Problem: Design a 8-input Butterfly diagram to calculate FFT of $x = [1, 2, -1, 3, 1, 2, 1, -3]$. Compare your result from the diagram. With that obtained from Matlab command fft (x, 8). Explain if you find any amplitudes in two results.

Solution:

Given, $x = [\underset{0}{1}, \underset{1}{2}, \underset{2}{-1}, \underset{3}{3}, \underset{4}{1}, \underset{5}{2}, \underset{6}{1}, \underset{7}{-3}]$

At first we should to note that, the Butterfly diagram builds on the Danieison Lenezes Lemma (D-L Lemma). [When D-L Lemma is expanded this naturally happens that, the order of index of input values, $x(n)$ is "reverse binary."]

So, we need to reverse the index of the input binary.

Decimal input values	Binary form of input values	Reverse Binary	Decimal of Reverse Binary
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

So, we can write,

$x(0) = 1$

$x(4) = 1$

$x(2) = -1$

$x(6) = 1$

$x(1) = 2$

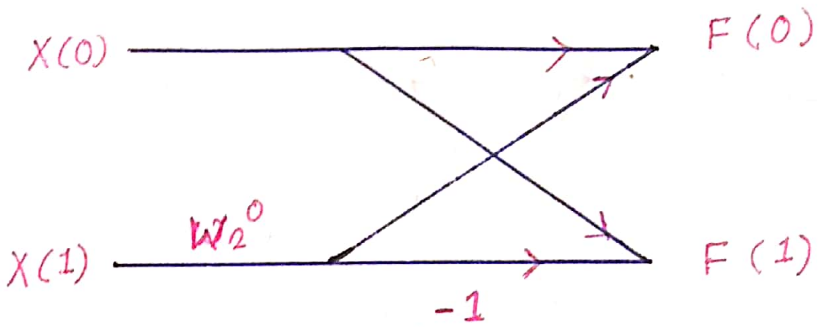
$x(5) = 2$

$x(3) = 3$

$x(7) = -3$

* The second thing to note is that the "twiddle factors" W, build up with each new expansion, so that we multiply more together. The Butterfly diagram deals with this by the adding of "Stages".

* The basic unit of butterfly diagram, consisting of just two inputs and two outputs.

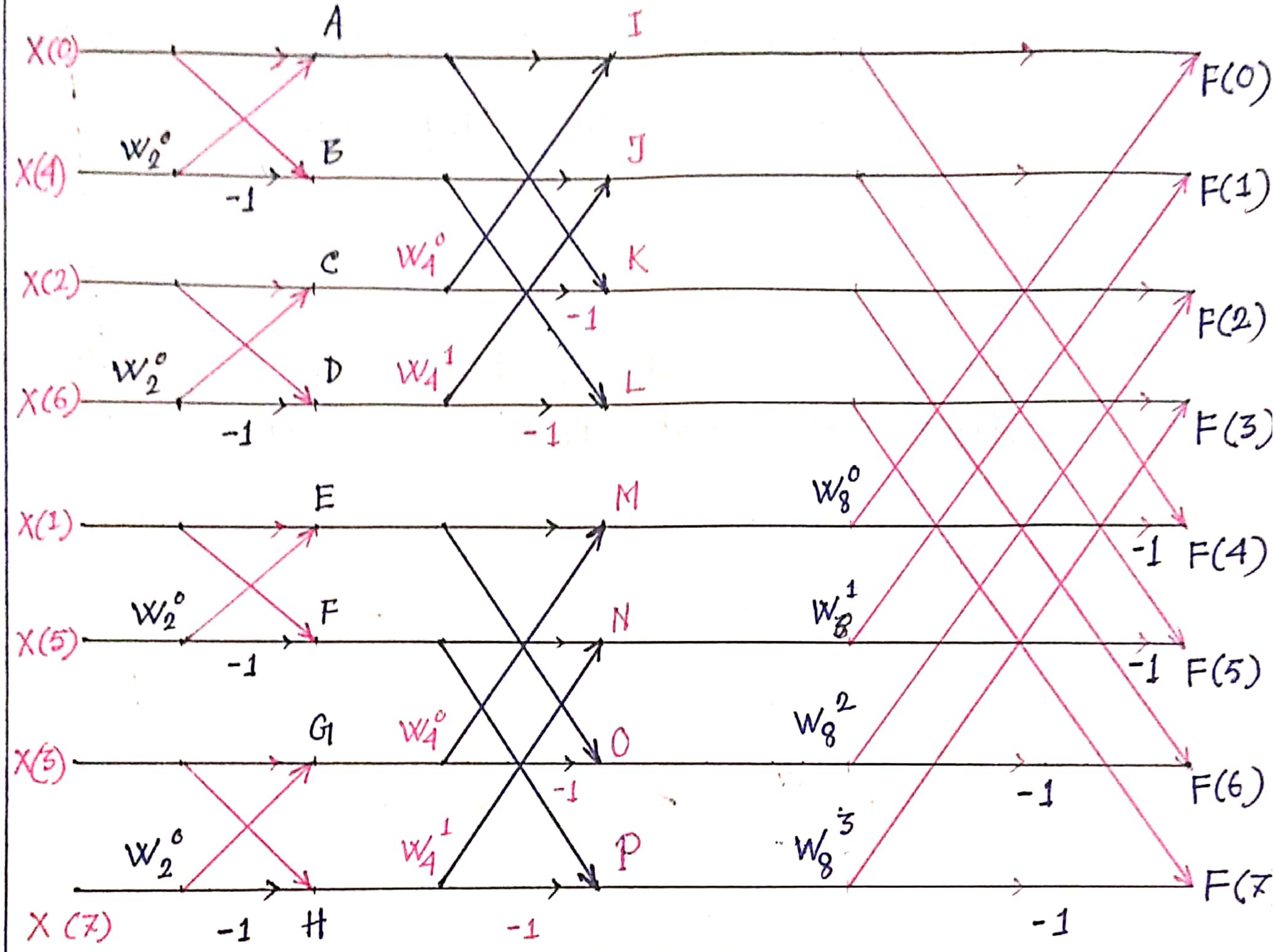


Above diagram has only stage 1 and W base is 2. If we add more stages then W base continues in binary fastion 2, 4, 8, 16 in order.

* Now, we can draw our desired 8 input Butterfly diagram.

2

An 8 input Butterfly



An 8 input Butterfly diagram has 12 - 2 input butterflies and thus $12 * 2 = 24$ multiplies.

Twiddle factors:

$$W_N^n = \cos\left(\frac{-2\pi n}{N}\right) + j\sin\left(\frac{-2\pi n}{N}\right)$$

$$W_2^0 = \cos\left(\frac{-2\pi(0)}{2}\right) + j\sin\left(\frac{-2\pi(0)}{2}\right)$$

$$= \cos(0) + j\sin(-0) = 1$$

$W_2^1 = -1$, $W_4^1 = -j$; $W_8^1 = \frac{1}{\sqrt{2}} + j\left(-\frac{1}{\sqrt{2}}\right)$

$W_2^2 = 1$; $W_4^2 = -1$; $W_8^2 = -j$

$W_2^3 = -1$; $W_4^3 = j$; $W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_2^4 = 1$; $W_4^4 = 1$; $W_8^4 = -1$

$W_2^5 = -1$; $W_4^5 = -j$; $W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$W_2^6 = 1$; $W_4^6 = -1$; $W_8^6 = j$

$W_2^7 = -1$; $W_4^7 = j$; $W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

* The equations derived from the 8-input Butterfly diagram are given below:

Stage 1:

$$A = X(0) + W_2^0 X(4)$$

$$B = X(0) - W_2^0 X(4)$$

$$C = X(2) + W_2^0 X(6)$$

$$D = X(2) - W_2^0 X(6)$$

$$E = X(1) + W_2^0 X(5)$$

$$F = X(1) - W_2^0 X(5)$$

$$G = X(3) + W_2^0 X(7)$$

$$H = X(3) - W_2^0 X(7)$$

Stage 2:

$$I = A + W_4^0 C$$

$$J = B + W_4^1 D$$

$$K = A - W_4^0 C$$

$$L = B - W_4^1 D$$

$$M = E + W_4^0 G$$

$$N = F + W_4^1 H$$

$$O = E - W_4^0 G$$

$$P = F - W_4^1 H$$

Stage 3 :

$$F(0) = I + W_8^0 M$$

$$F(1) = J + W_8^1 N$$

$$F(2) = K + W_8^2 O$$

$$F(3) = L + W_8^3 P$$

$$F(4) = I - W_8^0 M$$

$$F(5) = J - W_8^1 N$$

$$F(6) = K - W_8^2 O$$

$$F(7) = L - W_8^3 P$$

Now, substituting back in for I, J, K, L, M, N, O, P, A, B, C, D, E, F, G & H.

$$\therefore \quad F(0) = x(0) + W_2^0 x(4) + W_4^0 x(2) + W_4^0 W_2^0 x(6) + W_8^0 x(1) + W_8^0 W_2^0 x(5) + W_8^0 W_4^0 x(3) + W_8^0 W_2^0 W_4^0 x(7)$$

$$= 1 + (1)(1) + (1)(-1) + (1)(1)(1) + (1)(2) + (1)(1)(2) + (1)(1)(3) + (1)(1)(1)(-3)$$

$$= 1 + \cancel{1} - \cancel{1} + 1 + 2 + 2 + \cancel{3} - \cancel{3}$$

$$= 6.0000$$

$$F(1) = x(0) - W_2^0 x(4) + W_4^1 x(2) - W_4^1 W_2^0 x(6) + W_8^1 x(1) - W_8^1 W_2^0 x(5) + W_8^1 W_4^1 x(3) - W_2^0 W_8^1 W_4^1 x(7)$$

$$= 1 - 1 + j + j + \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(2) - (1)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(2) + \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-j)(3) - (1)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-j)(-3)$$

$$= 2j + \cancel{\sqrt{2}} - \cancel{j\sqrt{2}} - \cancel{\sqrt{2}} + \cancel{j\sqrt{2}} - \frac{3j}{\sqrt{2}} - \frac{3}{\sqrt{2}} - \frac{3j}{\sqrt{2}} - \frac{3}{\sqrt{2}}$$

$$= 2j - 3j\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - 3\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2j - (3\sqrt{2})j - 3\sqrt{2}$$

$$= -4.2426 - j\,2.2426$$

$$F(2) = x(0) + W_2^0 x(4) - W_4^0 x(2) - W_4^0 W_2^0 x(6) + W_8^2 x(1) + W_8^2 W_2^0 x(5) - W_8^2 W_4^0 x(3) - W_8^2 W_4^0 W_2^0 x(7)$$

$$= (1) + (1)(1) - (1)(-1) - (1)(1)(1) + (-j)(2) + (-j)(1)(2) - (-j)(1)(3) - (-j)(1)(1)(-3)$$

$$= 1 + 1 + \cancel{1} - \cancel{1} - 2j - 2j + \cancel{3j} - \cancel{3j}$$

$$= 2.0000 - j\,4.0000$$

$$F(3) = x(0) - W_2^0 x(4) - W_4^1 x(2) + W_4^1 W_2^0 x(6) + W_8^3 x(1) - W_8^3 W_2^0 x(5) - W_8^3 W_4^1 x(3) + W_2^0 W_4^1 W_8^3 x(7)$$

$$= 1 - (1)(1) - (-j)(-1) + (-j)(1)(1) + \left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(2) - \left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(1)(2) - \left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-j)(3) + (1)(-j)\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-3)$$

$$= 4.2426 - j\,6.2426$$

$$F(4) = x(0) + W_2^0 x(4) + W_4^0 x(2) + W_4^0 W_2^0 x(6) - W_8^0 x(1) - W_8^0 W_2^0 x(5) - W_8^0 W_4^0 x(3) - W_8^0 W_2^0 W_4^0 x(7)$$

$$= 1 + (1)(1) + (1)(-1) + (1)(1)(1) - (1)(2) - (1)(1)(2) - (1)(1)(3) - (1)(1)(1)(-3)$$

$$= -2.0000$$

$$F(5) = X(0) - W_2^0 X(4) + W_4^1 X(2) - W_4^1 W_2^0 X(6) -$$
$$W_8^1 X(1) + W_8^1 W_2^0 X(5) - W_8^1 W_4^1 X(3) +$$
$$W_8^1 W_4^1 W_2^0 X(7)$$

$$= (1) - (1)(1) + (-j)(-1) - (-j)(1)(1) - \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(2)$$
$$+ \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(1)(2) - \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-j)(3) +$$
$$\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-j)(1)(-3)$$

$=$

$$F(6) = X(0) + W_2^0 X(4) - W_4^0 X(2) - W_4^0 W_2^0 X(6)$$
$$- W_8^2 X(1) - W_8^2 W_2^0 X(5) + W_8^2 W_4^0 X(3) +$$
$$W_8^2 W_4^0 W_2^0 X(7)$$

$$= 1 + (1)(1) - (1)(-1) - (1)(1)(1) - (-j)(2) -$$
$$(-j)(1)(2) + (-j)(1)(3) + (-j)(1)(1)(-3)$$

$$= 1 + 1 + \cancel{1} - \cancel{1} + 2j + 2j - \cancel{3j} + \cancel{3j}$$

$$= 2 + 4j$$

$$F(7) = x(0) - W_2^0 x(4) - W_4^1 x(2) + W_4^1 W_2^0 x(6)$$

$$- W_8^3 x(1) + W_2^0 W_8^3 x(5) + W_8^3 W_4^1 x(3) -$$

$$W_8^3 W_4^1 W_2^0 x(7)$$

$$= 1 - (1)(1) - (-j)(-1) + (-j)(1)(1) - \left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(2)$$

$$+ (1)\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(2) + \left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-j)(3)$$

$$-\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-j)(-3)$$

$$= -2j + j(3\sqrt{2}) - 3\sqrt{2}$$

$$= -4.2426 + j\,2.2426$$

So,

$$F(0) = 6.0000$$

$$F(1) = -4.2426 - j\,2.2426$$

$$F(2) = 2.0000 - j\,4.0000$$

$$F(3) = 4.2426 - j\,6.2426$$

$$F(4) = -6.0000$$

$$F(5) = 4.2426 + j\,2.2426$$

$$F(6) = -2.0000 + j\,4.0000$$

$$F(7) = -4.2426 + j\,2.2426$$