ataraĵan.

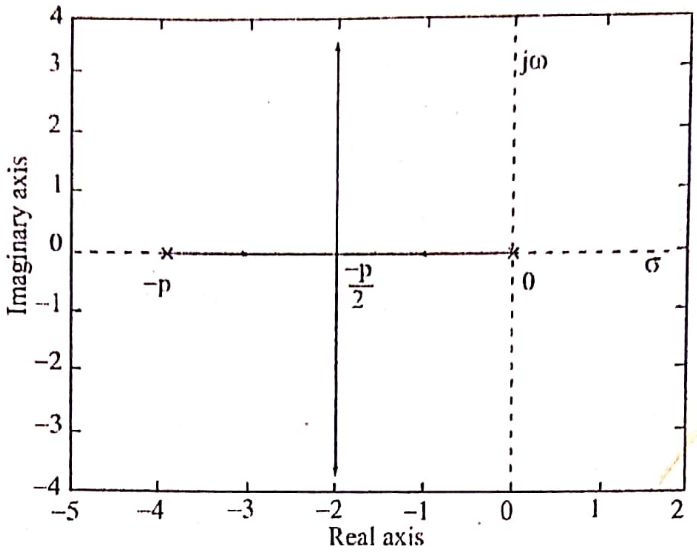


Fig 3.21

From Eq. (3.12) we can find that the real part of the roots are constant (since p is fixed) and the imaginary part varies as k varies. Therefore the roots are moving along the vertical line, one upwards and the other down wards as shown in Fig. 3.21

In summary when $k = 0$, the root loci start at open loop poles $s_1 = 0$ and $s_2 = -p$ and move in the opposite directions as k increases. At $k = p^2/4$, the two root loci meet at $-p/2$ (repeated roots at $-p/2$). For $k > p^2/4$ the root loci move vertically in opposite directions and tends to infinity as k tends to ∞.

Once the root locus is obtained, it is possible to determine the variation in system performance for the variation in the parameter k. Consider the open loop transfer function $G(s) = \frac{k}{s(s+p)}$. The roots of the characteristic equation $s^2 + 2s + k = 0$ when $p = 2$ are determined for different values of k and are tabulated in the table 3.1. When $0 < k < 1$, the roots lie on the negative real axis which corresponds to over damped response. When $k = 1$ the roots are $s_{1,2} = -1$. This corresponds at a critically damped system. For $k > 1$, the roots are complex conjugate and this corresponds to an under damped system.

Table 3.1 Location of the roots of the characteristic equation $s^2 + 2s + k = 0$

k	s_1	s_2
0	0	-2
0.5	- 0.293	-1.707
1.0	-1	-1
2.0	-1 + j	-1 - j
3.0	- 1+j 1.414	-1-j1.414
4.0	-1 + j 1.732	-1 -j1.732

Example of Root locus 1

Plot root locus for a unity feedback system

$$G(s) = \frac{k}{s(s+5)(s+10)}$$

→ **Step-1** : Identify loci/branches.

* Number of Poles $P = 3$. $[0, -5, -10]$.
* Number of zeros $Z = 0$

$\therefore$ Number of loci/Branches $= Max(P, Z) = 3$

→ **Step-2** : Number of Asymptotes. (A).

$$A = \#P - \#Z = 3 - 0 = 3$$

→ **Step-3** : Centroid of Asymptotes. (δ_c)

$$\delta_c = \frac{\sum Re|P| - \sum Re|Z|}{\#P - \#Z} = \frac{(0-5-10)-0}{3-0} = -5$$

→ **Step-4** : Angles of Asymptotes.

$$\theta = \frac{(2x+1).\,180^\circ}{P-Z}, \quad x = 0, 1, 2, \ldots (P-Z)$$
$$= 0, 1, 2 \text{ [In this case].}$$

$$\therefore \theta_1 = \frac{180^\circ}{3} = 60^\circ$$

$$\theta_2 = \frac{3 \times 180^\circ}{3} = 180^\circ$$

$$\theta_3 = \frac{5 \times 180^\circ}{3} = 300^\circ.$$

$$\therefore \theta = 60^\circ, 180^\circ, 300^\circ.$$

→ Step-5. Break Away Point.

Put $1+G(s)H(s)=0$ [characteristic equ^n] Put $\frac{dk}{ds}=0$

$$\Rightarrow 1+\frac{k}{s(s+5)(s+10)}=0$$

$$\Rightarrow s(s+5)(s+10)+k=0$$

$$\Rightarrow s^3+15s^2+50s+k=0$$

$$\therefore k=-(s^3+15s^2+50s)$$

$$\Rightarrow \frac{dk}{ds}=-(3s^2+30s+50)$$

$$\therefore 3s^2+30s+50=0$$

$$\therefore s=-2.11,\ -7.88$$

→ Step-6 Intersection to imaginary axis.

The characteristic equation,

$$s^3+15s^2+50s+k=0$$

s^3	1	50
s^2	15	k
s^1	$\frac{750-k}{15}$	0
s^0	k	

For stability $k>0$, $\frac{750-k}{15}>0$

Now,

$$\frac{750-k}{15}>0$$

$$\Rightarrow 750-k>0$$

$$\Rightarrow 750>k$$

$$\therefore k<750$$

Use the auxiliary equ^n

$$\Rightarrow A(s)=0$$

$$\Rightarrow 15s^2+k=0$$

$$\Rightarrow 15s^2+750=0$$

$$\Rightarrow s^2=-\frac{750}{15}$$

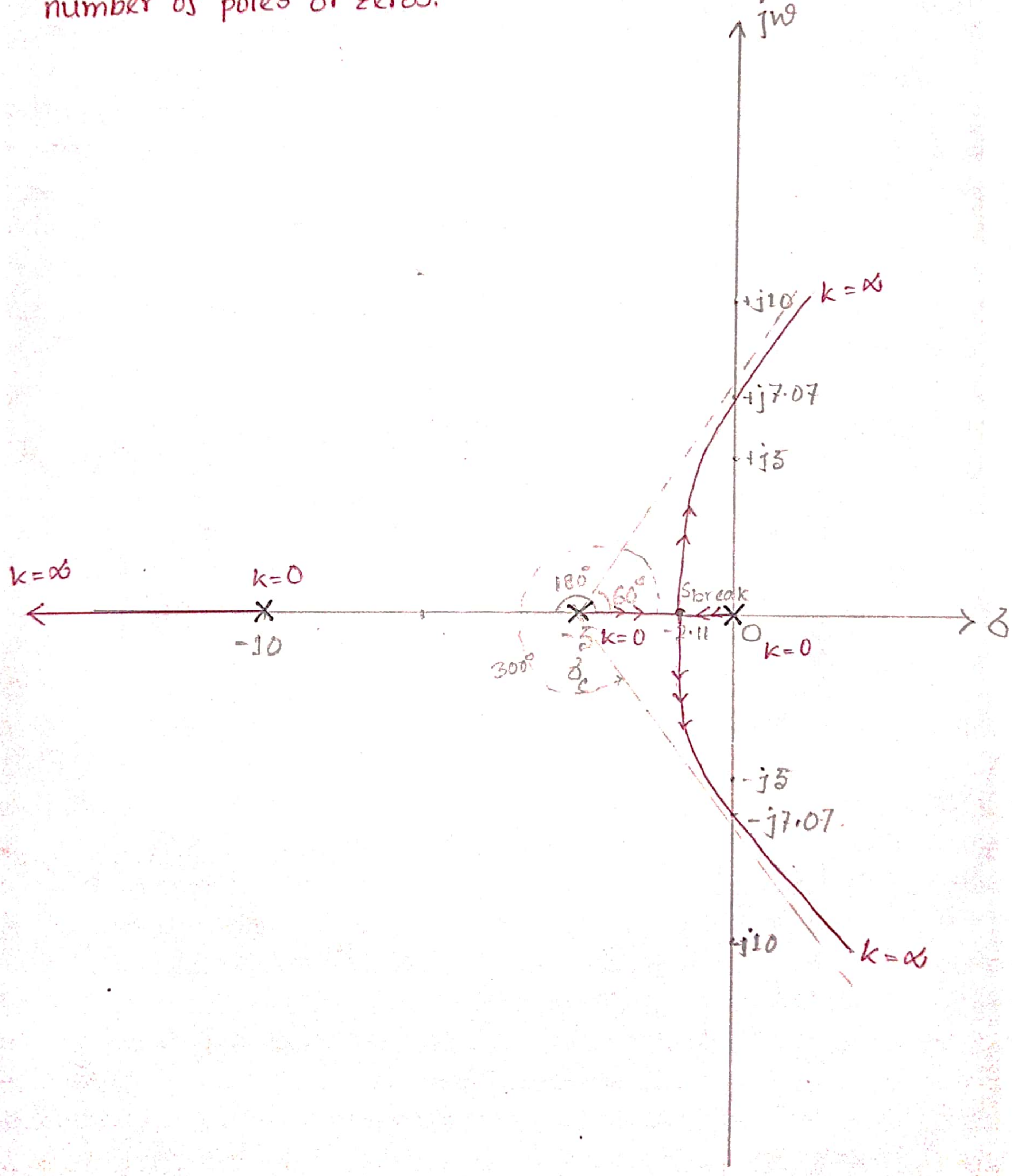
$$\therefore s=\pm j7.07.$$

* Step-7: Angle of Departure is for complex conjugate pol

~~Maximum Peak overshoot M_p, %OS~~

* At right side of S_{break}, the summation of poles & zeros should be odd.

* There is no root locus betⁿ even number of poles or zeros.



Example 2:

The open loop transfer fun^c with unity feedback is

$$G(s) = \frac{k}{s(s^2+4s+8)}.$$

Plot the root locus.

Sol^n :-

1. **No. of loci**

$\#P = 3 \; [0, -2 \pm j2]$. and $\#Z = 0$

$\therefore$ Number of branches $= \overset{Max}{(P, Z)} = 3$

2. **No. of Asymptotes (A)**

$(A) = P - Z = 3$

3. **Centroid of Asymptotes (δ_c)**

$$\delta_c = \frac{\Sigma Re|P| - \Sigma Re|Z|}{\#P - \#Z} = \frac{(0-2-2)-0}{3}$$

$$= \frac{-4}{3} = -1{\cdot}33.$$

4. **Angle of Asymptote θ**

$$\theta = \frac{(2x+1).180}{P-Z} \qquad x = 0, 1, 2 \text{ (as } A = 3)$$

$$\theta_x = 60^\circ, 180^\circ, 300^\circ$$

5. Break Away Point:

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(s^2+4s+8)} = 0$$

$$\Rightarrow s^3 + 4s^2 + 8s + k = 0$$

$$\Rightarrow k = -(s^3 + 4s^2 + 8s)$$

$$\Rightarrow \frac{dk}{ds} = -(3s^2 + 8s + 8)$$

$$\therefore \frac{dk}{ds} = -(3s^2 + 8s + 8) = 0$$

$$\therefore s = -1.33 \pm j0.94$$

* It is invalid BAP.

6. Intersection to imaginary axis:

The characteristic equ^n $s^3 + 4s^2 + 8s + k = 0$

s^3	1	8
s^2	4	k
s^1	$\frac{32-k}{4}$	0
s^0	k	

Hence for stability,

$k > 0$

and $\frac{32-k}{4} > 0$

$\Rightarrow 32 > k$

$\therefore k < 32$

$\therefore 0 < k < 32.$

Now, Auxiliary Equ^n,

$$A(s) = 4s^2 + k = 0$$

$$\Rightarrow 4s^2 + 32 = 0$$

$$\Rightarrow s = \sqrt{-8}$$

$$\therefore s = \pm j2.82.$$

7. Angle of Departure:

$$\phi_D = 180 - \phi$$

$= 180 - [\sum \text{Angle of remaining Pole} - \sum \text{Angle of zeros}]$.

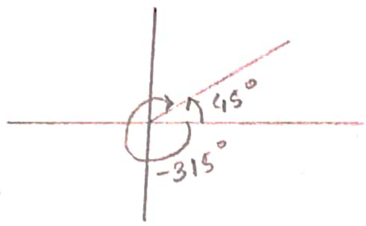
$\therefore \phi_{D_1} = 180^\circ - [(135^\circ + 90^\circ) - 0]$ {For pole $-2+j2$}

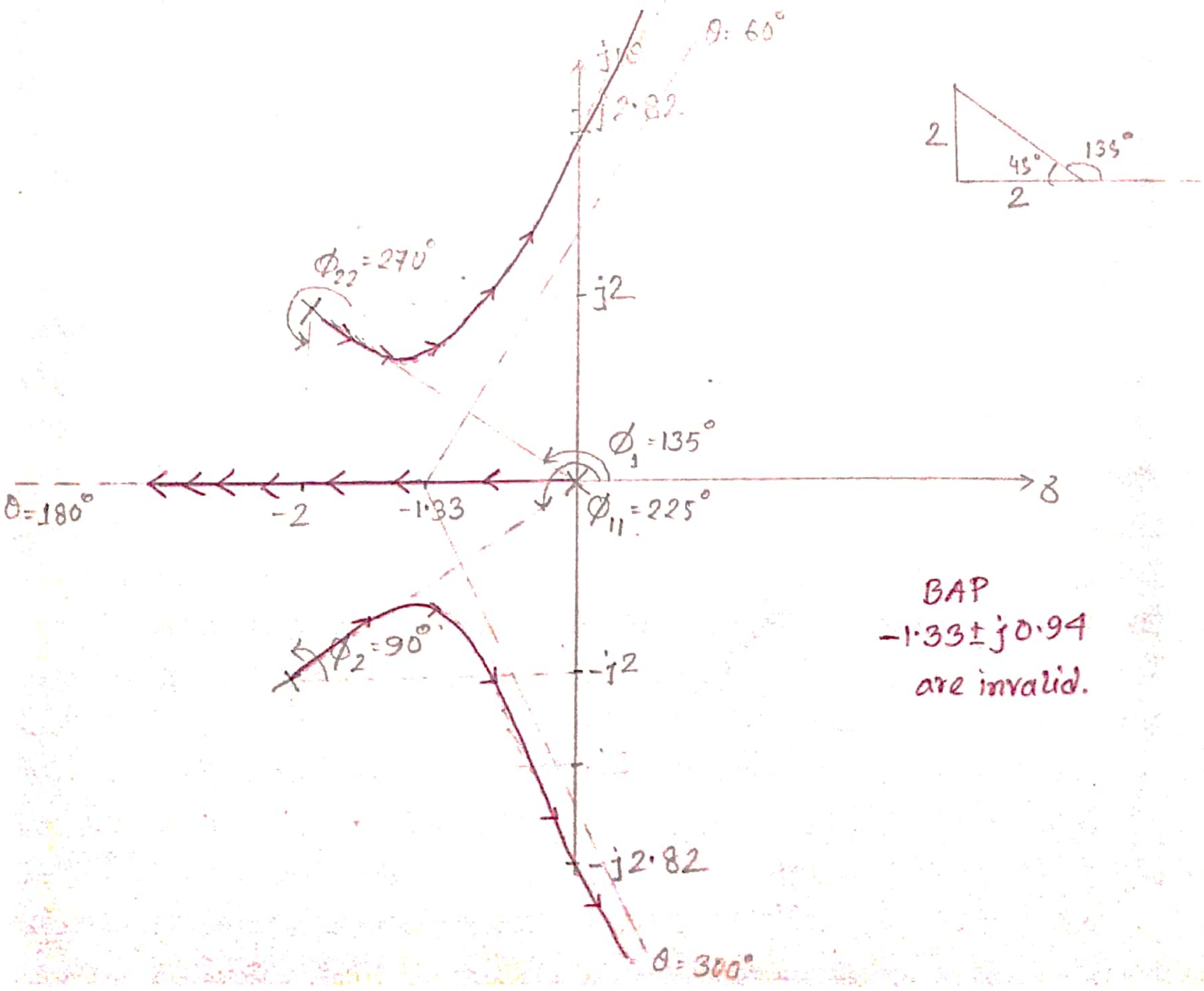
$= -45^\circ$

Now, $\phi_{D_2} = 180^\circ - [(225 + 270) - 0]$

$= -315^\circ$

$= +45^\circ$ (just reverse ϕ_{D_1}).





BAP
$-1.33 \pm j0.94$
are invalid.

Evan conditions

Let us consider a system with open loop transfer function $G(s)$ and feedback transfer function $H(s)$. Then the closed loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{3.14}$$

The characteristic equation of the system is given by

$$1 + G(s)H(s) = 0 \tag{3.15}$$

Let

$$P(s) = G(s)H(s), \tag{3.16}$$

Then

$$1 + P(s) = 0 \tag{3.17}$$

$$\Longrightarrow P(s) = -1 \tag{3.18}$$

since 's' is a complex variable and we can write the characteristic equation as follows

$$|p(s)| = 1 \quad \text{(magnitude criterion)} \tag{3.19}$$

and

$$\angle p(s) = \pm 180(2q + 1); \quad q = 0, 1, 2, \ldots \text{(angle criterion)} \tag{3.20}$$

The above two equations are known as Evan conditions. Using these Eqs. (3.19 and (3.20) we can define the locus as follows. The set of all points on the s-plane that satisfy the angle criterion of Eq. (3.20) is a root locus. The magnitude criterion (equation) gives the value of k corresponding to a point on the locus.

Determination of points on the root locus

Let us consider a system with open loop transfer function in pole zero form.

$$G(s)H(s) = \frac{k(s + z_1)(s + z_2)(s + z_3)(s + z_4)\ldots\ldots(s + z_m)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)\ldots\ldots(s + p_n)} \tag{3.21}$$

Where $n > m$, $z_i's$ and $p_i's$ are open loop zeros and poles.

The Evan's conditions for the existence of a point on the root locus are $|p(s)| = |G(s)H(s)| = 1$ (magnitude criterion) and

$$\angle p(s) = \pm 180(2q + 1); q = 0, 1, 2 \ldots \text{(angle criterion)}$$

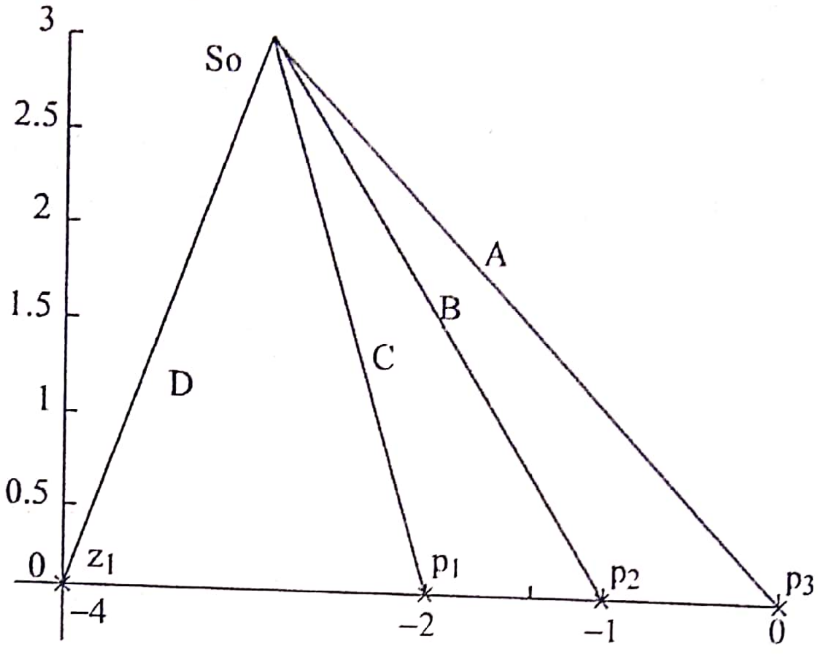


Fig 3.22

angle criterion is a very tedious procedure. By using a special tool called spirule, we can find the set of all points on the root locus. But this trial and error method of drawing root locus becomes obsolete. An approximate root locus plot can be plotted by using the set of hand calculations using construction rules. After the avilability of digital computer and software packages like MATLAB, plotting a root locus is reduced to a simple command to the computer. Any how it is important to know the construction rules to draw the approximate plot which is very useful in the design of control system. The following are the set of construction rules.

Construction Rules

Rule 1:

The root locus is symmetrical about the real axis and the number of branches equal to the order of the polynomial (Number of poles of the open loop transfer function).

The roots of the characteristic equations are either real, imaginary or complex conjugate or combination of the above; therefore the root locus is symmetrical about real axis. The root locus above the real axis is mirror image of the root locus below the real axis and vice versa. The number of branches of the root locus is equal to the order of the characteristic polynomial.

Rule 2:

All branches of root locus starts at open loop poles (when $k = 0$) and ends at either open loop zero or infinity (when $k = \infty$). The number of branches terminating at infinity equals to the difference between the number of poles and number of zeros.

Consider the characteristics equation of n^{th} order system

$$1+\frac{k\prod_{i=1}^{m}|(s+z_i)|}{\prod_{i=1}^{n}|(s+p_i)|}=0 \tag{3.27}$$

$$\prod_{i=1}^{n}|(s+p_i)|+k\prod_{j=1}^{m}|(s+z_j)|=0 \tag{3.28}$$

In the above equation, the set of all points for which the left hand side of equation is zero are on the root locus.

Let us consider the case $k=0$. For this, the characteristic equation yields

$$\prod_{i=1}^{n}|(s+p_i)|=0 \tag{3.29}$$

This shows that the set of all open loop poles are part of the root locus and all the branches of root locus originate from open loop poles.

Let us rewrite Eq. (3.27) as follows

$$\frac{1}{k}\prod_{i=1}^{n}|(s+p_i)|+\prod_{i=1}^{m}|(s+z_i)|=0 \tag{3.30}$$

when k tends to infinity, the first term approaches to zero and the equation becomes

$$\prod_{i=1}^{m}|(s+z_i)|=0 \tag{3.31}$$

Eq. (3.31) shows that set of all open loop zeros lie on the root locus branches and these open loop zeros are terminating points of root locus braches.

For a practical system the number of zeros (m) are less than number of poles (n). That is $m<n$. In the root locus plot n branches originates from open loop poles, out of which only m branches terminate at open loop zero. Therefore we have to find the terminating points of the remaining $(n-m)$ root locus branches.

Consider the equation (3.23)

$$\frac{\prod_{j=1}^{m}|(s+z_j)|}{\prod_{i=1}^{n}|(s+p_i)|}=\frac{1}{k} \tag{3.32}$$

when k tends to ∞, s tends to $\infty e^{j\theta}$ which implies that $(n-m)$ branches of the root locus terminate at infinity.

Rule 3:

A point on the real axis lies on the root locus if the sum of the poles and zeros on the real axis to the right of the point is an odd number.

Consider the open loop pole and zero configuration as shown in Fig. 3.23. Let s_0 be the test point. To check whether the test point s_0 is on the root locus or not, join all the poles and zeros to this point. At this point the angles made by the lines joining p_1, p_2, p_3 and p_4 with s_0 are $\angle s_0 + p_1, \angle s_0 + p_2, \angle s_0 + p_3$ and $\angle s_0 + p_4$ respectively. Similarly the angles made by the lines joining z_1, z_2, z_3 and z_3' are $\angle s_0 + z_1, \angle s_0 + z_2, \angle s_0 + z_3$ and $\angle s_0 + z_3'$ respectively. From Fig. 3.23 we can observe that the angles made by p_1, p_2 and z_1 are

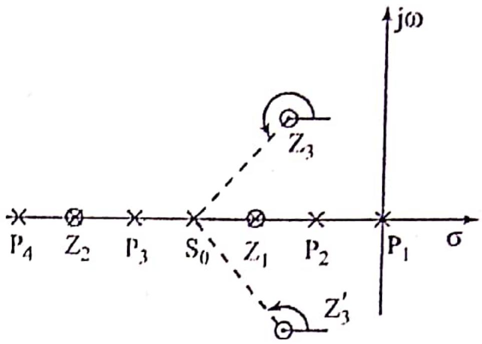


Fig. 3.23

$$\angle s_0 + p_1 = \angle s_0 + p_2 = \angle s_0 + z_1 = 180° \tag{3.33}$$

and the angle made by z_2, p_3 and p_4 are

$$\angle s_0 + z_2 = \angle s_0 + z_3 = \angle s_0 + p_4 = 0° \tag{3.34}$$

Angle made by $z_3$and $z_3^1$with s_0 are equal and opposite (ie) $\angle s_0 + z_3 + \angle s_0 + z_3^1 = 0$. Therefore it is not necessary to consider complex poles and zeros.

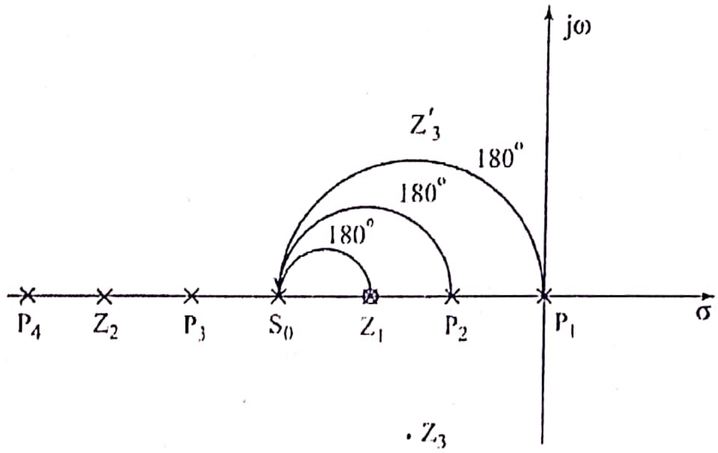


Fig. 3.24

The sum of the angles contributed by complex conjugate poles is zero. From this we conclude the following

1. The angle contribution of all the poles and zeros on the real axis to the right of the point is 180°.
2. The angle contribution of all the poles and zeros on the real axis to the left of the test point is 0°.

3. The angle contribution by complex conjugate poles and zeros is zero.

From Eq. (3.33) & Eq. (3.34) the angle of $G(s)H(s)$ with the point s_0 is given by

$$\begin{aligned}\Phi_{p1} + \Phi_{p2} + \Phi_{p3} + \Phi_{p4} + \Phi_{z1} + \Phi_{z2} + \Phi_{z3} + \Phi_{z3'} \\ = 180^\circ + 180^\circ + 0^\circ + 180^\circ + 0^\circ + 0^\circ + 0^\circ + 0^\circ = 180^\circ\end{aligned}$$

180° is odd multiple of 180° therefore s_0 is a point on the locus.

Similarly, For the test point s_1 the net angle contribution to all open loop poles and zeros are given by

$$\begin{aligned}\Phi_{p1} + \Phi_{p2} + \Phi_{p3} + \Phi_{p4} + \Phi_{z1} + \Phi_{z2} + (\Phi_{z3} + \Phi_{z3'}) \\ = 180^\circ + 180^\circ + 180^\circ + 0^\circ - 180^\circ + 0^\circ + 0^\circ + 0^\circ = 360^\circ \neq \pm 180^\circ(2q+1)\end{aligned}$$

Therefore s_1 is not a point on the root locus. Thus the necessary condition for determining the real axis locus is

$$(n_z - n_p)180^\circ = \pm(2q+1)180^\circ$$

Where n_p is the number of poles on the real axis to the right of the test point and n_z the number of zeros on the real axis to the right of the test point. Eq. (3.25) satisfies when $n_z - n_p$ must be an odd number. If $n_z - n_p$ is an odd number then $n_p + n_z$ also an odd number. Therefore we can conclude that if the total number of poles and zeros to the right of the test point s_0 on the real axis is odd then the test point lies on the root locus.

Example 3.13. Draw the root locus for the unity feedback system with open loop transfer function

$$G(s) = \frac{k(s+1)(s+3)}{s(s+2)(s+4)}$$

Solution .

The three rules so far we have seen are sufficient to draw the root locus of the given system.

Step 1. The number of open loop poles are three. Therefore the number of branches of the root locus are three. The plot of poles and zeros are shown in Fig. 3.25

Step 2. The three branches of the root locus starts from the open loop poles $s = 0, -2, -4$. Out of these three branches two branches of the root locus terminate at the two open loop zeros and one branch terminates at infinity.

Step 3. All the points between 0 and -1, -2 and -3, -4 and $-\infty$ lie on the root locus for which the sum of open loop poles and zeros to the right of test points are 1,3 and 5 respectively (all points are having odd number of poles and zeros to its right).

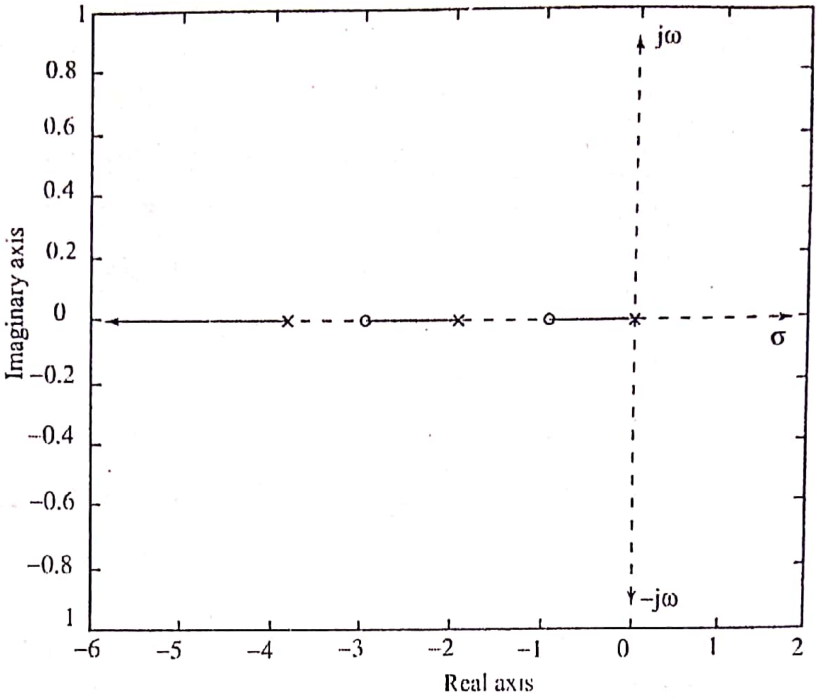


Fig. 3.25

Rule 4:

The $(n-m)$ root locus branches that proceed to infinity do so along the asymptotes with angles

$$\Phi_A = \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2, \ldots\ldots (n-m-1)$$

Consider a test point s_0 at infinity, the angles made by the line joining the test point s_0 and the open loop poles and zeros are equal to each other (say ϕ_A°). The total number of such angles is equal to $n-m$. Therefore the total angle made by the test point s_0 with all open loop poles and zeros is equal to $(n-m)\phi_A^\circ$. This angle must satisfy the angle criterion $(n-m)\phi_A^\circ = \pm 180^\circ(2q+1)$

$$(n-m)\Phi_A^\circ = (2q+1)180^\circ \tag{3.35}$$

$$\Phi_A^\circ = \frac{(2q+1)180^\circ}{(n-m)} \tag{3.36}$$

where $q = 0, 1, 2, 3 \ldots (n-m-1)$, since $(n-m)$ branches of the root locus tends to infinity along the asymptotes, the number of asymptotes is equal to $n-m$. Therefore q varies from 0 to $n-m-1$.

$$\Rightarrow \Phi_A^0 = \frac{(2q+1)180^\circ}{(n-m)} \quad q = 0, 1, 2, 3 \ldots n-m-1. \tag{3.37}$$

$$\Rightarrow \Phi_A^0 = \frac{(2q+1)180^\circ}{(\text{number of poles} - \text{number of zeros})} \quad q = 0, 1, 2, 3 \ldots n-m-1. \tag{3.38}$$

Rule 5:

The centroid, the point of intersection of the asymptotes with real axis is given by

$$\sigma_A = \frac{\text{sum of real parts of poles} - \text{sum of real parts of zeros}}{\text{sum of poles} - \text{sum of zeros}}$$

Consider the open loop transfer function

$$G(s)H(s) = \frac{k(s+z_1)(s+z_2)(s+z_3)\ldots\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}; m \leq n \tag{3.39}$$

$$= \frac{k\left[s^m + \left(\sum_{i=1}^{m} z_i\right)s^{m-1} + \ldots\ldots + \prod_{i=1}^{m} z_i\right]}{s^n + \left(\sum_{i=1}^{n} p_i\right)s^{n-1} + \ldots\ldots + \prod_{i=1}^{n} p_i} \tag{3.40}$$

Therefore the characteristic equation is

$$1 + G(s)H(s) = 1 + \frac{k\left[s^m + \left(\sum_{i=1}^{m} z_i\right)s^{m-1} + \ldots\ldots + \prod_{i=1}^{m} z_i\right]}{s^n + \left(\sum_{i=1}^{n} p_j\right)s^{n-1} + \ldots\ldots + \prod_{i=1}^{n} p_j} = 0 \tag{3.41}$$

Dividing the numerator and denominator by numerator polynomial, we have

$$\Rightarrow \quad 1 + \frac{k}{s^{n-m} + \left(\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i\right)s^{n-m-1} + \ldots\ldots + \ldots} = 0 \tag{3.42}$$

If we select the test point at infinity, that is for large values of s, the characteristic equation can be approximated to first two terms of the denominator polynomial.

$$\lim_{s\to\infty} 1 + G(s)H(s) = 1 + \frac{k}{s^{n-m} + \left(\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i\right)s^{n-m-1}} = 0 \tag{3.43}$$

Now let us consider following transfer function which has $n-m$ repeated poles at σ_A and no zeros.

$$G(s) = \frac{k}{(s+\sigma_A)^{n-m}} \tag{3.44}$$

The characteristic equation of the of the above transfer function with unity feedback is

$$1 + G(s)H(s) = 1 + \frac{k}{(s+\sigma_A)^{n-m}} \tag{3.45}$$

The above transfer function has $n-m$ root locus branches. All branches are originating at σ_A and terminating at infinity as straight lines drawn through σ_A. The binomial expansion of the above characteristic equation is

$$1+G(s)H(s)=1+\frac{k}{s^{n-m}+(n-m)\sigma_A s^{n-m-1}+\ldots} \tag{3.46}$$

For large values of s the equation (3.43) is identical to equation (3.46). This shows that the straight line root locus branches of the transfer function (3.45) are asymptotes of the transfer function (3.43) with centroid at σ_A. Equating (3.43) and (3.46) we have,

$$(n-m)\sigma_A=\sum_{j=1}^{n}p_j-\sum_{i=1}^{m}z_i \tag{3.47}$$

$$\Rightarrow \sigma_A=\frac{\sum_{j=1}^{n}p_j-\sum_{i=1}^{m}z_i}{n-m} \tag{3.48}$$

$$\Rightarrow -\sigma_A=\frac{\sum_{j=1}^{n}(-p_j)-\sum_{i=1}^{m}(-z_i)}{n-m} \tag{3.49}$$

The left hand side of the equation is a real number. Therefore the above equation may be modified as follows:

$$\Rightarrow -\sigma_A=\frac{\sum_{j=1}^{n}\text{Real part of }(-p_j)-\sum_{i=1}^{m}\text{Real part of }(-z_i)}{n-m} \tag{3.50}$$

$$\Rightarrow -\sigma_A=\frac{\sum_{j=1}^{n}\text{Real part of poles}-\sum_{i=1}^{m}\text{Real part of zeros}}{n-m} \tag{3.51}$$

Example 3.14. Sketch the root locus for a unity feedback system with open loop transfer function

$$G(s)=\frac{ks}{(s^2+4)(s+2)}$$

Solution .
The construction rules discussed so far are sufficient to sketch the root locus. The open loop poles are $s_{1,2}=\pm 2j$ and $s=-2$. The open loop zero is at $s=0$;

Step 1. The numbers of open loop poles are three therefore the numbers of branches in the root locus are three.

Step 2. The three branches of the root locus originate from the open loop poles $s=\pm 2j$ and -2 when $k=0$. There is only one zero therefore one of the branches terminates at open loop zero i.e. at $s=0$. Other two branches terminate at infinity.

Step 3. All points between 0 and -2 have odd number of poles and zero (only one zero) to its right side. Therefore all points between 0 and -2 are parts of root locus.

Step 4. The two root loci that proceed to infinity do so along the asymptotes with angles.

$$\phi_A = \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2, \ldots\ldots (n-m-1).$$

$$\phi_A = \frac{(2q+1)180^\circ}{2}; q = 0, 1.$$

$$\phi_A = 90^\circ, 270^\circ;$$

Step 5. The centroid, the point of intersection of the asymptotes on the real axis is given by

$$\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real part of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$

$$\sigma_A = \frac{-2-0}{2} = -1$$

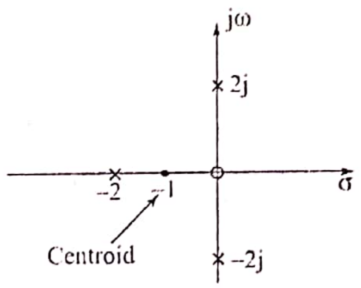


Fig. 3.26

Rule 6:

The break away points (points on which multiple roots of the characteristic equation exist) of the root locus are the solution of $\frac{dk}{ds} = 0$.

Let us assume that the characteristic equation has r multiple roots at $s = s_0$.

That is:

$$1 + G(s)H(s) = (s + s_0)^r x(s) \tag{3.52}$$

Here $x(s)$ does not contain the factor $s + s_0$.

Differentiating Eq. (3.52) on both sides with respect to s yields

$$\frac{d}{ds}[1 + G(s)H(s)] = r(s+s_0)^{r-1}x(s) + (s+s_0)^r x'(s) \tag{3.53}$$

where $x'(s)$ is the derivative of $x(s)$.

At $s = s_0$, RHS of the Eq. (3.53) becomes zero, which implies that the break away points (Multiple roots) are the roots of

$$\frac{d}{ds}[1 + G(s)H(s)] = 0 \tag{3.54}$$

Let the pole-zero form of the characteristic equation be

$$1 + G(s)H(s) = 1 + \frac{kP(s)}{Q(s)} = 0 \tag{3.55}$$

Differentiating Eq. (3.55) on both sides with respect to 's' we have

$$\frac{d}{ds}[1 + G(s)H(s)] = \frac{k[Q(s)P'(s) - P(s)Q'(s)]}{[Q(s)]^2} \tag{3.56}$$

$$\Rightarrow Q(s)P'(s) - P(s)Q'(s) = 0 \tag{3.57}$$

The roots of Eq. (3.55) are the roots of $\frac{d}{ds}[G(s)H(s)] = 0$.
From Eq. (3.55) we know that

$$k = -\frac{Q(s)}{P(s)} \tag{3.58}$$

Differentiating with respect to s yields

$$\frac{dk}{ds} = -\frac{[P(s)Q'(s) - Q(s)P'(s)]}{[P(s)]^2} \tag{3.59}$$

Comparing Eq. (3.56) and Eq. (3.59) we find that the numerators are equal and hence the break away points of the characteristic equation can be determined from the roots of $\frac{dk}{ds} = 0$.

Example 3.15. Sketch the root locus for a system with open loop transfer function

$$G(s)H(s) = \frac{k(s+1)}{(s+2)(s+3)(s+4)}$$

Solution .
The open loop poles are at $s = -2, -3$ and -4 and the open loop zero is at $s = -1$.
Therefore $n = 3; m = 1$.

Step 1. The number of branches in the root locus are three since $n = 3$.

Step 2. The three branches of the root locus originate from open loop poles at $s = -2, -3$ and -4 when $k = 0$. Since $m = 1$, out of the three root locus branches only one branch terminates at open loop zero and the remaining two branches terminating at infinity when $k = \infty$.

Step 3. All the points between 0 and $-2, -3$ and -4 lie on the root locus since the sum of poles and zeros to the right of these points is odd (1 and 3 respectively).

Step 4. The two root locus branches that proceed to infinity do so along the asymptotes with angles

$$\phi_A = \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2, \ldots\ldots (n-m-1).$$

$$\phi_A = \frac{(2q+1)180^\circ}{2}; q = 0, 1$$

$$= 90^\circ, 270^\circ$$

Step 5. The centroid, the point of intersection of the asymptotes on the real axis is given by

$$\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$

$$\sigma_A = \frac{(-2-3-4)-(-1)}{3-1} = \frac{-8}{2} = -4$$

Step 6. The break away points of the root locus are the solution of $\dfrac{dk}{ds} = 0$

$$G(s)H(s) = \frac{k(s+1)}{(s+2)(s+3)(s+4)}$$

$$k = -\frac{(s+2)(s+3)(s+4)}{(s+1)}$$

$$= \frac{-(s^3+9s^2+26s+24)}{(s+1)}$$

$$\frac{dk}{ds} = \frac{(s+1)(3s^2+18s+26)-(s^3+9s^2+26s+24)}{(s+1)^2}$$

$$(3s^3+21s^2+44s+26)-(s^3+9s^2+26s+24) = 0$$

$$2s^3+12s^2+18s+2 = 0$$

$$s^3+6s^2+9s+1 = 0$$

The roots are

$$-3.5321, -2.3473, -0.1206$$

The root -3.5321 alone lies on the root locus. Hence the break away point is at -3.5321.

The complete root locus plot is shown in Fig. 3.27.

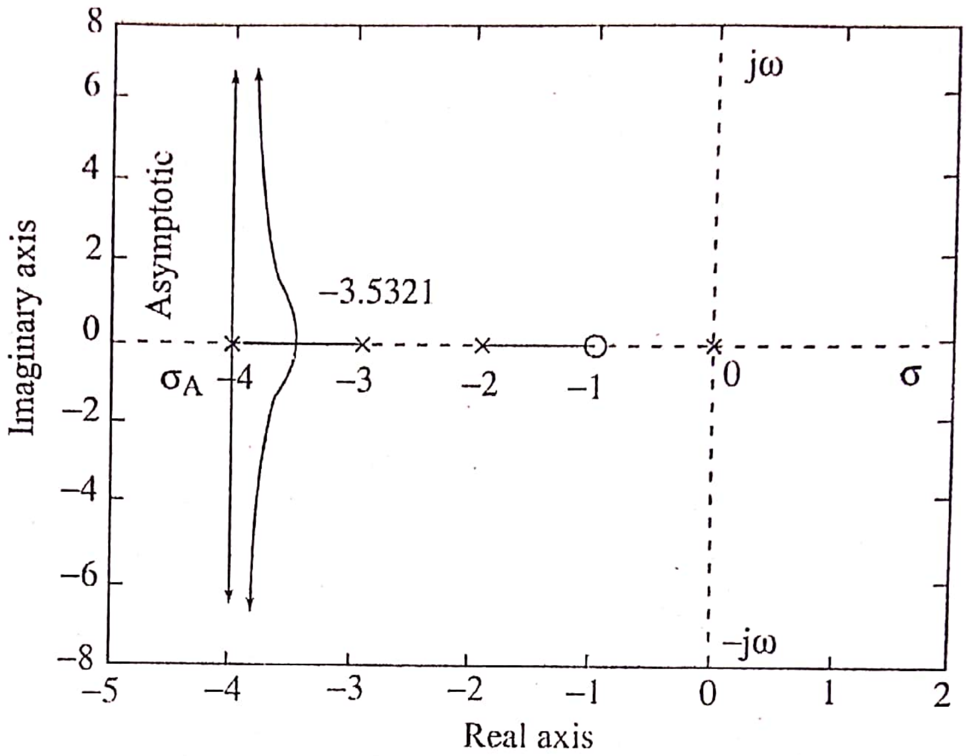


Fig. 3.27

Rule 7:

The angle of departure from an open loop pole is given by

$$\phi_p = \pm 180°(2q+1) + \phi; \quad q = 0, 1, 2 \ldots$$

where ϕ is net angle contribution to this pole by all other open loop poles and zeros. Similarly the angle of arrival at an open loop zero is given by

$$\phi_z = \pm 180°(2q+1) - \phi; \quad q = 0, 1, 2 \ldots\ldots$$

where ϕ is the net angle contribution to the zero under consideration by all other open loop poles and zeros.

The angle at which the root locus leaves one of the poles p_s is found by considering a point s_0 on the root locus branch very close to the pole p_s as shown in Figure. (3.28). The net angle contributed by all other poles and zeros to the pole p_s under consideration is almost equal to the net angle contribution to the arbitrary point s_0 since the pole p_s and the arbitrary point s_0 are very close to each other.

Let P_1, P_2, P_3, P_4, P_5 and P_6 are poles and Z_1 is the zero of the system. A point s_0 on the root locus is selected very close to P_5.

The net angle contribution of all poles and zeros to this point s_0 is

$$\phi_{net} = \phi_{z1} - (\phi_{p6} + \phi_{p5} + \phi_{p4} + \phi_{p3} + \phi_{p2} + \phi_1) \quad (3.60)$$

$$\phi_{net} = \phi_{p5} - \phi. \quad (3.61)$$

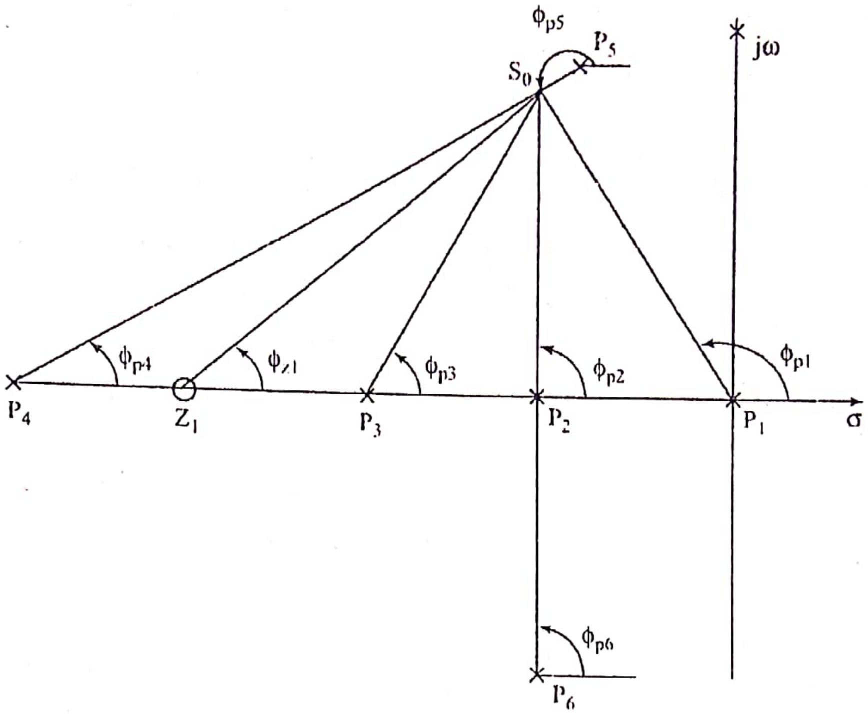


Fig. 3.28

where ϕ is the net angle contributed by all poles and zeros except P_5. We know that net angle contributed by all poles and zeros to a point on the root locus is

$$\phi_{net} = \pm 180°(2q+1); q = 0, 1, 2 \cdot \tag{3.62}$$

From Eq. (3.61) and Eq. (3.62), we have

$$\phi_{p5} - \phi = \pm 180(2q+1) \tag{3.63}$$

$$\Rightarrow \phi_{p5} = \pm 180(2q+1) + \phi \tag{3.64}$$

ϕ_{p5} is the angle made by the line joining p_5 and S_0. Since S_0 is very close to p_5, ϕ_{p5} is the anlge of departure.

$\therefore \phi$ is approximately equal to the netangle contribution by all poles and zeros with p_5. In general,

$$\phi_p = \pm 180(2q+1) + \phi \tag{3.65}$$

where $q = 0, 1, 2 \ldots$ and ϕ is the angle made by all poles and zeros with the pole under consideration.

Similarly the angle of arrival at an open loop zero is given by

$$\phi_z = \pm 180°(2q+1) - \phi \quad q = 0, 1, 2 \ldots\ldots$$

Where ϕ is the angle made by all poles and zeros with the zero under consideration.

Example 3.16. Sketch the root locus for a system with open loop transfer function

$$G(s)H(s) = \frac{k(s+1)}{s^2 + 4s + 13}$$

Solution .

The open loop poles are at $s_{1,2} = -2 \pm j3$ and open loop zero is at $s_3 = -1$. That is $n = 2; m = 1$.

Step 1. There are two root locus branches since the system has two open loop poles.

Step 2. The two branches of the root locus starts at open loop poles at $-2 \pm j3$. One branch terminates at open loop zero (Since it has only one zero) and the other terminates at infinity when $k = \infty$.

Step 3. All the points on the real axis between $-\infty$ and -1 are on the root locus branch.

Step 4. The root locus branch that terminates at infinity do so along the asymptote with angle

$$\begin{aligned}\phi_A &= \frac{(2q+1)180^\circ}{n-m};\ q = 0, 1, 2 \ldots\ldots n-m.\\ &= \frac{(2q+1)180^\circ}{1};\ q = 0\\ &= 180^\circ\end{aligned}$$

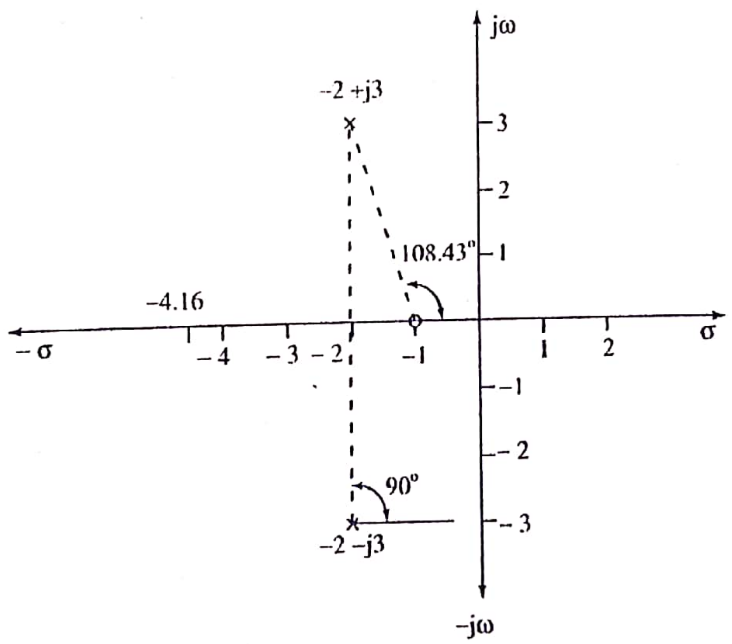


Fig. 3.29

Step 5. The break away points of the root locus are the solution of $\dfrac{dk}{ds} = 0$

$$k = \frac{-(s^2 + 4s + 13)}{s+1}$$

$$\frac{dk}{ds} = -\left[\frac{(s+1)(2s+4) - (s^2+4s+13)}{(s+1)^2}\right] = 0$$

$$(2s^2 + 6s + 4) - (s^2 + 4s + 13) = 0$$
$$\Rightarrow \quad s^2 + 2s - 9 = 0$$

The roots are at $s_1 = -4.16$ and $s_2 = 2.16$.

The break away point is at -4.16 since this point is on the root locus but the other root 2.16 is not on the root locus.

Step 6. The angle of departure from an open loop pole is given by

$$\phi_p = \pm 180°(2q + 1) + \phi; q = 0, 1, 2$$

For $q = 0$

$$\phi_p = \pm 180° + \phi$$

where ϕ is the net angle contribution at this pole due to the other open loop poles and zeros.

Let us consider the pole at $-2 + j3$.

The net angle contribution $\phi = \phi_{z1} - \phi_{p2} = 108.43° - 90° = 18.43°$.

The angle of department at pole p_1 is

$$\phi_p = \pm 180° + 18.43°$$
$$= 198.43°, \quad 161.57°$$

The complete root locus plot is shown in Fig. 3.30.

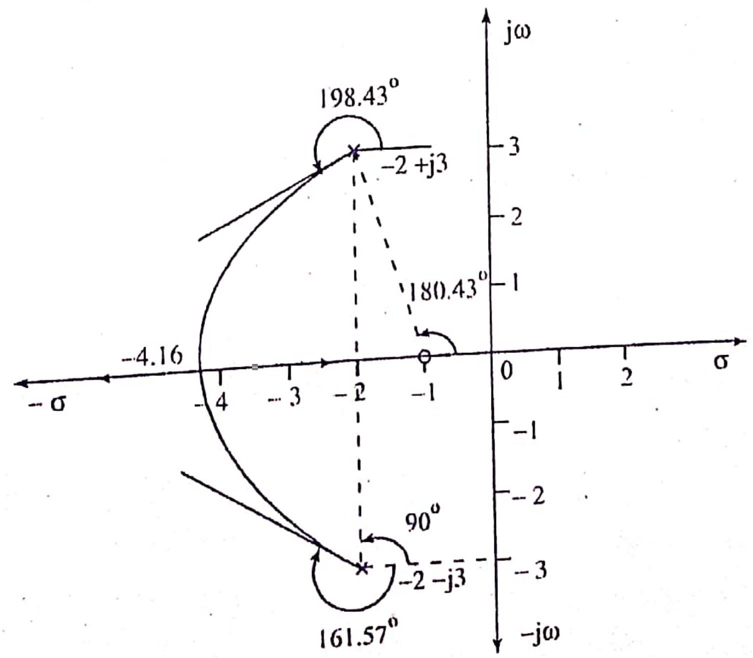


Fig. 3.30

Rule 8:

The intersection of root locus with imaginary axis can be determined using the Routl criterion.

Rule 9:

The open loop gain k (Transfer function in pole-zero form) at any point s_0 on the roo locus is given by

$$k = \frac{\prod |(s+p_i)|}{\prod |(s+z_i)}$$

$$k = \frac{\text{Product of phasor lengths from } s_0 \text{ to open loop poles}}{\text{Product of phasor lengths from } s_0 \text{ to open loop zeros}}$$

Example 3.17. Sketch the root locus of a feedback system whose open loop transfe function is given by

$$G(s)H(s) = \frac{k}{s(s+2)(s+3)}$$

Solution .
Using the rules discussed so far we can sketch the root locus. The open loop pole are at $s = 0, -2$ and -3 and there is no open loop zeros.

Step 1. The numbers of root locus branches are three since the number of open loo poles are three.

Step 2. The three branches of the root locus originate from the open loop poles a $s = 0, -2$ and -3 when $k = 0$ and all the three branches terminate at infinit when $k = \infty$.

Step 3. All the points between 0 and -2, -3 and $-\infty$ lies on the root locus for whic the sum of open loop poles and zeros to the right of the test point are 1 and . respectively.

Step 4. The three root locus branches that proceed to infinity do so along the asymptotes with angles

$$\phi_A = \frac{(2q+1)180^\circ}{3}; q = 0, 1, 2$$

$$\phi_A = 60^\circ, 180^\circ, 300^\circ$$

Step 5. The centroid, the point of intersection of the asymptotes on the real axis i given by

$$\sigma_A = \frac{\text{Sum of real part of poles} - \text{Sum of real part of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$

$$= \frac{0-2-3-0}{3} = \frac{-5}{3} = -1.667$$

Step 6. The break away points of the root locus are the solution of $\frac{dk}{ds} = 0$

$$G(s)H(s) = \frac{k}{s(s+2)(s+3)}$$

$$1 + G(s)H(s) = 0$$

$$\Longrightarrow G(s)H(s) = -1$$

$$\frac{k}{s(s+2)(s+3)} = -1$$

$$k = -s(s+2)(s+3) = -(s^3 + 5s^2 + 6s)$$

$$\frac{dk}{ds} = -[3s^2 + 10s + 6] = 0$$

The roots of $\frac{dk}{ds} = 0$ are -2.5485 and -0.7847 The point $s = -2.5485$ is not on the root locus. Therefore the breakaway point is -0.7847 which is on the root locus.

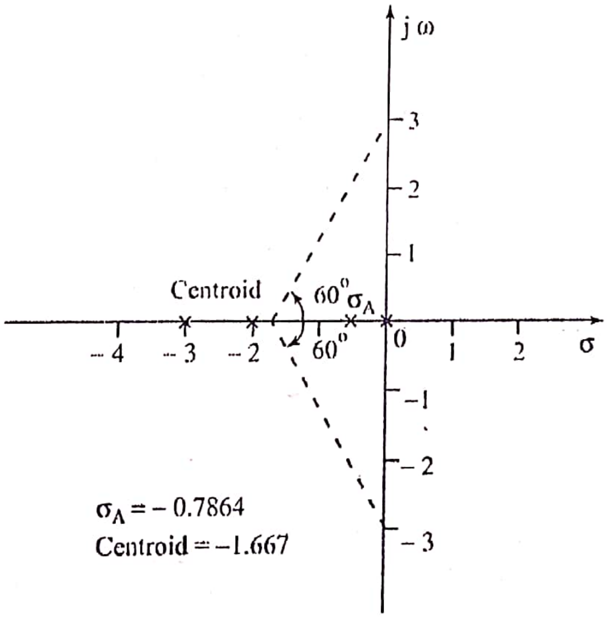


Fig. 3.31

Step 7. The intersection of the root locus with imaginary axis can be determined using Routh criterion. The characteristic equation is given by

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s+2)(s+3)} = 0$$

$$s(s+2)(s+3) + k = 0$$

$$s^3 + 5s^2 + 6s + k = 0$$

s^3	1	6
s^2	5	k
s	$(30-k)/5$	0
s^0	k	

$$(30-k)/5 \geq 0$$
$$0 < k < 30$$
$$s^2 = \pm j\sqrt{6}$$
$$s = \pm j2.449,$$

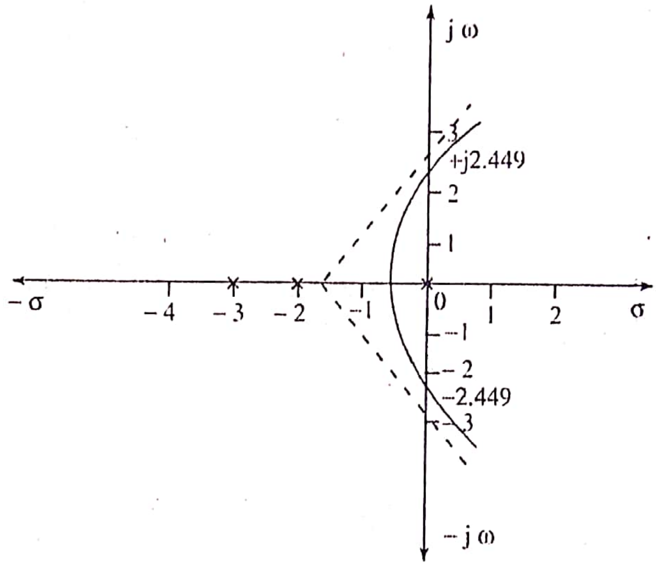


Fig. 3.32

where s is the point at which the root locus crosses imaginary axis. The complete root locus is shown in Fig. 3.32.

Example 3.18. Sketch the root locus for a unity feedback system with open loop transfer function

$$G(s) = \frac{k}{s(s^2 + 8s + 32)}$$

Solution .
The poles of the open loop transfer function are the roots of the denominator

$$s(s^2 + 8s + 32) = 0$$
$$\Longrightarrow P_1 = 0$$

$$P_{2,3} = \frac{-8 \pm \sqrt{64 - 4(32)}}{2} = \frac{-8 \pm \sqrt{-64}}{2} = -4 \pm j4$$

Mark the poles with × symbol on the graph sheet.

Step 1. There are three open loop poles, hence the number of branches in the root locus are three and no zeros.

Step 2. The three branches starts at $p_1 = 0, p_2 = -4 + j4$ and $p_3 = -4 - j4$ when $k = 0$ and terminate at infinity when $k = \infty$.

Step 3. All the points on the real axis between $-\infty$ to 0 lie on the root locus, since there is one pole to the right of these points.

Step 4. The three branches that terminates at infinity do so along the asymptotes with angles

$$\phi_n = \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2, \ldots (n - m - 1)$$

$$= \frac{(2q+1)180^\circ}{3} \quad q = 0, 1, 2$$

For $q = 0$ $\phi_{A1} = \frac{180^\circ}{3} = 60^\circ$

For $q = 1$ $\phi_{A2} = \frac{3(180^\circ)}{3} = 180^\circ$

For $q = 2$ $\phi_{A3} = \frac{5(180^\circ)}{3} = 300^\circ$

Step 5. The asymptotes meet at a point known as centroid

$$\sigma_A = \frac{\text{Sum of real parts of poles - Sum of real parts of zeros}}{\text{Number of poles - Number of zeros}}$$

$$= \frac{-4-4-0}{3} = -\frac{8}{3} = -2.667$$

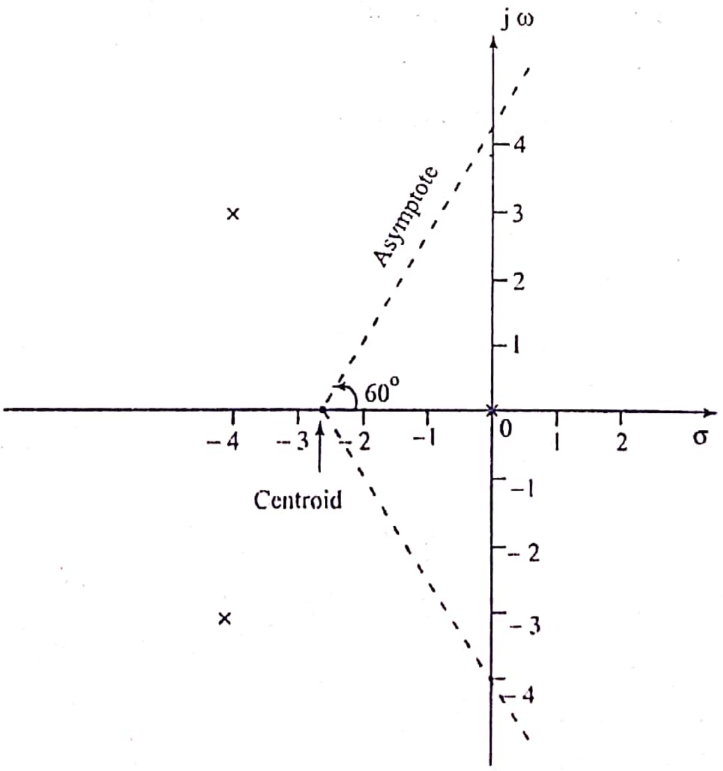


Fig. 3.33

Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4 using protractor.

Step 6. The break away point of root locus are the solution of $\frac{dk}{ds} = 0$

$$G(s)H(s) = \frac{k}{s(s^2+8s+32)}, \quad H(s) = 1$$

we know

$$1 + G(s)H(s) = 0$$

$$\Rightarrow \quad 1 + \frac{k}{s(s^2+8s+32)} = 0$$

$$k = -s(s^2+8s+32)$$

$$\frac{dk}{ds} = 0$$

$$\Rightarrow 3s^2 + 16s + 32 = 0$$

The roots are $\dfrac{-8 \pm j4\sqrt{2}}{3}$

The points are not on the root locus. Therefore there is no breakaway point.

Step 7. The angle of departure ϕ_p of a root locus from a complex open loop pole is

$$\phi_p = 180° + \phi$$

when ϕ is the net angle contribution at this pole by all other open loop poles and zero as shown in Fig. 3.34.

The angle of departure at pole p_2 is

$$\phi_{p_2} = 180° + \phi$$

where

$$\phi = -135° - 90°$$
$$= -225°$$

$$\phi_{p_2} = 180° - 225° = -45°$$

$$\boxed{\begin{array}{c} \tan^{-1}\left(\frac{4}{4}\right) = 45° \\ \phi_{p2} = 180° - 45° = 135° \\ \phi_{p3} = 90° \end{array}}$$

Similarly

$$\phi_{p_3} = -\phi_{p_2} = -(-45°) = 45°$$

Using protractor mark the angle of departure of complex pole

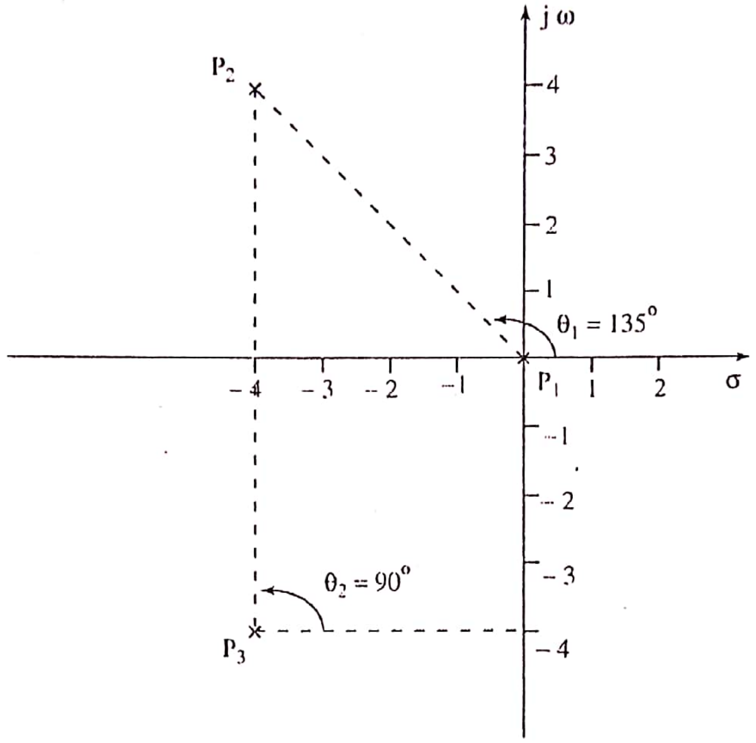


Fig. 3.34

Step 8. The crossing point on the imaginary axis can be found using Routh criterion.

The characteristic equation is given by

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s^2 + 8s + 32)} = 0$$

$$s^3 + 8s^2 + 32s + k = 0$$

s^3	1	32	0
s^2	8	k	0
s^1	$\frac{256 - k}{8}$	0	
s^0	k		

For stability

$$\frac{256 - k}{8} > 0 \text{ and } k > 0$$

$$\Rightarrow \quad 0 < k < 256$$

When $k = 256$, the root locus crosses the imaginary axis. The auxiliary equation is $8s^2 + k = 0 \Rightarrow 8s^2 + 256 = 0$. $\therefore s = \pm j\sqrt{32}$.

The complete root locus plot is shown in Fig. 3.35

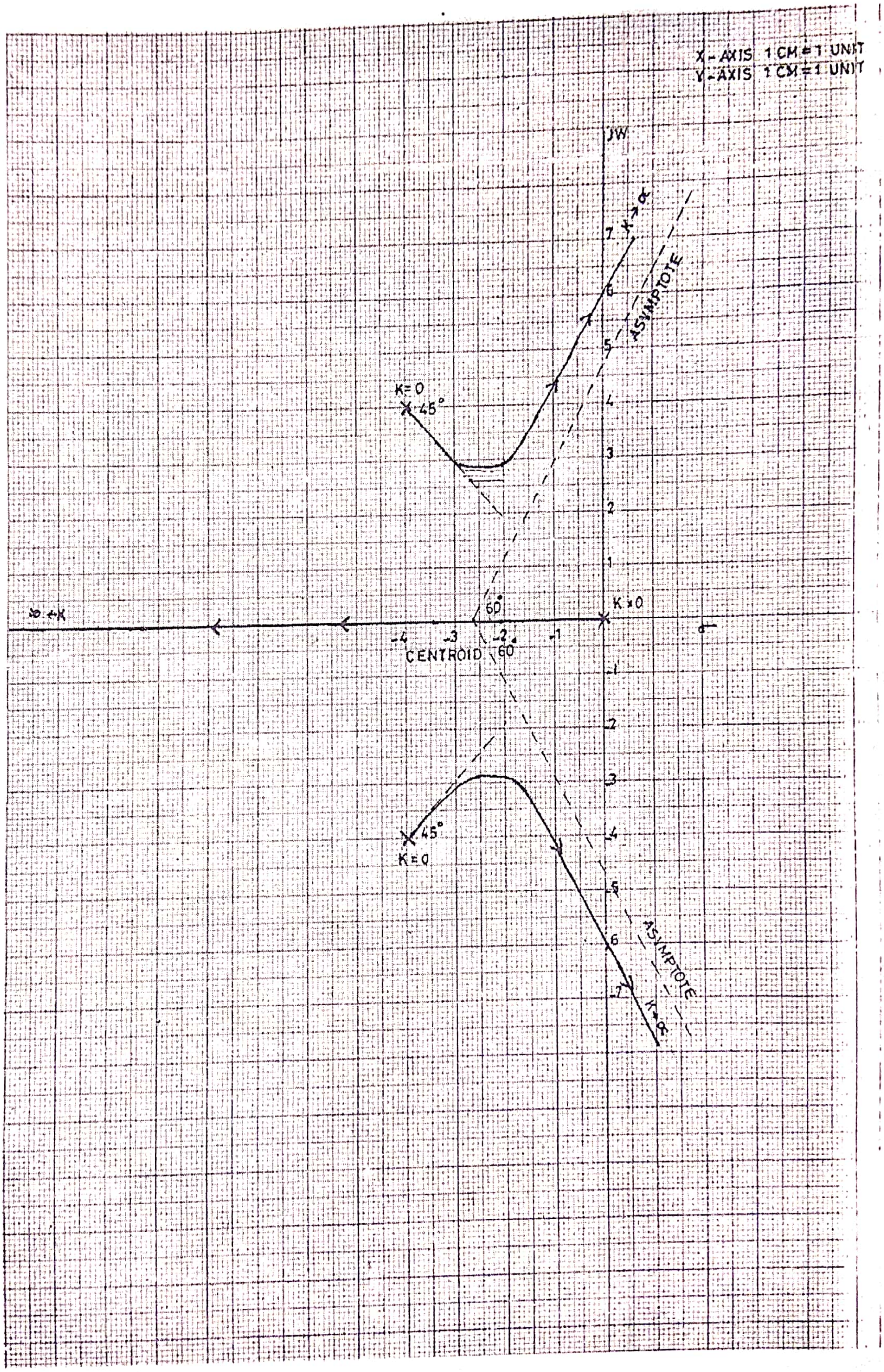

X - AXIS 1 CM = 1 UNIT
Y - AXIS 1 CM = 1 UNIT
JW
K → ∞
ASYMPTOTE
K = 0
45°
60°
K = 0
CENTROID
60
45°
K = 0
ASYMPTOTE
K → ∞
∞ ← K
σ

Fig. 3.35

Example 3.19. The poles of the with open loop transfer function

$$G(s) = \frac{k}{s(s+2)(s^2+2s+2)}$$

Solution .

The poles of the open loop transfer function are the roots of denominator. That is roots of

$$s(s+2)(s^2+2s+2) = 0$$

$$\Rightarrow \quad p_1 = 0$$

$$p_2 = -2$$

$$p_{3,4} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j1$$

Mark the poles with $\times$ symbol

Step 1. There are four open loop poles, hence the number of branches in the root locus are four.

Step 2. The four root locus branches start at $p_1 = 0, p_2 = -2, p_3 = -1 + j1$ and $p_4 = -1 - j1$ when $k = 0$ and terminates at infinity when $k = \infty$.

Step 3. All the points between -2 and '0' on the real axis lie on the root locus since there is one pole on the real axis to the right of these points.

Step 4. The four branches that terminate at infinity do so along the asymptotes with angles.

$$\phi_A = \frac{(2q+1)180°}{n-m}; q = 0, 1, 2 \ldots (n-m-1)$$

$$= \frac{(2q+1)180°}{4}; q = 0, 1, 2, 3$$

$$\text{For} \quad q = 0; \quad \phi_A = 45°$$

$$q = 1; \quad \phi_A = 135°$$

$$q = 2; \quad \phi_A = 225°$$

$$q = 3; \quad \phi_A = 335°$$

Step 5. The asymptotes meet at a point known as centroid

$$\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$

$$= \frac{0-2-1-1}{4} = -1$$

Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4 using protractor.

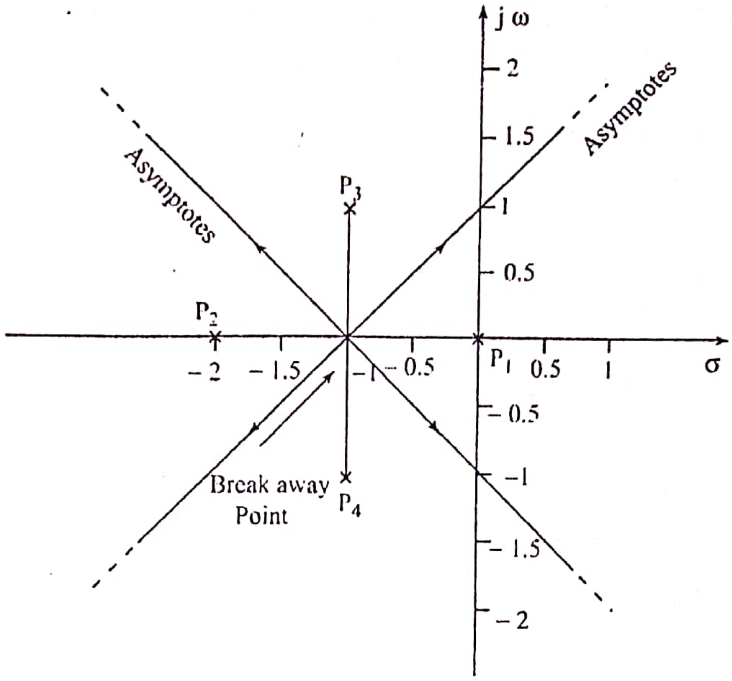


Fig. 3.36

Step 6. The break away points of the root locus are the solution of $\frac{dk}{ds} = 0$.

$$\text{Given} \quad G(s) = \frac{k}{s(s+2)(s^2+2s+2)}; H(s) = 1$$

The characteristic equation

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s+2)(s^2+2s+2)} = 0$$

$$\Rightarrow \quad k = -s(s+2)(s^2+2s+2)$$

$$\frac{dk}{ds} = -(4s^3 + 12s^2 + 12s + 4) = 0$$

$$= s^3 + 3s^2 + 3s + 1 = 0$$

$$= (s+1)^3 = 0 \Rightarrow \; s = -1$$

Step 7. The angle of departure ϕ_p of a root locus from a complex open loop pole is

$$\phi_p = 180° + \phi$$

where ϕ is the net angle contribution at this pole by all other open loop poles and zeros as shown in Fig. 3.37.

The angle of departure from p_3 is

$$\phi_{p3} = 180° + \phi$$

where

$$\phi = -\theta_1 - \theta_2 - \theta_4$$
$$= -135° - 45° - 90°$$
$$= 270°$$
$$\phi_{p3} = 180° - 270°$$
$$= -90°$$

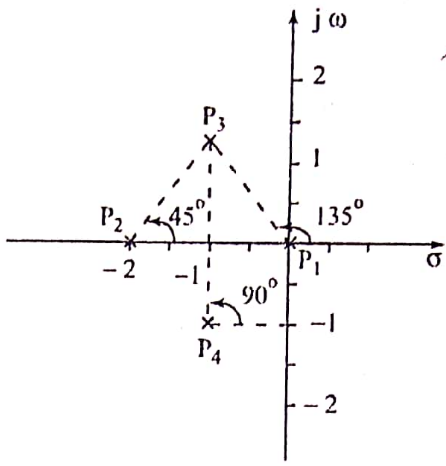


Fig. 3.37

Similarly the angle of departure from p_4 is

$$\bar{\phi}_{p3} = -(\phi_{p4}) = 90°$$

Step 8. The crossing point on the imaginary axis can be find using the Routh criterion. The characteristic equation is given by

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s+2)(s^2+2s+2)} = 0$$

$$\Rightarrow \quad s^4 + 4s^3 + 6s^2 + 4s + k = 0$$

s^4	1	6	k
s^3	4	4	0
s^2	$\frac{4(6)-1(4)}{4} = 5$	$\frac{4(k)}{4} = k$	
s^1	$\frac{20-4k}{5}$	0	
s^0	k		

For stability, all the elements in the first column of Routh array must be greater than zero. Therefore

$$\frac{20-4k}{5} > 0 \quad \text{and} \quad k > 0, \Rightarrow \quad 0 < k < 5$$

when $k = 5$ the root locus crosses imaginary axis. For $k = 5$, the auxiliary polynomial given by the coefficients of the s^2 row is

$$5s^2 + 5 = 0$$

$$\Rightarrow \quad s = \pm j1$$

The root locus has four branches start at the poles $p_1 = 0, p_2 = -2, p_3 = -1 + j1$ and $p_4 = -1 - j1$ and all the branches terminate at infinity. The root locus crosses the imaginary axis at $\pm j1$. The complete root locus plot is shown in Fig. 3.38.

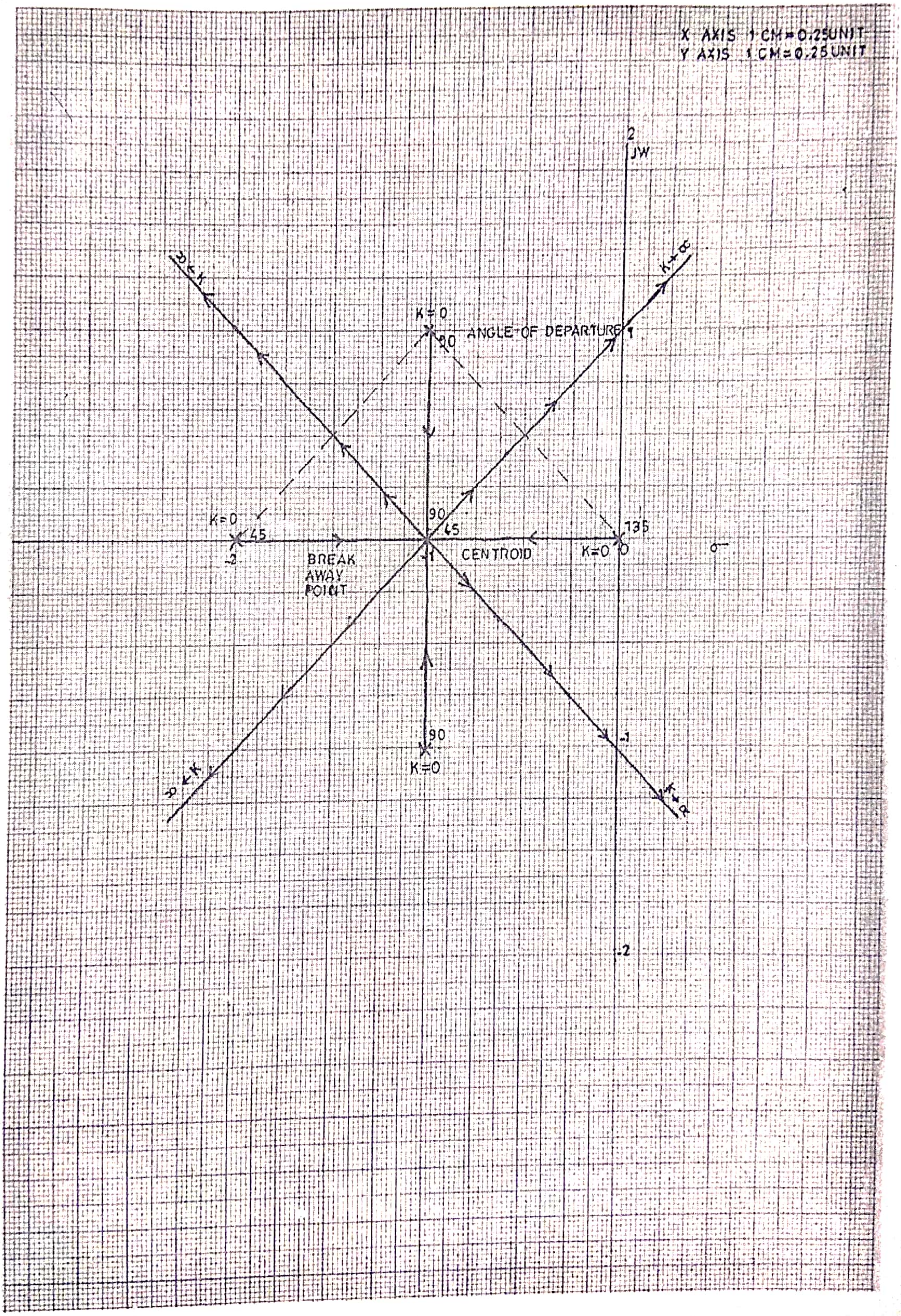

X AXIS 1 CM = 0.25 UNIT
Y AXIS 1 CM = 0.25 UNIT
2
JW
K → ∞
∞ ← K
K = 0
90
ANGLE OF DEPARTURE
1
K = 0
45
-2
90
45
135
BREAK
AWAY
POINT
-1
CENTROID
K = 0
0
σ
-1
90
K = 0
∞ ← K
K → ∞
-2

Fig. 3.38

Example 3.20. Sketch the root locus for a unity feedback system with open loop transfer function

$$G(s) = \frac{k(s+0.5)}{s^2(s+4.5)}$$

Solution .

The zeros are at $s = -0.5$. The poles are at $p_3 = -4.5$ and $p_{1,2} = 0$ (a double pole). Mark the poles with $\times$ symbol and mark the zero with 0 symbol .

Step 1 There are three open loop poles, hence the number of branches in the root locus are three.

Step 2 The three branches start at $p_{1,2} = 0$ and $p_3 = -4.5$ when $k = 0$. Since the number of zeros is one, out of these three branches, two branches terminate at infinity and one branch terminates at open loop zero when $k = \infty$.

Step 3 All the points between -4.5 and -0.5 are on the root locus since the sum of poles and zeros to the right of these points is equal to three.

Step 4 The two branches that terminate at infinity do so along the asymptotes with angles.

$$\begin{aligned}
\phi_A &= \frac{(2q+1)180^\circ}{n-m}; q = 0, 1, 2, \ldots (n-m-1) \\
&= \frac{(2q+1)180^\circ}{2}; q = 0, 1 \\
\Rightarrow \quad \phi_{A1} &= 90^\circ \\
\phi_{A2} &= 270^\circ
\end{aligned}$$

Step 5. The asymptotes meet at a point known as centroid

$$\begin{aligned}
\sigma_A &= \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{\text{Number of poles} - \text{Number of zeros}} \\
&= \frac{0.5 - 4.5}{2} = -2
\end{aligned}$$

Step 6. The break away point is the solution of $\dfrac{dk}{ds} = 0$.

$$1 + \frac{k(s+0.5)}{s^2(s+4.5)} = 0$$

$$\Rightarrow \quad k = \frac{-s^2(s+4.5)}{s+0.5}$$

$$\frac{dk}{ds} = -\left[\frac{(s+0.5)(3s^2+9s)-(s^3+4.5s^2)}{(s+0.5)^2}\right]$$
$$= \frac{-2s(s^2+3s+\frac{9}{4})}{(s+0.5)^2}$$
$$= \frac{-2s(s+\frac{3}{2})^2}{(s+0.5)^2}$$

Equating $\frac{dk}{ds} = 0$ yields

$$-2s\left(s+\frac{3}{2}\right)^2 = 0$$

$$\Rightarrow \quad s = 0, \frac{-3}{2}, \frac{-3}{2}$$

both points are on the root locus, hence both are breakaway points.

At $s = \frac{-3}{2}$; $\frac{dk}{ds}$ has double roots which means that not only $\frac{dk}{ds} = 0$ at this point but $\frac{d^2k}{ds^2}$ is also zero. That means the characteristic equation $1 + G(s)$ has a root of multiplicity three at $s = -\frac{3}{2}$. All the three loci originating from open loop poles will approach this point and then breakaway. The tangent to the three loci breaking away $s = \frac{-3}{2}$ is $\frac{360^\circ}{3} = 120$ apart. Similarly the tangent to the two loci originating from $s = 0$ is 180° apart.

Step 7. This step is not required for the given $G(s)$.

Step 8. From the asymptotic angles we can find that the root locus does not cross imaginary axis.

Applying Routh-Hururitz criterion to characteristic equation, we have

$$s^3 + 45s^2 + ks + 0.5k = 0$$

s^3	1	k	0
s^2	4.5	$0.5k$	0
s^1	$\frac{4k}{4.5}$	0	
s^0	$0.5k$		

$$0.5k > 0$$
$$\frac{4k}{4.5} > 0$$

The range of values of k is $0 < k < \infty$.

The first column of Routh-array does not become zero for any positive value of k. Therefore the root locus does not a cross the imaginary axis. Complete root locus plot is shown in Fig. 3.39.

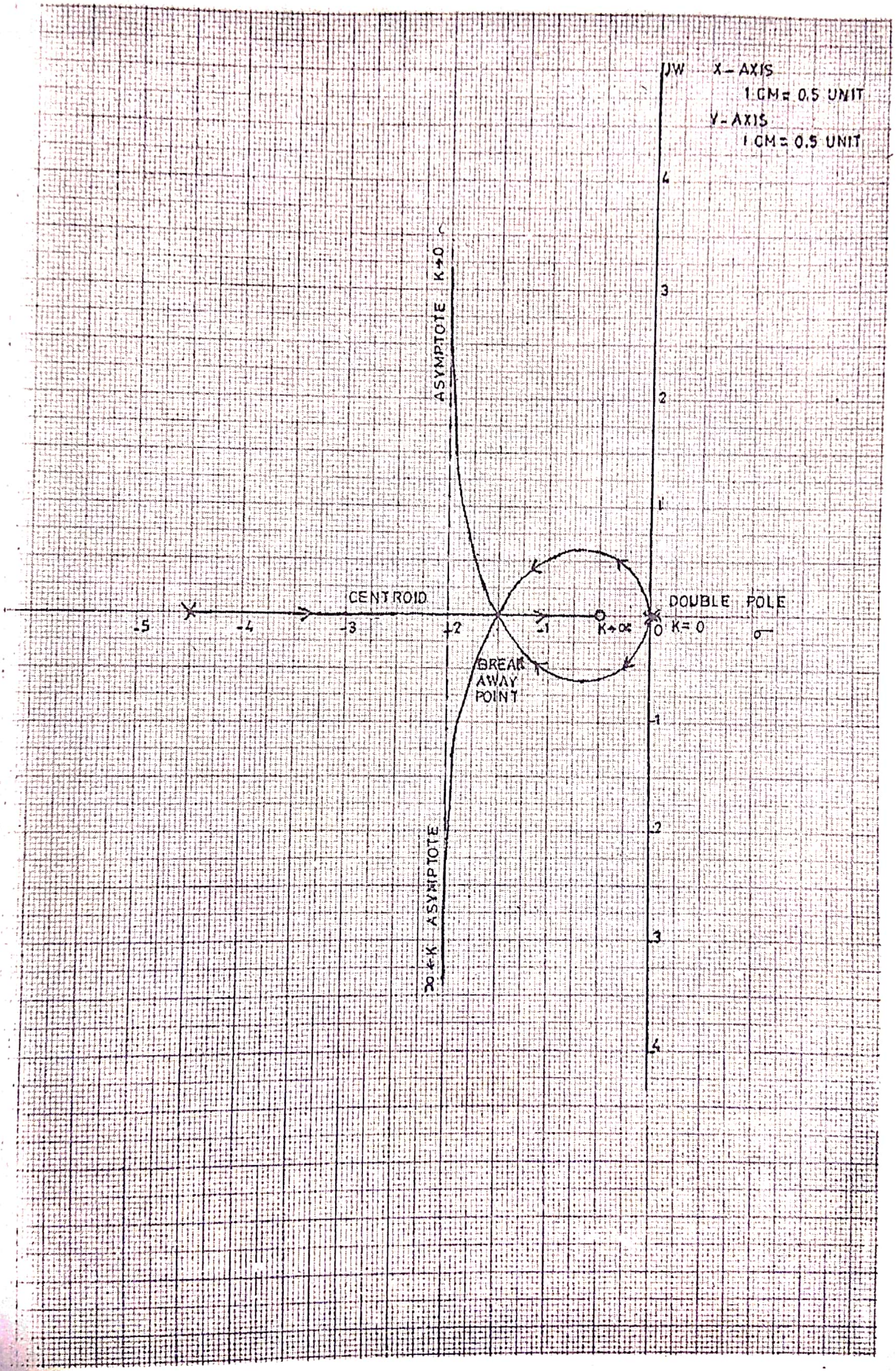

JW
X – AXIS
1 CM = 0.5 UNIT
Y – AXIS
1 CM = 0.5 UNIT
ASYMPTOTE K→0
CENTROID
DOUBLE POLE
K = 0
K→∞
σ
BREAK AWAY POINT
ASYMPTOTE
∞←K
-5
-4
-3
-2
-1
0
1
2
3
4
-1
-2
-3
-4

Fig. 3.39

Angle of Arrival

Example 3.21. Sketch the root locus of the unity feedback system whose open loop transfer function is

$$G(s) = \frac{k(s^2 - 2s + 2)}{(s+2)(s+3)(s+4)}$$

Solution .
The poles of the open loop transfer function are the roots of the denominator

$$(s+2)(s+3)(s+4) = 0$$
$$\Rightarrow \quad p_1 = -2; p_2 = -3; p_3 = -4$$

and the zeros are the roots of

$$s^2 - 2s + 2 = 0$$
$$\Rightarrow z_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = +1 \pm j1$$

Mark the poles with '×' symbol and zeros with ∘ symbol.

Step 1. There are three open loop poles, hence the number of branches in the root locus are three.

Step 2. The three branches of root locus start at open loop poles $p_1 = -2, p_2 = -3$ and $p_3 = -4$ when $k = 0$. Since the number of zeros are two, one of the branches terminates at infinity and remaining two branches terminate at open loop zeros when $k = \infty$.

Step 3. All the points between $-\infty$ to -4 and -3 to -2 on the real axis are on the root locus since the sum of zeros and poles to the right of these points are odd (3 and 1 respectively).

Step 4. The only one branch that terminates at infinity do so along the asymptote.

$$\begin{aligned}\phi_A &= \frac{(2q+1)180°}{n-m} q = 0, 1, 2 \ldots (n-m-1) \\ &= \frac{(2q+1)180°}{1}; q = 0 \\ &= 180°\end{aligned}$$

Step 5. The breakaway point on the root locus is the solution of $\frac{dk}{ds} = 0$. The characteristic equation is $1 + G(s)H(s) = 0$

$$1 + \frac{k(s^2 - 2s + 2)}{(s+2)(s+3)(s+4)} = 0$$
$$\Rightarrow \quad k = \frac{-(s^2 + 5s + 6)(s+4)}{s^2 - 2s + 2}$$

$$\frac{dk}{ds} = \frac{(s^2 - 2s + 2)(3s^2 + 18s + 26) - (s^3 + 9s^2 + 26s + 24)(2s - 2)}{(s^2 - 2s + 2)^2}$$

$$= 0$$

$$\Rightarrow s^4 - 4s^3 - 38s^2 - 12s + 100 = 0$$

The roots are

$$-2.3657, 1.4121, -3.529 \text{ and } 8.4821$$

The only point on the root locus is -2.3657.

Therefore the breakaway point is -2.3657.

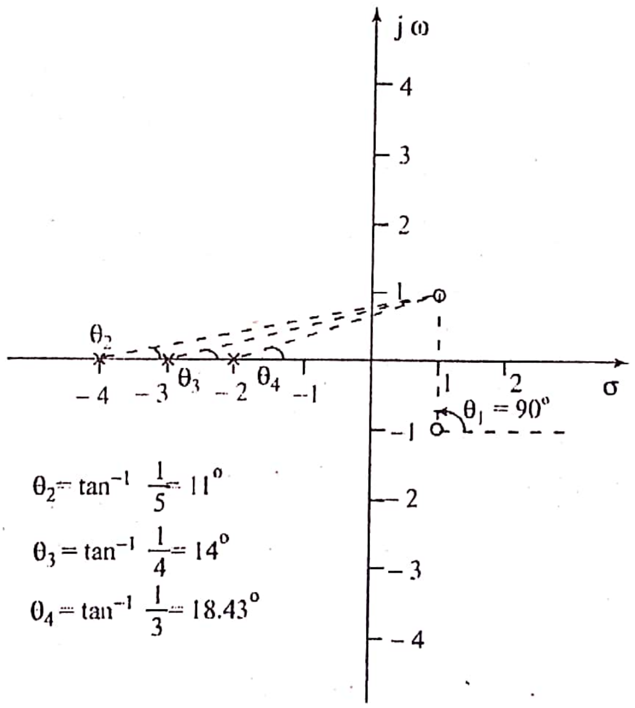


Fig. 3.40

Step 6. The angle of arrival at open loop zero is given by

$$\phi_2 = 180° - \phi$$

where ϕ is the net angle contribution at this zero by all other open loop poles and zeros as shown in Fig. 3.40.

$$\phi = \theta_1 - (\theta_2 + \theta_3 + \theta_4)$$
$$= 90° - (11° + 14° + 18.43°)$$
$$= 42.57°$$
$$\phi_{z1} = 180° - 42.57° = 137.43°$$
$$\phi_{z2} = \bar{\phi}_{z1} = -137.43°$$

Step 7. The intersection of the root locus with imaginary axis can be determined using Routh criterion

$$1 + k\frac{(s^2 - 2s + 2)}{(s+2)(s+3)(s+4)} = 0$$

The characteristic equation is

$$s^3 + (9+k)s^2 + (26-2k)s + 24 + 2k = 0$$

$$\begin{array}{c|cc}
s^3 & 1 & 26 - 2k \\
s^2 & 9 + k & 24 + 2k \\
s^1 & \dfrac{(9+k)(26-2k) - (24+2k)}{9+k} & 0 \\
s^0 & 24 + 2k &
\end{array}$$

$$(24 + 2k) > 0$$

$$\frac{(9+k)(26-2k) - (24+2k)}{9+k} > 0$$

$$\Rightarrow \quad k > -12$$

and

$$(9+k)(26-2k) - (24+2k) > 0$$

$$\Rightarrow \quad -2k^2 + 6k + 210 > 0$$

$$-k^2 + 3k + 105 > 0$$

$$k^2 - 3k - 105 < 0$$

$$\Rightarrow \quad (k - 11.8)(k + 88) < 0$$

$$\Rightarrow \quad k < 11.8 \text{ and } k < -8.8$$

Roots are

$$k_{1,2} = \frac{3 \pm \sqrt{6} + 420}{2}$$

$$k_{1,2} = \frac{3 \pm 20.6}{2}$$

$$\Rightarrow k_1 = 11.8$$

$$k_2 = -8.8$$

From above we can find that, the maximum value of k is 11.8. The auxiliary equation is

$$(9+k)s^2 + (24+2k) = 0$$

$$\Rightarrow \quad 20.8s^2 + 47.6s^2 = -\frac{47.6}{20.8} = -2.288$$

$$s = \pm j1.512$$

The points of intersection of the root locus with imaginary axis are $\pm j1.512$. Complete Root locus plot is shown in Fig. 3.41.

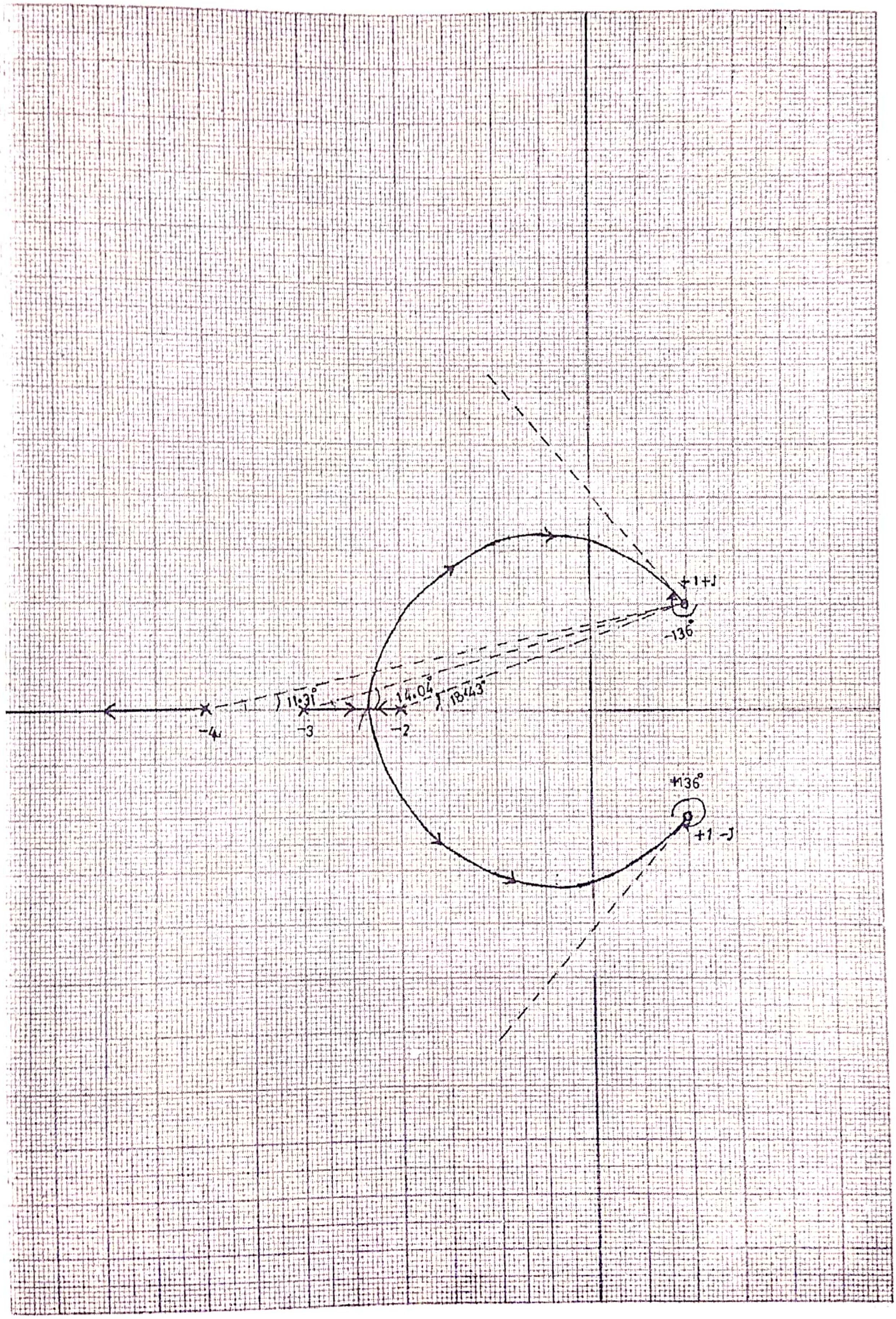


Fig. 3.41

Example 3.22. Sketch the root locus of the unity feedback system whose open loop transfer function is

$$G(s) = \frac{1}{s(s+2)(s^2+4s+13)}$$

Solution .

The poles of the open loop transfer function are the roots of the denominator

$$s(s+2)(s^2+4s+13) = 0$$
$$\Rightarrow p_1 = 0; p_2 = -2$$
$$p_{3,4} = \frac{-4 \pm \sqrt{16-52}}{2}$$
$$= -2 \pm j3$$

Step 1. There are four open loop poles, hence the number of branches in the root locus are four.

Step 2. The four branches of root locus start at open loop poles $p_1 = 0, p_2 = -2, p_3 = -2+j3$ and $p_4 = -2-j3$ when $k = 0$ and terminate at infinity when $k = \infty$.

Step 3. All the points between -2 and 0 on the real axis are on the root locus since there is only one pole to the right of these points.

Step 4. The four branches that terminate at infinity do so along the asymptotes ϕ_A.

$$\phi_A = \frac{(2q+1)180°}{n-m}; q = 0, 1, 2 \ldots (n-m-1)$$
$$= \frac{(2q+1)180°}{4}; q = 0, 1, 2, 3$$

when

$$q = 0 \quad \phi_{A1} = \frac{180°}{4} = 45°$$
$$q = 1 \quad \phi_{A2} = \frac{(3)180°}{4} = 135°$$
$$q = 2 \quad \phi_{A3} = \frac{5(180°)}{4} = 225°$$
$$q = 3 \quad \phi_{A4} = \frac{7(180°)}{4} = 315°$$

Step 5. The asymptotes meet at a point known as centroid σ_A

$$\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$
$$= \frac{0-2-2-2-0}{4} = -1.5$$

Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4 using protractor.

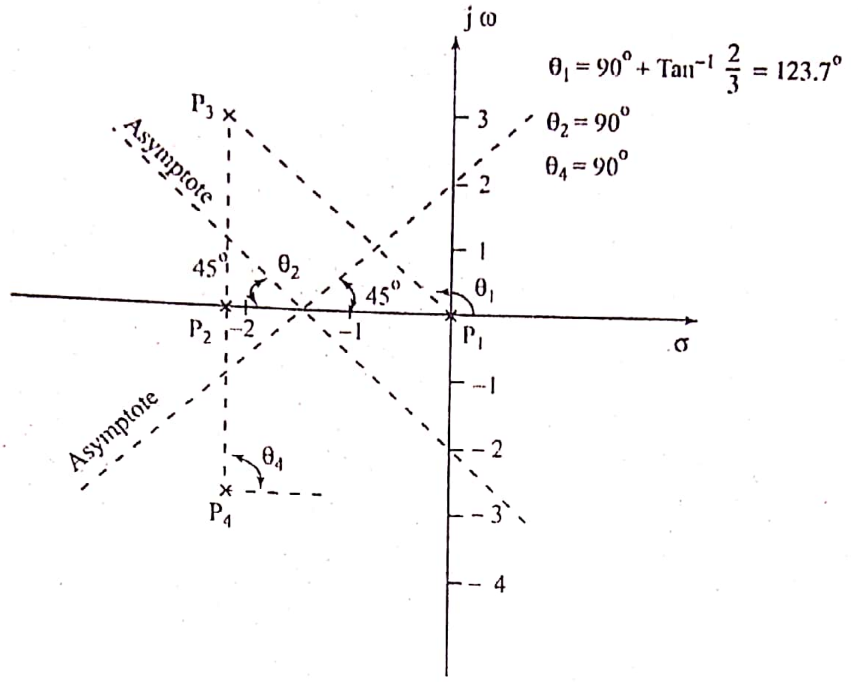


Fig. 3.42

Step 6. The breakaway points of the root locus are the solution of $\frac{dk}{ds} = 0$. Given

$$G(s) = \frac{k}{s(s+2)(s^2+4s+13)} \quad \text{and} \quad H(s) = 1$$

The characteristic equation is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s+2)(s^2+4s+13)} = 0$$

$$\Rightarrow \quad k = -s(s+2)(s^2+4s+13)$$

$$= -(s^4 + 6s^3 + 21s^2 + 26s)$$

$$\frac{dk}{ds} = 4s^3 + 18s^2 + 42s + 26 = 0$$

$$\Rightarrow \quad 2s^3 + 9s^2 + 21s + 13 = 0$$

From step 2 and step 3, we can find that two (out of four) root locus branches start at 0 and -2, approache each other and meet at a point and breakaway from there. Therefore there is a breakaway point between 0 and -2. Using trial and error procedure (see box) we can find that the breakaway point is at $s = -0.9$.

The $\frac{dk}{ds}$ can be written as

$$s^3 + 4.5s^2 + 10.55 + 6.5 = 0$$

at $s = -0.5$

$$(-0.5)^3 + 4.5(-0.5)^2 + 10.5(-0.5) + 6.5 = 2.25 \quad \text{(positive)}$$

at $s = -1$

$$(-1)^3 + 4.5(-1)^2 + 10.5(-1) + 6.5 = -0.5 \quad \text{(negative)}$$

For - 0.5 it is positive and for - 1 it is negative, therefore a root lies between -1 and -0.5 and is close to -1. Let us select $s = -0.9$

$$(-0.9)^3 + 4.5(-0.9)^2 + 10.5(-0.9) + 6.5 = -0.034$$
$$\simeq 0$$

The root is at $s = -0.9$. The remaining two roots can be found by dividing characteristic equation by $s + 0.9$

$$\begin{array}{r|l} & s^2 + 3.65 + 7.26 \\ \hline s + 0.9 & s^3 + 4.5s^2 + 10.5s + 6.5 \\ & s^3 + 0.9s^2 \\ \hline & \quad 3.6s^2 + 10.5s \\ & \quad 3.6s^2 + 3.24s \\ \hline & \qquad 7.26s + 6.5 \\ & \qquad 7.26s + 6.534 \\ \hline & \qquad\qquad \simeq 0 \end{array}$$

$s^3 + 3.6s + 7.26$. The roots are $\frac{-3.6 \pm \sqrt{(3.6)^2 - 4(7.26)}}{2} = -1.8 \pm j2$. These two points do not lie on root locus. Therefore the only breakaway point is at $s = -0.9$.

Step 7. The angle of departure from open loop pole is

$$\phi_p = 180^\circ + \phi$$

where ϕ is the net angle contribution at this pole by all other open loop poles and zeros.

$$\phi = -\theta_1 - \theta_2 - \theta_4$$

$$= -123.7^\circ - 90^\circ - 90^\circ \qquad \text{(refer to Fig. 3.42)}$$
$$= -303.7^\circ$$
$$\phi_{p3} = 180^\circ - 303.7$$
$$= -123.7^\circ$$
$$\phi_{p4} = \bar{\phi}_{p3} = -(-123.7^\circ) = 123.7$$

Step 8. The intersection of root locus with imaginary axis can be obtained using Routh's criterion.

The characteristic equation is given by

$$1 + \frac{k}{s(s+2)(s^2+4s+13)} = 0$$
$$s^4 + 6s^3 + 21s^2 + 26s + k = 0$$

$$\begin{array}{c|ccc}
s^4 & 1 & 21 & k \\
s^3 & 6 & 26 & 0 \\
s^2 & \frac{50}{3} & k & \\
s^1 & \dfrac{\frac{50}{3}(26) - 6k}{\frac{50}{3}} & 0 & \\
s^0 & k & &
\end{array}$$

for stability

$$k > 0$$

and

$$\frac{50}{3}(26) - 6k > 0$$
$$\Longrightarrow k < 72.22$$

The auxiliary equation is given by

$$\frac{50}{3}s^2 + k = 0$$
$$\frac{50}{3}s^2 + 72.22 = 0$$
$$\Longrightarrow s^2 = -4.333$$
$$s = \pm j2.08$$

Point of intersection of the root locus with imaginary axis is $\pm j2.08$. The complete root locus is shown in Fig. 3.43.

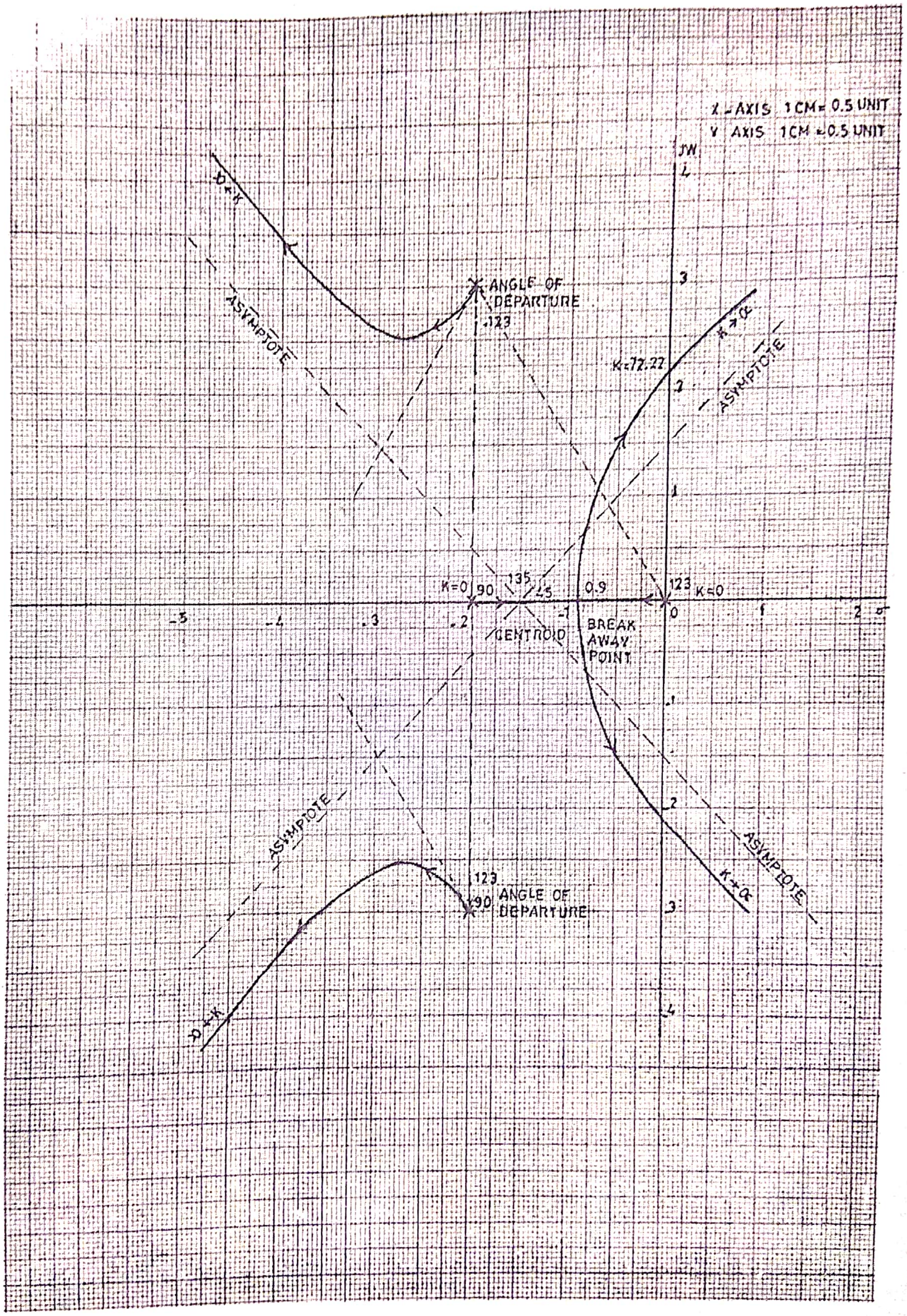

X - AXIS 1 CM = 0.5 UNIT
Y AXIS 1CM = 0.5 UNIT
jw
ASYMPTOTE
ANGLE OF DEPARTURE
123
K=72.27
K=0
90
135
45
0.9
123
K=0
CENTROID
BREAK AWAY POINT
ASYMPTOTE
ASYMPTOTE
ASYMPTOTE
123
90
ANGLE OF DEPARTURE
-5
-4
-3
-2
-1
0
1
2
σ

Fig. 3.43

Example 3.23. Sketch the root locus of the unity feedback system whose open loop transfer is

$$G(s) = \frac{1}{s(s+2)(s^2+2s+10)}$$

Solution .

The poles of the open loop transfer function are the roots of the denominator

$$s(s+2)(s^2+2s+10) = 0$$

$$\Longrightarrow p_1 = 0; \quad p_2 = -2; \quad P_{3,4} = \frac{-2 \pm \sqrt{4-40}}{2}$$

$$= -1 \pm j3$$

Step 1. There are four open loop poles, hence the number of branches in the root locus are four.

Step 2. The four branches of root locus start at open loop poles $P_1 = 0$; $P_1 = -2$; $P_3 = -1 + j3$ and $P_4 = -1 - j3$ when $k = 0$ and terminate at infinity when $k = \infty$

Step 3. All the points between -2 and 0 on the real axis are on the root locus since there is only one pole to the right of these points

Step 4. The four branches that terminate at infinity do so along the asymptote ϕ_A.

$$\phi_A = \frac{(2q+1)180°}{n-m}; \quad q = 0, 1, 2, \ldots (n-m-1)$$

$$= \frac{(2q+1)180°}{4}; \quad q = 0, 1, 2, 3$$

For $q = 0 \quad \phi_{A1} = 45°$

For $q = 1 \quad \phi_{A2} = 135°$

For $q = 2 \quad \phi_{A3} = 225°$

For $q = 3 \quad \phi_{A4} = 315°$

Step 5. The asymptotes meet at a point known as centroid σ_A

$$\sigma_A = \frac{\text{Sum of real parts of poles - Sum of real parts of zeros}}{\text{Number of poles - Number of zeros}}$$

$$= \frac{0-2-1-1-0}{4} = -1$$

Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4 using protractor.

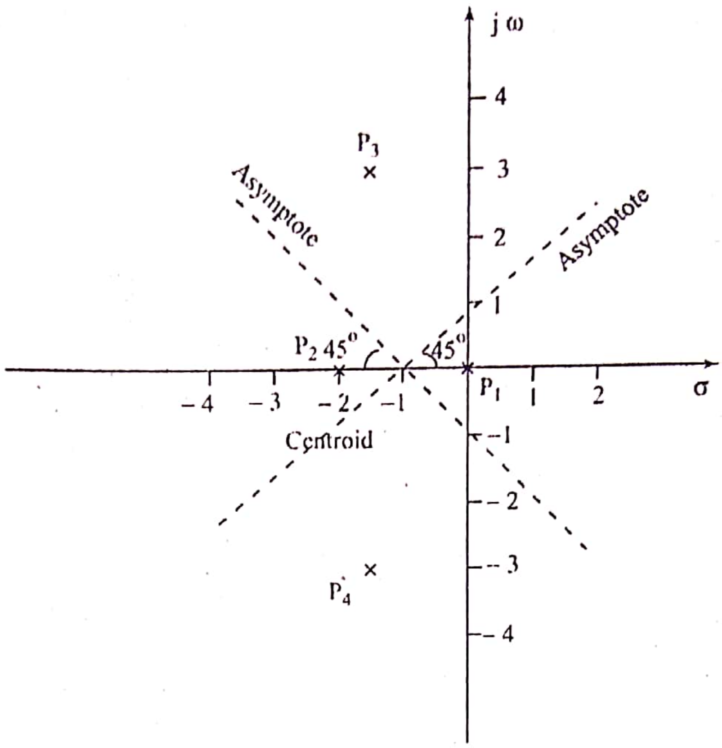


Fig. 3.44

Step 6. The break away points of the root locus are the solution of $\frac{dk}{ds} = 0$

Given $G(s) = \frac{k}{s(s+2)(s^2+2s+10)}$; $H(s) = 1$

The characteristic equation is

$$1 + G(s)H(s) = 1 + \frac{k}{s(s+2)(s^2+2s+10)} = 0$$

$$\Longrightarrow k = -s(s+2)(s^2+2s+10)$$

$$= -(s^2+2s)(s^2+2s+10)$$

$$= -(s^4+4s^3+14s^2+20s)$$

$$\frac{dk}{ds} = 4s^3 + 12s^2 + 28s + 20 = 0$$

$$\Longrightarrow s^3 + 3s^2 + 7s + 5 = 0$$

$$(s+1)(s^2+2s+5) = 0$$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 7 & 5 \\ & 0 & -1 & -2 & -5 \\ \hline & 1 & 2 & 5 & \boxed{0} \end{array}$$

$$s^2 + 2s + 5 = 0$$

$$\frac{-2 \pm \sqrt{4-20}}{2}$$

$$-1 \pm j2$$

The roots are

-1; $\quad -1 \pm j2$

The break away points are

-1; $\quad -1 \pm j2$

Step 7. The angle of departure from open loop pole is

$$\phi_p = 180° + \phi$$

where ϕ is the net angle contribution at this pole by all other open loop poles and zeros

$$\phi_{p3} = 180^\circ + \phi$$

$$\text{where} \quad \phi = -\theta_1 - \theta_2 - \theta_3 \qquad \text{(refer to Fig. 3.45)}$$

$$\theta_1 = 90^\circ + \tan^{-1}\frac{1}{3} = 108.27^\circ$$

$$\theta_2 = 90^\circ$$

$$\theta_3 = \tan^{-1} 3 = 71.56^\circ$$

$$\phi = -108.27^\circ - 90^\circ - 71.56^\circ$$

$$= -270^\circ 23' \simeq -270^\circ$$

$$\phi_{p3} = 180^\circ - 270^\circ = -90^\circ$$

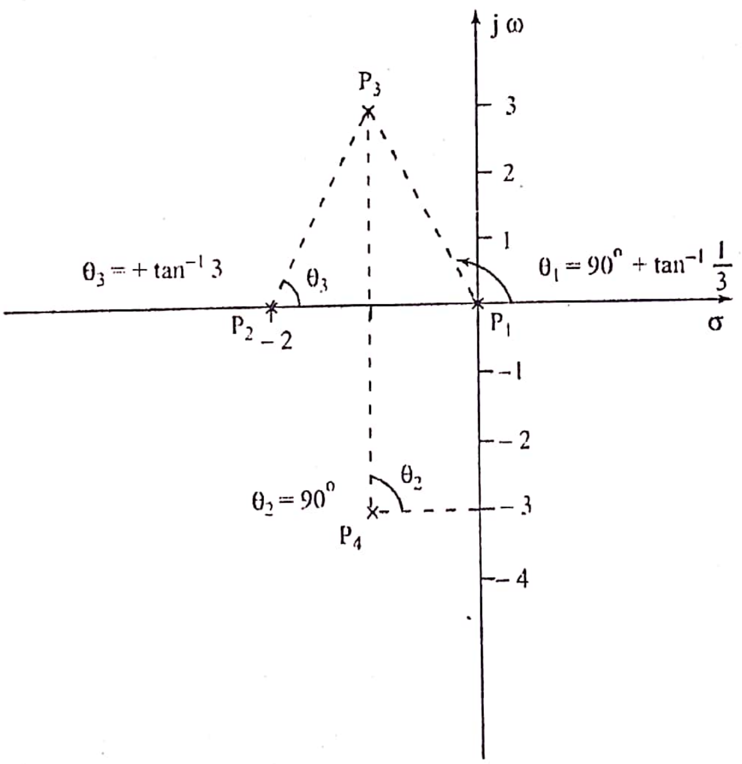


Fig. 3.45

Similarly

$$\phi_{p4} = \bar{\phi}_{p3} = -(-90^\circ) = 90^\circ$$

The intersection of root locus with imaginary axis can be obtained using Routh's criterion

Step 8. The characteristic equation is given by

$$1 + \frac{k}{s(s+2)(s^2+2s+10)} = 0$$

$$s^4 + 4s^3 + 14s^2 + 20s + k = 0$$

s^4	1	14	k
s^3	4	20	0
s^2	9	k	
s^1	$\frac{180-4k}{9}$	0	
s^0	k		

For stability

$k > 0$

$$\frac{180 - 4k}{9} > 0 \quad k < 45$$
$$\Longrightarrow 0 < k < 45$$

The auxiliary equation is given by

$$9s^2 + k = 0$$
$$9s^2 + 45 = 0$$
$$\Longrightarrow s = \pm j2.236$$

The points of intersection of root locus with imaginary axis are $\pm j2.236$. The complete root locus plot is shown in Fig. 3.46.

Example 3.24. Sketch the root locus of the unity feedback system whose open loop transfer functions is

$$G(s) = \frac{1}{s(s+2)(s+4)(s+5)}$$

Solution.
The poles of the open loop transfer function are the roots of the denominator of $G(s)$

$$\text{ie.,} \quad s(s+2)(s+4)(s+5) = 0$$
$$\Longrightarrow P_1 = 0; \quad P_2 = -2; \quad P_3 = -4; \quad P_4 = -5$$

Step 1. There are four open loop poles, hence the number of branches in the root locus are four.

Step 2. The four branches of root locus start at open loop poles $P_1 = 0$; $P_2 = -2$; $P_3 = -4$ $P_4 = -5$ when $k = 0$ and terminate at infinity when $k = \infty$.

Step 3. All the points between -5 and -4, -2 and 0 on the real axis are on the root locus since the sum of poles and zeros to the right of these points are odd (3 and 1 respectively).

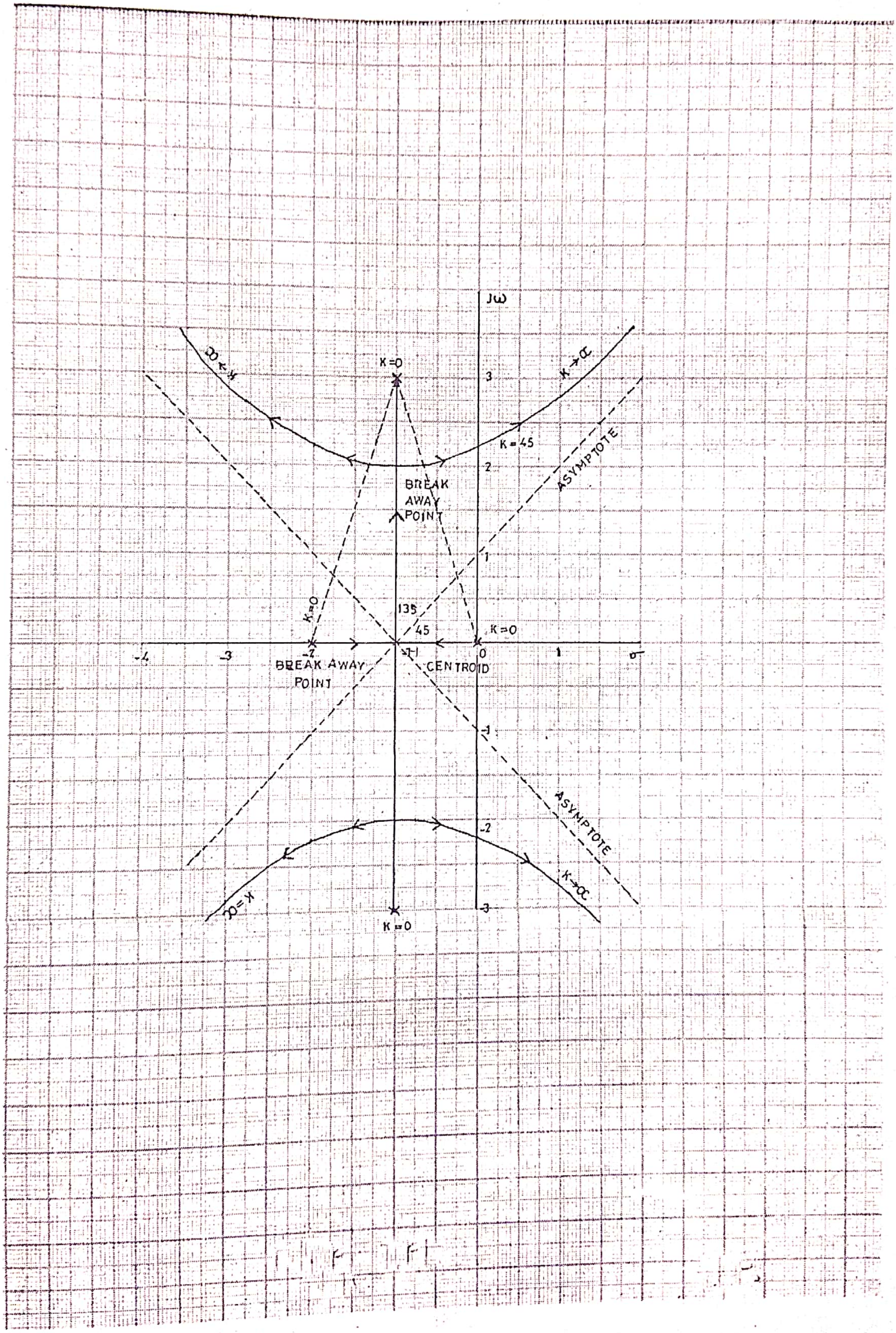

JW
K = 0
K → ∞
K = 45
ASYMPTOTE
BREAK
AWAY
POINT
K=0
135
45
K = 0
-4
-3
-2
-1
0
1
σ
BREAK AWAY
POINT
CENTROID
-1
-2
-3
ASYMPTOTE
K → ∞
K = ∞
K = 0

Fig. 3.46

Step 4. The four branches that terminate at infinity do so along the asymptote ϕ_A

$$
\begin{aligned}
\phi_A &= \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2 \ldots (n-m-1) \\
&= \frac{(2q+1)180^\circ}{4} \quad q = 0, 1, 2, 3 \\
\implies \phi_{A_1} &= 45^\circ \quad \phi_{A_2} = 135^\circ \quad \phi_{A_3} = 225^\circ \quad \text{and} \quad \phi_{A_4} = 315^\circ
\end{aligned}
$$

Step 5. The asymptotes meet at a point known as centroid σ_A

$$
\begin{aligned}
\sigma_A &= \frac{\text{Sum of real parts of poles - Sum of real parts of zeros}}{\text{Number of poles - Number of zeros}} \\
&= \frac{-0-2-4-5}{4} = -\frac{11}{4} = -2.75
\end{aligned}
$$

Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4 using protractor.

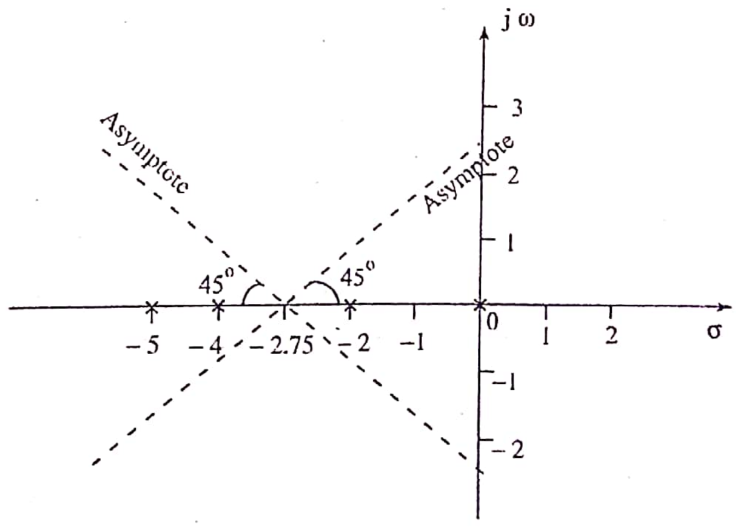


Fig. 3.47

Step 6. The break away points of the root locus are the solution of $\frac{dk}{ds} = 0$

$$
\begin{aligned}
k &= -s(s+2)(s+4)(s+5) \\
&= -(s^2+2s)(s^2+9s+20) \\
&= -(s^4+11s^3+38s^2+40s) \\
\frac{dk}{ds} &= 4s^3+33s^2+76s+40 = 0 \\
&= s^3+8.25s^2+19s+10 = 0
\end{aligned}
$$

From step 2 and step 3 and from Fig. 3.47, we can find that the break away points lie between 0 and -2; and between -4 and -5.

First break away point

At s = - 4.75

$$(-4.75)^3 + 8.25(-4.75)^2 + 19(-4.75) + 10 = 1.28$$

At s = - 4.5

$$(-4.5)^3 + 8.25(-4.5)^2 + 19(-4.5) + 10 = 0.4375$$

The root lies between -4.75 and -4.5

At s = - 4.6

$$(-4.6)^3 + 8.25(-4.6)^2 + 19(-4.6) + 10 = -0.166$$

At s = - 4.55

$$(-4.55)^3 + 8.25(-4.55)^2 + 19(-4.55) + 10 = 0.149$$

At s = - 4.57

$$(-4.57)^3 + 8.25(-4.57)^2 + 19(-4.57) + 10 = 0.026$$

At s = - 4.58

$$(-4.58)^3 + 8.25(-4.58)^2 + 19(-4.58) + 10 = -0.0366$$

Therefore the first break away point is at -4.575

Similarly the other break away point can be found at $s = -0.74$

The break away points are at -4.575 and at -0.74.

Step 7. The intersection of root locus with imaginary axis can be obtained using Routh's criterion.

The characteristic equation is given by

$$1 + \frac{k}{s(s+2)(s+4)(s+5)} = 0$$

$$s^4 + 11s^3 + 38s^2 + 40s + k = 0$$

s^4	1	38	k
s^3	11	40	0
s^2	34.36	k	
s^1	$\frac{1374.5-11k}{34.36}$	0	
s^0	k		

For stability

$k > 0$

$$\text{and} \quad \frac{1374.5 - 11k}{34.36} > 0$$

$$\Longrightarrow 0 < k < 124.9$$

When $k = 124.9$, the roots are on the imaginary axis.

The auxiliary equation is given by

$$34.36s^2 + 124.9 = 0$$

$$s = \pm j1.9$$

The points of intersection with imaginary axis are $\pm j1.9$. The complete root locus plot is shown in Fig. 3.48.

Example 3.25. Draw the root locus for the unity feedback system whose open loop transfer function is

$$G(s) = \frac{k(s+1)}{(s-1)(s+2)(s+4)}.$$

Find the range of k for which the system is stable.

Solution .

The poles are the roots of the denominator, that is the roots of

$$(s-1)(s+2)(s+4) = 0$$

$$\Longrightarrow p_1 = 1; \quad p_2 = -2; \quad p_3 = -4$$

The zeros are the root of the numerator

$$z_1 = -1$$

Step 1. There are three open loop poles, hence the number of branches in the root locus are three as shown in Fig. 3.49.

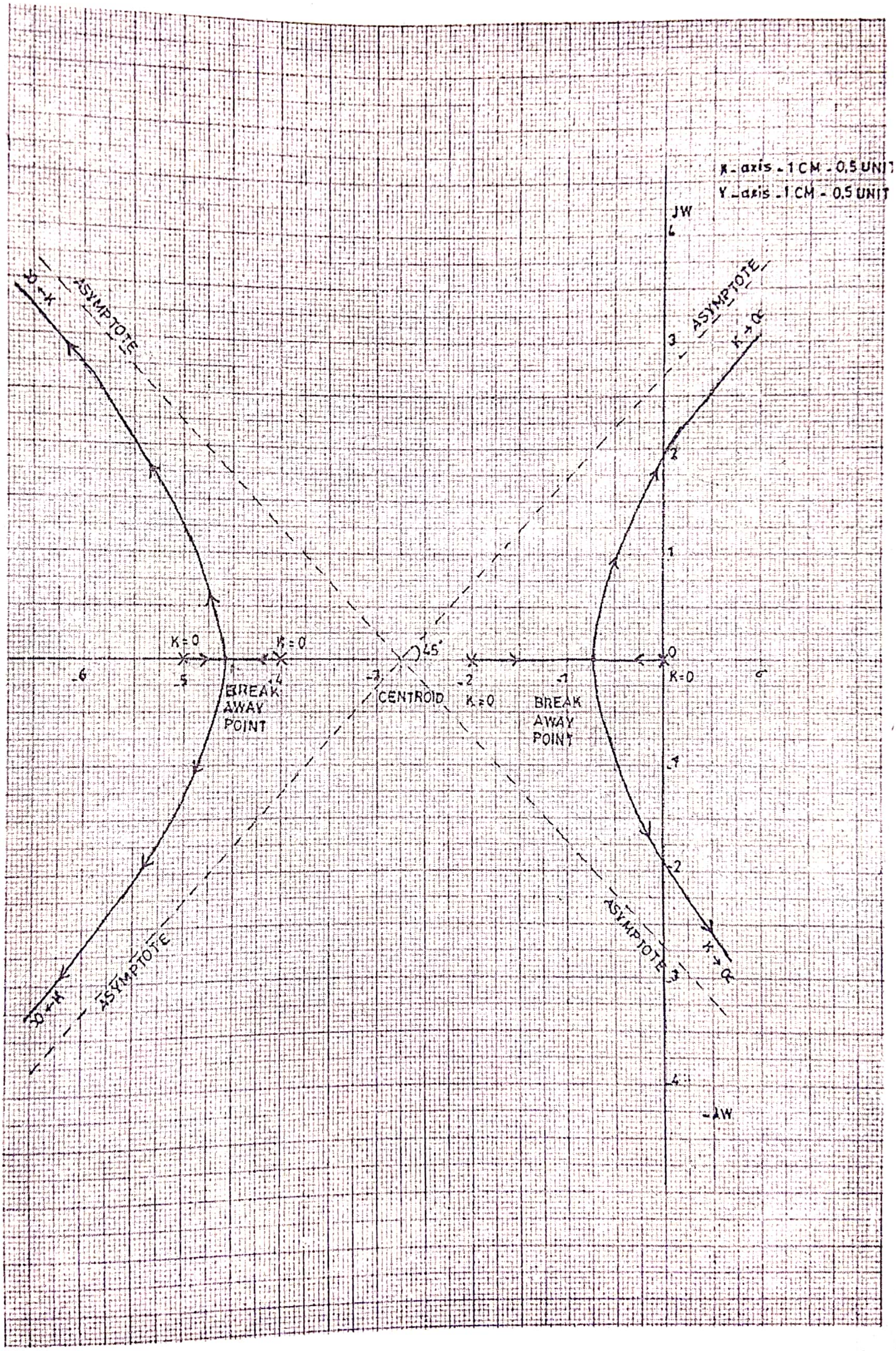

X - axis - 1 CM - 0.5 UNIT
Y - axis - 1 CM - 0.5 UNIT
JW
ASYMPTOTE
K → ∞
K = 0
BREAK AWAY POINT
CENTROID
45°
σ
-JW

Fig. 3.48

Step 2. The three branches of root locus starts at open loop poles $p_1 = 1$; $p_2 = -2$ and $p_3 = -4$ when $k = 0$. Since there is one open loop zero, out of the three branches one branch terminate at open loop zero and remaining two branches terminate at infinity when $k = \infty$.

Step 3. All the points between -4 and -2 as well as between -1 and 1 on the real axis are on the root locus since there are odd number of poles and zeros to the right of these points.

Step 4. The two branches that terminate at infinity do so along the asymptote at the angle

$$\begin{aligned}\phi_A &= \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2 \dots (n-m-1) \\ &= \frac{(2q+1)180^\circ}{2} \quad q = 0, 1 \\ \implies \phi_{A_1} &= 90^\circ \quad \phi_{A_2} = 270^\circ\end{aligned}$$

Step 5. The asymptotes meet at a point known as centroid σ_A

$$\begin{aligned}\sigma_A &= \frac{\text{Sum of real parts of poles - Sum of real parts of zeros}}{\text{Number of poles - Number of zeros}} \\ &= \frac{1-2-4}{2} = -2\end{aligned}$$

Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4 using protractor.

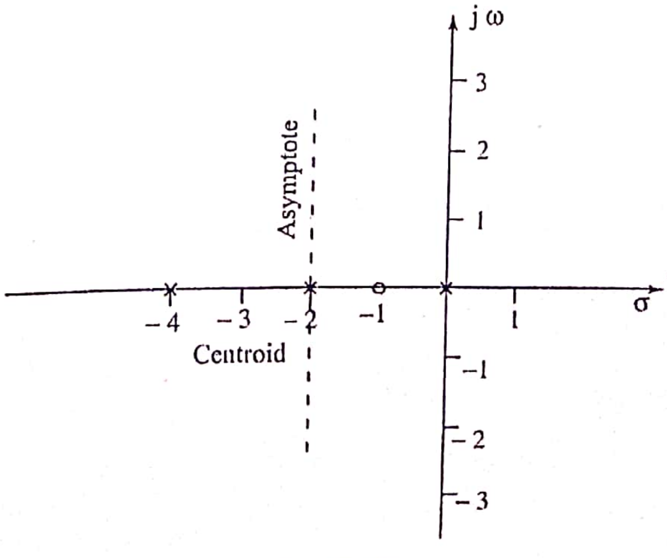


Fig. 3.49

Step 6. The break away points of the root locus are the solution of $\frac{dk}{ds} = 0$

$$k = \frac{-(s-1)(s+2)(s+4)}{(s+1)} = \frac{-(s^2+s-2)(s+4)}{s+1}$$
$$= \frac{-(s^3+5s^2+2s-8)}{s+1}$$
$$\frac{dk}{ds} = -\left[\frac{(s+1)[3s^2+10s+2]-[s^3+5s^2+2s-8]}{(s+1)^2}\right] = 0$$
$$= -\left[3s^3+3s^2+10s^2+10s+2s+2-s^3-5s^2-2s+8\right] = 0$$
$$= -[2s^3+8s^2+10s+10] = 0$$
$$\Rightarrow s^3+4s^2+5s+5 = 0$$

Step 7. is not required.

Step 8. The intersection of root locus with imaginary axis can be obtained using Routh's criterion.

The characteristic equation is given by

$$1 + \frac{k(s+1)}{(s-1)(s+2)(s+4)} = 0$$
$$\Longrightarrow (s-1)(s+2)(s+4) + k(s+1) = 0$$

$$s^3 + 5s^2 + (2+k)s + (-8+k) = 0$$

s^3	1	$2+k$
s^2	5	$(-8+k)$
s^1	$\frac{2+6k}{5}$	0
s^0	$-8+k$	

For stability

$$-8+k > 0 \quad \Longrightarrow k > 8$$
$$\frac{2+6k}{5} > 0 \Longrightarrow 2+6s > 0$$
$$6k > -2$$
$$k > -\frac{2}{6}$$
$$k > -\frac{1}{3}$$

The range of k for which the system being stable is $8 < k < \infty$. The asymptotes are 90° and 270° so the locus do not cross the $j\omega$-axis. The complete root locus plot is shown in Fig. 3.50.

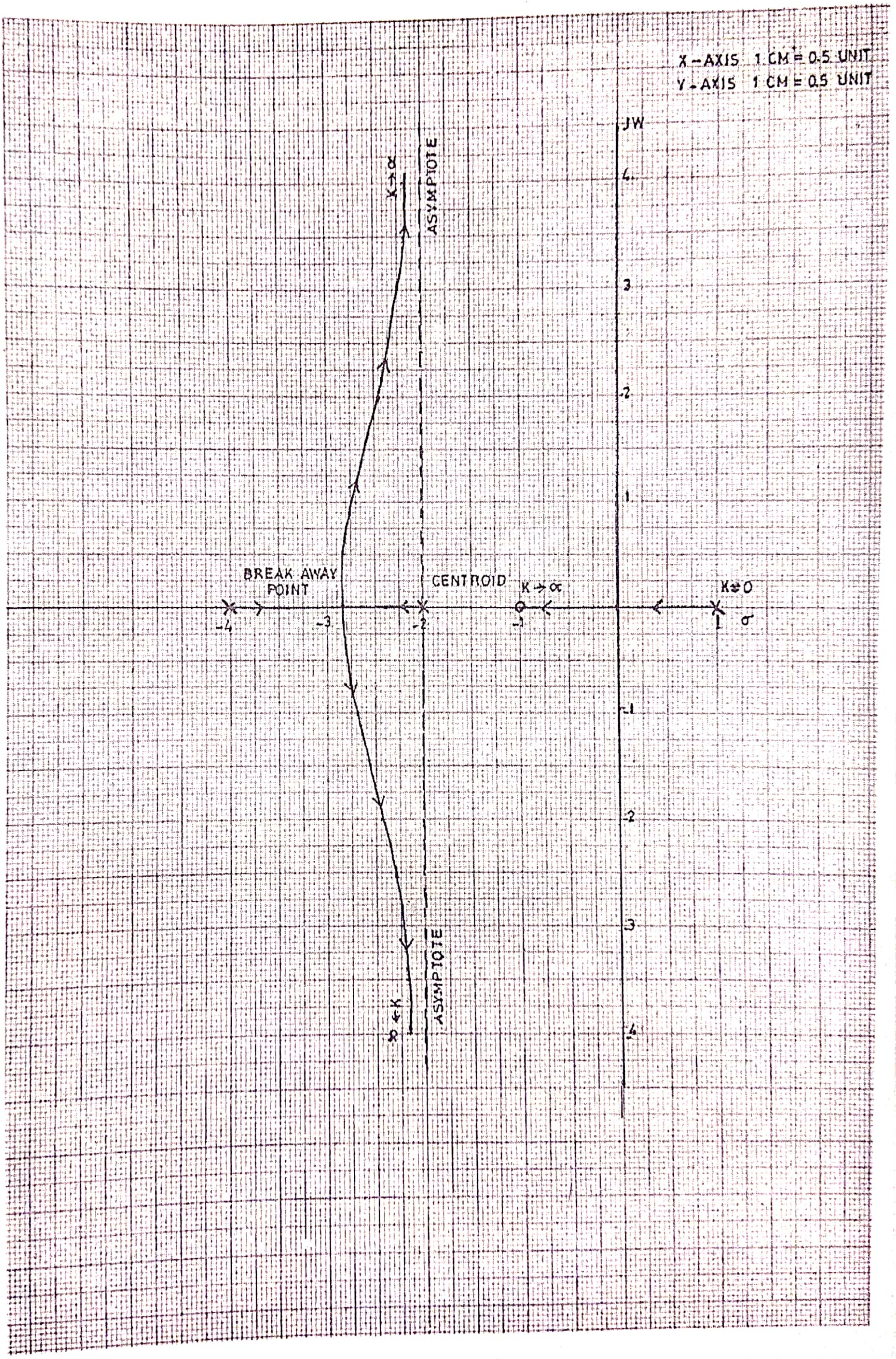

X – AXIS 1 CM = 0.5 UNIT
Y – AXIS 1 CM = 0.5 UNIT
JW
K→∝
ASYMPTOTE
BREAK AWAY POINT
CENTROID
K→∝
K=0
σ
-4
-3
-2
-1
1
2
3
4
-1
-2
-3
-4
ASYMPTOTE
∝←K

Fig. 3.50

Example 3.26. Draw the root locus of the unity feedback system whose open loop transfer function is

$$G(s) = \frac{s}{(s^2+4)(s+2)}$$

Solution .

The open loop poles are the roots of the denominator

$$(s^2+4)(s+2) = 0$$
$$\Longrightarrow p_1 = -2; \quad p_2 = +2j; \quad p_3 = -2j$$

and zero is at origin

Step 1. There are three open loop poles, therefore the number of branches in the root locus are three as shown in Fig. 3.51.

Step 2. All the three branches in the root locus start from open loop poles $p_1 = -2$; $p_2 = +2j$ and $p_3 = -2j$ out of these three branches one branch terminates at open loop zero and the remaining two branches terminate at infinity when $k = \infty$.

Step 3. All the points between -2 and 0 on the real axis are lying on the root locus.

Step 4. The two branches that terminate at infinity do so along the asymptote whose angles are

$$\phi_A = \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2 \ldots (n-m-1)$$
$$= \frac{(2q+1)180^\circ}{2} \quad q = 0, 1$$
$$\Longrightarrow = 90^\circ, \quad 270^\circ$$

Step 5. The asymptotes meet at a point known as centroid

$$\sigma_A = \frac{\text{Sum of real part of poles - Sum of real part of zeros}}{\text{Number of poles - Number of zeros}}$$
$$= \frac{-2}{2} = -1$$

Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4. The complete root locus is shown in Fig. 3.51.

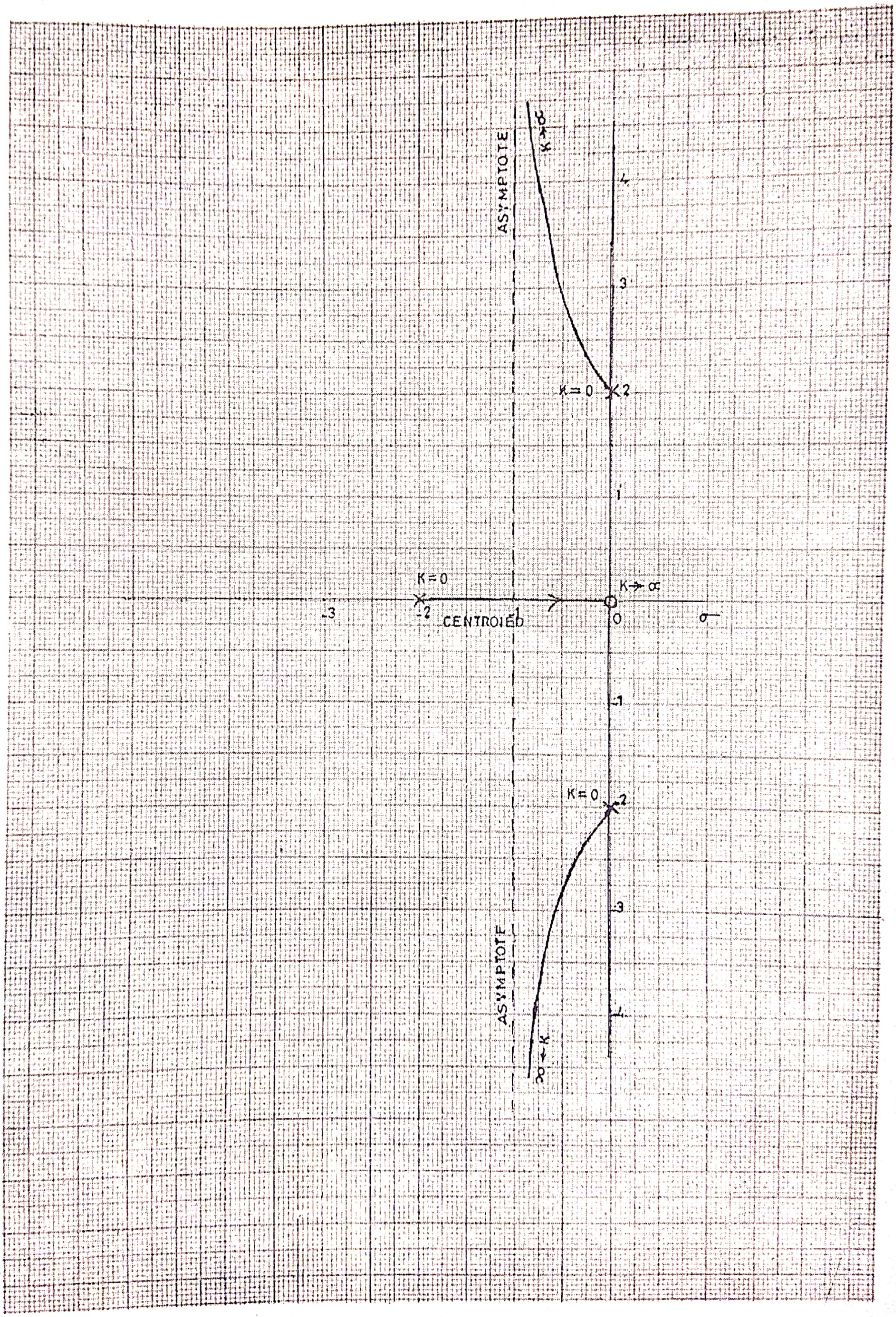


Fig. 3.51

Example 3.27. Plot the root locus for a unity feedback closed loop system whose open loop transfer function is

$$G(s) = \frac{1}{s(s+4)(s^2+2s+2)}$$

Solution .

The open loop poles are the roots of the denominator of $G(s)$

$$s(s+4)(s^2+2s+2) = 0$$

$$\Longrightarrow P_1 = 0; \quad P_2 = -4; \quad P_{3,4} = \frac{-2 \pm \sqrt{4-8}}{2}; \quad -1 \pm j1$$

Step 1. There are four open loop poles, therefore the number of branches in the root locus are four as shown in Fig. 3.52.

Step 2. All the four branches in the root locus start from open loop poles $P_1 = 0$; $P_2 = -4$; $P_3 = -1 + j1$ and $P_4 = -1 - j1$ when $k = 0$ and terminate at infinity when $k = \infty$.

Step 3. All the point between -4 and 0 lie on the root locus

Step 4. The four branches that terminate at infinity do so along the asymptote with the angles

$$\phi_A = \frac{(2q+1)180°}{n-m} \quad q = 0, 1, 2 \ldots (n-m-1)$$

$$= \frac{(2q+1)180°}{3} \quad q = 0, 1$$

$$\text{For } q = 0, \quad \phi_{A_1} = 45°$$
$$\text{For } q = 1, \quad \phi_{A_2} = 135°$$
$$\text{For } q = 2, \quad \phi_{A_3} = 225°$$
$$\text{For } q = 3, \quad \phi_{A_4} = 315°$$

Step 5. The asymptotes meet at a point known as centroid σ_A

$$\sigma_A = \frac{\text{Sum of real parts of poles - Sum of real parts of zeros}}{\text{Number of poles - Number of zeros}}$$

$$= \frac{0-4-1-1}{4} = -1.5$$

Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4.

Step 6. The break away points of the root locus are the solution of $\frac{dk}{ds} = 0$

$$\begin{aligned} k &= -s(s+4)(s^2+2s+2) \\ &= -(s^2+4s)(s^2+2s+2) \\ &= -(s^4+6s^3+10s^2+8s) \end{aligned}$$

$$\frac{dk}{ds} = 4s^3 + 18s^2 + 20s + 8 = 0$$

$$\Longrightarrow 2s^3 + 9s^2 + 10s + 4 = 0$$

$$\Longrightarrow s^3 + 4.5s^2 + 5s + 2 = 0$$

By trial and error procedure, we find the break away point at $s = -3.09$.

Step 7. The intersection of root locus with imaginary axis can be obtained using Routh's criterion.

The characteristic equation is given by

$$s^4 + 6s^3 + 10s^2 + 8s + k = 0$$

$$\begin{array}{c|ccc} s^4 & 1 & 10 & k \\ s^3 & 6 & 8 & 0 \\ s^2 & \frac{26}{3} & k & \\ s^1 & \dfrac{\frac{208}{3} - 6k}{\frac{26}{3}} & 0 & \\ s^0 & k & & \end{array}$$

For stability

$$k > 0$$

$$\text{and} \quad \frac{208}{3} - 6k > 0 \Longrightarrow 0 < k < \frac{107}{9}$$

The two branches of root locus cross the imaginary axis when $K = \frac{107}{9}$. To find the point of intersections with imaginary axis, consider the auxiliary equation is $\frac{26}{3}s^2 + \frac{104}{9} = 0$.

$$\therefore \quad \text{Points of intersection are} \quad s = \pm j1.154$$

The complete root locus is shown in Fig. 3.52.

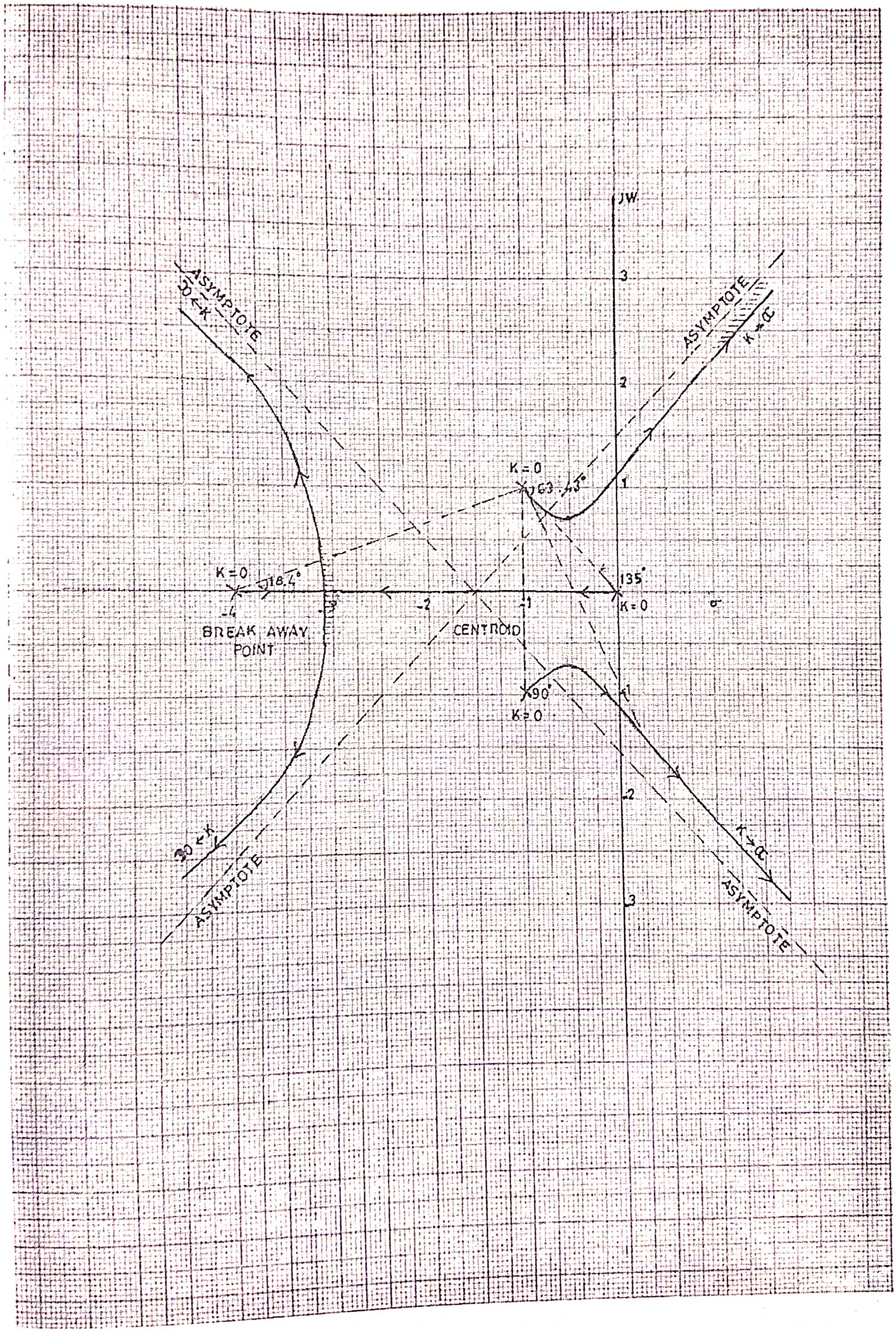

JW
ASYMPTOTE
ASYMPTOTE
ASYMPTOTE
ASYMPTOTE
K → ∞
K → ∞
K = 0
K = 0
K = 0
K = 0
63.43°
18.4°
135°
90°
BREAK AWAY POINT
CENTROID
σ
-4
-3
-2
-1
1
2
3
-1
-2
-3

Fig. 3.52

Example 3.28. A unity-feedback control system has an open loop transfer function

$$G(s) = \frac{k}{s(s+4)(s^2+8s+32)}; k \geq 0$$

Sketch the root locus of the system. Hence find the value of k so that the system has a damping factor of 0.707.

Solution .
The open loop poles are

$$p_1 = 0; p_2 = -4 \quad p_{3,4} = -4 \pm j4$$

Step 1. There are four open loop poles, hence the number of branches in the root locus are four.

Step 2. The four branches of the root locus originate from the open loop poles $p_1 = 0, p_2 = -4, p_3 = -4 + j4$ and $p_4 = -4 - j4$, when $k = 0$.
All the branches terminate at infinity when $k = \infty$.

Step 3. All the points between 0 and -4 are on the root locus since there is only one (odd) pole to the right of these points.

Step 4. The four root locus branches that proceed to infinity do so along the asymptotes with angles

$$\phi_A = \frac{(2q+1)180^\circ}{4}; q = 0, 1, 2, 3$$

$$\Rightarrow \quad \phi_{A1} = 45^\circ, \phi_{A2} = 135^\circ, \phi_{A3} = 225^\circ, \phi_{A4} = 315^\circ,$$

Step 5. The centroid, the point of intersection of the asymptotes on the real axis is given by

$$\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$

$$= \frac{-4-4-4-0}{4} = -3$$

Step 6. The breakaway point (point at which multiple roots of the characteristic equation occur) of the root locus are the solution of $\frac{dk}{ds} = 0$.

$$\begin{aligned} k &= -s(s+4)(s^2+8s+32) \\ &= -(s^2+4s)(s^2+8s+32) \\ &= -(s^4+12s^3+64s^2+128s) \end{aligned}$$

$$\begin{aligned} \frac{dk}{ds} &= 4s^3+36s^2+128s+128 = 0 \\ &= s^3+9s^2+32s+32 = 0 \end{aligned}$$

From step 2 and 3 we can easily find that the two root locus branches (out of four branches) starts from 0 and -4 move towards each other and meet at a point and then breakaway from them. Therefore the breakaway point lies between 0 and -4. The break away point by trial and error method is -1.57.

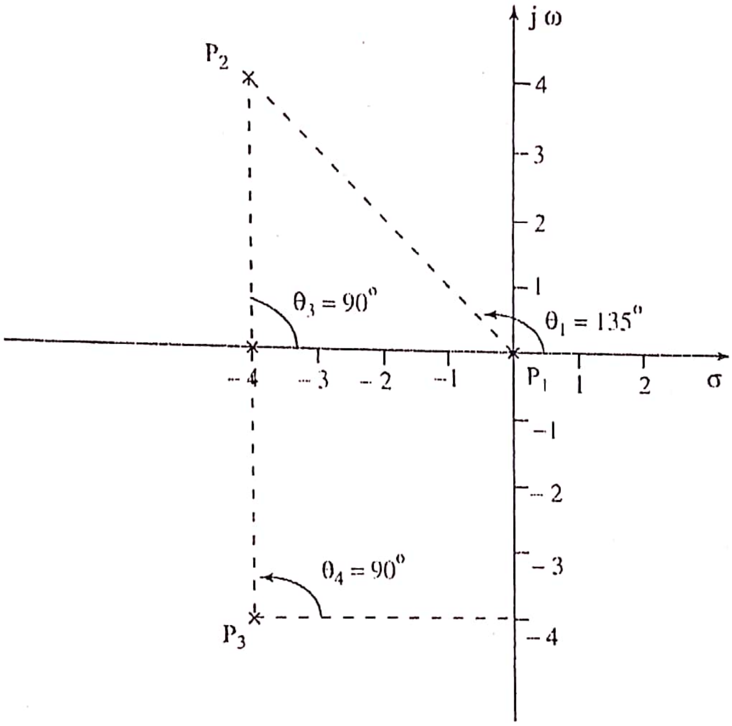


Fig. 3.53

Step 7. The angle of departure ϕ_p of a root locus from a complex open loop pole is

$$\phi_p = 180^\circ + \phi$$

where ϕ is the net angle contribution at this pole by all other open loop poles and zero

$$\phi = -\theta_1 - \theta_2 - \theta_4$$
$$= -135^\circ - 90^\circ - 90^\circ = -315^\circ$$
$$\phi_{p3} = 180^\circ - 315^\circ = -135^\circ$$
$$\phi_{p4} = \bar{\phi}_{p3} = 135^\circ$$

Step 8. The point at which the imaginary axis crosses the root locus can be obtained using Routh Huruwitz criterion.

The characteristic equation is

$$1 + G(s) = 0$$

$$1 + \frac{k}{s(s+4)(s^2+8s+32)} = 0$$

$$\Rightarrow \quad s^4 + 12s^3 + 64s^2 + 128s + k = 0$$

$$\begin{array}{c|ccc}
s^4 & 1 & 64 & k \\
s^3 & 12 & 128 & 0 \\
s^2 & \frac{160}{3} & k & \\
s^1 & \dfrac{\frac{160}{3}(128) - 12k}{\frac{160}{3}} & 0 & \\
s^0 & k & &
\end{array}$$

$$k > 0; k < \frac{(160)(128)}{3(12)}$$

$$\Rightarrow \quad 0 < k < 568.8$$

The critical value of k at which sustained oscillation occurs is $k = 568.9$.

The auxiliary equation is

$$\frac{160}{3}s^2 + 568.9 \Rightarrow s = \pm j3.266$$

The root locus plot is shown in Fig. 3.54. To find the value of k for $\zeta = 0.707$

$$\text{Draw a line with angle } \phi = \cos^{-1}(0.707)$$
$$= 45^\circ \text{ as shown Fig. 3.53}$$

The line cuts the root locus at a point s. Now measure the distances (with the help of scale) from this point s to all open loop poles and zeros and find the value of k using the relation

$$k = \frac{\text{Product of phasor length from point } s \text{ to open loop poles}}{\text{product of phasor length from point } s \text{ to open loop zeros}}$$
$$= (p_1 s)(p_2 s)(p_3 s)(p_4 s)$$
$$= (1.9)(3)(3.8)(6)$$
$$= 129.96$$
$$\simeq 130$$

This value is less than the maximum value 568.8 Complete root locus plot is shown in Fig. 3.54.

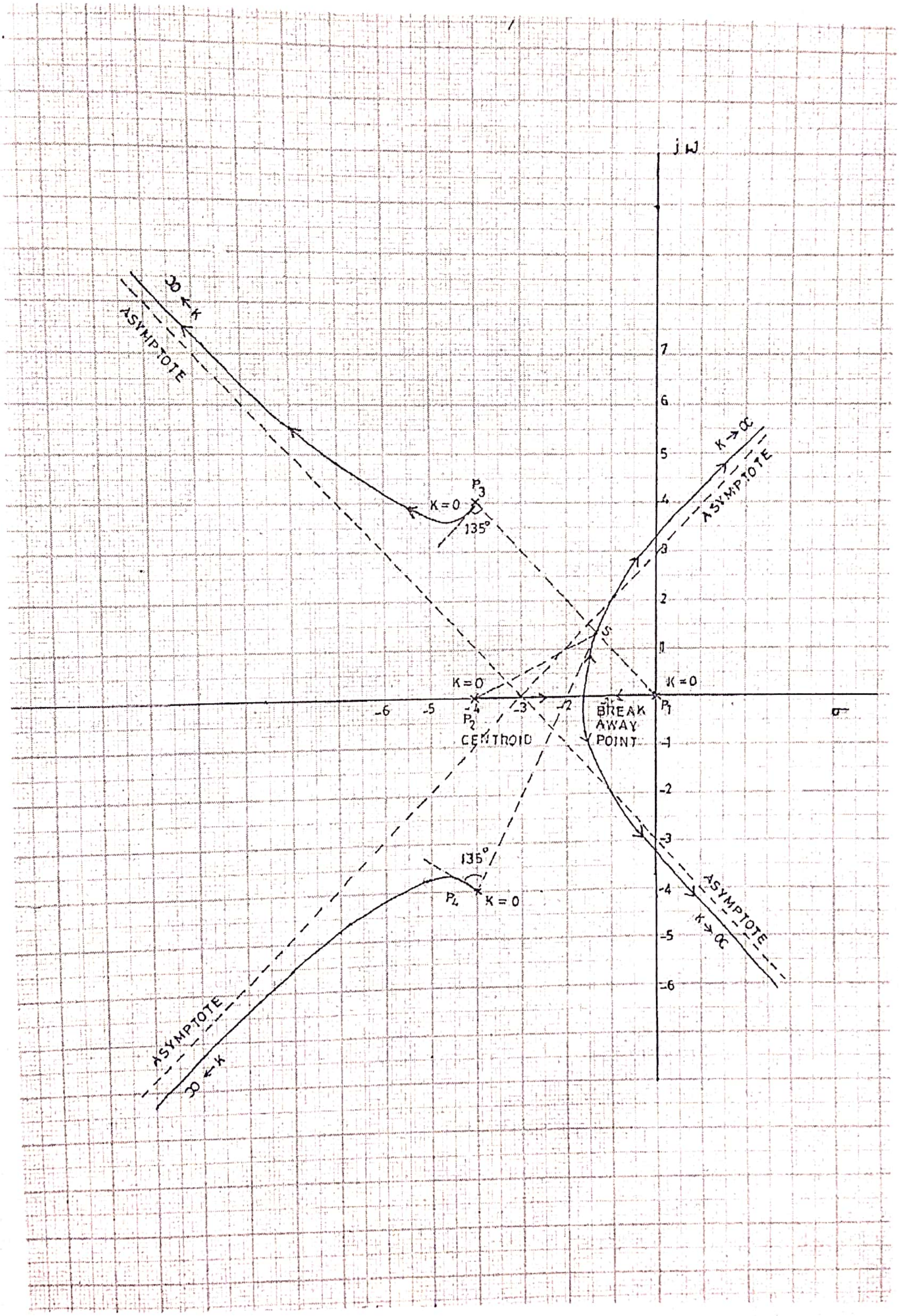

jω
σ
ASYMPTOTE
ASYMPTOTE
ASYMPTOTE
ASYMPTOTE
K = 0
K = 0
K = 0
K = 0
K → ∞
K → ∞
∞ ← K
∞ ← K
P3
P2
P1
P4
135°
135°
BREAK AWAY POINT
CENTROID

Fig. 3.54