EEE-2K15

Fourier Series

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Trigonometric series: The infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$ is called

Trigonometric series where a_0, a_n and b_n are constants.

Periodic function: If $f(x+T) = f(x)$ then $f(x)$ is called

eriodic function and T is called its period.

Example: Since $\tan(\pi + x) = \tan x$, so $\tan x$ is a

riodic function and its period is π.

Fourier series: Let, $f(x)$ be defined in the interval

$(-\pi, \pi)$ with period 2π which can be expanded in a

rigonometric series.

i.e. $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{——} (1)$$

then (1) is called a Fourier series where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx$

Here, a_0, a_n and b_n are Fourier Coefficients.

Important integrals:

(i) $\int_{-\pi}^{\pi} \sin nx \, dx = 0$

(ii) $\int_{-\pi}^{\pi} \cos nx \, dx = 0$

(iii) $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$

(iv) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$

if $m \neq n$

(v) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$

(vi) $\int_{-\pi}^{\pi} \cos^2 nx \, dx = 0$

(vii) $\int_{-\pi}^{\pi} \sin^2 nx \, dx = 0.$

Theorem: If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ then,

(i) $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$ (ii) $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$ and

(iii) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$

Proof: Given that,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— (1)}$$

(1) Now integrating both sides of (1) w.r. to x between the limits $-\pi$ and π then we get,

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}(x)_{-\pi}^{\pi} + \sum_{n=1}^{\infty} a_n \cdot 0 + \sum_{n=1}^{\infty} b_n \cdot 0$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = a_0\pi + 0$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

(ii) Again multiplying both sides of (1) by $\cos nx$ and then integrat[e] w.r.t. x between the limits $-\pi$ and π, we get,

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx + a_n\int_{-\pi}^{\pi}\cos^2 nx\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{a_n}{2}\int_{-\pi}^{\pi}(1+\cos 2nx)\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_n}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_n}{2}(2\pi) + 0$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

(iii). Also multiplying both sides of (1) by $\sin nx$ and then integrating w. r. to x between the limits $-\pi$ and π, we get,

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin nx \, dx + b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 + \frac{b_n}{2} \int_{-\pi}^{\pi} (1 - \cos 2nx) \, dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{b_n}{2} \cdot \left[x - \frac{\sin 2nx}{2n} \right]_{-\pi}^{\pi}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{b_n}{2} [2\pi]$$

$$\Rightarrow b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

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Theorem: If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

then show that,

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

and $$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Try

Even function: If $f(-x) = f(x)$ then $f(x)$ is called an even function.

Example: let, $f(x) = \cos x$

$\therefore f(-x) = \cos(-x) = \cos x = f(x)$

Thus, $\cos x$ is a even function.

Odd function: If $f(-x) = -f(x)$ then $f(x)$ is called an odd function.

Example: let, $f(x) = \sin x$

$\therefore f(-x) = \sin(-x) = -\sin x = -f(x)$

Thus, $\sin x$ is a ~~of~~ odd function.

(2) If $f(x)$ is an even function then show that,

(i) $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$ (ii) $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$

and (iii) $b_n = 0$.

Proof: (i) we know,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_0^\pi f(x)\,dx\right\} \text{ --- (1)}$$

In the first integral of (1) we put $x=-y$ then $dx=-dy$

Limit: if $x=-\pi$, then $-\pi=-y \Rightarrow y=\pi$
if $x=0$ " $0=-y \Rightarrow y=0$

$$\therefore \int_{-\pi}^{0} f(x)\,dx = -\int_\pi^0 f(-y)\,dy$$

$$= \int_0^\pi f(-y)\,dy$$

$$= \int_0^\pi f(-x)\,dx \; ; \left[\text{since } \int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right]$$

$$= \int_0^\pi f(x)\,dx \quad (\text{since } f(x) \text{ is a even function}) \text{ --- (2)}$$

Now from (1) and (2) we get,

$$a_0 = \frac{1}{\pi}\left\{\int_0^\pi f(x)\,dx + \int_0^\pi f(x)\,dx\right\}$$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$$

(ii) Again we know,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\} \quad \text{--- (3)}$$

in the first integral of (3), we put $x=-y$ then $dx=-dy$

$$\therefore \int_{-\pi}^{0} f(x)\cos nx\,dx = -\int_{\pi}^{0} f(-y)\cos(-ny)\,dy$$

$$= \int_{0}^{\pi} f(-y)\cos(-ny)\,dy$$

$$= \int_{0}^{\pi} f(-y)\cos ny\,dy$$

$$= \int_{0}^{\pi} f(-x)\cos nx\,dx \quad \text{--- (4)}$$

$$= \int_{0}^{\pi} f(x)\cos nx\,dx$$

Now from (3) and (4) we get,

$$a_n = \frac{1}{\pi}\left\{\int_{0}^{\pi} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\}$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

(iii) Also we know,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\} \quad \text{--- (5)}$$

In the first integral of (5), we put $x=-y$ then $dx=-dy$

$$\therefore \int_{-\pi}^{0} f(x)\sin nx\,dx = -\int_{\pi}^{0} f(-y)\sin(-ny)\,dy$$

$$= \int_0^{\pi} f(-y) \sin(-ny)\, dy$$

$$= -\int_0^{\pi} f(-y) \sin(ny)\, dy$$

$$= -\int_0^{\pi} f(-x) \sin nx\, dx$$

$$= -\int_0^{\pi} f(x) \sin nx\, dx \text{ ——— (6)}$$ [f(x) is even f^n]

from (5) and (6) we get,

$$b_n = \frac{1}{\pi}\left\{ -\int_0^{\pi} f(x) \sin nx\, dx + \int_0^{\pi} f(x) \sin nx\, dx \right\}$$

$$\Rightarrow b_n = 0$$

Pb. If f(x) is an odd function then show that,

(i) $a_0 = 0$ (ii) $a_n = 0$ and (iii) $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

Try

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Half-range fourier series:

If only cosine terms or only sine terms are present in fourier series then it is called Half-range fourier serie

① If only cosine terms are present in fourier series then it is called Half-range cosine series. (fourier cosine series)

i.e. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

(ii) If only sine terms are present in fourier series then it is called Half-range sine series, (Fourier sine se

i.e. $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

The function is generally defined in $(0,\pi)$ which is hal of $(-\pi,\pi)$ for this reason it is called Half-range.

13. Fourier series in different interval:

(i) If the function $f(x)$ is defined in the interval $(0, \pi)$ with Period π then the fourier series as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$

and $b_n = \frac{2}{\pi} \int_0^{\pi} \sin nx\, dx$

(ii) If the function $f(x)$ is defined in the interval $(0, 2\pi)$ with period 2π then the fourier series as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$, $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$

and $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$

(iii) If the function $f(x)$ is defined in the interval $(-l, l)$ with period $2l$ then the fourier series as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx$

and $b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\,dx.$

(iv) If the function $f(x)$ is defined in the interval (a, b) with period $b-a$ then the fourier series as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

where, $a_0 = \frac{2}{b-a}\int_{a}^{b} f(x)\,dx$

$$a_n = \frac{2}{b-a}\int_{a}^{b} f(x) \cos\frac{2n\pi x}{b-a}\,dx$$

and $b_n = \frac{2}{b-a}\int_{a}^{b} f(x) \sin\frac{2n\pi x}{b-a}\,dx$

(4) Expand the function $f(x) = x + x^2$ in the sine and cosine series in the interval $-\pi < x < \pi$. from this show that, $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$

Solution: Part (1): we have $f(x) = x + x^2$; $-\pi < x < \pi$ ——— (1)

In the interval $-\pi < x < \pi$, the fourier series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{——— (2)}$$

when, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2)\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ \left(\frac{\pi^2}{2} + \frac{\pi^3}{3} \right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3} \right) \right\}$$

$$= \frac{1}{\pi} \cdot \frac{2\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx\, dx$$

$$\Rightarrow a_n = \frac{1}{\pi} \left[(x + x^2) \cdot \frac{\sin nx}{n} - (1 + 2x) \cdot \left(-\frac{\cos nx}{n^2} \right) + (2) \left(-\frac{\sin nx}{n^3} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{1}{n} (x + x^2) \sin nx + \frac{1}{n^2} (1 + 2x) \cos nx - \frac{2}{n^3} \sin nx \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \cdot \frac{(-1)^n}{n^2} (1 + 2\pi - 1 + 2\pi)$$

$$= \frac{4\pi (-1)^n}{\pi n^2} = \frac{4(-1)^n}{n^2} \quad ; \left[\text{since, } \sin n\pi = 0,\ \cos n\pi = (-1)^n\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (1+2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(x+x^2)\cos nx + \frac{1}{n^2}(1+2x)\sin nx + \frac{2}{n^3}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{-2.(-1)^n}{n}$$

Now putting the values of a_0, a_n, b_n in (2) then we get

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\sin nx$$

$$\Rightarrow f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2} - 2\sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n}$$

$$\Rightarrow f(x) = \frac{\pi^2}{3} + 4\left(-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\cdots\right)$$
$$-2\left(-\frac{\sin nx}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \cdots\cdots\right)$$

$$\Rightarrow f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\cdots\right)$$
$$+2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\cdots\right) \quad (3)$$

At the end points $x = \pm\pi$

$$f(\pi) = \frac{1}{2}[f(-\pi+0) + f(\pi-0)]$$
$$= \frac{1}{2}[-\pi+\pi^2+\pi+\pi^2] \text{; since } f(x) = x+x^2$$
$$= \pi^2 \quad \text{——— (4)}$$

Now putting $x = \pi$ in (3), we get

$$f(\pi) = \frac{\pi^2}{3} - 4\left(\frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \cdots\cdots\right) + 2.0$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4.\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots\right)$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots\right)$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots\right)$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots$$

(Proved)

Q. If $f(x) = x$ in the interval $0 \le x \le \pi$, then expand the function in cosine series of the multiple angle of x and hence show that, $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\cdots$

Try