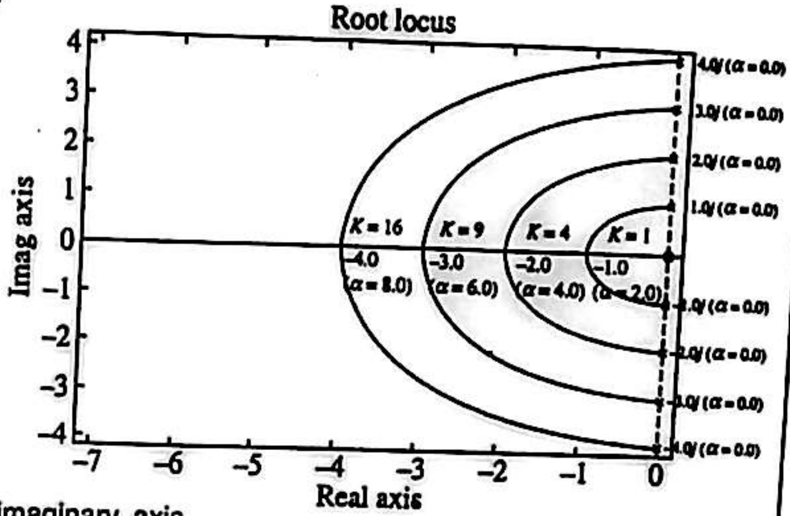


FIGURE 6.38 Root contours of the system of Figure 6.37.

It can be shown that the complex root branches are circular as we proved earlier in Example 6.2. MATLAB program to plot the root contours of a characteristic equation is given next.

MATLAB Program 6.3




```
%    rootcontour.m
%    To plot the root-contour of the characteristic  equation
%                         s^2 + αs + K = 0
clear;
pack;
clc;

axis([-5,2,-5,5]);
hold  on;
for K = 1 : 4,
   num = [1 0];
   den = [1 0 K^2];
    rlocus(num,den);
   l = length(den)-1;
   for α = 0:0.1:10,
       [R]= rlocus(num,den,α);
% To find the intersection with the imaginary axis.
       for n = 1:l,
           if abs(real(R(n)))<=0.001 & imag(R(n))~=0,
              str1 = sprintf(' %.1fj',imag(R(n))) ;
        text(0,imag(R(n)),str1);
        str1 = sprintf('          (α = %.1f)',α) ;
        text(0,imag(R(n)),str1);
          end;
       end;

% To find the Breakaway or Breakin Point.

     for p = 1:l-1,
     for q = p+1:l,
          if abs(R(p)-R(q))<= 0.1,
              str1 = sprintf(' %.1f',real(R(p))) ;
text(real(R(p)),imag(R(p))-0.4,str1);
        str1 = sprintf('(α = %.1f)',α) ;
        text(real(R(p)),imag(R(p))-0.8,str1);
          end;
        end;
      end;
    end;

   pop1 = sprintf(' K = %g',K^2);
   text(-K,0.5,pop1);
   hold on;

end;
title('Root-contour for the characteristic eqn. s^2 + αs + K = 0');
hold off;
```

EXAMPLE 6.15 Consider the feedback control system with open-loop transfer function

$$G(s)H(s) = \frac{K}{s(s+1)(s+\alpha)}$$

in which both K and α are variable in the range 0 to ∞.

Manipulating the characteristic equation

$$1+G(s)H(s) = 1+\frac{K}{s(s+1)(s+\alpha)} = 0 \qquad \text{or} \qquad s^2(s+1)+\alpha(s+1)s+K=0$$

in the form where α appears as a variable (gain) parameter of root-loci plot, we can write

$$1+\frac{\alpha s(s+1)}{s^2(s+1)+K}=0 \tag{i}$$

This is of the form where the root locus with respect to the parameter α can be drawn for an open-loop transfer function

$$\frac{\alpha s(s+1)}{s^2(s+1)+K}$$

The root-contours of Eq. (i) originate (i.e. for $\alpha = 0$) at open-loop poles of the reduced characteristic equation

$$s^2(s+1)+K = 0$$

and terminate (i.e. for $\alpha = \infty$) at 0, –2 and $-\infty$. The reduced characteristic equation is rewritten as

$$1+\frac{K}{s^2(s+1)} = 0$$

The root locus of the reduced characteristic equation with K as a variable parameter is plotted in Figure 6.39 with open-loop poles 0, 0 and –1.

The root contours for various values of K with varying α in the range $0 < K < \infty$ are drawn in Figure 6.40 following the root-locus technique. The value of α at which the root contours will cross the $j\omega$-axis are obtained by Routh criterion from the characteristic equation

$$s^3+(\alpha+1)s^2+\alpha s+K = 0$$

The Routh array is:

$$\begin{array}{lll}
s^3 & 1 & \alpha \\
s^2 & \alpha+1 & K \\
s^1 & \dfrac{\alpha(\alpha+1)-K}{\alpha+1} & \\
s^0 & K &
\end{array}$$

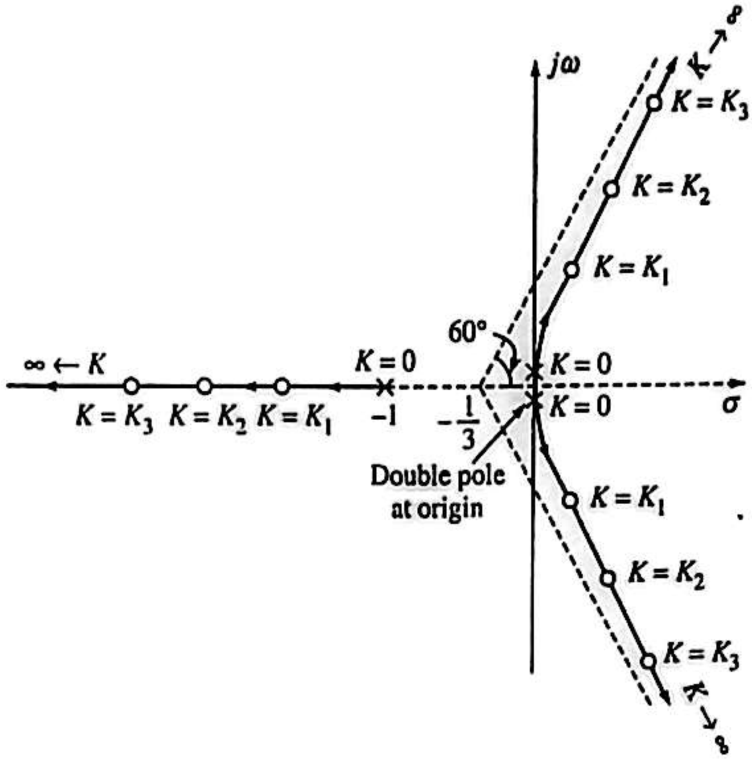


FIGURE 6.39 Example 6.15: root loci of $G(s)H(s) = K/s^2(s+1)$.

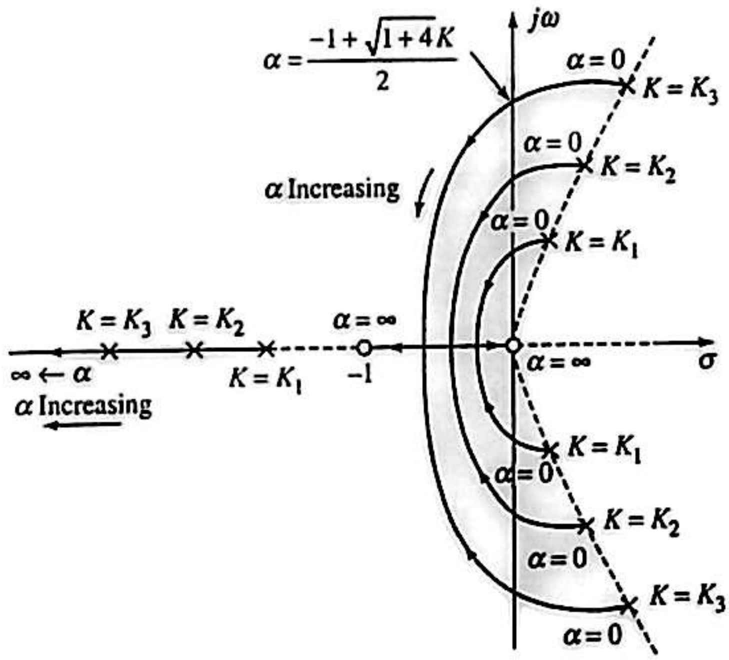


FIGURE 6.40 Example 6.15: root loci of $G(s)H(s) = \alpha s(s+1)/[s^2(s+1)+K]$.

The root contours cross the $j\omega$-axis for

$$\alpha(\alpha+1) - K = 0 \qquad \text{or} \qquad \alpha = \frac{-1+\sqrt{1+4K}}{2}$$

since only the positive value is permitted.

EXAMPLE 6.16 (i) For the system given in Figure 6.41, draw the root-loci first with $K_h = 0$ and K as variable. (ii) Obtain the value of K so that the system damping ratio is 0.158. (iii) For this value of the system gain, draw the root loci with K_h as variable. (iv) Find the value of K_h that improves the system damping ratio to 0.5, obtained using the root-loci plot. Grapical approximations are acceptable. Make qualitative plots of unit-step and unit-ramp responses of the above system with and without K_h feedback being operative.

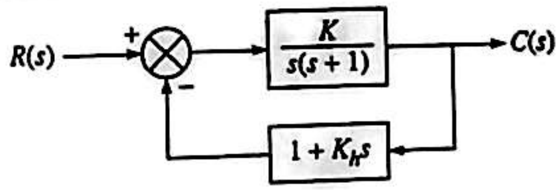


FIGURE 6.41 Example 6.16.

Solution: (i) $G(s)H(s) = \dfrac{K(1+K_h s)}{s(s+1)}$

The characteristic equation is

$$s(s + 1) + K(1 + K_h s) = 0$$

or

$$s^2 + s + KK_h s + K = 0$$

For part (i), $K_h = 0$. The reduced characteristic equation is therefore

$$s^2 + s + K = 0$$

and the equivalent characteristic equation $1 + G_1(s) = 0$

yields
$$G_1(s) = \frac{K}{s(s+1)}$$

The root-loci plot is shown in Figure 6.42.

(ii) For $\zeta = 0.158 = \cos\theta$ yields $\theta = 80.9°$. Further,

$$s^2 + s + K = 0 \quad \text{yeilds} \quad \omega_n^2 = K$$

This is equivalent to the standard form of equation with $\zeta = 0.158$, yielding

$$s^2 + 2(0.158)\sqrt{K}\, s + K = 0$$

Comparing, we get

$$2(0.158)\sqrt{K} = 1 \qquad \text{or} \qquad K = 10$$

Then $s^2 + s + 10 = 0$ yields $s_1 = -0.5 \pm j3.122$. The point s_1, in order to lie on root-loci needs the magnitude condition to be satisfied for a check.

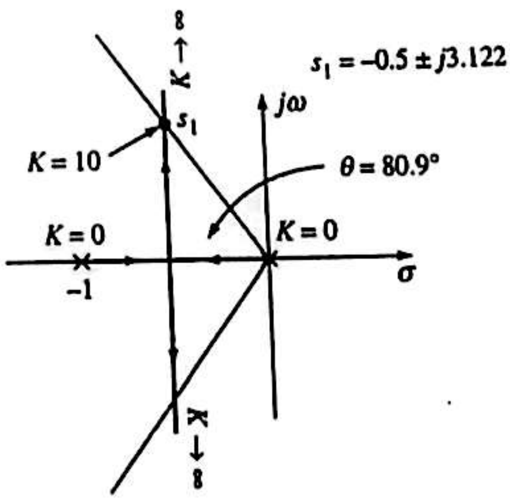


FIGURE 6.42 Example 6.16: root-loci with $K_h = 0$.

(iii) Root-loci for $K = 10$, the characteristic equation for variable K_h is

$$s^2 + (1 + 10K_h)s + 10 = 0$$

or

$$s^2 + s + 10 + 10K_h s = 0$$

or

$$1 + \frac{10K_h s}{s^2 + s + 10} = 0$$

Hence we have to find the root-loci for the variable K_h in $0 < K_h < \infty$ where the open-loop transfer function is

$$\frac{sK_h}{s^2 + s + 10}$$

The poles are at $-0.5 \pm j3.122$ and zero at origin. The number of root-loci = 2.

The break-in point $dK_h/ds = 0$ yeilds $s = -\sqrt{10} = -3.16$. The root-loci, for $K = 10$ are shown in Figure 6.43.

(iv) $\zeta = 0.5 = \cos\theta$ or $\theta = 60°$

The characteristic equation for the variable K_h with $K = 10$ is

$$s^2 + (1 + 10K_h)s + 10 = 0$$

Hence $\omega_n = \sqrt{10}$

Now with $\zeta = 0.5$ and $\omega_n = \sqrt{10}$, the characteristic equation becomes

$$s^2 + 2(0.5)\sqrt{10}\,s + 10 = 0$$

Comparing, we get

$$1 + 10K_h = \sqrt{10} = 3.16$$

or

$$K_h = 0.216$$

Let us have the value of s for $K_h = 0.216$ which becomes after solving

$$s^2 + 3.16s + 10 = 0$$

that is, $s_2 = -1.56 + j2.738$.

The root-loci is shown in Figure 6.44 indicating the s_2 point.

(v) Qualitative sketches for unit step and ramp responses are shown in Figures 6.45(a) and (b) respectively.

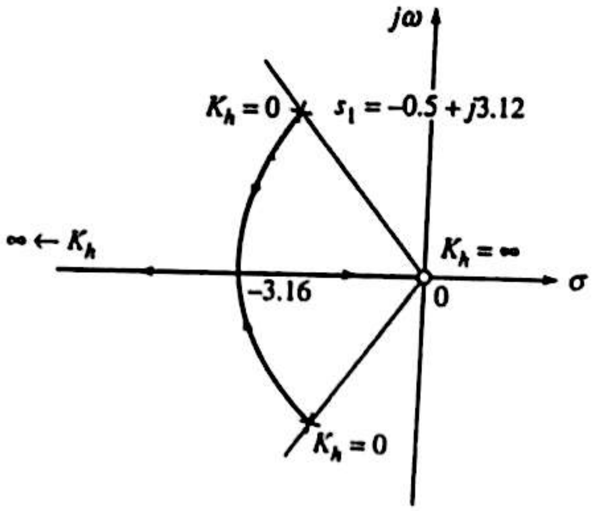


FIGURE 6.43 Example 6.16: root-loci with K_h as variable.

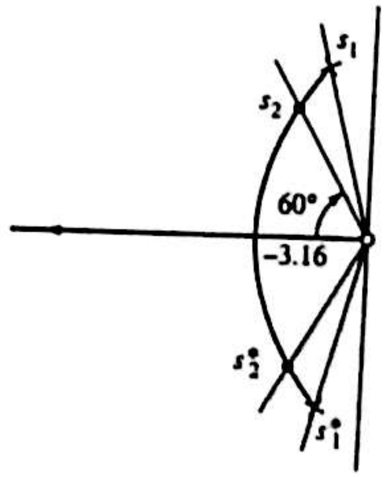


FIGURE 6.44 Example 6.16: root-loci with $\zeta = 0.5$.

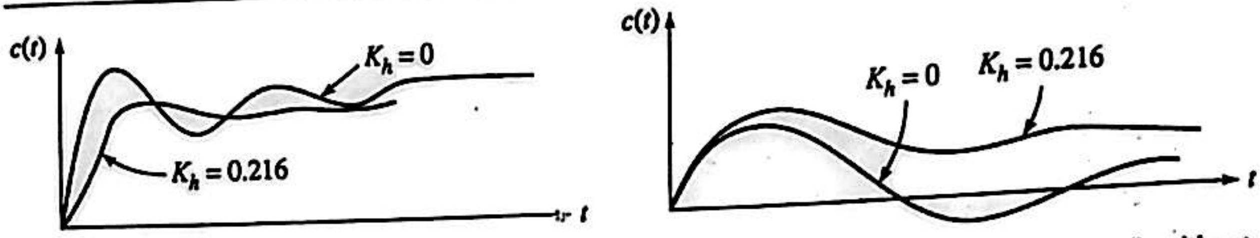


FIGURE 6.45 Example 6.16: (a) unit-step response and (b) unit-ramp response with and without K_h.

6.7.1 Multiple-loop system

Consider the system shown in Figure 6.46(a) where two feedback loops exist. The equivalent system is shown in Figure 6.46(b), where the open-loop transfer function

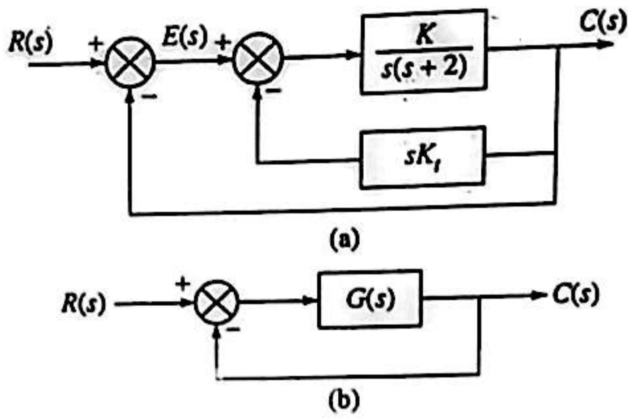


FIGURE 6.46 (a) System with two feedback loops and (b) its equivalent system.

$$G(s) = \frac{C(s)}{E(s)} = \frac{\dfrac{K}{s(s+2)}}{1+\dfrac{sKK_t}{s(s+2)}}$$

$$= \frac{K}{s(s+2)+sKK_t}$$

The characteristic equation of the system is $1 + GH = 0$, i.e.

$$s(s + 2) + sKK_t + K = 0$$

Rewriting this as

$$1+\frac{sKK_t}{s(s+2)+K} = 1+\frac{\alpha s}{s(s+2)+K} = 0 \tag{6.79}$$

where $\alpha = KK_t$.

The root contours can be plotted for various values of K with α varying from 0 to ∞. The root contours of Eq. (6.79) originate (i.e. for $\alpha = 0$) at open-loop poles of the reduced characteristic equation

$$s(s + 2) + K = 0$$

The reduced characteristic equation is rewritten as

$$1+\frac{K}{s(s+2)} = 0$$

The root locus of the reduced characteristic equation with K as a variable parameter can be plotted as dotted line in Figure 6.47 with open-loop poles at 0 and –2. The root contours plotted for various values of K with $\alpha = KK_t$ varying from 0 to infinity are shown in Figure 6.47.

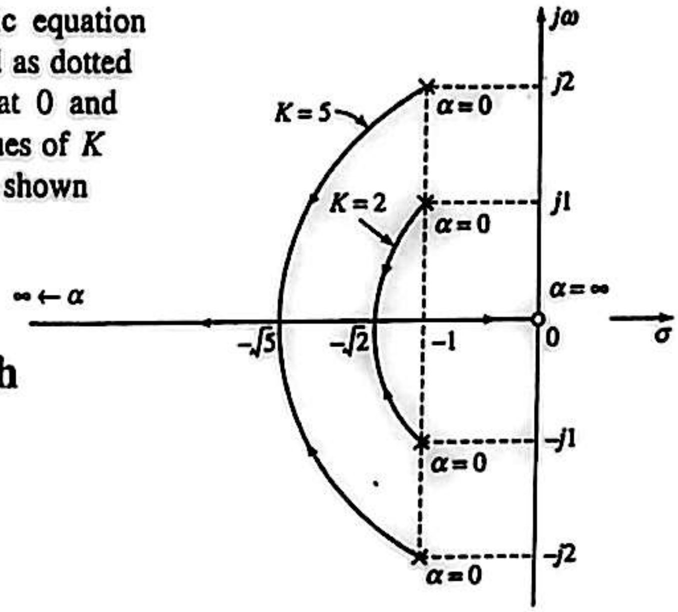


FIGURE 6.47

6.8 Root-locus for System with Transportation Lag

Consider a system having open-loop transfer function

$$G(s)H(s) = \frac{Ke^{-sT}}{s(s+2)}$$

where T is the transportation delay in seconds and is given as 1 s. Let us draw the root loci for K varying in the range $0 < K < \infty$.

If the transportation delay is small, then we can assume

$$e^{-sT} = 1 - sT$$

Rewriting $G(s)H(s)$ as

$$G(s)H(s) = \frac{K(1-s)}{s(s+2)}; \qquad T = 1$$

i.e.

$$G(s)H(s) = \frac{-K(s-1)}{s(s+2)}$$

The characteristic equation becomes

$$1 - G(s)H(s) = 0$$

Here the following two rules have to be modified:

(i) The angle condition instead of Eq. (6.13) becomes: $\angle\frac{K(s-1)}{s(s+2)} = 2k(\pi); k = 0, 1, 2, \ldots$

(ii) The existence of root loci for even (instead of odd as in Rule 6) number of poles and zeros to the right-hand side on the real axis, for the open-loop transfer function

$$\frac{K(s-1)}{s(s+2)}$$

The other rules are not to be modified.

The root loci is drawn as shown in Figure 6.48.

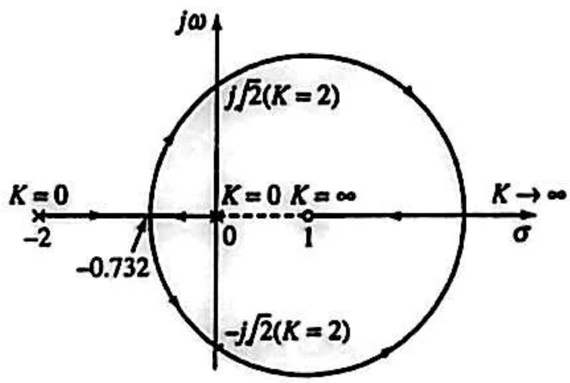


FIGURE 6.48 Root-loci plot of $1 + [Ke^{-s}/s(s+2)] = 0$ with e^{-s} is approximated as $(1 - s)$.

6.9 Design on Root Locus

To achieve a desired speed of response, the time constant of the system would have to be set below some specified value. Or, to obtain a desired oscillatory response, the damping characteristics of the system would have to be specified. The speed of response of the system is determined by the largest time constant appearing in the characteristic equation. On the root-locus plot, lines parallel to the imaginary axis represent lines that have constant $1/\tau$ value, where τ is the time constant. Hence, these can be considered as lines of fixed time constant value. Further, the undamped natural frequency ω_n and the damped natural frequency ω_d relate to the oscillatory behaviour of the system response. Lines that are parallel to the real axis, as in Figure 6.49, are lines of constant ω_d.

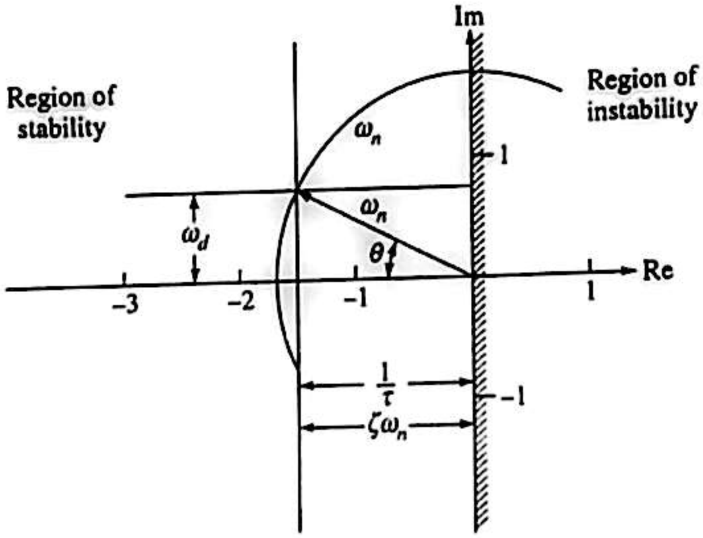


FIGURE 6.49 System design characteristics interpreted on the root-locus plot.

Constant ω_n values are indicated on the root locus plot by circles with the origin as the centre. We have already derived some important relations which are repeated here for ready reference such as the relation between ω_d and ω_n as

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Natural frequency can be related to the time constant by means of the damping ratio as

$$\frac{1}{\tau} = \zeta\omega_n$$

That means the vertical lines in complex plane represent constant $1/\tau$ or constant $\zeta\omega_n$ values. Further, the angle that relates the inclination of the damping ratio is given by

$$\cos\theta = \zeta$$

It is obviously clear that all the above three equations are applicable to underdamped systems, i.e. to systems with values of ζ less than unity.

EXAMPLE 6.17 For a feedback system of Figure 6.50:

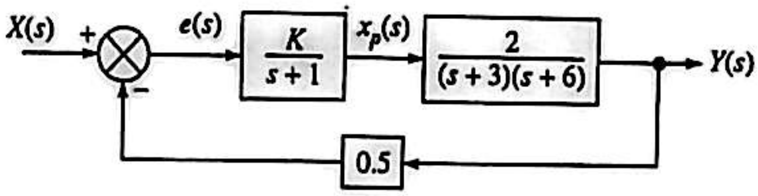


FIGURE 6.50 Example 6.17: control system.

(i) It is desired to have the maximum value of the time constant $\tau = 0.8$ s (for relatively fast response) without any oscillation of the closed-loop system response. Determine the value of gain K.

(ii) Further, check the response of the closed-loop system with the modified gain K for a step input $x(t) = 5$.

(iii) For $K = 10$, what will be the effect on system performance?

Solution: (i) The closed-loop transfer function

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{2K}{(s+1)(s+3)(s+6)}}{1+\dfrac{K}{(s+1)(s+3)(s+6)}} = \frac{2K}{s^3+10s^2+27s+18+K}$$

The open-loop transfer function is

$$G(s)H(s) = \frac{K}{(s+1)(s+3)(s+6)}$$

The time constant $\tau = 0.8$ s means that $1/\tau = 1.25$. To satisfy the design specification, the roots of the characteristic equation must lie on the root-locus plot to the left of $\sigma = -1.25$ line. Since the non-oscillatory response is required, the roots must lie on the real axis only. From the root-locus plot drawn as shown in Figure 6.51, the break-away point lies at -1.88, it means that a double pole occurs at -1.88 on the real axis for the particular value of gain K(say K_1). Further, it is quite obvious that the third pole of gain K_1 will be on the real axis away from -6. Now our task is to find the time constant τ corresponding to $\sigma = -1.88$ which comes out to be $1/1.88 = 0.53$ s which is within the design constraint of $\tau \le 0.8$ s and hence acceptable.

In order to determine the value of $K(= K_1)$ that corresponds to $s_1 = -1.88$, we make use of the magnitude condition as

$$G(s)H(s)\big|_{s=s_1} = \frac{K}{(s+1)(s+3)(s+6)} = -1$$

which leads to $$K = K_1 = -4.06$$

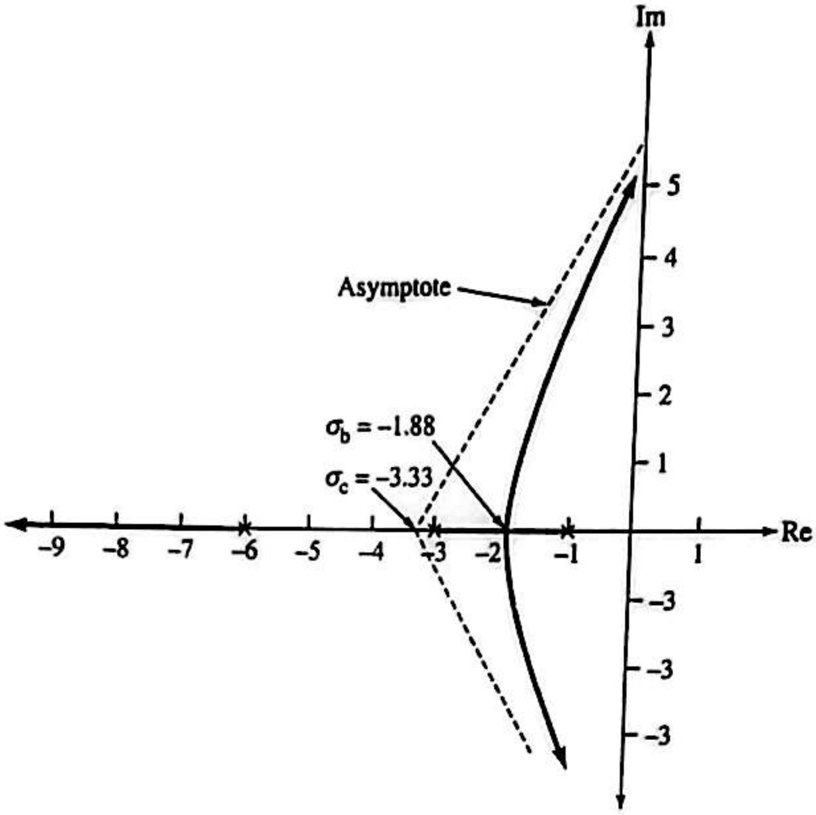


FIGURE 6.51 Example 6.17: root-loci plot.

The placement of the third root will be s_3 for $K = 4.06$ (obviously, s_3 will be to the left of the line $\sigma = -6$) and satisfy the magnitude condition and can be found out as

$$(s_3 + 1)(s_3 + 3)(s_3 + 6) = -1$$

which leads to $s_3 = -6.25$.

Now with $K = 4.06$, the double poles are at -1.88 ($t = 0.53$ s) and the third one at -6.24 ($t = 0.16$ s) which is < 0.8 s. Hence the speed of response is satisfactory as per the design constraint for $K = 4.06$.

(ii) Now the second part of the problem is to determine the closed-loop system response for a step of magnitude $x(t) = 5$.

$$\frac{Y(s)}{X(s)} = \frac{2K}{s^3+10s^2+27s+18+K} = \frac{2(4.06)}{s^3+10s^2+27s+18+4.06} = \frac{8.12}{s^3+10s^2+27s+22.06}$$

Hence the output response $y(t)$ becomes

$$y(t) = \mathcal{L}^{-1}\left[\frac{5(8.12)}{s(s^3+10s^2+27s+22.06)}\right] = 1.84 - 4.95te^{-1.88t} - 1.5e^{-1.88t} - 0.34e^{6.24t}$$

(iii) Now with $K = 10$, the gain condition to be satisfied, the characteristic equation is obtained as

$$\frac{10}{(s+1)(s+3)(s+6)} = -1$$

or

$$1 + G(s)H(s) = 1 + \frac{10}{(s+1)(s+3)(s+6)} = 0$$

eads to the roots as $-1.72 \pm j1.123$ and -6.515.

The closed-loop response is oscillatory and slower than case (i). Then nature of this esponse is determined by the complex roots whose imaginary part $\pm j1.123$ tells that the esponse is oscillatory, while the real part -1.732 determines the speed of response. The time constant associated with these roots is $\tau = 1/1.72 = 0.57$s and the time constant associated with the third root is $\tau = 1/6.515 = 0.15$ s. The undamped natural frequency $\omega_n = \sqrt{(1.742)^2 + (1.123)^2}$ $= 2.07$ rad/s. Again, ω_d can be determined as

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad \text{which leads to, } 1.123 = 2.07\sqrt{1-\zeta^2}\text{, hence } \zeta = 0.84$$

The closed-loop system is stable as all the roots have negative real part.

Summary

Root loci are the curves on the s-plane along which the roots of the characteristic equation move as a parameter (such as gain) is varied in the range from 0 to ∞. There is one locus for each root, that is, the root locus is unique. The beauty of the root-locus technique is that we can get the information about the roots of the characteristic equation of the closed-loop system from the open-loop pole-zero configuration when one parameter is varied in the positive range from 0 to ∞, without solving the characteristic equation of the closed-loop system.

The rules for constructing the root loci have been derived and their use has also been demonstrated, when one parameter (say gain) is varied in the positive range. Families of loci for more than one parameter varying can also be constructed as discussed in the construction of root contours. From the academic point of view, the construction of the inverse range from 0 to $-\infty$ has also been discussed.

The root loci may be used for analyzing stability and transient performances. The effects of parameter adjustment can be nicely visualized and hence the root-locus technique is of paramount importance in the design of control systems. It provides the designer with considerable insight into the system's stability, performance, and response characteristics. From the general shape of a root-loci diagram, one can get the idea of the type of controller or compensator needed to meet the particular design criterion.

The limitations of the root-locus technique lie in its inability to deal with more than one variable at a time, and the difficulty in dealing with time delays. Time delay or transportation delay cannot be handled conveniently using the standard Evan's rules of construction of root loci.

The root loci may be constructed by computer programming in the MATLAB platform. In order to evaluate the salient features of root-locus plots, such as break-away/break-in points, the intersection point with the imaginary axis, etc. one can develop the relevant program in C language in MATLAB platform as has been done here for better perception. Otherwise, using the Control Tool Box, these features can be evaluated.

The analytical expression for the break-away point cannot be obtained in MATLAB platform. One has to develop the relevant algorithm to sort out the problem. Each locus starts on a pole and terminates on a zero either at a finite point or at infinity. At overlapping points, multiplicity of root occurs and at that point loci depart. The root loci clashing at a point give the break-away or break-in point. This concept is useful in determining the break-away or break-in point in MATLAB platform for drawing the root loci.

Problems

6.1 Draw the rough sketch of the root loci for a unity-feedback system for each of the following forward transfer functions. Label all the critical points.

(a) $\dfrac{K}{s}$ (b) $\dfrac{K}{s^2}$

(c) $\dfrac{K}{s^3}$ (d) $\dfrac{K}{s(s+2)}$

(e) $\dfrac{K}{s(s+5)(s+9)}$ (f) $\dfrac{K}{s(s+4)(s^2+2s+2)}$

(g) $\dfrac{K}{s(s^2+2s+2)}$

6.2 The position control system with velocity feedback is shown in Figure P.6.2. Draw the root-loci for α varying in the range 0 to ∞. Then determine the value of α so that the damping coefficient of the closed-loop system may be 0.5.

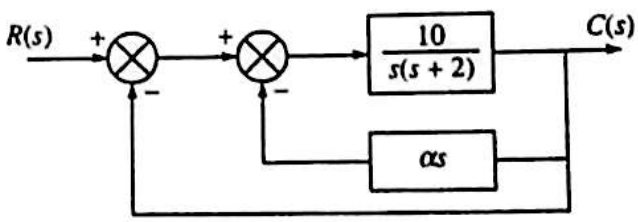


FIGURE P.6.2

6.3 For each of the unity-feedback systems having the transfer function $G(s)$ given below, where the variable parameter K is varying in the positive range from 0 to infinity, draw the root loci. Calculate the values of centroid of the asymptotes, the number of asymptotes and the values of ω at which root loci cross the imaginary axis.

(a) $\dfrac{K}{s(s+5)^2}$ (b) $\dfrac{K}{s(s+1)(s+2)}$

(c) $\dfrac{K(s+1)^2}{s^3}$ (d) $\dfrac{K(s+5)}{s^2(s+3)}$

(e) $\dfrac{K(s+7)}{s(s+1)(s+2)}$ (f) $\dfrac{K(s+2)}{s(s+1)(s+3+j10)(s+3-j10)}$

(g) $\dfrac{K(s^2+2s+25)}{s(s+1)(s^2+4s+25)}$ (h) $\dfrac{K(s+15)8}{(s+5)(s+2+j6)(s+2-j6)}$

6.4 For the open-loop transfer function

$$G(s)H(s) = \frac{K(s+10)}{s(s+5)(s+25)(s+50)}$$

draw the root loci. Determine the value of K for the system having damping coefficient as 0.707, and also find the value of the complex conjugate roots.

6.5 Using the root-locus technique, show how to estimate the roots of the characteristic polynomial

$$s^4 + 100s^2 + 10{,}000 = 0$$

6.6 Draw the the root loci for the system shown in Figure P.6.6.

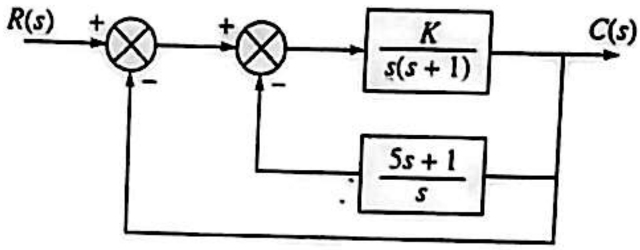


FIGURE P.6.6

6.7 For the open-loop transfer function

$$G(s)H(s) = \frac{K(s+1)(s+5)}{s^2(s+2)}$$

draw the root loci. Determine the values of s at which root loci leave the real axis and re-enter for the system to be underdamped.

6.8 For the open-loop transfer function

$$G(s)H(s) = \frac{K(s+2)}{s(s+1)(s+5)(s+8)}$$

draw the root loci. At some point the imaginary coordinate is $\omega = 1.5$, what is then the real value of s.

6.9 For the open-loop transfer function

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

draw the root loci. Determine the value of K for the closed-loop pole to be at $s = -5$. Determine the other two complex conjugate poles at this value of K.

6.10 For the open-loop transfer function

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)(s+4)}$$

draw the root loci. Determine the roots from the root loci for the value of $K = 10$.

6.11 For the open-loop transfer function

$$G(s)H(s) = \frac{K(s+5)}{s(s+1)}$$

draw the root loci. Then determine the location of roots for $K = 9$. Determine the range of K for which the closed-loop system is overdamped. Repeat the same for the system to be underdamped.

6.12 For a velocity feedback system shown in Figure P.6.12, draw the root loci for velocity constant K varying in the positive range from 0 to ∞. Determine the value of K for the closed-loop system to have damping coefficient as 0.707. Determine the roots at this value of K.

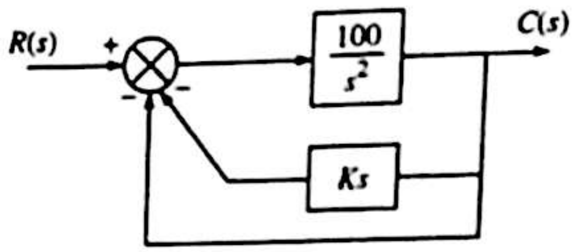


FIGURE P.6.12

6.13 For the system shown in Figure P.6.13, draw the root loci considering z as the variable parameter.

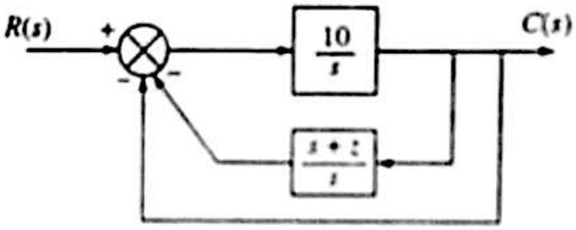


FIGURE P.6.13

6.14 For the system shown in Figure P.6.14, draw the root loci considering K as the variable parameter.

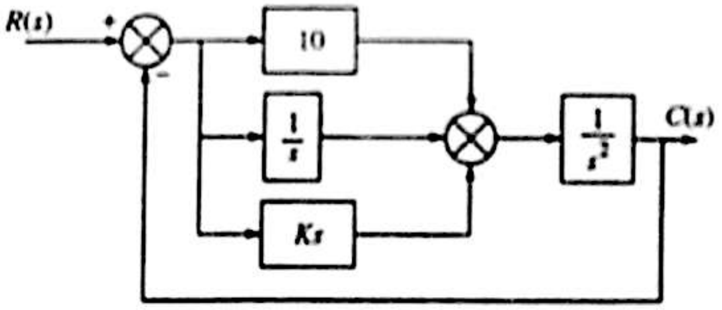


FIGURE P.6.14

6.15 For a unity-feedback system having the open-loop transfer function

$$\frac{20(s^2 + as + 10)}{s^2(s+1)}$$

draw the root loci considering the variable parameter as a.

6.16 From the root loci point of view, show the effect of variable parameter K on the transient response of the system shown in Figure P.6.16.

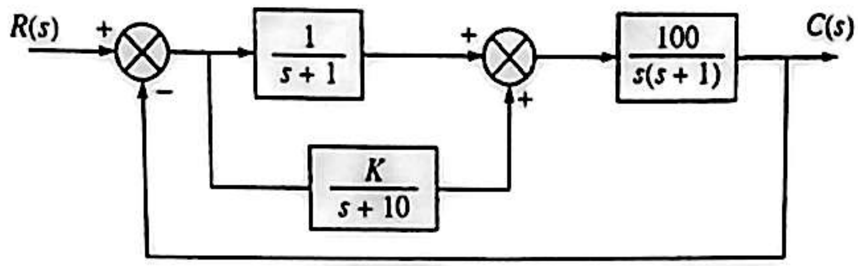


FIGURE P.6.16

6.17 For a unity-feedback system having the open-loop transfer function $G(s)$ as

$$G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$$

plot the root loci. Locate the closed-loop poles on the root loci such that the dominant closed-loop poles have a damping ratio equal to 0.5. Determine the range of gain K for stability.

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Process Control System

OBJECTIVE

By now, we have acquired some amount of knowledge about the classical control systems. The process control terminology though not introduced so far is mainly structured in the light of classical control theory. This chapter provides a glimpse of the design methodology of process control systems so that students get the confidence in handling the same.

CHAPTER OUTLINE

7.1 Introduction

One of the oldest and best known examples of a typical continuous controller is the original steam engine speed governor used by James Watt as depicted in Figure 7.1.

The engine drives the governor mechanism via a pulley. As the engine speed increases the centrifugal force tends to move the flyweights in an outward direction, thus actuating a lever in such a manner that the steam valve begins to close as the engine speed begins to increase.

As the steam valve can take up any position between fully open and fully closed extremities, such adjustment is termed *continuous control*.

Further, amongst the disciplines of continuous control there are two schools of thought. One school advocates pneumatic control and the other electronic control. However, depending upon the control application, one has to adopt either on-off control or continuous control. In the case of a process with large inertia, the continuous controller cannot react instantaneously to a fast changing input, whereas an on-off controller reacts immediately and brings the control output near to the set value.

Here we will discuss the philosophy of feedback control in process control systems. Any negative feedback control system is obviously a proportional control system. Our objective is to explain the continuous controller such as proportional (P), Proportional and Integral (PI), Proportional, Integral and Derivative (PID) for controlling a process control system. To begin with, we will consider the steam engine speed governor indeed as the process.

7.2 Proportional Control Action

The fly-ball governor was used for controlling the speed of James Watt's steam engine in 1770. After understanding its working, it should be an easy jump to the electronic equivalent, also shown in Figure 7.1.

In Figure 7.2 (the simplified block diagram version of Figure 7.1), the speed of the Watt's steam engine is detected by a tachogenerator which supplies voltage proportional to the speed. This input signal to the controller is compared with the set-value, which can be adjusted

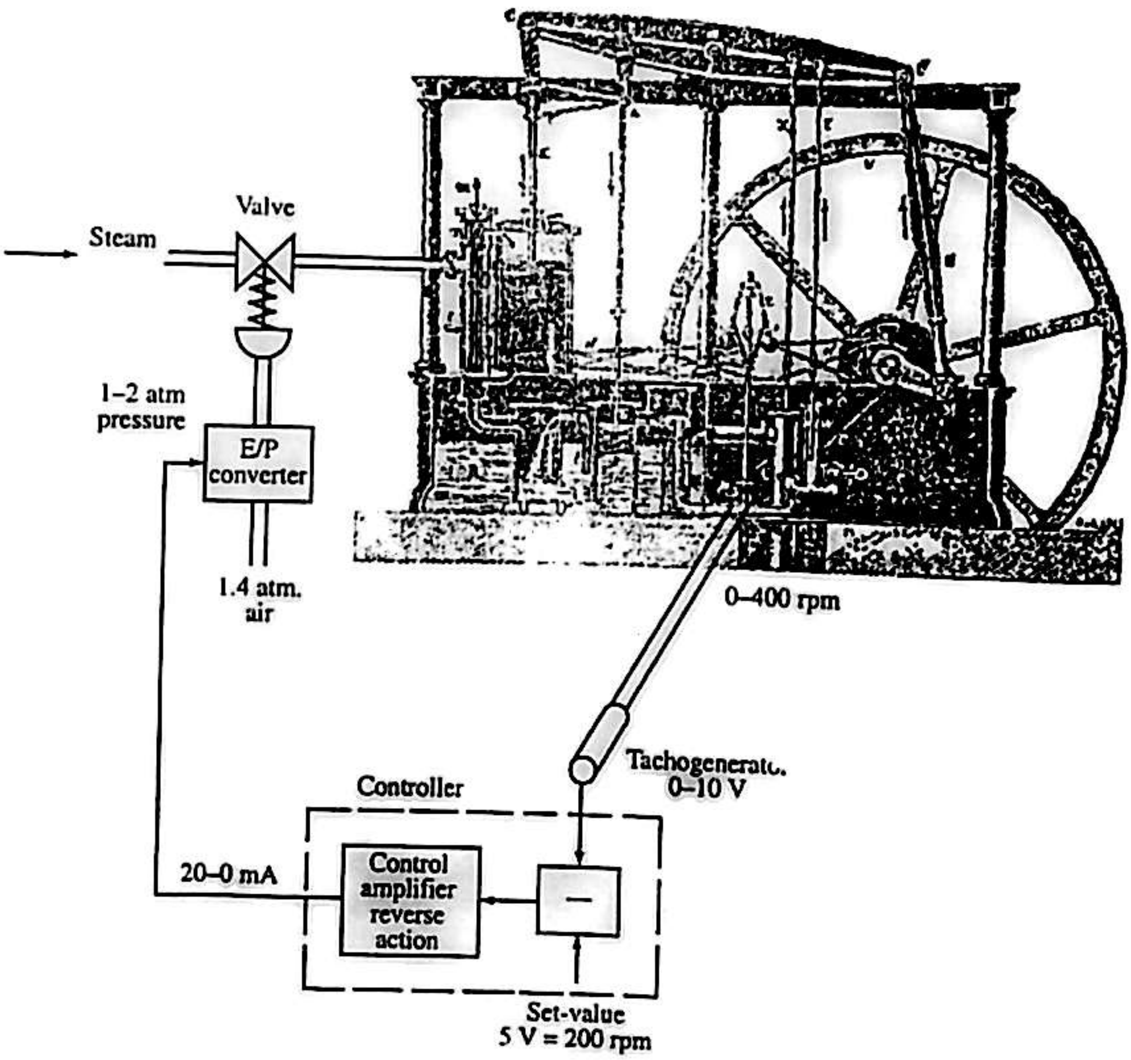


FIGURE 7.1 James Watt's steam engine with electronic speed controller.

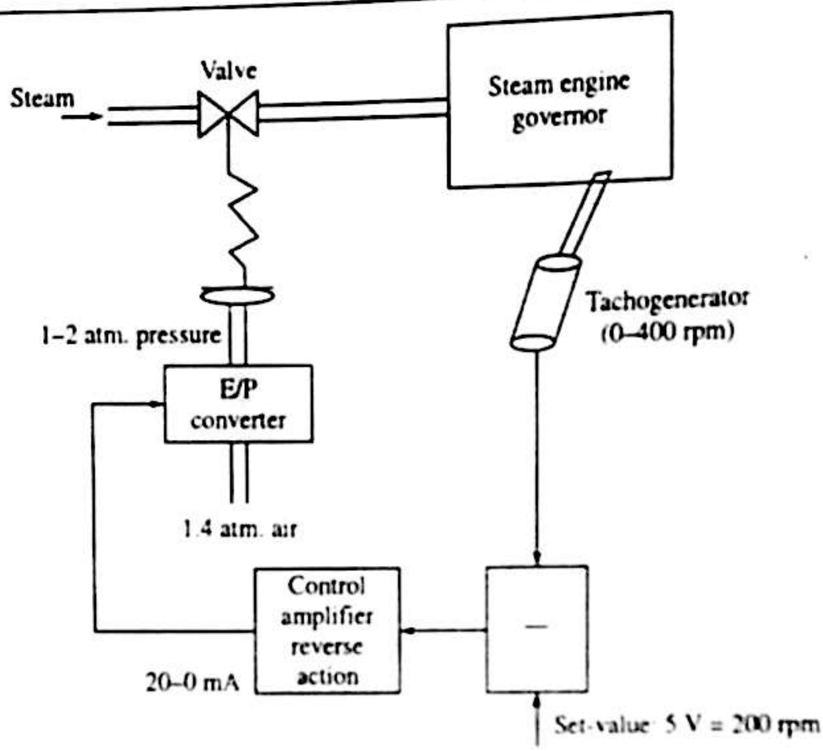


FIGURE 7.2 Simplified version of Figure 7.1.

manually to any required value. If the speed of the engine (the output of the tachogenerator) does not match the required set-value (the voltage represented by the set-point selected), the controller which supplies a correcting signal, moves the pneumatic valve in the steam line to another position.

Let us assume that the controller can supply as output a signal between 0 and 20 mA, the actual value depends on the difference existing between the set-value and the actual speed of the engine. Let us assume that the set-point speed of the engine is 200 rpm, and that the maximum permissible speed is 400 rpm. We adjust the controller in such a way that if the actual speed of the engine (i.e. the input to the controller) and the set-point coincide (i.e. no deviation), then the output signal of the controller will be 10 mA. This value of 10 mA is chosen in order to have sufficient correcting control action in both directions. This 10 mA is converted into a 0.6 kg/cm^2 gauge pressure signal by an electro-pneumatic (E/P) converter and then used to position a pneumatic valve to 50% opening. If, due to increased load, the speed of the engine decreases, the tachogenerator will produce less voltage and the controller will detect too low a signal at its input (negative deviation). The result will be that the controller will supply an increased output signal to the E/P converter. This increased signal will result in a higher pressure on the valve and this will increase its opening, allowing more steam to flow to the engine. The increased steam supply will again increase the speed of the engine. In principle, this is continuous proportional control and forms the basis of all feedback control schemes.

The control action may be either *reverse* or *direct*. When the controller's decreased input results in increased output, this is termed *reverse action*. However, there exist many applications which require that the increased controller's input should result in increased controller output. This is termed *direct action*.

Referring to Figure 7.3, we have the control valve with reverse control action where we assumed that at maximum speed (400 rpm) the controller will tend to close the steam valve completely and that at the nominal speed of 200 rpm (= controller set-point) the steam valve will be 50% open and that at 0 rpm the controller will tend to completely open the steam valve.

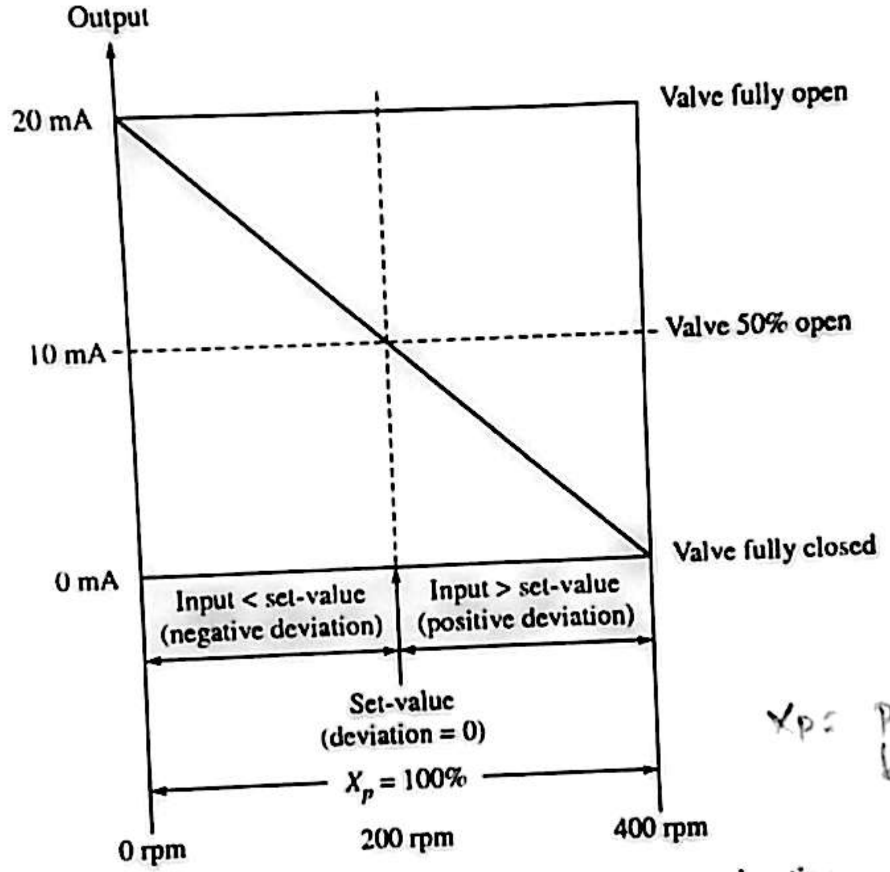


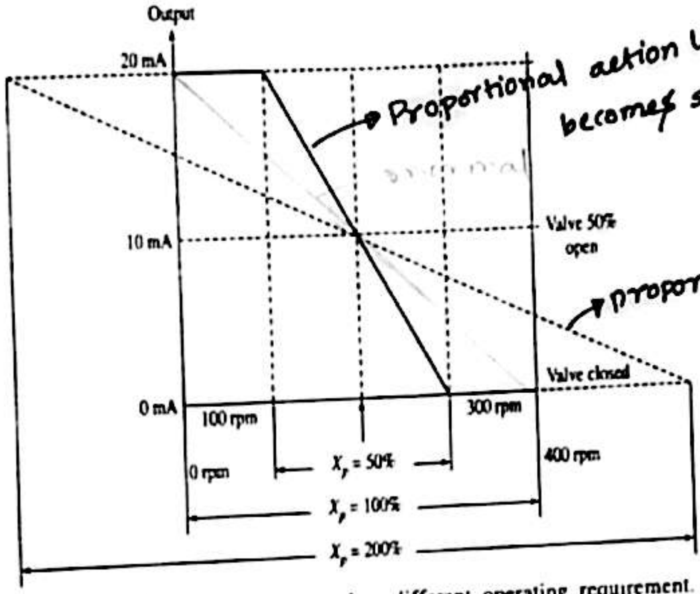
FIGURE 7.3 Control valve with reverse control action.

However, for different operating requirements, it might be necessary to have the valve fully closed at 300 rpm and to have it fully opened at 100 rpm. In this case, the output of the controller should be 0 mA at 300 rpm and 20 mA at 100 rpm. This is illustrated in Figure 7.4. The proportional action line of the controller is now steeper. In other words, the proportional band has become narrower (X_p small).

For some applications it might be necessary that the valve be never completely closed or never completely opened. In this case, for example, the output should be only 15 mA at minimum speed and 5 mA at maximum speed. In this case the proportional band becomes wider (X_p large) as illustrated in Figure 7.4.

In Figure 7.3, the proportional band is 100%, whereas in Figure 7.4, the proportional band is 50% as indicated by the continuous line and 200% as indicated by the dotted line.

It is also possible to make the proportional band 0% as shown in Figure 7.5(a) when the smallest deviation from the set-point would result in sudden complete opening or complete closing of the valve. This becomes the ON-OFF control. At the other extreme, the proportional



→ Proportional action line of the controller has becomes steeper / proportional band has become narrower. (Xp is small)

→ proportional band has become wider.

FIGURE 7.4 Control valve with a different operating requirement.

band would be infinite as shown in Figure 7.5(b), where, whatever the deviation may be from the set-point, the output of the controller would remain 10 mA. Thus there would be no control action at all.

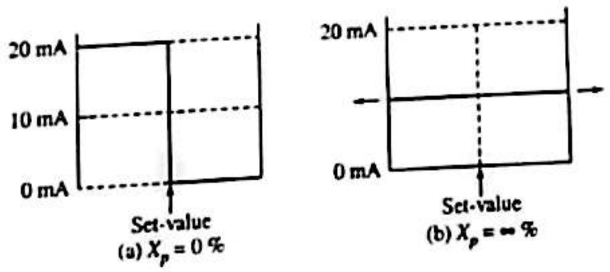


FIGURE 7.5 Proportional band: (a) ON-OFF control and (b) Nil control.

The proportional band is commonly used to refer to the sensitivity or gain of the proportional controller. Proportional band is the % of change in the input to the controller which is the error signal, required to cause 100% change in the output of the actuator. Thus a small (high) proportional band means high (small) gain of the controller or the controller with high (low) sensitivity. Thus a gain of 4%/1% means that there is a change of 4% in the output if the change in the input is 1%. Then

$$\text{Proportional band} = \frac{100\%}{\text{gain in percentage/\%}} = \frac{100\%}{4\%/1\%} = 25\%$$

Xp↓, gain of controller ↑ sensitivity ↑

proportional band $X_p \propto \frac{1}{\text{gain of controller}}$

$\propto \frac{1}{\text{sensitivity}}$

Xp↑; gain of controller ↓, sensitivity ↓.

7.2.1 Difficulty with proportional control

Proportional control seems to be most ideal, however, there is one major disadvantage invariably coupled with proportional control, i.e. offset or proportional offset. Under nominal load conditions, steam engine will run at 200 rpm when the steam valve is 50% opened (controller output is 10 mA). If the load on the engine is increased, the speed will drop. This results in negative control deviation which, in turn, will force the controller to increase its output signal because of the reverse action in the controller. Consequently, the steam valve will be opened wider and the engine will accelerate again. This, however, has the effect that the speed comes closer to the set-value and the control deviation decreases. If the proportional controller could manage to restore the process to the set-value, this would mean that the deviation would become zero again, which would however by definition coincide with an output signal of 10 mA. The increase in speed due to the larger output signal of the controller reduces the control deviation. The controller sensing this, tends now to close the valve again.

In fact, a proportional-controller can only supply sufficient steam to the engine for as long a duration as the load remain nominal. Any other load condition will force the controller to seek a new operating point along its proportional action line, automatically resulting in a deviation from the required set-point. This deviation is termed *proportional offset* or just *offset*. This offset increases in magnitude as the load conditions of the process differ more and more from their nominal value. The offset also increases as the width of the proportional band of the controller is made greater.

7.3 Integral Action or Reset

Offset is inherent in a proportional controller, which will either give too much or too little output as soon as the load conditions depart from nominal.

Therefore the P-controller must be equipped with additional means in order to get rid of this undesirable effect. We have to consider the inclusion of the unit which supplies sufficient additional output current to the control valve so that the offset is eliminated and the control deviation becomes zero. Let us try to describe this in detail with the help of Figure 7.6.

Due to an increased load on the engine, the P-controller tries to supply sufficient output current so that the set-value speed could be reached again. Let us assume that this would require 19 mA output. Due to its characteristics the P-controller cannot succeed in reaching this and will stabilize the process at 15 mA [Figure 7.6(b)], resulting in a proportional offset of 4 mA (which is equivalent to loss of some engine speed).

The additional unit (the integral or I-unit) will also detect the control deviation and will start to supply an ever increasing current to the control valve for as long as a deviation exists. The larger the deviation, the faster the increase in output current will be [Figure 7.6(d)].

This additional current will give the valve additional opening, resulting in an increased engine speed and a decreased deviation. This, in turn, will cause the output current of the P-controller to become smaller [Figure 7.6(c)]. After a certain period, due to the steadily increasing current from the I-unit, the offset will get nearer to zero and the P-controller will again reach its nominal 10 mA output current. In the meantime, the I-unit has constantly increased its output current and already produces some 9 mA. As soon as the set-point speed of the engine is reached, the I-unit will stop increasing the output current. Then, the sum of the currents from P and I-units would be exactly 19 mA, which is the current required to reach the set-point speed under the new load conditions [Figure 7.6(e)]

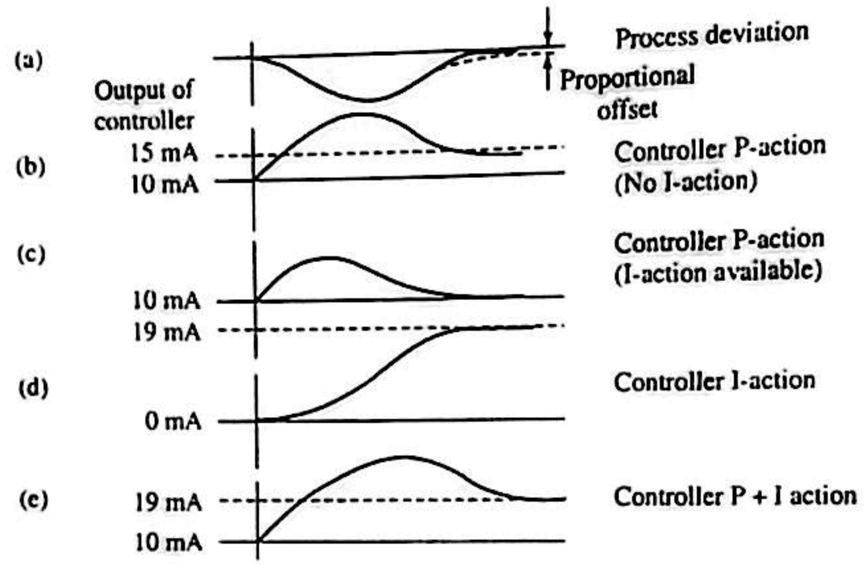


FIGURE 7.6 Response of P-I controller.

The response fully depends on the speed of the increase in current from the I-unit, i.e. whether it really stops at zero offset or whether the current continues to increase for a while. In the latter case we will be confronted with an overshoot and the same phenomenon takes place all over again in the other direction. It is important to understand that the combined action of P and I-unit, if adjusted to optimal values, can result in a smooth control action without overshoot and without proportional offset.

Briefly recapitulating, we see that:

- The P-unit tries to correct the position of the valve rapidly and proportionally to the control deviation but does not succeed completely.
- The I-unit starts to give a steadily increasing current as long as a control deviation continues to exist. The larger the deviation, the faster the increase or decrease in extra current. In the new balanced situation, the I-unit together with the 10 mA from the P-unit will supply all the current (positive or negative) necessary to form the required total output current to eliminate the control deviation.
- The more I-action (the faster the increase in current per time unit for a given deviation value), the less stable the control loop. At one extreme the current increases almost instantaneously to its maximum regardless of the magnitude of the deviation; on the other extreme the I-unit will not produce any current regardless of the size of deviation. Somewhere in between will be the optimum setting. For practical reasons, the value for the "I-action" is usually denoted by its reciprocal value $1/I$—action which is called T_i. Thus T_i represents the time interval necessary for integral action to cause

the same amount of correcting movement as that produced by proportional action due to a steady deviation from the set-value. This time interval T_i is usually expressed in minutes. Thus,

$$T_i = \infty \text{ means no I-action}$$

$$T_i = 0 \text{ means infinite I-action}$$

For steady process control, a continuous controller should include both types of control actions, i.e. P and I as shown in Figure 7.7.

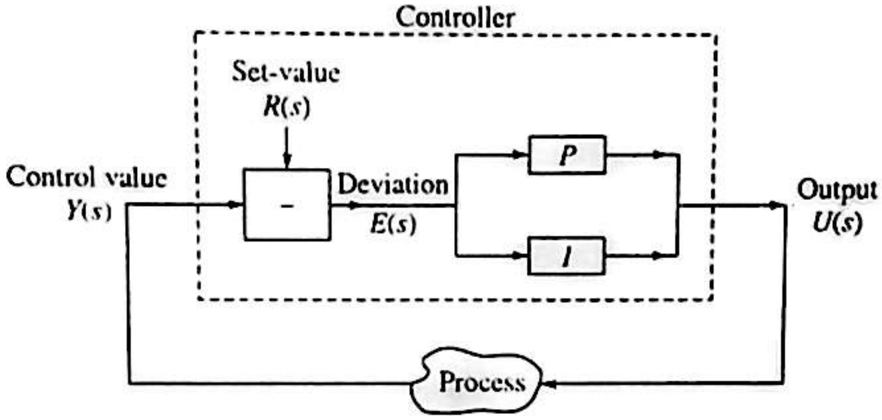


FIGURE 7.7 Continuous controller.

7.4 Differential Action: Derivative or Rate Control

As long as the process is running under steady-state conditions, the PI-controller will do fine. Even if the process disturbances occur, the PI-controller will handle these very well. However, the wider the proportional band (X_p) and the smaller the integral action (T_i), the less responsive the control system will become. Depending on the characteristics of the process, for instance, when running under steady-state conditions with small disturbances, the optimal PI setting would give optimum results, but as soon as too large a disturbance occurs, then the PI setting would be too slow. This especially can occur under starting-up conditions.

What in fact is required, is the extra amount of push in the beginning. As long as the speed of the steam engine is far below the set-value, it does no harm to give the control valve an additional lift; much more than is actually required for the set-value speed. This would result in the engine obtaining an excess of steam, which would accelerate the engine rapidly, enabling it to reach the set-value sooner. This situation must not, however, last longer than just a few moments, otherwise the speed of the engine might become too high and we would be confronted with an overshoot again. This extra push in the beginning is given by the derivative unit. It detects at its input the change in increase or decrease of the deviation. Thus it is not the magnitude of the deviation as such (as was the case with the P and the I control mode), but the speed with which the deviation value increases or decreases is what counts. The higher this speed (the quicker the control value runs away from the set-value) the higher the extra push signal or the D-action at the output as shown in Figure 7.8. This implies, on the other hand,

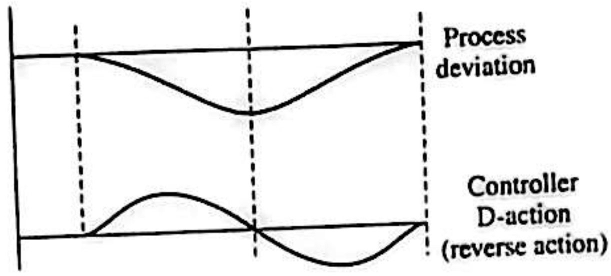


FIGURE 7.8 D-action.

that an existing deviation which remains constant will not cause any D-action. D-action is normally denoted as T_d in time-units and thus $T_d = 0$ implies no D-action and $T_d = \infty$ implies infinite D-action.

7.5 PID Control

In order to obtain optimal control for different processes, a continuous controller should include three different control modes: proportional action P, integral action I, and derivative action D as shown in Figure 7.9. The controller is of PID type. In Figure 7.10, it is shown how the process will react when:

(a) The controller acts as P-controller ($T_i = \infty$, $T_d = 0$)

(b) The controller acts as PI-controller ($T_d = 0$ and optimal setting for X_p and T_i)

(c) The controller acts as PID-controller (optimal settings for X_p, T_i and T_d)

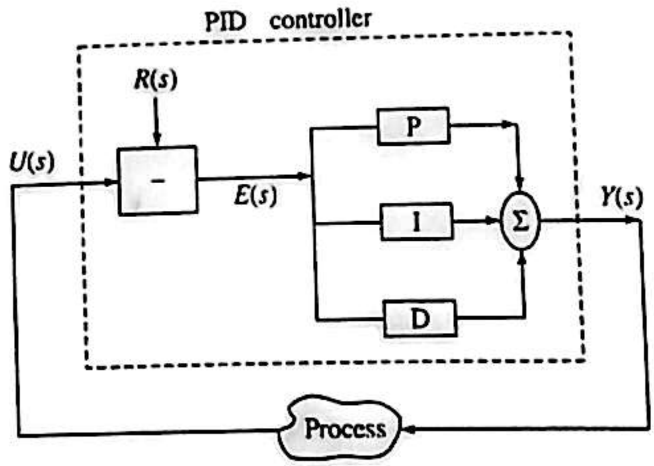


FIGURE 7.9 PID controller.

All three modes should be separately adjustable. It would require three independent operational amplifiers with control modes as shown in Figure 7.11. This scheme is costly and requires complex electronic circuitry. However, it is possible to achieve the same result cost-effectively by using one operational amplifier with the control modes as feedback circuit, as

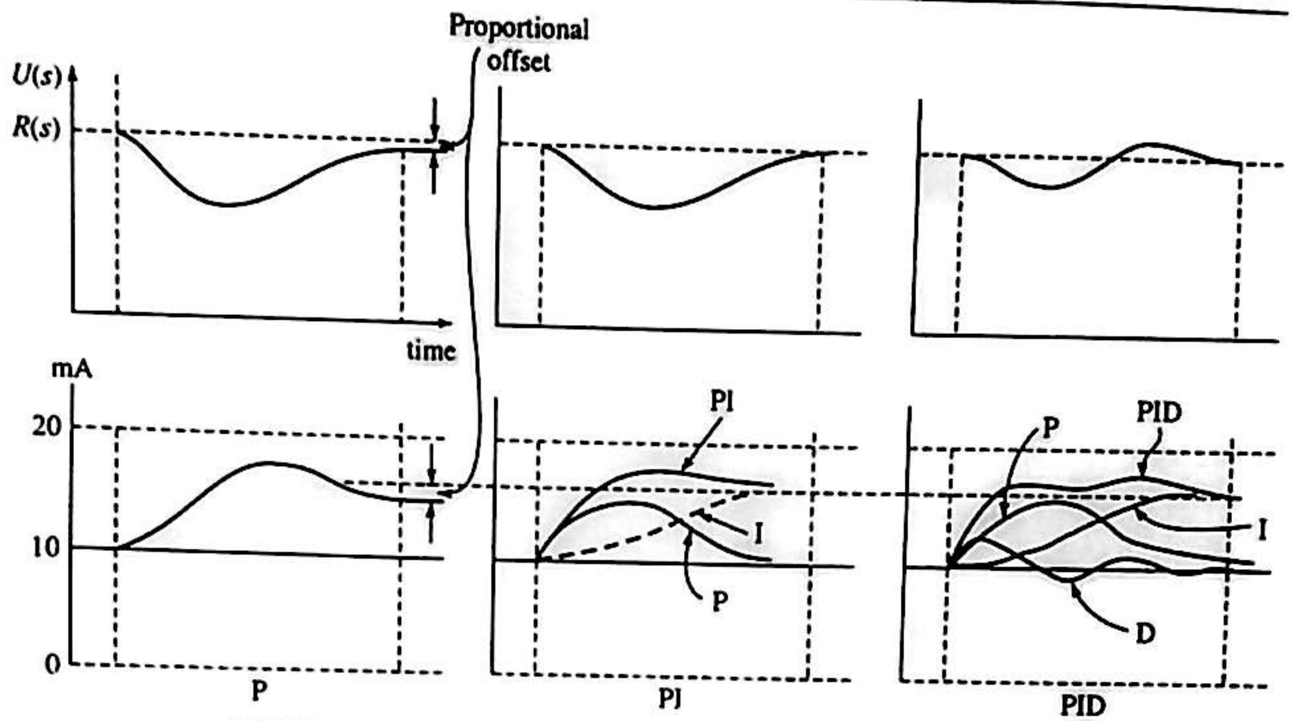


FIGURE 7.10 Process response to P-, PI-, and PID controllers.

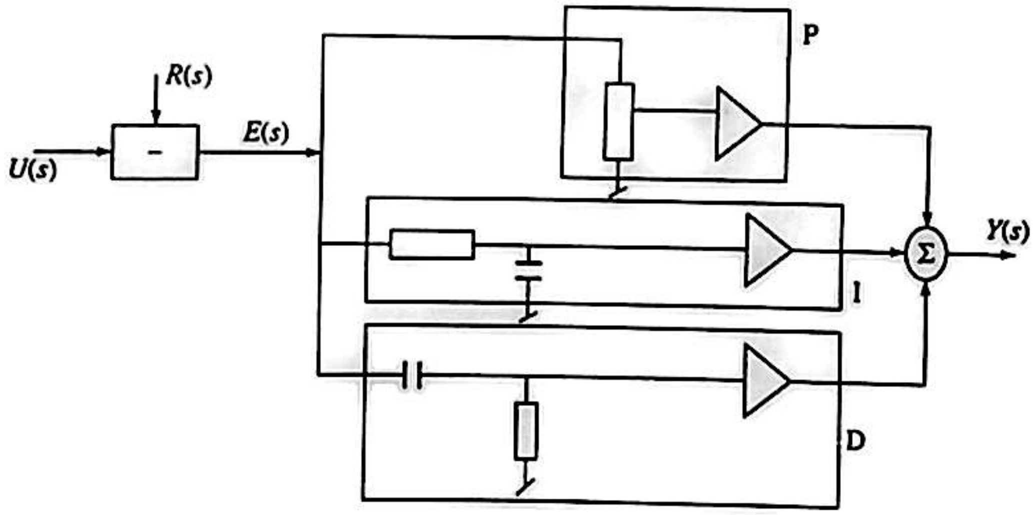


FIGURE 7.11 Control modes with three independent operational amplifiers.

shown in Figure 7.12. Note that because now we are considering controller in feedback, the *RC* networks for I and D are reversed.

The PID-control using one op-amp, as in Figure 7.12, would function well in normal operating conditions. However, the difficulty arises in the following cases:

(1) At start-up, when it takes a long time before the control valve reaches the set-value.

(2) When there is a sudden increase or decrease in the set-value, which is equivalent to case (1).

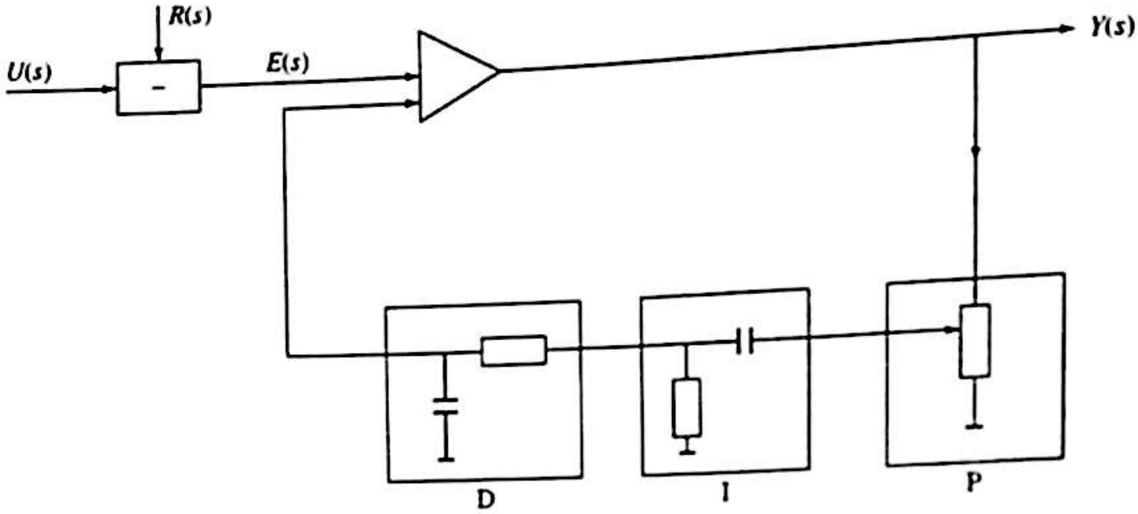


FIGURE 7.12 Control modes as feedback circuit.

Bearing in mind that we are now operating with a feedback arrangement, the P, I and D unit can no longer react to the control deviation as appearing at the input of the controller but can only be influenced by the output signal of the controller. A process normally requires a long period of time from start-up to reach the set-value. Due to the persistence of the control deviation, the controller will, after a while, produce maximum output (20 mA). This is due to the fact that the control deviation is larger than the proportional band and due to the action of the I-unit which has been given plenty of time to reach 20 mA. This saturated condition will continue until the process actually has reached the set-value. Only then will the controller start to reduce its output signal (much too late) and then only will the P, I and D unit be able to detect that some thing has happened at the input of the controller. The result will be that a serious overshoot would occur. In other words, if the deviation conditions last too long, the P, I and D unit will be unable to initiate corrective action.

There are two ways to overcome this drawback:

1. During start-up the I-action is switched off manually or automatically. By doing so, the I-unit cannot drive the controller output to its saturated condition, and thus the proportional action remains active.
2. The D-unit is not included in the feedback circuit but operates through a separate amplifier directly on the deviation signal at the input circuit of the controller. Note that the input signal to the controller keeps changing as long as the control valve does not reach the set-point. We agreed that the output signal to the controller during this period cannot change because it produced full scale signal (20 mA) Thus, with the D-unit acting in the input section, the PI section of the controller receives the sum of deviation signal at its input. Under normal running conditions, this works as decried before but at start-up conditions it offers a real advantage. We know that as long as the control valve changes, the D-unit will produce a signal. This signal will result in a breaking action, an early reduction in controller output signal, so that the control valve can reach the set-value smoothly without undue overshoot. We finally come to the definite block

diagram of a modern continuous PID controller in Figure 7.13, where we have the following components:

Proportional unit with X_p-setting

Integral unit with T_i-setting

Derivative unit with T_d-setting

Switch to eliminate I-action (start-up)

Rate before Reset (start-up)

Reverse action switch

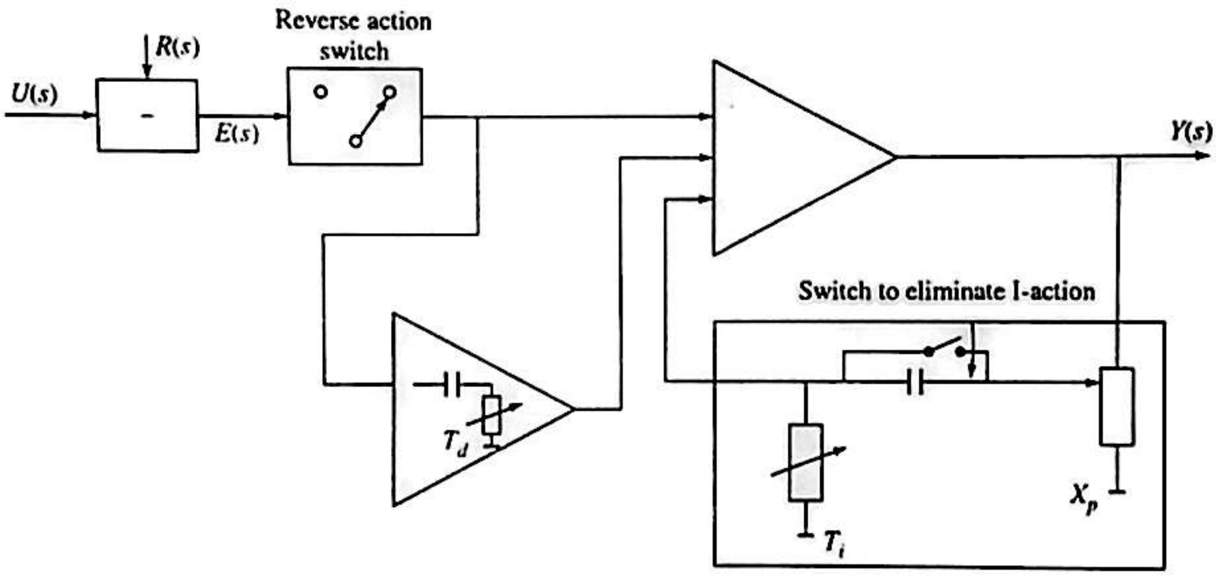


FIGURE 7.13 Block diagram of a modern PID controller.

The response of ideal controllers to standard inputs is given in Table 7.1.

7.5.1 Tuning rules for PID controllers

The PID controllers are very frequently used in industrial control systems:

Proportional (P) action : $u(t) = K_p e$ (7.1)

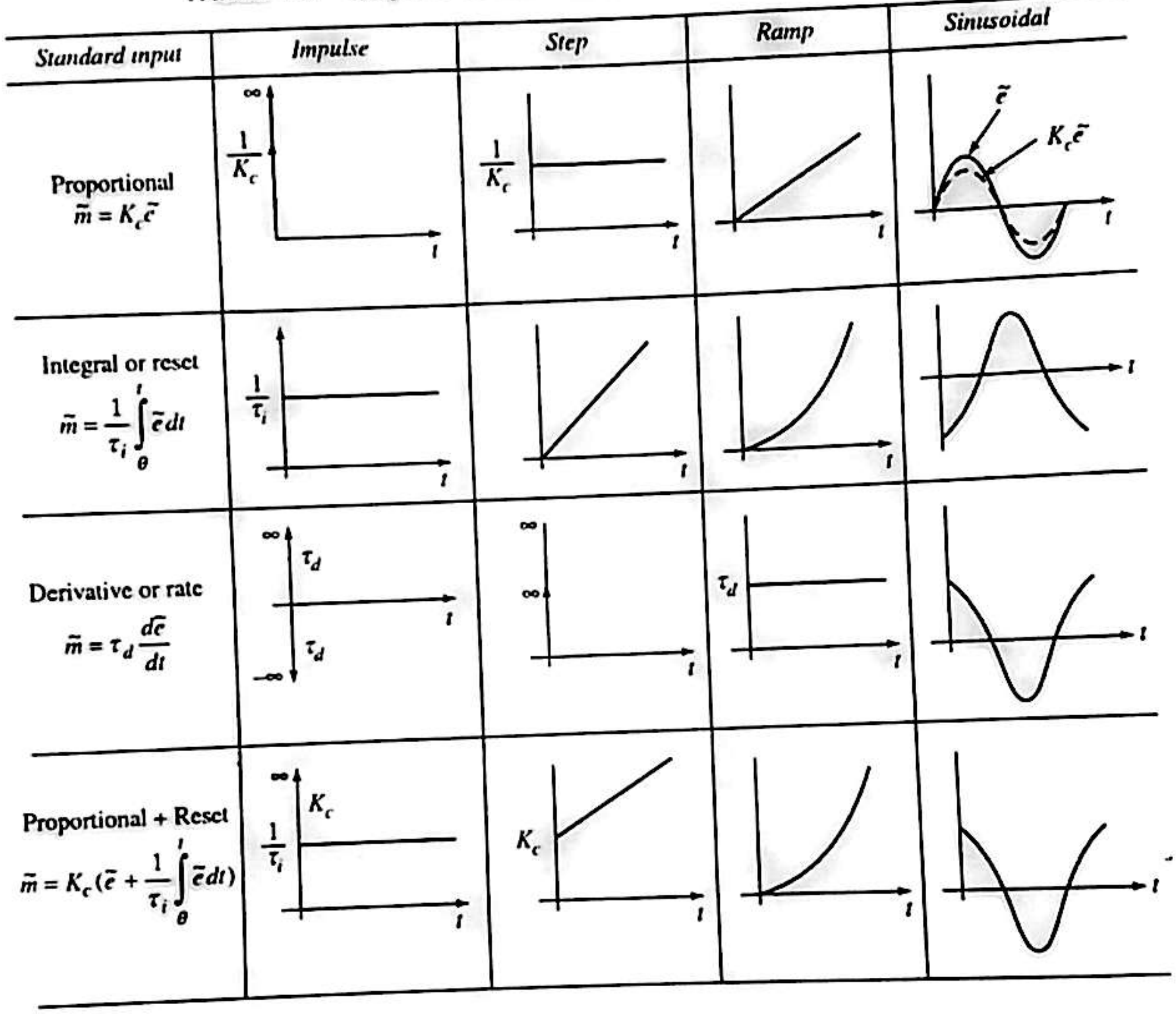
Integral (I) action : $u(t) = K_i \int e\,dt$ (7.2)

Derivative (D) action : $u(t) = K_d \dfrac{de}{dt}$ (7.3)

(P + I + D) action : $u(t) = K_p e + K_i \int e\,dt + K_d \dfrac{de}{dt}$ (7.4)

where, K_p is the proportional gain, K_i is the integral gain, and K_d is the derivative gain.

TABLE 7.1 Response of ideal controllers to standard inputs



In this case K_p, K_i and K_d become the controller parameters, that is, they are constants (adjustable on controller).

Let X_p be the proportional band where

$$K_p = \frac{1}{X_p} \tag{7.5}$$

The transfer function $G_c(s)$ of the PID controller is

$$G_c(s) = \frac{Y(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \tag{7.6}$$

And the output of the PID controller can be written as

$$y = \frac{1}{X_p}e + \frac{1}{X_p T_i}\int e\,dt + \frac{T_d}{X_p}\frac{de}{dt} \tag{7.7}$$

where

y is the output signal from the PID controller
e is the control deviation
X_p is the proportional band (= $1/K_p$)
T_i is the integral action time (= K_i/X_p) (7.8)
T_d is the derivative action time (= $K_d X_p$) (7.9)
(7.10)

It is noted that in actual PID controllers, instead of adjusting the proportional gain K_p, we adjust the proportional band X_p. The proportional band is $1/K_p$ and is expressed in per cent, for example, 25% band corresponds to $K_p = 4$. The control action of individual controllers is given in Table 7.2.

TABLE 7.2 P-, I-, and D-control actions

	Control action	*Main purpose*		*Stability*	*Reaction time control loop*
X_p (%) P	A control deviation will produce controller output proportional to the magnitude of the deviation	Produces bulk of corrective effort	X_p > (proportional band larger)	better	slower
			X_p < (proportional band smaller)	worse	quicker
T_i (time) I	The moment a control deviation exists, the controller starts to produce an ever-increasing output current. The larger the deviation, the faster this increase or decrease in output current	Elimination of proportional offset	T_i > (I-action smaller)	better	slower
			T_i < (I-action larger)	worse	quicker
T_d (time) D	Any *change* in control deviation will *result* in an output, proportional to the speed at which the deviation is changing	Extra push for quick return to set-value	T_d > (D-action larger	worse	quicker
			T_d = optimal	better	quicker
			T_d < (D-action smaller)	worse	slower

Figure 7.14 shows an electronic PID controller using operational amplifiers. The transfer function $E(s)/E_i(s)$ is given by

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1} = -\left(\frac{sR_2C_2+1}{sC_2}\right)\left(\frac{sR_1C_1+1}{R_1}\right) \tag{7.11}$$

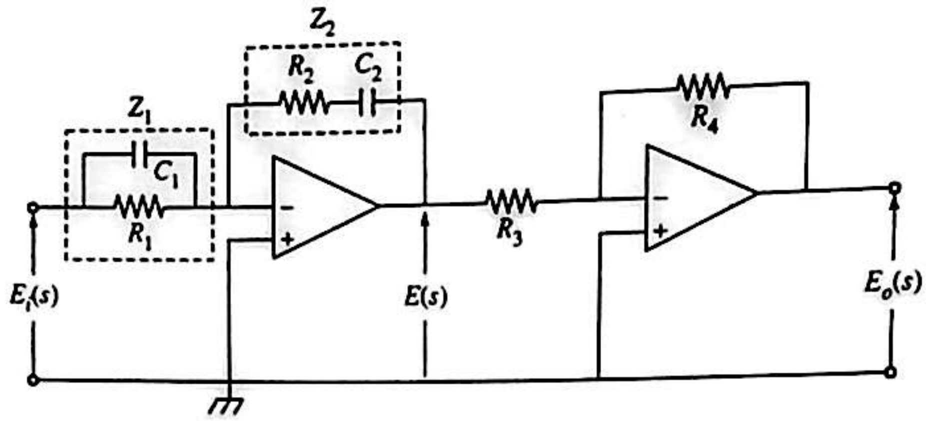


FIGURE 7.14 Electronic PID controller.

where $$Z_1 = \frac{R_1}{sR_1C_1+1} \quad \text{and} \quad Z_2 = \frac{sR_2C_2+1}{sC_2}$$

Further noting that

$$\frac{E_o(s)}{E(s)} = -\frac{R_4}{R_3} \tag{7.12}$$

we have

$$\begin{aligned}\frac{E_o(s)}{E_i(s)} &= \frac{E_o(s)}{E(s)}\frac{E(s)}{E_i(s)} \\ &= \frac{R_4(R_1C_1+R_2C_2)}{R_1R_3C_2}\left[1+\frac{1}{(R_1C_1+R_2C_2)s}+\frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2}s\right]\end{aligned} \tag{7.13}$$

Thus comparing with Eq. (7.6), we get

$$K_p = \frac{R_4(R_1C_1+R_2C_2)}{R_1R_3C_2} \tag{7.14}$$

$$T_i = R_1C_1 + R_2C_2 \tag{7.15}$$

$$T_d = \frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2} \tag{7.16}$$

Using Eq. (7.7), we can rewrite in terms of the proportional gain, integral gain, and derivative gain as

$$K_p = \frac{R_4(R_1C_1+R_2C_2)}{R_1R_3C_2} \tag{7.17}$$

$$K_i = \frac{R_4}{R_1R_3C_2} \tag{7.18}$$

$$K_d = \frac{R_2 R_4 C_1}{R_3} \tag{7.19}$$

Note that the second operational amplifier circuit (Figure 7.14) acts as a sign inverter as well as a gain adjuster.

7.6 PID Controller Design

We have seen that controllers for single-input-single-output systems consist of three elements: proportional (P), Integral (I), and Derivative (D) action. The transfer function of a controller which includes all three terms is called the three-term PID controller, given by

$$G_c = K_p\left(1 + T_d s + \frac{1}{T_i s}\right)$$

where, K_p, T_d, and T_i, have usual meaning as already explained.

Based on the three-term PID controller, there may be derived a number of others controllers. The majority of the industrial control elements are of the P or PI-type. These controllers are derived from the three-term PID controller $G_c(s)$ by making adjustments to T_d and T_i as

$T_d = 0$ and $T_i = \infty$ gives a P-controller

$T_d = 0$ and T_i = finite gives a PI-controller

Commercially available pneumatic or electronic controllers may be of non-interacting or interacting type depending on the principles of action. Only the derivative action is never implemented in practice because of noise problem. It may be noted that interacting controller means that an adjustment of any one parameter affects the other parameters, whereas non-interacting means the other way round.

7.7 Ziegler–Nichols Rules for Controller Tuning

When the process transfer function model is available, the following results can be used. They are named after J.G. Ziegler and Nathaniel Burgess Nichols, who developed them in the 1940s.

7.7.1 First approach

The process to be controlled is as shown in Figure 7.15. Under pure proportional control, the system is asymptotically stable in the range $0 \le K_p < K_c$, and goes unstable in an oscillatory manner when $K_p > K_c$. Now observe the following:

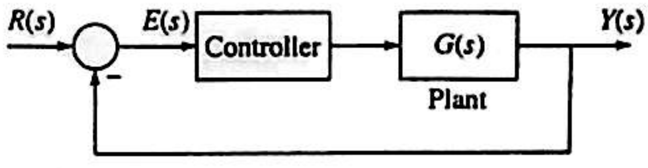


FIGURE 7.15 Controller tuning with provisions for P, PI, and PID controller settings.

1. Increase the gain K_p from 0 to K_c (decrease the proportional band X_p until the process starts to oscillate). At this critical gain K_c the closed-loop system is marginally stable so any gain adjustments must be carried out with extreme care. (If the output does not exhibit sustained oscillations for whatever value K_p may take, then this method does not apply).
2. Note the value K_c and the period of oscillation T.
3. The recommended settings of K_p, T_i, and T_d are given in Table 7.3 for different types of controller design.

TABLE 7.3 Ziegler–Nichols tuning rule based on critical gain K_c and critical period T

Type of controller	K_p	T_i	T_d
P	$0.5K_c$	∞	0
PI	$0.45K_c$	$0.83T$	0
PID	$0.6K_c$	$0.5T$	$0.125T$

Note that the PID controller as per Eq. (7.6) gives

$$G_c(s) = K_p\left(1+\frac{1}{T_is}+T_ds\right) = 0.6K_c\left(1+\frac{1}{0.5Ts}+0.125\,Ts\right)$$

$$= 0.075K_cT\,\frac{(s+4/T)^2}{s} \tag{7.20}$$

Thus, the PID controller has a pole at the origin and double zeros at $s = -4/T$.

As the transfer function model of the plant is available, Routh's array may be used to establish the critical gain K_c and the corresponding period of oscillation T. The procedure is:

1. Find the system's closed-loop characteristic equation under pure proportional control.
2. Form Routh's array and establish the critical gain K_c that produces an all-zero row. If the system goes unstable in an oscillatory manner, the all-zero row will be the row associated with s^1, the auxiliary polynomial will be of second order and there will be no roots of the remainder polynomial with positive real parts. Note that the system should remain stable for all positive values of K_p below the critical value.
3. Use the auxiliary polynomial to find the period of oscillation T, and apply the recommended settings given above.

EXAMPLE 7.1 As an illustration, the use of Ziegler–Nichols rules for finding the P, PI and PID controller settings for a plant whose transfer function model is available, is best understood by the following example. The unity-feedback system of Figure 7.15 having the plant transfer function

$$G(s) = \frac{6}{(s+1)(s+2)(s+3)}$$

and the controller in the forward path has provisions for P, PI and PID controller settings. Perform the controller settings and find the corresponding closed-loop transfer function and the response of the closed-loop system subjected to a unit-step input.

Solution: The characteristic equation of the closed-loop system under pure proportional control having proportional controller gain K_p as per the procedure discussed earlier, is

$$1 + K_pG(s) = 0$$

$$F(s) = 1 + G(s)H(s) = 1 + \frac{6 \times K_p}{(s+1)(s+2)(s+3)} = 0$$

or

$$s^3 + 6s^2 + 11s + 6(1 + K_p) = 0$$

The Routh array is:

$$\begin{array}{lcc} s^3 & 1 & 11 \\ s^2 & 6 & 6(K_p + 1) \\ s^1 & 11 - (K_p + 1) & \\ s^0 & 6(K_p + 1) & \end{array}$$

The range of stability is $0 < K_p \le 10$. The critical gain $K_c = 10$. The auxiliary polynomial is

$$6s^2 + 6(10 + 1) = s^2 + 11 = 0$$

Then the frequency of oscillation ω is obtained as

$$\omega = \sqrt{11}$$

The critical period, $T = 2\pi/\omega = 1.895$.

Therefore, the recommended settings as per Table 7.3 are as follows:

For P-control, $K_p = 0.5K_c = 0.5(10) = 5$, which gives the closed-loop transfer function as

$$\frac{Y(s)}{R(s)} = \frac{G'}{1+G'} = \frac{30}{s^3 + 6s^2 + 11s + 36}$$

where

$$G' = K_pG(s) = \frac{30}{(s+1)(s+2)(s+3)}$$

For PI-control, as per Table 7.3, $K_p = 0.45K_c = 4.5$ and $T_i = 0.83T = 0.83(1.895) = 1.572$, which gives the closed-loop transfer function as

$$\frac{Y(s)}{R(s)} = \frac{G'}{1+G'} = \frac{42.5s + 27}{1.572s^4 + 9.434s^3 + 17.3s^2 + 51.89s + 27}$$

where

$$G' = K_p\left(1 + \frac{1}{T_i s}\right)G(s) = \frac{4.5\left(1 + \frac{1}{1.572s}\right) \times 6}{(s+1)(s+2)(s+3)}$$

For PID-control, $K_p = 0.6K_c = 0.6(10) = 6$, $T_i = 0.5T = 0.5(1.895) = 0.947$ and

$$T_d = 0.125T = 0.125(1.895) = 0.237$$

which gives the closed-loop transfer function as

$$\frac{Y(s)}{R(s)} = \frac{G'}{1+G'} = \frac{8.525s^2 + 34.1s + 36}{0.947s^4 + 5.683s^3 + 18.5s^2 + 39.78s + 36}$$

where $$G' = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)G(s) = \frac{6\left(1 + \dfrac{1}{0.947s} + 0.237s\right)\times 6}{(s+1)(s+2)(s+3)}$$

Figure 7.16 shows the unit-step response under P, PI and PID control. Under proportional control, the response is relatively fast, and for reducing the steady-state error, a PI controller may be used. The introduction of integral action reduces the stability of the system. The introduction of derivative action has a stabilizing effect on the plant, but derivative action cannot be used if the controller is being fed with measurement noise (spurious signals).

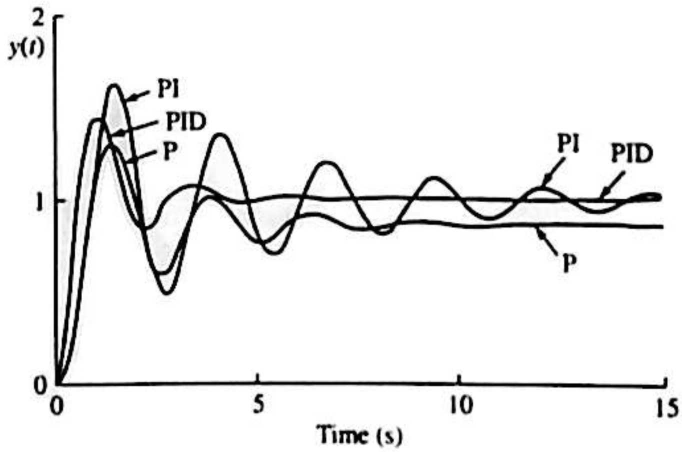


FIGURE 7.16 Time response under P PI, and PID control.

However, if the plant is complicated such that its mathematical model cannot be easily obtained, then analytical approach to the design of the PID controller is not possible. Then we must resort to experimental approaches to the design of PID controllers.

7.7.2 Second approach

Ziegler-Nichols tuning rules have been widely used to tune PID controllers in process control systems where the plant dynamics are not precisely known. Over many years, such tuning rules proved to be very useful. Ziegler–Nichols tuning rules can, of course, be applied to plants whose dynamics are known. Further, it may be noted that if plant dynamics are known, many analytical and graphical approaches to the design of PID controllers are available, in addition to Ziegler–Nichols tuning rules.

The test procedure is carried out as follows:

1. Obtain the unit-step transient response for the open-loop plant shown in Figure 7.17. If the plant involves neither the integrator nor the dominant complex conjugate poles, then a unit-step response curve may look like an S-shaped curve as shown in Figure 7.18. It may be noted that if the response does not exhibit an S-shaped curve, this method does not apply.

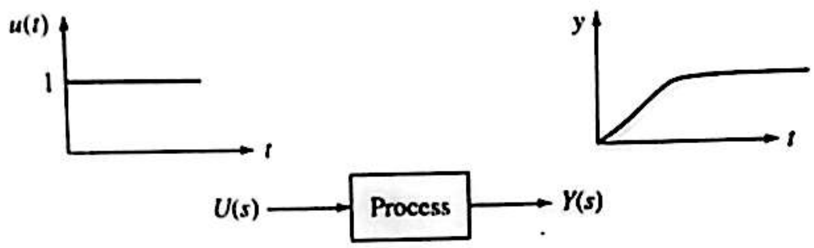


FIGURE 7.17 Open-loop plant.

2. The S-shaped curve may be characterized by the constants, delay time L and time constant T. Measure delay time L and time constant T by drawing a tangent line at the inflection point of the S-shaped curve (so that the slope R of the response becomes the maximum possible slope) and determine the intersections of the tangent line with the time axis and line $y(t) = K$, as shown in Figure 7.18. The transfer function $Y(s)/U(s)$ may then be approximated by a first-order system with a transportation lag. Then

$$\frac{Y(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts+1} \tag{7.21}$$

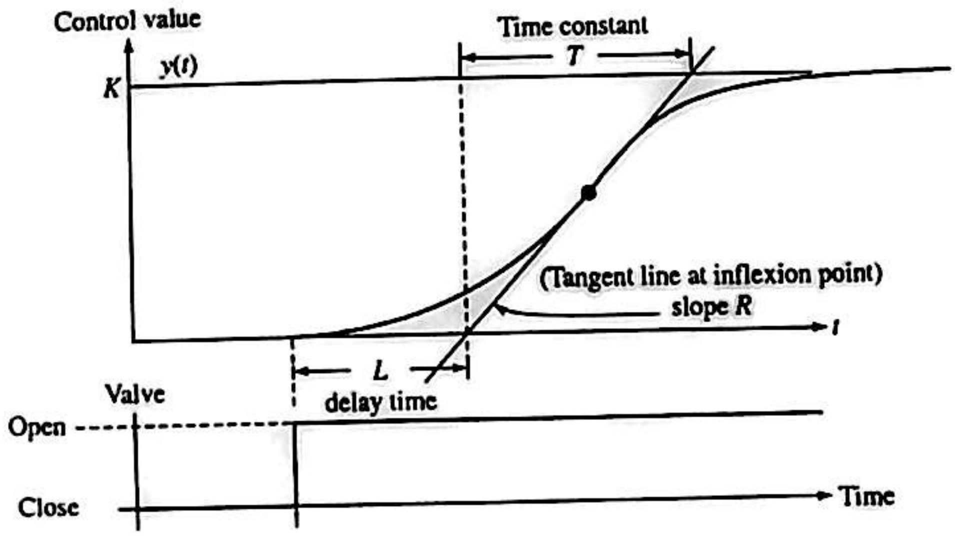


FIGURE 7.18 Unit-step response curve.

The transient response of a process to a step change of input is defined by two characteristics—delay time L and time constant T. Both characteristics together, more or less, define whether the controller can keep the systems within limits. As long as time constant T is relatively large compared with delay time L, the process is easily controllable with a simple controller. Difficulty arises with processes where the delay time L is large compared with time constant T.

3. The recommended Ziegler and Nichols settings to set the values of K_p, T_i and T_d are according to the formulae shown in Table 7.4.

TABLE 7.4 Ziegler–Nichols tuning rule based on step response of plant

Type of controller	$K_p = 1/X_p$	T_i	T_d
P	T/L	∞	0
PI	$0.9(T/L)$	$L/0.3$	0
PID	$1.2(T/L)$	$2L$	$0.5L$

Note that the PID controller as per Eq. (7.6) gives

$$G_c(s) = K_p\left(1+\frac{1}{T_i s}+T_d s\right)$$

$$= \frac{1.2T}{L}\left(1+\frac{1}{2Ls}+\frac{Ls}{2}\right)$$

$$= 0.6T\,\frac{\left(s+\frac{1}{L}\right)^2}{s} \tag{7.22}$$

Thus the PID controller has a pole at the origin and double zeros at $s = -1/L$.

7.8 Adjustment According to Process Characteristics in Ziegler–Nichols Method

Before being able to use any formula we have to measure the reaction of the complete loop to a disturbance. In Figure 7.19 such a control loop is shown. The output current u of the controller operates a valve via an electro-pneumatic (E/P) converter. The steam heats the process, the temperature of which is detected by a thermocouple. The mV signal from the thermocouple is converted by a transmitter to a mA signal which then serves as input signal to the controller.

Now we have to determine the combined response of all the elements which constitute the loop, i.e. the E/P converter, the valve, the process, the sensing element (transducer, the thermocouple) and the transmitter all in cascade as shown in Figure 7.19. This response is defined

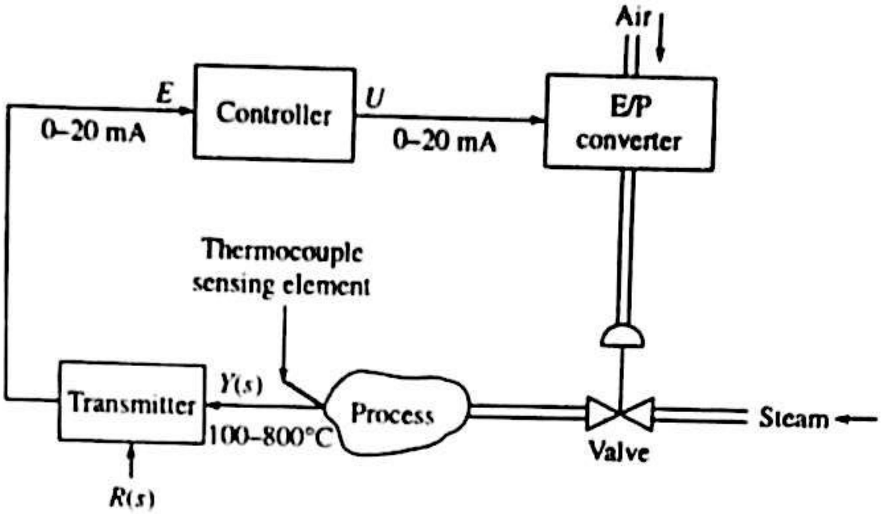


FIGURE 7.19 Process loop.

as the amplification factor of the control loop, $A = \Delta E/\Delta U$. If the time constant T, delay-time or dead-time L and amplification factor A of the loop are known, it is possible to calculate the approximate settings for P-control (i.e. K_p or X_p), I-control (T_i) and D-control (T_d) according to Table 7.4 as:

$$K_p = 1.2\left(\frac{1}{A}\right)\left(\frac{T}{L}\right)\% \tag{7.23}$$

or

$$X_p = 83(A)\left(\frac{L}{T}\right)\% \tag{7.24}$$

For unity amplification factor, K_p becomes $1.2(T/L)$ as given in Table 7.4.

$K_p = \left(\frac{1.2}{LR}\right)\%$; R is the slope of the S-shaped response curve.

$$T_i = 2L \text{ (min)} \tag{7.25}$$

$$T_d = 0.5L \text{ (min)} \tag{7.26}$$

The practical method for determining T, L, and A is given below:

For this test we will operate the controller in the manual mode setting, in order to be able to artificially create a step change in process input and then observe the reaction of the output of the process (i.e. input to the controller), by means of a recorder. Before doing so, however, we have to make sure that the process remains within the safe limits, that is, to say that we have to choose an operating level (controller output current I_n = nominal valve position) which still permits sufficient further output current change in either direction ($\pm I_d$) allowing the process to still remain within the safe operating limits but large enough to produce a noticeable effect on the recorder.

With this in mind, we continue as follows:

1. Record the output signal of the process (= output of the transmitter).

2. Select a nominal controller output current I_n.
3. Manually reduce the controller output current to a value $I_n - I_d$ and let the process recover.
4. Manually increase the controller output to the value $XI_n + I_d$ and let the process recover again.
5. Calculate $\Delta U = (I_n + I_d) - (I_n - I_d) = 2I_d$.
6. Determine from the recorder the jump in the process output current (= transmitter output current) = ΔE.
7. Calculate $A = \Delta E/\Delta U$.
8. Determine from the recording, T and L.
9. Calculate $X_p(=1/K_p)$, T_i, T_d.

EXAMPLE 7.2 In order to illustrate the practical settings and calculations of X_p, T_i and T_d, let us note the data from the experimental observation, as shown in Figure 7.20.

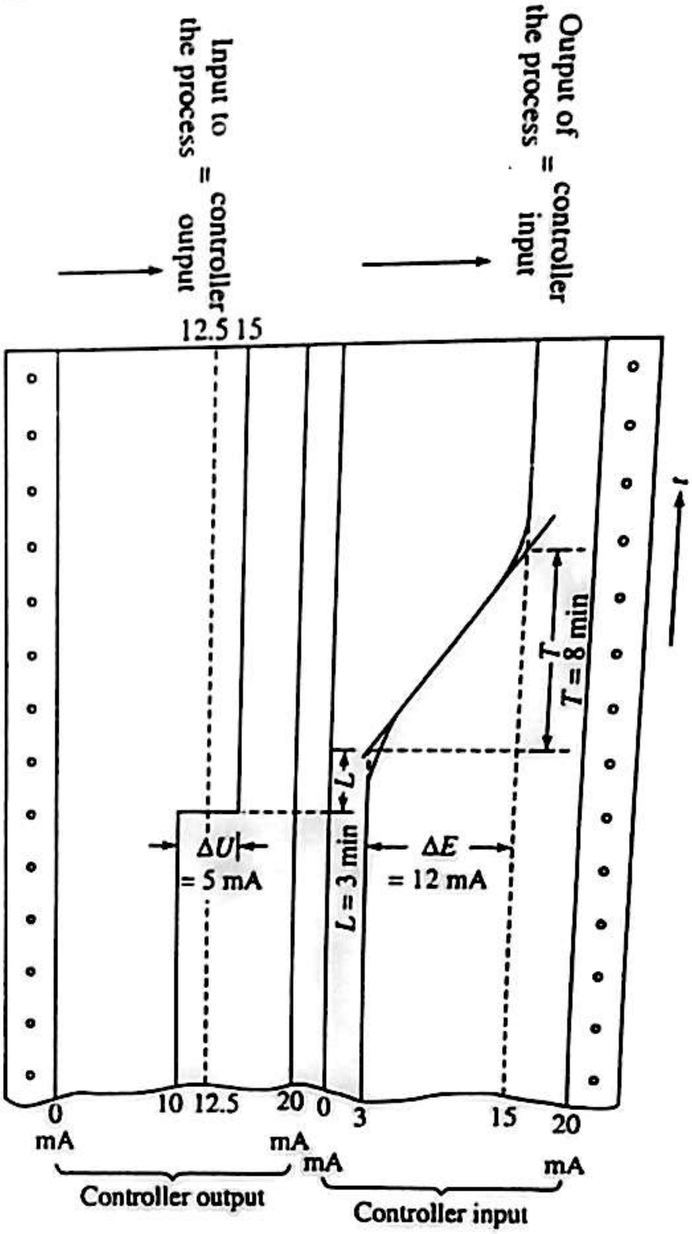


FIGURE 7.20 Input-output response.

- Output signal for nominal conditions = 12.5 mA
- A variation of ±2.5 mA is optimal for the process and the result can be clearly followed on the recorder.
- From the recording curve it is found that:

$$L = 3 \text{ min}, \quad T = 8 \text{ min}$$

Thus

$$\Delta E = 15 - 3 = 12 \text{ mA}; \quad \Delta U = 2 \times 2.5 = 5 \text{ mA}, \quad \text{and} \quad A = \frac{12}{5} = 2.4$$

From Eq. (7.24), $X_p = 83 \times 2.4 \times (3/8) = 74.7\% \approx 75\%$ or $K_p = 1.33$

From Eq. (7.26), $T_d = 0.5 \times 3 = 1.5$ min

From Eq. (7.25), $T_i = 2 \times 3 = 6$ min

Note that the values obtained in this way are only approximate and can therefore only serve as a guide to shorten the trial and error procedure.

7.9 Purpose of Ziegler–Nichols Tuning Method

The Ziegler–Nichols tuning method for PID controller for continuous-time systems is particularly useful if the transfer function of the system is not known. It was noted that the proportional part of the compensatory network improves the sensitivity to parameter variations, the integral part improves the steady-state accuracy if any steady-state error exists and the derivative part improves the stability of the system by increasing the damping. In practice, the Ziegler–Nichols tuning method provides a good set of initial values of the various constants K_p, T_i and T_d of the PID controller, which can often be improved by on-line tuning. The main advantage is that by using this method one can completely bypass the need to obtain an exact mathematical model for the process to be controlled.

7.10 Designing Controller Using Root Loci

In Example 7.2, the settings recommended by Ziegler and Nichols for PID controllers were used. It is, however, observed that for some applications, the resulting time response tends to be underdamped and no indication for the improvement of the performance is given through the Ziegler and Nichols methods. Now we will discuss through an example, how root loci method can be used to select controller settings.

Consider the plant transfer function of Example 7.1

$$G(s) = \frac{6}{(s+1)(s+2)(s+3)}$$

and the PID controller transfer function

$$G_c(s) = 1 + T_d s + \frac{1}{T_i s}$$

The scheme is shown in Figure 7.21.

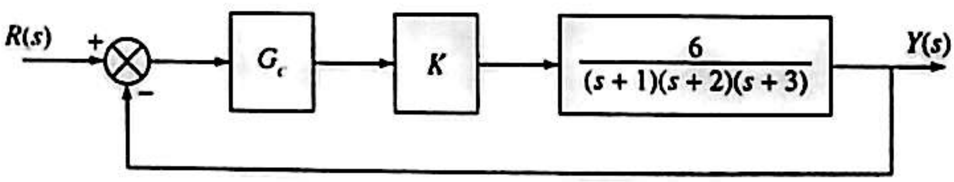


FIGURE 7.21 Plant and the PID controller.

Case A (Root loci for P control)

For proportional control, $T_d = 0$, $T_i = \infty$. The closed-loop characteristic equation with pure proportional controller is

$$1+\frac{6K}{(s+1)(s+2)(s+3)}=0$$

The root loci is drawn in Figure 7.22(a).

In the root loci, the closed-loop complex pole positions are identical for critical gain $K_c = 10$, Ziegler and Nichols gain (from Table 7.3) as $K_{ZN} = 0.5K_c = 5$. Clearly the response dynamics would be improved by reducing the controller gain K to 0.6. However, this increases the offset as the position error coefficient (as per the symbol used in earlier chapters)

$$K_p = \lim_{s\to 0} G(s)H(s) = 0.6$$

and the steady-state error, $e_{ss} = \dfrac{1}{1+K_p} = 0.625$. See Figure 7.22(b).

Case B (Root loci for PI control)

Under PI control, the closed-loop characteristic equation becomes

$$1+\frac{6K\left(1+\frac{1}{T_i s}\right)}{(s+1)(s+2)(s+3)}=1+\frac{6K\left(s+\frac{1}{T_i}\right)}{s(s+1)(s+2)(s+3)}=0$$

In the root-loci method either K or T_i can be varied with the other fixed. If T_i is fixed at the recommended Ziegler–Nichols value $T_i = 0.83T$ (from Table 7.2) where, from Routh–Hurwitz criterion we get $T_i = 1.57$, then the characteristic equation becomes

$$1+\frac{6K(s+0.637)}{s(s+1)(s+2)(s+3)}=0$$

The root loci is shown in Figure 7.22(c) for the variable parameter gain K in the range $0 < K < \infty$. Again the position error coefficient K_p with the recommended Ziegler–Nichols setting for $T_i = 1.57$ becomes

$$K_p = \lim_{s\to 0}\frac{6K(s+0.637)}{s(s+1)(s+2)(s+3)}=\infty$$

Hence the steady-state error for a unit-step input is

$$e_{ss}=\frac{1}{1+K_p}=0$$

It may be noted that the PI controller introduces a pole at the origin and a zero at $s = -0.637$. The PI-controller's pole at the origin reduces the error offset for unit step input to zero.

With only P-control the critical gain $K_c = 10$. For PI-control the Ziegler–Nichols gain $K_{ZN} = 0.45K_c = 4.5$ and at $K_{ZN} = 4.5$, the pair of dominant complex conjugate roots are also shown

The response for PI-controlled system is shown in Figure 7.22(d) for $K_{ZN} = 4.5$. From this plot it appears that the response could be improved by reducing the gain K to 2 as shown. Further reduction in gain K will reduce the effective dominance of the complex poles and in turn, damping coefficients of complex poles will increase as is shown in Figure 7.22(d). For two values of $K = K_{ZN} = 4.5$ and $K = 2$, the step responses are shown.

Now we will draw the root loci keeping K fixed at $K = 0.45$ and varying T_i in the range $0 < T_i < \infty$. To see the effect of adjustments to T_i, proceed as follows: Set the controller gain fixed at the recommended value $K = 0.45K_c = 0.45(10) = 4.5$ as specified in Ziegler–Nichols settings for PI control. Modify the characteristic equation as

$$1 + G(s)H(s) = 0$$

or

$$1+\frac{(4.5)(6)(s+1/T_i)}{s(s+1)(s+2)(s+3)} = 1+\frac{27(s+1/T_i)}{s^4+6s^3+11s^2+6s} = 0$$

Thus, the characteristic equation becomes

$$s^4 + 6s^3 + 11s^2 + 33s + \frac{27}{T_i} = 0$$

or

$$1+\frac{27/T_i}{s^4+6s^3+11s^2+33s} = 1+\frac{27/T_i}{s(s+5.11)(s+0.4666 \pm j5.109)} = 0$$

If $K_1 = 27/T_i$, then draw the root loci for the open-loop transfer function

$$\frac{K_1}{s(s+5.11)(s+0.4666+j5.109)(s+0.4666-j5.109)}$$

for $0 < K_1 < \infty$ and the root loci is shown in Figure 7.22(e). On the root loci it is indicated with arrows the increasing values of K_1, that is, decreasing values of T_i. The critical value of T_i is (108/121). Further, $T_i <$ (108/121) and with $K = 4.45$, the system becomes unstable. The root loci of Figure 7.22(e) for $T_i = 1.57$ (as stipulated by Ziegler–Nichols PI controller settings) should coincide with the root loci of Figure 7.22(c) for $K = 4.45 = K_{ZN}$.

Case C (Root loci for PID control)

The analysis of the PID controller using the root-locus approach where three parameters are to be adjusted, demands that two variables be fixed (T_d and T_i) and the other parameter K may be variable. Set $T_d = 0.125T$ and $T_i = 0.5T$ as recommended in Ziegler–Nichols Table 7.3 and for this particular example $T = 1.895$. Then $T_d = 0.237$ and $T_i = 0.947$.

The characteristic equation is

$$1+\frac{6K(1+0.237s+1/0.947s)}{(s+1)(s+2)(s+3)} = 1+\frac{6K(s+2.11\pm j0.07)}{s(s+1)(s+2)(s+3)} = 0$$

The root-loci plot for variations of K in the range $0 < K < \infty$ is drawn in Figure 7.22(f), from which the closed-loop system is seen to be stable for all values of gain K. For the proposed

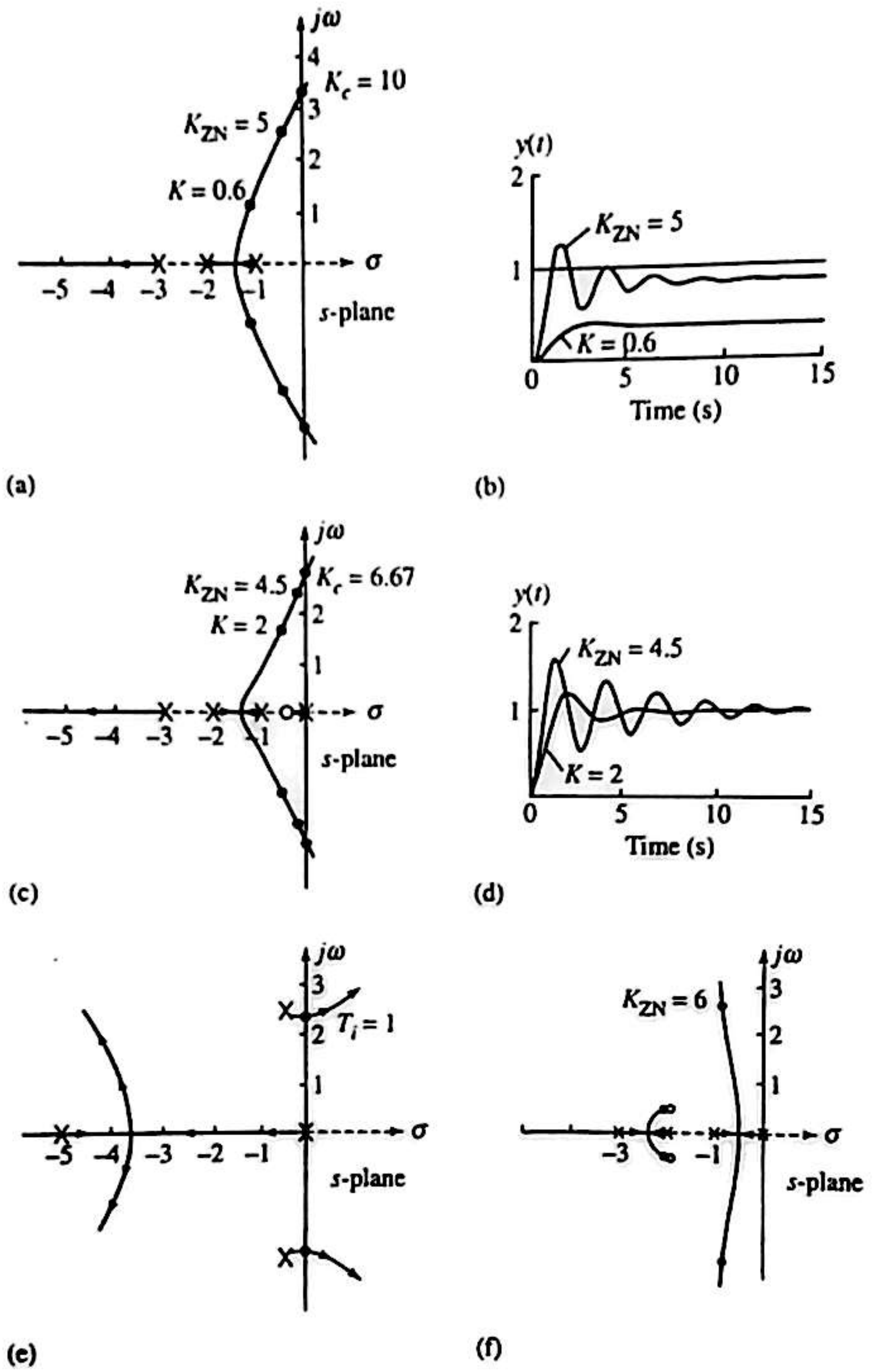


FIGURE 7.22 Designing controller using root loci.

Ziegler–Nichols gain $K = K_{ZN} = 0.6K_c = 0.6(10) = 6$ and the already proposed values set of $T_i = 0.5T$ and $T_d = 0.125T$, the unit-step response is shown in Figure 7.23. For comparison, see the response to a step input for P, PI and PID control as shown in Figure 7.16 (repeated below as Figure 7.24).

Problems

9.1 By use of the Nyquist criterion, determine whether the closed-loop systems having the following open-loop transfer functions are stable or not. If not, how many closed-loop poles lie in the right-half of s-plane?

(a) $G(s)H(s) = \dfrac{1+4s}{s^2(1+s)(1+2s)}$ (b) $G(s)H(s) = \dfrac{1}{s(1+2s)(1+s)}$

(c) $G(s)H(s) = \dfrac{1}{s^2+100}$

9.2 Check the stability of the systems by completing the corresponding Nyquist plots for the given polar plots in Figure P.10.2.

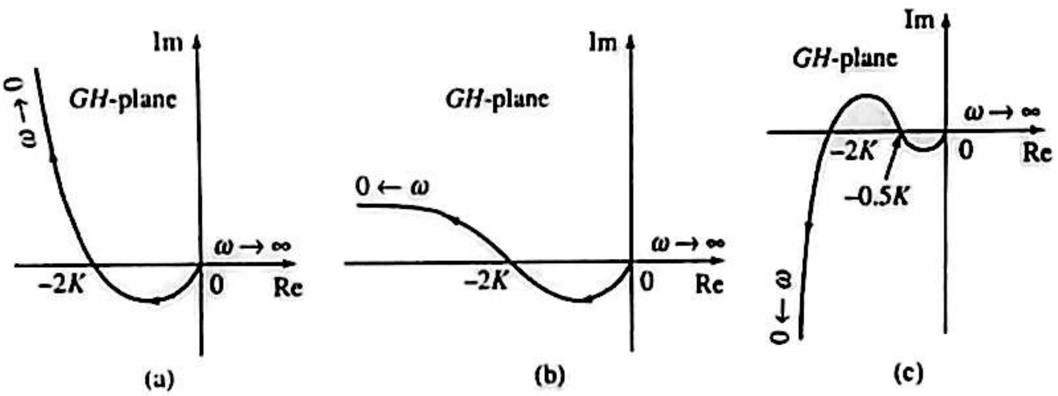


FIGURE P.10.2

9.3 Sketch a Nyquist plot for a system with the open-loop transfer function

$$\frac{K(1+0.5s)(s+1)}{(1+10s)(s-1)}$$

Determine the range of values of K for which the system is stable.

9.4 Consider a feedback system having the characteristic equation

$$1 + K = 0 \quad \frac{K}{(s-1)(s+1.5)(s+2)}$$

It is desired that all the roots of the characteristic equation have real parts less than -1. Extend the Nyquist stability criterion to find the largest value of K, satisfying this condition.

9.5 For the unity-feedback system having open-loop transfer function

$$\frac{100K(s+5)(s+40)}{s^3(s+100)(s+200)}$$

draw the Nyquist plot and find the stability taking $K = 1$.

Further verify the stability from Bode plot taking $K = 1$, and also from root-loci plot for $0 < K < \infty$.

Compensation Techniques

OBJECTIVE

The objective of this chapter is to present procedures for the design and compensation of single-input-single-output, linear, time-invariant control systems by the frequency-response approach. The root-locus method gives direct information on transient response and is useful in reshaping the transient response of closed-loop control systems whereas the frequency-response approach gives the information indirectly. The designer has to know both the approaches very well, in order to understand design specifications for realizing a successful system as different specifications leades to different approach to design.

CHAPTER OUTLINE

10.1 Introduction

The performance of a control system may be described either in terms of the time-domain performance-measures or in terms of the frequency-domain performance-measures. The performance of a system may be specified in terms of the time-domain performance-measures by specifying a certain peak time—for getting maximum overshoot and settling-time for a step input. Furthermore, it is usually necessary to specify the maximum allowable steady-state error for several standard test signal inputs. These performance specifications may be related in terms of the desirable locations of the poles and zeros of the closed-loop system transfer function. Thus the location of the closed-loop poles and zeros may be specified as per the performance-measures desired, for which the root locus approach may be considered. However, when the locus of the roots does not result in a suitable root configuration, we must add a compensating network in order to be able to alter the locus of the roots with the variation of one parameter.

Alternatively, we may describe the performance in terms of the frequency-domain performance-measures such as the peak of the closed-loop frequency response $M_{p\omega}$, the resonant frequency ω_r, the bandwidth, the gain and phase margin of the closed-loop system. We may add a suitable compensation network $G_c(s)$ and develop the design of the network in terms of the frequency response as portrayed on the polar

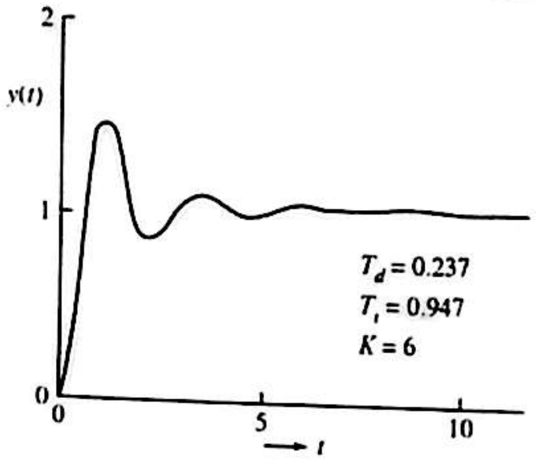


FIGURE 7.23 Time response under PID control.

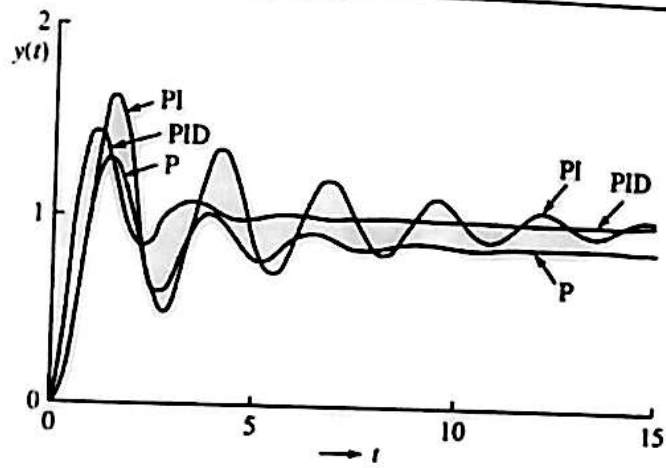


FIGURE 7.24 Time response under P, PI and PID control.

Summary

In this chapter we briefly explained the features of the process control system. The techniques used in process control are more or less the same that we are familiar with from classical control systems, only the terminologies are different somewhere. Here an attempt has been made to unify these two lines of control. The concept of feedback, the action of feedback principle in process control systems, the proportionality band and its relationship with dc gain, necessity of PI control and finally the PID control, and the circuitry have been discussed. The tuning procedure of PID-controller by the Ziegler–Nichols method, together with its analytical approach, has been discussed. The discussion is well supported by numerical problems. Though this chapter is an introduction to process control, it has been the endeavour to cover most of the basic principles of continuous control, without going into complex control loops such as multi-element control, ratio control, cascade control, computer control, fractioning column control, reactor control, and so on. This is only to say that this chapter is an humble approach to motivate conventional control engineers to the subject of process control systems.

Problems

7.1 A unity-feedback process control system having plant dynamics is given by

$$G_P(s) = \frac{4}{(s^2 + 8s + 80)(s + 1)}$$

and a PID (three-term) controller with transfer function is given by

$$G_c(s) = \frac{20(T_i T_d s^2 + T_i s + 1)}{T_i s}$$

Establish the values of T_i and T_d for the closed-loop system to be stable.

Ans. $T_i > 0$ and $T_d > -79/90$

7.2 Use the Ziegler–Nichol rules to design a three-term controller for a plant model having open-loop transfer function

$$G(s) = \frac{1}{s(s+1)(s+2)(s+3)}$$

Show that the resulting closed-loop system is stable.

Ans. $K_u = 10$, $T = 2\pi$

PID settings are: $K = 6$, $T_i = \pi$, and $T_d = \pi/4$

The closed-loop characteristic equation of the model and PID controller is

$$s^5 + 6s^4 + 11s^3 + (6 + 1.5\pi)s^2 + 6s + 6/\pi = 0$$

and by Routh's array it is stable.

7.3 Consider the electronic PID controller of Figure 7.14. Determine the values of R_1, R_2, and R_4 for the given values of $C_1 = C_2 = 10$ μF and $R_3 = 10$ kΩ.

Ans. $R_1 = R_2 = 153.85$ kΩ and $R_4 = 197.1$ kΩ

Problems

9.1 By use of the Nyquist criterion, determine whether the closed-loop systems having the following open-loop transfer functions are stable or not. If not, how many closed-loop poles lie in the right-half of s-plane?

(a) $G(s)H(s) = \dfrac{1+4s}{s^2(1+s)(1+2s)}$ (b) $G(s)H(s) = \dfrac{1}{s(1+2s)(1+s)}$

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9.2 Check the stability of the systems by completing the corresponding Nyquist plots for the given polar plots in Figure P.10.2.

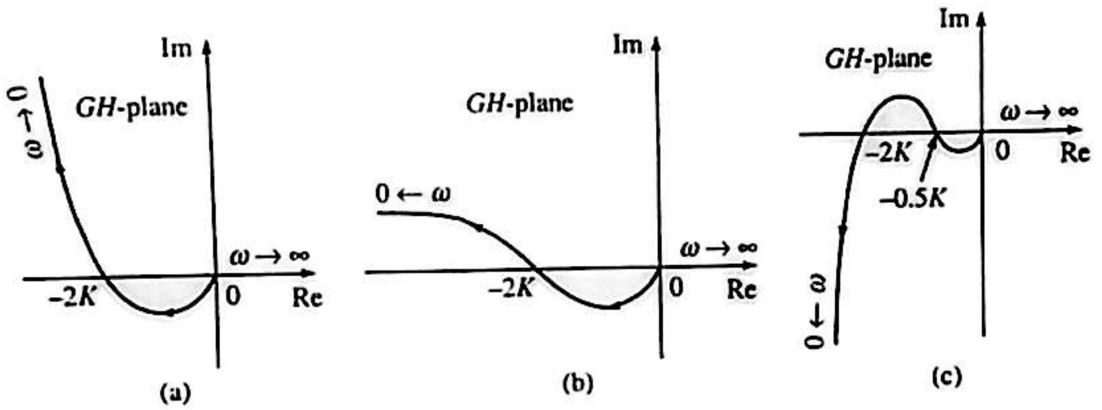

FIGURE P.10.2

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Compensation Techniques

OBJECTIVE

The objective of this chapter is to present procedures for the design and compensation of single-input-single-output, linear, time-invariant control systems by the frequency-response approach. The root-locus method gives direct information on transient response and is useful in reshaping the transient response of closed-loop control systems whereas the frequency-response approach gives the information indirectly. The designer has to know both the approaches very well, in order to understand design specifications for realizing a successful system as different specifications leades to different approach to design.

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Alternatively, we may describe the performance in terms of the frequency-domain performance-measures such as the peak of the closed-loop frequency response $M_{p\omega}$, the resonant frequency ω_r, the bandwidth, the gain and phase margin of the closed-loop system. We may add a suitable compensation network $G_c(s)$ and develop the design of the network in terms of the frequency response as portrayed on the polar plot, Bode diagram, or Nichols chart. A cascade transfer function is readily accounted for, on a Bode plot by adding the frequency response of the network, hence the Bode diagram in frequency response approach is preferred.

The frequency response and root locus may be used as tools for designing any of the compensation schemes, except that the root locus is not convenient when more than one or possibly two variable parameters are introduced.

It is the purpose of this chapter to further describe the addition of several compensation networks to a feedback control system. First, we shall consider the addition of a so-called phase-lead compensating network and describe the design of the network by root locus and frequency response techniques. Then, using both the root locus and frequency response techniques, we shall describe the design of the lag compensation networks in order to obtain a suitable system performance. Finally, we shall describe a lag–lead compensator in order to get a suitable performance criterion both in low and high frequency range.

The design aspect of the state-feedback compensator will be discussed in the concerned chapter on state variable approach.

While the design of linear control systems is carried out in the frequency domain, the frequency response design provides information on the steady-state response, stability margin and system bandwidth. The transient response performance can be estimated indirectly in terms of the phase margin, the gain margin and resonant peak magnitude. The percent overshoot is reduced with an increase in the phase margin, and the speed of response is increased with an increase in the bandwidth. Thus, the gain-crossover frequency, the resonant frequency and the bandwidth give a rough estimate of the speed of transient response.

A common approach to the frequency response design is to adjust the open-loop gain so that the requirement on the steady-state accuracy is achieved. If the specifications concerning the phase margin and gain margin are not satisfied, then it is necessary to reshape the open-loop transfer function by adding an additional controller to the open-loop transfer function. When only the gain is varied, the phase angle plot will not be affected. The Bode magnitude curve is shifted up or down to correspond to the increase or decrease in K_p, the gain of the proportional controller. Similarly, the effect of changing K_p on the polar plot is to enlarge or reduce it, but the shape of the polar plot cannot be changed.

Several compensation schemes are shown in Figure 10.1 for a simple single-loop control system. When the compensator $G_c(s)$ is placed in the forward path as in Figure 10.1(a), it is called a *series* or *cascade* compensator. The other compensation schemes are *feedback* compensation as in Figure 10.1(b), *output* or *load* compensation as in Figure 10.1(c), and *input* compensation as in Figure 10.1(d). The choice of the compensation scheme depends upon the specification, power level at various signal modes in the system and the availability of components. Suppose we want to alter the damping coefficient as per the requirement of our specification. We can put (i) a high-pass RC circuit or an electronic circuit using operational amplifiers or mechanical spring-dashpot systems in the forward path, or (ii) a tachogenerator in the feedback path. Obviously, a power amplifier would be required in the forward path because of the low level of error signal. On the contrary, tachogenerator feedback does not require any amplification because of the high level of output signal. Both the forward path high-pass RC circuit (derivative control) and tachogenerator feedback control do change the damping ratio; but the use of a proper compensation circuit depends on the availability of components and signal level

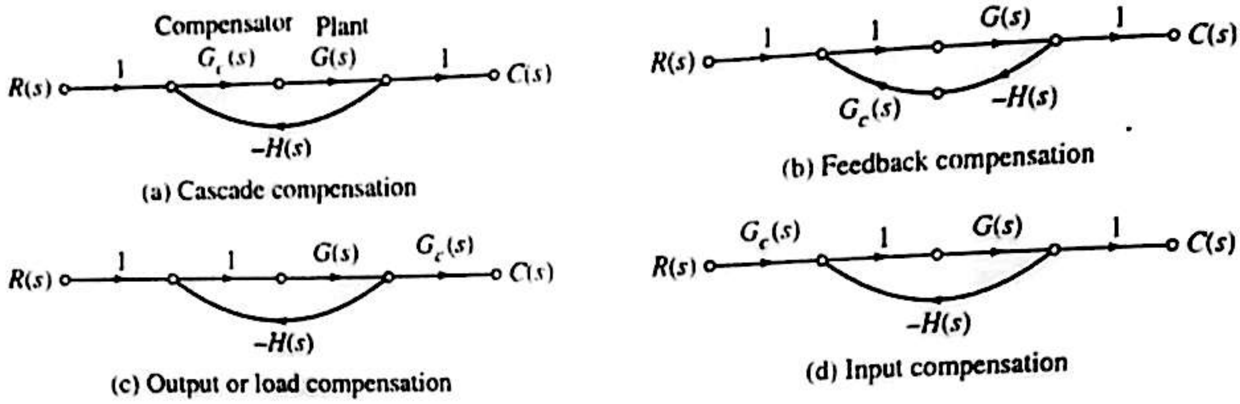

FIGURE 10.1 Compensation schemes.

10.2 Cascade Compensation Networks

A system is cascade compensated when the controller (compensator) is placed in the main forward transmission path as shown in Figure 10.2. The compensation network $G_c(s)$ is cascaded with the unalterable process $G(s)$ in order to provide a suitable loop transfer function $G_c(s)G(s)$. Clearly, the compensator $G_c(s)$ may be chosen to alter the shape of the root locus or the frequency response.

Mathematically, $G_c(s)$ is a ratio of two polynomials, i.e.

$$G_c(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{N}(s+p_j)} \tag{10.1}$$

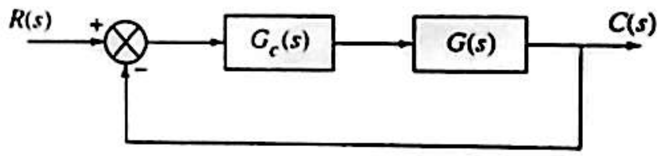

FIGURE 10.2 Block diagram of cascade compensation scheme.

Then the problem reduces to the judicious selection of the poles and zeros of the compensator. When graphical design methods are used, the problem of selecting suitable locations of the poles and zeros is solved by trial and error.

To illustrate the properties of the compensation network, we shall consider a first-order compensator. The compensation approach developed on the basis of a first-order compensator may then be extended to higher-order compensators.

To understand the use of compensators, it is both necessary and desirable that the effects of the compensating networks on both the Bode curves and the root locus be clearly understood.

Consider
$$G_c(s) = K\frac{s+z}{s+p} \tag{10.2}$$

where both z and p are real. This transfer function may have:

$|z| < |p|$ = high-pass filter, also called the phase-lead compensator (by taking the phase of the numerator and denominator polynomials).

$|z| > |p|$ = low-pass filter, also called the phase-lag compensator.

The Bode plot approach is applied readily when a multisection compensating network (having several zeros and poles) is required.

The root-locus method, while providing an excellent insight into complex system problems, is a good computational tool only when the compensator requirements are very modest.

10.3 Lead Compensating Networks

The arrangement of R and C in Figure 10.3 causes a phase-lead action. An alternating input signal v_{in} may produce an output v_{out} that leads v_{in} by an angle ϕ. The transfer function becomes

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{RCs}{RCs+1} \tag{10.3}$$

The phase angle $\phi = 90° - \tan^{-1}\omega RC$ is a positive acute angle. However, this simple circuit is not used in a servo system, for its output v_{out} becomes zero as the signal frequency ω is decreased towards zero, that is, the steady-state condition; a loop containing this circuit has no gain at standstill and therefore has low accuracy.

Zero at the origin is not permitted in a cascade compensator because that would block the zero-frequency (dc) component of the error signal and the system would then be incapable of reaching the desired steady state.

If another resistor R_1 is added across C as shown in Figure 10.4, this becomes a useful phase-lead circuit. At low frequency (or when $\omega = 0$), the current in C is negligible; R_1 and R_2 act as a simple voltage divider so that $v_{out} = v_{in}(R_2/(R_1 + R_2))$. The loss of steady-state gain may be offset (compensated) by increasing the amplifier gain in the loop. This circuit provides no leading angle when ω is very large or very small. At very high frequencies, C becomes a short-circuit around R_1, so v_{out} becomes as large as v_{in}.

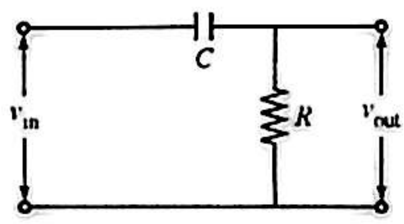

FIGURE 10.3 Phase-lead circuit having no gain at low frequencies.

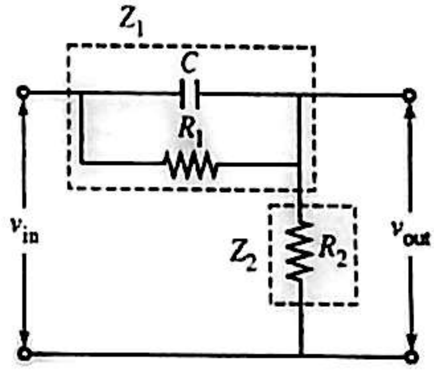

FIGURE 10.4 A useful RC phase-lead circuit.

Using the symbols defined in Figure 10.4, we find that the complex impedances Z_1 and Z_2 are

$$Z_1 = \frac{R_1}{R_1Cs+1}, \qquad Z_2 = R_2$$

The transfer function between the output $V_{out}(s)$ and the input $V_{in}(s)$ becomes

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2}{R_1 + R_2} \frac{R_1 Cs + 1}{\dfrac{R_1 R_2}{R_1 + R_2} Cs + 1} \tag{10.4}$$

Let us define:

$$R_1 C = T; \ \frac{R_2}{R_1 + R_2} = \alpha < 1 \tag{10.5}$$

Then the transfer function becomes

$$\frac{V_{out}(s)}{V_{in}(s)} = \alpha\left(\frac{Ts+1}{\alpha Ts+1}\right) = \frac{s + 1/T}{s + 1/\alpha T} \tag{10.6}$$

$$= K_c\left(\frac{s+1/T}{s+1/\alpha T}\right) = K_c\alpha\left(\frac{Ts+1}{\alpha Ts+1}\right)$$

with $K_c = 1$ to make the uniformity with Eq. (10.8).

The phase angle is

$$\phi = \tan^{-1} \omega T - \tan^{-1} \alpha\omega T = \text{+ve acute angle} \quad \text{for } 0 < \alpha < 1 \tag{10.7}$$

It is thus obvious that the output leads the input as $0 < \alpha < 1$.

There are many ways to realize a continuous time compensator. Let us consider an electronic circuit using operational amplifiers as in Figure 10.5. The transfer function for this circuit is obtained as follows.

$$\frac{E_{out}(s)}{E_{in}(s)} = \left(-\frac{R_4}{R_3}\right)\left(-\frac{Z_2}{Z_1}\right) = \left(\frac{R_4 C_1}{R_3 C_2}\right)\left(\frac{s + 1/R_1 C_1}{s + 1/R_2 C_2}\right)$$

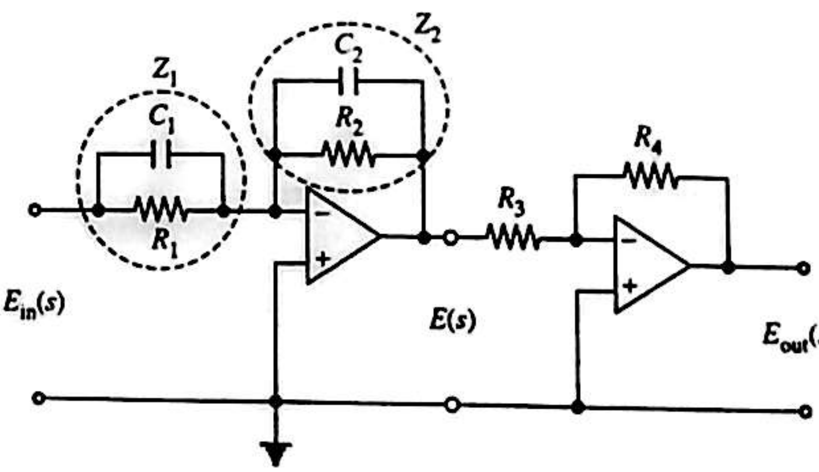

FIGURE 10.5 Electronic circuit which is a lead network if $R_1C_1 > R_2C_2$ and a lag network if $R_1C_1 < R_2C_2$.

where $$Z_2 = \frac{R_2}{R_2C_2s+1}; \quad Z_1 = \frac{R_1}{R_1C_1s+1}$$

or $$\frac{E_{out}(s)}{E_{in}(s)} = K_c\alpha\left(\frac{Ts+1}{\alpha Ts+1}\right) = K_c\left(\frac{s+1/T}{s+1/\alpha T}\right) \tag{10.8}$$

where $$T = R_1C_1, \quad \alpha T = R_2C_2, \quad K_c = \frac{R_4C_1}{R_3C_2}$$

Note that, $$K_c\alpha = \frac{R_4C_1}{R_3C_2}\cdot\frac{R_2C_2}{R_1C_1} = \frac{R_2R_4}{R_1R_3}; \quad \alpha = \frac{R_2C_2}{R_1C_1}$$

The electronic op-amp. circuit has a dc gain of $K_c\alpha$. We see that this network is a lead network if $R_1C_1 > R_2C_2$, i.e. $0 < \alpha < 1$. It is a lag network if $R_1C_1 < R_2C_2$, i.e. $\alpha > 1$.

10.4 Characteristics of Lead Networks

A lead network as per Eq. (10.8) has a zero at $s = -1/T$ and a pole at $s = -1/\alpha T$. Since $0 < \alpha < 1$, we see that the zero is always located to the right of the pole in the complex s-plane. For a small value of α (which is in practice) the pole is located far to the left. The minimum value of α is limited by the physical construction of the lead network and is usually taken to be about 0.05. If the value of α is small, it is necessary to cascade an amplifier in order to compensate for the attenuation caused by the lead network.

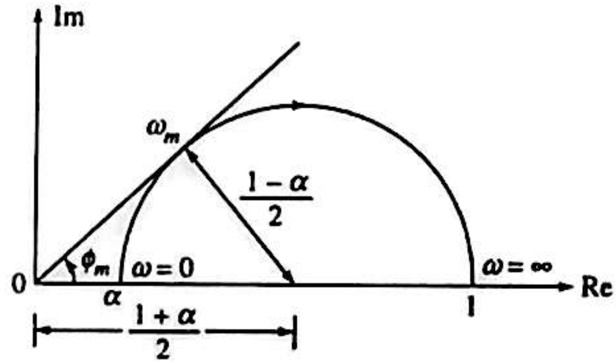

FIGURE 10.6 Polar plot of a lead network.

Figure 10.6 show the polar plot of

$$K_c\alpha\left(\frac{j\omega T+1}{j\omega\alpha T+1}\right); \quad \text{for } 0 < \alpha < 1 \tag{10.9}$$

For a given value of α, the angle between the positive real axis and the tangent line drawn from the origin to the semicircle gives the maximum phase lead angle ϕ_m. We will denote the frequency at the tangent point to be ω_m. From Figure 10.6, the phase angle at $\omega = \omega_m$ is

$$\sin\phi_m = \frac{(1-\alpha)/2}{(1+\alpha)/2} = \frac{1-\alpha}{1+\alpha} \tag{10.10}$$

Equation (10.10) relates the maximum phase lead angle ϕ_m with the value of α.

Figure 10.7 shows the Bode diagram of a lead network when $\alpha = 0.1$. The corner frequencies for the lead network are $\omega = 1/T$ and $\omega = 1/\alpha T = 10/T$. By examining this figure, we see that ω_m is the geometric mean of the two corner frequencies, i.e.

$$\log \omega_m = \frac{1}{2}[\log (1/T) + \log (1/\alpha T)]$$

or

$$\omega_m = \frac{1}{\sqrt{\alpha}\, T} \tag{10.11}$$

As seen from Figure 10.7, the lead network is basically a high-pass filter. The high frequencies are passed but the low frequencies are attenuated. Therefore, an additional gain elsewhere is needed to increase the low frequency gain.

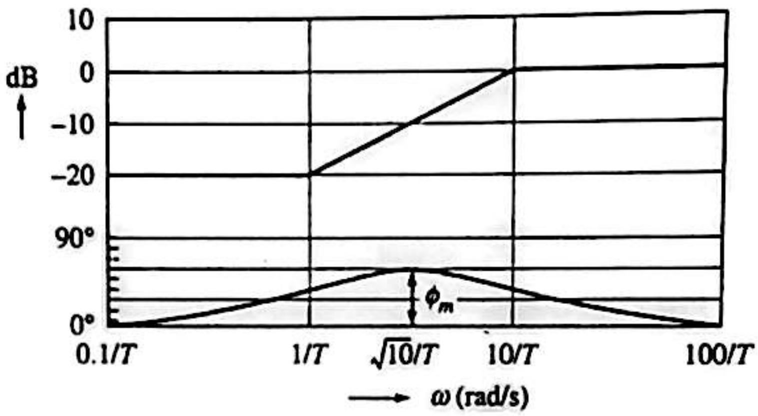

FIGURE 10.7 Bode diagram of a lead compensator $\alpha(j\omega T + 1)/(j\omega\alpha T + 1)$, where $\alpha = 0.1$.

EXAMPLE 10.1 Consider the system shown in Figure 10.8. It is desired to modify the closed loop so that an undamped natural frequency $\omega_n = 4$ rad/s is obtained without changing the value of the damping ratio. What type of compensation would you propose?

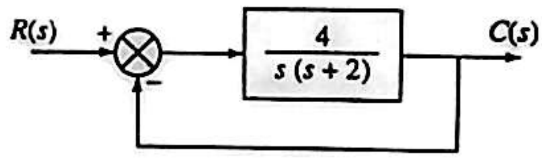

FIGURE 10.8 Example 10.1.

Solution: The characteristic equation of the unmodified closed-loop system is

$$1 + G(s)H(s) = 1 + \frac{1}{s(s+2)} = s^2 + 2s + 4 = 0$$

Then, $\omega_n = 2$ and $\zeta = 0.5$. We know $\zeta = \cos\theta = 0.5$, which gives $\theta = \pm 60°$. A damping ratio of 0.5 implies that the complex poles lie on the lines drawn through the origin making angles of $\pm 60°$ with the negative real axis.

The desired locations of the closed-loop poles are

$$s = -2 \pm j2\sqrt{3}$$

as the given $\omega_n = 4$ and $\zeta = 0.5$.

In the present system, the angle of $G(s)H(s)$ at the desired closed-loop poles is

$$\angle \frac{4}{s(s+2)}\bigg|_{s=-2+j2\sqrt{3}} = -\ 210^\circ$$

Hence we propose to have a lead compensator in the forward path to provide a leading angle of at least +30°.

EXAMPLE 10.2 In continuation of Example 10.1 where we concluded that a lead compensator is required in order to satisfy the given specifications, let us now design a suitable lead compensator for the same.

Solution: A general procedure for determining the lead compensator is as follows:

First, find the sum of the angles at the desired location of one of the dominant closed-loop poles with the open-loop poles and zeros of the original system, and then determine the necessary angle ϕ to be added so that the total sum of the angles is equal to $\pm 180^\circ(2k + 1)$. The lead network must contribute this angle. (If the angle is quite large, then two or more lead networks may be needed rather than a single one.)

If the original system has the open-loop transfer function $G(s)H(s)$, then the compensated system will have the open-loop transfer function

$$G_1(s)H(s) = \left(\alpha \frac{Ts+1}{\alpha Ts+1}\right) K_c G(s)H(s)$$

where the first term on the right-hand side corresponds to the lead network, the second term K_c is the gain of the amplifier, and the last term $G(s)H(s)$ is the original open-loop transfer function. Note that the amplifier provides the desired impedance matching as well as the desired gain K_c. Also note, there are many possible values for T that will yield the necessary angle contribution at the desired closed-loop poles. The next step is to determine the locations of the pole and zero of the lead network; in other words, the value of T. In choosing the value of T, we shall introduce a procedure to obtain the largest possible value for α so that the additional gain required of the amplifier is as small as possible. First draw a horizontal line passing through point P, the desired location for one of the dominant closed-loop poles at $-2 \pm j2\sqrt{3}$. This is shown as line PA in Figure 10.9. Draw also a line, connecting the point P and the origin. Bisect the angle between the lines AP and PO. Draw two lines PC and PD which make angles $\pm\phi/2$ with the bisector PB. The intersections of PC and PD with the negative real axis give the necessary location for the pole and zero of the lead network. The compensator thus designed will make point P, a point on the root locus of the compensated system as shown in Figure 10.10. The open-loop gain is determined by means of the magnitude condition.

$$\text{Transfer function} = \frac{400}{745}\left(\frac{0.345s+1}{0.185s+1}\right) = \frac{s+2.9}{s+5.4}.$$

We determine the pole and zero of the lead network as in Figure 10.10 to be

$$\text{pole at } s = -5.4 \qquad \text{and} \qquad \text{zero at } s = -2.9$$

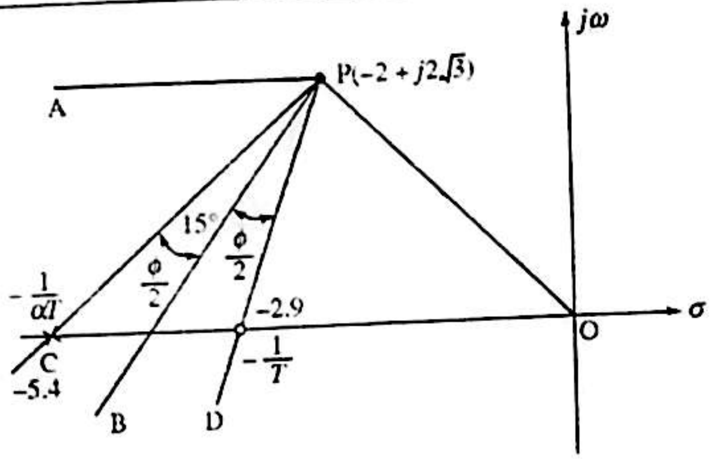

FIGURE 10.9 Example 10.2.

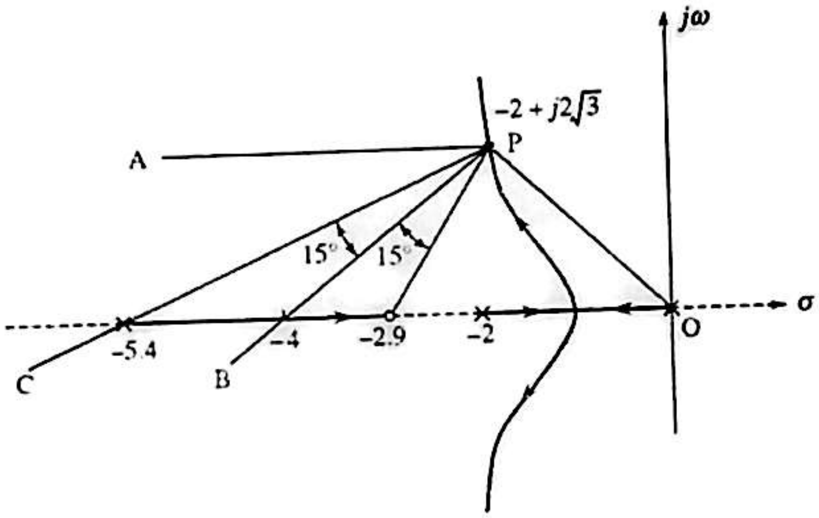

FIGURE 10.10 Example 10.2: root locus of the compensated system.

Hence the transfer function of the lead compensator $G_c(s)$ is

$$G_c(s) = \frac{(s + 2.9)}{(s + 5.4)} K_c$$

where the gain K_c of the amplifier is for gain adjustment for satisfying the magnitude condition of the root locus approach.

Thus the open-loop transfer function of the compensated system becomes

$$G_c(s)G(s) = \frac{s + 2.9}{s + 5.4} K_c \frac{4}{s(s + 2)} = \frac{K(s + 2.9)}{s(s + 2)(s + 5.4)} \quad \text{where} \quad K = 4K_c$$

The root-locus plot for the compensated system is shown in Figure 10.10. The gain K is evaluated from the magnitude condition as follows:

$$\left| \frac{K(s + 2.9)}{s(s + 2)(s + 5.4)} \right|_{s=-2+j2\sqrt{3}} = 1 \quad \text{or} \quad K = 18.7$$

It therefore follows that

$$G_c(s)G(s) = \frac{18.7(s+2.9)}{s(s+2)(s+5.4)}$$

The gain constant K_c of the amplifier is

$$K_c = \frac{K}{4} = 4.68$$

The static velocity error coefficient K_v is obtained from the expression

$$K_v = \lim_{s\to 0} sG_c(s)G(s)H(s); \qquad H(s) = 1$$

$$= \lim_{s\to 0} \frac{18.7(s+2.9)s}{s(s+2)(s+5.4)} = 5.02 \text{ s}^{-1}$$

The third closed-loop pole is found by dividing the characteristic equation by the known factors as follows:

$$s(s + 2)(s + 5.4) + 18.7(s + 2.9) = (s + 2 - j2\sqrt{3})(s + 2 + j2\sqrt{3})(s + 3.4)$$

The foregoing compensation method enables us to place the dominant closed-loop poles at the desired points $-2 \pm j2\sqrt{3}$ in the complex s-plane. The third pole at $s = -3.4$ is close to the added zero at $s = -2.9$. Therefore, the effect of this pole on the transient response is relatively small. Since no restriction has been imposed on the non-dominant pole and no specification has been given concerning the value of the static velocity error coefficient, we conclude that the present design of the lead compensating network is satisfactory either by *RC* network of Figure 10.4 with R_1 = 345 kΩ, R_2 = 400 kΩ, C = 1 μF or by electronic operational amplifier circuit of Figure 10.5 with values either (i) first choice: R_1 = 345 kΩ, R_2 = 400 kΩ, C_1 = 1 μF, C_2 = 0.47 μF, R_4 = 10 kΩ and R_3 = 4.7 kΩ; or (ii) second choice: R_1 = 34.5 kΩ, R_2 = 18.5 kΩ, $C_1 = C_2$ = 10 μF, R_4 = 46.8 kΩ and R_3 = 10 kΩ.

EXAMPLE 10.3 Consider the unity-feedback system having open-loop transfer function $\frac{4K}{s(s+2)}$. It is desired to find a compensator for the system so that the static velocity error coefficient K_v is 20 s^{-1}, the phase margin is at least 50°, and the gain margin is at least 10 dB.

Solution: In the example, the phase and gain margins have been specified. We shall therefore employ Bode diagrams. Adjusting the gain K to meet the steady-state performance specification or providing the required static velocity error coefficient which is given as 20 s^{-1}, we obtain

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} \frac{4Ks}{s(s+2)} = 2K = 20$$

or

$$K = 10$$

With K = 10 the given system satisfies the steady-state requirement.

We next plot the Bode diagram of

$$G(j\omega)H(j\omega) = \frac{40}{j\omega(j\omega+2)} = \frac{20}{j\omega(0.5j\omega+1)}$$

Figure 10.11 shows the magnitude and phase-angle curves of Bode diagram of $G(j\omega)H(j\omega)$. From this plot, the phase and gain margins of the system are found to be 17° and $+\infty$ dB respectively. The specification calls for a phase margin of at least 50°. We thus find that the additional phase lead necessary to satisfy the relative stability requirement is 33°. In order to achieve a phase margin of 50° without decreasing the value of K, it is necessary to insert a suitable lead compensator into the system.

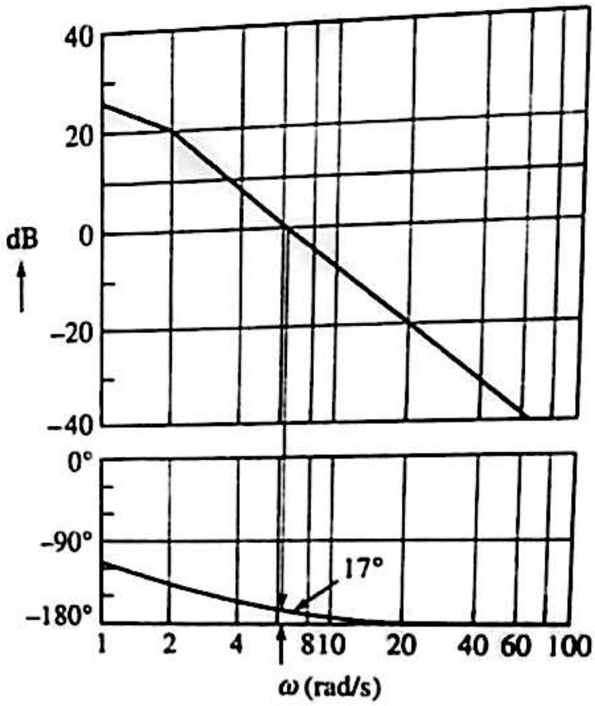

FIGURE 10.11 Example 10.3: Bode diagram for $G(j\omega)H(j\omega)$.

Noting that the addition of a lead compensator modifies the magnitude curve in the Bode diagram, we realize that the gain-crossover frequency will be shifted to the right. We must offset the increased phase lag of $G(j\omega)H(j\omega)$ due to this increase in the gain-crossover frequency. Considering the shift of the gain crossover frequency, we may assume that ϕ_m, the maximum phase lead required, as approximately 38° with margin of tolerance as 5° added to the relative stability requirement of 33°.

Since,
$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \tag{10.12}$$

$\phi_m = 38°$ corresponds to $\alpha = 0.24$.

Once the attenuation factor α has been determined on the basis of the required phase lead angle, the next step is to determine the corner frequencies $\omega = 1/T$ and $\omega = 1/(\alpha T)$ of the lead network. To do so, we first note that the maximum phase lead angle ϕ_m occurs at the geometric mean of the two corner frequencies which makes $\omega = 1/(T\sqrt{\alpha})$. The amount of the modification in the magnitude curve at $\omega = 1/(T\sqrt{\alpha})$ due to the inclusion of the term $(Ts+1)/(\alpha Ts+1)$ is

$$\left|\frac{1+j\omega T}{1+j\alpha\omega T}\right|_{\omega=1/\sqrt{\alpha}T} = \frac{1}{\sqrt{\alpha}}$$

where
$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.24}} = \frac{1}{0.49} = 6.2 \text{ dB}$$

Note that $|G(j\omega)| = 6.2$ dB corresponds to $\omega = 9$ rad/s. We shall select this frequency to be the new gain crossover frequency ω_c. Noting that this frequency corresponds to $1/(T\sqrt{\alpha})$, or $\omega_c = 1/(T\sqrt{\alpha})$, we obtain

$$\frac{1}{T} = \sqrt{\alpha}\,\omega_c = 4.41 \qquad \text{and} \qquad \frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = 18.4$$

The lead network thus determined is

$$\frac{s+4.41}{s+18.4} = \frac{0.24(0.227s+1)}{0.054s+1}$$

We increase the amplifier gain by a factor of $1/0.24 = 4.17$ in order to compensate for the attenuation due to the lead network. Then the transfer function of the compensator which consists of the lead network and the amplifier becomes

$$G_c(s) = (4.17)K\,\frac{s+4.41}{s+18.4} = 10\,\frac{0.227s+1}{0.054s+1}$$

Then the compensated system has the open-loop transfer function as $41.7\left(\dfrac{s+4.41}{s+18.4}\right)\left(\dfrac{4}{s(s+2)}\right)$.

EXAMPLE 10.4 The forward transfer function of a unity-feedback system is given by

$$\frac{28}{s(s+3)(s+6)}$$

It is desired that the real part of the dominant poles of the closed-loop system be not less than 4, keeping the damping ratio ζ unchanged. Also the static velocity error coefficient K_v must be at least 10. Design a suitable compensator as needed by this system.

Solution: Given the open-loop transfer function as

$$G(s)H(s) = \frac{28}{s(s+3)(s+6)}$$

the closed-loop poles are obtained from the characteristic equation $1 + GH(s) = 0$, as -7, $-1 \pm j\sqrt{3}$.

The damping coefficient of the dominant closed-loop poles $-1 \pm j\sqrt{3}$ is $\zeta = 0\ 5$. Hence the desired complex conjugate closed-loop poles are $-4 \pm j4\sqrt{3}$.

In the present system, the angle $G(s)H(s)$ at the desired closed-loop pole is

$$\angle \left.\frac{28}{s(s+3)(s+6)}\right|_{s=-4+j4\sqrt{3}} = -292.11°$$

Hence, we need a lead compensator in the forward path to provide a leading angle of at least $(292.11° - 180°) = 112.11°$.

In order to satisfy the criterion for velocity error coefficient $K_v \geq 10$, we need an amplifier in cascade of gain $K \geq 6.43$ as

$$K_v = 10 = \lim_{s \to 0} s[KG(s)H(s)] = \frac{28}{18}K, \text{ i.e. } K = 6.43$$

EXAMPLE 10.5 Let us consider a single-loop compensated feedback control system as shown in Figure 10.12 where $G(s) = K/s^2$ and $H(s) = 1$.

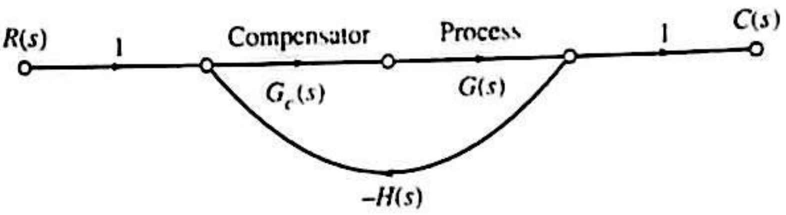

FIGURE 10.12 Example 10.5.

A lead compensating network has to be designed to meet the following specifications:

(i) Settling time, $t_s \leq 4$ s
(ii) Percent overshoot for a step input < 20 %
(iii) Acceleration error coefficient, $K_a \geq 2$

Solution: From Figure 10.13, the percent overshoot versus damping ratio curve of the second-order underdamped system, we get for percent overshoot ≤ 20%, the damping ratio as $\zeta \geq 0.45$.

The settling time (2% criterion) requirement is, $t_s = 4 = 4/\zeta\omega_n$. Hence

$$\omega_n = 1/\zeta = 1/0.45 = 2.22.$$

Therefore, the desired dominant complex conjugate roots for $\zeta = 0.45$ and $\omega_n = 2.22$ are $-1 \pm j2$.

We have to design a lead compensator

$$G_c(s) = \frac{s+z}{s+p} \quad \text{where } |z| < |p|$$

such that the points $-1 \pm j2$ lie on the root loci of the compensated system whose open-loop transfer function is

$$G_c(s)G(s)H(s) = \frac{K(s+z)}{s^2(s+p)}$$

Place the zero of the compensator directly below the desired location at $s = -z = -1$ as shown in Figure 10.14. We have to place the

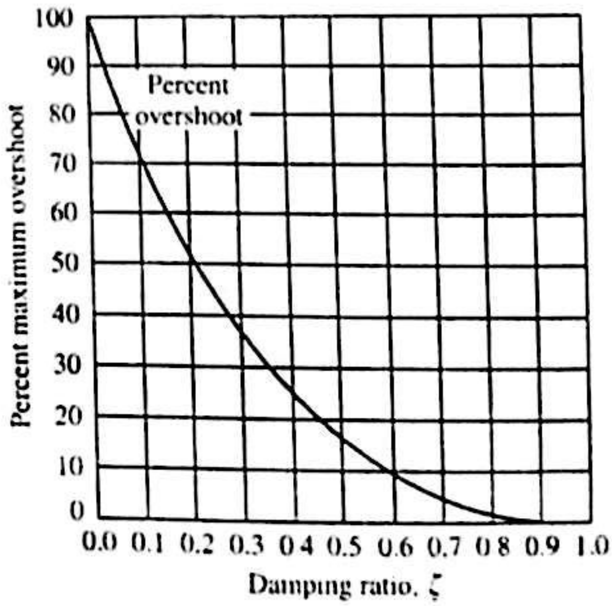

FIGURE 10.13 Example 10.5 percent overshoot vs damping ratio ζ for a second-order underdamped system.

undetermined pole location of the compensator so that the angle condition of the root loci is satisfied for the compensated system. Hence the angle condition for

$$\left|\frac{K(s-1)}{s^2(s+p)}\right|_{s=-1+j2} = -180^\circ$$

i.e. $90^\circ - 2(116.56^\circ) - \theta_p = -180^\circ$

or $\theta_p = 36.88^\circ$

Then a line is drawn at an angle $\theta_p = 36.88^\circ$ intersecting the desired root location at the real axis as shown in Figure 10.14. The point of intersection with the real axis is then $s = -p = -3.6$. Therefore, the compensator is

$$G_c(s) = \frac{s+1}{s+3.6}$$

The open-loop transfer function of the compensated system is

$$G(s)H(s)G_c(s) = \frac{K(s+1)}{s^2(s+3.6)}$$

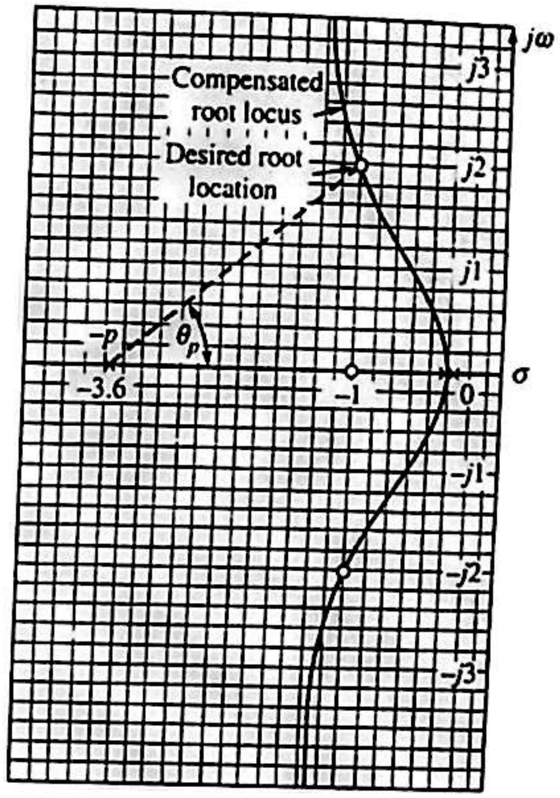

FIGURE 10.14 Example 10.5: phase-lead compensation.

From the magnitude condition of the root loci, we get the value of gain K by measuring the vector lengths from the poles and zeros to the desired root location at $-1 + j2$ as $K = \frac{(2.23)^2(3.25)}{2} = 8.1$. For type-2 system, K_p and K_v are infinity, hence it results in zero steady-state errors for step and ramp inputs. However, the acceleration constant K_a is

$$K_a = \lim_{s\to 0} s^2 G(s)H(s)G_c(s) = \frac{8.1}{3.6} = 2.25 > 2$$

The steady-state error for acceleration input is finite.

The design of the compensating network is complete as it satisfies all the specifications.

10.5 Lag Compensation

Figure 10.15 shows an electrical lag network. The name "lag network" comes from the fact that when the input voltage v_{in} is sinusoidal, the output voltage

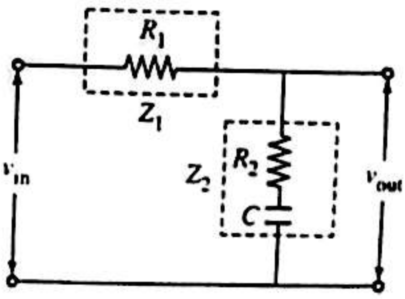

FIGURE 10.15 Electrical lag network

v_{out} is sinusoidal but lags the input by an angle which is a function of the frequency of the input sinusoid. The complex impedances Z_1 and Z_2 are

$$Z_1 = R_1, \quad Z_2 = R_2 + \frac{1}{sC}$$

The transfer function between the output voltage $V_{out}(s)$ and the input voltage $V_{in}(s)$ is given by

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2Cs}{(R_1 + R_2)Cs + 1}$$

Let us define

$$R_2C = T; \qquad \frac{R_1 + R_2}{R_2} = \beta > 1$$

Then the transfer function becomes

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{Ts + 1}{\beta Ts + 1} = \frac{1}{\beta}\left(\frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}\right) \tag{10.13}$$

10.5.1 Characteristics of lag networks

An RC-lag network of Figure 10.15 has the following transfer function

$$\frac{Ts + 1}{\beta Ts + 1} = \frac{1}{\beta}\left(\frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}\right); \qquad (\beta > 1)$$

In the complex plane, a lag network has a pole at $s = -1/\beta T$ and a zero at $s = -1/T$, i.e., the pole is located to the right of the zero.

Figure 10.16 shows the polar plot of a lag network. Figure 10.17 shows the Bode diagram of a lag network when $\beta = 10$. The corner frequencies of the lag network are $\omega = 1/T$ and $\omega = 1/\beta T$. The lag network is essentially a low-pass filter.

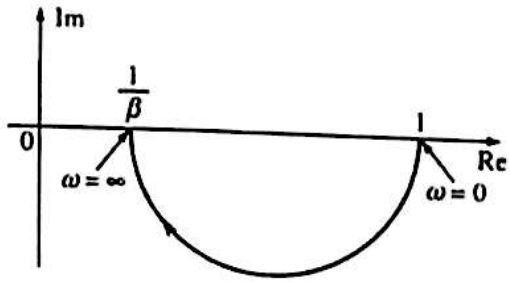

FIGURE 10.16 Polar plot of a lag network.

One can use the electronic op-amp lag circuit of Figure 10.5 just to make it different. In the RC circuit, β has been used for the lag circuit whereas the electronic circuit, with proper choice of values, can be used as lag circuit where $\alpha > 1$

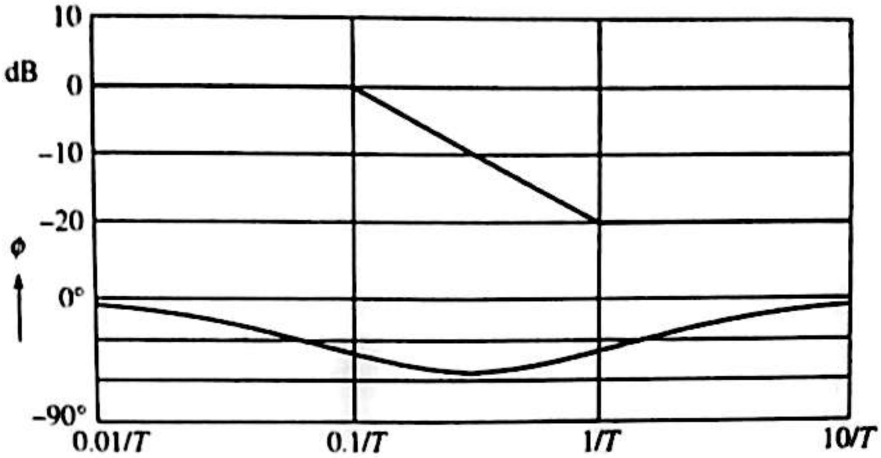

FIGURE 10.17 Bode plot of a lag network with $\beta = 10$.

10.5.2 Lag compensation techniques based on the root-locus approach

Consider the problem of finding a suitable compensation network for the case where the system exhibits satisfactory transient response characteristics but unsatisfactory steady-state characteristics. Compensation in this case essentially consists of increasing the open-loop gain without appreciably changing the transient-response characteristics. This means that the root locus in the neighbourhood of the dominant closed-loop poles should not be changed appreciably, but the open-loop gain should be increased as much as needed.

To avoid an appreciable change in the root loci, the angle contribution of the lag network should be limited to a small amount, say 5°. To assure this, we place the pole and zero of the lag network relatively close together and near the origin of the s-plane. Then the closed-loop poles of the compensated system will be shifted only slightly from their original locations. Hence the transient response characteristics will essentially be unchanged.

Note that if we place the pole and zero of the lag network very close to each other, then $(s_1 + 1/T)$ and $(s_1 + 1/\beta T)$ are almost equal, where s_1 is the closed-loop pole. Thus

$$\left|\frac{1}{\beta}\left(\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}\right)\right| \approx \left|\frac{1}{\beta}\left(\frac{s_1+\frac{1}{T}}{s_1+\frac{1}{\beta T}}\right)\right| \approx \frac{1}{\beta} \tag{10.14}$$

This implies that the open-loop gain can be increased approximately by a factor of β without altering the transient-response characteristics. If the pole and zero are placed very close to the origin, the value of β can be made large. Usually, $1 < \beta < 15$, and $\beta = 10$ is a good choice.

An increase in gain means an increase in the static error coefficients. If the open-loop transfer function of the uncompensated system is $G(s)$, then the static velocity error coefficient K_v is

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

If the compensator is chosen as

$$G_c(s) = K_c \frac{Ts+1}{\beta Ts+1} = \frac{K_c}{\beta}\left(\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}\right) \tag{10.15}$$

Then for the compensated system with the open-loop transfer function $G_c(s)G(s)H(s)$, the static velocity error coefficient $\hat{K}_v$ becomes

$$\hat{K}_v = \lim_{s\to 0} sG_c(s)G(s)H(s) = \lim_{s\to 0} G_c(s)\,K_v; \qquad \text{as } H(s) = 1$$

$$= K_c K_v$$

Thus if the compensator is given by Eq. (10.15), then the static velocity error coefficient is increased by a factor of K_c.

EXAMPLE 10.6 Consider the system shown in Figure 10.18. It is desired that the dominant pole of the closed-loop system should have the damping ratio of 0.5 and the magnitude of the real part of the pole be less than unity. Also, the velocity error coefficient should be at least 10. Design a suitable compensator.

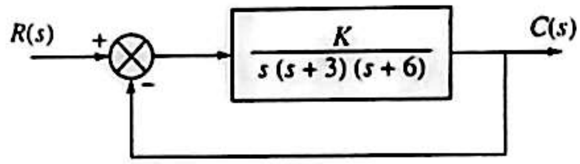

FIGURE 10.18 Example 10.6.

Solution: Let us draw the root loci for the open-loop transfer function

$$G(s)H(s) = \frac{K}{s(s+3)(s+6)};\ 0 < K < \infty$$

The root loci is drawn as shown in Figure 10.19. Draw $\zeta = \cos\theta = 0.5$ or $\theta = 60°$ line. The intersection of root loci in quardant II with $\theta = 60°$ line gives the value of $K = 28$. The characteristic equation of the closed-loop system at this value of $K = 28$ is

$$1 + GH(s) = 1 + \frac{28}{s(s+3)(s+6)}$$

$$= \frac{s(s+3)(s+6)+28}{s(s+3)(s+6)} = \frac{(s+7)(s^2+2s+4)}{s(s+3)(s+6)}$$

The roots of the closed-loop system are

$$(s+7)(s^2+2s+4) = 0$$

or

$$(s+7)(s+1+j\sqrt{3})(s+1-j\sqrt{3}) = 0$$

The real part of the complex poles is −1, which just meets our specification. Further, for

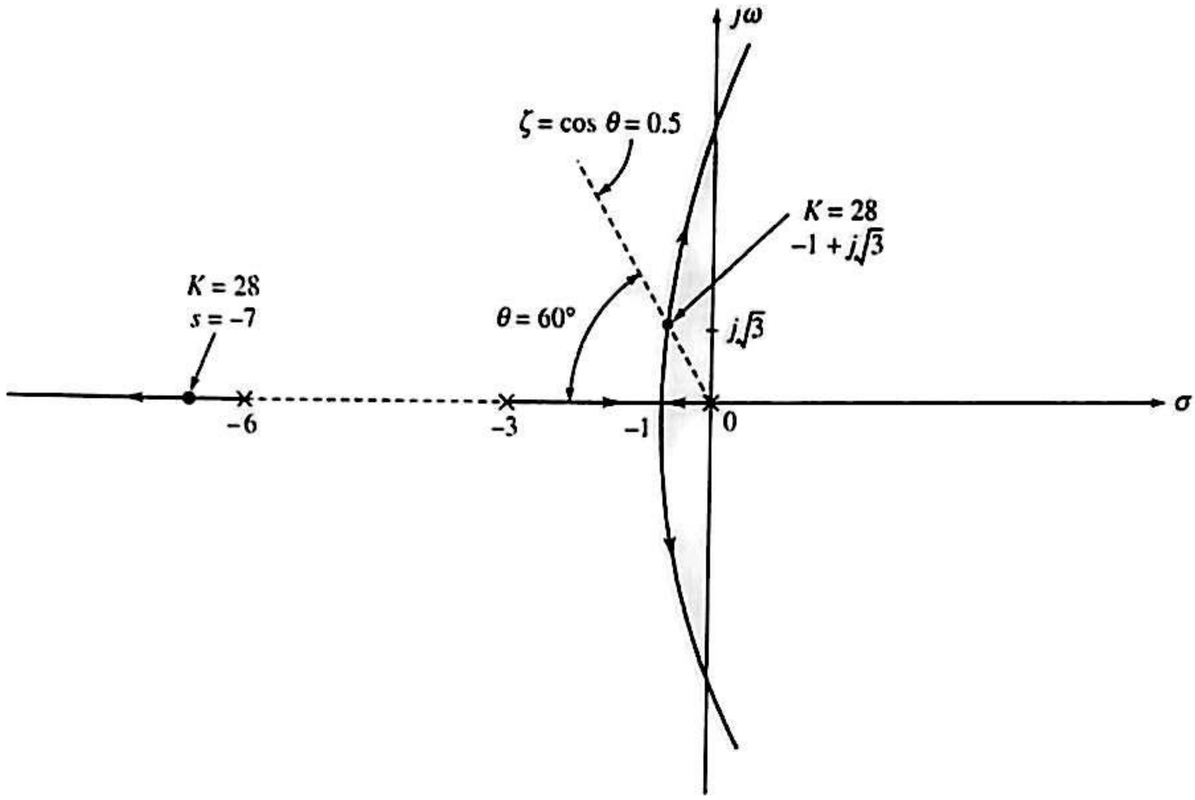

FIGURE 10.19 Example 10.6: root loci.

$K = 28$, the static velocity error coefficient K_v becomes

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{28s}{s(s+3)(s+6)} = \frac{28}{18} = 1.55$$

which is less than the desired value of $K_v = 10$.

In order to increase the value of K_v to 10, keeping in mind that without appreciably changing the location of the dominant closed-loop poles (i.e. keeping the constraints intact), let us insert a lag compensator which consists of a lag network and an amplifier, in cascade with the given feed-forward transfer function.

Let us place the pole and zero of the lag network at $s = -0.01$ and $s = -0.1$ respectively. Then the structure of the lag network is

$$\frac{sT+1}{s\beta T+1}; \qquad \beta > 1$$

from which we get as

$$\frac{1}{\beta} \frac{\left(s + \frac{1}{T}\right)}{\left(s + \frac{1}{\beta T}\right)} = \frac{1}{10}\left(\frac{s+0.1}{s+0.01}\right)$$

For the attenuation due to the lag network, we cascade an amplifier of gain K_c. The feed-forward transfer function of the compensated system would then be

$$G_1(s) = \frac{1}{10}\left(\frac{s+0.1}{s+0.01}\right)(K_c)\frac{28}{s(s+3)(s+6)} = \frac{K(s+0.1)}{s(s+0.01)(s+3)(s+6)}; \quad \text{where } K = 2.8K_c$$

It may be noted that the distances of the compensator pole and zero from the origin of the s-plane are chosen to be small compared with the distance of the dominant poles from the origin so that the compensator will not affect the locus near the dominant poles.

The lag compensator is only to modify the static error coefficient, in this case K_v. The block diagram of the compensated system is shown in Figure 10.20.

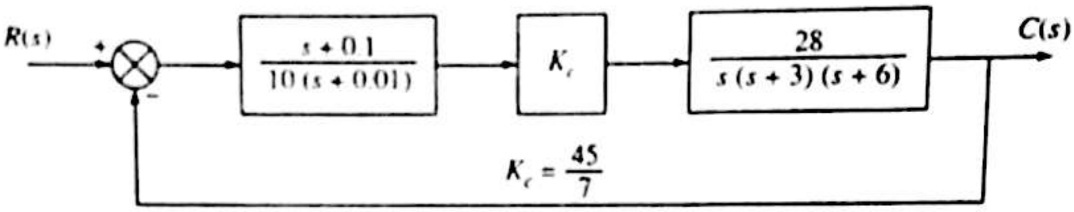

FIGURE 10.20 Example 10.6: block diagram of the compensated system.

The desired static velocity error coefficient K_v of the compensated unity-feedback system is 10, then by definition the modified static velocity error coefficient is given by

$$\lim_{s\to 0} sG_1(s)H(s) = 10 \qquad \text{or} \qquad \lim_{s\to 0}\left[\frac{s(2.8K_c)(s+0.1)}{s(s+0.01)(s+3)(s+6)}\right] = 10$$

which leads to

$$K_c = \frac{45}{7}$$

A plot of the root loci for the system with the lag compensator is shown in Figure 10.21. It may be seen that the root-locus plot is almost unchanged near the dominant poles. Although we are adding a pole and a zero with the transfer function of the forward path, their contribution to the argument near the dominant poles is negligible, so that the angle condition is still satisfied at that point.

EXAMPLE 10.7 For a unity-feedback system having forward transfer function

$$G(s) = \frac{100}{s(s+1)(0.1s+1)}$$

design a suitable compensator so that the phase margin is at least 45°.

Solution: The Bode plot (with error correction) of the uncompensated system

$$G(s)H(s) = \frac{100}{s(s+1)(0.1s+1)}$$

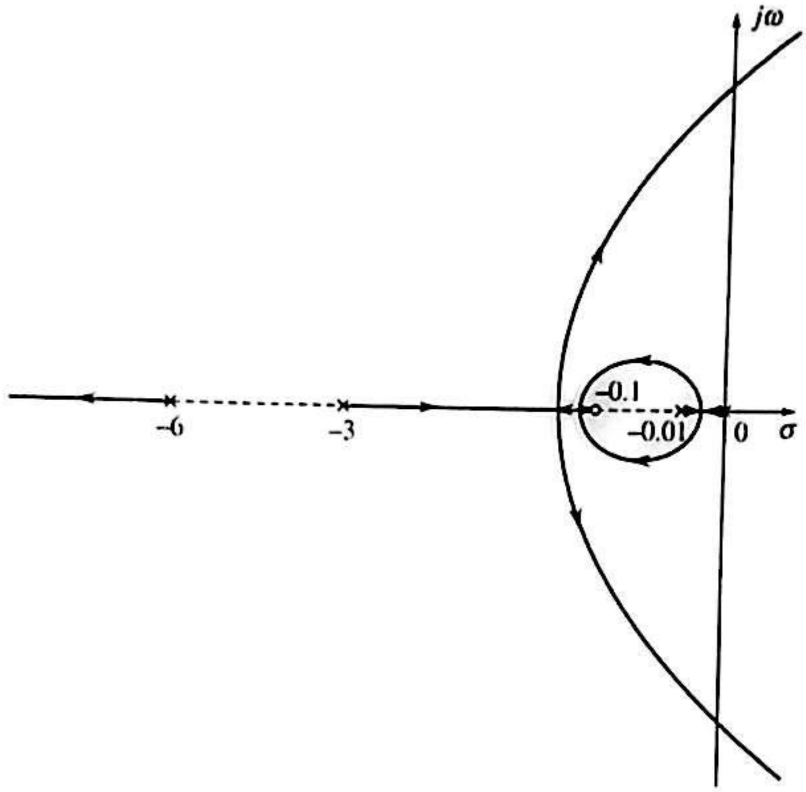

FIGURE 10.21 Example 10.6: root loci of the compensated system of Figure 10.20.

is drawn in Figure 10.22. The gain-crossover frequency ω_g is 8.5 rad/s. The phase-crossover frequency ω_p is 3.2 rad/s. The gain margin (GM) is −20 dB and the phase margin (PM) is −35°. The system is unstable. If the gain-crossover frequency were moved to a new value $\omega_{gd} = 0.7$ by providing the required gain reduction which comes out to be 43 dB from the Bode plot of Figure 10.22, the phase margin would have been 50°. A phase lag network can provide the required attenuation (gain reduction) of 43 dB which comes out to be gain reduction by a factor 140 [i.e. 20 log 140 = 42.92 dB ≈ 43 dB] without affecting the phase curve of the uncompensated system. The usual lag compensating network structure is

$$G_c(s) = \frac{1}{\alpha}\left(\frac{s+(1/T)}{s+(1/\alpha T)}\right)$$

that is, a pole at $-1/(\alpha T)$ and a zero at $-1/T$ and $1/\alpha$ is the attenuation. The proposed lag network should have $\alpha = 140$. Then the lag compensator network with pole at $-1/140T$ and zero at $-1/T$ is separated by slightly more than two decades.

The compensator zero is introduced at $\omega_{gd}/10 = 0.07$ and the phase curve is corrected. From the Bode diagram, it can be seen that the net phase margin is about 50° which is in accordance with the demand.

The compensator is then

$$G_c(s) = \frac{1}{140}\left(\frac{s+0.07}{s+0.0005}\right) = \frac{\dfrac{s}{0.07}+1}{\dfrac{s}{0.0005}+1} = \frac{14.29s+1}{2000s+1}$$

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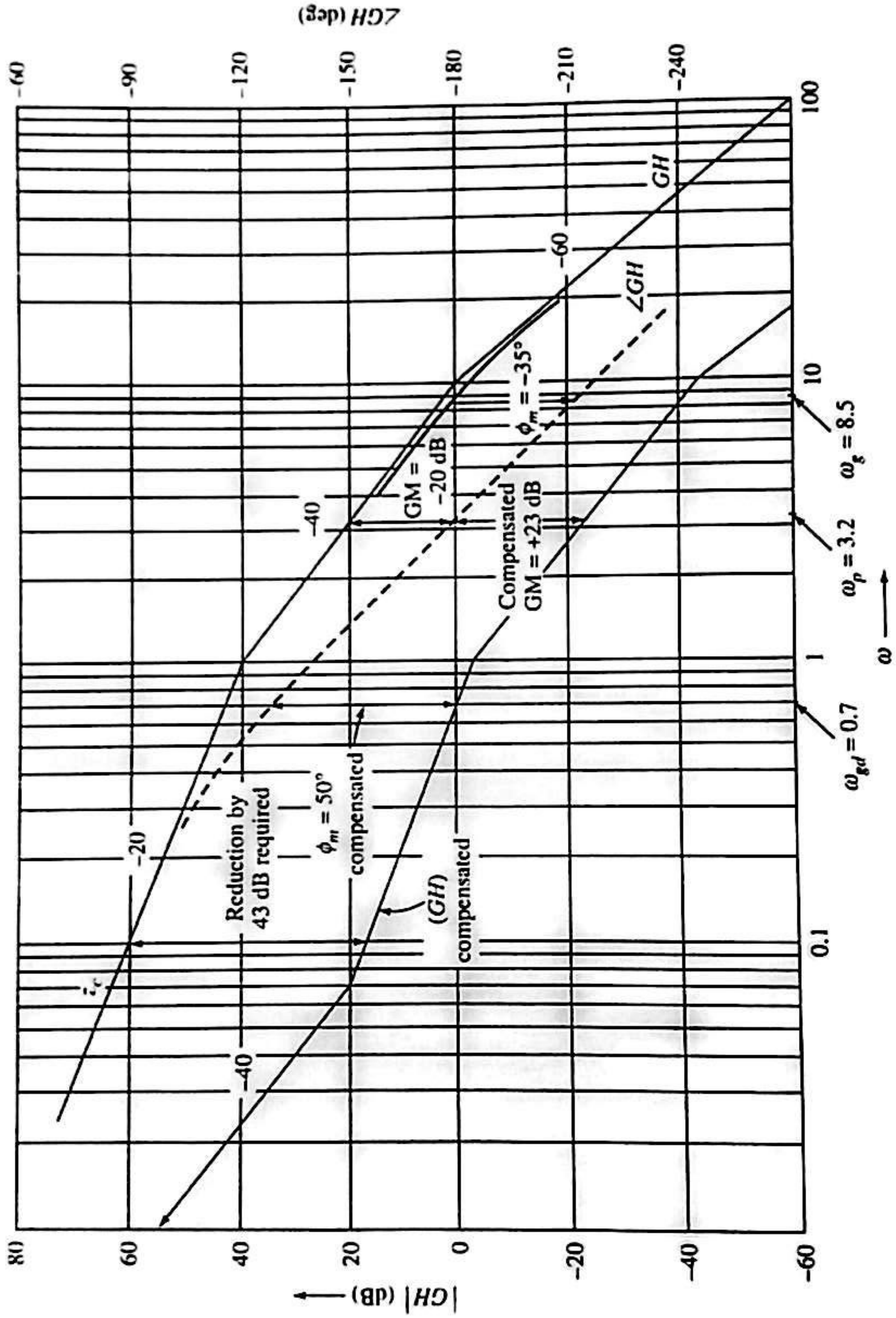

FIGURE 10.22 Example 10.7: Bode plots.

The compensated open-loop transfer function is therefore given by

$$G_c(s)G(s) = \frac{100(14.29s+1)}{s(s+1)(0.1s+1)(2000s+1)}$$

The magnitude and phase plots of the compensated system are shown in Figure 10.22. The phase margin is about 50° and as per the specification the gain margin is +23 dB. The compensated system is stable.

10.6 Lag–lead Compensation

Lead compensation increases the bandwidth, which improves the speed of response, and also reduces the amount of overshoot. However, improvement in steady-state performance is rather small. Basically, derivative control has this type of property that is effective in transient part and ineffective in steady-state part of the response. Lag compensation results in a large improvement in steady-state performance but results in slower response due to the reduced bandwidth. This lag network has similarity with I-control.

If improvements in both transient and steady-state response (namely, large increases in the gain and bandwidth) are desired, then both a lead network and a lag network may be used simultaneously. Rather then introducing both lead network and lag networks as separate elements, it is economical to use a single lag–lead network. The lag–lead network combines the advantages of the lag and lead networks.

The lag–lead network possesses two poles and two zeros. Therefore, such compensation increases the order of the system by two, unless cancellation of a pole and a zero occurs in the compensated system.

10.6.1 Lag–lead networks

Figure 10.23 shows an RC electrical lag–lead network. For a sinusoidal input, the output is sinusoidal with a phase shift which is a function of the input frequency. This phase angle varies from lag to lead as the frequency is increased from zero to infinity. Note that phase lead and lag occur in different frequency bands.

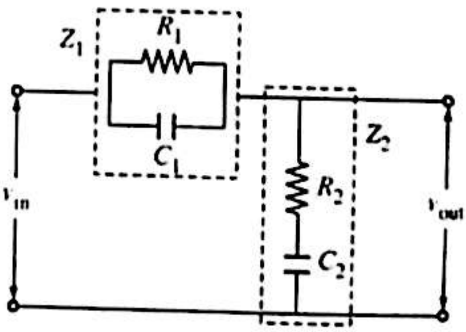

FIGURE 10.23 RC lag–lead network.

Let us obtain the transfer function of the RC lag–lead network. The complex impedances Z_1 and Z_2 are

$$Z_1 = \frac{R_1}{R_1C_1s+1}, \quad Z_2 = R_2 + \frac{1}{C_2s}$$

The transfer function between $V_{out}(s)$ and $V_{in}(s)$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{Z_2}{Z_1+Z_2} = \frac{(R_1C_1s+1)(R_2C_2s+1)}{(R_1C_1s+1)(R_2C_2s+1)+R_1C_2s}$$

The denominator of this transfer function can be factored into two real terms. Let us define

$$R_1C_1 = T_1 \qquad \text{and} \qquad R_2C_2 = T_2$$

$$R_1C_1 + R_2C_2 + R_1C_2 = \frac{T_1}{\beta} + \beta T_2 \qquad (\beta > 1)$$

Then $V_{out}(s)/V_{in}(s)$ can be simplified to

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{(T_1s+1)(T_2s+1)}{\left(\frac{T_1}{\beta}s+1\right)(\beta T_2s+1)} = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)} \tag{10.16}$$

The electronic circuit using operational amplifiers is shown in Figure 10.24. The transfer function for this compensator is obtained as follows:

$$Z_1 = R_3 \| (R_1 + C_1) = \frac{(R_1C_1s+1)R_3}{(R_1+R_3)C_1s+1}$$

$$Z_2 = R_4 \| (R_2 + C_2) = \frac{(R_2C_2s+1)R_4}{(R_2+R_4)C_2s+1}$$

Again, $$\frac{V(s)}{V_{in}(s)} = -\frac{Z_2}{Z_1} \qquad \text{and} \qquad \frac{V_{out}(s)}{V(s)} = -\frac{R_6}{R_5}$$

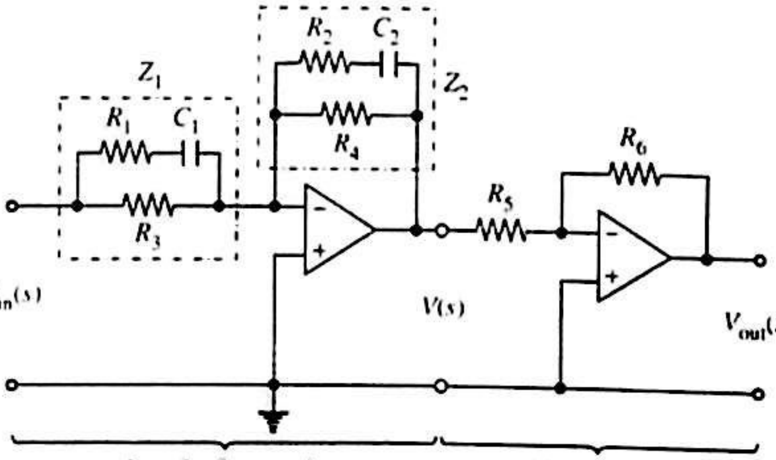

FIGURE 10.24 Lag–lead compensator.

Hence, the transfer function of the circuit is

$$\frac{V_{out}(s)}{V_{in}(s)} = K_c \frac{\beta}{\gamma}\left(\frac{T_1s+1}{\frac{T_1}{\gamma}s+1}\right)\left(\frac{T_2s+1}{\beta T_2s+1}\right)$$

$$= K_c \frac{(s+1/T_1)(s+1/T_2)}{(s+\gamma/T_1)(s+1/\beta T_2)} \tag{10.17}$$

where
$$\beta = \frac{R_2+R_4}{R_2} > 1, \quad \gamma = \frac{R_1+R_3}{R_1} > 1$$

and
$$K_c = \left(\frac{R_2 R_4 R_6}{R_1 R_3 R_5}\right)\left(\frac{R_1+R_3}{R_2+R_4}\right)$$

Note that usually $\gamma = \beta$ is the normal choice. The transfer functions for the *RC* circuit and the electronic circuit with op-amps have the same structure as is evident from Eqs. (10.16) and (10.17).

10.6.2 Characteristics of lag–lead networks

Consider the simplified version of transfer function of the lag–lead network with $K_c = 1$, i.e.

$$\frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)}$$

The first term
$$\frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}} = \frac{1}{\beta}\frac{T_1 s+1}{\frac{T_1 s+1}{\beta}} \qquad (\beta > 1)$$

produces the effect of the lead network and the second term

$$\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} = \beta\frac{T_2 s+1}{\beta T_2 s+1} \qquad (\beta > 1)$$

produces the effect of the lag network.

If the Bode plot or the polar plot of the lag–lead network is drawn, it would be seen that for $0 < \omega < \omega_1$ the network acts as a lag network, while for $\omega_1 < \omega < \infty$ it acts as a lead network, where the frequency ω_1 at which the phase angle is zero is given by

$$\omega_1 = \frac{1}{\sqrt{T_1 T_2}}$$

The magnitude curve will have the value 0-dB at the low-frequency and high-frequency regions. This is because the transfer function of the lag–lead network as a whole does not contain β as a factor.

EXAMPLE 10.8 Consider a unity-feedback control system whose forward transfer function is

$$G(s) = \frac{10K}{s(s+2)(s+8)}$$

Design a compensator so that $K_v = 80\ s^{-1}$ and the dominant closed-loop poles are located at $-2 \pm j2.3$.

Solution: The characteristic equation is: $s^3 + 10s^2 + 16s + 10 = 0$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \frac{10K}{(2)(8)} = 80$$

Hence, $$K = 128$$

If we use the lag–lead compensator given be Eq. (10.16), then the open-loop transfer function of the compensated system becomes

$$G_c(s)G(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)} \cdot \frac{1280}{s(s+2)(s+8)}$$

Now if T_2 is chosen large enough so that

$$\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} \approx 1$$

then for closed-loop poles to lie at $s = -2 \pm j2.3$, the magnitude condition becomes unity.

The phase lead portion of the lag–lead network must contribute 136.102° to make the root locus of the compensated system pass through the $-2 \pm j2.3$ points, i.e. the

$$\text{phase condition is } \angle \left.\frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}}\right|_{s=-2+j2.3} = 136.102°$$

This correction cannot be done with a single lead network, so we must connect two identical lead networks and hence conditions for each of them become

$$\left|\frac{s+\frac{1}{T_1'}}{s+\frac{\beta'}{T_1'}}\right|_{s=-2+j2.3} = \frac{1}{13.33} = 0.274 \quad \text{and} \quad \angle \left.\frac{s+\frac{1}{T_1'}}{s+\frac{\beta'}{T_1'}}\right|_{s=-2+j2.3} = 68.051°$$

So the transfer function of the lead network after detailed calculations becomes

$$\frac{s+2.39}{s+14.411}\,\frac{s+2.39}{s+14.411}$$

Now for each of the two lag networks, we require that

$$\left.\frac{s+\dfrac{1}{T_2}}{s+\dfrac{1}{\beta T_2}}\right|_{s=-2+j2.3} = 1$$

and the compensator transfer function becomes

$$G_c(s) = \frac{s+0.05}{s+0.0083}\cdot\frac{s+2.39}{s+14.411}$$

Hence the compensated transfer function becomes

$$G_c(s)G(s) = \frac{1280}{s(s+1)(s+2)}\,\frac{s+0.05}{s+0.0083}\,\frac{s+2.39}{s+14.411}$$

10.7 Compensation of Operational Amplifier

A compensation scheme for operational amplifiers has been implemented using the root-locus and Bode plot techniques. A practical case has been taken for the 741 op-amp.

The high frequency model of an op-amp with a single corner frequency is shown in Figure 10.25. The open-loop voltage gain is then obtained as follows:

$$V_o = \frac{-jX_C}{R_o - jX_c} A_{OL} V_d \qquad (10.18)$$

or $$A = \frac{V_o}{V_d} = \frac{A_{OL}}{1 + j2\pi f R_o C}$$

or $$A = \frac{A_{OL}}{1 + j(f/f_1)} \qquad (10.19)$$

where $$f_1 = \frac{1}{2\pi R_o C} \qquad (10.20)$$

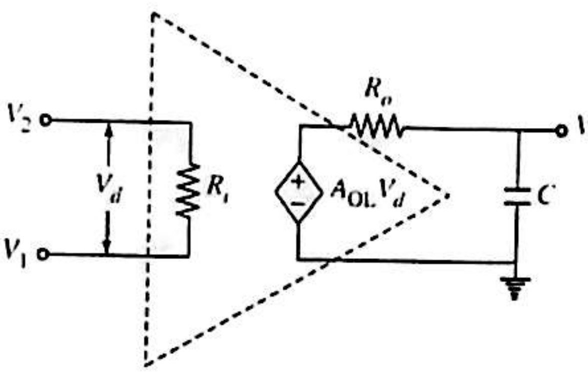

FIGURE 10.25 High frequency model of an op-amp with a single corner frequency

is the corner frequency of the op-amp.

The voltage transfer function in s-domain can be written as

$$G(s) = A(s) = \frac{A_{OL}}{1 + j(f/f_1)} = \frac{A_{OL}}{1 + j(\omega/\omega_1)}$$

$$= \frac{A_{OL} \cdot \omega_1}{j\omega + \omega_1} = \frac{A_{OL} \cdot \omega_1}{s + \omega_1}$$

where

A_{OL} is the open-loop voltage gain

V_o is the output voltage = $A_{OL} V_d = A_{OL}(V_1 - V_2)$

R_i is the input impedance

R_o is the output impedance

$V_d = (V_1 - V_2)$ is the differential input voltage

V_1 is the non-inverting input signal

V_2 is the inverting input signal

The magnitude and the phase angle of the open-loop voltage transfer function are functions of frequency and can be written as

$$|A| = \frac{A_{OL}}{\sqrt{(1 + (f/f_1)^2)}} \tag{10.21}$$

$$\phi = -\tan^{-1}(f/f_1) \tag{10.22}$$

The magnitude and phase characteristics from Eq. (10.21) and Eq. (10.22) are shown in Figures 10.26(a) and (b) respectively. It can bee seen that:

(i) For frequency $f \ll f_1$, the magnitude of the gain is $20 \log A_{OL}$ in dB.
(ii) At frequency $f = f_1$, the gain is 3-dB down from the dc value of A_{OL} in dB. This frequency f_1 is called the corner frequency.
(iii) For $f \gg f_1$, the gain rolls off at the rate of –20 dB/decade or –6 dB/octave.

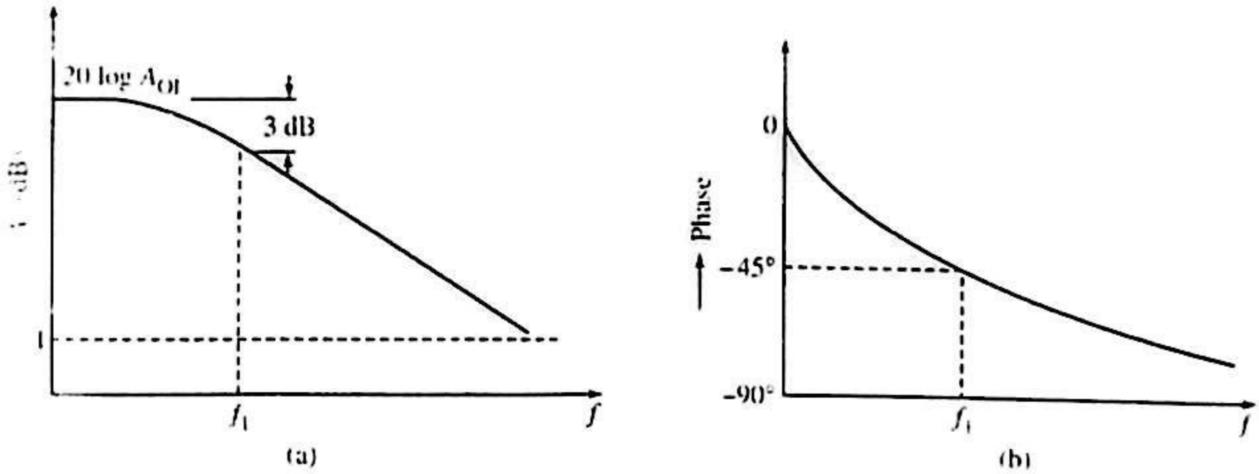

FIGURE 10.26 (a) Open-loop magnitude characteristic of Figure 10.25 and (b) open-loop phase characteristic of Figure 10.25.

It can further be seen from the phase characteristics that the phase angle is zero at frequency f = 0. At corner frequency f_1 the phase angle is –45° (lagging) and at infinite frequency the phase angle is $-90°$. This shows that a maximum of 90° phase change can occur in an op-amp with a single capacitor. It may be mentioned here that zero frequency does not occur in log scale. For all practical purposes, the zero frequency is taken as one decade below the corner frequency and the infinite frequency is taken one decade above the corner frequency.

10.7.1 Transfer function of a practical op-amp

Ideally, an op-amp should have an infinite bandwidth. This means that if its open-loop gain is 90 dB with dc signal, its gain should remain the same 90 dB through audio and on to high radio frequencies. The practical op-amp gain, however, decreases (i.e. rolls off) at higher frequencies.

What causes the gain of the op-amp to roll off after a certain frequency is reached? Obviously, there must be a capacitive component in the equivalent circuit of the op-amp. This capacitance is due to the physical characteristics of the device (BJT or FET) used and the internal construction of op-amp. For an op-amp with only one break (corner) frequency, all the capacitor effects can be represented by a single capacitor C as shown in Figure 10.25. This figure represents the high frequency model of the op-amp with a single corner frequency. It may be observed that the high frequency model of Figure 10.25 is a modified version of the low frequency model with a capacitor C at the output. There is one pole due to RC and obviously one -20 dB/decade roll-off comes into effect.

A practical op-amp, however, has a number of stages and each stage produces a capacitive component. Thus due to a number of RC-pole pairs, there will be a number of different break frequencies. The transfer function of an op-amp with three break frequencies can be assumed as

$$A = \frac{A_{OL}}{\left(1 + j\dfrac{f}{f_1}\right)\left(1 + j\dfrac{f}{f_2}\right)\left(1 + j\dfrac{f}{f_3}\right)}; \qquad 0 < f_1 < f_2 < f_3 \tag{10.23}$$

or

$$A = \frac{A_{OL}\cdot\omega_1\cdot\omega_2\cdot\omega_3}{(s+\omega_1)(s+\omega_2)(s+\omega_3)}; \quad 0 < \omega_1 < \omega_2 < \omega_3 \tag{10.24}$$

EXAMPLE 10.9 For a minimum-phase function, the straight line approximation of open-loop voltage transfer function vs frequency is shown in Figure 10.27. Determine the voltage transfer function.

Solution: The open-loop frequency response is flat (90 dB) from low frequencies (including dc) to 200 kHz, the first break frequency. From 200 kHz to 2 MHz, the gain drops from 90 dB to 70 dB which is at a -20 dB/decade or -6 dB/octave rate. At frequencies from 2 MHz to 20 MHz, the roll-off rate is -40 dB/decade or -12 dB/octave. Accordingly, as the frequency increases, the cascading effect of RC pairs (poles) comes into effect and the roll-off rate increases successively by -20 dB/decade at each corner frequency. Each RC pole pair also introduces a lagging phase of maximum up to $-90°$. Hence the corner frequencies are

$$f_1 = 200 \text{ kHz}; \quad f_2 = 2 \text{ MHz}; \quad \text{and} \quad f_3 = 20 \text{ MHz}$$

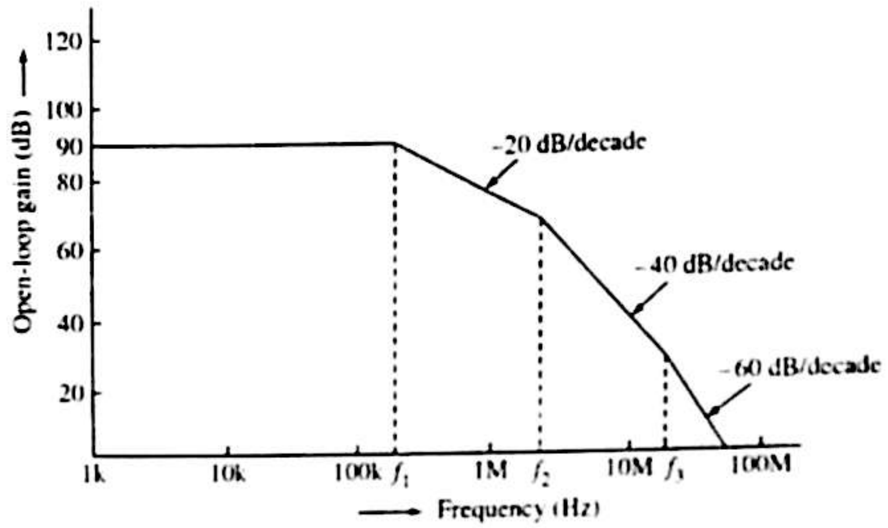

FIGURE 10.27 Approximation of open-loop gain vs frequency curve.

At each successive corner frequency the voltage gain falls by –20 dB/decade. The dc gain is 90 dB which is equal to

$$90 \text{ dB} = 20 \log A_{OL} \quad \text{or} \quad A_{OL} = 31623$$

Therefore, the voltage transfer function

$$A(s) = \frac{A_{OL}\,\omega_1\,\omega_2\,\omega_3}{(s+\omega_1)(s+\omega_2)(s+\omega_3)} = \frac{(31623)(0.2\times10^6)(2\times10^6)(20\times10^6)}{(s+0.2\times10^6)(s+2\times10^6)(s+20\times10^6)}$$

EXAMPLE 10.10 For the circuit of Figure 10.28, determine the closed-loop transfer function, where the open-loop frequency response for minimum-phase function of the op-amp is shown in Figure 10.27.

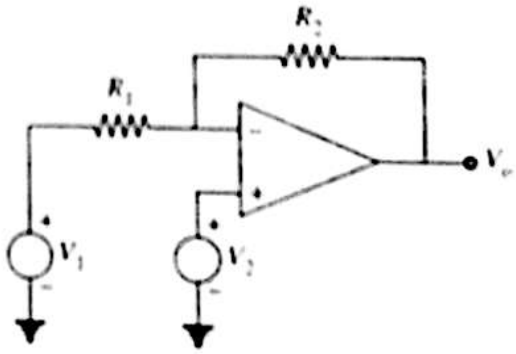

FIGURE 10.28 Resistive feedback in op-amp.

Solution: From the given open-loop frequency response of Figure 10.27, the open-loop voltage transfer function is

$$A(s) = \frac{A_{OL}\,\omega_1\,\omega_2\,\omega_3}{(s+\omega_1)(s+\omega_2)(s+\omega_3)},\ 0 < \omega_1 < \omega_2 < \omega_3$$

where $A_{OL} = 31623$; $f_1 = 0.2$ MHz, $f_2 = 2$ MHz, $f_3 = 20$ MHz and $\omega_i = 2\pi f_i$; $i = 1, 2, 3$.

From the negative feedback concepts, we may write the closed-loop transfer function of the circuit of Figure 10.29 as

$$A_{CL} = \frac{A}{1 + A\beta} \tag{10.25}$$

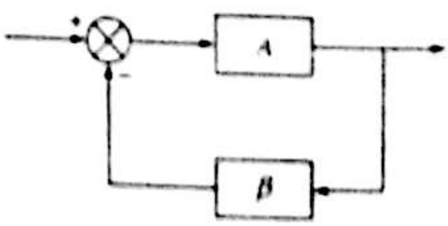

FIGURE 10.29 Feedback loop.

where A is the open-loop voltage gain and β is the feedback ratio. In Eq. (10.25), if the characteristic equation $(1 + A\beta) = 0$, the circuit will become just unstable, that is it will lead into sustained oscillations.

Rewriting the characteristic equation as, $1 + A\beta = 0$ leads to

$$\text{loop gain, } A\beta = -1 \tag{10.26}$$

Since $A\beta$ is a complex quantity, the magnitude condition become

$$|A\beta| = 1 \tag{10.27}$$

and the phase condition is

$$\angle A\beta = \pi (\text{or odd multiple of } \pi) \tag{10.28}$$

In the given circuit, the feedback network is a resistive network, so it does not provide any phase shift. The op-amp is used in the inverting mode and hence provides negative feedback. At high frequencies, due to each corner frequency, an additional phase shift of maximum $-90°$ can take place in open-loop gain A. So for two corner frequencies, a maximum of phase shift that can be associated with gain $A\beta$ is $-180°$ which makes the total phase shift $\angle A\beta$ equal to odd multiples of π. In this case, there is every possibility that the amplifier may begin to oscillate as both the magnitude and phase conditions laid down by Eqs. (10.27) and (10.28) are satisfied. It may be noted that oscillation is just the starting point of instability, or, to be more precise, it is just at the verge of instability. The instability means unbounded output: which can arise from Eq. (10.25), when

$$(1 + A\beta) < 1 \qquad \text{or} \qquad A\beta < 0 \text{ (i.e. negative)}$$

and then $A_{CL} > A$, i.e. the closed-loop gain increases and leads to instability. The phase contribution by the resistive feedback network is zero. At low frequencies, the additional phase contribution of A is zero, so $A\beta > 0$ and obviously $A_{CL} < A$ and the system is stable. But at high frequencies, the system A having three corner frequencies or three RC-pole pairs, there is a chance of open-loop transfer function $A\beta$ to contribute a maximum of $-270°$ phase shift and for which $A\beta$ may become negative and instability would occur at high frequencies. This is further elaborated in Figure 10.30.

Let us say that a closed-loop gain of 80 dB ($|(A_{OL})| = 10{,}000$) is desired. The projection of the 80-dB curve upon the open-loop frequency response curve intersects it at a -20 dB/decade rate of closure (point A) as shown in Figure 10.30. The bandwidth is approximately 600 kHz and a maximum of $-90°$ phase shift is added to the open-loop transfer function $A\beta$. The amplifier will remain stable.

Now, if the feedback resistors are so chosen that the op-amp has a closed-loop gain of 1,000 or 60 dB, the bandwidth is about 3.5 MHz. However, now the 60-dB projection on to the open-loop curve intersects at a -40 dB/decade rate of closure (point B). The maximum phase shift that may get added to is now $(-90° - 90°)$, that is, $-180°$. This circuit is likely to be unstable and should not be used without modification. Similarly, a closed-loop gain of 20 dB causes a -60 dB/decade rate of closure (point C). A maximum $-270°$ phase shift is added to the open-loop transfer function $A\beta$ to cause unstable operation. Thus, we may conclude that

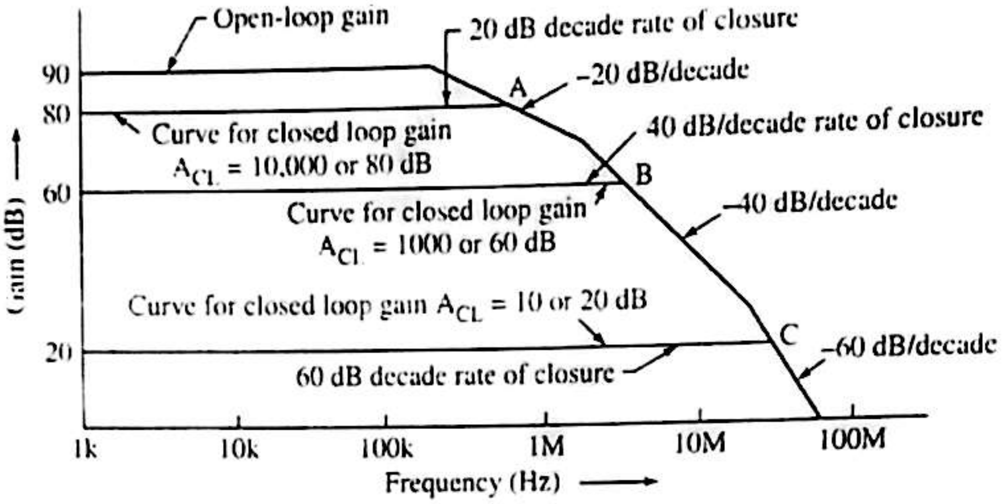

FIGURE 10.30 Effect of feedback on open-loop gain vs. frequency curve.

for stable operation, the rate of closure between the closed-loop gain projection and the open-loop curve should not exceed –20 dB/decade. At higher frequencies for lower closed-loop gains, the feedback becomes significant and regenerative, and may result in sustained oscillations.

So far, we have discussed stability of an op-amp qualitatively. To provide a quantitative discussion on stability, let us rewrite the transfer function of an op-amp characterized by three poles, as

$$A = \frac{A_{OL}\,\omega_1\,\omega_2\,\omega_3}{(s+\omega_1)(s+\omega_2)(s+\omega_3)}; \qquad 0 < \omega_1 < \omega_2 < \omega_3$$

Obviously the poles of the open-loop transfer function are at $-\omega_1$, $-\omega_2$ and $-\omega_3$. The closed-loop poles, that is, the poles of A_{CL} in Eq. (10.25) will be given by the roots of the characteristic equation

$$1 + A\beta = 0$$

Putting the value of A from Eq. (10.24), we get

$$1 + \frac{\beta A_{OL}\,\omega_1\,\omega_2\,\omega_3}{(s+\omega_1)(s+\omega_2)(s+\omega_3)} = 0$$

or $$(s + \omega_1)(s + \omega_2)(s + \omega_3) + \beta A_{OL}\,\omega_1\omega_2\omega_3 = 0$$

or $$s^3 + s^2(\omega_1 + \omega_2 + \omega_3) + s(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3) + \omega_1\omega_2\omega_3(1 + \beta A_{OL}) = 0 \quad (10.29)$$

The roots of this cubic equation depend upon βA_{OL}, the dc loop gain and therefore, βA_{OL} becomes the critical parameter that determines the new pole location. Further βA_{OL} can take any value between zero for no feedback ($\beta = 0$) and A_{OL} for maximum feedback ($\beta = 1$). For variable β in the range $0 < \beta A_{OL} < \infty$, the root loci is shown in Figure 10.31. When $\beta A_{OL} = 0$, the roots are at $-\omega_1$, $-\omega_2$ and $-\omega_3$ and lie on the negative real axis. For small values of βA_{OL}, the roots still lie on the left-half of the s-plane with one real root and two complex conjugate roots (a, a^*). If βA_{OL} is increased further beyond a critical value $(\beta A_{OL})_c$, the two roots will move to the right half of the s-plane causing instability. We will find out the critical

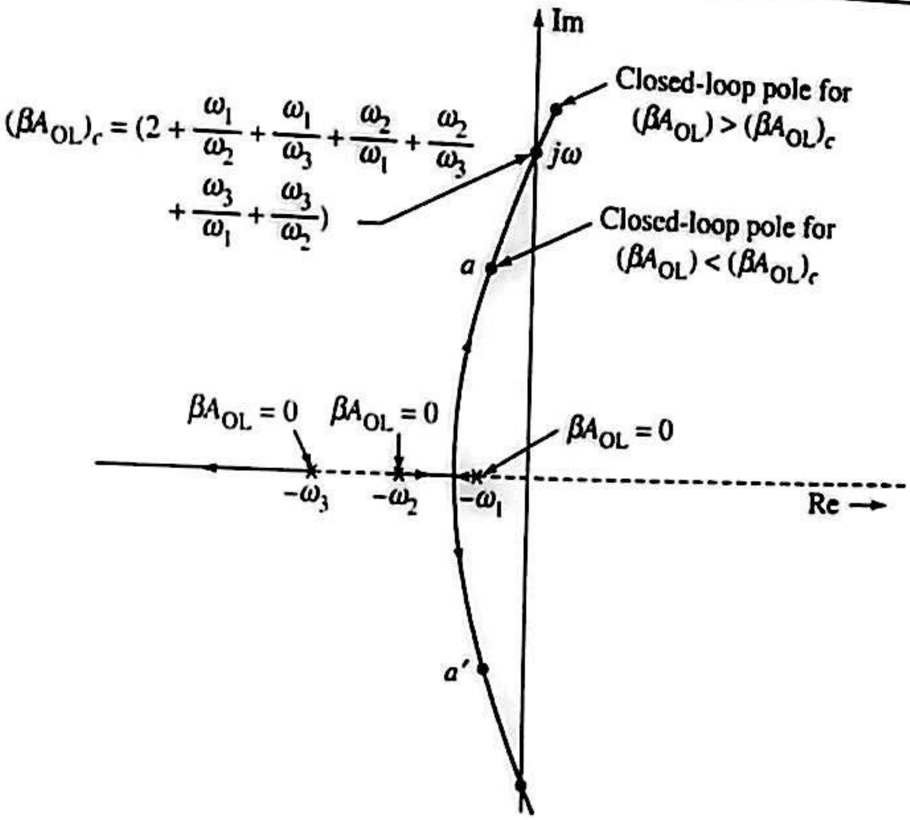

FIGURE 10.31 Root loci as a function βA_{OL}.

value of βA_{OL} if increased further beyond a critical value (βA_{OL}); the two roots will now move to the right-half of the s-plane causing instability. We will find out the critical value of βA_{OL} for which the closed-loop system becomes just unstable. Rewrite Eq. (10.29) as

$$a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0 = 0 \tag{10.30}$$

where

$a_3 = 1$; $a_2 = \omega_1 + \omega_2 + \omega_3$; $a_1 = \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3$; and $a_0 = \omega_1\omega_2\omega_3(1 + \beta A_{OL})$

In order to find the critical value of βA_{OL}, apply Routh's stability criterion to Eq. (10.30), that is

(i) All the coefficients a_3, a_2, a_1 and a_0 should be positive.
(ii) $a_2 a_1 - a_3 a_0 \geq 0$ (10.31)

Putting $s = j\omega$ in Eq. (10.30)

$$a_3(j\omega)^3 + a_2(j\omega)^2 + a_1(j\omega) + a_0 = 0$$

or

$$(a_0 - a_2\omega^2) + j\omega(a_1 - a_3\omega^2) = 0$$

Equating the real and imaginary parts to zero, we get

$$a_0 - a_2\omega^2 = 0 \tag{10.32}$$

$$a_1 - a_3\omega^2 = 0 \tag{10.33}$$

Thus, the frequency of oscillations is given by

$$\omega_{osc} = \pm\sqrt{\frac{a_0}{a_2}} = \pm\sqrt{\frac{a_1}{a_3}} \tag{10.34}$$

Putting the values of coefficients

$$\omega_{osc} = \sqrt{\frac{a_1}{a_3}} = \sqrt{\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3} \tag{10.35}$$

Also from Eq. (10.31), we get

$$a_0 = \frac{a_2 a_1}{a_3} \tag{10.36}$$

or $$\omega_1\omega_2\omega_3\{1 + (\beta A_{OL})_c\} = (\omega_1 + \omega_2 + \omega_3)(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)$$

or $$(\beta A_{OL})_c = 2 + \frac{\omega_1}{\omega_2} + \frac{\omega_1}{\omega_3} + \frac{\omega_2}{\omega_1} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_1} + \frac{\omega_3}{\omega_2} \tag{10.37}$$

It is obvious that $(\beta A_{OL})_c$ depends upon the ratio of the open-loop pole locations. The minimum value of $(\beta A_{OL})_c$ will occur when all the poles are located at the same place giving $(\beta A_{OL})_c = 8$.

As an example, if $A_{OL} = 10^5$ and $\omega_1 = \omega_2 = \omega_3 = 10^7$ rad/s, then the circuit will oscillate at a frequency of

$$\omega_{osc} = \omega_1\sqrt{3} \text{ rad/s} = 10^7\sqrt{3} \text{ rad/s}$$

On the other hand, if $1000\,\omega_1 = \omega_2 = \omega_3$, the critical loop gain is approximately

$$(\beta A_{OL})_c \approx 2\left(\frac{\omega_2}{\omega_1}\right) \approx 2000 \quad \text{and} \quad \beta_c < \frac{2000}{A_{OL}} \approx \frac{2000}{10000} = 0.2$$

In Figure 10.28, $\beta = R_1/(R_1 + R_2)$

For $$\beta < 0.2 \qquad \frac{R_1 + R_2}{R_1} > \frac{1}{0.2} > 5 \quad \text{or} \quad \frac{R_2}{R_1} \geq 4$$

This means that if the op-amp is used as an inverting amplifier in Figure 10.28, the inverting gain magnitude should be greater than 4 and if used as a non-inverting amplifier, the non-inverting gain should be greater than 5 for oscillations to sustain.

If it is desired that the amplifier should remain stable for any resistive network, that is, $0 < \beta < 1$, then A_{OL} must satisfy the most stringent condition for $\beta = 1$, that is,

$$A_{OL} < \left(2 + \frac{\omega_1}{\omega_2} + \frac{\omega_1}{\omega_3} + \frac{\omega_2}{\omega_1} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_1} + \frac{\omega_3}{\omega_2}\right)$$

10.7.2 Frequency compensation

Two types of frequency compensating techniques are used: (i) external compensation and (ii) internal compensation.

External frequency compensation

Two common methods for accomplishing this goal are: Dominant-pole compensation and Pole-zero (lag) compensation.

Dominant-pole compensation: Suppose A is the uncompensated transfer function of the op-amp in the open-loop condition as given by Eq. (10.24). Let us introduce a dominant pole by adding an RC network in series with the op-amp as in Figure 10.32 or by connecting a capacitor C from a suitable high resistance point to ground. The compensated transfer function A' becomes

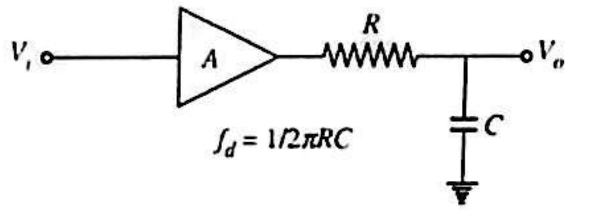

FIGURE 10.32 Dominant pole compensation.

$$A' = \frac{V_o}{V_i} = A\,\frac{\dfrac{-j}{\omega C}}{R - \dfrac{j}{\omega C}} = \frac{A}{1 + j\dfrac{f}{f_d}} \qquad (10.38)$$

where
$$f_d = \frac{1}{2\pi RC}$$

using Eq. (10.23), we get

$$A' = \frac{A_{OL}}{\left(1 + j\dfrac{f}{f_d}\right)\left(1 + j\dfrac{f}{f_1}\right)\left(1 + j\dfrac{f}{f_2}\right)\left(1 + j\dfrac{f}{f_3}\right)}; \qquad f_d < f_1 < f_2 < f_3$$

The capacitance C is chosen so that the modified loop gain drops to 0-dB with a slope of –20 dB/decade at a frequency where the poles of the uncompensated transfer function A contribute negligible phase shift. Usually $f_d = \omega_d/2\pi$ is selected so that the compensated transfer function A' passes through the 0-dB line at the pole f_1 of the uncompensated A. The frequency can be found graphically by pole f_1 of the uncompensated A. The frequency can be found graphically by having A' pass through the 0-dB line at the frequency f_1 with a slope of –20 dB per decade as shown in Figure 10.33. The value of capacitor C now can be calculated since $f_d = 1/2\pi RC$. The dominant-pole compensation technique reduces the open-loop bandwidth drastically. But the noise immunity of the system is improved.

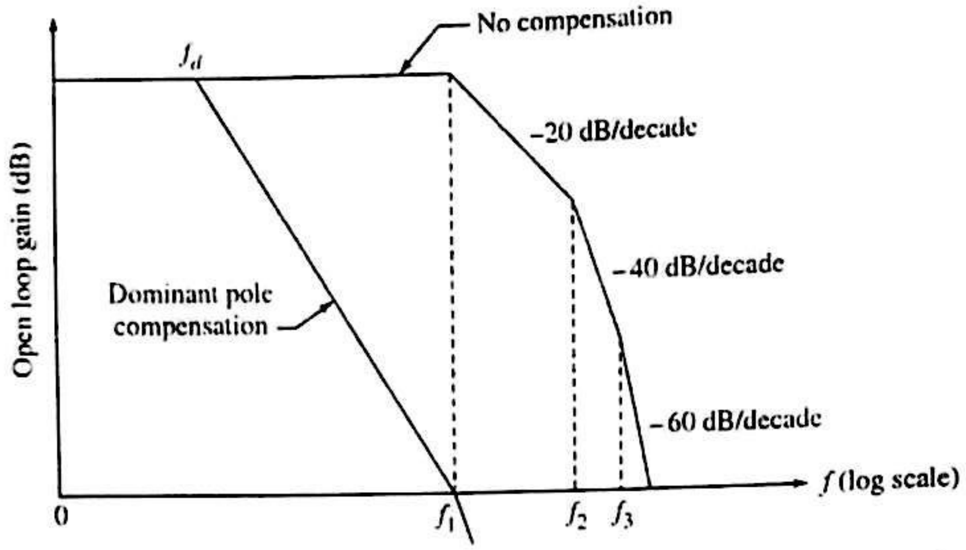

FIGURE 10.33 Gain vs. frequency curve for dominant pole compensation.

Pole-zero compensation: Here the uncompensated transfer function A is altered by adding both pole and zero as shown in Figure 10.34. The zero should be at a higher frequency than the pole. The transfer function of the compensating network alone is

$$\frac{V_o}{V_2} = \frac{Z_2}{Z_1 + Z_2} = \left(\frac{R_2}{R_1 + R_2}\right)\frac{1 + j\dfrac{f}{f_1}}{1 + j\dfrac{f}{f_0}} \tag{10.39}$$

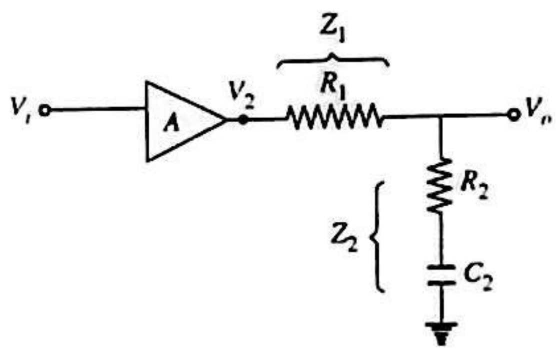

FIGURE 10.34 Pole-zero compensation.

where $\quad Z_1 = R_1;\ Z_2 = R_2 + \dfrac{1}{j\omega C_2};\ f_1 = \dfrac{1}{2\pi R_2 C_2};\ f_0 = \dfrac{1}{2\pi(R_1 + R_2)C_2}$

The compensating network is designed to produce a zero at the first corner frequency f_1 of the uncompensated transfer function A. This zero will cancel the effect of the pole at f_1. The pole of the compensating network at $f_0 = \omega_0/2\pi$ is selected so that the compensated transfer function A' passes through 0-dB at the second corner frequency f_2 of the uncompensated transfer function A in Eq. (10.23). The frequency can be found graphically by having A' pass through 0-dB at the frequency f_2 with a slope of –20 dB/decade as shown in Figure 10.35. Assuming that the compensating network does not load the amplifier, i.e. $R_2 \gg R_1$, then the overall transfer function becomes

$$A' = \frac{V_0}{V_2} = \frac{V_0}{V_2}\cdot\frac{V_2}{V_i} = A\left(\frac{R_2}{R_1 + R_2}\right)\frac{1 + j\dfrac{f}{f_1}}{1 + j\dfrac{f}{f_0}} \tag{10.40}$$

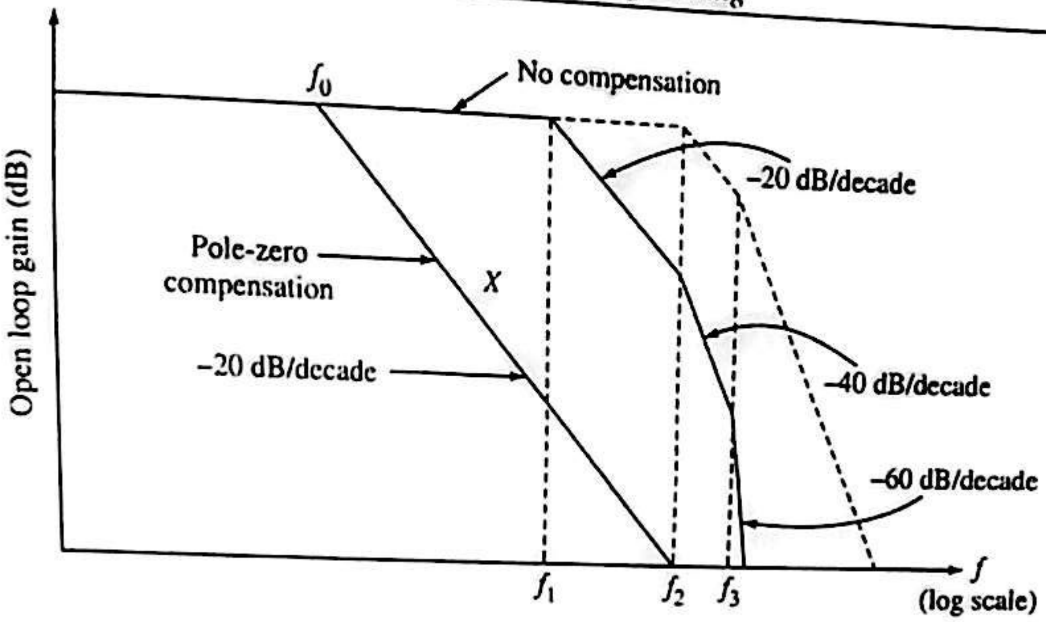

FIGURE 10.35 Open-loop gain vs. frequency for pole-zero compensation.

and note that $R_2 >> R_1$, so that $R_2/(R_1 + R_2) \approx 1$, then

$$A' = \frac{A_{OL}}{\left(1 + j\dfrac{f}{f_0}\right)\left(1 + j\dfrac{f}{f_2}\right)\left(1 + j\dfrac{f}{f_3}\right)} \quad ; \qquad 0 < f_0 < f_1 < f_2 < f_3$$

Consider again the frequency response (Bode plot) for the uncompensated op-amp having three poles at frequencies f_1, f_2 and f_3. Now select R_2 and C_2 so that the zero of the compensating network is equal to the pole at the frequency f_1 (lowest). If there had been no pole added by the compensating network, the response would have changed to that of the dotted curve in Figure 10.35. However, because of the predominance of the pole of the compensating network at f_0, the rate of closure will be −20 dB/decade throughout as shown in the curve X of Figure 10.35. The pole at f_0 should be selected so that the −20 dB/decade fall should meet the 0-dB line at f_2 which is the second pole of A.

Internally compensated operational amplifier

The type 741 op-amp is compensated and has an open-loop gain vs. frequency response as shown in Figure 10.36. The op-amp IC 741 contains a capacitance C_1 of 30 pF (see the manual), that internally shunts off the signal current and thus reduces the available output signal at higher frequencies. This internal capacitance, which is an internal compensating component, causes the open-loop gain to roll off at −20 dB/decade rate and thus assures for a stable circuit. The 741 op-amp has a 1 MHz gain bandwidth product. This means that the product of the coordinates, gain and

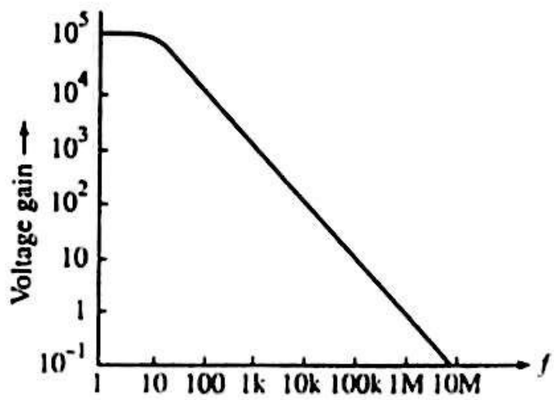

FIGURE 10.36 Frequency response of internally compensated MA741 op-amp

frequency, of any point on the open-loop gain vs. frequency curve is about 1 MHz. If the 741 op-amp is wired for a closed-loop gain of 10^4 or 80 dB, its bandwidth is 100 Hz as can be seen by projecting to the right from 10^4 in the curve of Figure 10.36. For gain of 10^2, the bandwidth increases to 10 kHz and for gain 1, the bandwidth is 1 MHz. For 741 op-amp, unity gain-bandwidth product is specified as 1 MHz in the data sheet. This simply means that op-amp 741 has 1 MHz bandwidth with unity gain as seen in Figure 10.36. Some internally compensated op-amps are Fairchild's μA741, National Semiconductor's LM741, LM107 and LM112, and Motorala's MC1558. Internally-compensated op-amp 741 is widely used and its compensation can be well understood only through the control systems point of view. That is why we have discussed the compensation of this op-amp in detail.

Summary

As we mentioned earlier, the design and compensation is limited to the classical approach and is applicable to the single-input-single-output linear time-invariant system. One has to be concerned about the environmental changes and accordingly take into account the margin of tolerances in the design. In actual design problems, we must choose the hardware where the design constraints such as cost, size, weight and reliability factors are addressed. For the multivariable system, we apply the modern control approaches, one of which (observability design) is discussed in Chapter 12.

Problems

10.1 Determine the transfer function of a lead compensator that will provide a phase lead of 45° and gain of 10 dB at $\omega = 8$ rad/s.

10.2 Determine the transfer function of a lead compensator that will provide a phase lead of 50° and gain of 8 dB at $\omega = 5$ rad/s.

10.3 While designing a suitable control system for a missile, it was found necessary to introduce a lead of 35° and gain 6.5 dB at $\omega = 2.8$ rad/s. What will be the transfer function of the lead compensator that will satisfy the above requirements?

10.4 Design a lag compensator that will provide a phase lag of 50° and an attenuation of 15 dB at $\omega = 2$ rad/s.

10.5 Design a lag compensator that will provide a phase lag of 45° and antenuation of 10 dB at $\omega =$ rad/s.

10.6 A lag compensator required for a position control system must provide a phase lag of 35° and attenuation of 8 dB at $\omega = 1.9$ rad/s. Determine the transfer function.

10.7 Determine the transfer function of a lag–lead compensator that will provide a phase lead of 50° and attenuation of 15 dB at $\omega = 6$ rad/s.

10.8 Determine the transfer function of a lag–lead compensator that will provide a phase lead of 55° and attenuation of 20 dB at $\omega = 4$ rad/s.