Error in Measurement

Errors in measurement systems can be divided into those that arise during the measurement process and those that arise due to later corruption of the measurement signal by induced noise during transfer of the signal from the point of measurement to some other point.
Errors arising during the measurement process can be divided into two groups:

- Systematic errors
- Random errors.

Systematic error: Systematic errors describe errors in the output readings of a measurement system that are consistently on one side of the correct reading, i.e. either all the errors are positive or they are all negative. Two major sources of systematic errors are system disturbance during measurement and the effect of environmental changes (modifying inputs). Other sources of systematic error include bent meter needles, the use of uncalibrated instruments, drift in instrument characteristics and poor cabling practices.

Random error: Random errors are perturbations of the measurement either side of the true value approximately equal numbers for a series of measurements made of the same quantity. Such perturbations are mainly small, but large perturbations occur from time to time, again unpredictably. Random errors often arise when measurements are taken by human observation of an analogue meter, especially where this involves interpolation between scale points. Electrical noise can also be a source of random errors. To a large extent, random errors can be overcome by taking the same measurement a number of times and extracting a value by averaging or other statistical techniques.

Sources of systematic error

Systematic errors in the output of many instruments are due to factors inherent in the manufacture of the instrument arising out of tolerances in the components of the instrument. They can also arise due to wear in instrument components over a period of time. In other cases, systematic errors are introduced either by the effect of environmental disturbances or through the disturbance of the measured system by the act of measurement. These various sources of systematic error, and ways in which the magnitude of the errors can be reduced, are discussed below.

a) System disturbance due to measurement

Disturbance of the measured system by the act of measurement is a common source of systematic error. If we were to start with a beaker of hot water and wished to measure its temperature with a mercury-in-glass thermometer, then we would take the thermometer, which would initially be at room temperature, and plunge it into the water. In so doing, we would be introducing a relatively cold mass (the thermometer) into the hot water and a heat transfer would take place between the water and the thermometer. This heat transfer would lower the temperature of the water. Whilst the reduction in temperature in this case would be so small as to be undetectable by the limited measurement resolution of such a thermometer, the effect is finite and clearly establishes the principle that, in nearly all measurement situations, the process of measurement disturbs the system and alters the values of the physical quantities being measured.
Thus, as a general rule, the process of measurement always disturbs the system being measured. The magnitude of the disturbance varies from one measurement system to the next and is affected particularly by the type of instrument used for measurement. Ways of minimizing disturbance of measured systems is an important consideration in instrument design. However, an accurate understanding of the mechanisms of system disturbance is a prerequisite for this.

Measurements in electric circuits

In analyzing system disturbance during measurements in electric circuits, Thevenin's theorem is often of great assistance. For instance, consider the circuit shown in Figure (a) in which the voltage across resistor R_5 is to be measured by a voltmeter with resistance R_m. Here, R_m acts as a shunt resistance across R_5, decreasing the resistance between points AB and so disturbing the circuit. Therefore, the voltage E_m measured by the meter is not the value of the voltage E_0 that existed prior to measurement. The extent of the disturbance can be assessed by calculating the open-circuit voltage E0 and comparing it with E_m.

Thevenin's theorem allows the circuit of Figure (a) comprising two voltage sources and five resistors to be replaced by an equivalent circuit containing a single resistance and one voltage source, as shown in Figure (b). For the purpose of defining the equivalent single resistance of a circuit by Thevenin's theorem, all voltage sources are represented just by their internal resistance, which can be approximated to zero, as shown in Figure (c). Analysis proceeds by calculating the equivalent resistances of sections of the circuit and building these up until the required equivalent resistance of the whole of the circuit is obtained. Starting at C and D, the circuit to the left of C and D consists of a series pair of resistances (R_1 and R_2) in parallel with R_3, and the equivalent resistance can be written as:

$$\frac{1}{R_{CD}} = \frac{1}{R_1 + R_2} + \frac{1}{R_3} \quad \text{or} \quad R_{CD} = \frac{(R_1 + R_2)R_3}{R_1 + R_2 + R_3}$$

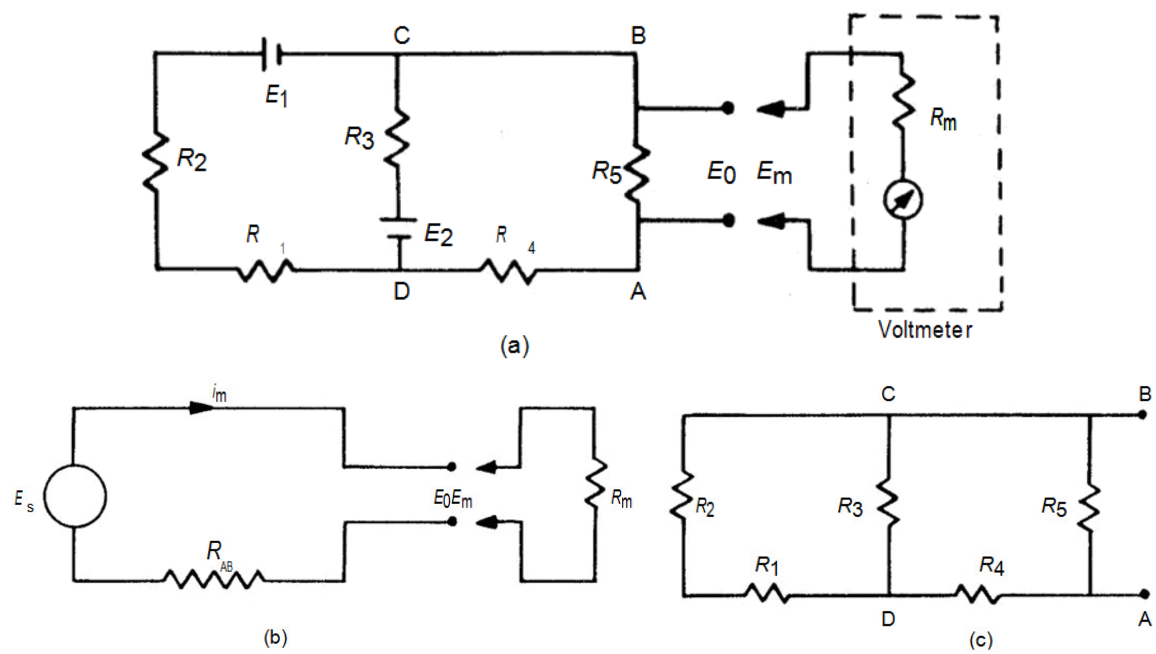


Fig. Analysis of circuit loading: (a) a circuit in which the voltage across R_5 is to be measured; (b) equivalent circuit by Thevenin's´ theorem; (c) the circuit used to find the equivalent single resistance R_{AB}.

Moving now to A and B, the circuit to the left consists of a pair of series resistances (R_{CD} and R_4) in parallel with R_5. The equivalent circuit resistance R_{AB} can thus be written as

$$\frac{1}{R_{AB}} = \frac{1}{R_{CD} + R_4} + \frac{1}{R_5} \quad \text{or} \quad R_{AB} = \frac{(R_4 + R_{CD})R_5}{R_4 + R_{CD} + R_5}$$

Substituting for R_{CD} using the expression derived previously, we obtain:

$$R_{AB} = \frac{\left[\frac{(R_1 + R_2)R_3}{R_1 + R_2 + R_3} + R_4\right] R_5}{\frac{(R_1 + R_2)R_3}{R_1 + R_2 + R_3} + R_4 + R_5}$$

Defining I as the current flowing in the circuit when the measuring instrument is connected to it, we can write:

$$I = \frac{E_0}{R_{AB} + R_m},$$

And the voltage measured by the meter is then given by:

$$E_m = \frac{R_m E_0}{R_{AB} + R_m}.$$

In the absence of the measuring instrument and its resistance R_m, the voltage across AB would be the equivalent circuit voltage source whose value is E_0. The effect of measurement is therefore to reduce the voltage across AB by the ratio given by:

$$\frac{E_m}{E_0} = \frac{R_m}{R_{AB} + R_m}$$

It is thus obvious that as R_m gets larger, the ratio E_m/E_0 gets closer to unity, showing that the design strategy should be to make R_m as high as possible to minimize disturbance of the measured system.

b) Errors due to environmental inputs

An environmental input is defined as an apparently real input to a measurement system that is actually caused by a change in the environmental conditions surrounding the measurement system. The fact that the static and dynamic characteristics specified for measuring instruments are only valid for particular environmental conditions (e.g. of temperature and pressure). These specified conditions must be reproduced as closely as possible during calibration exercises because, away from the specified calibration conditions, the characteristics of measuring instruments vary to some extent and cause measurement errors. The magnitude of this environment induced variation is quantified by the two constants known as sensitivity drift and zero drift, both of which are generally included in the published specifications for an instrument. Such variations of environmental conditions away from the calibration conditions are sometimes described as modifying inputs to the measurement system because they modify the output of the system. When such modifying inputs are present, it is often difficult to determine how much of the output change in a measurement system is due to a change in the measured variable and how much is due to a change in environmental conditions. This is illustrated by the following example.

Suppose we are given a small closed box and told that it may contain either a mouse or a rat. We are also told that the box weighs 0.1 kg when empty. If we put the box onto bathroom scales and observe a reading of 1.0 kg, this does not immediately tell us what is in the box because the reading may be due to one of three things:

i. a 0.9 kg rat in the box (real input)
ii. an empty box with a 0.9 kg bias on the scales due to a temperature change (environmental input)
iii. a 0.4 kg mouse in the box together with a 0.5 kg bias (real + environmental inputs).

Thus, the magnitude of any environmental input must be measured before the value of the measured quantity (the real input) can be determined from the output reading of an instrument.
In any general measurement situation, it is very difficult to avoid environmental inputs, because it is either impractical or impossible to control the environmental conditions surrounding the measurement system. System designers are therefore charged with the task of either reducing the susceptibility of measuring instruments to environmental inputs or, alternatively, quantifying the effect of environmental inputs and correcting for them in the instrument output reading.

c) Wear in instrument components

Systematic errors can frequently develop over a period of time because of wear in instrument components. Recalibration often provides a full solution to this problem.

d) Connecting leads

In connecting together the components of a measurement system, a common source of error is the failure to take proper account of the resistance of connecting leads (or pipes in the case of pneumatically or hydraulically actuated measurement systems). For instance, in typical applications of a resistance thermometer, it is common to find that the thermometer is separated from other parts of the measurement system by perhaps 100 meters. The resistance of such a length of 20 gauge copper wire is 7, and there is a further complication that such wire has a temperature coefficient of 1 $m\Omega/^{0}C$.
Therefore, careful consideration needs to be given to the choice of connecting leads. Not only should they be of adequate cross-section so that their resistance is minimized, but they should be adequately screened if they are thought likely to be subject to electrical or magnetic fields that could otherwise cause induced noise. Where screening is thought essential, then the routing of cables also needs careful planning.

Reduction of systematic errors

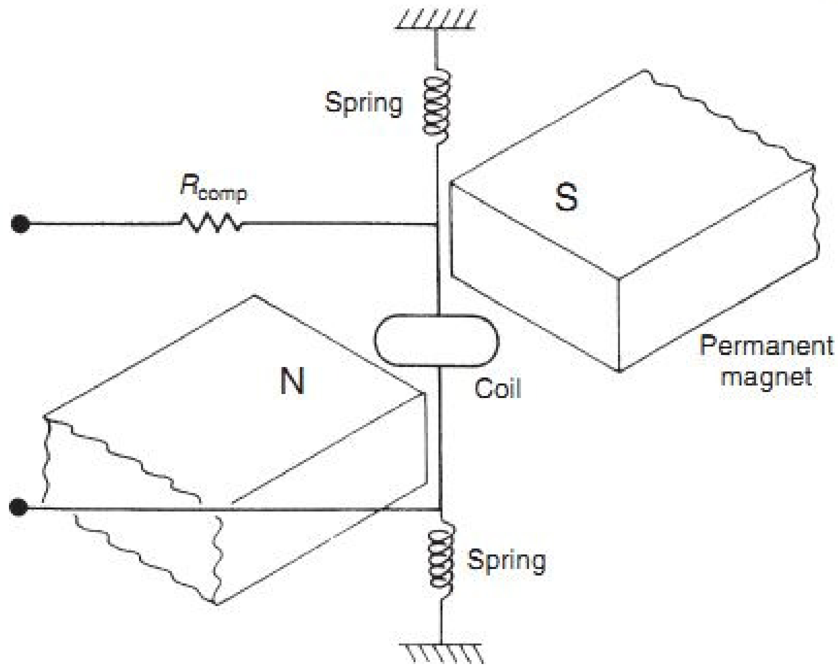
The prerequisite for the reduction of systematic errors is a complete analysis of the measurement system that identifies all sources of error. Simple faults within a system, such as bent meter needles and poor cabling practices, can usually be readily and cheaply rectified once they have been identified. However, other error sources require more detailed analysis and treatment. Various approaches to error reduction are considered below.

i. Careful instrument design

Careful instrument design is the most useful weapon in the battle against environmental inputs; by reducing the sensitivity of an instrument to environmental inputs to as low a level as possible. For instance, in the design of strain gauges, the element should be constructed from a material whose resistance has a very low temperature coefficient (i.e. the variation of the resistance with temperature is very small). However, errors due to the way in which an instrument is designed are not always easy to correct, and a choice often has to be made between the high cost of redesign and the alternative of accepting the reduced measurement accuracy if redesign is not undertaken.

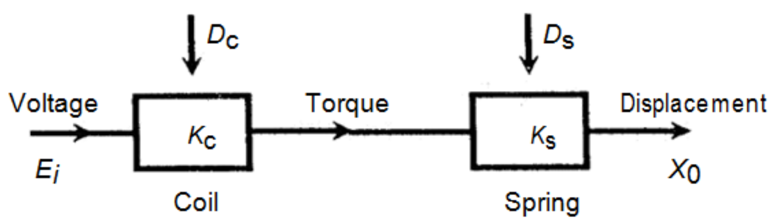
ii. Method of opposing inputs

The method of opposing inputs compensates for the effect of an environmental input in a measurement system by introducing an equal and opposite environmental input that cancels it out. One example of how this technique is applied is in the type of milli-voltmeter shown in Figure. This consists of a coil suspended in a fixed magnetic field produced by a permanent magnet. When an unknown voltage is applied to the coil, the magnetic field due to the current interacts with the fixed field and causes the coil (and a pointer attached to the coil) to turn. If the coil resistance R_{coil} is sensitive to temperature, then any environmental input to the system in the form of a temperature change will alter the value of the coil current for a given applied voltage and so alter the pointer output reading. Compensation for this is made by introducing a compensating resistance R_{comp} into the circuit, where R_{comp} has a temperature coefficient that is equal in magnitude but opposite in sign to that of the coil. Thus, in response to an increase in temperature, R_{coil} increases but R_{comp} decreases, and so the total resistance remains approximately the same.



iii. High-gain feedback

The benefit of adding high-gain feedback to many measurement systems is illustrated by considering the case of the voltage-measuring instrument whose block diagram is shown in Figure. In this system, the unknown voltage E_i is applied to a motor of torque constant K_m, and the induced torque turns a pointer against the restraining action of a spring with spring constant K_s. The effect of environmental inputs on the motor and spring constants is represented by variables D_m and D_s.



Block diagram for voltage-measuring instrument.

In the absence of environmental inputs, the displacement of the pointer X_0 is given by: $X_0=K_mK_sE_i$. However, in the presence of environmental inputs, both K_m and K_s change, and the relationship between X_0 and E_i can be affected greatly. Therefore, it becomes difficult or impossible to calculate E_i from the measured value of X_0. Consider now what happens if the system is converted into a high-gain, closed-loop one, as shown in Figure, by adding an amplifier of gain constant K_a and a feedback device with gain constant K_f. Assume also that the effect of environmental inputs on the values of K_a and K_f are represented by D_a and D_f. The feedback device feeds back a voltage E_0 proportional to the pointer displacement X_0. This is compared with the unknown voltage E_i by a comparator and the error is amplified. Writing down the equations of the system, we have:

$$E_0 = K_f X_0; \quad X_0 = (E_i - E_0) K_a K_m K_s = (E_i - K_f X_0) K_a K_m K_s$$

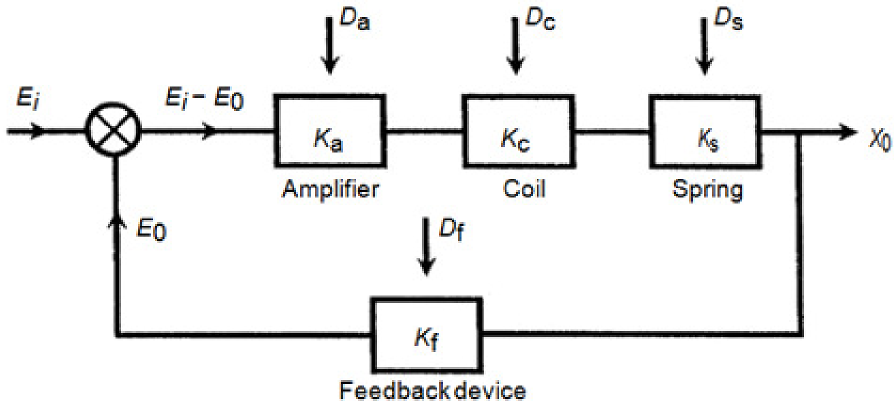


Fig. Block diagram of voltage-measuring instrument with high-gain feedback.

Thus:

$$E_i K_a K_m K_s = (1 + K_f K_a K_m K_s) X_0$$

i.e.

$$X_0 = \frac{K_a K_m K_s}{1 + K_f K_a K_m K_s} E_i$$

Because K_a is very large (it is a high-gain amplifier), $K_f.K_a.K_m.K_s >> 1$, and equation reduces to:

$$X_0 = E_i / K_f$$

This is a highly important result because we have reduced the relationship between X_0 and E_i to one that involves only K_f. The sensitivity of the gain constants Ka, Km and Ks to the environmental inputs D_a, D_m and D_s has thereby been rendered irrelevant, and we only have to be concerned with one environmental input D_f. Conveniently, it is usually easy to design a feedback device that is insensitive to environmental inputs: this is much easier than trying to make a motor or spring insensitive. Thus, high-gain feedback techniques are often a very effective way of reducing a measurement system's sensitivity to environmental inputs. However, one potential problem that must be mentioned is that there is a possibility that high-gain feedback will cause instability in the system. Therefore, any application of this method must include careful stability analysis of the system.

iv. Calibration

Instrument calibration is a very important consideration in measurement systems. All instruments suffer drift in their characteristics, and the rate at which this happens depends on many factors, such as the environmental conditions in which instruments are used and the frequency of their use. Thus, errors due to instruments being out of calibration can usually be rectified by increasing the frequency of recalibration.

v. Manual correction of output reading

In the case of errors that are due either to system disturbance during the act of measure-ment or due to environmental changes, a good measurement technician can substantially reduce errors at the output of a measurement system by calculating the effect of such systematic errors and making appropriate correction to the instrument readings. This is not necessarily an easy task, and requires all disturbances in the measurement system to be quantified. This procedure is carried out automatically by intelligent instruments.

vi. Intelligent instruments

Intelligent instruments contain extra sensors that measure the value of environmental inputs and automatically compensate the value of the output reading. They have the ability to deal very effectively with systematic errors in measurement systems, and errors can be attenuated to very low levels in many cases.

Quantification of systematic errors

Once all practical steps have been taken to eliminate or reduce the magnitude of systematic errors, the final action required is to estimate the maximum remaining error that may exist in a measurement due to systematic errors. Unfortunately, it is not always possible to quantify exact values of a systematic error, particularly if measurements are subject to unpredictable environmental conditions. The usual course of action is to assume mid-point environmental conditions and specify the maximum measurement error as ±x% of the output reading to allow for the maximum expected deviation in environmental conditions away from this mid-point. Data sheets supplied by instrument manufacturers usually quantify systematic errors in this way, and such figures take account of all systematic errors that may be present in output readings from the instrument.

Random Error

Random errors in measurements are caused by unpredictable variations in the measurement system. They are usually observed as small perturbations of the measurement either side of the correct value, i.e. positive errors and negative errors occur in approximately equal numbers for a series of measurements made of the same constant quantity. Therefore, random errors can largely be eliminated by calculating the average of a number of repeated measurements, provided that the measured quantity remains constant during the process of taking the repeated measurements. This averaging process of repeated measurements can be done automatically by intelligent instruments. The degree of confidence in the calculated mean/median values can be quantified by calculating the standard deviation or variance of the data, these being parameters that describe how the measurements are distributed about the mean value/median.

Statistical analysis of measurements subject to random errors

- **Mean and median values**

The average value of a set of measurements of a constant quantity can be expressed as either the mean value or the median value. As the number of measurements increases, the difference between the mean value and median values becomes very small. However, for any set of n measurements $x_1, x_2, x_3 \ldots x_n$ of a constant quantity, the most likely true value is the mean given by:

$$x_{\text{mean}} = \frac{x_1 + x_2 + \cdots x_n}{n}$$

This is valid for all data sets where the measurement errors are distributed equally about the zero error value, i.e. where the positive errors are balanced in quantity and magnitude by the negative errors.

The median is an approximation to the mean that can be written down without having to sum the measurements. The median is the middle value when the measurements in the data set are written down in ascending order of magnitude. For a set of n measurements $x_1, x_2, x_3 \ldots x_n$ of a constant quantity, written down in ascending order of magnitude, the median value is given by:

$$x_{\text{median}} = x_{n+1}/2$$

Thus, for a set of 9 measurements $x_1, x_2, x_3 \ldots x_9$ arranged in order of magnitude, the median value is x_5. For an even number of measurements, the median value is mid-way between the two centre values, i.e. for 10 measurements $x_1, x_2, x_3 \ldots x_{10}$, the median value is given by: $(x_5+x_6)/2$.

Suppose that the length of a steel bar is measured by a number of different observers and the following sets of 11 measurements are recorded. We will call this measurement set A.

398 420 394 416 404 408 400 420 396 413 430 (Measurement set A)

So mean = 409.0 and median = 408. Suppose now that the measurements are taken again using a better measuring rule, and with the observers taking more care, to produce the following measurement set B:

409 406 402 407 405 404 407 404 407 407 408 (Measurement set B)

For these measurements, mean = 406.0 and median = 407. Which of the two measurement sets A and B, and the corresponding mean and median values, should we have most confidence in? Intuitively, we can regard measurement set B as being more reli-able since the measurements are much closer together. In set A, the spread between the smallest (396) and largest (430) value is 34, whilst in set B, the spread is only 6.

- Thus, the smaller the spread of the measurements, the more confidence we have in the mean or median value calculated.

Let us now see what happens if we increase the number of measurements by extending measurement set B to 23 measurements. We will call this measurement set C.

409 406 402 407 405 404 407 404 407 407 408 406 410 406 405 408
406 409 406 405 409 406 407 (Measurement set C)

Now, mean = 406.5 and median = 406.

- This confirms our earlier statement that the median value tends towards the mean value as the number of measurements increases.

Standard deviation and variance

Expressing the spread of measurements simply as the range between the largest and smallest value is not in fact a very good way of examining how the measurement values are distributed about the mean value. A much better way of expressing the distribution is to calculate the variance or standard deviation of the measurements. The starting point for calculating these parameters is to calculate the deviation (error) d_i of each measurement x_i from the mean value x_{mean}:

$$d_i = x_i - x_{\mathrm{mean}}$$

The variance (V) is then given by:

$$V = \frac{d_1^2 + d_2^2 \cdots d_n^2}{n-1}$$

The standard deviation is simply the square root of the variance. Thus:

$$\sigma = \sqrt{V} = \sqrt{\frac{d_1^2 + d_2^2 \cdots d_n^2}{n-1}}$$

We have observed so far that random errors can be reduced by taking the average (mean or median) of a number of measurements. However, although the mean or median value is close to the true value, it would only become exactly equal to the true value if we could average an infinite number of measurements. As we can only make a finite number of measurements in a practical situation, the average value will still have some error. This error can be quantified as the standard error of the mean, which will be discussed in detail a little later. However, before that, the subject of graphical analysis of random measurement errors needs to be covered.

Estimation of random error in a single measurement

In many situations where measurements are subject to random errors, it is not practical to take repeated measurements and find the average value. Also, the averaging process becomes invalid if the measured quantity does not remain at a constant value, as is usually the case when process variables are being measured. Thus, if only one measurement can be made, some means of estimating the likely magnitude of error in it is required. The normal approach to this is to calculate the error within 95% confidence limits, i.e. to calculate the value of the deviation D such that 95% of the area under the probability curve lies within limits of ±D. These limits correspond to a deviation of $\pm 1.96\sigma$. Thus, it is necessary to maintain the measured quantity at a constant value whilst a number of measurements are taken in order to create a reference measurement set from which can be calculated. Subsequently, the maximum likely deviation in a single measurement can be expressed as: Deviation = $\pm 1.96\sigma$. However, this only expresses the maximum likely deviation of the measurement from the calculated mean of the reference measurement set, which is not the true value as observed earlier. Thus the calculated value for the standard error of the mean has to be added to the likely maximum deviation value. Thus, the maximum likely error in a single measurement can be expressed as:

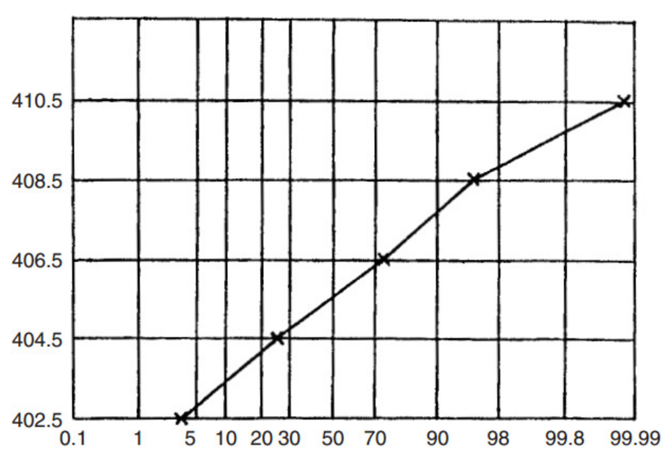
$$\text{Error} = \pm(1.96\sigma + \alpha)$$

Distribution of manufacturing tolerances

Many aspects of manufacturing processes are subject to random variations caused by factors that are similar to those that cause random errors in measurements. In most cases, these random variations in manufacturing, which are known as tolerances, fit a Gaussian distribution, and the analysis of random measurement errors can be applied to analyze the distribution of these variations in manufacturing parameters.

Rogue data points

In a set of measurements subject to random error, measurements with a very large error sometimes occur at random and unpredictable times, where the magnitude of the error is much larger than could reasonably be attributed to the expected random variations in measurement value. Sources of such abnormal error include sudden transient voltage surges on the mains power supply and incorrect recording of data (e.g. writing down 146.1 when the actual measured value was 164.1). It is accepted practice in such cases to discard these rogue measurements, and a threshold level of a $\pm 3\sigma$ deviation is often used to determine what should be discarded. It is extremely rare for measurement errors to exceed $\pm 3\sigma$ limits when only normal random effects are affecting the measured value.



Normal probability plot.

Special case when the number of measurements is small

When the number of measurements of a quantity is particularly small and statistical analysis of the distribution of error values is required, problems can arise when using standard Gaussian tables in terms of z because the mean of only a small number of measurements may deviate significantly from the true measurement value. In response to this, an alternative distribution function called the Student-t distribution can be used which gives a more accurate prediction of the error distribution when the number of samples is small. This is discussed more fully in Miller (1990).

Aggregation of measurement system errors

Errors in measurement systems often arise from two or more different sources, and these must be aggregated in the correct way in order to obtain a prediction of the total likely error in output readings from the measurement system. Two different forms of aggregation are required. Firstly, a single measurement component may have both systematic and random errors and, secondly, a measurement system may consist of several measurement components that each have separate errors.

Combined effect of systematic and random errors

If a measurement is affected by both systematic and random errors that are quantified as ±x (systematic errors) and ±y (random errors), some means of expressing the combined effect of both types of error is needed. One way of expressing the combined error would be to sum the two separate components of error, i.e. to say that the total possible error is e = ±(x+y). However, a more usual course of action is to express the likely maximum error as follows:

$$e = \sqrt{(x^2 + y^2)}$$

It can be shown (ANSI/ASME, 1985) that this is the best expression for the error statistically, since it takes account of the reasonable assumption that the systematic and random errors are independent and so are unlikely to both be at their maximum or minimum value at the same time.

Aggregation of errors from separate measurement system components

A measurement system often consists of several separate components, each of which is subject to errors. Therefore, what remains to be investigated is how the errors associated with each measurement system component combine together, so that a total error calculation can be made for the complete measurement system. All four mathematical operations of addition, subtraction, multiplication and division may be performed on measurements derived from different instruments/transducers in a measurement system. Appropriate techniques for the various situations that arise are covered below.

- **Error in a sum**

If the two outputs y and z of separate measurement system components are to be added together, we can write the sum as S = y + z. If the maximum errors in y and z are ±ay and ±bz respectively, we can express the maximum and minimum possible values of
S as:

$$S_{max} = (y + ay) + (z + bz); \quad S_{min} = (y - ay) + (z - bz); \quad \text{or } S = y + z \pm (ay + bz)$$

This relationship for S is not convenient because in this form the error term cannot be expressed as a fraction or percentage of the calculated value for S. Fortunately, statistical analysis can be applied (see Topping,

1962) that expresses S in an alternative form such that the most probable maximum error in S is represented by a quantity e, where e is calculated in terms of the absolute errors as

$$e = \sqrt{(ay)^2 + (bz)^2}$$

Thus S = (y+z) ± e. This can be expressed in the alternative form:

$$S = (y + z)(1 \pm f) \quad \text{where } f = e/(y + z)$$

It should be noted that equations are only valid provided that the measurements are uncorrelated (i.e. each measurement is entirely independent of the others).

- **Error in a difference**

If the two outputs y and z of separate measurement systems are to be subtracted from one another, and the possible errors are ±ay and ±bz, then the difference S can be expressed (using statistical analysis as for calculating the error in a sum and assuming that the measurements are uncorrelated) as:

$$S = (y - z) \pm e \quad \text{or} \quad S = (y - z)(1 \pm f)$$

Where e is calculated as above, and f = e/(y-z)

- **Error in a product**

If the outputs y and z of two measurement system components are multiplied together, the product can be written as P = yz. If the possible error in y is ±ay and in z is ±bz, then the maximum and minimum values possible in P can be written as:

$$P_{\max} = (y + ay)(z + bz) = yz + ayz + byz + aybz;$$

$$P_{\min} = (y - ay)(z - bz) = yz - ayz - byz + aybz$$

For typical measurement system components with output errors of up to one or two percent in magnitude, both a and b are very much less than one in magnitude and thus terms in aybz are negligible compared with other terms. Therefore, we have P_{max} = yz(1+a +b); P_{min} = yz (1-a-b) . Thus the maximum error in the product P is ± (a+b). Whilst this expresses the maximum possible error in P, it tends to overestimate the likely maximum error since it is very unlikely that the errors in y and z will both be at the maximum or minimum value at the same time. A statistically better estimate of the likely maximum error e in the product P, provided that the measurements are uncorrelated, is given by Topping (1962):

$$e = \sqrt{a^2 + b^2}$$

Note that in the case of multiplicative errors, e is calculated in terms of the fractional errors in y and z (as opposed to the absolute error values used in calculating additive errors).

Error in a quotient

If the output measurement y of one system component with possible error ±ay is divided by the output measurement z of another system component with possible error ±bz, then the maximum and minimum possible values for the quotient can be written as:

$$Q_{\max} = \frac{y + ay}{z - bz} = \frac{(y + ay)(z + bz)}{(z - bz)(z + bz)} = \frac{yz + ayz + byz + aybz}{z^2 - b^2 z^2};$$

$$Q_{\min} = \frac{y - ay}{z + bz} = \frac{(y - ay)(z - bz)}{(z + bz)(z - bz)} = \frac{yz - ayz - byz + aybz}{z^2 - b^2 z^2}$$

For a<<1 and b<<1, terms in ab and b^2 are negligible compared with the other terms.

Hence:

$$Q_{\max} = \frac{yz(1+a+b)}{z^2}; \quad Q_{\min} = \frac{yz(1-a-b)}{z^2}; \quad \text{i.e. } Q = \frac{y}{z} \pm \frac{y}{z}(a+b)$$

Thus the maximum error in the quotient is ± (a+b). However, using the same argument as made above for the product of measurements, a statistically better estimate (see Topping, 1962) of the likely maximum error in the quotient Q, provided that the measurements are uncorrelated.

Total error when combining multiple measurements

The final case to be covered is where the final measurement is calculated from several measurements that are combined together in a way that involves more than one type of arithmetic operation. For example, the density of a rectangular-sided solid block of material can be calculated from measurements of its mass divided by the product of measurements of its length, height and width. The errors involved in each stage of arithmetic are cumulative, and so the total measurement error can be calculated by adding together the two error values associated with the two multiplication stages involved in calculating the volume and then calculating the error in the final arithmetic operation when the mass is divided by the volume.