INTRODUCTION

→ Signal: A signal is a function of independent variables such as time, distance, position, temperature and pressure. A signal carries information, and the objective of signal processing is to extract useful information carried by the signal. It is can be generated by a single source or by multiple sources.

→ Digital signal processing: Digital signal processing is concerned with the digital representation of signals and the use of digital processors to analyze, modify or extract information ~~form~~ from signals. The signals used in most popular forms of DSP are derived from analog signals which have been sampled ~~by~~ at regular intervals and converted into digital form.

→ Advantages of DSP: The attraction of DSP comes from of key advantages such as the following:

→ Guaranteed accuracy: Accuracy is only

determined by the number of bits used.

→ Perfect reproducibility : Identical performance from unit to unit is obtained since there are no variations due to component tolerances. For examples, by using DSP, a digital recording can be copied or reproduced ~~nor~~ several times over without any degradation in the signal quality.

→ No drift in performance with temperature or age.

→ Advantage is always taken of the tremendous advances in semiconductor technology to achieve greater reliability, smaller size, lower cost, low power consumption and higher speed.

→ Greater flexibility : DSP systems can be programmed and reprogrammed to perform a variety of functions, without modifying the hardware.

→ Superior performance : DSP can be used to perform functions not possible with analog signal processing.

→ In some cases information may already be in a digital form and DSP offers the only viable option.

→ Disadvantages of DSP: The DSP system can not overcome some disadvantages:

→ Greater cost: DSP designs can be expensive. At the present, fast ADCs/DACs are too expensive or do not have sufficient resolution. Currently, only specialized ICs can be used to process signal and these are quite expensive.

→ less speedy: Most DSP devices are still not fast enough and can only process signals of moderate bandwidths. Bandwidths in the 100MHz range are still processed only by analog methods. Nevertheless, DSP devices are becoming faster and faster.

→ Design time: DSP designs can be time

consuming and in some cases almost impo-
-ssible. Here is the shortage of suitable engineers.

→ Finite wordlength problems: DSP uses only a limited number of bits. If an insufficient number of bits is used to represent variables serious degradation in system performance may result.

→ Applications areas:

- → Image processing.
- → Instrumentation/control.
- → Speech/audio.
- → Military.
- → Telecommunications.
- → Biomedical.
- → consumer applications.

→ Building Blocks of a digital signal processore:

Consider an analog signal containing noise being received by a receiving instrument. To remove noise, we intend to use digital signal processing ~~signal~~ technique.

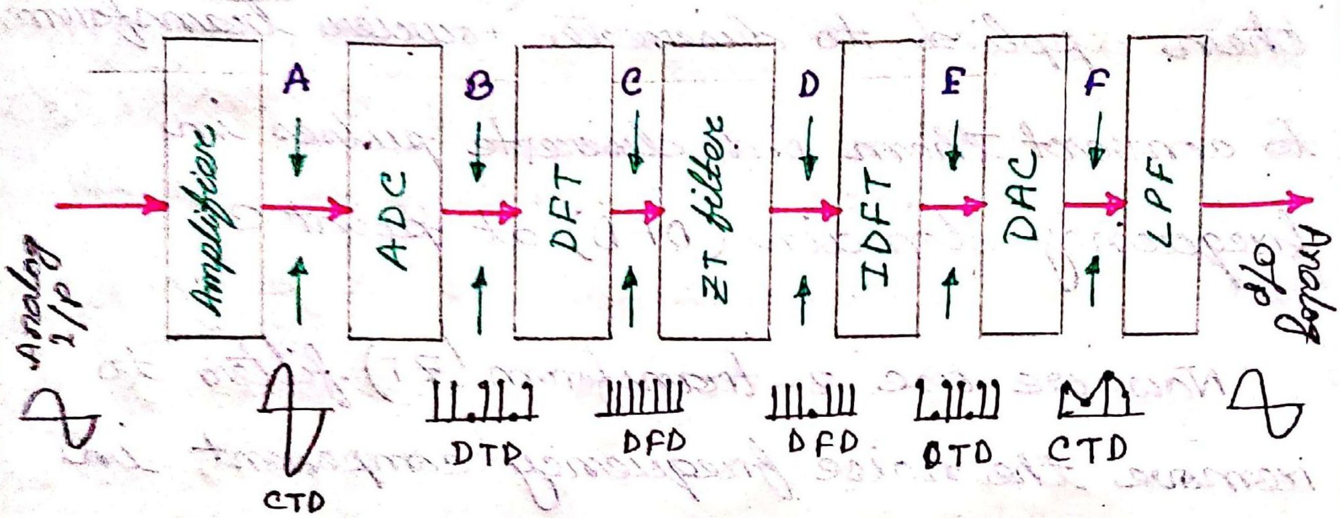

Fig: Block diagram of a typical digital signal processore.

Here the input signal, which contains noise and hence to be processed, is first amplified by an amplifier. After amplification we have the smooth analog waveform in continuous time domain (CTD) at point A. Now we have to convert this amplified analog signal in time domain into its digital counterpart in time domain.

By using ADC we get the digital signal in time domain at point B. Now the signal at point B is in discreate time domain (DTD) and is represented in binary format of 1's and 0's. These discreate pulses in time domain then applied to discreate Fourier transformer to convert them into discreate pulses in frequency domain (DFD) at point C.

Now we use z-transform (ZT) filter to remove the noise frequency component in discrete signal.

The filtered DFD components are now applied to an inverse discrete Fourier transformer to convert them back into DTD signal pulses.

The output of IDFT is at point E, is then applied to a digital to ~~anag~~ analog converter (DAC) to convert DTD to analog signal,

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which is now devoid of noise. This is developed at point F.

The CTD signal developed at point F is quantized in nature. This is smoothened by using low pass filter (LFP) and amplified if required.

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Analog I/O interface for real-time DSP systems

Analog to digital Conversion :

Analog signals are converted into digital signals by a process called analog to digital conversion, for which we use sampling.

In the sampling process, we take samples of the amplitude of the given analog signal at regular intervals of time using sampling pulses.

The sampling operation converts the given signal (Analog) into discrete pulses of varying amplitudes. It can be seen that these discrete amplitude pulses occur at points where the sampling pulses exist, and have amplitudes that are proportional to the amplitude of the original analog signal at that point. This discrete waveform ~~are~~ is called pulse-amplitude-modulated (PAM) waveforms.

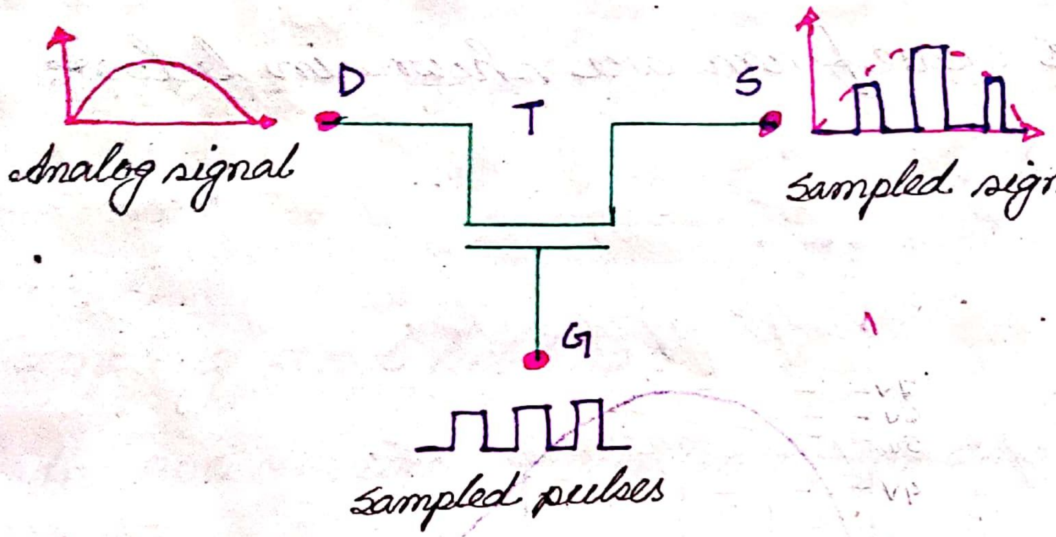


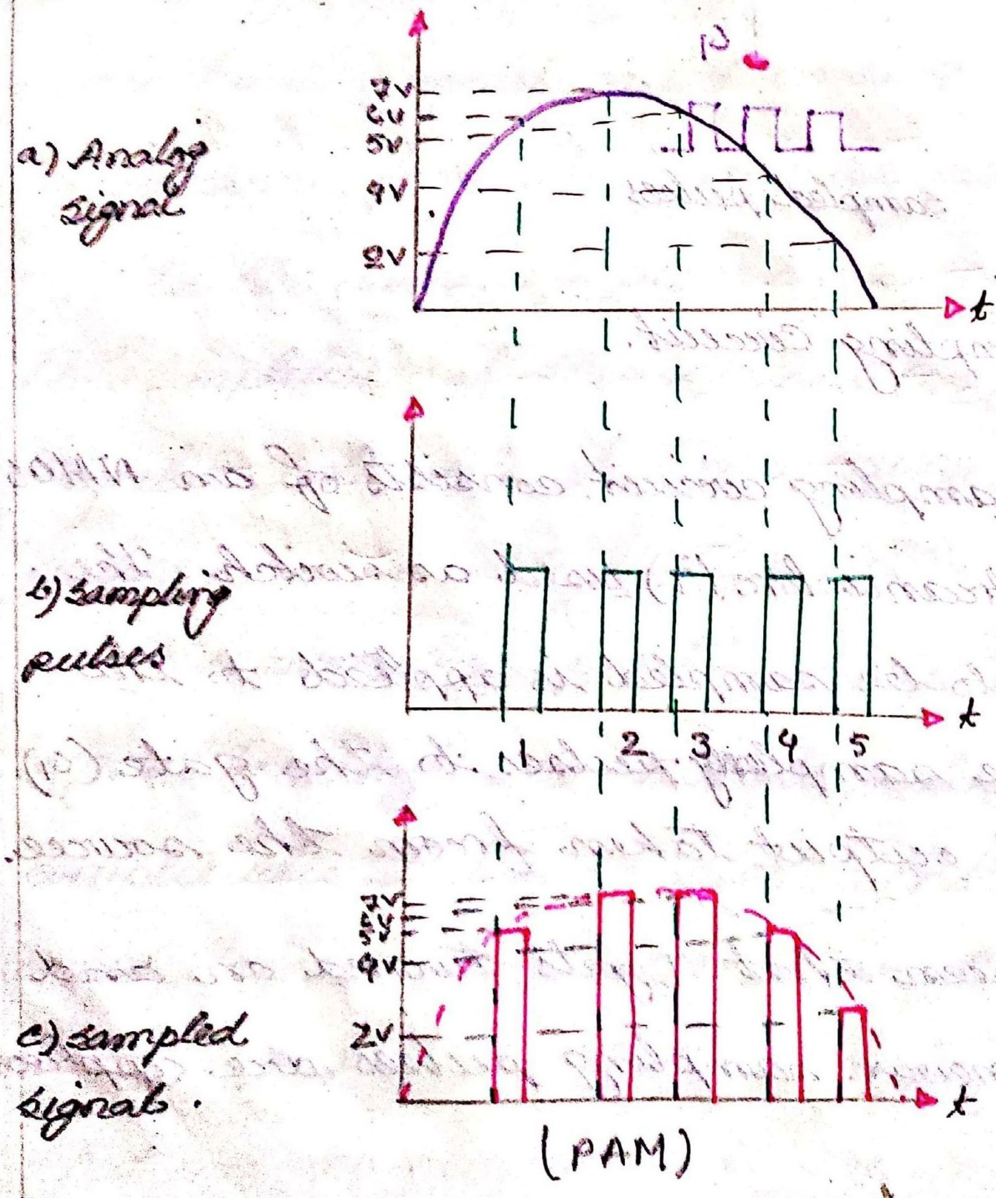
Fig: Sampling circuit.

Here the sampling circuit consists of an N.MOS field effect transistor (T) used as switch. The analog signal to be sampled is applied to the drain (D), the sampling pulses to the gate (G) and sampled output taken from the source.

It can be seen that T gets turned on and conducts whenever sampling pulses are applied to the gates.

when it get turned on, it conducts the input analog signal from its drain to the source.

The waveform are shown in below:



Here, a) shows the ~~smoth~~ smooth analog signal to be sampled.

(b) shows the impulse train.

And c) " " Sampled (PAM) waveform.

→ **Sampling and holding:**

An ADC requires finite time for analog to digital conversion. Because, if the amplitude varies during the conversion period, then the ADC will be in confusion as to the exact value of the input signal amplitude that it has to convert into binary.

For keeping the analog input constant at the required levels during the conversion periods, we use the sample and hold circuit.

The circuit is briefly described below:

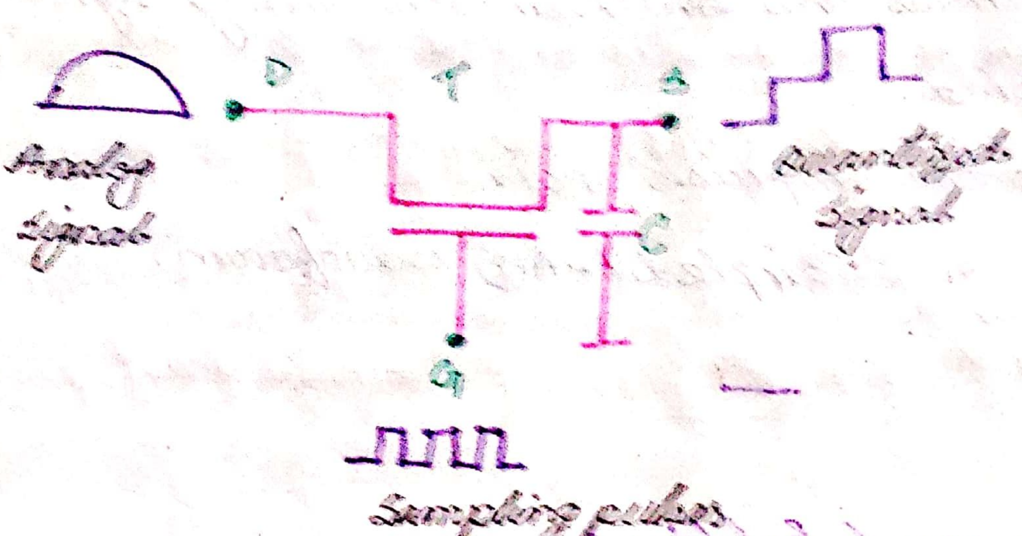


Fig: Sample and hold ckt.

This figure shows an elementary sample and hold circuit. In that a capacitor has been connected to the source terminal of the NMOS. This capacitor will act as the holding device of the circuit.

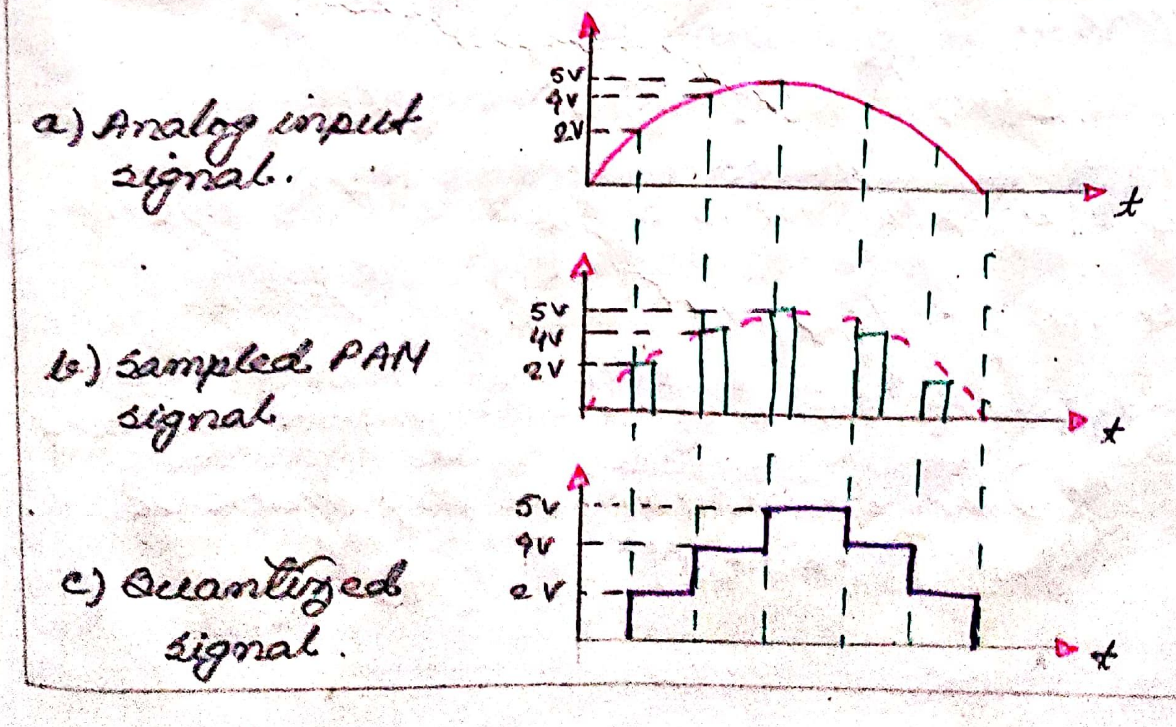
The working principle of this circuit is:

The analog signal to be sampled be applied to the drain terminal of the NMOS (T) and the sampling pulses is applied to the gate.

When T is turned on, the input analog signals get transmitted to the ~~a~~ output terminal of T through its drain-source path. At that time C is charged to the max.m value of the amplitude available across the output terminals.

During the off period of T, C retains (hold) this charge steadily without discharging. Thus, the sampled amplitude is held constant by C, which helps in the smooth working of ADC.

The figure is shown below:



We find that the sampling and holding operation converts a PAM signal into a quantized signal as shown in fig: (a), (b), and (c). This quantized signal, which is the discret-ized version of the analog input signal, is then applied to ADC for A/D.

→ The Nyquist Sampling Theorem:

Sampling is performed based on the theorem, called the sampling theorem or Nyquist sampling theorem. It states that,

"A signal can be recovered from its samples completely only if sampling is done on the condition that the sampling frequency is equal to or greater than twice the maximum frequency of the analog signal, which is subjected to sampling."

If, f_s = Sampling frequency, and

f_m = Maximum frequency of the analog signal

Then from sampling theorem we have,

$$f_s \geq 2 f_m .$$

Now consider our voice signal as an example. Our voice signal be band limited to 4 KHz ∴ $f_m = 4\,KHz$. So we have to sample this signal at a rate, $f_s \geq 8\,KHz$.

→ Proof of the sampling theorem:

Sampling is a convolution operation. If,

$x(t)$ = Input analog signal.

$s(t)$ = Sampling pulses.

and, $y(t)$ = sampled output then this operation can be expressed as,

$$y(t) = x(t) \circledast s(t)$$

OR, $y(t) = x(t) \circledast s(t)$

We know that, convolution in time domain can be seen to be equal to multiplication in frequency domain. So, after Fourier transform,

$$Y(\omega) = X(\omega) \cdot S(\omega) \quad \text{--- (i)}$$

where $Y(\omega)$, $X(\omega)$ and $S(\omega)$ are the Fourier transform of $y(t)$, $x(t)$ and $s(t)$.

Now, for the sampling, we have to use an impulse train. which can be expressed as,

$$s(t) = \delta(t - nT) \quad \text{--- (ii)}$$

where, $n = 0, 1, 2 \ldots\ldots$

T = Fixed time period.

In case of sampling, we take T as the sampling interval, $T_s = \frac{1}{f_s}$, The Fourier transform of the impulse train,

$$S(\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

$$= \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \text{ ---- (iii)}$$

where, $\omega_s = \frac{2\pi}{T_s}$ = Amplitude of the sampled pulses.

Now from eqn (i) and (iii) we have,

$$Y(\omega) = X(\omega)\left[\omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right] \text{ --- (iv)}$$

$$\Rightarrow Y(\omega) = \omega_s \left[X(\omega)\,\delta(\omega) + X(\omega)\,\delta(\omega - \omega_s) + \text{----} + X(\omega)\,\delta(\omega - n\omega_s) + X(\omega)\,\delta(\omega + \omega_s) + X(\omega)\,\delta(\omega + 2\omega_s) + \text{-------} + X(\omega)\,\delta(\omega + n\omega_s)\right] \text{ --- (v)}$$

From eqn (v), we find that sampling operation has produced a sample of order zero, and an infinite number of sidebands samples of orders 1, 2, --- and so on.

We also notice that the zeroth order sample is located at $\omega = 0$, the first order at $\omega = \pm\omega_s$, the second order at $\omega = \pm 2\omega_s$ and so on.

Here, $\omega_s [X(\omega)\delta(\omega)]$ = Amplitudes of all these samples (Maximum value).

The below figures give the pictorial represen--tations related to this situation:

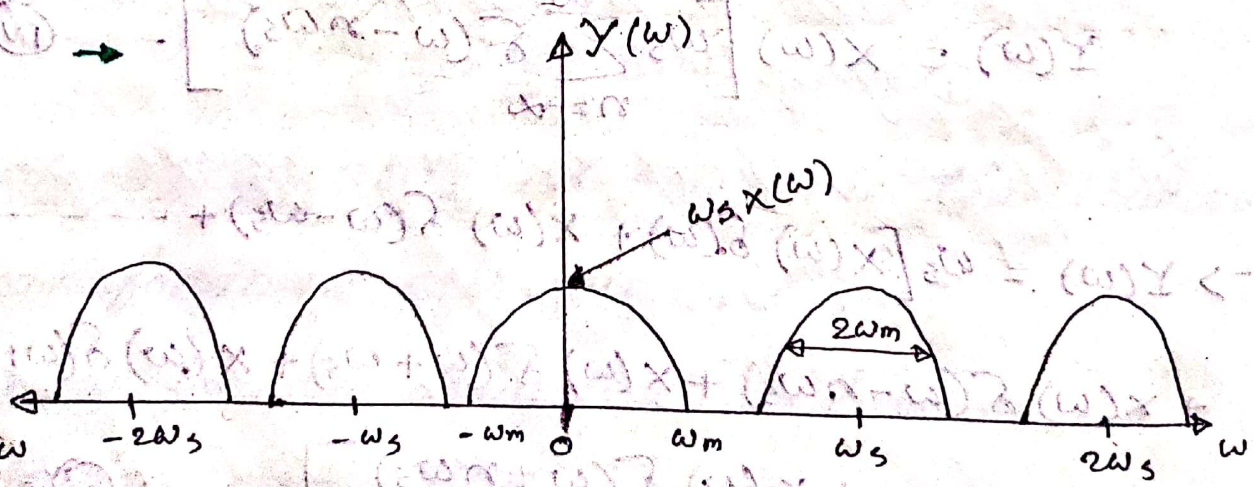


Fig 1 : Frequency components when $\omega_s > 2\omega_m$

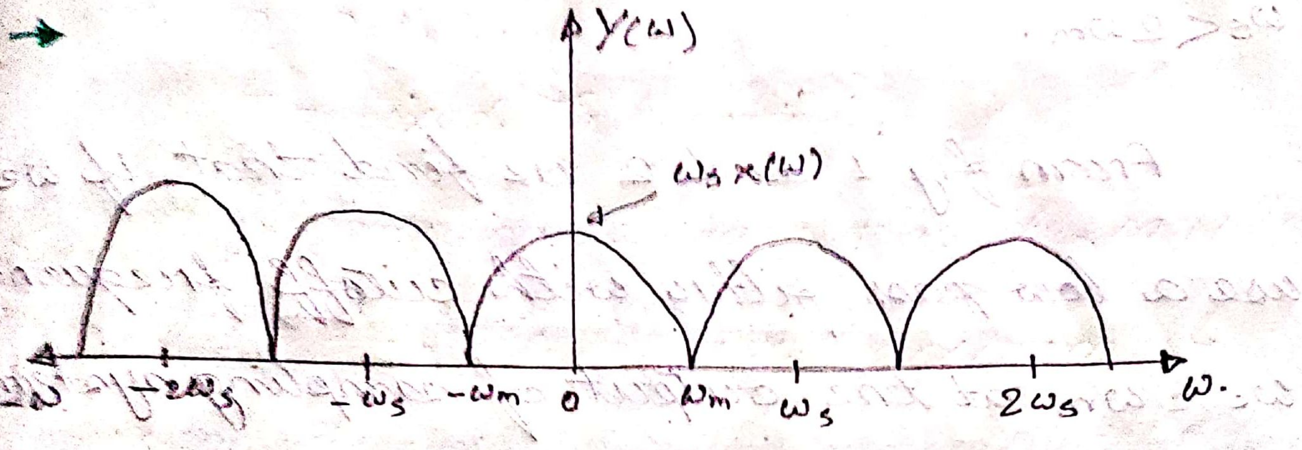


Fig 2: Frequency components when $\omega_s = 2\omega_m$

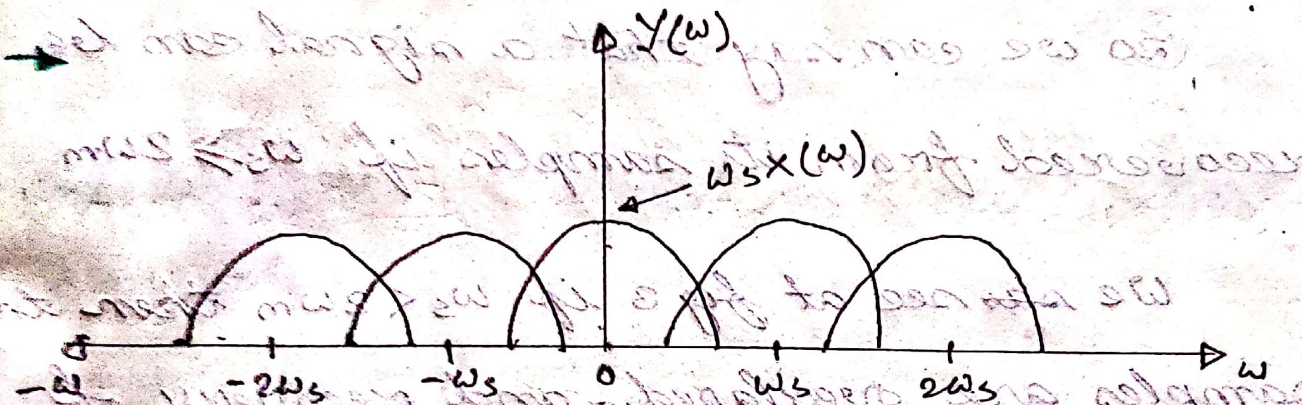


Fig 3: Frequency components when $\omega_s < 2\omega_m$.

Here, fig1 shows that the samples remain seperated from each other if $\omega_s > 2\omega_m$.

Fig 2 shows the samples remain just touching each other if $\omega_s = 2\omega_m$

Fig 3 shows the samples overlap if

$\omega_s < 2\omega_m$.

From fig 1 and 2 we find that, if we use a low pass filter with cutoff frequency $\omega_c = \omega_m$ at the output of sampling system, we can recover the original signal back from its sampled version.

So we can say that a signal can be recovered from its samples if $\omega_s \geq 2\omega_m$

We ~~saw~~ see at fig 3 if $\omega_s < 2\omega_m$ then the samples are overlaped, and recovery of the original signal back from its sampled version is impossible. Overlaping of the samples ~~are~~ is called "aliasing."

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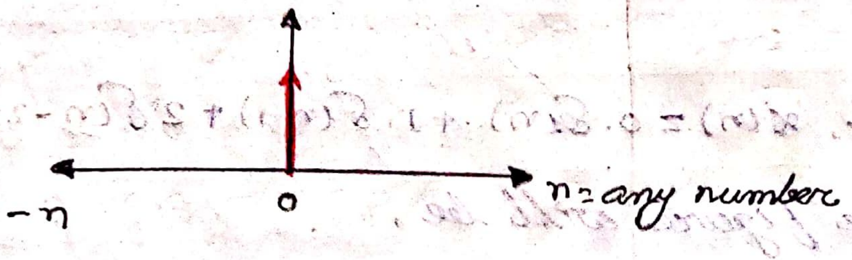
Elementary Signals

In discrete-time signals and systems are a number of basic signals. They are :

- Impulse / Delta / Impulse train / Digital Impulse.
- Step function / Digital step.
- Ramp function.
- Parabolic function.
- Sinusoidal function.

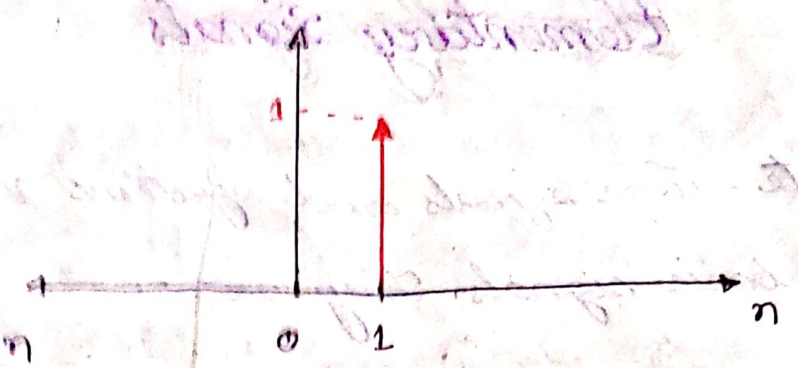
→ **Impulse** : The unit sample sequence is denoted as $\delta(n)$ where,

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



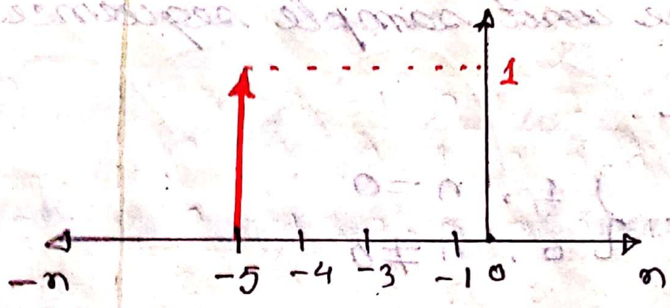
For better understand we need to observe some examples :

- $$\delta(n-1) = \begin{cases} 1, & n-1 = 0 \\ 0, & n-1 \neq 0 \end{cases}$$



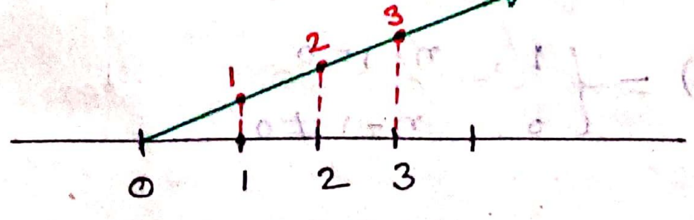
• $\delta(n+5) = \begin{cases} 1, & n+5=0 \\ 0, & n+5 \neq 0. \end{cases}$

so here only $\delta(n+5) = 1$ when $n = -5$.



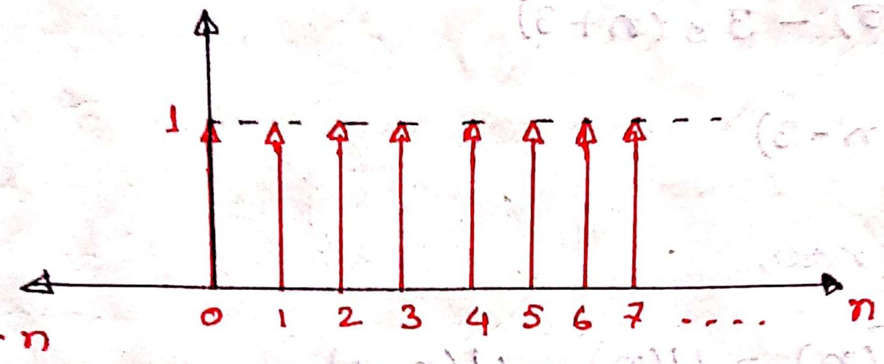
• If, $x(n) = 0 \cdot \delta(n) + 1 \cdot \delta(n-1) + 2\delta(n-2) + 3\delta(n-3) + \cdots$

Then the figure will be,



→ **Digital step / Unit step:** The digital step signal is denoted by $u(n)$ and defined as,

$$u(n) = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

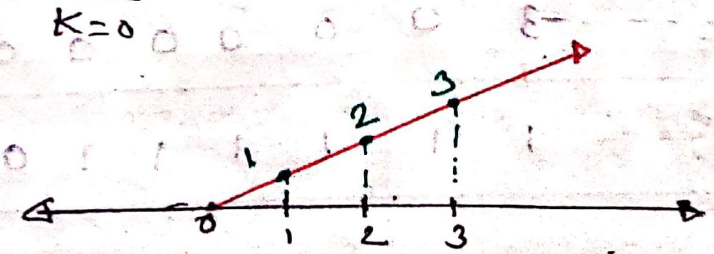


* Impulse can be defined by the unit step, i.e.

$$\delta(n) = u(n) - u(n-1)$$

* Similarly unit step can be defined by the delta i.e.

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k)$$



This can be defined by unit step as,

$$x(n) = n[u(n) - u(n-4)]$$

Prob: Plot the following delta functions:

(a) $-\delta(n-3)$

(b) $2\delta(n+3)$

(c) $\delta(n-5) - 3\delta(n+3)$

(d) $1-\delta(n-3)$

Soln: We know,

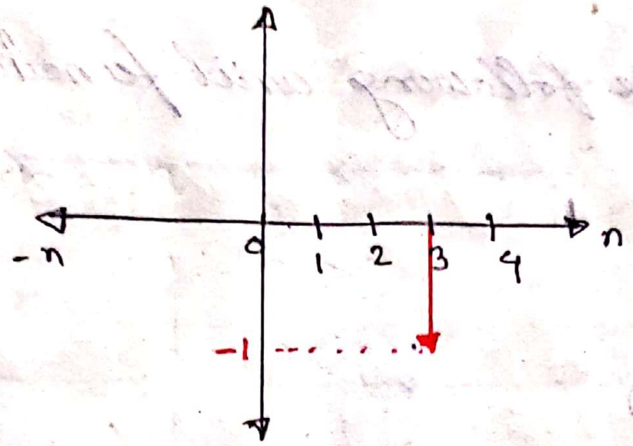
$$\delta(n) = u(n) - u(n-1)$$

The Table, required for ploting $\delta(n)$:

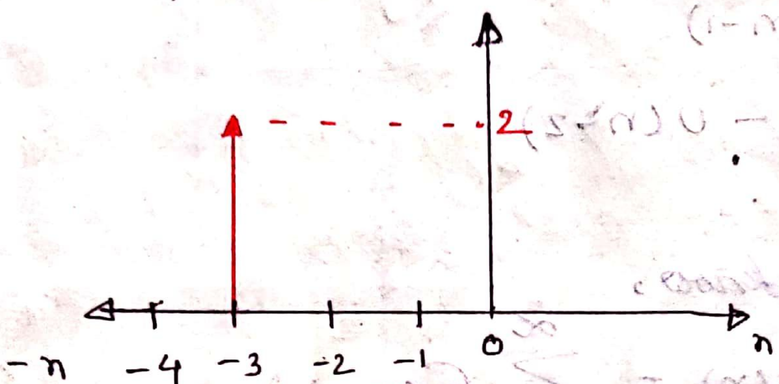
n	$-n$	------	-3	-2	-1	0	1	2	3	4	5	------	n
$\delta(n)$	0	---	0	0	0	1	0	0	0	0	0	----	0
$-\delta(n-3)$	0	---	0	0	0	0	0	0	-1	0	0	----	0
$2\delta(n+3)$	0	---	2	0	0	0	0	0	0			----	0
$3\delta(n+3)$	0	---	3	0	0	0	0	0	0	0	0	----	0
$\delta(n-5)$	0	-----	0	...	...	0	...	...	0	0	1	----	0
$\delta(n-5) -3\delta(n+3)$	0	----	-3	0	0	0	0	0	0	0	1	----	0
$1-\delta(n-3)$	1	----	1	1	1	1	1	1	0	1	1	-----	1

Now the figures are shown in below:

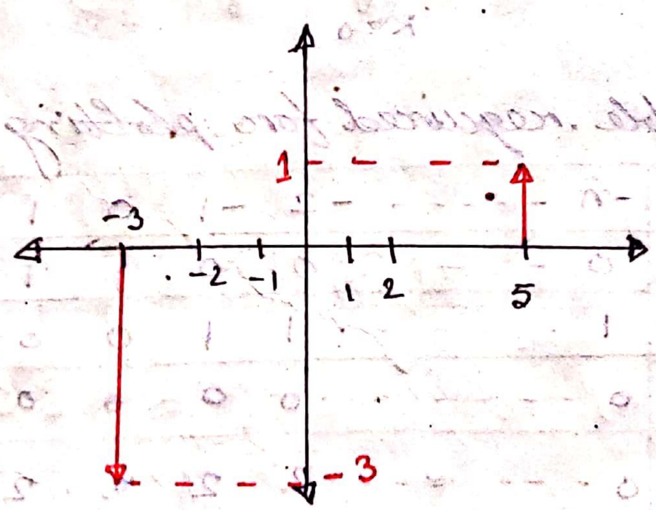
(a)



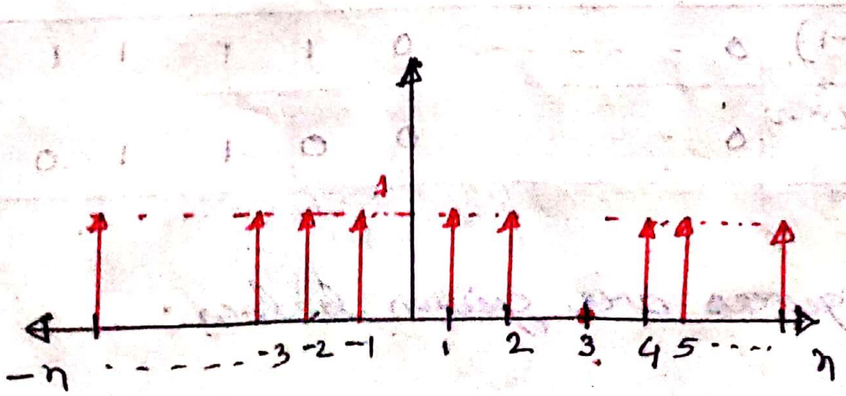
(b)



(c)



(d)



Prob: Plot the following unit functions:

(a) $1 - U(n)$

(b) $-U(n-3)$

(c) $2U(n+2)$

(d) $U(-n-1)$

(e) $U(n) - U(n-2)$

Soln: We know,

$$U(n) = \sum_{K=0}^{\infty} \delta(n-K)$$

The table required for plotting $U(n)$:

n	$-n$ ----	-2	-1	0	1	2	3	----	n
$U(n)$	0 ----	0	0	1	1	1	1	----	1
$1-U(n)$	1 ----	1	1	0	0	0	0	----	0
$-U(n-3)$	0 ----	0	0	0	0	0	-1	----	-1
$2U(n+2)$	0 ----	2	2	2	2	2	2	----	2
$U(-n-1)$	0 ----	0	1	1	1	1	1	----	1
$U(n)-U(n-2)$	0 ----	0	0	1	1	0	0	----	0

The figures are given below:

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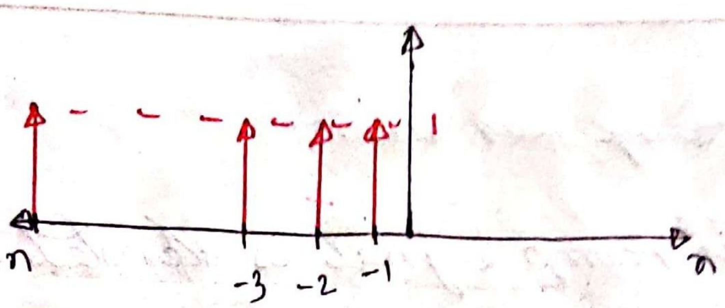
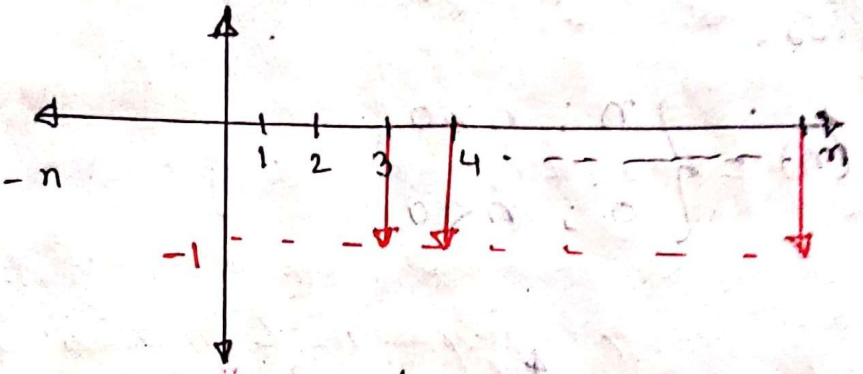
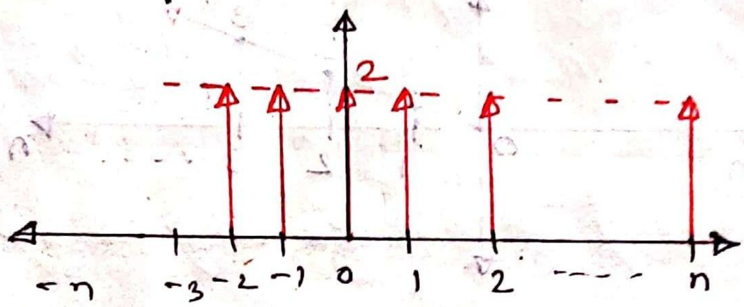
(a)



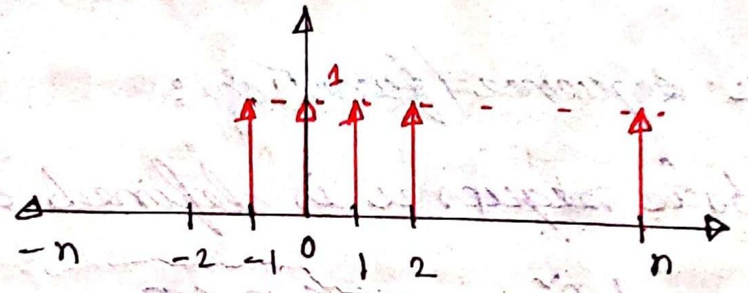
(b)



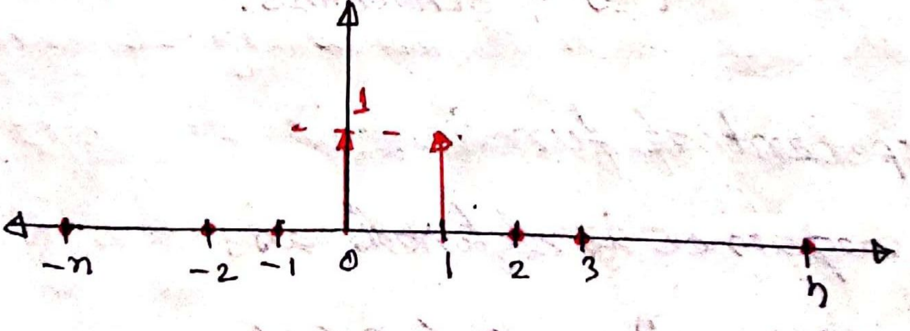
(c)



(d)



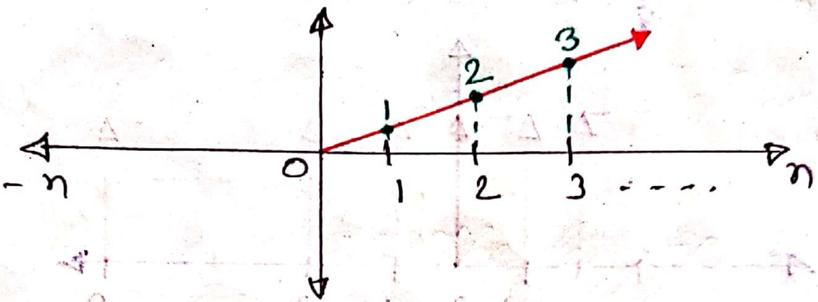
(e)



→ Unit ramp function:

The unit ramp signal is denoted as $U_r(n)$ and defined as,

$$U_r(n) = \begin{cases} n ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$$



→ Parabolic sequence/function:

The parabolic sequence is defined as,

$$P(n) = \begin{cases} n^2 ; & -\infty \leq n \leq \infty \\ 0 ; & \text{otherwise.} \end{cases}$$

→ Exponential function:

this sequence is defined as,

$$e(n) = a^n ; \quad -\alpha \leq n \leq \alpha$$

$$\Rightarrow \quad x[n] = A\alpha^n ; \quad -\alpha \leq n \leq \alpha .$$

Here, A, a or α are real/complex number.

Digital Signal

- A digital signal obtain from an analog signal by sampling and by the help of analog to digital converter.

$$x_a(t) \longrightarrow \text{switch} \longrightarrow x_a(nT) \longrightarrow \boxed{A/D} \longrightarrow x(n)$$

- $x_a(nT)$: Analog time discreet signal.
- $x_a(t)$: Analog signal.
- $x(n)$: Digital signal.
- $T = \frac{1}{f_s}$; f_s : Sampling freq.

[* the mathematical techniques for handlling in discrete signal and digital signal are same.]

So we can write digital signal as,

$$\{x(n)\} = \{-1, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 2, -2, 1 \ldots \ldots\}$$

This is a sequence. (Real).

We can write the sequence by considering the imaginary part as,

$\{x(n)\} = \{1+j, 2+j. \text{------}\}$

Actually imaginary signal is also a real signal but it just orthogonal of real part.

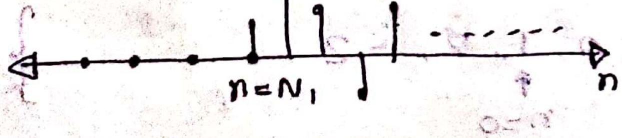
• **Length of sequence:** The length of the sequence is equal to the samples number.

suppose, $x(n) = \{\underset{\uparrow n=0}{1}, 2, 3, 4\}$

Here length $= N = 4$.

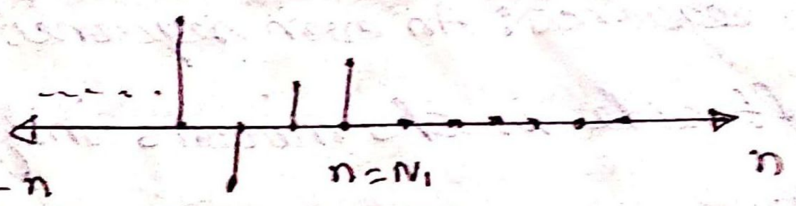
So $n = 0$ to $N-1$.

• **Right sided signal:** If a signal exists from $n = N_1$ to the right side and the left side remain zero, then that is called R.S.S.



Here, N_1 = Can be anything (number)

• Left sided signal : If a signal exists from $n = N_1$ & the left side and the right side remain zero then that is called L.S.S.

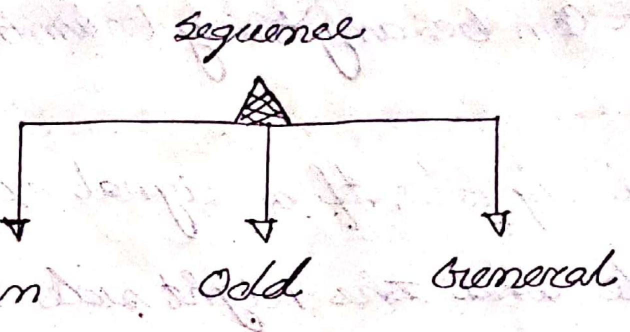


[* In general a signal is the combination of L.S.S. and R.S.S.]

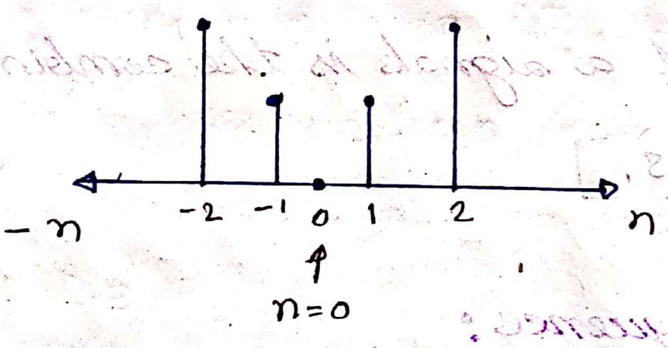
→ Types of sequence:

Generally sequence is devided into three types. They are:

- Even sequence
- Odd sequence
- The mixture of even and odd sequence.



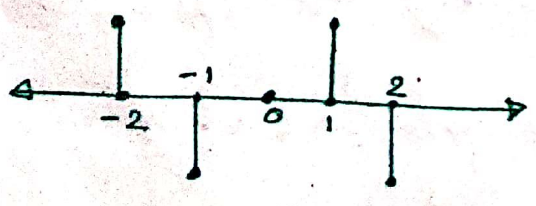
• **Even sequence:** An even sequence is one which left side is the mirror image of the right side.



So for even signal, $x_e(n) = x_e(-n)$

• **Odd sequence:** A sequence or signal is odd if,

$$x_o(n) = -x_o(-n)$$



• General sequence: A signal is neither odd nor is even. It is the decomposition of even and odd. so,

$$x(n) = x_e(n) + x_o(n)$$

Here, $x_e(n) = \frac{x(n) + x(-n)}{2}$.

$$x_o(n) = \frac{x(n) - x(-n)}{2}.$$

→ Another types:

• Periodic sequence: A function $x(n)$ is a periodic sequence if, $x(n) = x(n + kN)$ where the smallest N is refered as period.

• Bounded sequence: If the value of $x(n)$ is bounded then this sequence is called bounded sequence. i.e.

$$|x(n)| \leq M < \alpha \; ; \; \forall n.$$

Here, M = some number.

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Now if the sum of the all values of $x(n)$ in the limit of $-\alpha$ to α is bounded i.e.

$$\sum_{n=-\alpha}^{\alpha} |x(n)| \leq P < \alpha \quad \text{where, } P = \text{some number}$$

Then $x(n)$ is said to be absolutely summable sequence.

• Square Summable sequence: square summable is in which the magnitude of $x(n)$ is squared and summed up from $-\alpha$ to α and is less than or ~~eqal~~ equal to some number Q. i.e.

$$\underbrace{\sum_{n=-\alpha}^{\alpha} |x(n)|^2}_{} \leq Q < \alpha$$

Energy of this sequence.

→ Average power of the sequence: we know, Power is the energy per unit time.

We have the energy of the sequence over the length $(2K+1)$ is,

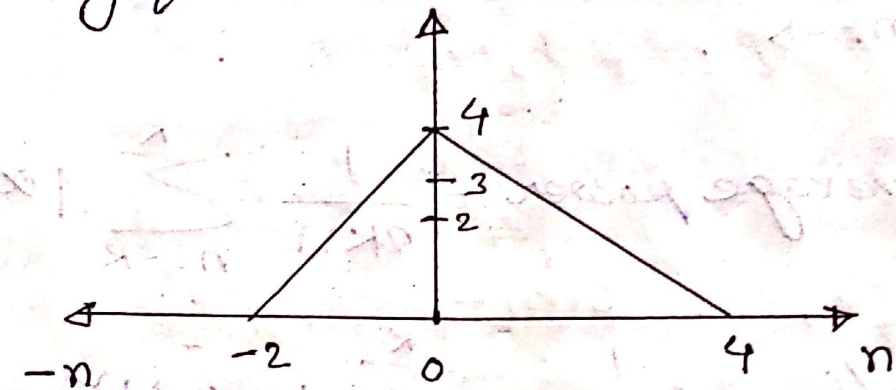
$$\sum_{n=-K}^{K} |x(n)|^2 .$$

So the average power $= \dfrac{1}{2K+1} \displaystyle\sum_{n=-K}^{K} |x(n)|^2$

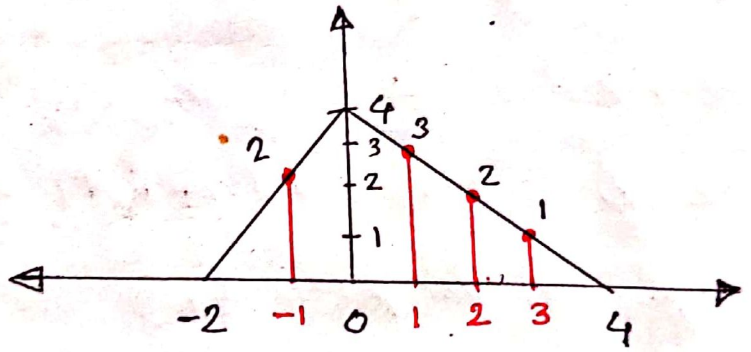
So. $P_{av} = \lim\limits_{K \to \infty} \dfrac{1}{2K+1} \displaystyle\sum_{n=-K}^{K} |x(n)|^2$.

Some Examples

→ Problem: Find the odd and even part of the following function:



Soln: We have sampled the function given in fig:



We know,

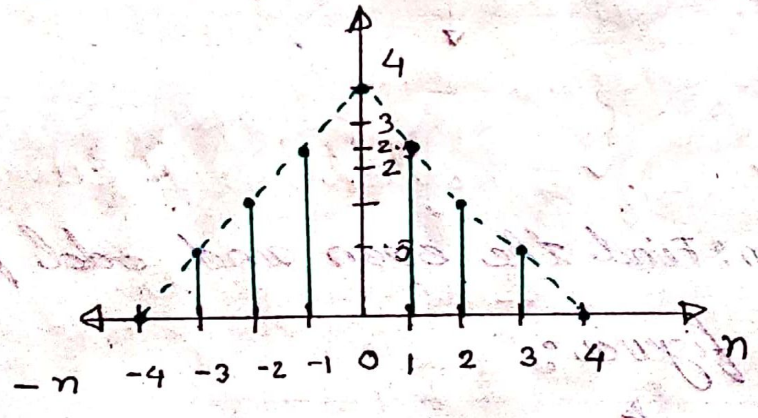
$$x_e(n) = \frac{x(n) + x(-n)}{2}$$

$$x_o(n) = \frac{x(n) - x(-n)}{2}$$

→ Tabulation of $x_e(n)$:

n	-4	-3	-2	-1	0	1	2	3	4
$x(n)$	0	0	0	2	4	3	2	1	0
$x(-n)$	0	1	2	3	4	2	0	0	0
$x(n)+x(-n)$	0	1	2	5	8	5	2	1	0
$x_e(n)$	0	.5	1	2.5	4	2.5	1	.5	0

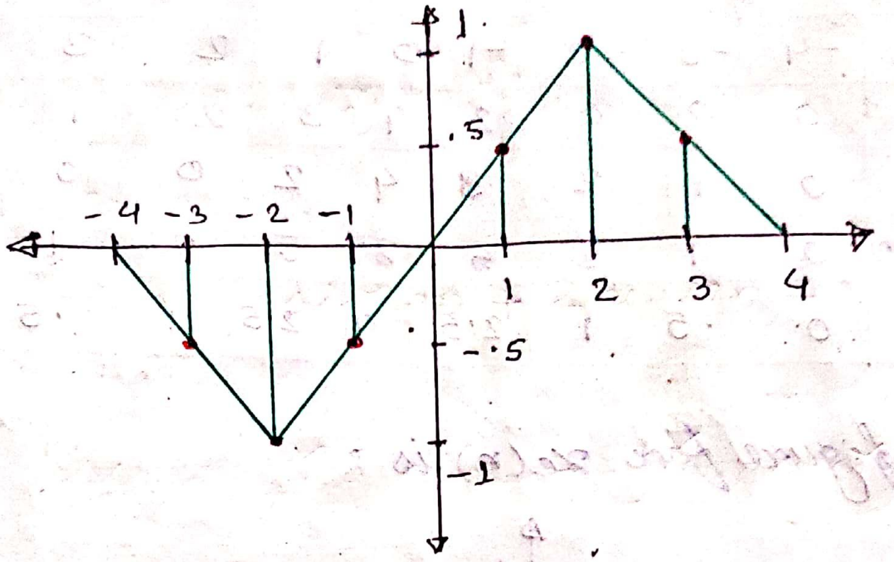
So the figure for $x_e(n)$ is :



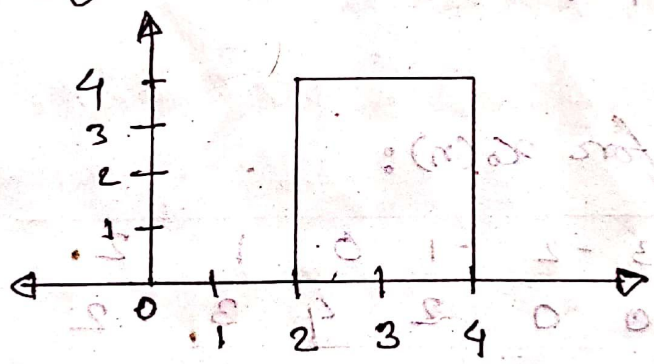
→ Tabulation for $x_o(n)$:

n	-4	-3	-2	-1	0	1	2	3	4
$x(n)$	0	0	0	2	4	3	2	1	0
$x(-n)$	0	1	2	3	4	2	0	0	0
$x(n)-x(-n)$	0	-1	-2	-1	0	1	2	1	0
$x_o(n)$	0	-.5	-1	-.5	0	.5	1	.5	0

So the figure for $x_o(n)$:



→ **Problem:** Find the even and odd part for the following figure :



Soln: We know,

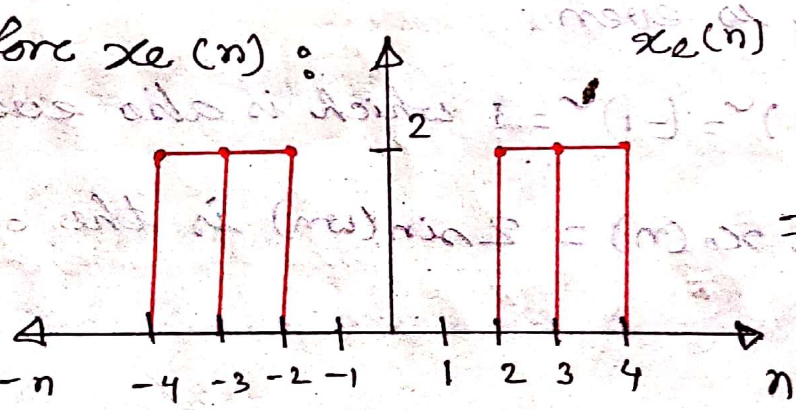
$$x_e(n) = \frac{x(n) + x(-n)}{2}$$

$$x_o(n) = \frac{x(n) - x(-n)}{2}$$

→ Tabulation for $x_e(n)$ and $x_o(n)$:

n	−4	−3	−2	−1	0	1	2	3	4
$x(n)$	0	0	0	0	0	0	4	4	4
$x(-n)$	4	4	4	0	0	0	0	0	0
$x(n)+x(-n)$	4	4	4	0	0	0	4	4	4
$x_e(n)$	2	2	2	0	0	0	2	2	2
$x(n)-x(-n)$	−4	−4	−4	0	0	0	4	4	4
$x_o(n)$	−2	−2	−2	0	0	0	2	2	2

→ figure for $x_e(n)$:

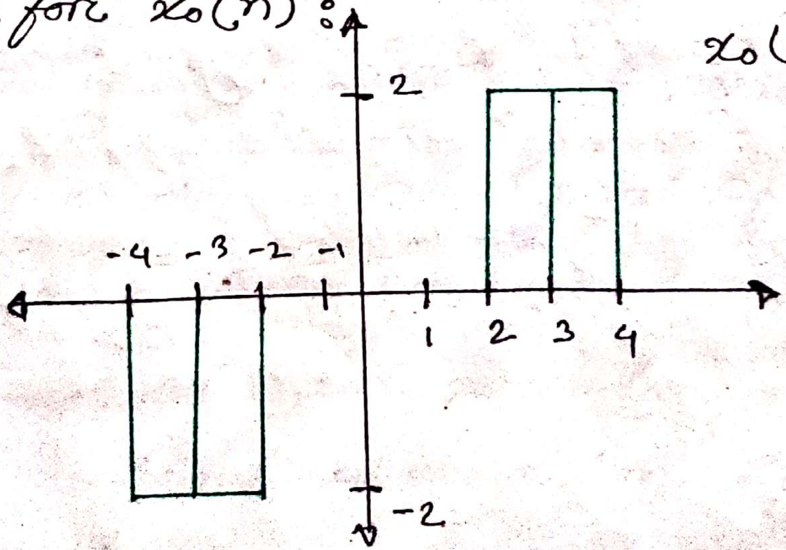


$$x_e(n) = 2\sum_{k=2}^{4} \delta(n-k)$$

$= 2[U(n-2) - U(r$

$+2[U(n+4) -$

→ figure for $x_o(n)$:



$$x_o(n) = 2\sum_{k=2}^{4} \delta(n-k)$$

$= 2[U(n-2) - U(n$

$-2[U(n+2)$

→ Problem: Find odd and even part of:

$x(n) = (\sin \omega n + 1)^2$

Soln: We have,

$$x(n) = \sin^2 \omega n + 2 \sin \omega n + 1$$

Here,

$$(\sin \omega n)^2 = [\sin(-\omega n)]^2 = \sin^2 \omega n$$

which is even.

$(1)^2 = (-1)^2 = 1$ which is also even.

So, $x_o(n) = 2\sin(\omega n)$ is the odd part.

Sinusoidal Signals

→ Prove that, while all analog sinusoidal signals are periodic but digital sinusoidal signals may or may not be periodic.

Proof: Let a sinusoidal analog signal is,

$$x(t) = A \cos(\omega_0 t + \phi)$$

Here. $\omega_0 t = \frac{2\pi t}{T_0}$

$$\Rightarrow \omega_0 T_0 = 2\pi$$

which refers the periodicity of the $x(t)$.

It can be write as,

$$\omega_0 T_0 = 2\pi N$$

where N is the number of period.

Now let a sinusoidal digital signal is

$$x(n) = A \cos(\omega_0 n + \phi)$$

It cannot be said periodic or non periodic without a condition. If it is periodic then,

$x(n+N) = x(n)$

$\therefore \cos\{\omega_0(n+N)+\phi\} = \cos(\omega_0 n+\phi)$

so for the periodicity we can write as,

$\omega_0 N = 2\pi r$ [r = Integer]

$\Rightarrow \frac{\omega_0}{2\pi} = \frac{r}{N}$ = It must have to be rational number.

so if cos 3n then,

$\frac{\omega_0}{2\pi} = \frac{3}{2\pi}$ = which is not rational number

so this is not be periodic.

So to say, while all analog sinusoidal signals are periodic but all digital signal may or may not be periodic.

(Proved)

→ Prove that, a linear combination of some signals (Digital) is periodic.

Proof: Suppose three periodic signals are $x_1(n)$, $x_2(n)$, $x_3(n)$ where the periods of these are N_1, N_2 and N_3.

So the linear sum of these signals are,

$$x_4(n) = \alpha x_1(n) + \beta x_2(n) + \gamma x_3(n).$$

The period of this signal is,

$$N_4 = LCM(N_1, N_2, N_3)$$

Let, $N_1 = 3$, $N_2 = 5$, $N_3 = 12$ then,

$$N_4 = 60$$

Then we can say, after 60 cycles $x_1(n)$ executes 20 complete cycles, where $x_2(n)$ executes 12 and $x_3(n)$ executes 5 cycles.

So, at this no. of 60, each of them must have completed an integral no. of.

cycles. So the sum of digital periodic signals is also periodic.

(Proved)

→ Why the range of vision is 0 to π?

Ans: We can explain an analog signal to be digital signal as,

$$\underset{\text{Analog}}{A\cos(\Omega t + \phi)} \xrightarrow{\text{(sampling)}} A\cos(\Omega n T + \phi)$$

$$\rightarrow \boxed{A/D} \rightarrow \underset{\text{Digital.}}{A\cos(\omega n + \phi)}$$

Where, $\omega = \Omega T = \frac{\Omega}{f_s} = \frac{2\pi f}{f_s}$

Here, f = Actual frequency.

f_s = Sampling "

Here ω is unitless because f/f_s has no unit.

But we give ω in radian because it is angle after all and,

Ω = Radian/sec.

ω = Normalised digital frequency.

we know, $f = f_s/2$

so, $\omega_{max} = 2\pi \dfrac{f_s/2}{f_s} = \pi$ so,

$0 < \omega < \pi$.

so for the +ve frequency the range of vision is 0 to π.

But for -ve freq. it is, 0 to $-\pi$.

So on spectrum basis the range of vision of a digital system is $-\pi$ to π. where it is for analog is $-\alpha$ to α.

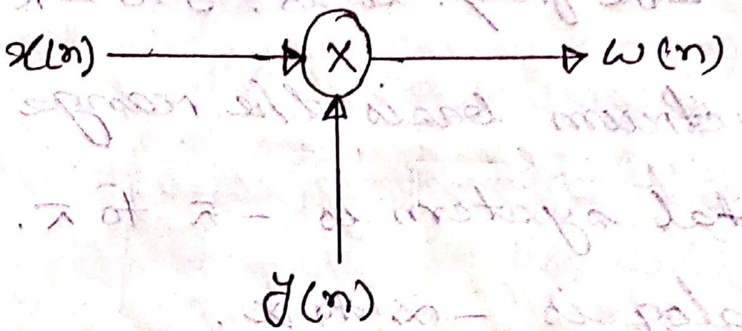
[** Radian has no dimension because the angle = $\dfrac{\text{Radian}}{\text{Radius}}$. so only radian is not a dimension.]

Operations on Sequences

→ **Multiplication:** The multiplication of two sequences is,

$$w(n) = x(n)\, y(n)$$

The symbol of this,



→ It means corresponding samples are multi-plied.

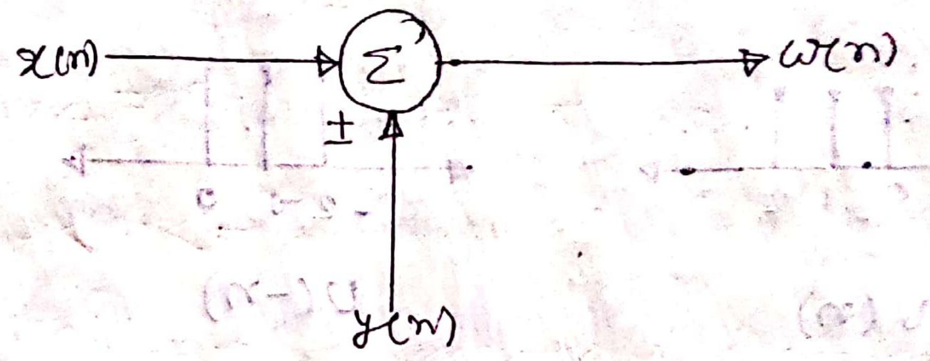
→ It is a non linear operation.

→ If the length of two sequences are not same, we need to pad zeros.

→ **Addition/subtraction:** The operation of two sequences is,

$$w(n) = x(n) \pm y(n)$$

The symbol is,



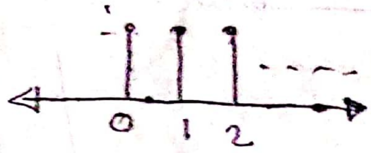
* If we want to add three sequences we need two summers.

→ Scalar multiplication: If we multiply a sequence with a scalar α is called scalar multiplication,

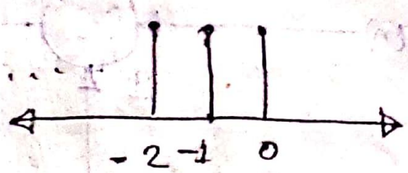
$$x(n) = \alpha\, x(n)$$

The symbol:

→ Time reversal: The time reversal of $x(n)$ is $x(-n)$ which is flip over. The figure of $u(n)$ and $u(-n)$ is given below:



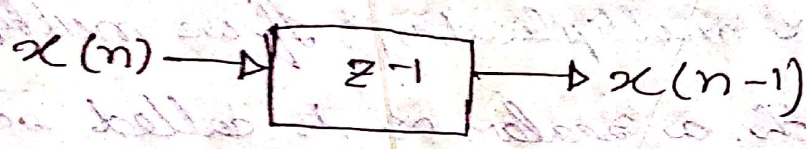
$u(n)$



$u(-n)$

So, $u(n) + u(-n) = 1 + \delta(n) \; ; \forall n$

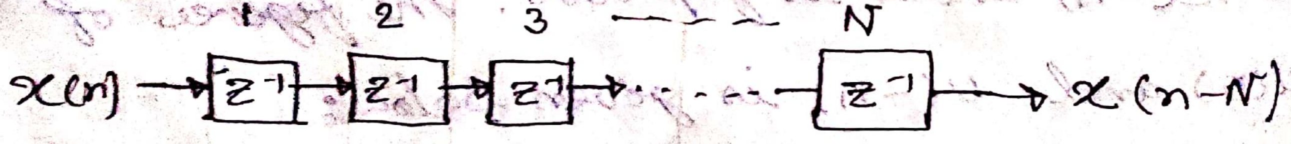
→ **Delay:**



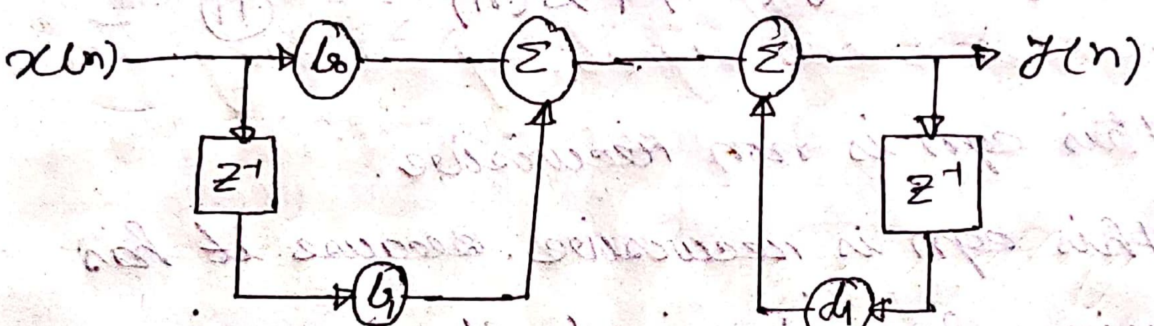
z^{-1}: It is the symbol of delay.

* For hardware it is 1 delay.
* For software, retrieving the immediate past sample. the immediate past sample is stored somewhere.

→ For delaying $x(n)$ to $x(n-N)$ then it executes as,



→ **Digital systems :** Digital systems handle digital signals and operates on digital signals. to produce another digital signal which is someways better ~~to~~ than at the begining we had. Symbolic diagram for this system is,

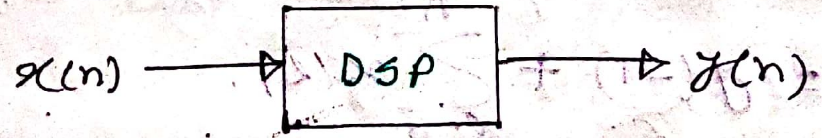


so the equation,

$$y(n) = b_0 x(n) + b_1 x(n-1) + d_1 y(n-1)$$

This is called difference equation.

In general expression is,



→ Some examples :

• Accumulator : An accumulator simply adds up.

$$y(n) = \sum_{l=-\alpha}^{n} x(l) \quad ---- (1)$$

$$\Rightarrow y(n) = y(n-1) + x(n) \quad --- (11)$$

(1) This eqn is non recursive.

(11) This eqn is recursive. Because it has recursion. Recursion is feedback.

We have to compute the present output, we have to have the immediate past output, i.e. recursion. This eqn can be written as,

$$y(n) = \sum_{l=-\alpha}^{-1} x(l) + \sum_{l=0}^{n} x(l)$$

$$= y(-1) + \sum_{l=0}^{n} x(l).$$

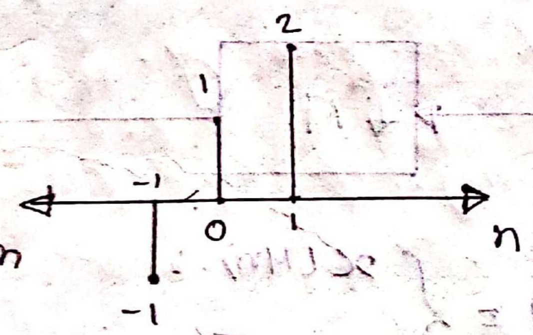
• **Up sampler:** In up sampling the relation between $y(n)$ and $x(n)$ is given by,

$$y(n) = \begin{cases} x\left(\frac{n}{L}\right) ; & n = 0, \pm L, \pm 2L \ldots \\ 0 , & \text{otherwise.} \end{cases}$$

So, symbolically:

$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow y(n)$$

Example:



$x(n)$

Let, $L = 3$, then, $y(0) = x(0) = 1$.

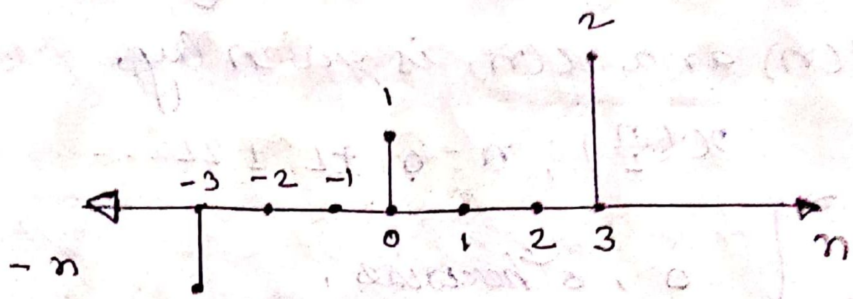
$\cancel{y\left(\frac{1}{3}\right) = 0 ; \; y\left(\frac{2}{3}\right) = 0}$.

$\cancel{y\left(\frac{3}{3}\right) =}$

$\therefore y(1) = x(1/3) = 0 ; \; y(2) = x(2/3) = 0$

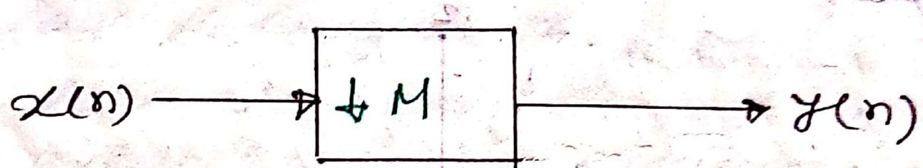
$y(3) = x(3/3) = 2$.

So; $y(n)$ is :



so, it not change the picture of the signal, only expand the signal, by padding two zeros in between two samples.

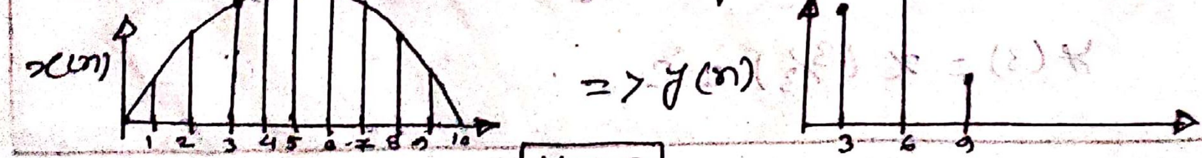
• **Down sampler:**



$$\text{where, } y(n) = \begin{cases} x(Mn) ; \\ 0, \text{ otherwise} . \end{cases}$$

this mean sampling exists only $-2M, -M, 0, M$ and $2M$.

In between two samples as like 0 and M_{th} sample there is no sample as like $M_{th}-1$ sample.



$\boxed{M = 3}$

• M-pt moving average system : It is described as,

$$y(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k)$$

Here we can described as,

$$x(n) \; ; \; x(n-1) \; ; \; x(n-2) \; . \; . \; . \; . \; . \; x(n-(M-1)).$$

$x(n)$ ↓ Present Input

$x(n-1)$ ↓ Past Input.

It is moving, so it is called moving average system. It is added present value and past value. It is also called "delta smoothing system".

Let,

$$x(n) = s(n) + d(n)$$

$$= \text{True signal} + \text{Noise}.$$

Here, $d(n) = 0$ by moving avg. system.

Classification of DS

- **Linear and Non linear system:**

A system is linear iff it obeys two rules:

→ Homogenity.

→ Superposition.

- **Homogenity:** Here x_1 leads y_1. Suppose,

$$x_1(n) \longrightarrow y_1(n) \text{ then.}$$

$$\alpha x_1(n) \longrightarrow \alpha y_1(n)$$

It is called homogenity.

- **Superposition:** If two inputs are super-imposed then the output is also be superposition of the individual output. i.e.

$$\alpha x_1(n) + \beta x_2(n) \longrightarrow \alpha y_1(n) + \beta y_2(n).$$

otherwise the system is non linear.

Ex: An accumulator, linear system,

$$y(n) = \sum_{l=-\alpha}^{\alpha} x(l)$$

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The non linear system,

$$y(n) = y(-1) + \sum_{l=0}^{\infty} x(l).$$

• Time (shift) invariance: If $x(n)$ leads to $y(n)$ in time invariance $x(n-n_0)$ leads to $y(n-n_0)$. so.

$$x(n) \longrightarrow y(n)$$

$$\Rightarrow x(n-n_0) \longrightarrow y(n-n_0)$$

The shifts, shifting in integer but the shape is preserved.

• Upsampler: We know,

$$y(n) = \begin{cases} x(\frac{n}{L}) ; & n = 0, \pm L, \pm 2L \ldots \\ 0 , & \text{elsewhere}. \end{cases}$$

Here, $x(n) \xrightarrow{\text{leads to}} x(\frac{n}{L})$ or $x(P)$

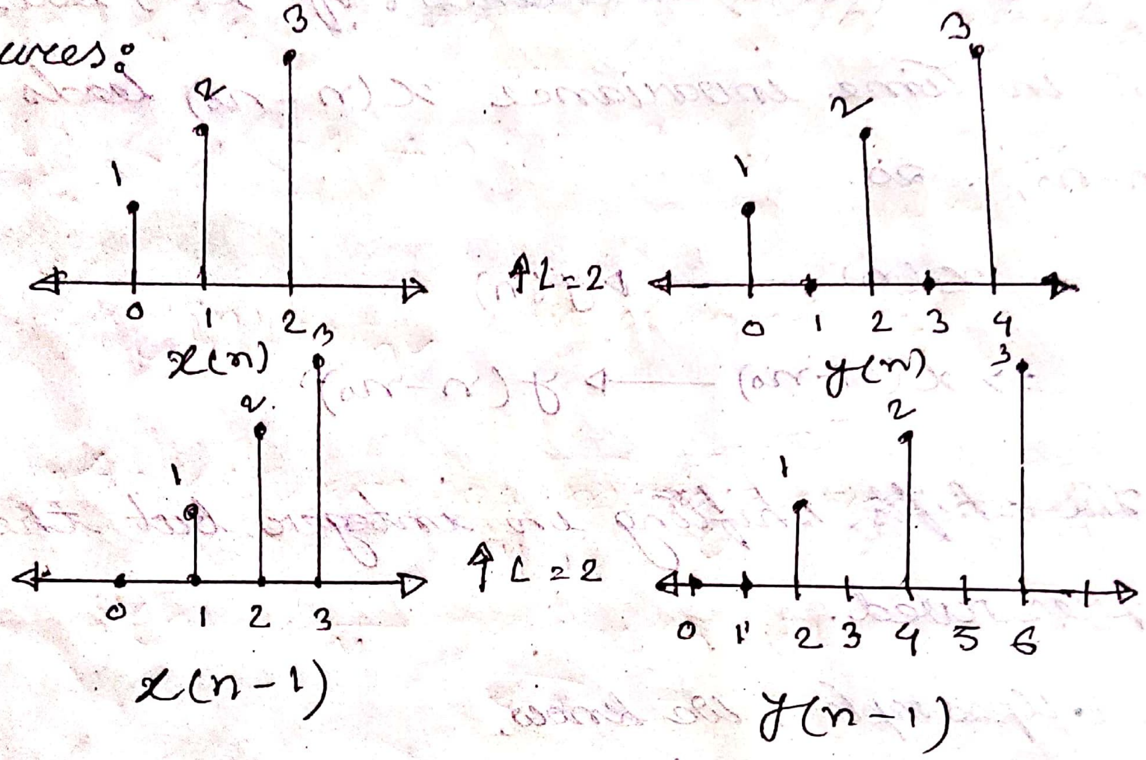
where, $P = \frac{n}{L}$.

$\therefore x(n-n_0) \xrightarrow{\text{leads to}} x(P-n_0)$ or $x(\frac{n}{L}-n_0)$

But, $y(n-n_0) = x\left(\frac{n-n_0}{L}\right)$

It is clear that, $x\left(\frac{n}{L}-n_0\right)$ and $x\left(\frac{n-n_0}{L}\right)$ are not same. So this is a time varying system.

Figures:



It is a linear system.

→ **Causality:** A system is causal if the future of this system can not be predicted. It is also said "Realizable system". So,

"If the present output $y(n)$ is depends on the present input of $x(n)$, its past values

$x(n-i)$ ~~dep~~ and perhaps past ~~of~~ values of $y(n-j)$ then the system is causal."

i, j = +ve integer.

$\therefore$ A digital system is causal iff,

$$x_1(n) \equiv x_2(n) \text{ for } n \leq N.$$

$$\Rightarrow y_1(n) \equiv y_2(n) \text{ for } n \leq N.$$

[$\equiv$: Identically equal to].

Otherwise the system is non-causal.

Example: $y(n) = x(n^2)$

This system is non causal. Because, when $n=2$, $y(2) = x(4)$ so for causal $y(2)$ depends on $x(2)$ but here $y(2)$ depends on $x(4)$.

* so $x(n)$ is causal if, $x(n)=0$; $n<0$.

$x(-1)$, $x(0)$ are non causal signal.

* so the signal will be causal if there is no initial value. otherwise non-causal.

→ Stability: A digital system is stable iff, for a bounded input we will get a bounded output. i.e. for,

$|x(n)| < B_x < \alpha$ $\forall n$ we get the output as

$|y(n)| < B_y < \alpha$ $\forall n$.

Now, if $y(n) = \frac{1}{n^2}$ this is unstable.

Example: we have for moving avrg. system,

$$y(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k)$$

Here, $|x(n)| \leq B_x < \alpha$ and for this,

$$|y(n)| \leq \frac{1}{M} \sum_{k=0}^{M-1} B_x = B_x.$$

This is a BIBO system.

~~By~~ But, $y(n) = \sum_{k=0}^{\alpha} x(n-k)$ is unstable.

→ **Passivity:**

Here, the energy of the output can not exceed the energy of the input. So a digital system is said to be passive if,

$$\sum_{n=-\infty}^{\infty} |y(n)|^2 \leq \sum_{n=-\infty}^{\infty} |x(n)|^2$$

i.e. a network can not generates energy.

Impulse and Step response

- Impulse response:

$$\delta(n) \longrightarrow \boxed{DS} \longrightarrow h(n)$$

Here, if $h(n) = 0$ for $n < 0$ then the system is causal. Otherwise it is noncausal system.

- Step response:

$$u(n) \longrightarrow \boxed{DS} \longrightarrow s(n)$$

- Example:

  - Accumulator: It can be various types.

Lets, ① $y(n) = \sum_{l=-\infty}^{\infty} x(l)$ Then the.

$$h(n) = \begin{cases} 1, & n \geq 0 \\ 0, & 0 < n \end{cases}$$ [we replace x by δ]

$$= u(n)$$

② Suppose, $y(n) = \sum_{K=0}^{\infty} x(n-K)$

Then, $h(n) = \sum_{K=0}^{\infty} \delta(n-K)$

$= u(n)$

Although the accumulator is different but the impulse response is same. Now take the another accumulator we have,

③ $y(n) = y(-1) + \sum_{l=0}^{n} x(l)$ then the response, $h(n) = \begin{cases} y(-1)+1 & ; n \geq 0 \\ y(-1) & ; n < 0 \end{cases}$

This is a non linear system.

• **Upsampler:** We know for up sampler,

$y(n) = \begin{cases} x(\frac{n}{L}) & ; n = 0, \pm L, \pm 2L \ldots\ldots \\ 0 & , \text{otherwise} \end{cases}$

so, $h(n) = \delta(n)$

And, $s(n) = \delta(n) + \delta(n-L) + \delta(n-2L) + \cdots$

$$= \sum_{k=0}^{\infty} \delta(n-kL)$$

LTI SYSTEMS

• **Convolution:**

Let $\delta(n)$ leads $h(n)$ i.e.

$$\delta(n) \longrightarrow h(n) \text{ then.}$$

$$\delta(n-k) \longrightarrow h(n-k) \quad [\because \text{Time invariance}]$$

$$x(k)\,\delta(n-k) \longrightarrow x(k)\,h(n-k) \quad [\text{Homogenity}; \; x(k) = \text{const}]$$

$$\underbrace{\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)}_{x(n)} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x(k)\,h(n-k)}_{y(n)} \quad [\text{Superposition}]$$

which is the famous convolution theorem.

i.e. for LTI system,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

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$$= \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

$$= x(n) * h(n)$$

$$= h(n) * x(n)$$

→ The operation of convolution is commutative.

→ Special case,

In general convolution is applied for LTI system,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Here $h(n)$ is causal if $h(n) = 0$; $n < 0$ then,

$$y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n-k)$$

And $x(n)$ is also causal : $x(k) = 0$, $k < 0$ then,

$$y(n) = \sum_{k=0}^{n} x(k)\, h(n-k)$$

Here all these are causal. (LTIC(SBS))

* Convolution is associative,

$$x_1 * x_2 * x_3 = x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$$

* Convolution is also distributive,

$$x_1 * (x_2 + x_3) = x_1 * x_2 + x_1 * x_3 .$$

→ Computing convolution:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

- step by step method:

$x(k)$ [Find this put the value]

x_{-1} x_0 x_1

$h(k)$

$h(-k)$

$h(1-k)$ [shift to right]

$h(-1-k)$ [shift left]

so for $y(0)$ we should multiply $x(k)$, $h(-k)$

" $y(1)$ " " " $x(k)$, $h(1-k)$

" $y(-1)$ " " " $x(k)$, $h(-1-k)$

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- *Example:*

-2	-1	0	1	2	0	$\rightarrow k$
	x_{-1}	x_0	x_1	x_2		$\rightarrow x(k)$
h_{-2}	h_{-1}	h_0	h_1			$\rightarrow h(k)$
	h_1	h_0	h_{-1}	h_{-2}		$\rightarrow h(-k)$
		h_1	h_0	h_{-1}	h_{-2}	$\rightarrow h(1-k)$

$\therefore$ $\cancel{y(0) = x_{-1}h_{-1} + x_0 h_0 + x_1 h_1}$

$$y(0) = x_{-1}h_1 + x_0 h_0 + x_1 h_{-1} + x_2 h_{-2} = -6$$

$$y(1) = x_0 h_1 + x_1 h_0 + x_2 h_{-1}$$

$$= 0 \times 1 + 1 \times 0 + 2 \times (-1) = -2$$

** The length of $y(n)$ is,

$$N = N_1 + N_2 - 1$$

where, N_1 = Length of x.

N_2 = Length of h.

• Eazy way :

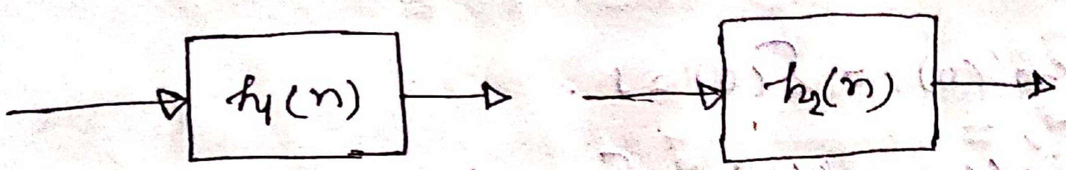
		-2	-1	0	1	2	$\rightarrow K$
			x_{-1}	x_0	x_1	x_2	$\rightarrow x(k)$
		h_{-2}	h_{-1}	h_0	h_1	ϕ	$\rightarrow h(k)$
			$h_1 x_{-1}$	$h_1 x_0$	$h_1 x_1$	$h_1 x_2$	
		$h_0 x_{-1}$	$h_0 x_0$	$h_0 x_1$	$h_0 x_2$	$\times$	
	$h_{-1} x_{-1}$	$h_{-1} x_0$	$h_{-1} x_1$	$h_{-1} x_2$	$\times$		
$h_{-2} x_{-1}$	$h_{-2} x_0$	$h_{-2} x_1$	$h_{-2} x_2$	$\times$			
$y(-3)$	$y(-2)$	$y(-1)$	$y(0)$	$y(1)$	$y(2)$	$y(3)$	

$$y(0) = h_1 x_{-1} + h_0 x_0 + h_{-1} x_1 + x_{-2} h_{-2}$$

$$y(1) = h_1 x_0 + h_0 x_1 + h_{-1} x_2$$

→ Inverse system :

If a system with impulse response $h_1(n)$ and a inverse of another system $h_2(n)$ then,



iff $h_1(n) * h_2(n) = \delta(n)$

• Accumulator :

We have for accumulator,

$$y(n) = \sum_{k=0}^{\infty} x(n-k)$$

$$h(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

If an inverse system exists and if its impulse response is $h'(n)$ then,

$$h'(n) * \sum_{k=0}^{\infty} \delta(n-k) = \delta(n)$$

$$\Rightarrow \sum_{k=0}^{\infty} h'(n-k) = \delta(n) \quad \text{--- (1)}$$

Now, by expanding eqn ① we have,

$$h'(n) + h'(n-1) + h'(n-2) + \cdots = \delta(n)$$

Now putting the values of n,

$n = 0$, $h'(0) = \delta(0) = 1$.

$n = 1$, $h'(1) + h'(0) = \delta(1) = 0$

$\therefore h'(1) = -h'(0) = -1$

$n = 2$, $h'(2) + h'(1) + h'(0) = \delta(2) = 0$

$\therefore h'(2) = -h'(1) - h'(0) = 1 - 1 = 0$

$\therefore h'(n) = 0, \ n \geq 2$.

$$h'(n) = \delta(n) - \delta(n-1)$$

It is the inverse of an accumulator.

Discrete Time Fourier Transformation

→ DTFT:

$$x_d(e^{J\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-J\omega n}$$

→ IDTFT: $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{Jn\omega}\, d\omega$$

Ex: Find DTFT for $\delta(n)$?

Soln: Here, $x(n) = \delta(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$

$$\therefore F_d[\delta(n)] = X(e^{J\omega}) = \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-Jn\omega}$$

$$= 1 \;;\; n = 0.$$

Ex: Find DTFT for $U(n)$?

Soln: Here, $x(n) = U(n) = \begin{cases} 1, & n \geq 0. \\ 0, & n < 0. \end{cases}$

$$\therefore F_d[U(n)] = \sum_{n=-\infty}^{\infty} U(n)\, e^{-J\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-J\omega n}$$

$$\therefore X(e^{j\omega}) = 1 + e^{-j\omega} + e^{-2\omega j} + e^{-3j\omega} + \cdots \quad (i)$$

$(i) \times e^{-j\omega} \Rightarrow$

$$e^{-j\omega} X(e^{j\omega}) = e^{-j\omega} + e^{-2\omega j} + e^{-3j\omega} + \cdots \quad (ii)$$

$(i) - (ii) \Rightarrow$

$$X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}}$$

$$\therefore X(\omega) = \frac{1}{1 - e^{-j\omega}}.$$

Ex: Find DTFT for $a^n u(n)$?

Soln: Here, $x(n) = a^n u(n)$

$$\therefore X(\omega) = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$= 1 + a e^{-j\omega} + a^2 e^{-2j\omega} + a^3 e^{-3j\omega} + \cdots \quad (i)$$

$(i) \times a e^{-j\omega} \Rightarrow$

$$a e^{-j\omega} X(\omega) = a e^{-j\omega} + a^2 e^{-2j\omega} + a^3 e^{-3j\omega} + \cdots \quad (ii)$$

(i) − (ii) =>

$$X(\omega) = \frac{1}{1 - a e^{-j\omega}}.$$

Ex: Find DTFT for $\left(\frac{1}{2}\right)^n u(n) + 2^n u(n)$

Solⁿ: Here, $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$

$x_2(n) = 2^n u(n)$

$$X_1(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$X_2(\omega) = \frac{1}{1 - 2 e^{-j\omega}}$$

$$\text{So, } X(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} + \frac{1}{1 - 2 e^{-j\omega}}$$

$$= \frac{2}{2 - e^{-j\omega}} + \frac{1}{1 - 2 e^{-j\omega}}$$

$$= \frac{2 - 4e^{-j\omega} + 2 - e^{-j\omega}}{(2 - e^{-j\omega})(1 - 2e^{-j\omega})}$$

$$= \frac{4 - 5e^{-j\omega}}{(2 - e^{-j\omega})(1 - 2e^{-j\omega})}.$$

** A DFT is a discrete Fourier Transformation; An FFT is a fast Fourier Transform. An FFT is a DFT, but is must faster for calculations. The whole point of the FFT is speed in calculating a DFT.

→ The DFT:

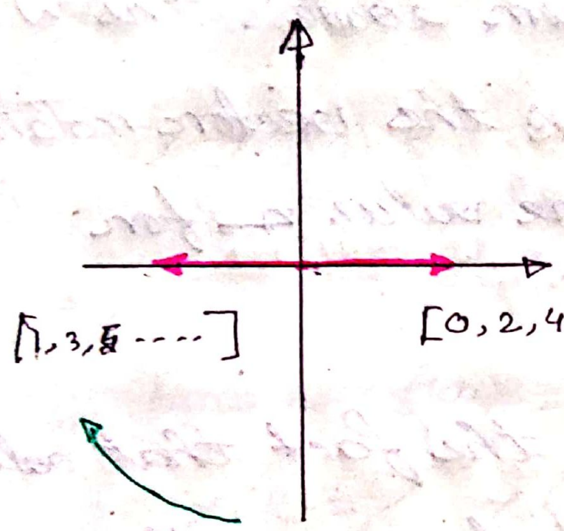
The DFT converts discrete data from a time wave into a frequency spectrum. Using the DFT implies that the finite segment that is analyzed is one period of an infinitely extended periodic signal.

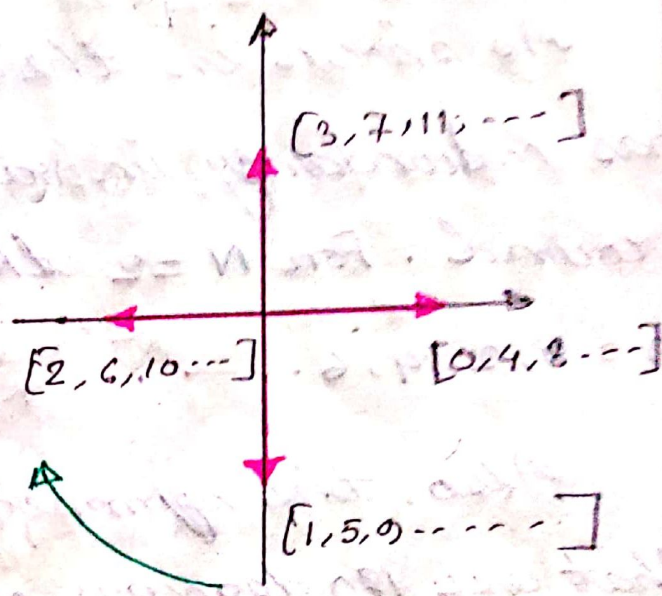
$$F(n) = \sum_{K=0}^{N-1} x(K)\, e^{-J(2\pi/N)Kn}$$

Here, $W_N^n = e^{-J(2\pi/N)Kn}$ which is named Twiddle Factor. The speed is depends on twiddle factor.

→ The twiddle factor:

The Twiddle factor, W, describes a "rotating vector", which rotates in increments according to the number of samples, N. Here are graphs for N = 2, 4 and 8:





N = 2

$$W_2^n = e^{-\frac{j2\pi n}{2}}$$

N = 4

$$W_4^n = e^{-\frac{j2\pi n}{4}}$$

The same set of W values repeat over and over for different values of n. Also, those that are 180 degrees apart are the negative of each other. These facts are used to make calculating the DFT efficient and help to make the FFT possible.

- The ~~next~~ redundancy and symmetry of the twiddle factor :

As shown in the diagram above, the W has redundancy in values as the vector rotates around. For $N = 2$ the same value ~~n~~ for $n = 0, 2, 4, 6 \ldots$

Also, the symmetry is the fact that values that are 180 degrees out of phase are the negative of each other. So for example, W for $N = 4$ samples where $n = 0, 4, 8$ etc are the $-ve$ of $n = 2, 6, 10$ etc.

- Computation of DFT:

(i) $$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-J(2\pi/N)Kn}$$

$$= x(0)\, e^{0} + x(1)\, e^{-J(2\pi/N)K} + x(2)\, e^{-J(2\pi/N)2K} + \cdots + x(N-1)\, e^{-J(2\pi/N)K(N-1)}$$

(ii) $K = 0 \Rightarrow$

$$X(0) = x(0) + x(1) + \cdots + x(N-1)$$

(iii) $k = 1$;

$$x(1) = x(0) + x(1)\, e^{-j(2\pi/N)} + \cdots\cdots$$

→ **Example:** Find DFT for,

$$x(n) = (\underset{\uparrow\, n=0}{1}, 0, -1, 2)$$

Soln: Here we have,

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j(2\pi/4)n} \quad [\because N = 4]$$

$$\Rightarrow X(0) = x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 0 - 1 + 2 = 2$$

$$\Rightarrow X(1) = x(0) + x(1)\, e^{-j(2\pi/4)} + x(2)\, e^{-j(2\pi/4)2}$$

$$+ x(3)\, e^{-j(2\pi/4)3}$$

$$= 1 - e^{-j\pi} + 2\, e^{-j\, 3\pi/2}$$

$$= 2 + 2j$$

$$\left[\because e^{-j\pi/2} = \cos \pi/2 - j \sin \pi/2 = -j \right.$$

$$\left. e^{-j\, 3\pi/2} = \cos 3\pi/2 - j \sin 3\pi/2 = \right]$$

similarly we have,

$$x(2) = -2$$

$$x(3) = 2 - 2j$$

→ IDFT: For IDFT we have,

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} x(K)\, e^{j(2\pi/N)nK}$$

Example: Find IDFT for,

$$x(K) = (\underset{\uparrow_{n=0}}{2}, 2+2j, -2, 2-2j)$$

Soln: We know,

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} x(K)\, e^{j(2\pi/N)nK}.$$

$$= \frac{1}{4}\left[x(0) + x(1)\, e^{j\pi/2\, n} + x(2)\, e^{j\pi n} + x(3)\, e^{j\, 3n\pi/2}\right]$$

$\therefore x(0) = 1$

$x(1) = 0$

$x(2) = -1$

$x(3) = 2$

• How fast is an FFT versus a straight DFT?

Ans: For DFT N^2 operations are required.

For FFT only $N \log_2 N$ operations are required.

For 1024 samples a DFT requires,

$1024^2 = 1048576$ operations.

Where FFT requires,

$1024 \log_2 1024 = 10240$ operations.

This means that a 1024 samples FFT is 102.4 times faster than DFT.

→ To learn FFT we need to learn:

1. Danielson-Lanczos Lemma.
2. The twiddle factor $= W_N^n$.
3. The Butterfly diagram.

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→ The Danielson-Lanczos Lemma:

We have, $F(n) = \sum_{k=0}^{N-1} x(n)\, e^{-J(2\pi/N)kn}$.

This equation is expanded further more and more in even and odd terms by the D-L lemma

$\therefore F(n) = E + O$ where,

$$E = \sum_{K=0}^{N/2-1} x(2K)\, e^{-J\frac{2\pi(2K)n}{N}}$$

$$= \sum_{K=0}^{N/2-1} x(2K)\, e^{-\frac{J2\pi kn}{N/2}}$$

$$O = \sum_{K=0}^{\frac{N}{2}-1} x(2K+1)\, e^{-\frac{J2\pi(2K+1)n}{N}}$$

$$= \sum_{K=0}^{N/2-1} x(2K+1)\, e^{-\frac{J2\pi Kn}{N/2}}\, e^{-\frac{J2\pi n}{N}}$$

$$= W_N^n \sum_{K=0}^{N/2-1} x(2K+1)\, e^{-\frac{J2\pi kn}{N/2}}$$

Where, Twiddle factor,

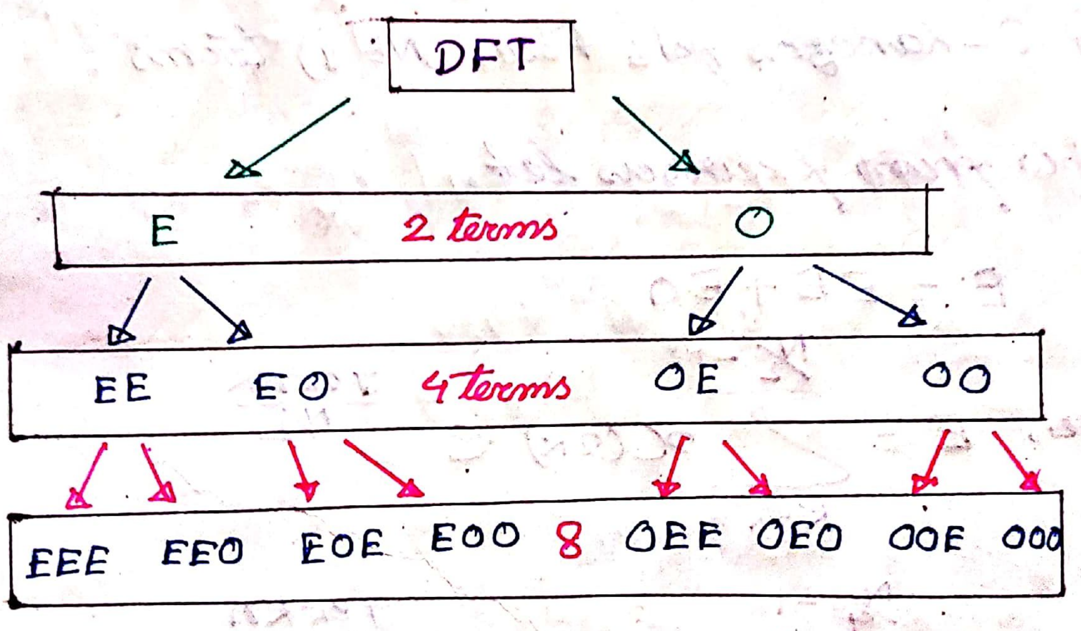
$$W_N^n = e^{-\frac{j2\pi n}{N}}$$

Again we can devide:

$$E = EE + EO$$

$$O = OE + OO$$

So,



The $F(n) = E + O$ is the first level breakdown. We can continue this terms untill we run out of samples and only have one values in the sum. as,

$$\sum_{k=0}^{0} = 1$$

For, $N = 2$ we have,

$$F(n) = \sum_{k=0}^{0} x(2k)\, e^{-\frac{j2\pi nk}{\frac{2}{2}}} + W_2^n \sum_{k=0}^{0} x(2k+1)\, e^{-j2\pi kn}$$

∴ when $k = 0$ then,

$$F(n) = x(0) + W_2^n\, x(1).$$

→ Now D-Lanczos for Four ($N = 4$) terms :

Now from previous let,

$$E = EE + EO$$

where, $$E = \sum_{k=0}^{N/2 - 1} x(2k)\, e^{-\frac{j2\pi nk}{N/2}}$$

$$\therefore EE = \sum_{k=0}^{N/4 - 1} x(4k)\, e^{-\frac{j2\pi kn}{N/4}}$$

$$EO = \sum_{k=0}^{N/4 - 1} x(2(2k+1))\, e^{-\frac{j2\pi(2k+1)n}{N/2}}$$

$$= W_{N/2}^{n} \sum_{k=0}^{N/4 - 1} x(4k+2)\, e^{-\frac{j2\pi nk}{N/4}}$$

similarly, $O = OE + OO$

$$\text{where, } OE = W_N^{n} \sum_{k=0}^{N/4 - 1} x(4k+1)\, e^{-\frac{j2\pi k n}{N/4}}$$

$[\because k = 2k]$

$$\text{And } OO = W_N^{n} W_{N/2}^{n} \sum_{k=0}^{\frac{N}{4} - 1} x(4k+3)\, e^{-\frac{j2\pi k n}{N/4}}$$

[Here, $k = 2k+1$]

So, $N = 4$ so,

$$F(n) = \sum_{k=0}^{\frac{N}{4} - 1} x(4k)\, e^{-\frac{j2\pi nk}{N/4}} + W_{N/2}^{n} \sum_{k=0}^{\frac{N}{4} - 1} x(4k+2)\, e^{-\frac{j2\pi kn}{N/4}} + W_N^{n} \sum_{k=0}^{\frac{N}{4} - 1} x(4k+1)\, e^{-\frac{j2\pi kn}{N/4}}$$

$$+ W_N^{n} W_{N/2}^{n} \sum_{k=0}^{N/4 - 1} x(4k+3)\, e^{-\frac{j2\pi kn}{N/4}}$$

$\therefore K = 0,$

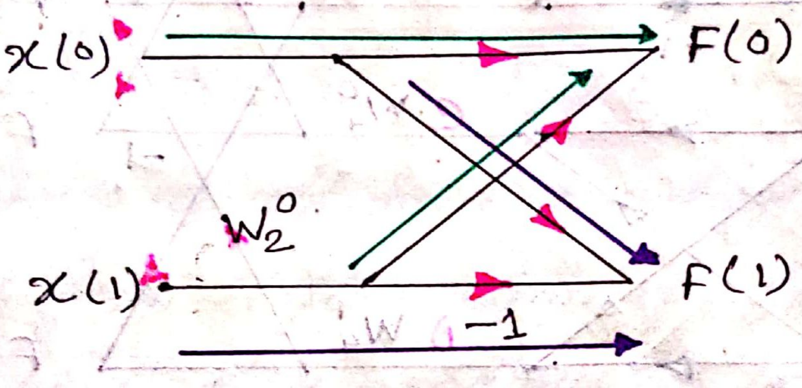
$$F(n) = x(0) + W_2^n x(2) + W_4^n x(1) + W_4^n W_2^n \cdot x(3).$$

→ The Danielson - Lanczos for 8 input values $(K=0)$:

$$F(n) = x(0) + W_2^n x(4) + W_4^n x(2) + W_4^n W_2^n x(6) + W_8^n W_2^n x(5) + W_8^n W_4^n x(3) + W_8^n W_4^n W_2^n x(7).$$

→ The Butterfly Diagram:

The butterfly diagram builds on the D-L Lemm
nd the twiddle factors to create an efficient
lgorithm.



$$\therefore F(0) = x(0) + W_2^0\, x(1)$$

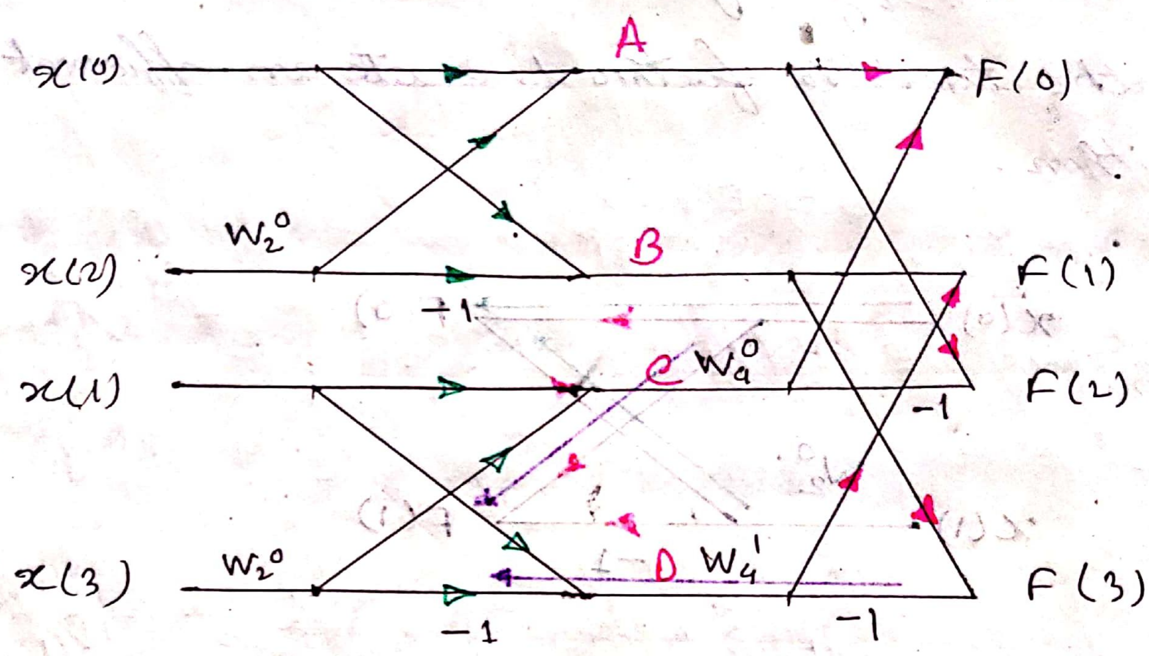
$$F(1) = x(0) - W_2^0\, x(1)$$

$$\left[\because W_2^1 = -W_2^0\right]$$

$$\left[\text{And} \therefore F(1) = x(0) + W_2^1\, x(1)\right]$$

- Create a 4 input butterfly:

Here we need 2, 2-input butterflies. i.e.



Now, $A = x(0) + W_2^0\, x(2)$

$B = x(0) - W_2^0\, x(2)$

$C = x(1) + W_2^0\, x(3)$

$D = x(1) - W_2^0\, x(3)$

$\therefore F(0) = A + W_4^0\, C$

$= x(0) + W_2^0\, x(2) + W_4^0\, x(1) + W_4^0 W_2^0\, x(3)$

$F(1) = B + W_4^1\, D.$

$= x(0) - W_2^0 x(2) + W_4^1 x(1) - W_4^1 W_2^0 x(3)$

$F(2) = A - W_4^0 C$

And, $F(3) = B - W_4^1 D$

→ Example: Design a 8 input Butterfly diagram to calculate FFT of $x = [1, 2, -1, 3, 1, 2, 1, -3]$. Compare your result from the diagram with that obtained from Matlab command fft(x, 8). Explain if you find any ambiguities in two results.

Soln: Here given up,

$$x = [1, 2, -1, 3, 1, 2, 1, -3]$$

We know,

$$W_N^n = e^{-j\frac{2\pi n}{N}}$$

$$\therefore W_2^0 = W_4^0 = W_8^0 = 1$$

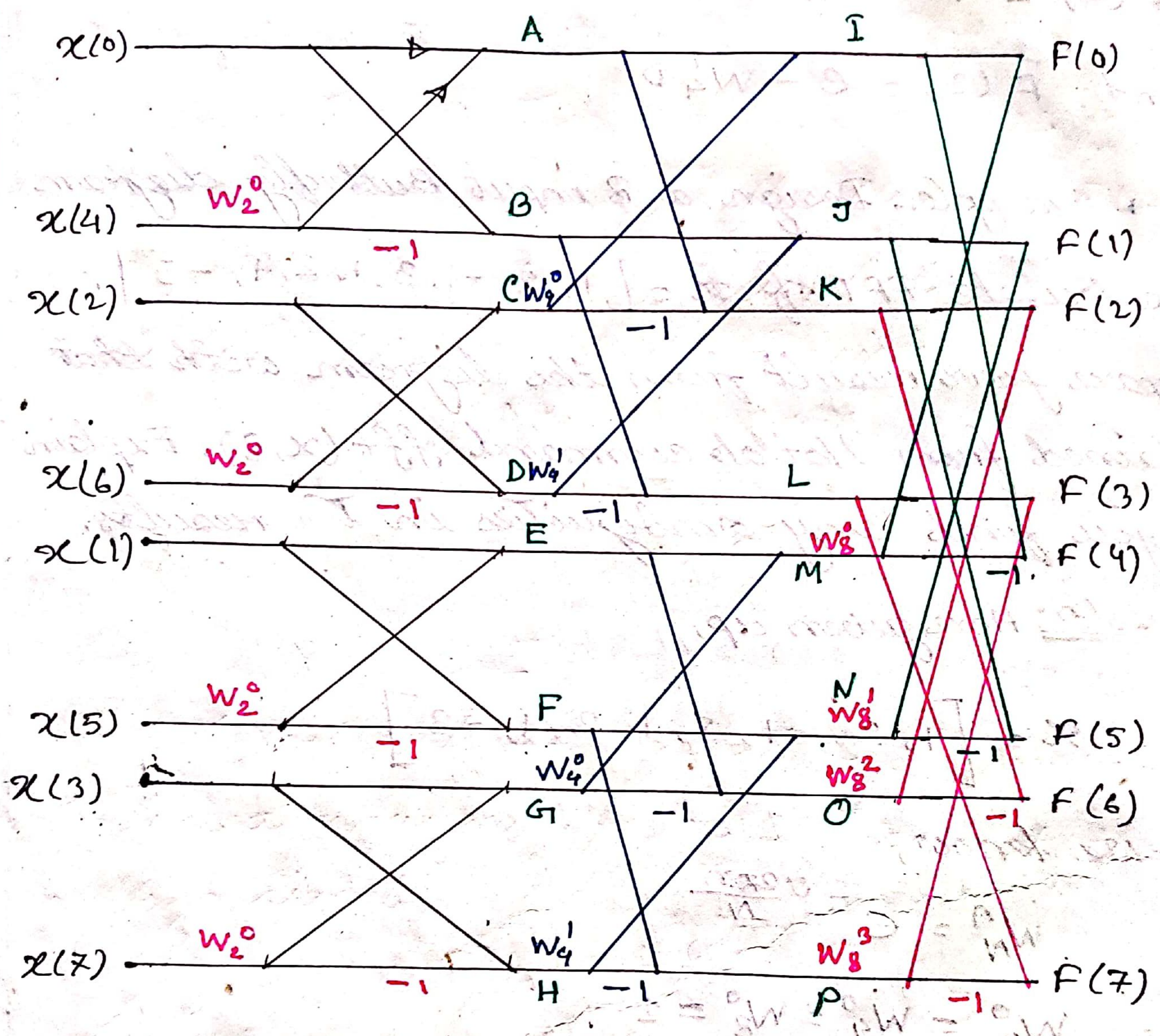
$$W_4^1 = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$$

$$W_8^1 = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4 = \frac{1-j}{\sqrt{2}}$$

$$W_8^2 = e^{-\frac{j2\pi 2}{8}} = e^{-j\pi/2} = -j$$

$$W_8^3 = e^{-j3\pi/4} = \cos 3\pi/4 - j\sin 3\pi/4 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$= \frac{-1-j}{\sqrt{2}}$$



From figure we have,

$A = x(0) + W_2^0 x(4)$

$= 1 + 1 = 2$

$B = x(0) - W_2^0 x(4) = 1 - 1 = 0$

$C = x(2) + W_2^0 x(6)$

$= -1 + 1 = 0$

$D = x(2) - W_2^0 x(6)$

$= -2$

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$E = x(1) + W_2^0 x(5) = 2+2 = 4$

$F = x(1) - W_2^0 x(5) = 2-2 = 0$

$G = x(3) + W_2^0 x(7) = 3-3 = 0$

$H = x(3) - W_2^0 x(7) = 3+3 = 6$

$I = A + W_4^0 C = 2+0 = 2$

$J = B + W_4^1 D = 0 + 2j = 2j$

$K = A - W_4^0 C = 2-0 = 2$

$L = B - W_4^1 D = 0 - 2j = -2j$

$M = E + W_4^0 G = 4+0 = 4$

$N = F + W_4^1 H = 0 - j6 = -6j$

$O = E - W_4^0 G = 4-0 = 4$

$P = F - W_4^1 H = 0 + j6 = 6j$

$\therefore F(0) = I + W_8^0 M = 2+4 = 6$

$F(1) = J + W_8^1 N = 2j + \left(\frac{1-j}{\sqrt{2}}\right)(-6j) = -4.243 - 2.243j$

$F(2) = K + W_8^2 O = 2 - 4j$

$F(3) = L + W_8^3 P = -2j + \frac{(-1-j)}{\sqrt{2}} \times 6j = 4.2426 - 6.2426j$

$F(4) = I - W_8^0 M = -2$

$F(5) = J - W_8^1 \cancel{M} N = 4.2426 + 6.2426j$

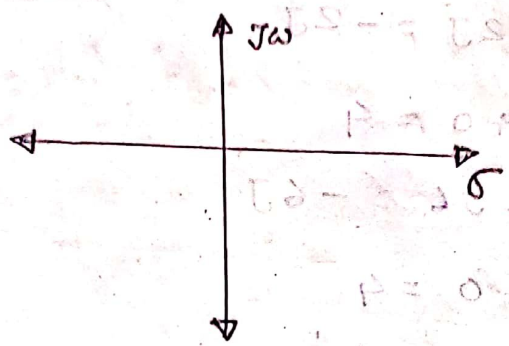
$F(6) = K - W_8^2 O = 2 + 4j$

$F(7) = L - W_8^3 P = -4.2426 + 2.2426j$

Z-Transformation

→ Why we need z-transformation?

Ans: In continuous time domain, we have fourier transform and have some difficulties i.e. FT of any CT signal is only $j\omega$ axis. And there are signals for which FT does not exist.



For those signals we instead of $e^{j\omega t}$ ~~where~~ we introduce a convergence factor $e^{-\sigma t}$ so that the resulting signal,

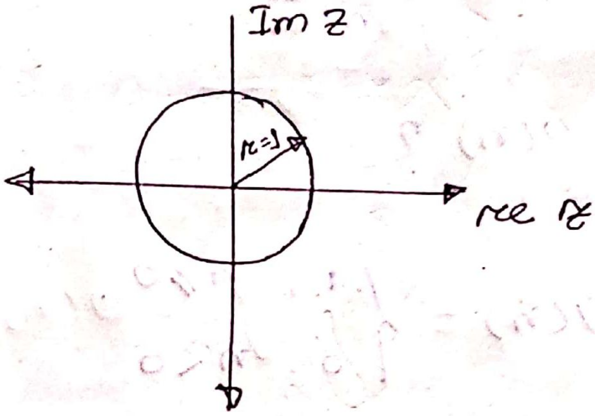
$$x(t)\, e^{-\sigma t} e^{-j\omega t}$$

Here the convergence factor become $x(t)\, e^{-\sigma t}$ i.e. LT and we cover the whole plane.

z-transfer has some kinds of interpretation

we also introduce a convergence factor r^{-n}

in DTFT. So that time the resulting signal becomes, $x(n) r^{-n} e^{-j\omega n}$ which as like F.T. but here r is only 1. So we can not cover whole z-plane.



To remove this restriction we need to introduce a new variable i.e. z by which we can encompass the whole z-plane. So the signal become,

$$x(z) = \sum x(n)\, z^{-n}$$

where, $z = re^{j\omega}$ and this r can be varied.

So this is the necessity of z in DT domain.

→ Find ZT of delta function?

Ans: We have, $x(n) = \delta(n) = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$

$$\therefore x(z) = Z[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = 1, \quad n=0$$

→ Find ZT of $U(n)$?

Ans: We have,

$$x(n) = U(n) = \begin{cases} 1, & n\geq 0 \\ 0, & n<0 \end{cases}$$

$$\therefore X(z) = Z[U(n)] = \sum_{n=-\infty}^{\infty} U(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \cdots + 0 \quad \text{(i)}$$

$$\text{(i)} \times z^{-1} \Rightarrow z^{-1}X(z) = z^{-1} + z^{-2} + z^{-3} + \cdots \quad \text{(ii)}$$

$$\text{(i)} - \text{(ii)} \Rightarrow X(z) = \frac{1}{1-z^{-1}}$$

→ Find ZT of $a^n U(n)$:

Ans: $\frac{1}{1-az^{-1}}$

Inverse of z-transform:

Inverse of ZT as like as inverse of FT. i.e.

$$g(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(z) z^n \, d\omega$$

* Example: Find out the IZT of,

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+.6)}$$

Soln: we have,

$$H(z) = \frac{1+2z^{-1}}{(1-.2z^{-1})(1+.6z^{-1})}$$

$$= \frac{A}{1-0.2z^{-1}} + \frac{B}{1+0.6z^{-1}}$$

$$A = (1-0.2z^{-1})\, H(z)\Big|_{.2z^{-1}=1} = 2.75$$

$$B = -1.75.$$

$$\therefore H(z) = \frac{2.75}{1-0.2z^{-1}} - \frac{1.75}{1+0.6z^{-1}}$$

$$h(n) = 2.75(.2)^n u(n) - 1.75(-.6)^n u(n)$$

→ Long Division Technique:

→ $$G(z) = \frac{1}{1 - z^{-1}}$$

$$= \frac{z}{z - 1}$$

$$z - 1 \,\big)\, z \,\big(\, 1 + z^{-1} + z^{-2} + \cdots$$

$$\underline{z - 1}$$

$$1$$

$$\underline{1 - z^{-1}}$$

$$z^{-1}$$

$$\underline{z^{-1} - z^{-2}}$$

$$\vdots$$

$$\therefore g(n) = \{\underset{\uparrow}{1}, 1, 1, \cdots\} = u(n)$$

→ $$G(z) = \frac{1}{1 - z^{-1}}$$

$$= \frac{1}{-z^{-1} + 1}$$

$$-z^{-1} + 1 \,\big)\, 1 \,\big(\, -z - z^2 - z^3 - \cdots$$

$$\underline{1 - z}$$

$$z$$

$$\underline{z - z^2}$$

$$z^2$$

$$\vdots$$

$$\therefore g(n) = \{\underset{\uparrow}{0}, -1, -1, -1, \cdots\}$$

$$= -u(-n-1)$$

→ Prove that ZT is a periodic transformation.

OR

Find ZT of $x(z) = \dfrac{1}{1+z^{-1}+z^{-2}}$

<u>Ans</u>: We have,

$$x(z) = \frac{1}{1+z^{-1}+z^{-2}}$$

So,

$$
\begin{array}{r|l|l}
1+z^{-1}+z^{-2} & 1 & 1-z^{-1}+z^{-3}-\cdots \\
 & 1+z^{-1}+z^{-2} & \\
\hline
 & -z^{-1}-z^{-2} & \\
 & -z^{-1}-z^{-2}-z^{-3} & \\
\hline
 & z^{-3} & \\
 & z^{-3}+z^{-4}+z^{-5} & \\
\hline
 & -z^{-4}-z^{-5} & \\
 & \vdots &
\end{array}
$$

So, $x(z) = 1 - z^{-1} + z^{-3} - z^{-4} + z^{-6} \cdots\cdots$

$\therefore x(n) = \{1, -1, 0, 1, -1, 0, 1, -1, 0 \cdots\cdots\}$

Here, we see that there is a period after every (1, −1, 0). So we obtain a periodic function after transformation. So ZT is a periodic transformation. (Proved).

FIR AND IIR

→ FIR (Finite impulse Response):

If a system has,

$$h(n) \neq 0 \quad \text{for } N_1 \leq n \leq N_2$$
$$= 0 \quad \text{everywhere else.}$$

so the range of vision is,

i.e. our range of vision is limited or finite on 'n' axis then the system is FIR.

Here the output is,

$$y(n) = \sum_{k=N_1}^{N_2} x(k)\, h(n-k)$$

so the system has $(N_2 - N_1 + 1)$ no. of samples. i.e. finite.

→ IIR (Infinite impulse response):

If a systems range of vision is $-\infty$ to 0 or 0 to ∞ or $-\infty$ to ∞ then the system is IIR.

→ Example : the moving average system i.e.

$$y(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k)$$, is a FIR

and here, $h(n) = \{ \underset{\substack{\uparrow \\ n=0}}{\frac{1}{M}}, \frac{1}{M}, \text{-------} \underset{\substack{\uparrow \\ n=M-1}}{\frac{1}{M}} \}$

** FIR and IIR filters can be computed non-recursively.

** Recursive and non-recursive this two terms is needed to describe the process of computation.

** Recursive computation requires feedback.

** Non-recursive " doesnot require " .

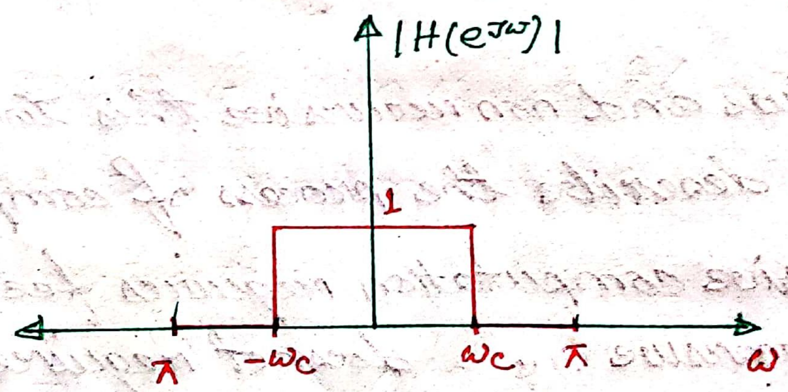
** IIR filters can also be computed recursively.

Simple Digital Filters

→ **Filter:** A filter is an electrical device which selects an appropriate frequency and rejects others.

→ **Q:** Show that an Ideal low pass filter is not realizable.

Ans: Let an ideal low pass filter's characteristic curve is :



For this curve range of vision is $-\pi$ to π. And the magnitude is 1 in the limit $-\omega_c$ to ω_c.

So its impulse response is,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \quad [\because H(e^{j\omega}) = 1]$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-jn\omega_c}}{jn} \right]$$

$$= \frac{\sin n\omega_c}{n\pi}$$

$\neq 0$ for $n<0$

so the system is non causal and therefore not realizable.

Also, $h(n) = \frac{\sin n\omega_c}{n\pi}$ is not ~~a~~ absolutely summable. so.

$\sum |h(n)| \not< \infty$. so it is not unstable.

(showed)

→ Q: Find that, what type of filter it is the function,

$$H(z) = \frac{1}{2}(1+z^{-1})$$

Ans: We know, $z = e^{j\omega}$.

$$\therefore H(e^{j\omega}) = \frac{1}{2}(1+e^{-j\omega})$$

$$= \begin{cases} 1, & \omega = 0 \\ 0, & \omega = \pi \end{cases}$$

So this is a low pass filter. Now, when $\omega = 0$ then $z = 1$ and when $\omega = \pi$, $z = -1$.

Now the frequency response is,

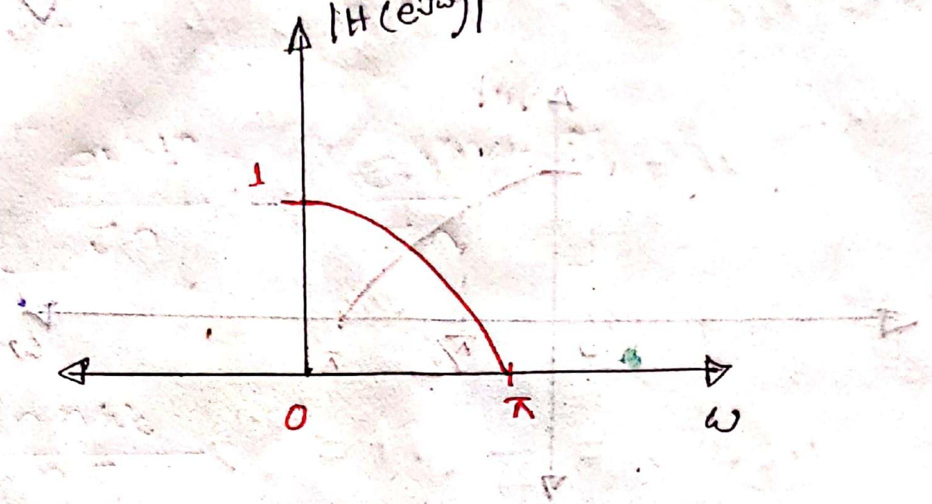
$$H(e^{j\omega}) = \frac{1}{2}(1+e^{-j\omega})$$

$$= e^{-j\omega/2}\left\{\frac{\frac{1}{e^{-j\omega/2}} + e^{-j\omega/2}}{2}\right\}$$

$$= e^{-j\omega/2}\left\{\frac{e^{j\omega/2} + e^{-j\omega/2}}{2}\right\}$$

$$= e^{-j\omega/2}\cos\omega/2$$

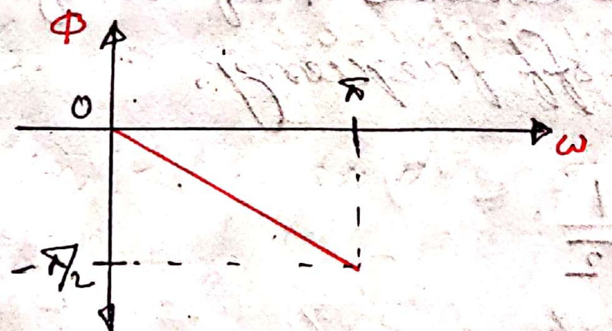
so the graph is,



Now the angle,

$$\phi = -\frac{\omega}{2} \quad [\because e^{-j\omega/2} = e^{-j\phi}]$$

$$= \begin{cases} 0, & \omega = 0 \\ -\pi/2, & \omega = \pi \end{cases}$$

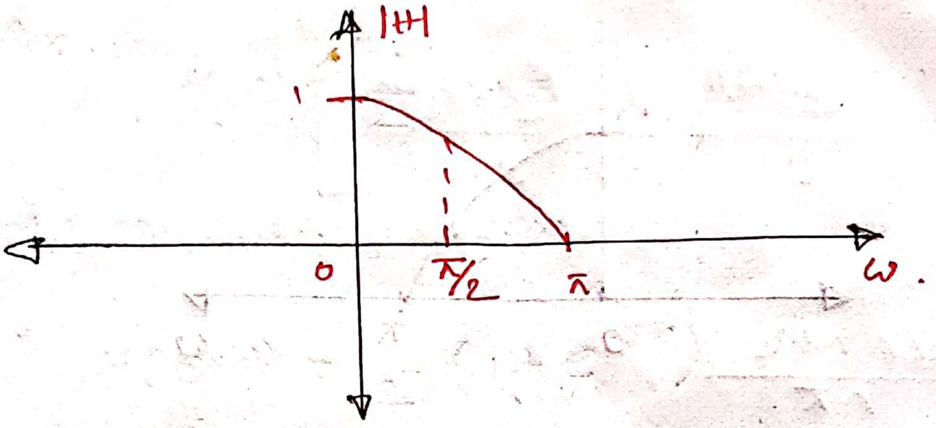


Now the cut off frequency.

$\cos \omega/2 = \frac{1}{\sqrt{2}}$ [Because we want to make it at 0.707]

$\therefore \omega_c = \pi/2$

So the characteristics figure showing cut-off.



So this is a Bad LPF because it has large bandwidth and large stop band.

If we want to make it better we need to cascade these filters.

So, suppose we cascade M ~~nof~~ no. of. $H(z)$ then we have, the cut-off frequency,

$$\left(\cos\frac{\omega_c}{2}\right)^M = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\omega_c/2 = 2^{-M/2}$$

$$\therefore \omega_c \approxeq 0.3 \text{ for } M = 3$$

So the cut of frequency comes closer.

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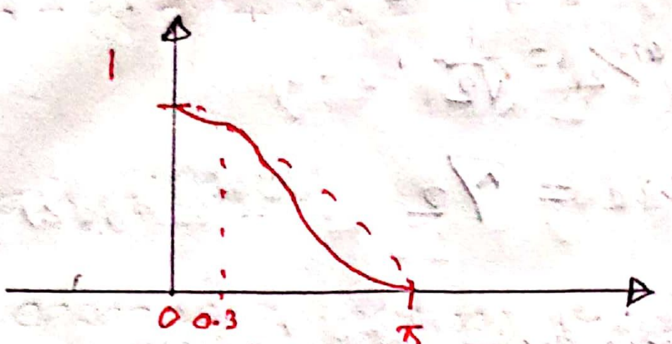
then the characteristics figure:



→ If a function is, $H(z) = \frac{1}{2}(1 - z^{-1})$.

<u>Ans:</u> Here, $H(z) = \frac{1}{2}(1 - z^{-1})$

when, $z = 1$ then, $H(z) = 0$

$z = -1$ " , $H(z) = 1$

So this is a highpass filter.

Now, the frequency response is,

$$H(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega})$$

$$= j\, e^{-j\omega/2} \sin \omega/2$$

And the angle, $\phi = j - \omega/2 = \frac{\pi}{2} - \frac{\omega}{2}$

$$= \begin{cases} \pi/2, & \omega = 0 \\ 0, & \omega = \pi \end{cases}$$

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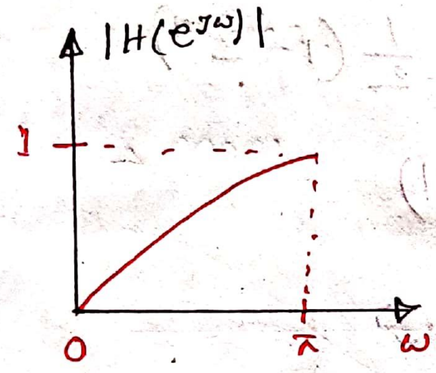
And the cut-off, ~~cos~~

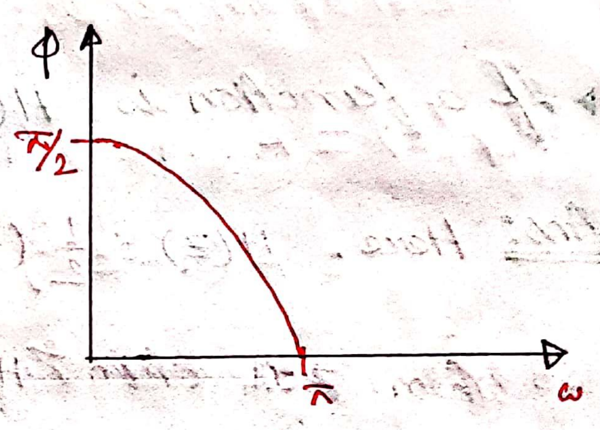
$$\sin \omega_c/2 = \frac{1}{\sqrt{2}},$$

$$\therefore \omega_c = \pi/2$$

The characteristics curves are,



Magnitude



Phase

From figure we see that $H_{HP}(z)$ is just flip over of $H_{LP}(z)$.

$$\therefore H_{HP}(z) = H_{LP}(-z) \text{ and vice versa.}$$

If, $z = e^{-j\omega}$ then.

$$-z = e^{-j(\omega-\pi)}$$

Now, if, $H_{LP}(z) = a_0 + a_1 z^{-1} + \text{---} + a_N z^{-N}$

then,

$H_{HP}(-z) = a_0 - a_1 z^{-1} + a_2 z^{-2} + \cdots\cdots$

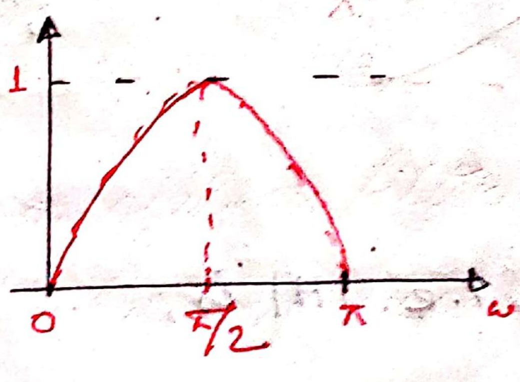
→ Band pass filter:

If we cascade a lowpass with highpass then we get Bandpass filter:

$H_{BP}(z) = A(1+z^{-1})(1-z^{-1})$

$= A(1-z^{-2})$

so the characteristic curve,



$H(e^{j\omega}) = 2Aje^{-j\omega}\sin\omega$

so the bandwidth $= \pi/2$

Here, $A = \frac{1}{2}$ [cause, the max$^m = 1 = 2A$]

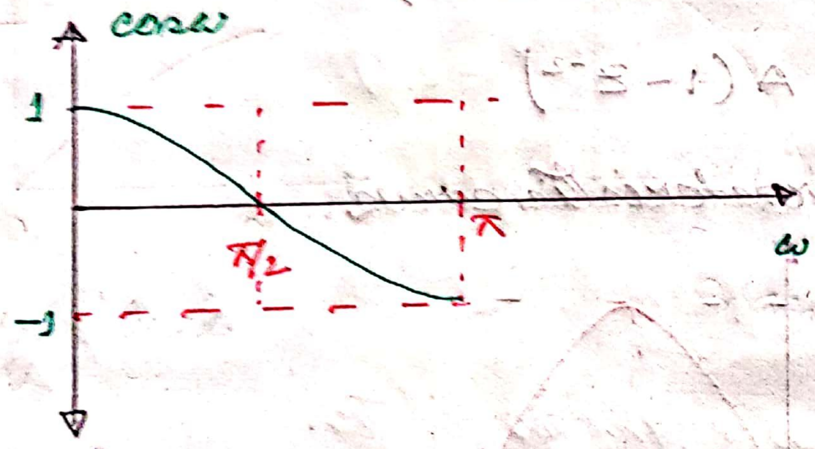
→ Bandstop filter:

$$H_{BS}(z) = \frac{1}{2}(1+z^{-2})$$

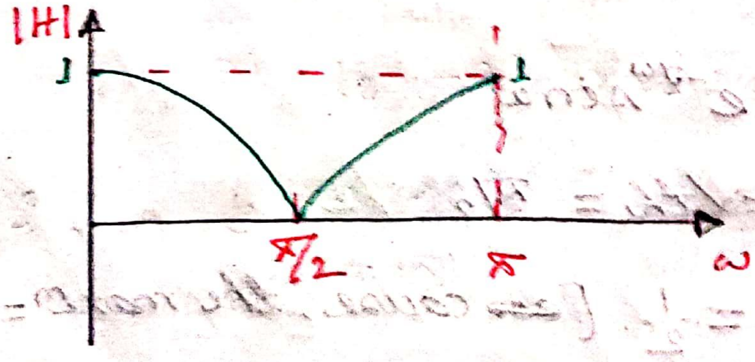
so, $H_{BS}(e^{j\omega}) = \frac{1}{2} e^{-j\omega} 2\cos\omega$

$$= e^{-j\omega}\cos\omega$$

so the curve.

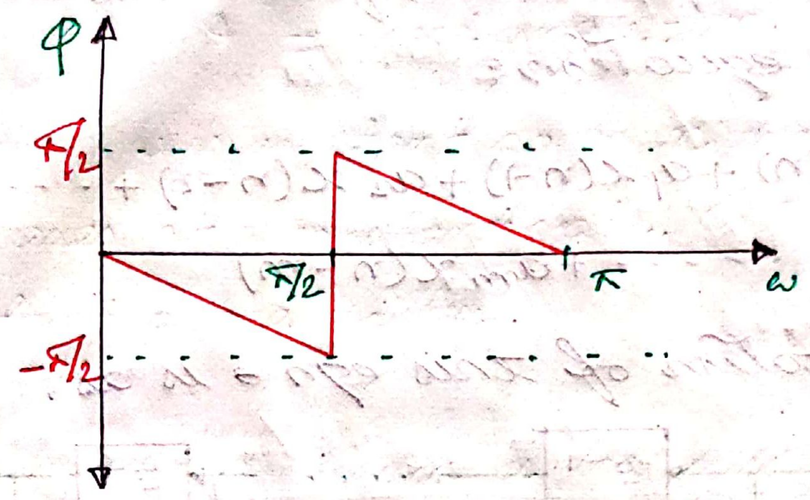


so the magnitude i.e. |H| is,



Now the angle.

$$\phi = -\omega + \angle \cos\omega$$



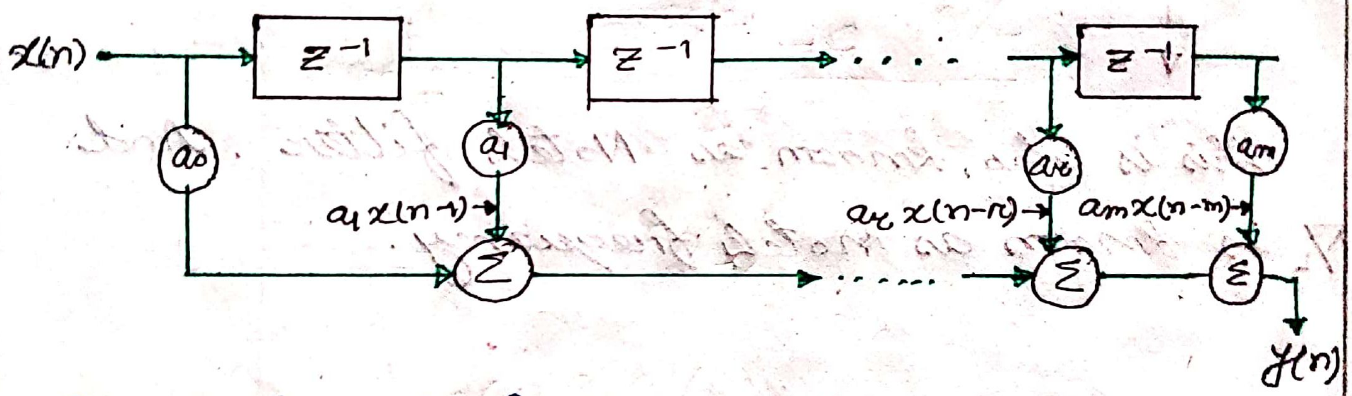
This is also known as Notch filter. And $\pi/2$ is known as notch frequency.

FIR Filter Design

Finite impulse response (FIR) filters are characterized by the equation:

$$y(n) = a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) + \cdots$$
$$\cdots + a_m x(n-m)$$

The implementation of this eqn is as,



Basic principle of FIR filter design:

FIR filters are designed by assuming that the magnitude of transfer function is unity. i.e.

$$|H(\omega)| = 1$$

$$\therefore |Y(\omega)| = |X(\omega)|.$$

i.e. An FIR filter does not introduces any losses. It may be noted that all types of FIR filters are designed by choosing a suitable and finite value of $|H(\omega)|$ which is less than 1.

$\therefore H(\omega) = M e^{-j\omega n} = M e^{-j\theta}$

where, $M(=1)$ = Magnitude.

θ = Phase angle = ωn

since ω is constant so as n increases $H(\omega)$ is θ decreases at a constant rate. so FIR filters called constant phase filter.

→ **Problem:** Design a low pass FIR filters for the following specifications:

Cut-off frequency = 500 Hz.
sampling " = 2 KHz
Order of the filter, $N = 10$.
Length, $L = 11$ [$\because N+1$].

solution:

step 1: Normalization of cut-off frequency:

$$\omega_c = 2\pi \frac{f_c}{f_s} = 2\pi \frac{500}{2000} = \pi/2$$

step 2: Fixing $H(\omega)$ to be used:

$$H(\omega) = \begin{cases} 1, & -\pi/2 \le \omega \le \pi/2, \\ 0, & \text{elsewhere}. \end{cases}$$

Step 3: Determine the impulse response:

From IDFT we get,

$$h(n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} H(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega$$

$$= \frac{1}{n\pi} \cdot \frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j}$$

$$= \frac{1}{n\pi} \sin(n\pi/2)$$

$$= \frac{1}{2} \operatorname{sinc}(n\pi/2) \text{ ------- (1)}$$

Step 4: Determine the coefficients:

For $n=0$, eqn (1) gives, $h(0) = 0.5$.

$n=1$ " " " , $h(1) = 0.3183$

$n=2$ " " " , $h(2) = 0$

$n=3$ " " " , $h(3) = -0.1061$

$n=4$ " " " , $h(4) = 0$

$n=5$ " " " , $h(5) = 0.0637$.

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We stop our computation at this point here because the length, $L = N+1 = 11$ and $h(n) = h(-n)$

Thus the,

$$h(n) = \left(\frac{1}{5\pi}, 0, -\frac{1}{3\pi}, 0, \frac{1}{\pi}, \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}}, \frac{1}{\pi}, 0, -\frac{1}{3\pi}, 0, \frac{1}{5\pi}\right)$$

Step 5: Determining the transfer function back from the impulse response sequence:

Here we write,

$$H(z) = \frac{1}{5\pi} z^5 + \left(-\frac{1}{3\pi}\right) z^3 + \frac{1}{\pi} z + \frac{1}{2} + \frac{1}{\pi} z^{-1} + \left(-\frac{1}{3\pi}\right) z^{-3} + \frac{1}{5\pi} z^{-5} \quad \text{------ (ii)}$$

Here the -ve values of frequencies (+ve power of z terms) indicating that we are going to realize a noncausal filter.

To make the filter causal, (ii) $\times z^{-5}$,

$$\therefore H(z) = \frac{1}{5\pi} - \frac{1}{3\pi} z^{-2} + \frac{1}{\pi} z^{-4} + \frac{1}{2} z^{-5} + \frac{1}{\pi} z^{-6} + \left(-\frac{1}{3\pi}\right) z^{-8} + \frac{1}{5\pi} z^{-10} \quad \text{------ (iii)}$$

Step6: Implementation of the filter:

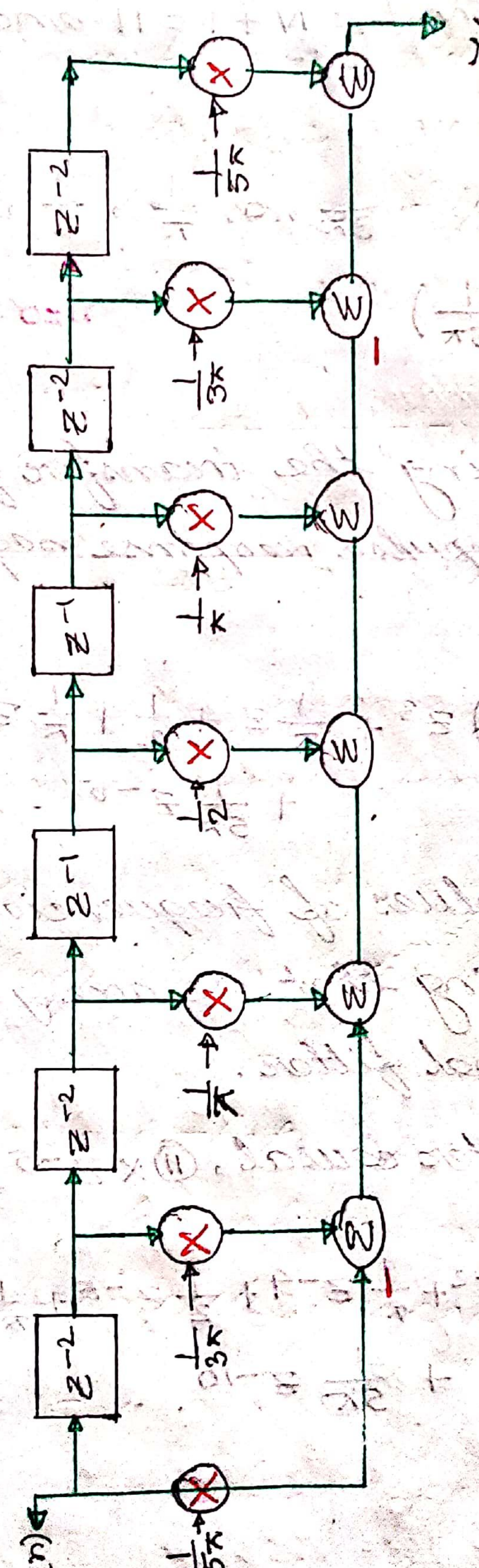


Fig: Implementation of eqn (iii)

Step 7: Adding more filter coefficients:

The accuracy of FIR filter can be increased by increasing the number of filter co-efficients.

Assuming that we want to find filter coefficients up to n, where n is odd, so, $h(\pm n) = \frac{1}{n\pi}$

And for even number, $h(\pm 4) = h(\pm 6) = \cdots = 0$.

Thus we find that there will be infinite number of values of filter coefficients as $n \to \pm\infty$.

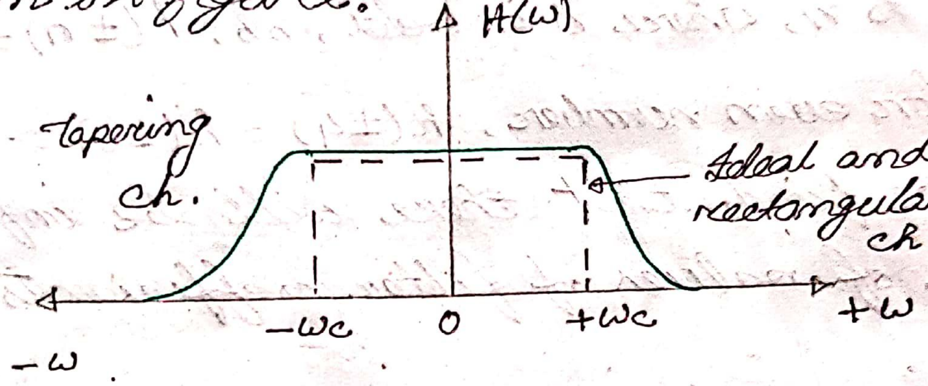
We can also see that the larger the n, the smaller the value of the filter coefficients.

→ Design of FIR filters, using window functions:

Finite register length of computers necessitates abrupt termination of FIR filter coefficients at some finite value of n. This gives rise to the Gibb's phenomenon which is dangerous in many situations, as it gives rise to sharp transients.

In many cases, these transients may destroy the hardware used for the construction of the filter.

To prevent the occurance of the Gibb's phenomenon, we must avoid abrupt termination of filter coefficients. The way to avoid abrupt termination is to use a tapering characteristic as shown in figure:



To get gradually tapering transfer cha., we make use of window function.

"Window functions are mathmatical functions that are designed to have tapering characteristics."

Thus, we see that, to prevent the occurance of the ~~gi~~ Gibb's phenomenon, we must use a modified impulse response given by,

$$h'(n) = h(n) \times \omega(n) \quad \text{---} \textcircled{1}$$

$h(n)$ = Impulse response for $H(\omega)$.

$\omega(n)$ = Window function.

The rectangular window:

The rectangular window defined as,

$$w_{REC}(n) = \begin{cases} 1, & -M \le n \le M. \\ 0, & \text{elsewhere.} \end{cases}$$

$$\therefore h'(n) = h(n) \times w_{REC}(n)$$

$$= \begin{cases} h(n), & -M \le n \le M \\ 0, & \text{elsewhere.} \end{cases}$$

This equation says, that the modified impulse response $h'(n)$ is the same as $h(n)$. Hence this window is never used for the practical design of FIR filters.

The Hann (Hanning) window:

The Hann window defined as,

$$w_{HAN}(n) = 0.5 + 0.5 \cdot \cos\left(\frac{2\pi n}{N}\right) ; -\frac{N}{2} \le n \le N/2$$

$$= 0.5 + 0.5 \cos\left(\frac{2\pi n}{M}\right) ; -M \le n \le M \left[\because M = \frac{N}{2}\right]$$

By changing the limits the eqn also ~~a~~ written

as: $w_{HAN}(n) = 0.5 - 0.5 \cos\left(\frac{n\pi}{N}\right)$; $0 \le n \le N$.

For, $-M \le n \le M$, $w_{HAN}(n) = w_{HAN}(-n)$

Now, $N = 10$ and $L = 11$ so,

so, for, $n=0$, $W_{HAN}(0)=1$

$n=1$, $W_{HAN}(1)=0.9045$

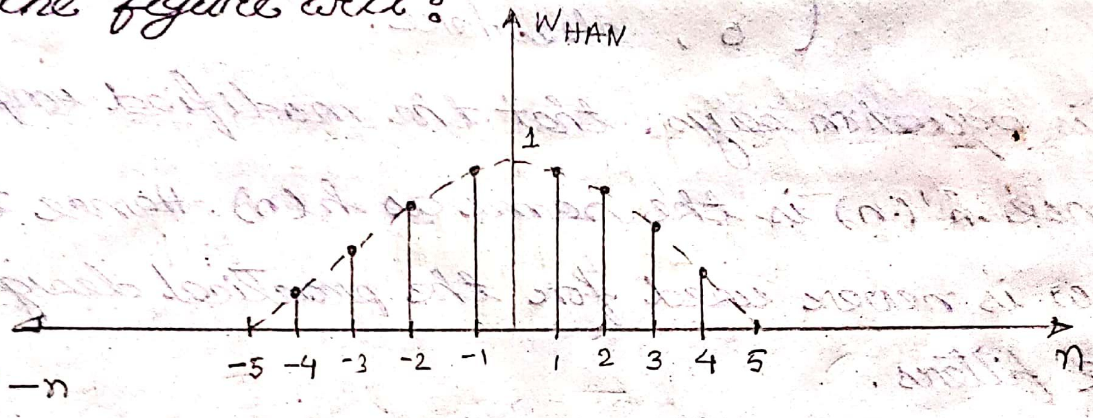
$n=2$, $W_{HAN}(2)=0.6545$

$n=3$, $W_{HAN}(3)=0.3455$

$n=4$, $W_{HAN}(4)=0.0955$

$n=5$, $W_{HAN}(5)=0$.

so the figure will:



This wave form is known as raised-cosine wave form, as it looks like the +ve half cycle of a cosine wave raised in the middle portion and falling off sharply.

The Hamming window:

The Hamming window is defined as,

$$w_{HAM}(n) = 0.54+0.46\cos(2\pi n/N) \begin{cases} -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2} & \text{(odd)} \\ -N/2 \le n \le N/2 & \text{(even)} \end{cases}$$

$= 0$, elsewhere.

The Blackman window:

It is defined as,

$$w_{BLA}(n) = 0.42 - 0.5\cos\left(\frac{n\pi}{M}\right) + 0.08\cos\left(\frac{2\pi n}{M}\right); \quad -M \le n \le M.$$

This is the modification of Hann window.

The ~~Bart~~ Barlett window:

It is defined as,

$$w_{BAR}(n) = \begin{cases} n/M & ; 0 \le n \le M. \\ 2 - n/M & ; M \le n \le \cancel{4}N \end{cases}$$

→ Problem: Use (a) Hamming window; (b) Hann window to the solⁿ given in previous example to determine modified impulse response.

Solution: We know from previous example,

$$h(n) = \frac{\sin(n\pi/2)}{2(n\pi/2)}$$

(a) Hamming window:

We know, $w_{HAM}(n) = 0.54 + 0.46\cos\left(\frac{n\pi}{M}\right); \quad -M \le n \le M.$

$$\therefore h'(n) = h(n) \times w_{HAM}(n)$$

$$= \frac{\sin(n\pi/2)}{2(n\pi/2)}\left[0.54 + 0.46\cos\left(\frac{n\pi}{M}\right)\right]$$

$\therefore N=10$ so, $M=5$.

so for, $n=0,1,2,3,4,5$.

$h'(0) = 0.5$.

$h'(1) = 0.3111$

$h'(2) = 0$

$h'(3) = -0.0622$.

$h'(4) = 0.0051$.

$$\therefore H(z) = 0.0051 - 0.0178 z^{-2} + 0.3111 z^{-4} + 0.5 z^{-5}$$
$$+ 0.3111 z^{-6} - 0.0178 z^{-8} + 0.0051 z^{-10}$$

This can be implemented by using an appropriate FIR structure.

(b) Hann window:

Here we have,

$$h'(n) = \frac{\sin(n\pi/2)}{2(n\pi/2)} \left[0.5 + 0.5\cos\left(\frac{n\pi}{5}\right)\right]$$

so, $h'(0) = 0.5$; $h'(1) = 0.2879$; $h'(2) = 0$

$h'(3) = -0.0363$; $h'(4) = h'(5) = 0$

The causal filter is given by,

$$H(z) = -0.0367 z^{-2} + 0.2879 z^{-4} + 0.5 z^{-5}$$
$$+ 0.2870 z^{-6} - 0.0367 z^{-8}.$$