$$\phi_{FM}(t) = A\cos\left(w_c t + k_f\int_{-\alpha}^{t} m(t)dt\right)$$

$$\phi_{PM}(t) = A\cos\left(w_c t + k_p m(t)\right).$$

see, ch-5, (P-203-205) (B.P. Lathi).

Bandwidth Analysis of Angle Modulated Waves: (B.P. Lathi P-209)

$$\phi_{FM} = A\cos\left(w_c t + k_f\int_{-\alpha}^{t} m(\alpha)d\alpha\right) \quad \cdots (i)$$

$$= A\cos\left(w_c t + k_f a(t)\right) \quad \left[a(t) = \int_{-\alpha}^{t} m(\alpha)d\alpha\right] \quad \cdots (ii).$$

Expanding (ii) in polar form,

$$\hat{\phi}_{FM} = A e^{j(w_c t + k_f a(t))} \longrightarrow \phi_{FM}(t) = Re\left[\hat{\phi}_{FM}(t)\right]$$

$$= A e^{jk_f a(t)} \cdot e^{jw_c t} \quad \cdots (iii)$$

[$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$, sine term is avoided as it is imaginary, that's why ϕ_{FM} is written as $\hat{\phi}_{FM}$].

we need,

$$\therefore \phi_{FM} = Re(\hat{\phi}_{FM})$$

Expanding equ^n (iii) we get,

$$\left[e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\right].$$

$$\hat{\phi}_{FM} = A\left[1 + jk_f a(t) + \frac{(jk_f a(t))^2}{2!} + \frac{(jk_f a(t))^3}{3!} + \cdots + \frac{(jk_f a(t))^n}{n!}\right](\cos w_c t + j\sin w_c t)$$

$$= A\left[1 + jk_f a(t) - \frac{k_f^2 a^2(t)}{2!} - \frac{jk_f^3 a^3(t)}{3!} + \cdots\cdots + \frac{j^n k_f^n a^n(t)}{n!}\right](\cos w_c t + j\sin w_c t)$$

$$= A\left[\cos w_c t - k_f a(t)\sin w_c t - \frac{k_f^2 a^2(t)}{2!}\cos w_c t \cdots\cdots\right]$$ (Omitting all imaginary terms).

Finally,

$$\phi_{FM} = A\left[\boxed{\cos\omega_c t - k_f a(t)\sin\omega_c t} - \frac{k_f^2 a^2(t)}{2!}\cos\omega_c t + \cdots\right]$$

(we don't need it)

This term looks like a traditional DSB-SC AM wave

If the (~~Bandwidth~~ of ~~red box~~ $a(t)$ or freq) is B then the total BW will be 2B.

So, the red box is called $\boxed{\text{narrowband FM (NBFM)}}$.

The BW of others will be high.

If let, $a(t) = \cos\theta \Rightarrow a^2(t) = \cos^2\theta = \frac{(1+\cos 2\theta)}{2}$. [Angle increased two times so BW will also increase].

Similarly $a^3(t) = \cos^3\theta = \frac{\cos 3A + 3\cos A}{4}$ [BW tripled]

Thus the others is called wide-band FM (WBFM).

Q Does that mean that the BW of FM is infinity?

No, as $\frac{k_f^n a^n(t)}{n!} \approx 0$ very small compared to $n!$

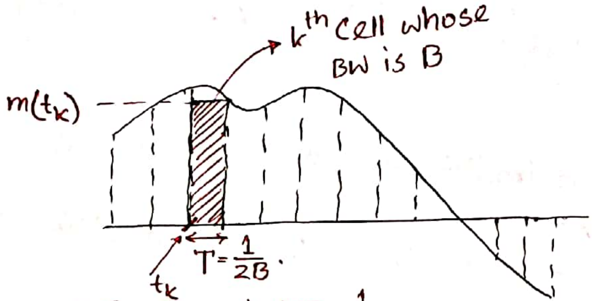
$$\frac{2^{100}}{100!} = 1.35 \times 10^{-128} \quad \text{(Matlab)}$$

It show that the last term will tend to zero and the BW will become finite eventually.

(*) PM equ^n also similar.

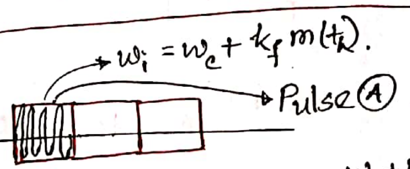
$$\phi_{PM} = \cos(\omega_c t + k_p m(t)).$$

* Divide into ~~k no. of~~ cells (Pulses).



* Time period $T = \frac{1}{2B}$ (Nyquist Theorem)

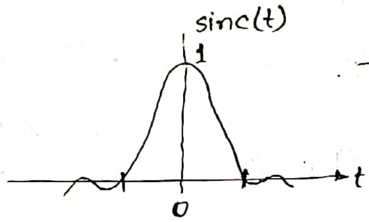
*

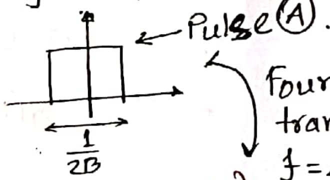


Ins. freq. at t_k is $\omega_i = \omega_c + k_f m(t)$.

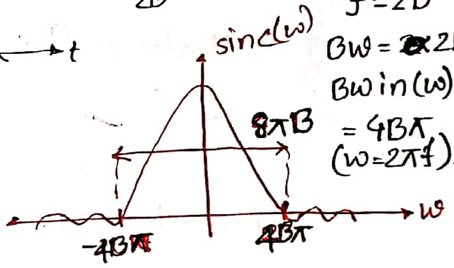
* The frequency domain representation of a rectangular pulse is a sinc function.

$$\text{sinc}(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin t}{t} & t \neq 0 \end{cases}$$



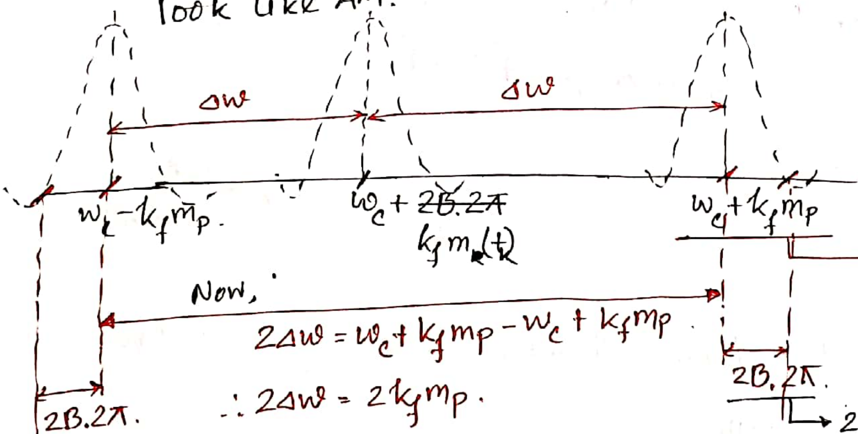


$f = 2B$



BW = ~~2x~~2B
BW in (ω) = 4Bπ (ω=2πf).

Thus, the pulse (A) is Amp. modulated with freq. $\omega_i = \omega_c + k_f m(t_k)$. The freq. domain will look like AM.



$\pm m_p$ = maximum and minimum peak to peak value at $m(t_k)$/Deviation.

see page-16 equⁿ (V)
$f_{imax} = A_m k_f + f_c$.

Now,

$2\Delta\omega = \omega_c + k_f m_p - \omega_c + k_f m_p$

$\therefore 2\Delta\omega = 2k_f m_p$.

$\therefore \Delta\omega = k_f m_p$.

2B in normal freq.
To transform into ang. freq (ω) multiply 2π.

Total BW_{FM} (in ang. freq) $= 2\Delta\omega + 4B\pi + 4B\pi$
$= 2\Delta\omega + 8B\pi = 2k_f m_p + 8B\pi$

In normal freq total BW_{FM} $= \frac{BW_{FM(ang.)}}{2\pi}$

$= \frac{2k_f m_p + 8B\pi}{2\pi}$

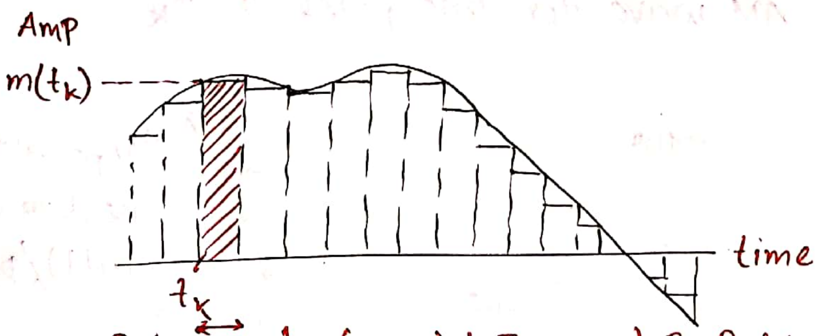
$= 2\left[\frac{k_f m_p}{2\pi} + \frac{\cancel{4}^{2} B\cancel{\pi}}{\cancel{2\pi}}\right]$

$= 2\left[\frac{k_f m_p}{2\pi} + 2B\right]$

$= 2\left[\frac{\Delta\omega}{2\pi} + 2B\right]$ $[\because \Delta\omega = k_f m_p]$

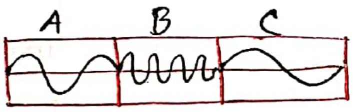
$= 2\left[\Delta f + 2B\right]$ $[\because \omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi}]$

Consider the Shape of the message Signal m(t)

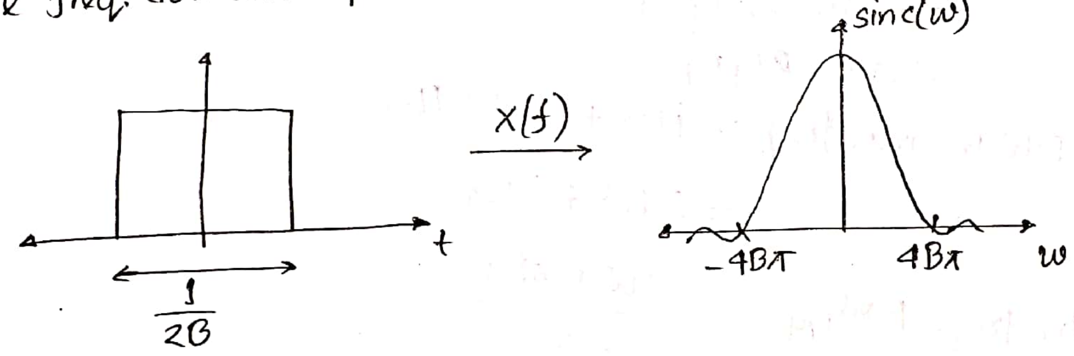


Time Period $T = \frac{1}{2B}$ (Nyquist Theorem). B = BW.

Sample into Pulses/cells. Thus each cell has a certain freq. or instantaneous freq. $w_i = w_c + k_f m(t_k)$



The freq. domain representation of a rectangular pulse is Sinc fun^c.

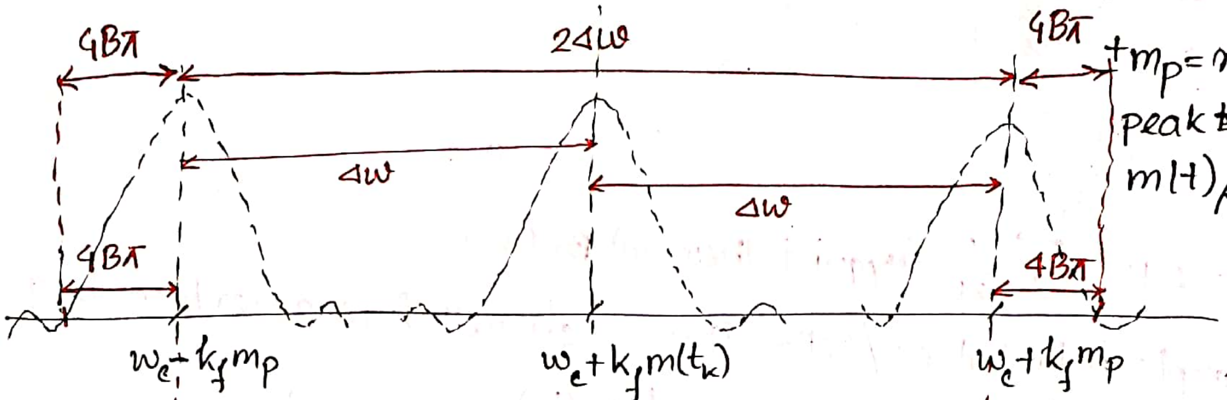


Here, time period $T = \frac{1}{2B}$.

$\therefore$ frequency $f = \frac{1}{T} = 2B$.

$\therefore$ In angular freq. $w = 2\pi f$
$= 4\pi B$.

The pulses are multiplied with sinosoids. Thus the pulse will behave like an AM wave for this pulse at t_k.



m_p = max^m & min^m peak to value of $m(t)$/Deviation

↳ we know

$f_{i\,max} = A_m k_f + f_c$.

$f_{i\,min} = f_c - A_m k_f$.

$2\Delta\omega$ = max + (−min)

$\Rightarrow 2\Delta\omega = \omega_c + k_f m_p - \omega_c + k_f m_p$

$\Rightarrow 2\Delta\omega = 2k_f m_p$.

$\therefore \Delta\omega = k_f m_p$.

Total BW in ang. freq $= 4B\pi + 2\Delta\omega + 4B\pi$.

$= 2\Delta\omega + 8B\pi$

$\therefore$ In freq. $BW_{FM} = \dfrac{2\Delta\omega + 8B\pi}{2\pi}$

$= \dfrac{2k_f m_p + 8B\pi}{2\pi}$

$= 2\left[\dfrac{k_f m_p}{2\pi} + \dfrac{\overset{2}{\cancel{4}}B\pi}{\cancel{2\pi}}\right]$.

$= 2\left[\dfrac{\Delta\omega}{2\pi} + 2B\right]$.

$= 2\left[\Delta f + 2B\right]$. $[\because \omega = 2\pi f]$.

Now we get;

$$BW_{FM} = 2[\Delta f + 2B]$$

This is the ideal case, but practically the required BW is much smaller.

So, Carson came up with a formula,

$$BW_{FM} = 2[\Delta f + B]$$

Also can be written as $\longrightarrow = 2B\left[\frac{\Delta f}{B} + 1\right]$.

$= 2B(\beta + 1)$ $\left[\text{recall } \beta = \frac{\Delta f}{f_m}\right]$.

β = modulation index $= \frac{\Delta f}{B} = \frac{\Delta f}{f_m}$, Δf = frequency deviation

f_m/B = Message frequency/BW

$\Delta f = \frac{k_f m_p}{2\pi}$, k_f = frequency Sensitivity (Hz/V)

m_p = max. peak voltage (V).

(Read B.P. Lathi P-213)

—x—

Spectral Analysis of Tone frequency Modulation (B.P Lathi P-214)
(Verification of Carson's Formula).

Let, message $m(t) = \alpha \cos w_m t$ [sinusoids are called tone modulated signals]

Now,

$$a(t) = \int_0^t m(t) = \frac{\alpha}{w_m} \cdot \sin w_m t \quad [\text{assume } a(-\infty) = 0]$$

We know,

$$\phi_{FM}(t) = A\cos\left(w_c t + k_f \int_{-\infty}^{t} m(t)\,dt\right) \cdots (i)$$

From (i) we get, the estimated value of $\phi_{FM}(t)$

$$\hat{\phi}_{FM}(t) = A e^{j(w_c t + k_f a(t))}$$

$$= A e^{j(w_c t + k_f \int_{-\infty}^{t} \alpha \cos w_m t\, dt)}$$

$$= A e^{j(w_c t + k_f \cdot \frac{\alpha}{w_m} \cdot \sin w_m t)}$$

The deviation from carrier frequency $\Delta w = k_f m_p = k_f \alpha$ [In this case].

$$\Rightarrow \Delta f = \frac{\alpha k_f}{2\pi} \cdots (ii)$$

Now, modulation index/deviation ratio $\beta = \frac{\Delta f}{B}$ [B = Bandwidth of message signal or ~~so~~ also written as message frequency f_m] $\cdots$ (iii)

Equating (ii) and (iii) we get,

$$\beta = \frac{\Delta f}{B} = \frac{\alpha k_f}{2\pi B} = \frac{\alpha k_f}{w_m} \quad [\because 2\pi B = 2\pi f_m = w_m] \cdots (iv)$$

Scanned by CamScanner

Recall,

$$\hat{\phi}_{FM}(t) = A e^{j(\omega_c t + k_f a(t))}$$

$$= A e^{j(\omega_c t + k_f \cdot \frac{\alpha}{\omega_m} \cdot \sin\omega_m t)}$$

Putting the value of β from (iv) in here we get,

$$\hat{\phi}_{FM}(t) = A e^{j(\omega_c t + \beta \sin\omega_m t)} \quad \text{---} (v)$$

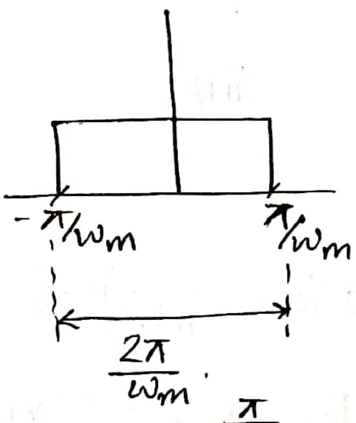
$$= A e^{j\omega_c t} \cdot e^{j\beta \sin\omega_m t}$$

Here $e^{j\beta\sin\omega_m t}$ looks like a periodic signal with period $2\pi/\omega_m$ and can be expanded by the expotential Fourier series.

$$e^{j\beta \sin\omega_m t} = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_m t} \quad \text{----} (vi)$$

where $D_n = \dfrac{1}{\frac{2\pi}{\omega_m}} \displaystyle\int_{\text{initial Value}}^{\text{Final Value}} e^{j\beta \sin\omega_m t} \cdot e^{-jn\omega_m t} \, dt$

Time period



If the bandwidth $\frac{1}{T} = \frac{2\pi}{\omega_m}$.

So, the initial value $= -\frac{\pi}{\omega_m}$.

& final value $= \frac{\pi}{\omega_m}$.

$$\therefore D_n = \frac{\omega_m}{2\pi} \int_{-\frac{\pi}{\omega_m}}^{\frac{\pi}{\omega_m}} e^{j\beta \sin\omega_m t} \cdot e^{-jn\omega_m t} \, dt \quad \text{------} (vii)$$

Scanned by CamScanner

The equ^n (vii) was,

$$D_n = \frac{\omega_m}{2\pi}\int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta \sin \omega_m t} \cdot e^{-jn\omega_m t}\, dt$$

Let, $\omega_m t = x$

$\Rightarrow \omega_m dt = dx$

$\Rightarrow dt = \frac{1}{\omega_m} dx.$

If, (for limits) $t \to -\frac{\pi}{\omega_m}$,

then, $\boxed{x \to -\pi}$

and $t \to \frac{\pi}{\omega_m}$.

then, $\boxed{x \to \pi}$

$dt = \frac{1}{\omega_m} dx.$

$\Rightarrow \frac{\pi}{\omega_m} = \frac{1}{\omega_m} \cdot x$

$\therefore x = \pi.$

Putting limits, x, dx in equ^n (vi) we get,

$$D_n = \frac{\omega_m}{2\pi}\int_{-\pi}^{\pi} e^{j\beta \sin x} \cdot e^{-jnx} \cdot \frac{dx}{\omega_m}.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\beta \sin x} \cdot e^{-jnx} \cdot dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} \cdot dx \quad \text{------ (viii)}$$

The equ^n (viii) looks like Bessel function $J_n(\beta)$ of first kind and n^{th} order. Putting the value of D_n in equ^n (vi) we get,

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t} \qquad [D_n = J_n(\beta)] \text{ --- (ix)}$$

we know from equ^n (v):

$$\hat{\phi}_{FM}(t) = A e^{j\omega_c t} . e^{j\beta \sin n\omega_m t}$$

Putting the value of (ix) in (v) we get,

$$\hat{\phi}_{FM}(t) = A . e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j\omega_m t n}.$$

$$= A \sum_{n=-\infty}^{\infty} J_n(\beta) . e^{j(\omega_c t + n\omega_m t)}$$

Finally, $\phi_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$ [Return to angular form from polar form Like equ^n (1)].

We can see that the lone modulated FM signal has a carrier component and infinite number of side-band frequencies. Like $\omega_c \pm \omega_m$, $\omega_c \pm 2\omega_m$, ------, $\omega_c \pm n\omega_m$ as $J_{-n}(\beta) = (-1)^n J_n(\beta)$.

To make the BW finite we will cancel where $J_n(\beta)$ is negligible.

Generally $J_n(\beta)$ is negligible for $n > \beta + 1$

∴ The number of significant side-band impulses $= \beta + 1$.

∴ The Bandwidth of FM is given by $B_{FM} = 2(\beta+1)f_m$

$$= 2(\beta f_m + f_m)$$

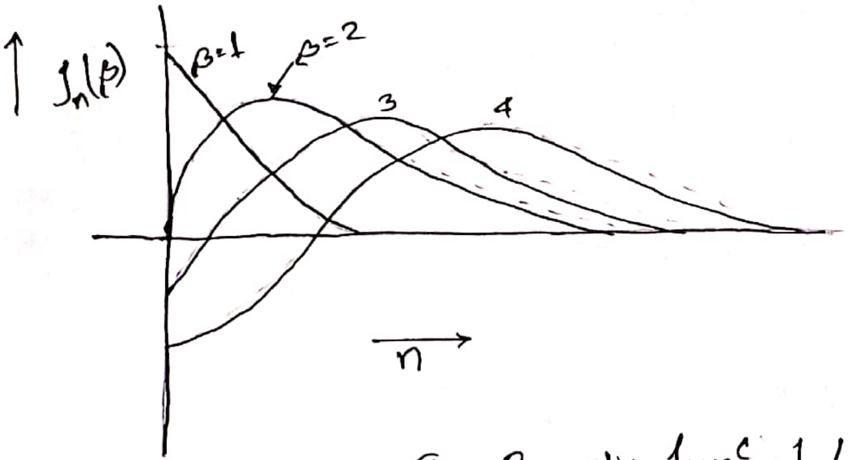
Here, $f_m = B =$ message freq./BW

$$= 2\left(\frac{\Delta f}{f_m} . f_m + f_m\right)$$

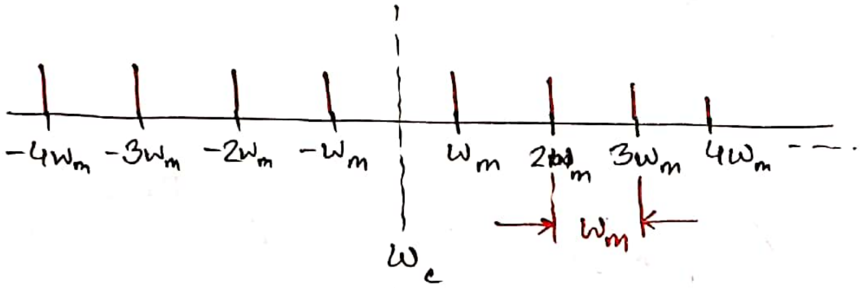
$$= 2(\Delta f + f_m)$$

$$= 2(\Delta f + B).$$

which verifies Carson's formula.



The Bessel's func $J_n(\beta)$.



Notes for $B_{FM} = 2(\beta+1) f_m$.

* when $\beta \ll 1$ $B_{FM} \cong 2 f_m$, only one significant side band called Narrowband FM (NBFB).

* But $B_{FM} = 2\beta(1+\frac{1}{\beta}) f_m$

for $\beta \gg$ high, $\frac{1}{\beta} \approx 0$, and $B_{FM} \cong 2\beta . f_m$

$= 2 . \frac{\Delta f}{f_m} . f_m$

$= 2\Delta f$. (called wide-band FM).