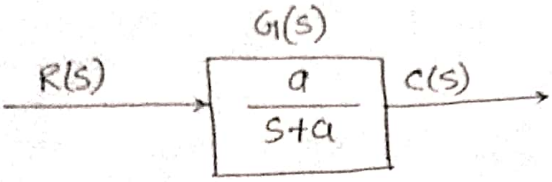
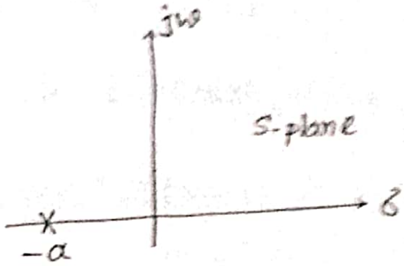
Time Response

Ch-4 (Nise) Time Response

ch-5 (ogata) Transient & Steady State Response.

4.3 First Order System (Nise P-166)





Here, $G(s) = \frac{C(s)}{R(s)} = \frac{a}{a+s}$.

$\therefore$ O/p $C(s) = G(s)\,R(s) = \frac{a}{s(s+a)}$ [for unit step i/p $R(s) = \frac{1}{s}$].

$\therefore C(s) = \frac{1}{s} - \frac{1}{s+a}$

$\therefore c(t) = 1 - e^{-at} = c_f(t) - c_n(t)$ . . . (i)

forced response (1); Natural Response (e^{-at})

Time constant $(=\frac{1}{a})$.

Put $t = \frac{1}{a}$ in (i) we get,

$c(t)\big|_{t=\frac{1}{a}} = 1 - e^{-1} = 1 - 0.37 = 0.63$

Time constant is the time taken for the step response to rise to 63% of its final value.

Rise time, T_r

T_r is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

We know, $c(t) = 1 - e^{-at}$

$\therefore \cancel{c(0.9) - c(0.1) =}$

Now, $c(t) = 0.9$

$\Rightarrow 1 - e^{-at} = 0.9$

$\Rightarrow e^{-at} = 0.1$

$\Rightarrow -at = \ln 0.1 = -2.31$

$\therefore t_{(0.9)} = \dfrac{2.31}{a}$

Again, $c(t) = 0.1$

$\Rightarrow 1 - e^{-at} = 0.1$

$\Rightarrow e^{-at} = 0.9$

$\Rightarrow -at = \ln 0.9 = -0.11$

$\therefore t_{(0.1)} = \dfrac{0.11}{a}$

$$\therefore T_r = t_{0.9} - t_{0.1}$$

$$= \frac{2.31}{a} - \frac{0.11}{a}$$

$$= \frac{2.2}{a}$$

Settling Time, T_s

T_s is the time to reach 98% of its final value.

$\therefore c(t) = 0.98$

$\Rightarrow 1 - e^{-at} = 0.98$

$\Rightarrow e^{-at} = 0.02$

$\Rightarrow -at = \ln 0.02$

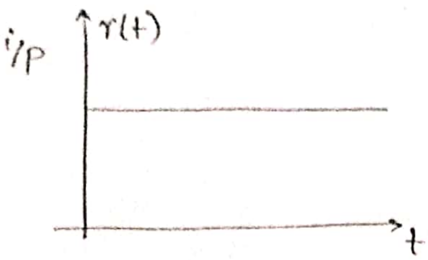
$\Rightarrow -at \cong -3.91 \approx -4$

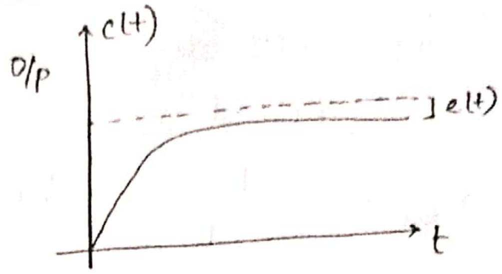
$\Rightarrow t = \dfrac{4}{a}$

$\therefore T_s = \dfrac{4}{a}$

Steady - State Error for

Unit Step Response of First Order Systems. (Ogata, Ch-5, P-161)





Error signal $e(t) = r(t) - c(t)$

$$= 1 - (1 - e^{-at})$$

$$= e^{-at}$$

Unit-ramp response of First-Order Systems. (Ogata, P-162)

Reference
Input $r(t) = t$

$$\therefore R(s) = \frac{1}{s^2}$$

Output $c(s) = \frac{1}{\tau s + 1} \cdot \frac{1}{s^2}$ $\left[\tau = \frac{1}{a}, \text{ Ogata} \Leftrightarrow \text{Nise}\right]$.

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

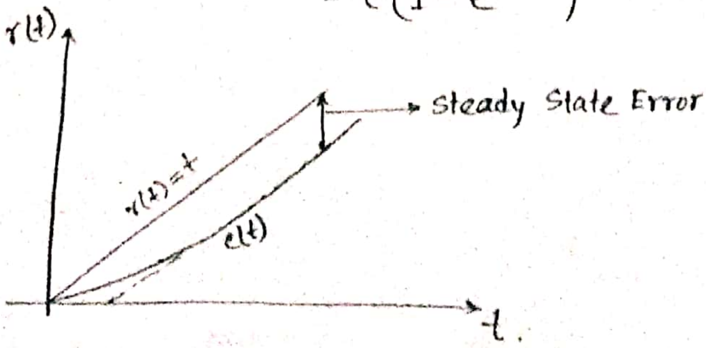
$$\Rightarrow c(t) = t - \tau + \tau e^{-t/\tau}, \quad \text{for } t \geq 0$$

$\therefore$ The error signal $e(t) = r(t) - c(t)$

$$= t - (t - \tau + \tau e^{-t/\tau})$$

$$= \tau - \tau e^{-t/\tau}$$

$$= \tau(1 - e^{-t/\tau})$$



At $e(\infty) = \tau$.

So, the smaller the time constant, the smaller the steady state error in ramp input.

Unit Impulse Response (Ogata, P-163)

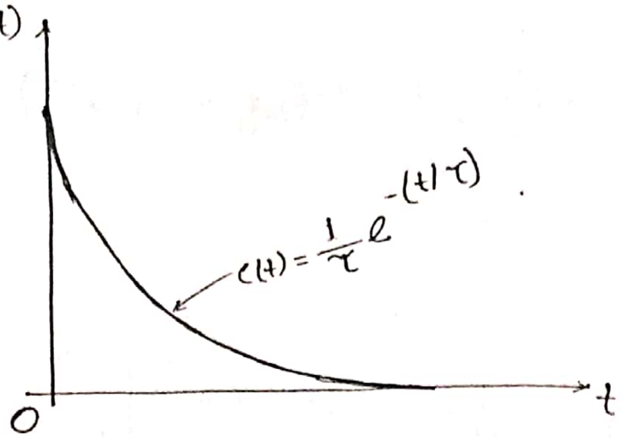
Reference input $r(t) = \delta(t)$

$$\therefore R(s) = 1$$

$$\therefore \text{Output } C(s) = \frac{1}{\tau s + 1} = \frac{1/\tau}{s + 1/\tau}$$

$$= \frac{1}{\tau} e^{-\frac{t}{\tau}} \quad \text{for } t \geq 0.$$

Response Curve.



An important Property:

For ramp-input output $c(t)_{ramp} = t - \tau + \tau e^{-t/\tau}$

Now, $\frac{dc(t)_{ramp}}{dt} = 1 - e^{-t/\tau} = 1 - e^{-at} = c(t)_{step}$

Again, $\frac{dc(t)_{step}}{dt} = \frac{1}{\tau} e^{-t/\tau} = c(t)_{ramp}.$

Second Order System:

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

ω_n = undamped natural frequency.

ξ = damping ratio = $\frac{\text{Expotential decay frequency}}{\text{Natural Frequency}}$.

If $\xi = 0 \rightarrow$ undamped system.

$\xi = 1 \rightarrow$ Critically damped.

$0 < \xi < 1 \rightarrow$ Under-damped system.

$\xi > 1 \rightarrow$ Over-damped system.

Let us relate ω_n & ξ in pole location,

$$\underset{a}{s^2} + \underset{b}{2\xi\omega_n} s + \underset{c}{\omega_n^2} = 0$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}.$$

For $\xi = 0$, $s_{1,2} = \pm j\omega_n$, Two imaginary

$\xi = 1$, $s_{1,2} = -\omega_n$, Purely Real but same.

$0 < \xi < 1$, $s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$

$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$= -\xi\omega_n \pm j\omega_d$

= Two complex conjugates

$\xi > 1$, $s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$.

= Two real & distinct

Notes on ξ :

Consider $G(s) = \dfrac{K}{Js^2 + Bs + k}$ (ogata P-165, Nise P-174)

$$= \frac{k/J}{\left[s + B/2J + \sqrt{(B/2J)^2 - k/J}\,\right]\left[s + B/2J - \sqrt{(B/2J)^2 - k/J}\,\right]}$$

Poles are complex conjugate if $B^2 - 4Jk < 0$ **

and real if $B^2 - 4Jk > 0$

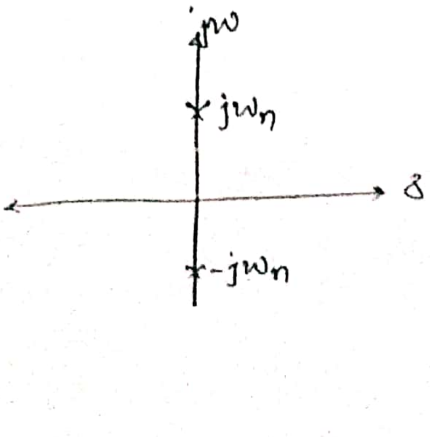
and critical $B^2 - 4Jk = 0$

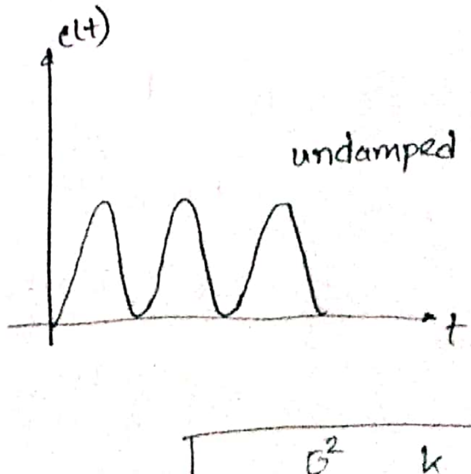
$\therefore B_c = 2\sqrt{Jk}$.

Damping ratio ξ is the ratio of actual damping B to critical damping $B_c = 2\sqrt{2k}$.

$$\therefore \xi = \frac{B}{B_c} = \frac{B}{2\sqrt{Jk}}.$$

* For $G(s) = \dfrac{b}{s^2 + as + b}$, the natural frequency $\omega_n = \sqrt{b}$ is found when $a = 0$. The roots are on $j\omega$ (imaginary) axis as $\pm j\sqrt{b}$.





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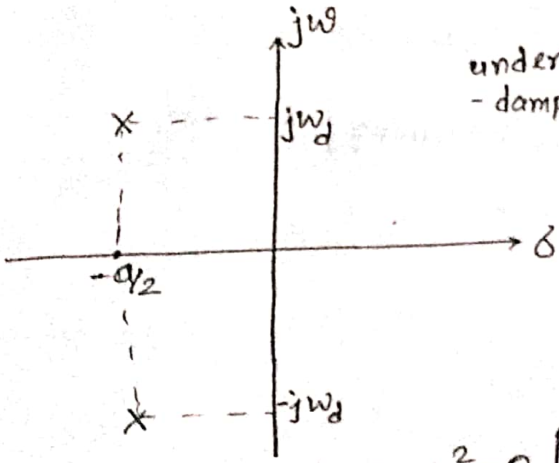
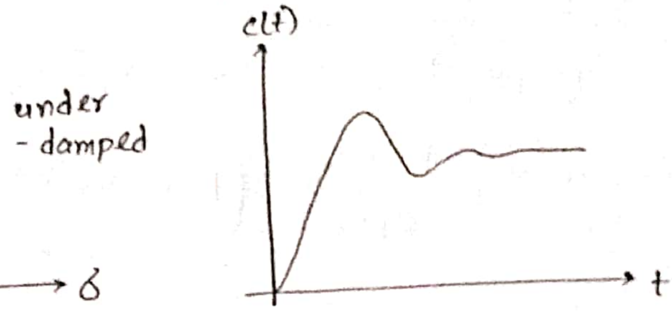
$$\boxed{\begin{aligned} &\frac{B^2}{4J^2} - \frac{k}{J} \\ &= \frac{B^2 - 4Jk}{4J^2} \end{aligned}}$$

Again the ~~pole~~ system $G(s) = \dfrac{b}{s^2+as+b}$ becomes underdamped the pole roots have two complex conjugates having real part at $-a/2$. The magnitude of this value is the expotenatial decay frequency (δ)

* For roots $-\delta_d \pm j\omega_d$.

$$c(t)_{\text{under damped}} = A e^{-\delta_d t} \cos(\omega_d t - \phi)$$





we have, $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ $[\Rightarrow a = 2\xi\omega_n]$

Here, $\xi = \dfrac{a/2}{\omega_n} = \dfrac{\text{Expotential Decay Frequency}}{\text{Natural Frequency}}$

Time Response of Undamped 2nd Order System for step input

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2}$$

$$\Rightarrow C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + \omega_n^2} \qquad [\xi = 0]$$

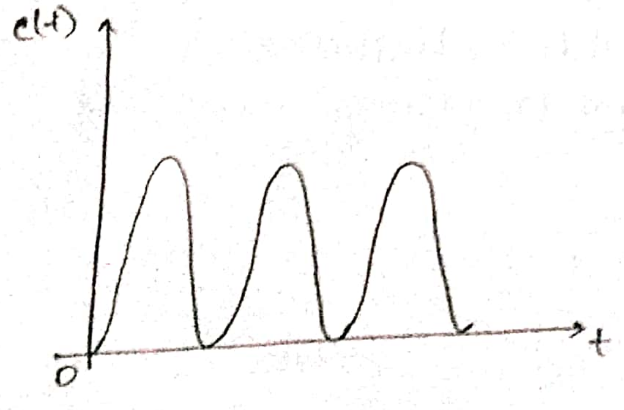
$$\Rightarrow C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + \omega_n^2} \qquad [R(s) = \text{step input}]$$

$$= \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 + \omega_n^2} \qquad [\text{Roots are imaginary}]$$

$$= \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$$\Rightarrow c(t) = 1 - \cos\omega_n t. \quad \text{for } t \geq 0$$



Time Response of
Underdamped 2nd order System (For unit step input) [ogata, P-166].

We have, $G(s) = \frac{C(s)}{R(s)} = \frac{w_n^2}{s^2+2\xi w_n s + w_n^2}$. $[0<\xi<1]$.

$$\Rightarrow C(s) = \frac{w_n^2}{s(s^2+2\xi w_n s + w_n^2)} = \frac{k_1}{s} + \frac{k_2 s + k_3}{s^2+2\xi w_n s + w_n^2}$$ [Roots are imaginary complex conjugate]

$$\Rightarrow C(s) = \frac{1}{s} - \frac{s+2\xi w_n}{\boxed{s^2+2\xi w_n s + w_n^2}}$$

$s_{1,2} = -\xi w_n \pm j w_n\sqrt{1-\xi^2}$
$= -\xi w_n \pm j w_d \; [w_d = w_n\sqrt{1-\xi^2}]$.

$$\Rightarrow C(s) = \frac{1}{s} - \frac{s+\xi w_n + \xi w_n}{(s+\xi w_n + j w_d)(s+\xi w_n - j w_d)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{s+\xi w_n + \xi w_n}{(s+\xi w_n)^2 + w_d^2}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{s+\xi w_n}{(s+\xi w_n)^2 + w_d^2} - \frac{\xi w_n}{(s+\xi w_n)^2 + w_d^2}$$

We know,

$$\mathcal{L}\left[Ae^{-at}\cos wt\right] = \frac{A(s+a)}{(s+a)^2+w^2}$$

$$\mathcal{L}^{-1}\left[\frac{1.(s+\xi w_n)}{(s+\xi w_n)^2+w_d^2}\right] = e^{-\xi w_n t}\cos w_d t.$$

Again.

$$\mathcal{L}\left[Be^{-at}\sin wt\right] = \frac{Bw}{(s+a)^2+w^2}.$$

Now,

$$A(s) = \frac{\xi w_n}{(s+\xi w_n)^2+w_d^2} = \frac{\xi w_n\left(\frac{w_d}{w_d}\right)}{(s+\xi w_n)^2+w_d^2} = \frac{\xi w_n}{w_d}\cdot\frac{w_d}{(s+\xi w_n)^2+w_d^2}.$$

$$\therefore \mathcal{L}^{-1}[A(s)] = \frac{\xi w_n}{w_d}\cdot e^{-\xi w_n t}\sin w_d t.$$

Hence,

$$c(t) = \mathcal{L}^{-1}[C(s)]$$

$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}\sin\omega_d t \ast e^{-\xi\omega_n t} \quad \text{for } t \geq 0$$

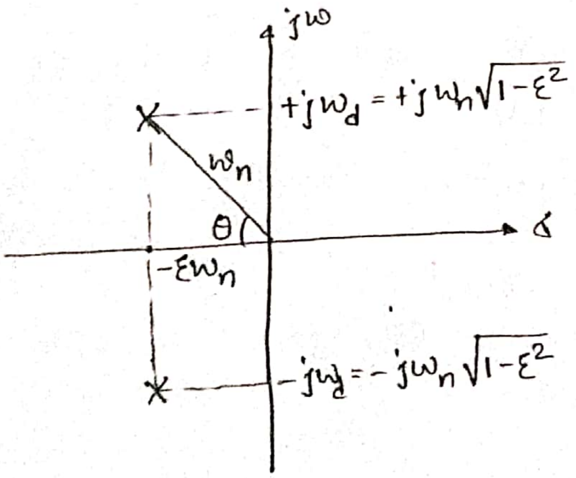
$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t} \cdot \sin\omega_d t.$$

$$= 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$\cancel{= 1 - e^{-\xi\omega_n t}/\sqrt{1-\xi^2}\,(\cos\omega_d t}$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cdot\cos\omega_d t + \xi\sin\omega_d t\right) \text{ ---- (A).}$$

Now we observe roots of the poles $s_{1,2} = -\xi\omega_n \pm j\omega_d$.



$$\tan\theta = \frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n} = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\text{Hypotenuse} = \sqrt{\xi^2\omega_n^2 + \omega_n^2(1-\xi^2)}$$

$$= \omega_n.$$

$$\sin\theta = \frac{\omega_n\sqrt{1-\xi^2}}{\omega_n} = \sqrt{1-\xi^2}.$$

$$\cos\theta = \frac{\xi\omega_n}{\omega_n} = \xi.$$

we have, $\tan\theta = \dfrac{\sqrt{1-\xi^2}}{\xi}$

$\sin\theta = \sqrt{1-\xi^2}$

$\cos\theta = \xi.$

Putting the value of $\sin\theta, \cos\theta$ in (A) we get,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\theta . \cos\omega_d t + \cos\theta . \sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \text{ for } t \geq 0 \text{ where } \theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

and $\omega_d = \omega_n\sqrt{1-\xi^2}$.

which is the unit step response for 2^{nd} order system. $(0<\xi<1)$

Eventually the response becomes undamped for $\xi = 0$.

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2} . t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right).$$

$$= 1 - \sin\left(\frac{\pi}{2} + \omega_n t\right)$$

$= 1 - \cos\omega_n t$. which is ~~an~~ undamped response.

* See figure 5-10 (Ogata, P-173) The response curve [Print]

Parameters associated with underdamped response.

1. Rise Time T_r
2. Peak Time T_P
3. Percentage Overshoot $\%OS$
4. Settling time T_s..
5. Delay Time T_d.

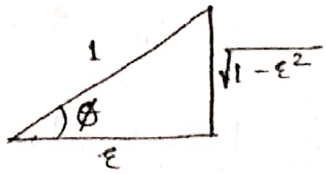
Rise Time T_r: (Ogata, P-171)

Time to reach 0% to 100% of its final value.

We have

$$c(t) = c(t_r) = 1 - \frac{e^{-\xi w_n t}}{\sqrt{1-\xi^2}} \sin(w_d t + \phi)$$

$$\Rightarrow 1 = 1 - \frac{e^{-\xi w_n t_r}}{\sqrt{1-\xi^2}} \sin(w_d t_r + \phi).$$



$$\Rightarrow \sin(w_d t_r + \phi) = 0$$

$$\Rightarrow \sin(w_d t_r + \phi) = \sin\pi \quad (\text{taking } n\pi, n=1).$$

$$\Rightarrow w_d t_r + \phi = \pi$$

$$\Rightarrow w_d t_r = \pi - \phi$$

$$\Rightarrow t_r = \frac{\pi - \phi}{w_d}$$

$$\Rightarrow t_r = \frac{\pi - \phi}{w_n\sqrt{1-\xi^2}}.$$

Peak Time T_p (Nise, P-179)

Step i/p.

$c(s) = T(s)R(s)$

$= \frac{1}{s} \cdot T(s)$.

$\therefore T(s) = sC(s)$.

Time to reach first / maximum peak.

T_p can be found by differentiating $c(t)$.

We know,

$$\mathcal{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \cdot = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \xi^2\omega_n^2 + \omega_n^2 - \xi^2\omega_n^2}$$

$$\Rightarrow \mathcal{L}[\dot{c}(t)] = \frac{\omega_n^2}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)} = \frac{\frac{\omega_n}{\sqrt{1-\xi^2}} \cdot \omega_n\sqrt{1-\xi^2}}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}$$

$$\therefore \dot{c}(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cdot \sin\omega_n\sqrt{1-\xi^2}\, t.$$

At peak, $\dot{c}(t) = 0$

$$\Rightarrow \sin\omega_n\sqrt{1-\xi^2}\, t = 0 = \sin n\pi$$

$$\Rightarrow \omega_n\sqrt{1-\xi^2}\, t = n\pi$$

$$\therefore t = \frac{n\pi}{\omega_n\sqrt{1-\xi^2}}$$

Hence peak time $T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$ [$n = 1$] The first Peak.

$\%OS = e^{-(\xi\pi/\sqrt{1-\xi^2})} \times 100$ (Nise, P-180) [self]

$$= \frac{c_{max} - c_{final}}{c_{final}} \times 100.$$

Settling Time T_s : Stays within $\pm 2\%$ of steady-state or final value.

$$T_s = \frac{4}{\xi\omega_n} \quad \text{[Nise, P-181] [self]}$$

$$* \; e^{-\xi\omega_n T_s} \cdot \frac{1}{\sqrt{1-\xi^2}} = 0.02$$

Maximum Peak Overshoot M_p, %OS

Maximum overshoot occurs at $t = T_p$, where $T_p = \frac{\pi}{\omega_n\sqrt{1-\varepsilon^2}} = \frac{\pi}{\omega_d}$

And $c_{final} = c(\infty) = 1 - \frac{e^{-\varepsilon\omega_n t}}{\sqrt{1-\varepsilon^2}}\left[\sin(\omega_d t+\theta)\right]$

$$= 1 - \frac{e^{-\infty}}{\sqrt{1-\varepsilon^2}}\left[\sin(\infty)\right] \quad [\because t \to \infty]$$

$$= 1.$$

$$M_p = \frac{c_{max} - c_{final}}{c_{final}} \times 100\%$$

$$= \frac{c(T_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$= \frac{c(T_p) - 1}{1} \times 100\%$$

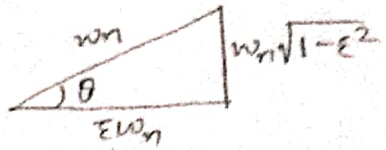
$$= (c(T_p) - 1) \times 100\%$$

$$\therefore M_p = 1 - \frac{e^{-\varepsilon\omega_n T_p}}{\sqrt{1-\varepsilon^2}}\sin(\omega_d T_p + \theta) - 1.$$

$$= 1 - \frac{e^{-\varepsilon\omega_n \cdot \frac{\pi}{\omega_n\sqrt{1-\varepsilon^2}}}}{\sqrt{1-\varepsilon^2}}\sin\left(\omega_d \cdot \frac{\pi}{\omega_d} + \theta\right) - 1.$$

$$= \cancel{1} - \frac{e^{-\frac{\varepsilon\pi}{\sqrt{1-\varepsilon^2}}}}{\sqrt{1-\varepsilon^2}}\sin(\pi+\theta) - \cancel{1}.$$

$$= -\frac{e^{-\frac{\varepsilon\pi}{\sqrt{1-\varepsilon^2}}}}{\sqrt{1-\varepsilon^2}}\left[-\sin\theta\right].$$



$\sin\theta = \sqrt{1-\varepsilon^2}$.

$$= \frac{e^{-\varepsilon\pi/\sqrt{1-\varepsilon^2}}}{\sqrt{1-\varepsilon^2}} \times \sin\theta$$

$$= e^{-\varepsilon\pi/\sqrt{1-\varepsilon^2}} \quad [\because \sin\theta = \sqrt{1-\varepsilon^2}].$$

$$\therefore \%OS = e^{-\varepsilon\pi/\sqrt{1-\varepsilon^2}} \times 100\%$$

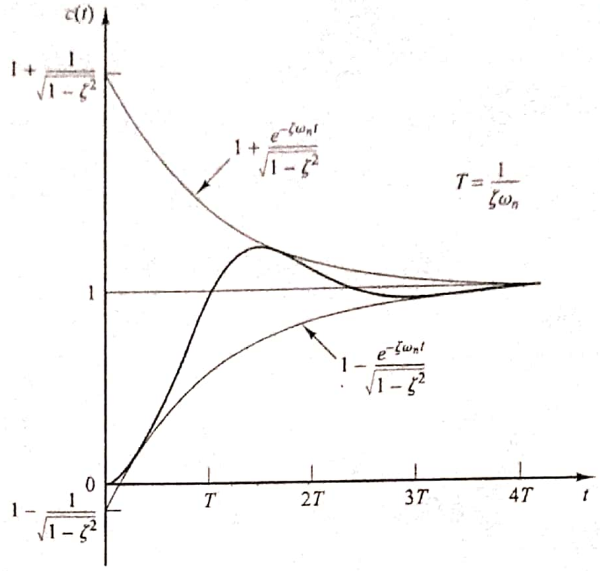


Figure 5–10 Pair of envelope curves for the unit-step response curve of the system shown in Figure 5–6.

The curves $1 \pm (e^{-\zeta\omega_n t}/\sqrt{1-\zeta^2})$ are the envelope curves of the transient response to a unit-step input. The response curve $c(t)$ always remains within a pair of the envelope curves, as shown in Figure 5–10. The time constant of these envelope curves is $1/\zeta\omega_n$.

The speed of decay of the transient response depends on the value of the time constant $1/\zeta\omega_n$. For a given ω_n, the settling time t_s is a function of the damping ratio ζ. From Figure 5–7, we see that for the same ω_n and for a range of ζ between 0 and 1 the settling time t_s for a very lightly damped system is larger than that for a properly damped system. For an overdamped system, the settling time t_s becomes large because of the sluggish response.

The settling time corresponding to a $\pm 2\%$ or $\pm 5\%$ tolerance band may be measured in terms of the time constant $T = 1/\zeta\omega_n$ from the curves of Figure 5–7 for different values of ζ. The results are shown in Figure 5–11. For $0 < \zeta < 0.9$, if the 2% criterion is used, t_s is approximately four times the time constant of the system. If the 5% criterion is used, then t_s is approximately three times the time constant. Note that the settling time reaches a minimum value around $\zeta = 0.76$ (for the 2% criterion) or $\zeta = 0.68$ (for the 5% criterion) and then increases almost linearly for large values of ζ. The discontinuities in the curves of Figure 5–11 arise because an infinitesimal change in the value of ζ can cause a finite change in the settling time.

For convenience in comparing the responses of systems, we commonly define the settling time t_s to be

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad \text{(2\% criterion)} \tag{5–22}$$

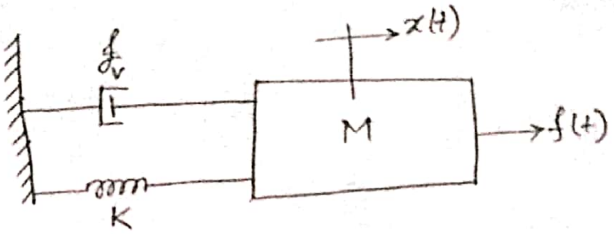
or

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad \text{(5\% criterion)} \tag{5–23}$$

Note that the settling time is inversely proportional to the product of the damping ratio and the undamped natural frequency of the system. Since the value of ζ is usually determined from the requirement of permissible maximum overshoot, the settling time

Exercise 70 (Nise, P-226)

Condider the translational mechanical system as shown in figure. A 1lb force $f(t)$ is applied at $t=0$. If $f_v=1$, find K and M such that the response is characterized by a 4sec settling time and a 1sec peak time. Also, what is the resulting %OS ?



Sol^n:- we know the transfer function $G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$

[Ex 2.16 (P-63)].

Given,

$f(t) = 1$ lb

$f_v = 1$

$T_s = 4$sec

$T_p = 1$sec

$K = ?$

$M = ?$

$\%OS = ?$

Now,

$$G(s) = \frac{1}{Ms^2 + f_v s + k}$$

$$= \frac{\frac{1}{M}}{s^2 + \frac{f_v}{M}s + \frac{K}{M}}$$

$$= \frac{\frac{1}{M}}{s^2 + \frac{1}{M}s + \frac{K}{M}} \quad [\because f_v = 1] \quad \text{--- (i)}$$

Now the roots of G(s) denominator (poles)

$$s_{1,2} = \frac{-\frac{1}{M} \pm \sqrt{\frac{1}{M^2} - \frac{4k}{M}}}{2}$$

$$= -\frac{1}{2M} \pm \frac{1}{2}\sqrt{\frac{1}{M^2} - \frac{4k}{M}} \quad \text{--- (ii)}$$

we know,

Settling Time $T_s = \frac{4}{\xi\omega_n} = \frac{4}{|Re|} = \frac{4}{1/2M} = 4sec.$

$\Rightarrow 4 = 8M$

$\therefore M = \frac{1}{2}$ (Ans)

Putting $M = \frac{1}{2}$ in equ^n (i) we get,

$$G(s) = \frac{2}{s^2 + 2s + 2k} \quad \text{--- (iii)}$$

$\therefore$ Poles in (ii)

$$S_{1,2} = -\frac{1}{2M} \pm \frac{1}{2}\sqrt{\frac{1}{M^2} - \frac{4k}{M}}$$

$$= -\frac{1}{2\times\frac{1}{2}} \pm \frac{1}{2}\sqrt{\frac{1}{1/4} - \frac{4k}{1/2}}$$

$$= -1 \pm \frac{1}{2}\sqrt{4-8k}$$

$$= -1 \pm \frac{1}{2}\sqrt{(-4)(2k-1)}$$

$$= -1 \pm \frac{1}{2}\sqrt{(j)^2(4)(2k-1)}$$

$$= -1 \pm \frac{2j}{2}\sqrt{(2k-1)}$$

$$= -1 \pm j\sqrt{2k-1}.$$

Given, peak time $T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{|Im|} = \frac{\pi}{\sqrt{2k-1}} = 1sec.$

$\Rightarrow \pi = \sqrt{2k-1}.$

$\Rightarrow 2k-1 = \pi^2$

$\therefore k = 5.43.$ (Ans)

Scanned by CamScanner

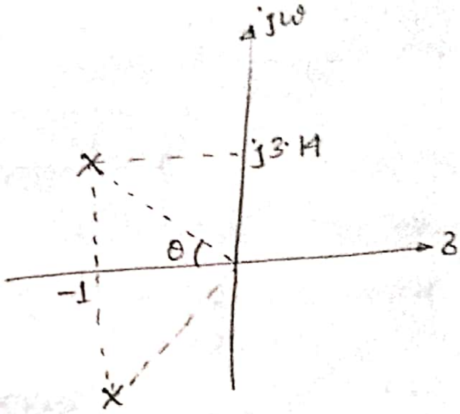
Hence the poles are,

$$S_{1,2} = -\frac{1}{2M} \pm \frac{1}{2}\sqrt{\frac{1}{M^2} - \frac{4k}{M}}$$

$$= -1 \pm \frac{1}{2}\sqrt{4 - 8k} \quad [M = \tfrac{1}{2}].$$

$$= -1 \pm \frac{1}{2}\sqrt{-39.44}$$

$$= -1 \pm j3.14.$$



We know, $\xi = \cos\theta$

$$\Rightarrow \xi = \frac{1}{\sqrt{1^2 + 3.14^2}} = 0.303.$$

The system is underdamped.

$$\therefore \%OS = e^{-(\pi\xi/\sqrt{1-\xi^2})} \times 100$$

$$= 36.83\% \text{ (Ans.)}.$$

*** Alternate:

Denominator of $G(s)$ at equ^n (iii) for $k = 5.43$ is $s^2 + 2s + 10.86$

Now equating $s^2 + 2\xi\omega_n s + \omega_n^2$ we get,

$$\omega_n = \sqrt{10.86} = 3.295$$

And, $2\xi\omega_n = 2$

$$\therefore \xi = \frac{1}{\omega_n}$$

$$= 0.303 \text{ (Ans.)}.$$

Exercise 56. P-222 (Fluid Filled Catheter) (Nise).

The systems seems to be underdamped system. (Figure P4.13)

We know $c(t) = 1 - \frac{e^{-\varepsilon\omega_n t}}{\sqrt{1-\varepsilon^2}} \cdot \sin(\omega_d t + \theta)$.

Here, $\omega_d = \omega_n\sqrt{1-\varepsilon^2}$ is the period of oscillation $\left(\frac{2\pi}{T}\right)$.

Hence, Period of oscillation $\frac{2\pi}{T} = \omega_n\sqrt{1-\varepsilon^2}$.

From figure we get, $\frac{T}{2} = (0.0674 - 0.0505)\,sec = 0.0169\,sec$

$$\boxed{\therefore T = 0.0338\ sec}$$

The peaks of the response occur when "sin" term is ± 1.

Hence from figure we get,

$$1 + \frac{e^{-\varepsilon\omega_n(0.0505)}}{\sqrt{1-\varepsilon^2}} = 1.15$$

$$\Rightarrow e^{-\varepsilon\omega_n(0.0505)} = 0.15\left(\sqrt{1-\varepsilon^2}\right)$$

And,

$$1 - \frac{e^{-\varepsilon\omega_n(0.0674)}}{\sqrt{1-\varepsilon^2}} = 0.923$$

$$\Rightarrow e^{-\varepsilon\omega_n(0.0674)} = 0.077\left(\sqrt{1-\varepsilon^2}\right).$$

From which we get,

$$\frac{e^{-\varepsilon\omega_n(0.0505)}}{e^{-\varepsilon\omega_n(0.0674)}} = \frac{0.15}{0.077}.$$

$$\Rightarrow e^{+\varepsilon\omega_n(0.0169)} = 1.95$$

$$\Rightarrow +0.0169\,\varepsilon\omega_n = 0.6678$$

$$\Rightarrow +\varepsilon\omega_n = 39.51. \Rightarrow \boxed{\omega_n = \frac{39.51}{\varepsilon}}.$$

We have, $T = 0.0338$ sec.

$$\therefore \omega_n\sqrt{1-\varepsilon^2} = \frac{2\pi}{T} = 185.89$$

$$\Rightarrow \frac{39.51}{\varepsilon}\sqrt{1-\varepsilon^2} = 185.89$$

$$\Rightarrow \frac{1561}{\varepsilon^2}(1-\varepsilon^2) = (185.89)^2$$

$$\Rightarrow 1561(1-\varepsilon^2) = (185.89\varepsilon)^2$$

$$\Rightarrow 1561 - 1561\varepsilon^2 - (185.89\varepsilon)^2 = 0.$$

$$\Rightarrow 1561\varepsilon^2 + (185.89\varepsilon)^2 - 1561 = 0$$

$$\Rightarrow 36116.1\,\varepsilon^2 = 1561.$$

$$\therefore \boxed{\varepsilon = 0.208}.$$

And, $\boxed{\omega_n = \frac{39.51}{\varepsilon} = 190.8}$

$$\therefore T(s) = \frac{\omega_n^2}{s^2 + 2\varepsilon\omega_n s + \omega_n^2}$$

$$= \frac{(190.8)^2}{s^2 + 79s + (190.8)^2}.\ (Ans).$$

MATLAB

ML

Students who are using MATLAB should now run ch4p1 in Appendix B. You will learn how to generate a second-order polynomial from two complex poles as well as extract and use the coefficients of the polynomial to calculate T_p, %OS, and T_s. This exercise uses MATLAB to solve the problem in Example 4.6.

Example 4.7

Transient Response Through Component Design

Design

D

PROBLEM: Given the system shown in Figure 4.21, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque $T(t)$.

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100.$$

$$\Rightarrow \frac{20}{100} = e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$$

$$\Rightarrow -\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(0.2)$$

$$\Rightarrow -\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = -1.6$$

$$\therefore \zeta = 0.45.$$

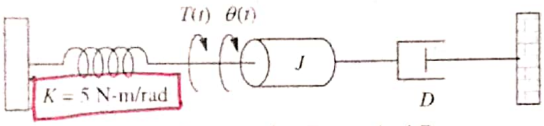


FIGURE 4.21 Rotational mechanical system for Example 4.7

SOLUTION: First, the transfer function for the system is

$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}} \tag{4.49}$$

From the transfer function,

$$\omega_n = \sqrt{\frac{K}{J}} \tag{4.50}$$

and

$$2\zeta\omega_n = \frac{D}{J} \tag{4.51}$$

But, from the problem statement,

$$T_s = 2 = \frac{4}{\zeta\omega_n} \tag{4.52}$$

or $\zeta\omega_n = 2$ Hence,

$$2\zeta\omega_n = 4 = \frac{D}{J} \tag{4.53}$$

Also, from Eqs. (4.50) and (4.52),

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}} \tag{4.54}$$

From Eq. (4.39), a 20% overshoot implies $\zeta = 0.456$. Therefore, from Eq. (4.54),

$$\zeta = 2\sqrt{\frac{J}{K}} = 0.456 \tag{4.55}$$

Hence,

$$\frac{J}{K} = 0.052 \tag{4.56}$$

From the problem statement, $K = 5$ N-m/rad. Combining this value with Eqs. (4.53) and (4.56), $D = 1.04$ N-m-s/rad, and $J = 0.26$ kg-m^2.

Second-Order Transfer Functions via Testing

Just as we obtained the transfer function of a first-order system experimentally, we can do the same for a system that exhibits a typical underdamped second-order response. Again, we can measure the laboratory response curve for percent overshoot and settling time, from which we can find the poles and hence the denominator. The numerator can be found, as in the first-order system, from a knowledge of the measured and expected steady-state values. A problem at the end of the chapter illustrates the estimation of a second-order transfer function from the step response.

Skill-Assessment Exercise 4.5

TryIt 4.1

Use the following MATLAB statements to calculate the answers to Skill-Assessment Exercise 4.5. Ellipses mean code continues on next line.

```
numg=361;
deng=[1 16 361];
omegan=sqrt(deng(3)...
 /deng(1))
zeta=(deng(2)/deng(1))...
 /(2*omegan)
Ts=4/(zeta*omegan)
Tp=pi/(omegan*sqrt...
 (1-zeta^2))
pos=100*exp(-zeta*...
 pi/sqrt(1-zeta^2))
Tr=(1.768*zeta^3 -...
 0.417*zeta^2+1.039*...
 zeta+1)/omegan
```

WileyPLUS WPCS Control Solutions

PROBLEM: Find ζ, ω_n, T_s, T_p, T_r, and $\%OS$ for a system whose transfer function is $G(s) = \dfrac{361}{s^2 + 16s + 361}$.

ANSWERS:

ζ = Apago 1PDF 0.Enhancer = 0.079 s, and $\%OS = 23.3\%$.

The complete solution is located at www.wiley.com/college/nise.

$\zeta = 0.421$, $\omega_n = 19$, $T_s = 0.5s$, $T_p = 0.182s$, $T_r = 0.079s$, $\%OS = 23.3\%$.

Now that we have analyzed systems with two poles, how does the addition of another pole affect the response? We answer this question in the next section.

4.7 System Response with Additional Poles

In the last section, we analyzed systems with one or two poles. It must be emphasized that the formulas describing percent overshoot, settling time, and peak time were derived only for a system with two complex poles and no zeros. If a system such as that shown in Figure 4.22 has more than two poles or has zeros, we cannot use the formulas to calculate the performance specifications that we derived. However, under certain conditions, a system with more than two poles or with zeros can be

FIGURE 4.22 Robot follows input commands from a human trainer

e^{-at}

$= \frac{1}{e^{at}}$

$= \frac{1}{e^{\infty}} \Big|_{a \to \infty}$

$= 0$

approximated as a second-order system that has just two complex *dominant poles*. Once we justify this approximation, the formulas for percent overshoot, settling time, and peak time can be applied to these higher-order systems by using the location of the dominant poles. In this section, we investigate the effect of an additional pole on the second-order response. In the next section, we analyze the effect of adding a zero to a two-pole system.

Let us now look at the conditions that would have to exist in order to approximate the behavior of a three-pole system as that of a two-pole system. Consider a three-pole system with complex poles and a third pole on the real axis. Assuming that the complex poles are at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ and the real pole is at $-\alpha_r$, the step response of the system can be determined from a partial-fraction expansion. Thus, the output transform is

$c(s) = \frac{\omega_n^2}{s(s+\alpha_r)(s^2+2\xi\omega_n s+\omega_n^2)}$

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r} \tag{4.57}$$

or, in the time domain,

will cancel out if $a_r \gg \xi\omega_n$

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t} \tag{4.58}$$

* Better $a_r > 5\xi\omega_n$.

The component parts of $c(t)$ are shown in Figure 4.23 for three cases of α_r. For Case I, $\alpha_r = \alpha_{r_1}$ and is not much larger than $\zeta\omega_n$; for Case II, $\alpha_r = \alpha_{r_2}$ and is much larger than $\zeta\omega_n$; and for Case III, $\alpha_r = \infty$.

Let us direct our attention to Eq. (4.58) and Figure 4.23. If $\alpha_r \gg \zeta\omega_n$ (Case II), the pure exponential will die out much more rapidly than the second-order underdamped step response. If the pure exponential term decays to an insignificant value at the time of the first overshoot, such parameters as percent overshoot, settling time, and peak time will be generated by the second-order underdamped step response component. Thus, the total response will approach that of a pure second-order system (Case III).

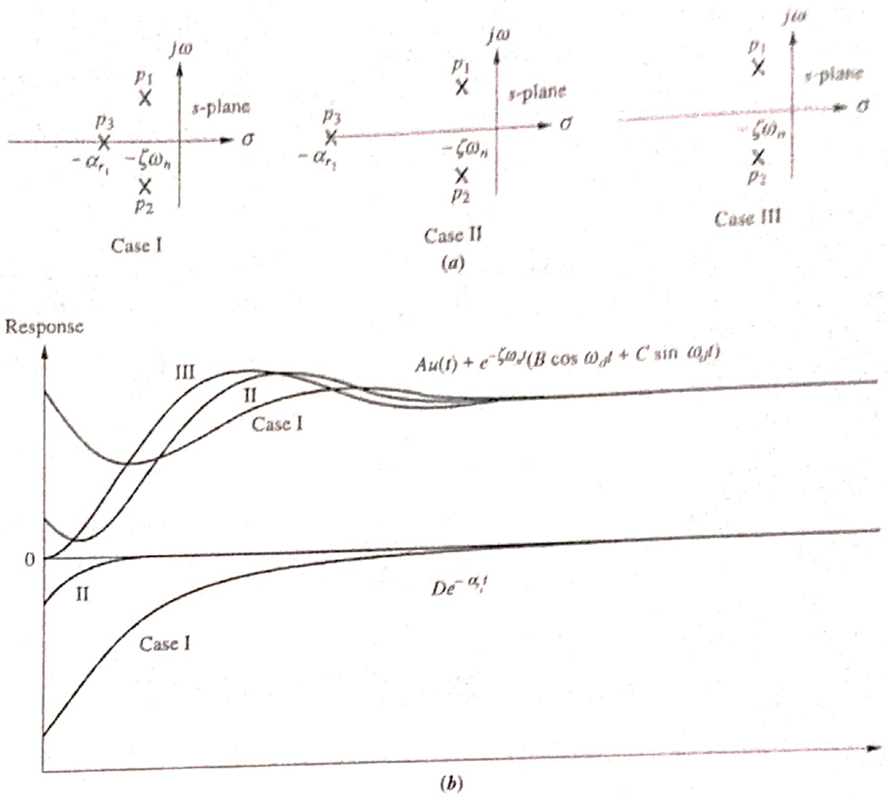


FIGURE 4.23 Component responses of a three-pole system: **a.** pole plot; **b.** component responses: Nondominant pole is near dominant second-order pair (Case I), far from the pair (Case II), and at infinity (Case III)

If α_r is not much greater than $\zeta\omega_n$ (Case I), the real pole's transient response will not decay to insignificance at the peak time or settling time generated by the second-order pair. In this case, the exponential decay is significant, and the system cannot be represented as a second-order system.

The next question is, How much farther from the dominant poles does the third pole have to be for its effect on the second-order response to be negligible? The answer of course depends on the accuracy for which you are looking. However, this book assumes that the exponential decay is negligible after five time constants. Thus, if the real pole is five times farther to the left than the dominant poles, we assume that the system is represented by its dominant second-order pair of poles.

What about the magnitude of the exponential decay? Can it be so large that its contribution at the peak time is not negligible? We can show, through a partial-fraction expansion, that the residue of the third pole, in a three-pole system with dominant second-order poles and no zeros, will actually decrease in magnitude as the third pole is moved farther into the left half-plane. Assume a step response, $C(s)$, of a three-pole system:

$$C(s) = \frac{bc}{s(s^2 + as + b)(s + c)} = \frac{A}{s} + \frac{Bs + C}{s^2 + as + b} + \frac{D}{s + c} \tag{4.59}$$

where we assume that the nondominant pole is located at $-c$ on the real axis and that the steady-state response approaches unity. Evaluating the constants in the numerator of each term,

$$A = 1; \qquad B = \frac{ca - c^2}{c^2 + b - ca} \tag{4.60a}$$

$$C = \frac{ca^2 - c^2a - bc}{c^2 + b - ca}; \quad D = \frac{-b}{c^2 + b - ca} \tag{4.60b}$$

As the nondominant pole approaches ∞, or $c \to \infty$,

$$A = 1;\ B = -1;\ C = -a;\ D = 0 \tag{4.61}$$

Thus, for this example, D, the residue of the nondominant pole and its response, becomes zero as the nondominant pole approaches infinity.

The designer can also choose to forgo extensive residue analysis, since all system designs should be simulated to determine final acceptance. In this case, the control systems engineer can use the "five times" rule of thumb as a necessary but not sufficient condition to increase the confidence in the second-order approximation during design, but then simulate the completed design.

Let us look at an example that compares the responses of two different three-pole systems with that of a second-order system.

Example 4.8

Comparing Responses of Three-Pole Systems

PROBLEM: Find the step response of each of the transfer functions shown in Eqs. (4.62) through (4.64) and compare them.

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542} \tag{4.62}$$

$$T_2(s) = \frac{245.42}{(s+10)(s^2 + 4s + 24.542)} \tag{4.63}$$

$$T_3(s) = \frac{73.626}{(s+3)(s^2 + 4s + 24.542)} \tag{4.64}$$

SOLUTION: The step response, $C_i(s)$, for the transfer function, $T_i(s)$, can be found by multiplying the transfer function by $1/s$, a step input, and using partial-fraction expansion followed by the inverse Laplace transform to find the response, $c_i(t)$. With the details left as an exercise for the student, the results are

$$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8°) \tag{4.65}$$

$$c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34°) \tag{4.66}$$

$$c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63°) \tag{4.67}$$

The three responses are plotted in Figure 4.24. Notice that $c_2(t)$, with its third pole at -10 and farthest from the dominant poles, is the better approximation of $c_1(t)$,

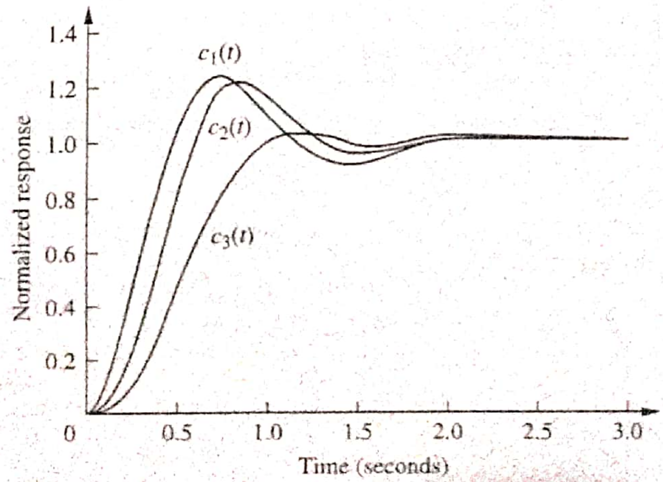


FIGURE 4.24 Step responses of system $T_1(s)$, system $T_2(s)$, and system $T_3(s)$

the pure second-order system response; $c_3(t)$, with a third pole close to the dominant poles, yields the most error.

MATLAB
ML

Students who are using MATLAB should now run ch4p2 in Appendix B. You will learn how to generate a step response for a transfer function and how to plot the response directly or collect the points for future use. The example shows how to collect the points and then use them to create a multiple plot, title the graph, and label the axes and curves to produce the graph in Figure 4.24 to solve Example 4.8.

Simulink
SL

System responses can alternately be obtained using Simulink. Simulink is a software package that is integrated with MATLAB to provide a graphical user interface (GUI) for defining systems and generating responses. The reader is encouraged to study Appendix C, which contains a tutorial on Simulink as well as some examples. One of the illustrative examples, Example C.1, solves Example 4.8 using Simulink.

Gui Tool
GUIT

Another method to obtain systems responses is through the use of MATLAB's LTI Viewer. An advantage of the LTI Viewer is that it displays the values of settling time, peak time, rise time, maximum response, and the final value on the step response plot. The reader is encouraged to study Appendix E at www.wiley.com/college/nise, which contains a tutorial on the LTI Viewer as well as some examples. Example E.1 solves Example 4.8 using the LTI Viewer.

Skill-Assessment Exercise 4.6

TryIt 4.2

Use the following MATLAB and Control System Toolbox statements to investigate the effect of the additional pole in Skill-Assessment Exercise 4.6(a). Move the higher-order pole originally at −15 to other values by changing "a" in the code.

```
a=15
numga=100*a;
denga=conv([1 a],...
 [1 4 100]);
Ta=tf(numga,denga);
numg=100;
deng=[1 4 100];
T=tf(numg,deng);
step(Ta,'.',T,'-')
```

PROBLEM: Determine the validity of a second-order approximation for each of these two transfer functions:

a. $G(s) = \dfrac{700}{(s+15)(s^2+4s+100)}$

b. $G(s) = \dfrac{360}{(s+4)(s^2+2s+90)}$

ANSWERS:

a. The second-order approximation is valid.

b. The second-order approximation is not valid.

The complete solution is located at www.wiley.com/college/nise.

4.8 System Response With Zeros

Now that we have seen the effect of an additional pole, let us add a zero to the second-order system. In Section 4.2, we saw that the zeros of a response affect the residue, or amplitude, of a response component but do not affect the nature of the response—exponential, damped sinusoid, and so on. In this section, we add a real-axis zero to a two-pole system. The zero will be added first in the left half-plane and then in the right half-plane and its effects noted and analyzed. We conclude the section by talking about pole-zero cancellation.

Starting with a two-pole system with poles at $(-1 \pm j2.828)$, we consecutively add zeros at -3, -5, and -10. The results, normalized to the steady-state value, are plotted in Figure 4.25. We can see that the closer the zero is to the dominant poles, the greater its effect on the transient response. As the zero moves away from the dominant poles, the response approaches that of the two-pole system. This analysis can be reasoned via the partial-fraction expansion. If we assume a group of poles and a zero far from the poles, the residue of each pole will be affected the same by the zero. Hence, the relative amplitudes remain appreciably the same. For example, assume the partial-fraction expansion shown in Eq. (4.68):

TryIt 4.3

Use the following MATLAB and Control System Toolbox statements to generate Figure 4.25.

```
deng=[1 2 9];
Ta=tf([1 3]*9/3,deng);
Tb=tf([1 5]*9/5,deng);
Tc=tf([1 10]*9/10,deng);
T=tf(9,deng);
step(T,Ta,Tb,Tc)
text(0.5,0.6,'no zero')
text(0.4,0.7,...
 'zero at -10')
text(0.35,0.8,...
 'zero at -5')
text(0.3,0.9,'zero at -3')
```

$$
\begin{aligned}
T(s) &= \frac{(s+a)}{(s+b)(s+c)} = \frac{A}{s+b} + \frac{B}{s+c} \\
&= \frac{(-b+a)/(-b+c)}{s+b} + \frac{(-c+a)/(-c+b)}{s+c}
\end{aligned} \tag{4.68}
$$

If the zero is far from the poles, then a is large compared to b and c, and

$$
T(s) \approx a\left[\frac{1/(-b+c)}{s+b} + \frac{1/(-c+b)}{s+c}\right] = \frac{a}{(s+b)(s+c)} \tag{4.69}
$$

Hence, the zero looks like a simple gain factor and does not change the relative amplitudes of the components of the response.

Another way to look at the effect of a zero, which is more general, is as follows (*Franklin, 1991*): Let $C(s)$ be the response of a system, $T(s)$, with unity in the

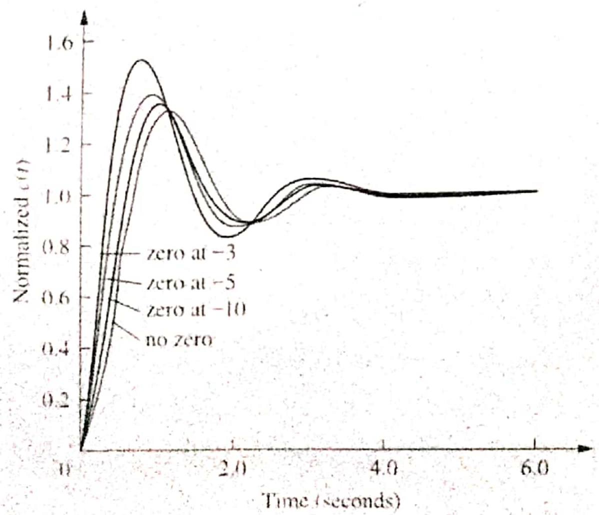


FIGURE 4.25 Effect of adding a zero to a two-pole system

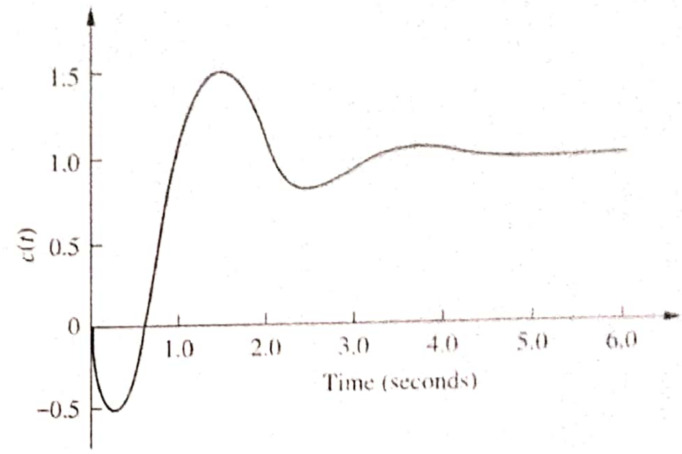


FIGURE 4.26 Step response of a nonminimum-phase system

numerator. If we add a zero to the transfer function, yielding $(s+a)T(s)$, the Laplace transform of the response will be

$$\boxed{(s+a)C(s) = sC(s) + aC(s)} \tag{4.70}$$

Thus, the response of a system with a zero consists of two parts: the derivative of the original response and a scaled version of the original response. If a, the negative of the zero, is very large, the Laplace transform of the response is approximately $aC(s)$, or a scaled version of the original response. If a is not very large, the response has an additional component consisting of the derivative of the original response. As a becomes smaller, the derivative term contributes more to the response and has a greater effect. For step responses, the derivative is typically positive at the start of a step response. Thus, for small values of a, we can expect more overshoot in second-order systems because the derivative term will be additive around the first overshoot. This reasoning is borne out by Figure 4.25.

when additional zero is at right half plane

An interesting phenomenon occurs if a is negative, placing the zero in the right half-plane. From Eq. (4.70) we see that the derivative term, which is typically positive initially, will be of opposite sign from the scaled response term. Thus, if the derivative term, $sC(s)$, is larger than the scaled response, $aC(s)$, the response will initially follow the derivative in the opposite direction from the scaled response. The result for a second-order system is shown in Figure 4.26, where the sign of the input was reversed to yield a positive steady-state value. Notice that the response begins to turn toward the negative direction even though the final value is positive. A system that exhibits this phenomenon is known as a *nonminimum-phase* system. If a motorcycle or airplane was a nonminimum-phase system, it would initially veer left when commanded to steer right.

Let us now look at an example of an electrical nonminimum-phase network.

Example 4.9

Transfer Function of a Nonminimum-Phase System

PROBLEM:

a. Find the transfer function, $V_o(s)/V_i(s)$ for the operational amplifier circuit shown in Figure 4.27.

b. If $R_1 = R_2$, this circuit is known as an all-pass filter, since it passes sine waves of a wide range of frequencies without attenuating or amplifying their magnitude (*Dorf, 1993*). We will learn more about frequency response in Chapter 10. For now, let $R_1 = R_2$, $R_3C = 1/10$, and find the step response of the filter. Show that component parts of the response can be identified with those in Eq. (4.70).

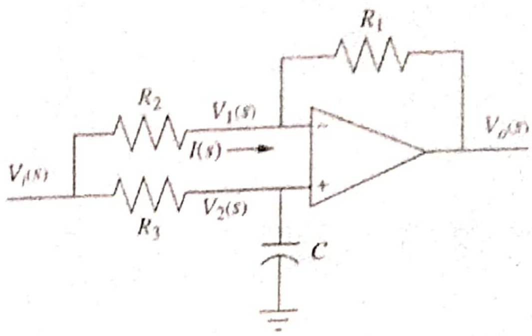


FIGURE 4.27 Nonminimum-phase electric circuit (Reprinted with permission of John Wiley & Sons, Inc.)

SOLUTION:

a. Remembering from Chapter 2 that the operational amplifier has a high input impedance, the current, $I(s)$, through R_1 and R_2, is the same and is equal to

$$I(s) = \frac{V_i(s) - V_o(s)}{R_1 + R_2} \tag{4.71}$$

Also,

$$V_o(s) = A(V_2(s) - V_1(s)) \tag{4.72}$$

But

$$V_1(s) = I(s)R_1 + V_o(s) \tag{4.73}$$

Substituting Eq. (4.71) into (4.73),

$$V_1(s) = \frac{1}{R_1 + R_2}(R_1 V_i(s) + R_2 V_o(s)) \tag{4.74}$$

Using voltage division,

$$V_2(s) = V_i(s)\frac{1/Cs}{R_3 + \dfrac{1}{Cs}} \tag{4.75}$$

Substituting Eqs. (4.74) and (4.75) into Eq. (4.72) and simplifying yields

$$\frac{V_o(s)}{V_i(s)} = \frac{A(R_2 - R_1R_3Cs)}{(R_3Cs + 1)(R_1 + R_2(1 + A))} \tag{4.76}$$

Since the operational amplifier has a large gain, A, let A approach infinity. Thus, after simplification

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 - R_1R_3Cs}{R_2R_3Cs + R_2} = -\frac{R_1}{R_2}\frac{\left(s - \dfrac{R_2}{R_1R_3C}\right)}{\left(s + \dfrac{1}{R_3C}\right)} \tag{4.77}$$

b. Letting $R_1 = R_2$ and $R_3C = 1/10$,

$$\frac{V_o(s)}{V_i(s)} = -\frac{\left(s - \dfrac{1}{R_3C}\right)}{\left(s + \dfrac{1}{R_3C}\right)} = -\frac{(s - 10)}{(s + 10)} \tag{4.78}$$

For a step input, we evaluate the response as suggested by Eq. (4.70):

$$C(s) = -\frac{(s-10)}{s(s+10)} = -\frac{1}{s+10} + 10\frac{1}{s(s+10)} = sC_o(s) - 10C_o(s) \quad (4.79)$$

where

$$C_o(s) = -\frac{1}{s(s+10)} \quad (4.80)$$

is the Laplace transform of the response without a zero. Expanding Eq. (4.79) into partial fractions,

$$C(s) = -\frac{1}{s+10} + 10\frac{1}{s(s+10)} = -\frac{1}{s+10} + \frac{1}{s} - \frac{1}{s+10} = \frac{1}{s} - \frac{2}{s+10} \quad (4.81)$$

or the response with a zero is

$$c(t) = -e^{-10t} + 1 - e^{-10t} = 1 - 2e^{-10t} \quad (4.82)$$

Also, from Eq. (4.80),

$$C_o(s) = -\frac{1/10}{s} + \frac{1/10}{s+10} \quad (4.83)$$

or the response without a zero is

$$c_o(t) = -\frac{1}{10} + \frac{1}{10}e^{-10t} \quad (4.84)$$

The normalized responses are plotted in Figure 4.28. Notice the immediate reversal of the nonminimum-phase response, $c(t)$.

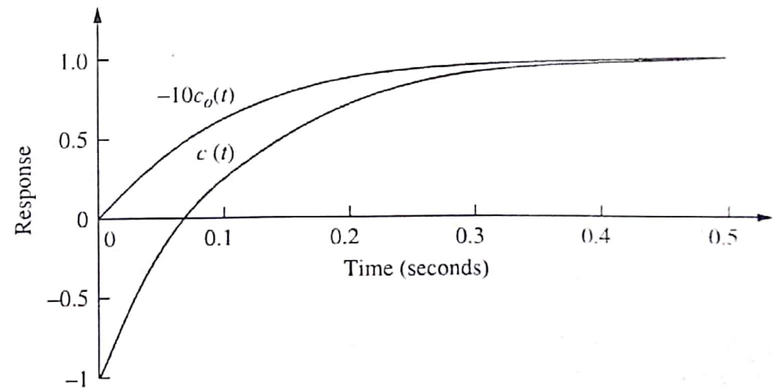


FIGURE 4.28 Step response of the nonminimum-phase network of Figure 4.27 ($c(t)$) and normalized step response of an equivalent network without the zero ($-10c_o(t)$)

We conclude this section by talking about pole-zero cancellation and its effect on our ability to make second-order approximations to a system. Assume a three-pole system with a zero as shown in Eq. (4.85). If the pole term, $(s + p_3)$, and the zero term, $(s + z)$, cancel out, we are left with

$$T(s) = \frac{K\cancel{(s+z)}}{\cancel{(s+p_3)}\,(s^2 + as + b)} \quad (4.85)$$

as a second-order transfer function. From another perspective, if the zero at $-z$ is very close to the pole at $-p_3$, then a partial-fraction expansion of Eq. (4.85) will show that the residue of the exponential decay is much smaller than the amplitude of the second-order response. Let us look at an example.

Example 4.10

Evaluating Pole-Zero Cancellation Using Residues

PROBLEM: For each of the response functions in Eqs. (4.86) and (4.87), determine whether there is cancellation between the zero and the pole closest to the zero. For any function for which pole-zero cancellation is valid, find the approximate response.

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)} \tag{4.86}$$

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)} \tag{4.87}$$

SOLUTION: The partial-fraction expansion of Eq. (4.86) is

$$C_1(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5} \tag{4.88}$$

The residue of the pole at -3.5, which is closest to the zero at -4, is equal to 1 and is not negligible compared to the other residues. Thus, a second-order step response approximation cannot be made for $C_1(s)$. The partial-fraction expansion for $C_2(s)$ is

$$C_2(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{0.033}{s+4.01} \tag{4.89}$$

near to zero.

The residue of the pole at -4.01, which is closest to the zero at -4, is equal to 0.033, about two orders of magnitude below any of the other residues. Hence, we make a second-order approximation by neglecting the response generated by the pole at -4.01:

$$C_2(s) \approx \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} \tag{4.90}$$

and the response $c_2(t)$ is approximately

$$c_2(t) \approx 0.87 - 5.3e^{-5t} + 4.4e^{-6t} \tag{4.91}$$

TryIt 4.4

Use the following MATLAB and Symbolic Math Toolbox statements to evaluate the effect of higher-order poles by finding the component parts of the time response of $c_1(t)$ and $c_2(t)$ in Example 4.10.

```
syms s
C1=26.25*(s+4)/...
  (s*(s+3.5)*...
  (s+5)*(s+6));
C2=26.25*(s+4)/...
  (s*(s+4.01)*...
  (s+5)*(s+6));
c1=ilaplace(C1);
c1=vpa(c1,3);
'c1'
pretty(c1)
c2=ilaplace(C2);
c2=vpa(c2,3);
'c2'
pretty(c2);
```

Skill-Assessment Exercise 4.7

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PROBLEM: Determine the validity of a second-order step-response approximation for each transfer function shown below.

a. $G(s) = \dfrac{185.71(s+7)}{(s+6.5)(s+10)(s+20)}$

b. $G(s) = \dfrac{197.14(s+7)}{(s+6.9)(s+10)(s+20)}$

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we have

$$F(s) = \frac{1}{1+G(s)} = \frac{1}{1+\frac{10}{s(s+6)}} = \frac{s(s+6)}{s^2+6s+10}$$

$$C_0 = \lim_{s\to 0} F(s) = \lim_{s\to 0} \frac{s(s+6)}{s^2+6s+10} = 0$$

Note: The relationship between k_p and C_0 is

$$C_0 = \frac{1}{1+k_p}.$$

$$C_1 = \lim_{s\to 0} \frac{dF(s)}{ds} = \lim_{s\to 0} \frac{(s^2+6s+10)(2s+6) - s(s+6)(2s+6)}{(s^2+6s+10)^2}$$

$$= \frac{(10)(6)-0}{100} = \frac{60}{100} = 0.6$$

$$C_2 = \lim_{s\to 0} \frac{d^2F(s)}{dt^2}$$

$$= \lim_{s\to 0} \frac{d}{dt}\left[\frac{10(2s+6)}{(s^2+6s+10)^2}\right]$$

$$= \lim_{s\to 0}\left[\frac{(s^2+6s+10)^2(2) - 2(2s+6)(s^2+6s+10)(2s+6)}{(s^2+6s+10)^4}\right]$$

$$\Longrightarrow \quad C_2 = -0.052$$

$$e(t) = \cos(t) + C_1\dot{r}(t) + C_2\ddot{r}(t) + \ldots$$

$$e(t) = 0.6\dot{r}(t) + -0.052\dot{r}t$$

$$e(t) = 0.6(1+2t) + -0.052(2)$$

$$= 0.496 + 1.2t$$

$$e(\infty) = \infty$$

2.18 Effect of Adding a Zero to a System

Let us consider a second order system with transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{2.150}$$

Now let us introduce a zero at $s = -z$ to the above second order closed loop system.

Then we get

$$\frac{C(s)}{R(s)} = \frac{(s+z)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{2.151}$$

Adding a zero to the transfer function affects the gain of the system. Therefore, to make the gain unaffected (to bring gain to unity) divide the numerator by Z. Now,

$$\frac{C(s)}{R(s)} = \frac{(s+z)\omega_n^2/z}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{2.152}$$

To analyse the effect of adding a zero to the system, we have to find the time domain specifications. We can write Eq. (2.150) as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} + \frac{s}{z}\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right) \tag{2.153}$$

Let $c(t)$ is the step response of the system $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, then the step response of $\frac{s}{z}\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$ is $\frac{1}{z}\frac{d}{dt}[c(t)]$. Therefore the step response of the system with added zero is

$$c_z(t) = c(t) + \frac{1}{z}\frac{d}{dt}c(t) \tag{2.154}$$

The response of the system with added zero $c_z(t)$ and the response of the original system $c(t)$ are shown in Fig.2.26

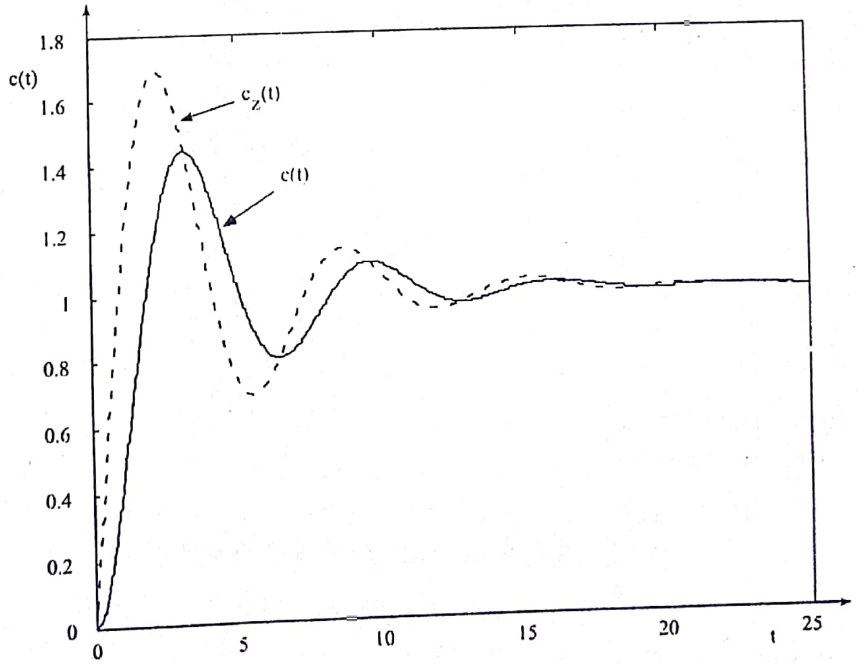

Fig. 2.26 Effect of adding zero to the system

From the response we can observe the following

1. The rise time of $c_z(t)$ is less than that of $c(t)$.
2. The Maximum peak overshoot of $c_z(t)$ is greater than that of $c(t)$.

In summary the addition of zero to a system decreases risetime and increases the maximum peak over shoot. More over, the values of rise time and peak over shoot depends on the selection of value of z. Note that from the equation 2.153, the smaller values of zero, i.e., the zero close to origin produces large over shoot where as the large values of zero i.e., the zero away from the origin produces negligible effect on transient response as shown in Fig. 2.27. Therefore, selecting the zeros near the origin should be avoided in design.

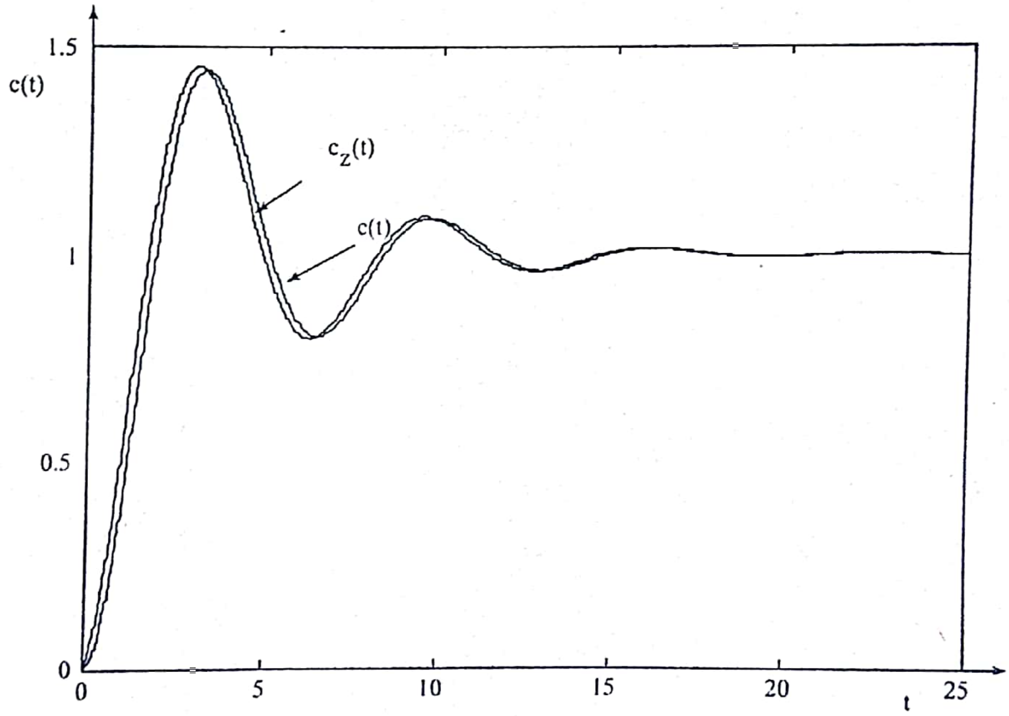

Fig. 2.27

2.19 Compensation

Adding some zeros and poles to the original system to improve it's dynamic performance is known as compensation. Some of the compensation schemes are discussed below.

2.19.1 *Derivative error compensation*

A system is said to have derivative error compensation if the generation of its actuating signal depends on the rate of change of error signal. For the system shown in Fig. 2.28 the type of compensation is introduced by using an amplifier which provides an output signal containing two terms, one proportional to derivative of the error signal and