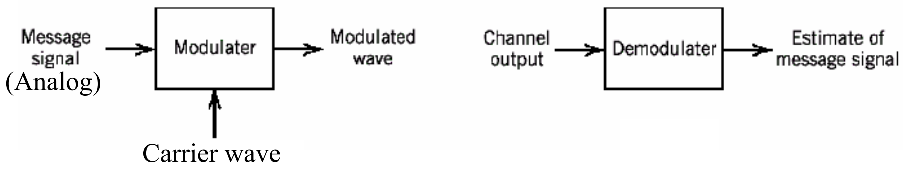
Chapter 2 Continuous-Wave Modulation

"Analog Modulation" is the subject concerned in this chapter.

2.1 Introduction

- Analog communication system
  - The most common carrier is the sinusoidal wave.



2.1 Introduction

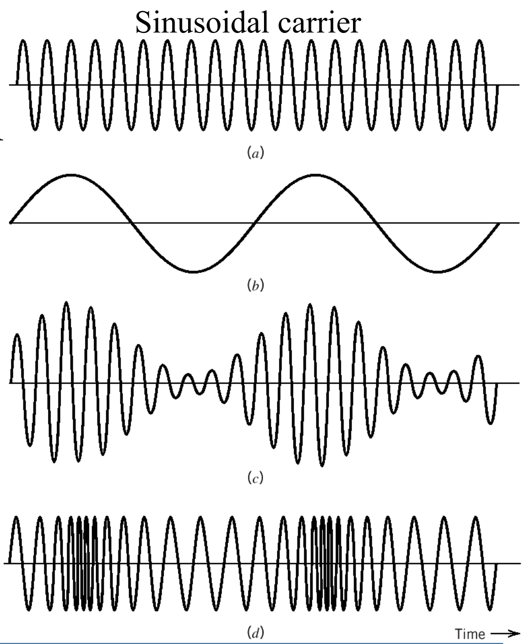
- Modulation
  - A process by which *some characteristic of a carrier* is varied in accordance with a *modulating wave* (baseband signal).
- Sinusoidal Continuous-Wave (CW) modulation
  - Amplitude modulation
  - Angle modulation

2.1 Introduction

Baseband signal

Amplitude Modulation

Frequency Modulation



2.2 Double-Sideband with Carrier or simply Amplitude Modulation

Carrier $c(t) = A_c \cos(2\pi f_c)$

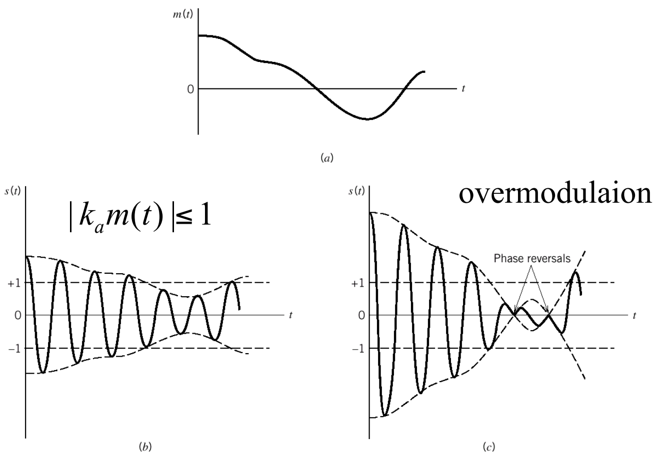
Baseband $m(t)$

Modulated Signal $s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$,

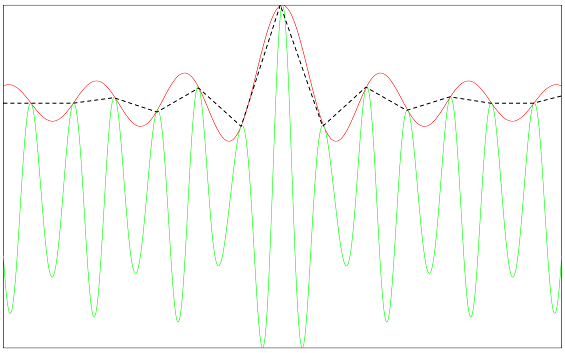
where k_a is amplitude sensitivity or modulation index

- Two required conditions on amplitude sensitivity
  - $1 + k_a\, m(t) \geq 0$, which is ensured by $|k_a\, m(t)| \leqq 1$.
    - The case of $|k_a\, m(t)| > 1$ is called *overmodulation*.
    - The value of $|k_a\, m(t)|$ is sometimes represented by "percentage" (because it is limited by 1), and is named ($|k_a\, m(t)|$ $\times 100$)% modulation.
  - $f_c >> W$, where W is the message bandwidth.
    - Violation of this condition will cause **nonvisualized envelope**.

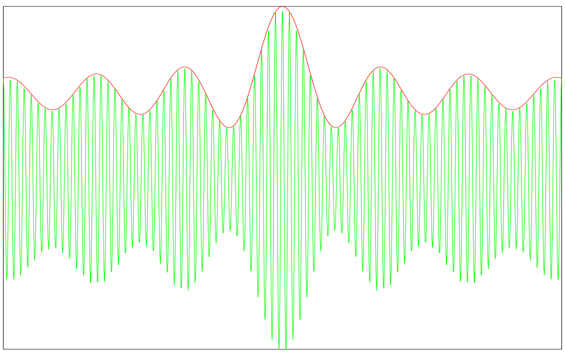
2.2 Overmodulation



2.2 Example of *Non-Visualized Envelope*



2.2 Example of *Visualized Envelope*

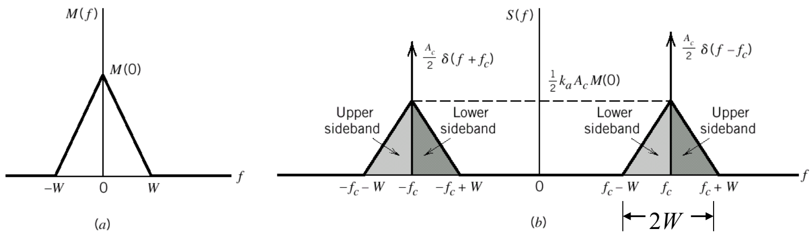


2.2 Transmission Bandwidth

$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$\Rightarrow S(f) = \frac{A_c}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] + \frac{k_a A_c}{2}\left[M(f - f_c) + M(f + f_c)\right]$$

Transmission bandwidth $B_T = 2W$.

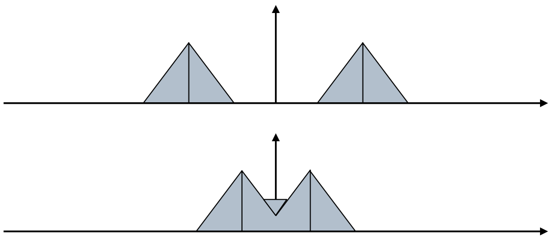


2.2 Transmission Bandwidth

- Transmission bandwidth of an AM wave
  - For positive frequencies, the highest frequency component of the AM wave equals $f_c + W$, and the lowest frequency component equals $f_c - W$.
  - The difference between these two frequencies defines the transmission bandwidth B_T for an AM wave.

2.2 Transmission Bandwidth

- The condition of $f_c > W$ ensures that the sidebands do not overlap.



2.2 Negative Frequency

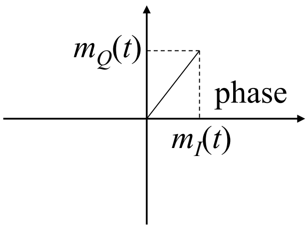
- Operational meaning of "negative frequency" in spectrum
  - If time-domain signal is *real-valued*, the *negative frequency spectrum* is simply a mirror of the *positive frequency spectrum*.
  - We may then define a one-sided spectrum as

$$S_{\text{one-sided}}(f) = 2S(f) \quad \text{for } f \geq 0.$$

  - Hence, if only real-valued signal is considered, it is unnecessary to introduce "negative frequency."

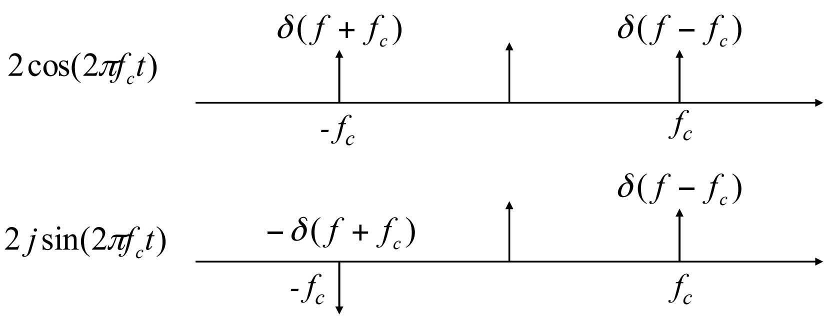
2.2 Negative Frequency

- So the introduction of *negative frequency part* is due to the need of *imaginary signal part*.
- Signal phase information is embedded in *imaginary signal part* of the signal.



2.2 Negative Frequency

- As a result, the following two spectrums contain the same *frequency components* but *different phases* (90 degree shift in complex plane).

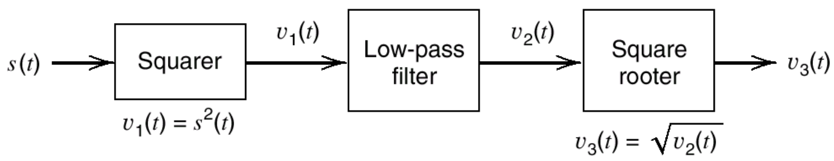


2.2 Negative Frequency

- Summary
  - *Complex-valued* baseband signal consists of information of amplitude and phase; while *real-valued* baseband signal only contains amplitude information.
  - One-sided spectrum only bears *amplitude information*, while two-sided spectrum (with negative frequency part) carries also *phase information*.

2.2 Virtues of Amplitude Modulation

- AM receiver can be implemented in terms of *simple circuit* with *inexpensive electrical components*.
  - E.g., AM receiver

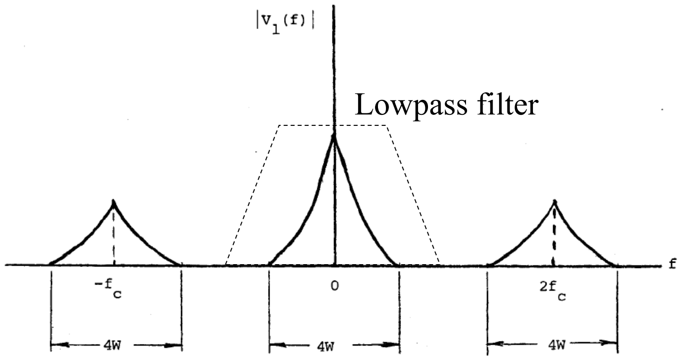


$$v_1(t) = s^2(t) = A_c^2[1 + k_a m(t)]^2 \cos^2(2\pi f_c t)$$

$$= \frac{A_c^2}{2}[1 + k_a m(t)]^2[1 + \cos(4\pi f_c t)]$$

2.2 Virtues of Amplitude Modulation

- The bandwidth of $m^2(t)$ is twice of $m(t)$. (So to speak, the bandwidth of $m(f)*m(f)$ is twice of $m(f)$.)



2.2 Virtues of Amplitude Modulation

- So if $2f_c > 4W$,

$$\Rightarrow v_2(t) = \frac{A_c^2}{2}[1 + k_a m(t)]^2$$

$$\Rightarrow v_3(t) = \frac{A_c}{\sqrt{2}}[1 + k_a m(t)]$$

$$\text{if } m(t) \text{ is zero mean} \overset{\text{block DC}}{\Rightarrow} \frac{A_c k_a}{\sqrt{2}} m(t)$$

By means of a squarer, the receiver can recover the information-bearing signal **without the need of a local carrier**.

2.2 Limitations of Amplitude Modulation (DSB-C)

- Wasteful of power and bandwidth

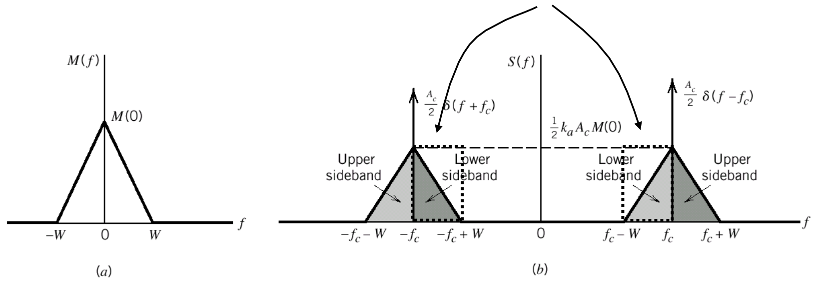
$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$
$$= \underbrace{A_c\cos(2\pi f_c t)}_{\text{with carrier}} + k_a m(t)\cos(2\pi f_c t)$$

Waste of power in the information-less "with-carrier" part.

2.2 Limitations of Amplitude Modulation

- Wasteful of power and bandwidth

Only requires half of bandwidth after modulation



2.3 Linear Modulation

- **Definition**
  - Both $s_I(t)$ and $s_Q(t)$ in

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

are *linear* function of $m(t)$.

2.3 Linear Modulation

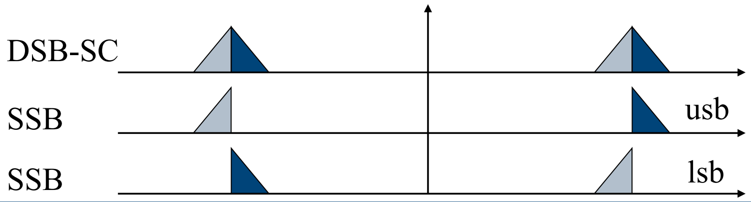
- For a single real-valued $m(t)$, three types of modulations can be identified according to how $s_Q(t)$ are *linearly* related to $m(t)$, at the case that $s_I(t)$ is exactly $m(t)$:
  - (*Some modulation may have $m_I(t)$ and $m_Q(t)$ that respectively bear independent information.*)

1. Double SideBand-Suppressed Carrier modulation (DSB-SC)
2. Single SideBand (SSB) modulation
3. Vestigial SideBand (VSB) modulation

2.3 DSB-SC and SSB

Type of modulation	$s_I(t)$	$s_Q(t)$	
DSB-SC	$m(t)$	0	
SSB	$m(t)$	$\hat{m}(t)$	Upper side band transmission
SSB	$m(t)$	$-\hat{m}(t)$	Lower side band transmission

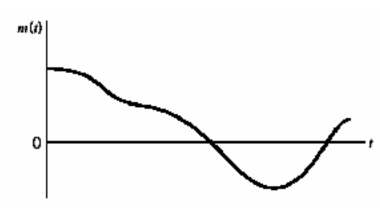
* $\hat{m}(t)$ = Hilbert transform of $m(t)$, which is used to completely "suppress" the other sideband

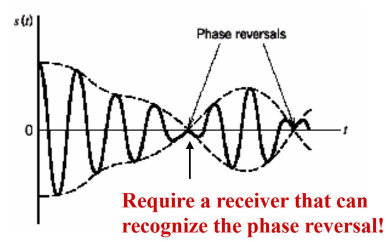


2.3 DSB-SC

- Different from DSB-C, DSB-SC $s(t)$ undergoes a *phase reversal* whenever $m(t)$ crosses zero.

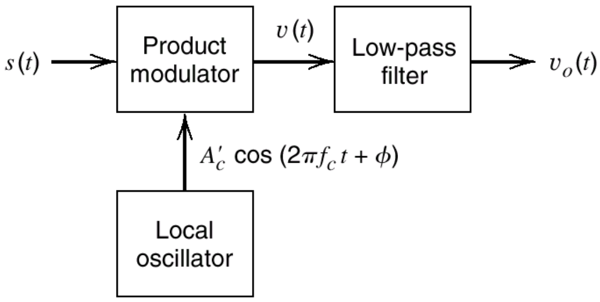
$$s(t) = m(t)\cos(2\pi f_c t)$$





2.3 Coherent Detection for DSB-SC

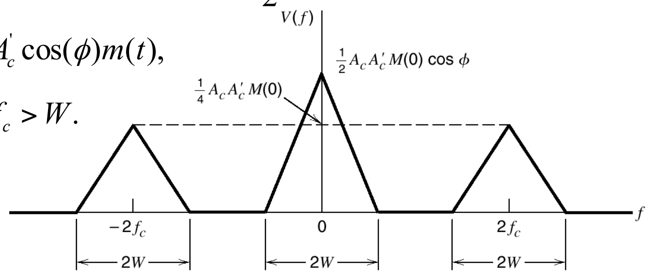
- For DSB-SC, we can no longer use the “envelope detector” (as used for DSB-C), in which no local carrier is required at the receiver.
- The *coherent detection* or *synchronous demodulation* becomes necessary.



2.3 Coherent Detection for DSB-SC

$$
\begin{aligned}
v(t) &= A_c' \cos(2\pi f_c t + \phi) s(t) \\
&= A_c A_c' \cos(2\pi f_c t) \cos(2\pi f_c t + \phi) m(t) \\
&= \frac{1}{2} A_c A_c' \cos(4\pi f_c t + \phi) m(t) + \frac{1}{2} A_c A_c' \cos(\phi) m(t) \\
&\xrightarrow{LowPass} \frac{1}{2} A_c A_c' \cos(\phi) m(t),
\end{aligned}
$$

provided $f_c > W$.



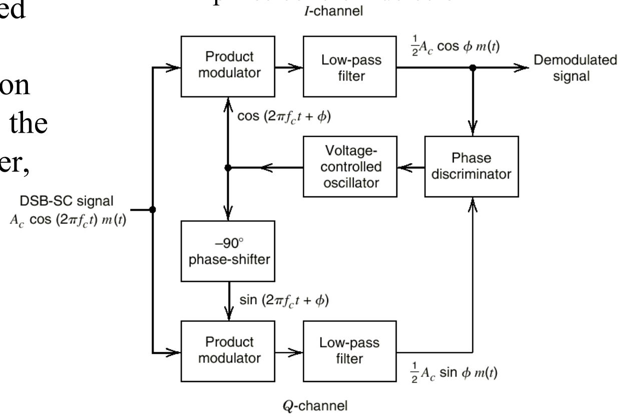
2.3 Coherent Detection for DSB-SC

- *Quadrature null effect* of the coherent detector.
  - If $\phi = \pi/2$ or $-\pi/2$, the output of coherent detector for DSB-SC is *nullified*.
- If ϕ is not equal to either $\pi/2$ or $-\pi/2$, the output of coherent detector for DSB-SC is simply attenuated by a factor of $\cos(\phi)$, **if ϕ is a constant, independent of time.**
- However, in practice, ϕ often varies with time; therefore, it is necessary to have an additional mechanism to maintain the local carrier in the receiver in perfect synchronization with the local carrier in the transmitter.
- Such an additional mechanism adds the system complexity of the receiver.

2.3 Costas Receiver for DSB-SC

- An exemplified design of synchronization mechanism is the Costas receiver, where two coherent detectors are used.

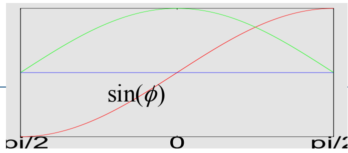
In-phase coherent detector



Quadrature-phase coherent detector

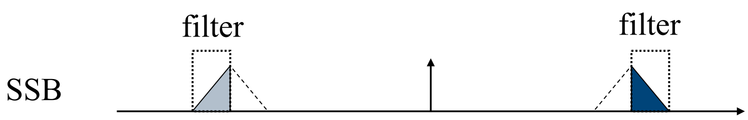
2.3 Costas Receiver for DSB-SC

- Conceptually, the Costas receiver adjusts the phase ϕ so that it is close to 0.
  - When ϕ drifts away from 0, the Q-channel output will have *the same polarity* as the I-channel output for *one direction* of phase drift, and *opposite polarity* for *another direction* of phase drift.
  - The phase discriminator then adjusts ϕ through the voltage controlled oscillator.



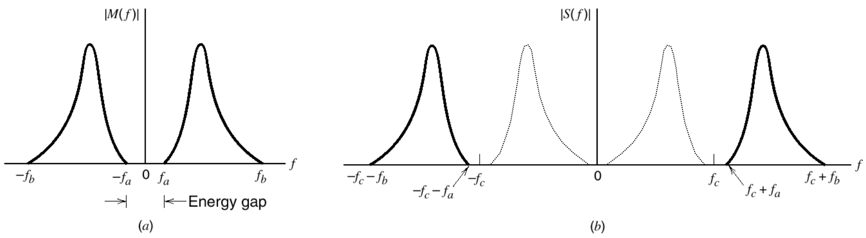
2.3 Single-Sideband Modulation

- How to generate SSB signal?
  - 1. Product modulator to generate DSB-SC signal
  - 2. Band-pass filter to pass only one of the sideband and suppress the other.
- The above technique may not be applicable to a DSB-SC signal like below. Why?



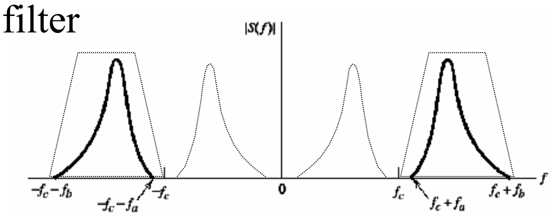
2.3 Single-Sideband Modulation

- For the generation of a SSB modulated signal to be possible, the message spectrum must have an *energy gap* centered at the origin.



2.3 Single-Sideband Modulation

- Example of signal with -300 Hz ~ 300 Hz energy gap
  - Voice : A band of 300 Hz to 3100 Hz gives good articulation
- Also required for SSB modulation is a highly selective filter

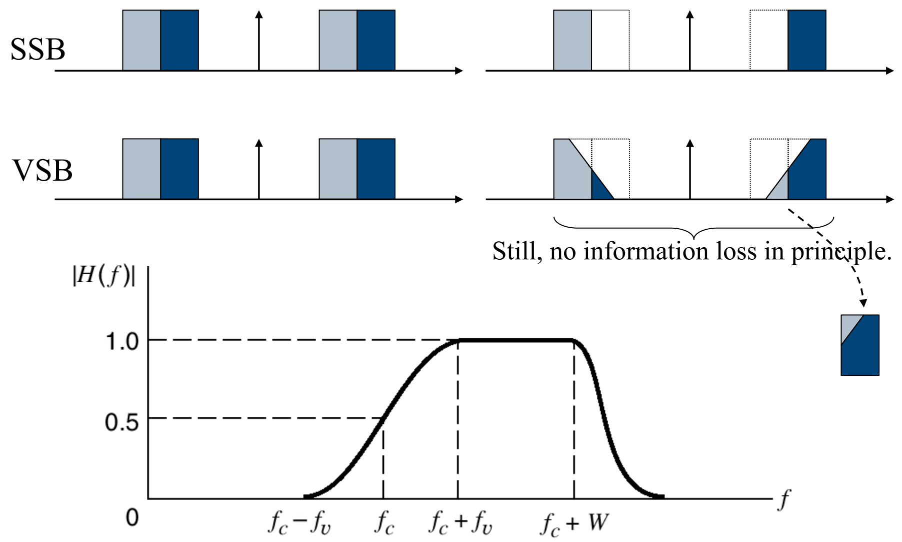


2.3 Single-Sideband Modulation

- Phase synchronization is also an important issue for SSB demodulation. This can be achieved by:
  - Either a separate low-power pilot carrier
  - Or a highly stable local oscillator (for voice transmission)
    - Phase distortion that gives rise to a *Donald Duck* voice effect is relatively insensitive to human ear.

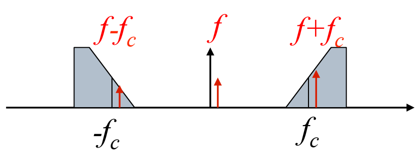
2.3 Vestigial Sideband Modulation

- Instead of transmitting only one sideband as SSB, VSB modulation transmits a partially suppressed sideband and a vestige of the other sideband.



2.3 Requirements for VSB filter

1. The sum of values of the magnitude response $|H(f)|$ at any two frequencies equally displaced above and below f_c is **unity**. I.e., $|H(f_c - f)| + |H(f_c + f)| = 1$ for $-f_v < f < f_v$.
2. $H(f - f_c) + H(f + f_c) = 1$ for $-W < f < W$.



So the transmission band of VSB filter is $B_T = W + f_v$.

2.3 Generation of VSB Signal

- Analysis of VSB
  - Give a real baseband signal $m(t)$ of bandwidth W.
    - Then, $M(-f) = M^*(f)$ and $M(f) = 0$ for $|f| > W$.
  - Let $M_{VSB}(f) = M(f)[1 + H_Q(f)/j]/2$, where

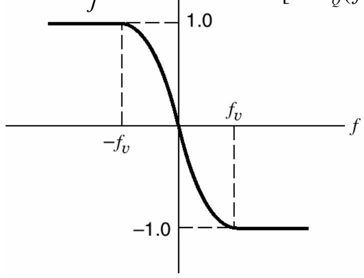
$$\frac{1}{j}H_Q(f) = \begin{cases} 1, & f \le -f_v \\ \in(0,1), & -f_v < f < 0, \text{ and } H_Q(-f) = H_Q^*(f). \\ 0, & f = 0 \end{cases}$$

The filter is denoted by H_Q because it is used to generate $s_Q(t)$ (cf. Slide 2-23)

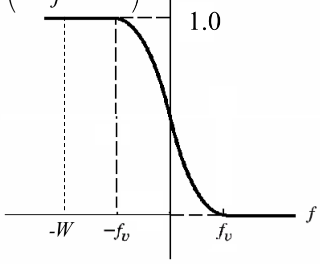
2.3 Generation of VSB Signal

$$L_Q(f) = \frac{1}{j}H_Q(f) \text{ is real.} \Rightarrow L_Q(-f) = \frac{1}{j}H_Q(-f) = \frac{1}{j}H_Q^*(f) = \left[-\frac{1}{j}H_Q(f)\right]^* = -L_Q^*(f)$$
$$= -L_Q(f).$$

$L_Q(f) = \frac{1}{j}H_Q(f)$



$[1 + L_Q(f)]/2 = \left(1 + \frac{1}{j}H_Q(f)\right)/2$



2.3 How to recover from VSB signal?

$$M_{VSB}(f) + M^*_{VSB}(-f)$$

$$= \frac{1}{2}\left(M(f)[1 + L_Q(f)] + M^*(-f)[1 + L_Q(-f)]^*\right)$$

$$= \frac{1}{2}\left(M(f)[1 + L_Q(f)] + M(f)[1 + L_Q(-f)]\right),$$

since $[1 + L_Q(-f)]$ is real, and $M^*(-f) = M(f)$

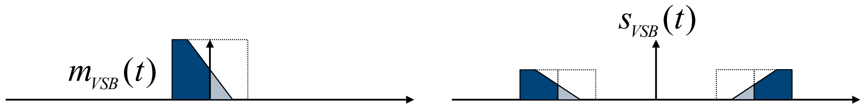
$$= \frac{1}{2}\left(M(f)[2 + L_Q(f) + L_Q(-f)]\right)$$

$$= M(f), \text{because } L_Q(-f) = -L_Q(f).$$

2.3 VSB upper sideband transmission

$$S_{DSB}(f) = \tfrac{1}{2}[M(f+f_c) + M(f-f_c)] = \tfrac{1}{2}[M(f+f_c) + M^*(-f+f_c)]$$

$$\begin{aligned}
S_{VSB}(f) &= \frac{1}{2}[M_{VSB}(f+f_c) + M^*_{VSB}(-f+f_c)] \\
&= \frac{1}{2}\left(M(f+f_c)\frac{[1+L_Q(f+f_c)]}{2} + M^*(-f+f_c)\frac{[1+L_Q(-f+f_c)]}{2}\right) \\
&= \frac{1}{2}\left(M(f+f_c)\mathbf{1}\{-f_c - W \le f \le -f_c + W\} \times \frac{[1+L_Q(f+f_c)]}{2}\mathbf{1}\{f \le -f_c + W\}\right. \\
&+ \left. M^*(-f+f_c)\mathbf{1}\{f_c - W \le f \le f_c + W\} \times \frac{[1+L_Q(-f+f_c)]}{2}\mathbf{1}\{f \ge f \ge f_c - W\}\right)
\end{aligned}$$

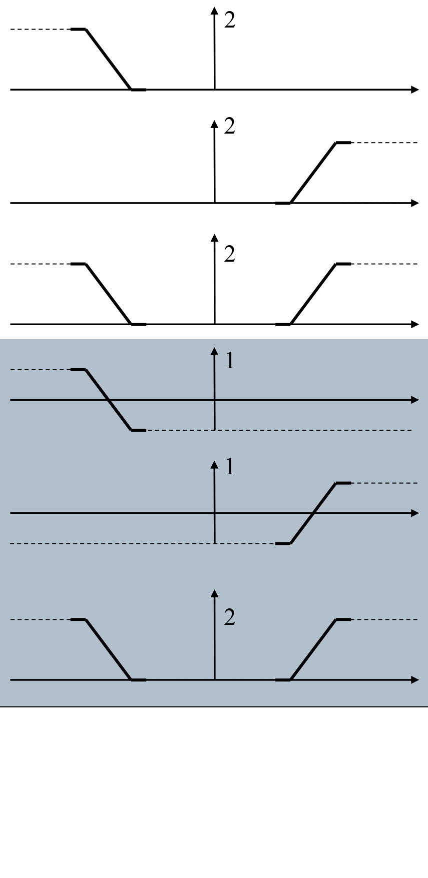


$$\boxed{\begin{array}{l} M_L(f)F_L(f)+M_R(f)F_R(f)=[M_L(f)+M_R(f)][F_L(f)+F_R(f)] \\ \text{if } M_L(f)F_R(f)=M_R(f)F_L(f)=0. \end{array}}$$

cont.

$$= \frac{1}{2}\left(M(f+f_c)\mathbf{1}\{-f_c-W \le f \le -f_c+W\} + M^*(-f+f_c)\mathbf{1}\{f_c-W \le f \le f_c+W\}\right)$$
$$\times\left(\frac{[1+L_Q(f+f_c)]}{2}\mathbf{1}\{f \le -f_c+W\} + \frac{[1+L_Q(-f+f_c)]}{2}\mathbf{1}\{f \ge f_c-W\}\right)$$
$$= \frac{1}{2}\left(M(f+f_c)+M^*(-f+f_c)\right)$$
$$\times\left(\frac{[1+L_Q(f+f_c)]}{2}\mathbf{1}\{f \le -f_c+W\} + \frac{[1+L_Q(-f+f_c)]}{2}\mathbf{1}\{f \ge f_c-W\}\right)$$
$$= s_{DSB}(f)\times\left(\frac{[1+L_Q(f+f_c)]}{2}\mathbf{1}\{f \le -f_c+W\} + \frac{[1+L_Q(-f+f_c)]}{2}\mathbf{1}\{f \ge f_c-W\}\right)$$
$$= s_{DSB}(f)\times\frac{1}{2}\left([1+L_Q(f+f_c)]\mathbf{1}\{f+f_c \le W\} + [1+L_Q(-f+f_c)]\mathbf{1}\{-f+f_c \le W\}\right)$$
$$\boxed{= s_{DSB}(f)\times\frac{1}{2}\left(2+L_Q(f+f_c)+L_Q(-f+f_c)\right)}$$

(See next slide.)



$[1+L_Q(f+f_c)]\mathbf{1}\{f+f_c \le W\}$

$[1+L_Q(-f+f_c)]\mathbf{1}\{-f+f_c \le W\}$

$[1+L_Q(f+f_c)]\mathbf{1}\{f+f_c \le W\}$
$+[1+L_Q(-f+f_c)]\mathbf{1}\{-f+f_c \le W\}$

$L_Q(f+f_c)$

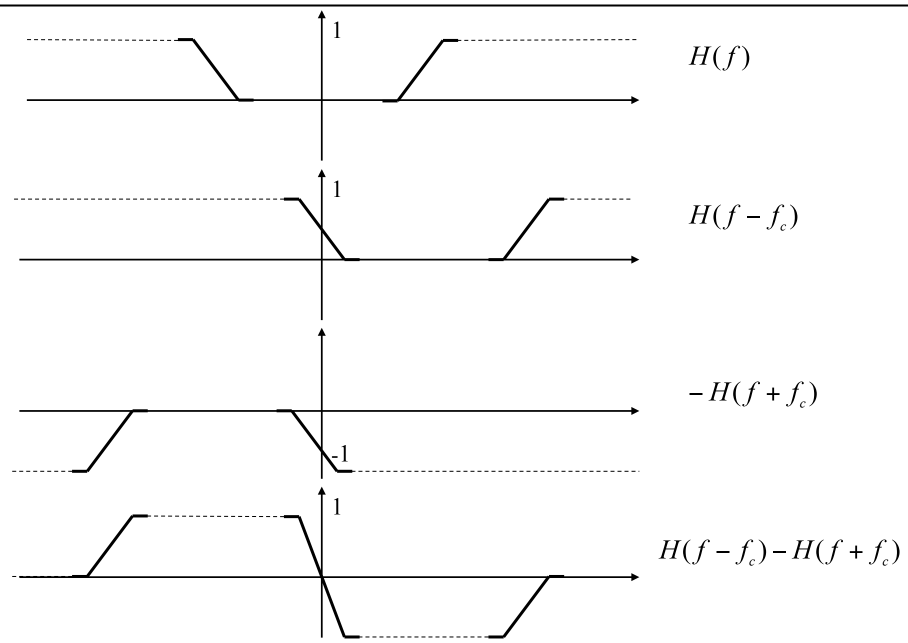
$L_Q(-f+f_c)$

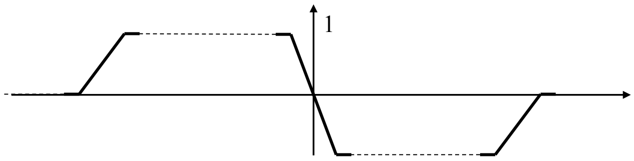
$L_Q(f+f_c)+L_Q(-f+f_c)+2$

Consequently,

$$\begin{cases} s_{VSB}(f) = s_{DSB}(f)\frac{1}{2}\left(L_Q(f+f_c) + L_Q(-f+f_c) + 2\right) \\ s_{VSB}(f) = s_{DSB}(f)H(f) \end{cases}$$

$$\Rightarrow H(f) = \frac{1}{2}\left(L_Q(f+f_c) + L_Q(-f+f_c) + 2\right)$$





$\Rightarrow L_Q(f) = H(f-f_c) - H(f+f_c)$ for $|f| \le W$

$\Rightarrow H_Q(f) = j[H(f-f_c) - H(f+f_c)]$ for $|f| \le W$

2.3 Mathematical Representation of VSB signal

$$M_{VSB}(f) = M(f)[1 - jH_Q(f)]/2$$
$$= \frac{1}{2}M(f) - \frac{1}{2}jM(f)H_Q(f)$$
$$= \frac{1}{2}M(f) + \frac{1}{2}jM'(f)$$

where $M'(f) = -M(f)H_Q(f) = -jM(f)L_Q(f)$.

Notably, $m'(t)$ is real. This is an extension of Hilbert Transform.

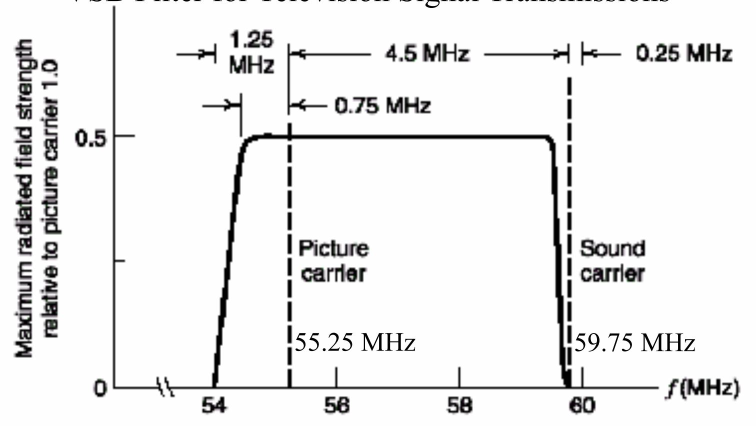
$$\begin{aligned} M'(-f) &= -jM(-f)L_Q(-f) = jM^*(f)L_Q(f) \\ &= (-j)^* M^*(f)L_Q^*(f) = [-jM(f)L_Q(f)]^* = [M'(f)]^*. \end{aligned}$$

2.3 Application of VSB Modulation

- Television Signals
  1. The video signal exhibits a *large* bandwidth and *significant low-frequency* content.
     - Hence, no *energy gap* exists (SSB becomes impractical).
     - VSB modulation is adopted to save bandwidth.
     - Notably, since a rigid control of the transmission VSB filter at the very high-power transmitter is expensive, a "not-quite" VSB modulation is used instead (a little waste of bandwidth to save cost).

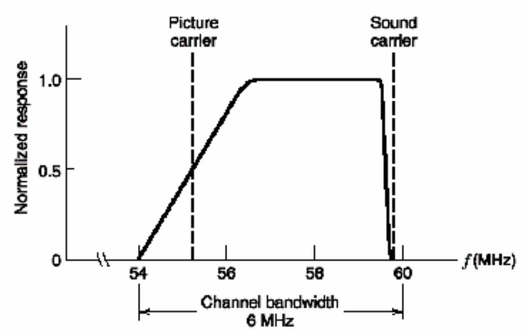
2.3 Application of VSB Modulation

VSB Filter for Television Signal Transmissions



2.3 Application of VSB Modulation

- As the transmission signal is not quite VSB modulated, the receiver needs to “re-shape” the received signal before feeding it to a VSB demodulator.



2.3 Application of VSB Modulation

2. In order to save the cost of the receiver (i.e., in order to use envelope detector at the receiver), an additional carrier is added.
   - Notably, additional carrier does not increase bandwidth, but just add transmission power.

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2} A_c k_a \big(m(t)\cos(2\pi f_c t) \pm m'(t)\sin(2\pi f_c t)\big)$$

$$= A_c \left[1 + \frac{1}{2} k_a m(t)\right] \cos(2\pi f_c t) \pm \frac{1}{2} k_a A_c m'(t)\sin(2\pi f_c t)$$

2.3 Application of VSB Modulation

- Distortion of envelope detector

$$s^2(t) = A_c^2\left[1+\frac{1}{2}k_a m(t)\right]^2\cos^2(2\pi f_c t) + \frac{1}{4}k_a^2 A_c^2 (m'(t))^2 \sin^2(2\pi f_c t)$$

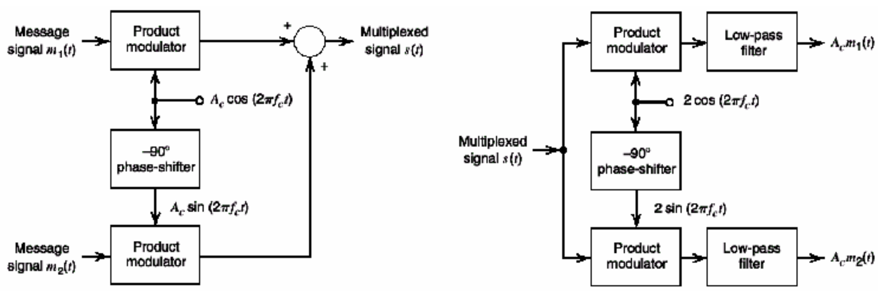
$$\pm\frac{1}{2}k_a A_c^2 m'(t)\left[1+\frac{1}{2}k_a m(t)\right]\sin(2\pi f_c t)\cos(2\pi f_c t)$$

$$\xrightarrow{\text{LowPass}} \frac{1}{2}A_c^2\left(\left[1+\frac{1}{2}k_a m(t)\right]^2 + \underbrace{\frac{1}{4}k_a^2 (m'(t))^2}_{\text{Distortion}}\right)$$

The distortion can be compensated by reducing the amplitude sensitivity k_a or increasing the width of the vestigial sideband. Both methods are used in the design of Television broadcasting system.

2.3 Extension Usage of DSB-SC

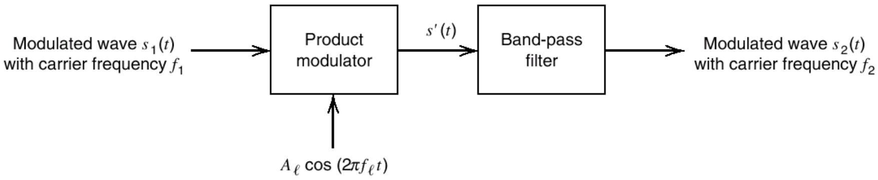
- Quadrature-Carrier Multiplexing or Quadrature Amplitude Modulation (QAM)



Synchronization is critical in QAM modulation, which is often achieved by a separate low-power pilot tone outside the passband of the modulated signal.

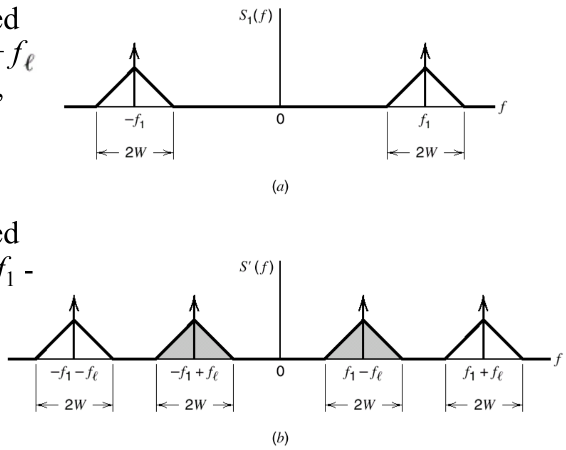
2.4 Frequency Translation

- The basic operation of SSB modulation is simply a special case of *frequency translation*.
  - So SSB modulation is sometimes referred to as *frequency changing*, *mixing*, or *heterodyning*.
  - The mixer is a device that consists of a product modulator followed by a band-pass filter, which is exactly what SSB modulation does.



2.4 Frequency Translation

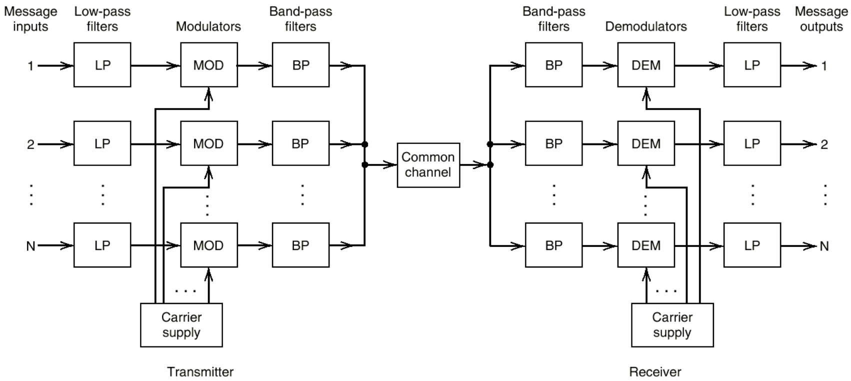
- The process is named *upconversion*, if $f_1 + f_\ell$ is the wanted signal, and $f_1 - f_\ell$ is the unwanted image signal.
- The process is named *downconversion*, if $f_1 - f_\ell$ is the wanted signal, and $f_1 + f_\ell$ is the unwanted image signal.



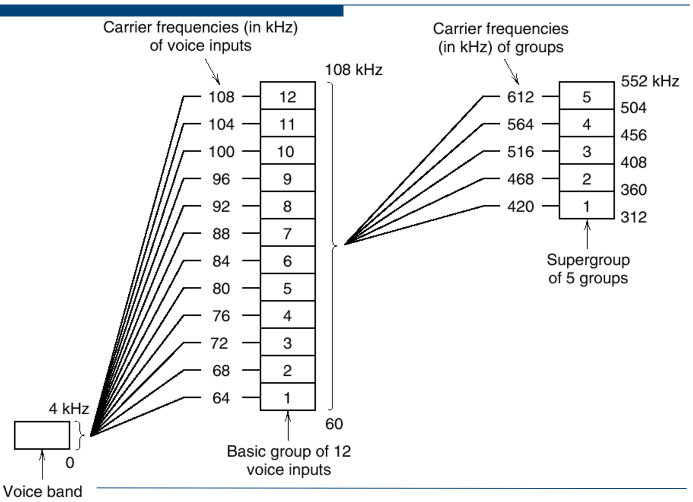
2.5 Frequency-Division Multiplexing

- *Multiplexing* is a technique to combine a number of independent signals into a composite signal suitable for transmission.
- Two conventional multiplexing techniques
  - Frequency-Division Multiplexing (FDM)
  - Time-Division Multiplexing (TDM)
    - Will be discussed in Chapter 3.

2.5 Frequency-Division Multiplexing



Example 2.1



2.6 Angle Modulation

- Angle modulation
  - The angle of the carrier is varied in accordance with the baseband signal.
- Angle modulation provides us with a practical means of exchanging *channel bandwidth* for improved *noise performance*.
  - So to speak, angle modulation can provide better discrimination against noise and interference than the amplitude modulation, at the expense of increased transmission bandwidth.

2.6 Angle Modulation

- Commonly used angle modulation
  - Phase modulation (PM)

    $s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$, where k_p is phase sensitivity.
  - Frequency modulation (FM)

$$\begin{aligned} s(t) &= A_c \cos\left[2\pi \int_0^t (f_c + k_f m(\tau)) d\tau\right] \\ &= A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau\right] \end{aligned}$$

where k_f is frequency sensitivity.

2.6 Angle Modulation

- Main differences between Amplitude Modulation and Angle Modulation
  1. Zero crossing spacing of angle modulation no longer has a perfect regularity as amplitude modulation does.
  2. Angle modulated signal has constant envelope; yet, the envelope of amplitude modulated signal is dependent on the message signal.

2.6 Angle Modulation

- Similarity between PM and FM
  - PM is simply an FM with $\int_0^t m(\tau)d\tau$ in place of $m(t)$.

$$s_{PM}(t) = A_c \cos[2\pi f_c t + k_p m(t)]$$

$$s_{FM}(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

  - Hence, the text only discusses FM in this chapter.

2.7 Frequency Modulation

- $s(t)$ of FM modulation is a **non-linear** function of $m(t)$.

$$s(t) = A_c \cos\left[2\pi \int_0^t f_i(\tau)d\tau\right] = A_c \cos\left[2\pi \int_0^t (f_c + k_f m(\tau))d\tau\right]$$

$$= A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

- So its general analysis is hard.
- To simplify the analysis, we may assume a single-tone transmission, where

$$m(t) = A_m \cos(2\pi f_m t)$$

- From the formula in the previous slide,

$$
\begin{aligned}
f_i(t) &= f_c + k_f m(t) \\
&= f_c + k_f A_m \cos(2\pi f_m t) \\
&= f_c + \Delta f \cdot \cos(2\pi f_m t)
\end{aligned}
$$

where $\Delta f = k_f A_m$ is the frequency deviation.

$$
\begin{aligned}
\Rightarrow s(t) &= A_c \cos\left[2\pi \int_0^t f_i(\tau) d\tau\right] \\
&= A_c \cos\left[2\pi \int_0^t [f_c + \Delta f \cdot \cos(2\pi f_m \tau)] d\tau\right] \\
&= A_c \cos\left[2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t)\right]
\end{aligned}
$$

where $\beta = \Delta f / f_m$ is often called the modulation index of FM signal.

- *Modulation index* β is the largest deviation from $2\pi f_c t$ in FM system.

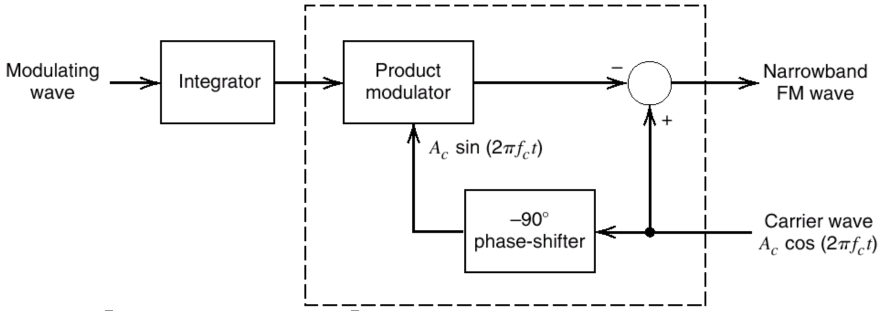
$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

- As a result,

$$f_c - \beta f_m = f_c - \Delta f \le f_i(t) = f_c + \Delta f \cdot \cos(2\pi f_m t) \le f_c + \Delta f = f_c + \beta f_m$$

1. A *small* β corresponds to a *narrowband* FM.
2. A *large* β corresponds to a *wideband* FM.

2.7 Narrowband Frequency Modulation



$$
\begin{aligned}
s(t) &= A_c \cos\left[2\pi f_c t + \beta \sin(2\pi f_m t)\right] \\
&= A_c \cos(2\pi f_c t)\cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t)\sin[\beta \sin(2\pi f_m t)] \\
&\approx A_c \cos(2\pi f_c t) - A_c \sin(2\pi f_c t)\beta \sin(2\pi f_m t) \quad \text{(Often, } \beta < 0.3.\text{)}
\end{aligned}
$$

2.7 Narrowband Frequency Modulation

- Comparison between approximate narrowband FM modulation and AM (DSB-C) modulation

$$
\begin{aligned}
s_{FM}(t) &\approx A_c \cos(2\pi f_c t) - A_c \sin(2\pi f_c t)\beta \sin(2\pi f_m t) \\
&= A_c \cos(2\pi f_c t) + \frac{\beta A_c}{2}\cos(2\pi(f_c + f_m)t) - \frac{\beta A_c}{2}\cos(2\pi(f_c - f_m)t)
\end{aligned}
$$

$$
\begin{aligned}
s_{AM}(t) &= A_c[1 + k_a m(t)]\cos(2\pi f_c t) \\
&= A_c[1 + k_a A_m \cos(2\pi f_m t)]\cos(2\pi f_c t) \\
&= A_c \cos(2\pi f_c t) + \frac{k_a A_m}{2}\cos(2\pi(f + f_m)t) + \frac{k_a A_m}{2}\cos(2\pi(f - f_m)t)
\end{aligned}
$$

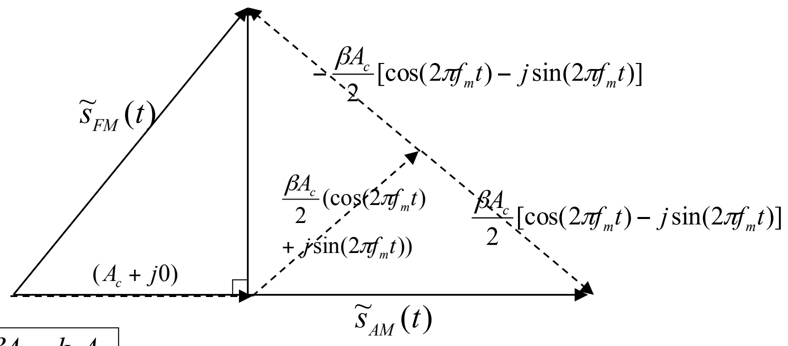
2.7 Narrowband Frequency Modulation

- Represent them in terms of their low-pass isomorphism.

$$\tilde{s}_{FM}(t) = (A_c + j0) + \frac{\beta A_c}{2}[\cos(2\pi f_m t) + j\sin(2\pi f_m t)]$$
$$- \frac{\beta A_c}{2}[\cos(2\pi f_m t) - j\sin(2\pi f_m t)]$$
$$\tilde{s}_{AM}(t) = (A_c + j0) + \frac{k_a A_m}{2}[\cos(2\pi f_m t) + j\sin(2\pi f_m t)]$$
$$+ \frac{k_a A_m}{2}[\cos(2\pi f_m t) - j\sin(2\pi f_m t)]$$

2.7 Narrowband Frequency Modulation

- Phaser diagram



Let $\beta A_c = k_a A_m$.

2.7 Spectrum of Single-Tone FM Modulation

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$
$$= \mathrm{Re}\{A_c \exp(j[2\pi f_c t + \beta \sin(2\pi f_m t)])\}$$
$$= \mathrm{Re}\{\tilde{s}(t)\exp(j2\pi f_c t)\}$$
$$\Rightarrow \tilde{s}(t) = A_c \exp(j[\beta \sin(2\pi f_m t)])$$

(See Slide 1-247)

$$\boxed{\sum_{n=-\infty}^{\infty} J_n(x) e^{jn\phi} = e^{jx\sin(\phi)}}$$

$$\Rightarrow \tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi f_m n t}$$

where $J_n(\cdot)$ is the nth order Bessel function of the first kind.

2.7 Spectrum of Single-Tone FM Modulation

$$\Rightarrow \tilde{S}(f) = \int_{-\infty}^{\infty} \tilde{s}(t) e^{-j2\pi f t} dt$$
$$= \int_{-\infty}^{\infty} \left(A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi f_m n t} \right) e^{-j2\pi f t} dt$$
$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \int_{-\infty}^{\infty} e^{-j2\pi (f - n f_m) t} dt$$
$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \delta(f - n f_m)$$

2.7 Spectrum of Single-Tone FM Modulation

Consequently,

$$
\begin{aligned}
S(f) &= \frac{1}{2}\left[\tilde{S}(f-f_c)+\tilde{S}^*(-f-f_c)\right]\\
&= \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f-f_c-nf_m)+\delta(-f-f_c-nf_m)\right]\\
&= \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)\right]
\end{aligned}
$$

2.7 Spectrum of Single-Tone FM Modulation

- The power of $s(t)$
  - By definition, the time-average autocorrelation function is given by:

$$
\bar{R}_s(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[s(t)s(t+\tau)]dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} s(t)s(t+\tau)dt
$$

  - Hence, the power of $s(t)$ is equal to:

$$
\begin{aligned}
\bar{R}_s(0) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} s^2(t)dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A_c^2\cos^2\left[2\pi f_c t+\beta\sin(2\pi f_m t)\right]dt\\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A_c^2\,\frac{1+\cos\left[4\pi f_c t+2\beta\sin(2\pi f_m t)\right]}{2}dt \approx \frac{A_c^2}{2}
\end{aligned}
$$

2.7 Spectrum of Single-Tone FM Modulation

- The time-average power spectral density of a deterministic signal $s(t)$ is given by (cf. Slide 1-117)

$$\overline{PSD}(f) = \lim_{T\to\infty} \frac{1}{2T} S(f) S_{2T}^*(f)$$

where $S_{2T}(f)$ is the Fourier transform of $s(t)\cdot \mathbf{1}\{|t|\le T\}$.

  - From

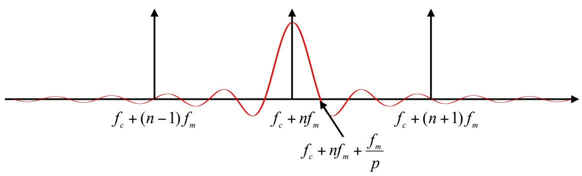
$$s_{2T}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_c + nf_m)t)\cdot \mathbf{1}\{|t|\le T\}$$

we obtain:

$$S_{2T}(f) = A_c T \sum_{n=-\infty}^{\infty} J_n(\beta)\left[\mathrm{sinc}(2T(f - f_c - nf_m)) + \mathrm{sinc}(2T(f + f_c + nf_m))\right]$$

For simplicity, assume that $2T$ *increases along the multiple of* $1/f_m$, i.e., $2T = p/f_m$, where p is an integer. Also assume that f_c is a multiple of f_m, i.e., $f_c = qf_m$, where q is an integer. Then

$$\begin{aligned}
&\overline{PSD}(f)\\
&= \lim_{T\to\infty} \frac{1}{2T} S(f) S_{2T}^*(f)\\
&= \lim_{p\to\infty} \frac{A_c^2}{4} \sum_{k=-\infty}^{\infty} J_k(\beta)\left[\delta(f - f_c - kf_m) + \delta(f + f_c + kf_m)\right]\\
&\times \sum_{n=-\infty}^{\infty} J_n(\beta)\left[\mathrm{sinc}(p(f - f_c - nf_m)/f_m) + \mathrm{sinc}(p(f + f_c + nf_m)/f_m)\right]
\end{aligned}$$



$$\overline{PSD}(f)=\frac{A_c^2}{4}\lim_{p\to\infty}\left\{\sum_{n=-\infty}^{\infty}J_n(\beta)\text{sinc}(p(f-f_c-nf_m)/f_m)\sum_{k=-\infty}^{\infty}J_k(\beta)\delta(f-f_c-kf_m)\right.$$
$$+\sum_{n=-\infty}^{\infty}J_n(\beta)\text{sinc}(p(f-f_c-nf_m)/f_m)\sum_{k=-\infty}^{\infty}J_k(\beta)\delta(f+f_c+kf_m)$$
$$+\sum_{n=-\infty}^{\infty}J_n(\beta)\text{sinc}(p(f+f_c+nf_m)/f_m)\sum_{k=-\infty}^{\infty}J_k(\beta)\delta(f-f_c-kf_m)$$
$$\left.+\sum_{n=-\infty}^{\infty}J_n(\beta)\text{sinc}(p(f+f_c+nf_m)/f_m)\sum_{k=-\infty}^{\infty}J_k(\beta)\delta(f+f_c+kf_m)\right\}$$
$$=\frac{A_c^2}{4}\left\{\sum_{n=-\infty}^{\infty}J_n^2(\beta)\delta(f-f_c-nf_m)+\sum_{n=-\infty}^{\infty}J_n(\beta)J_{-n-2q}(\beta)\delta(f-f_c-nf_m)\right.$$
$$\left.+\sum_{n=-\infty}^{\infty}J_n(\beta)J_{-n-2q}(\beta)\delta(f+f_c+nf_m)+\sum_{n=-\infty}^{\infty}J_n^2(\beta)\delta(f+f_c+nf_m)\right\}$$
$$\approx\frac{A_c^2}{4}\left\{\sum_{n=-\infty}^{\infty}J_n^2(\beta)[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)]\right\}$$

2.7 Average Power of Single-Tone FM Signal

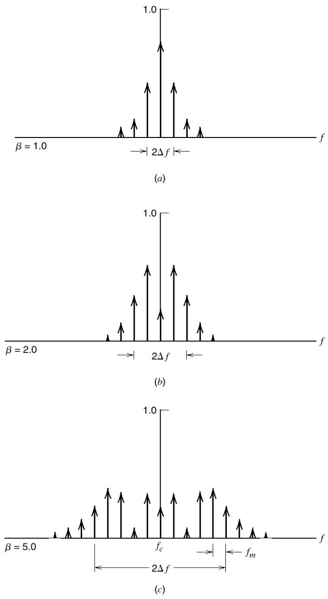
- Hence, the power of a single-tone FM signal is given by:

$$\int_{-\infty}^{\infty}\overline{PSD}(f)df=\frac{A_c^2}{2}\left(\sum_{n=-\infty}^{\infty}J_n^2(\beta)+\sum_{n=-\infty}^{\infty}J_n(\beta)J_{-n-2q}(\beta)\right)$$
$$=\frac{A_c^2}{2}\left(1+\sum_{n=-\infty}^{\infty}(-1)^nJ_n(\beta)J_{n+2q}(\beta)\right)$$
$$\approx\frac{A_c^2}{2}$$

- **Question**: Can we use $2\Delta f$ to be the bandwidth of a single-tone FM signal?

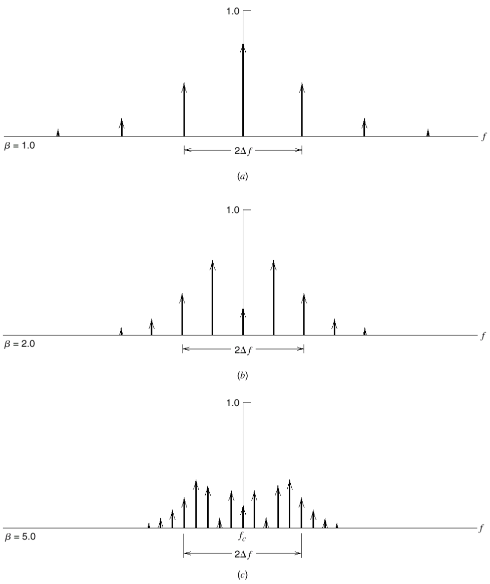
Example 2.2

- Fix f_m and k_f, but vary $\beta = \Delta f/f_m = k_f A_m/f_m$.

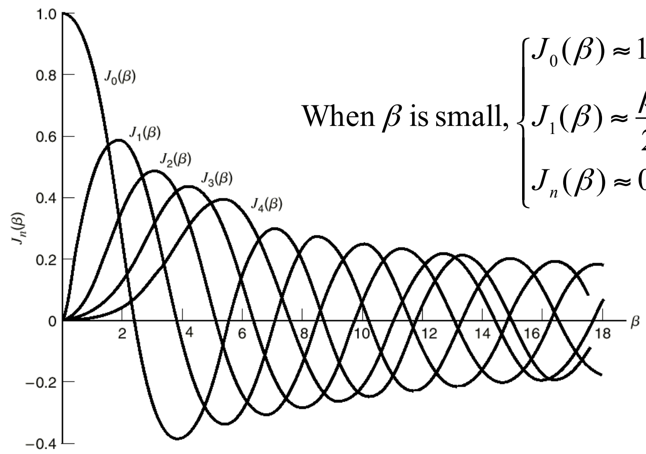


Example 2.2

- Fix A_m and k_f, but vary $\beta = \Delta f/f_m = k_f A_m/f_m$.



2.7 Spectrum of Narrowband Single-Tone FM Modulation



When β is small, $\begin{cases} J_0(\beta) \approx 1 \\ J_1(\beta) \approx \dfrac{\beta}{2} \\ J_n(\beta) \approx 0 \text{ for } n > 2 \end{cases}$.

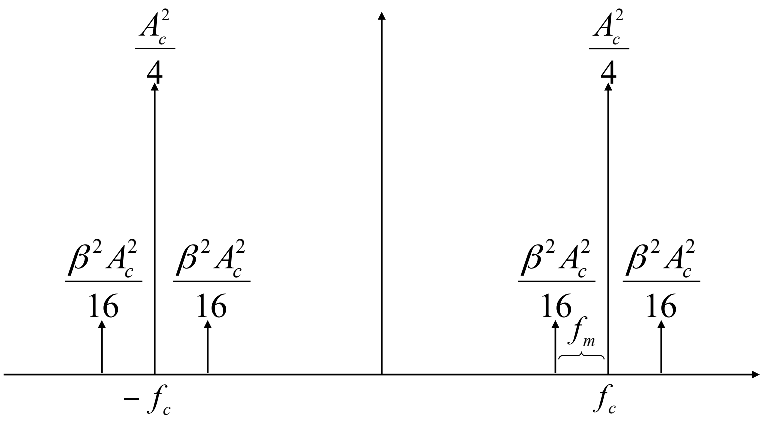
2.7 Spectrum of Narrowband Single-Tone FM Modulation

- This results in an approximate spectrum for narrowband single-tone FM signal spectrum as

$$\begin{aligned}\overline{PSD}(f) &\approx \frac{A_c^2}{4}\sum_{n=-\infty}^{\infty} J_n^2(\beta)\left[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)\right]\\ &\approx \frac{A_c^2}{4}J_{-1}^2(\beta)\left[\delta(f-f_c+f_m)+\delta(f+f_c-f_m)\right]\\ &\quad+\frac{A_c^2}{4}J_0^2(\beta)\left[\delta(f-f_c)+\delta(f+f_c)\right]\\ &\quad+\frac{A_c^2}{4}J_1^2(\beta)\left[\delta(f-f_c-f_m)+\delta(f+f_c+f_m)\right]\end{aligned}$$

$$J_n(\beta) = (-1)^n J_{-n}(\beta) \Rightarrow J_n^2(\beta) = J_{-n}^2(\beta)$$

$$\begin{aligned}
\overline{PSD}(f) &= \frac{A_c^2}{4} J_1^2(\beta)\left[\delta(f - f_c + f_m) + \delta(f + f_c - f_m)\right] \\
&\quad + \frac{A_c^2}{4} J_0^2(\beta)\left[\delta(f - f_c) + \delta(f + f_c)\right] \\
&\quad + \frac{A_c^2}{4} J_1^2(\beta)\left[\delta(f - f_c - f_m) + \delta(f + f_c + f_m)\right] \\
&= \frac{\beta^2 A_c^2}{16}\left[\delta(f - f_c + f_m) + \delta(f + f_c - f_m)\right] \\
&\quad + \frac{A_c^2}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right] \\
&\quad + \frac{\beta^2 A_c^2}{16}\left[\delta(f - f_c - f_m) + \delta(f + f_c + f_m)\right]
\end{aligned}$$



2.7 Transmission Bandwidth of FM signals

- Carson's rule – An empirical bandwidth
  - An empirical rule for Transmission Bandwidth of FM signals
    - For large β, the bandwidth is essentially $2\Delta f$.
    - For small β, the bandwidth is effectively $2f_m$.
    - So Carson proposes that:

$$B_T \approx 2\Delta f + 2f_m = 2\Delta f\left(1 + \frac{1}{\beta}\right)$$

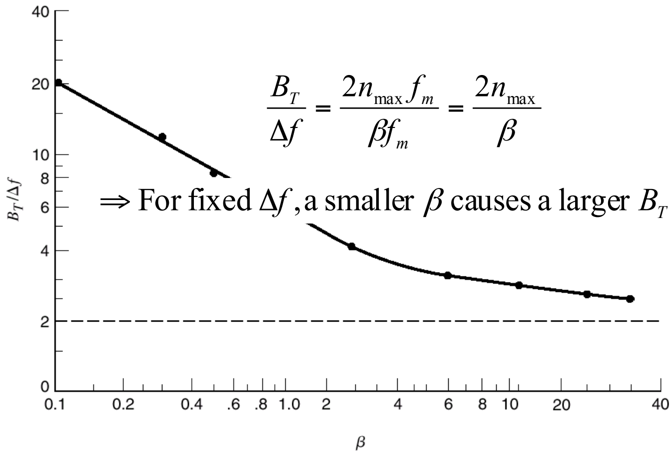
2.7 Transmission Bandwidth of FM signals

- "Universal-curve" transmission bandwidth
  - The transmission bandwidth of an FM wave is the *minimum* separation between two frequencies beyond which none of the side frequencies is greater than 1% of the carrier amplitude obtained when the modulation is removed.

$$S(f) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]$$

$$A_c\cos(2\pi f_c t) \rightarrow \frac{A_c}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

$\Rightarrow B_T = 2n_{\max} f_m$, where $n_{\max} = \max\left\{n : \frac{A_c}{2}|J_n(\beta)| > 0.01\frac{A_c}{2}\right\}$.



$$\frac{B_T}{\Delta f} = \frac{2n_{\max} f_m}{\beta f_m} = \frac{2n_{\max}}{\beta}$$

$\Rightarrow$ For fixed Δf, a smaller β causes a larger B_T.

β	$2n_{\max}$	$B_T/\Delta f$
0.1	2	20.0
0.3	4	13.3
0.5	4	8.0
1.0	6	6.0
2.0	8	4.0
5.0	16	3.2
10.0	28	2.8
20.0	50	2.5
30.0	70	2.3

2.7 Bandwidth of a General FM wave

- Now suppose $m(t)$ is no longer a single tone but a general message signal of bandwidth W.
  - Hence, the "worst-case" tone is $f_m = W$.
    - For nonsinusoidal modulation, the *deviation ratio* $D = \Delta f / W$ is used instead of the *modulation index* β.
    - The *derivation ratio* D plays the same role for nonsinusoidal modulation as the *modulation index* β for the case of sinusoidal modulation.
  - We can then use Carson's rule or "universal curve" to determine the transmission bandwidth B_T.

2.7 Bandwidth of a General FM wave

- Final notes
  - Carson's rule usually underestimates the transmission bandwidth.
  - Universal curve is too conservative in bandwidth estimation.
  - So, a choice of a transmission bandwidth in-between is acceptable for most practical purposes.

Example 2.3

- FM radio in North America requires the maximum frequency derivation $\Delta f = 75$ kHz.
  - If some message signal has bandwidth $W = 15$ kHz, then the deviation ratio $D = \Delta f / W = 75/15 = 5$.
  - Then

$$B_{T,\text{Carson}} = 2\Delta f\left(1 + \frac{1}{D}\right) = 2 \times 75\left(1 + \frac{1}{D}\right) = 180\,\text{kHz}$$

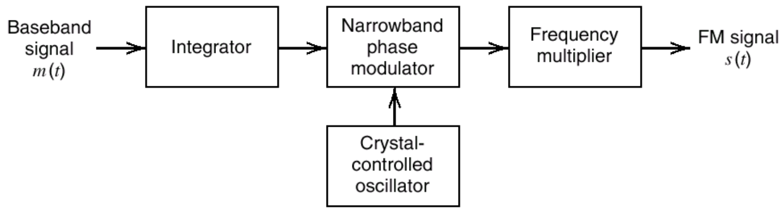
$$B_{T,\text{Universal Curve}} = \frac{2n_{\max}}{D}\Delta f = \frac{16}{5}75 = 240\,\text{kHz}$$

Example 2.3

- In practice, a bandwidth of 200 kHz is allocated to each FM transmitter.
- So Carson's rule underestimates B_T, while "Universal Curve" overestimates B_T.

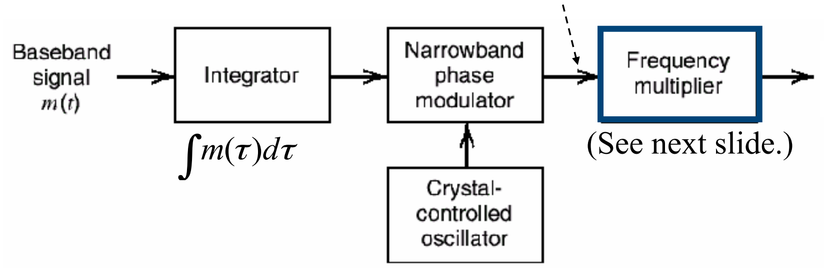
2.7 Generation of FM Signals

- Direct FM
  - Carrier frequency is directly varied in accordance with the message signal as accomplished using a voltage-controlled oscillator.
- Indirect FM
  - The message is first integrated and sent to a phase modulator.
  - So, the carrier frequency is not directly varied in accordance to the message signal.



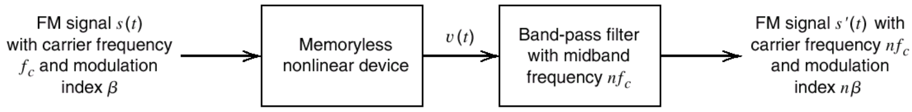
2.7 Generation of FM Signals

$$s(t) = A_c \cos[2\pi f_c t + 2\pi k_f \int m(\tau)d\tau]$$



2.7 Generation of FM Signals

- Frequency multiplier

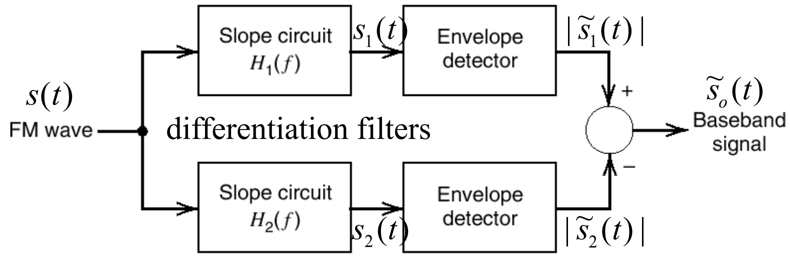


$$\begin{aligned} v(t) &= a_1 s(t) + a_2 s^2(t) + \cdots + a_n s^n(t) \\ &= \sum_{i=1}^{n} A_i \cos\left(2\pi i f_c t + 2\pi i k_f \int_0^t m(\tau)d\tau\right) \quad \text{(See the next slide)} \\ &\xrightarrow{\text{Bandpass}} A_n \cos\left(2\pi n f_c t + 2\pi n k_f \int_0^t m(\tau)d\tau\right) \\ &= A_c' \cos\left(2\pi f_c' t + 2\pi k_f' \int_0^t m(\tau)d\tau\right) \end{aligned}$$

$$
\begin{cases}
\cos^2(x) = \frac{1}{2}(\cos(2x) + 1) \\
\cos^3(x) = \frac{1}{4}(\cos(3x) + 3\cos(x)) \\
\cos^4(x) = \frac{1}{8}(\cos(4x) + 4\cos(2x) + 3\cos(x)) \\
\cos^5(x) = \frac{1}{16}(\cos(5x) + 5\cos(3x) + 10\cos(x)) \\
\cdots
\end{cases}
$$

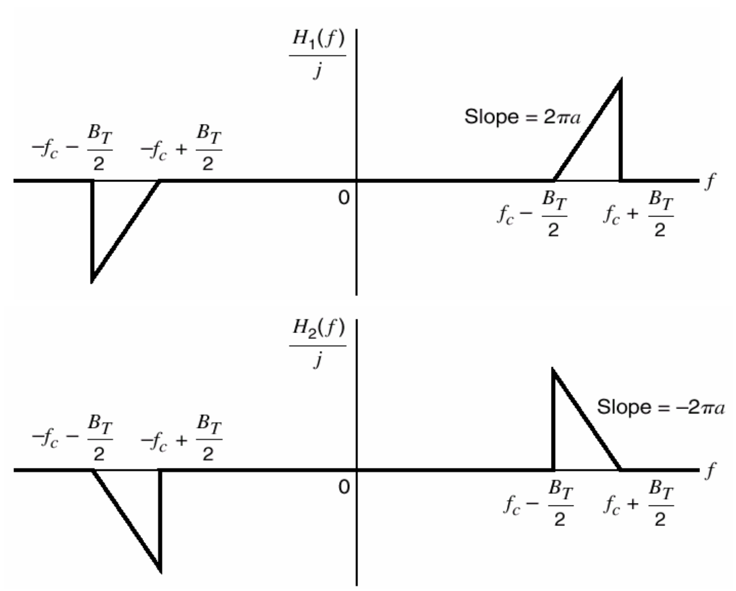
2.7 Demodulation of FM Signals

- Indirect Demodulation – Phase-locked loop
  - Will be introduced in Section 2.14
- Direct Demodulation
  - Balanced frequency discriminator



$$H_1(f) = \begin{cases} j2\pi a\left(f - f_c + \dfrac{B_T}{2}\right), & |f - f_c| \le \dfrac{B_T}{2} \\ j2\pi a\left(f + f_c - \dfrac{B_T}{2}\right), & |f + f_c| \le \dfrac{B_T}{2} \\ 0, & \text{elsewhere} \end{cases}$$

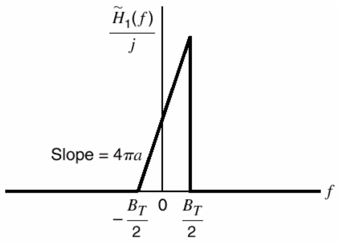
$$H_2(f) = \begin{cases} -j2\pi a\left(f + f_c + \dfrac{B_T}{2}\right), & |f + f_c| \le \dfrac{B_T}{2} \\ -j2\pi a\left(f - f_c - \dfrac{B_T}{2}\right), & |f - f_c| \le \dfrac{B_T}{2} \\ 0, & \text{elsewhere} \end{cases}$$



2.7 Analysis of Direct Demodulation in terms of Low-Pass Equivalences

$$\tilde{H}_1(f) = \begin{cases} 2H_1(f+f_c), & |f| \le \dfrac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} j4\pi a\left(f + \dfrac{B_T}{2}\right), & |f| \le \dfrac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$



$$\Rightarrow \tilde{S}_1(f) = \frac{1}{2}\tilde{H}_1(f)\tilde{S}(f) = \begin{cases} j2\pi a\left(f + \dfrac{B_T}{2}\right)\tilde{S}(f), & |f| \le \dfrac{B_T}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$\Rightarrow \tilde{s}_1(t) = a\left[\frac{d\tilde{s}(t)}{dt} + j\pi B_T \tilde{s}(t)\right]$$

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

$$\Rightarrow \tilde{s}(t) = A_c \exp\left[j2\pi k_f \int_0^t m(\tau)d\tau\right]$$

$$\Rightarrow \tilde{s}_1(t) = a\left[\frac{d\tilde{s}(t)}{dt} + j\pi B_T \tilde{s}(t)\right]$$

$$= a\left[\left(jA_c 2\pi k_f m(t)\exp\left[j2\pi k_f \int_0^t m(\tau)d\tau\right]\right) + j\pi B_T\left(A_c \exp\left[j2\pi k_f \int_0^t m(\tau)d\tau\right]\right)\right]$$

$$= j\pi B_T a A_c \left[1 + \frac{2k_f}{B_T} m(t)\right] \exp\left[j2\pi k_f \int_0^t m(\tau)d\tau\right]$$

$$\Rightarrow s_1(t) = \mathrm{Re}\{\tilde{s}_1(t)\exp(j2\pi f_c t)\}$$

$$= \mathrm{Re}\left\{ j\pi B_T a A_c \left[1 + \frac{2k_f}{B_T} m(t)\right] \exp\left[j2\pi k_f \int_0^t m(\tau)d\tau \right] \exp(j2\pi f_c t) \right\}$$

$$= -\pi B_T a A_c \left[1 + \frac{2k_f}{B_T} m(t)\right] \sin\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right)$$

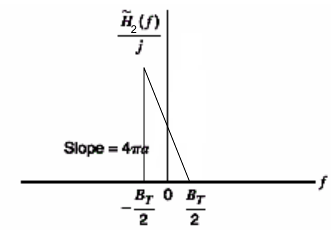
$$= \pi B_T a A_c \left[1 + \frac{2k_f}{B_T} m(t)\right] \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau + \frac{\pi}{2}\right)$$

$\Rightarrow$ If $\left|\frac{2k_f}{B_T} m(t)\right| < 1$ and $f_c >> W$, then envelope detector can be used to obtain the amplitude of the lowpass equivalent message.

$$|\tilde{s}_1(t)| = \pi B_T a A_c \left[1 + \frac{2k_f}{B_T} m(t)\right]$$

Similarly,

$$\tilde{H}_2(f) = \begin{cases} -j4\pi a\left(f - \frac{B_T}{2}\right), & |f| \le \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$



$$\Rightarrow \tilde{S}_2(f) = \frac{1}{2}\tilde{H}_2(f)\tilde{S}(f) = \begin{cases} -j2\pi a\left(f - \frac{B_T}{2}\right)\tilde{S}(f), & |f| \le \frac{B_T}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$\Rightarrow \tilde{s}_2(t) = -a\left[\frac{d\tilde{s}(t)}{dt} - j\pi B_T \tilde{s}(t)\right]$$

$$= j\pi B_T a A_c \left[1 - \frac{2k_f}{B_T} m(t)\right] \exp\left[j2\pi k_f \int_0^t m(\tau)d\tau \right]$$

$$\Rightarrow s_2(t) = \mathrm{Re}\{\tilde{s}_2(t)\exp(j2\pi f_c t)\}$$

$$= \pi B_T a A_c \left[1 - \frac{2k_f}{B_T} m(t)\right] \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau + \frac{\pi}{2}\right)$$

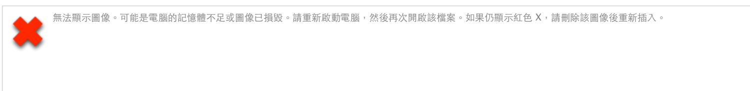
$$\Rightarrow |\tilde{s}_2(t)| = \pi B_T a A_c \left[1 - \frac{2k_f}{B_T} m(t)\right]$$

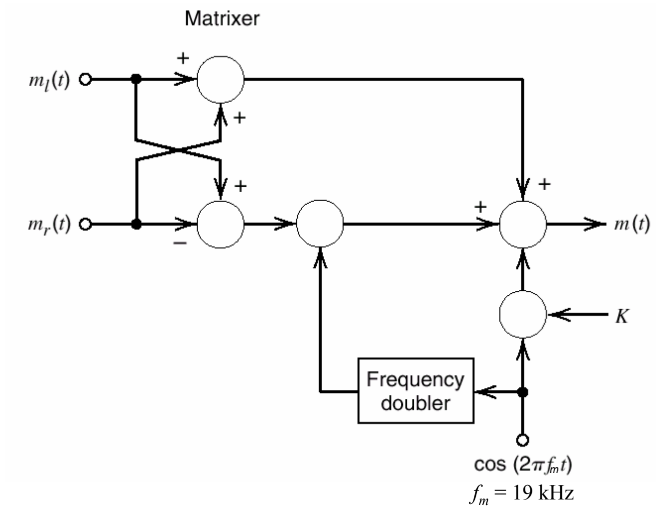
$$\Rightarrow \tilde{s}_o(t) = |\tilde{s}_1(t)| - |\tilde{s}_2(t)| = 4\pi k_f a A_c m(t)$$

Final Note: a is a parameter of the filters, which can be used to adjust the amplitude of the resultant output.

2.7 FM Stereo Multiplexing

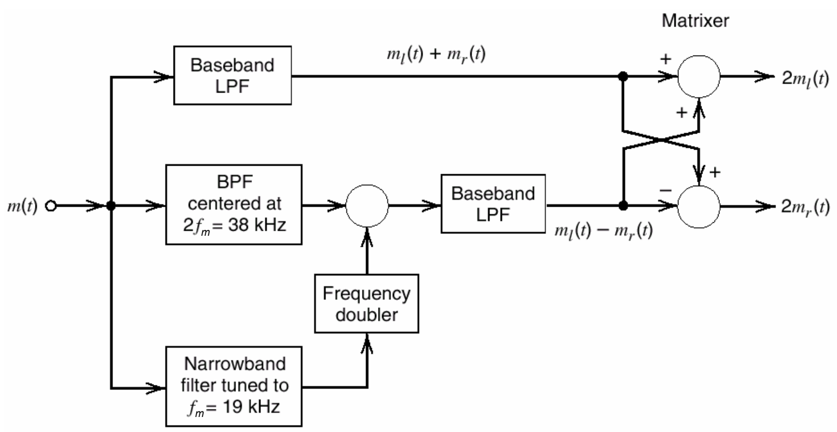
- How to do *Stereo Transmission* in FM radio?
  - Two requirements:
    - Backward compatible with monophonic radio receivers
    - Operate within the allocated FM broadcast channels
  - To fulfill these requirements, the baseband message signal has to be re-made.





$$m(t) = \underbrace{[m_l(t) + m_r(t)]}_{\text{For monophonic reception}} + [m_l(t) - m_r(t)]\cos(4\pi f_m t) + \underbrace{K\cos(2\pi f_m t)}_{\text{For coherent detection}}$$

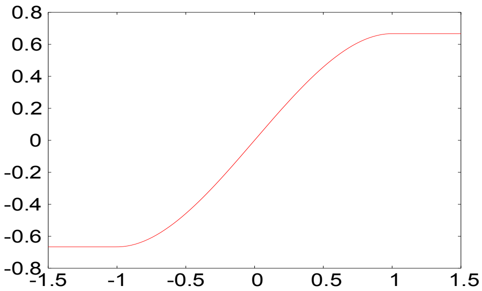
Demultiplexer in receiver of FM stereo.



2.8 Nonlinear Effects in FM Systems

- The channel (including background noise, interference and circuit imperfection) may introduce non-linear effects on the transmission signals.
  - For example, non-linearity due to amplifiers.

$$v_o(t) = \begin{cases} v_i(t) - \frac{1}{3}v_i^3(t), & |v_i(t)| \le 1 \\ \frac{2}{3}, & v_i(t) > 1 \\ -\frac{2}{3}, & v_i(t) < 1 \end{cases}$$



2.8 Nonlinear Effects in FM Systems

- Suppose

$$\begin{cases} v_o(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t) \\ v_i(t) = A_c \cos\left[2\pi f_c t + \phi(t)\right] \\ \phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau \end{cases}$$

Then
$$\begin{aligned} v_o(t) &= a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t) \\ &= a_1 A_c \cos[2\pi f_c t + \phi(t)] + a_1 A_c^2 \cos^2[2\pi f_c t + \phi(t)] \\ &\quad + a_1 A_c^3 \cos^3[2\pi f_c t + \phi(t)] \end{aligned}$$

$$v_o(t) = a_1 A_c \cos[2\pi f_c t + \phi(t)] + \frac{1}{2} a_2 A_c^2 \left(1 + \cos[4\pi f_c t + 2\phi(t)]\right)$$

$$+ \frac{1}{4} a_3 A_c^3 \left(3\cos[2\pi f_c t + \phi(t)] + \cos[6\pi f_c t + 3\phi(t)]\right)$$

$$= \frac{1}{2} a_1 A_c^2 + \underbrace{\left(a_1 A_c + \frac{3}{4} a_3 A_c^3\right) \cos[2\pi f_c t + \phi(t)]}_{B_{T,\text{Carson}} = 2\Delta f + 2W}$$

(Check Slide 2-64.)

$$+ \underbrace{\frac{1}{2} a_2 A_c^2 \cos[4\pi f_c t + 2\phi(t)]}_{B_{T,\text{Carson}} = 4\Delta f + 2W} + \underbrace{\frac{1}{4} a_1 A_c^3 \cos[6\pi f_c t + 3\phi(t)]}_{B_{T,\text{Carson}} = 6\Delta f + 2W}$$

Thus, in order to recover $s(t)$ from $v_o(t)$ using band-pass filter (i.e., to remove $2f_c$ and $3f_c$ terms from $v_o(t)$), it requires:

$$2f_c - (4\Delta f + 2W)/2 > f_c + (2\Delta f + 2W)/2$$

$$\text{or equivalently, } f_c > 3\Delta f + 2W.$$

The filtered output is therefore:

$$v_{o,\text{filtered}}(t) = \left(a_1 A_c + \frac{3}{4} a_3 A_c^3\right) \cos[2\pi f_c t + \phi(t)]$$

- **Observations**
  - Unlike AM modulation, FM modulation is not affected by distortion produced by transmission through a channel with amplitude nonlinearities.
  - So the FM modulation allows the usage of highly non-linear amplifiers and power transmitters.

2.8 AM-to-PM Conversion

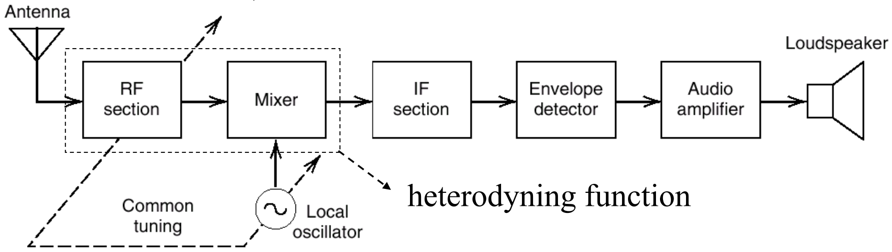
- Although FM modulation is insensitive to amplitude nonlinearity, it is indeed very sensitive to phase nonlinearity.
  - A common type of phase nonlinearity encountered in microwave radio transmission is the *AM-to-PM conversion*.
  - The AM-to-PM conversion is owing to that the *phase characteristic* of amplifiers (or repeaters) also depends on the instantaneous amplitude of the input signal.
    - Notably, the nonlinear amplifiers discussed previously will leave the phase of the input unchanged.
  - Often, it requires that the peak phase change for a 1-dB change in input envelope is less than 2%.

2.9 Superheterodyne Receiver

- A commercial radio communication system contains not only the "transmission" but also some other functions, such as:
  - Carrier-frequency tuning, to select the desired signals
  - Filtering, to separate the desired signal from other unwanted signals
  - Amplifying, to compensate for the loss of signal power incurred in the course of transmission

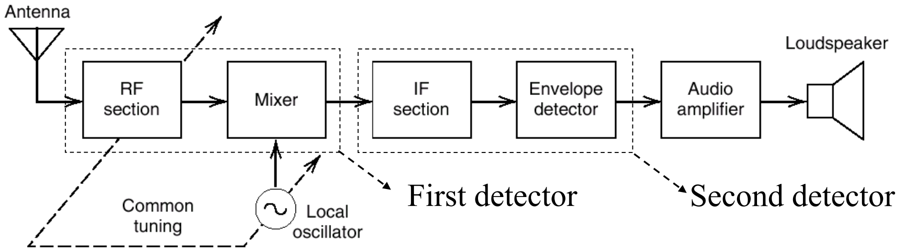
2.9 Superheterodyne Receiver

- A *superheterodyne* receiver or *superhet* is designed to facilitate the fulfillment of these functions, especially the first two.
  - It overcomes the difficulty of having to build a *tunable highly selective and variable* filter (rather a fixed filter is applied on IF section).



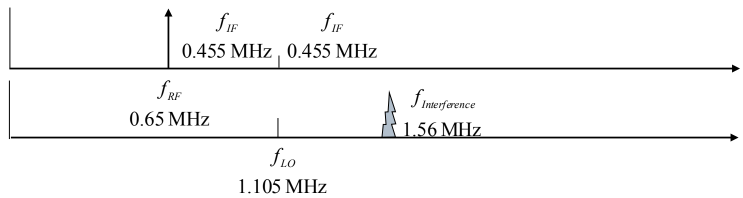
2.9 Superheterodyne Receiver

Example	*AM Radio*	*FM Radio*
RF carrier range	0.535-1.605 MHz	88-108 MHz
Midband frequency of IF section	0.455 MHz	10.7 MHz
IF bandwidth	10 kHz	200 kHz



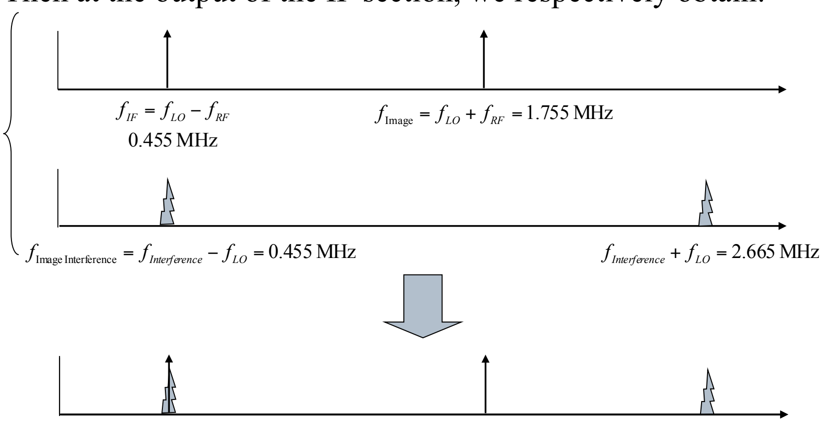
2.9 Image Interference

- Fix f_{IF} and f_{LO} at the receiver end. What is the f_{RF} that will survive at the IF section output?
  - Answer: $f_{RF} = |f_{LO} \pm f_{IF}|$
  - Example. Suppose the receiver use 1.105MHz local oscillator, and receives two RF signals respectively centered at 0.65MHz and 1.56MHz.



2.9 Image Interference

Then at the output of the IF section, we respectively obtain:



2.9 Image Interference

- A cure of image interference is to employ a highly selective stages in the RF session in order to favor the desired signal (at f_{RF}) and discriminate the undesired signal (at $f_{RF} + 2f_{IF}$ or $f_{RF} - 2f_{IF}$).

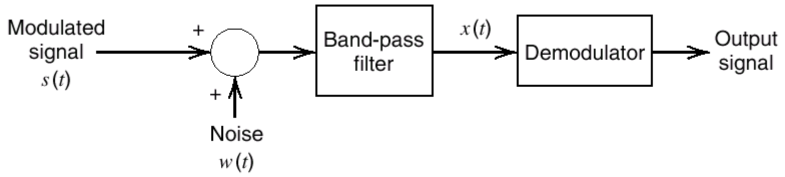
2.9 Advantage of Constant Envelope for FM modulation

- Observations
  - For FM modulation, any variation in amplitude is caused by noise or interference.
  - For FM modulation, the information is resided on the variations of the instantaneous frequency.
  - So we can use an *amplitude limiter* to remove the amplitude variation, but to retain the frequency variation after the IF section.

2.9 Advantage of Constant Envelope for FM modulation

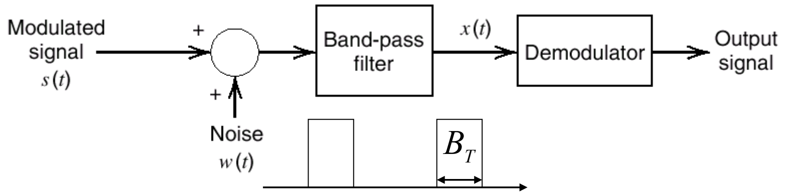
- Amplitude limiter
  - Clipping the modulated wave at the IF section output (almost to the zero axis) to result in a near-rectangular wave.
  - Pass the rectangular wave through a bandpass filter centered at f_{IF} to suppress harmonics (due to clipping).
  - Then the filter output *retains the frequency variation with constant amplitude*.

2.10 Noise in CW Modulation Systems



- To simplify the system analysis, we assume:
  - ideal band-pass filter (that is just wide enough to pass the modulated signal $s(t)$ without distortion),
  - ideal demodulator,
  - Gaussian distributed white noise process.
- So the only source of imperfection is from the noise.

2.10 Noise in CW Modulation Systems



So after passing through the ideal bandpass filter, $s(t)$ is unchanged but $w(t)$ becomes a narrowband noise $n(t)$. Hence,

$$x(t) = s(t) + n(t),$$

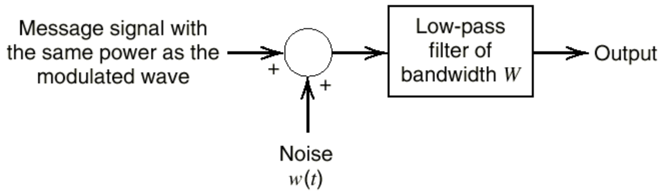
where $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$.

2.10 Noise in CW Modulation Systems

- Input signal-to-noise ratio (SNR_I)
  - The ratio of the average power of the modulated signal $s(t)$ to the average power of the filtered noise $n(t)$.
- Output signal-to-noise ratio (SNR_O)
  - The ratio of the average power of the demodulated message signal to the average power of the noise, measured at the receiver output.

2.10 Noise in CW Modulation Systems

- It is sometimes advantageous to look at the lowpass equivalent model.
- Channel signal-to-noise ratio (SNR_C)
  - The ratio of the average power of the modulated signal $s(t)$ to the average power of the channel noise in the message bandwidth, measured at the receiver input (as illustrated below).



2.10 Noise in CW Modulation Systems

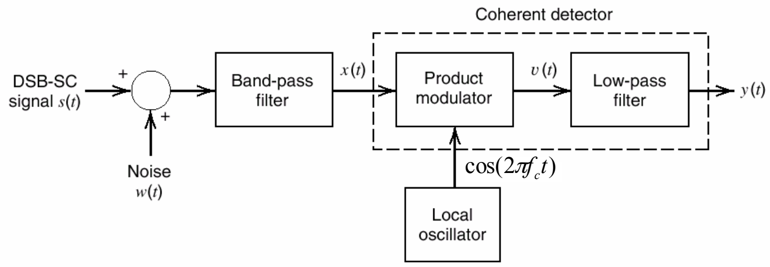
- Notes
  - SNR_C is nothing to do with the receiver structure, but depends on the channel characteristic and modulation approach.
  - SNR_O is however receiver-structure dependent.
- Finally, define the *figure of merit* for the receiver as:

$$\text{figure of merit} = \frac{SNR_O}{SNR_C}$$

2.11 Noise in Linear Receivers Using Coherent Detection

- Recall that for AM demodulation
  - when the carrier is suppressed, **linear** coherent detection is used. (Section 2.11)
  - when the carrier is additionally transmitted, **nonlinear** envelope detection is used. (Section 2.12)
- The noise analysis of the above two cases are respectively addressed in Sections 2.11 and 2.12.

2.11 Noise in Linear Receivers Using Coherent Detection



$m(t)$: stationary with zero meam and PSD $S_M(f)$ bandlimited to W.

$s(t) = CA_c m(t)\cos(2\pi f_c t)$

2.11 Noise in Linear Receivers Using Coherent Detection

- Average signal power

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[s^2(t)]dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E\left[C^2A_c^2\cos^2(2\pi f_c t)m^2(t)\right]dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}C^2A_c^2\cos^2(2\pi f_c t)E\left[m^2(t)\right]dt$$

$$= C^2A_c^2P\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos^2(2\pi f_c t)dt$$

$$= \frac{C^2A_c^2P}{2}$$

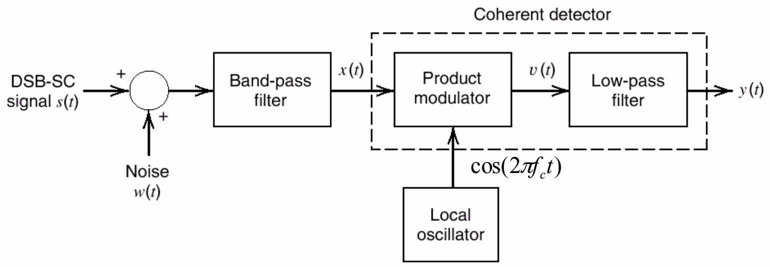
where $P = E[m^2(t)] = \int_{-W}^{W}S_M(f)df$ is the message power.

2.11 Noise in Linear Receivers Using Coherent Detection

- Noise power in the message bandwidth

$$\int_{-W}^{W}S_w(f)df = \int_{-W}^{W}\frac{N_0}{2}df = WN_0$$

2.11 Noise in Linear Receivers Using Coherent Detection



- Channel SNR for DSB-SC and coherent detection (observed at $x(t)$)

$$SNR_{C,\text{DSB-SC}} = \frac{C^2A_c^2P/2}{WN_o} = \frac{C^2A_c^2P}{2WN_o}$$

- Next, we calculate output SNR (observed at $y(t)$) under the condition that the transmitter and the receiver are perfectly synchronized.

$$\begin{aligned} x(t) &= s(t) + w(t) \\ &= CA_c m(t)\cos(2\pi f_c t) + w_I(t)\cos(2\pi f_c t) - w_Q(t)\sin(2\pi f_c t) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad v(t) &= x(t)\cos(2\pi f_c t) \\ &= [CA_c m(t)\cos(2\pi f_c t) + w_I(t)\cos(2\pi f_c t) - w_Q(t)\sin(2\pi f_c t)]\cos(2\pi f_c t) \\ &= CA_c m(t)\cos^2(2\pi f_c t) + w_I(t)\cos^2(2\pi f_c t) - w_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\ &\xrightarrow{\text{LowPass}} \frac{1}{2}CA_c m(t) + \frac{1}{2}w_I(t) \end{aligned}$$

(See Slide 1-219.)

$$\Rightarrow y(t) = \frac{1}{2}CA_c m(t) + \frac{1}{2}w_I(t)$$

(Recall $E[w^2(t)] = E[w_I^2(t)] = E[w_Q^2(t)]$.)

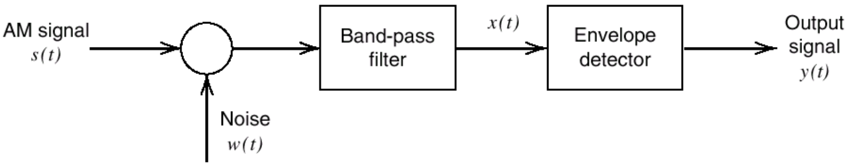
$$\Rightarrow SNR_{O,\text{DSB-SC}} = \frac{E\left[C^2A_c^2m^2(t)/4\right]}{E\left[w_I^2(t)/4\right]} = \frac{C^2A_c^2P}{E[w^2(t)]} = \frac{C^2A_c^2P}{2WN_0}$$

$\Rightarrow$ Figure of merit for DSB-SC and coherent detection = 1.

Similar derivation on SSB and coherent detection yields the same *figure of merit*.

- Conclusions
  - Coherent detection for SSB performs the same as coherent detection for DSB-SC.
  - There is no SNR degradation for SSB and DSB-SC coherent receivers. The only effect of these modulation and demodulation processes is to translate the message signal to a different frequency band to facilitate its transmission over a band-pass channel.
  - No trade-off between noise performance and bandwidth. This may become a problem when high quality transceiving is required.

2.12 Noise in AM Receivers Using Envelope Detection



$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[s^2(t)]dt = A_c^2 E\left[(1 + k_a m(t))^2\right]\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos^2(2\pi f_c t)dt$$

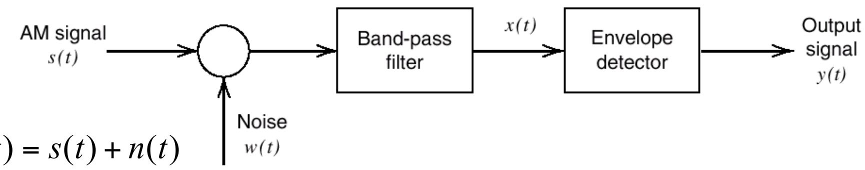
$$= \frac{A_c^2}{2}(1 + k_a^2 P) \quad \text{(Assume } m(t) \text{ zero mean.)}$$

Also, $\int_{-W}^{W} S_w(f)df = \int_{-W}^{W} \frac{N_0}{2} df = WN_0$

- Hence, channel SNR for DSB-C is equal to:

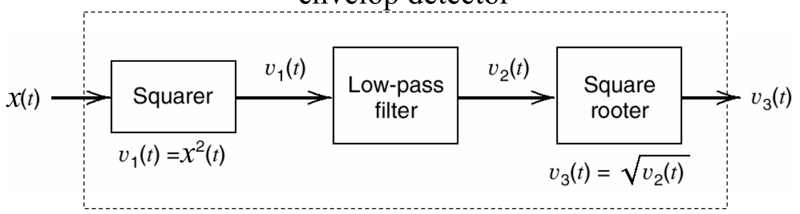
$$\Rightarrow SNR_{C,\text{AM}} = \frac{A_c^2(1+k_a^2 P)}{2WN_o}$$

- Next, we calculate output SNR (observed at $y(t)$) under the condition that the transmitter and the receiver are perfectly synchronized.



$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c[1 + k_a m(t)]\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= [A_c(1 + k_a m(t)) + n_I(t)]\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}$$

envelop detector



$$\begin{aligned} y(t) &= \sqrt{\left(x^2(t)\right)_{\text{LowPass}}} \\ &= \frac{1}{\sqrt{2}}\sqrt{[A_c(1 + k_a m(t)) + n_I(t)]^2 + n_Q^2(t)} \\ &\approx \frac{1}{\sqrt{2}}[A_c(1 + k_a m(t)) + n_I(t)] \quad \text{if } A_c[1 + k_a m(t)] >> |\tilde{n}(t)| \\ &\overset{\text{block DC}}{=} \frac{1}{\sqrt{2}}[A_c k_a m(t) + n_I(t)] \end{aligned}$$

(Refer to Slide 2-136 and 2-138.)

$$\Rightarrow SNR_{O,\mathrm{AM}} = \frac{E\left[A_c^2 k_a^2 m^2(t)/2\right]}{E\left[n_I^2(t)/2\right]} = \frac{A_c^2 k_a^2 P}{E[n^2(t)]} = \frac{A_c^2 k_a^2 P}{2WN_0}$$

$$\Rightarrow \frac{SNR_{O,\mathrm{AM}}}{SNR_{C,\mathrm{AM}}} \approx \frac{A_c^2 k_a^2 P/(2WN_0)}{A_c^2(1+k_a^2 P)/(2WN_o)} = \frac{k_a^2 P}{1+k_a^2 P} < 1$$

- Conclusion
  - Even if the noise power is small compared to the average carrier power at the envelope detector output, the noise performance of a full AM receiver is inferior to that of a DSB-SC receiver due to the wastage of transmitter power.

Example 2.4 Single-Tone Modulation

- Assume $m(t) = A_m \cos(2\pi f_m t)$

$$\Rightarrow s(t) = A_c[1 + k_a A_m \cos(2\pi f_m t)]\cos(2\pi f_c t)$$

Hence,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[s^2(t)]dt = A_c^2 \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[1 + k_a A_m \cos(2\pi f_m t)\right]^2 \cos^2(2\pi f_c t)dt$$

$$= A_c^2 \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\cos^2(2\pi f_c t) + 2k_a A_m \cos(2\pi f_m t)\cos^2(2\pi f_c t)$$

$$+ k_a^2 A_m^2 \cos^2(2\pi f_m t)\cos^2(2\pi f_c t)\Big)dt$$

$$= A_c^2\left(\frac{1}{2} + 0 + \frac{k_a^2 A_m^2}{4}\right) = \frac{A_c^2}{2}(1 + k_a^2 P)$$

where $P = \lim_{T\to\infty} \frac{1}{T}\int_0^T A_m^2 \cos^2(2\pi f_m t)dt = \frac{A_m^2}{2}$.

$\Rightarrow$ Following similar procedure as previous discussion,

$$\frac{SNR_{O,\mathrm{AM}}}{SNR_{C,\mathrm{AM}}} \approx \frac{k_a^2 P}{1+k_a^2 P} = \frac{k_a^2 A_m^2/2}{1+k_a^2 A_m^2/2}.$$

So even if for 100% percent modulation ($k_a A_m = 1$), the figure of merit = 1/3. This means that *an AM system with envelope detection* must transmit **three** times as much average power as *DSB-SC with coherent detector* to achieve the same quality of noise performance.

2.12 Threshold Effect

- What if $A_c[1+k_a m(t)] >> |\tilde{n}(t)|$ is violated (in *AM modulation with envelope detection*)?

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c[1+k_a m(t)]\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= [A_c(1+k_a m(t)) + n_I(t)]\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}$$

$$B >> |\tilde{n}| \Rightarrow \sqrt{(B+n_I)^2 + n_Q^2} = \sqrt{B^2 + 2n_I B + n_I^2 + n_Q^2} \approx \sqrt{B^2 + 2n_I B + n_I^2} \approx B + n_I$$

$$B << |\tilde{n}| \Rightarrow \sqrt{(B+n_I)^2 + n_Q^2} = \sqrt{B^2 + 2n_I B + |\tilde{n}|^2} \approx \sqrt{B^2 + 2|\tilde{n}|B + |\tilde{n}|^2} \approx B + |\tilde{n}|$$

$$y(t)=\sqrt{\left(x^2(t)\right)_{\text{LowPass}}}$$

$\boxed{\text{Assume } A_c[1+k_am(t)] << |\tilde{n}(t)|.}$

$$=\frac{1}{\sqrt{2}}\sqrt{[A_c(1+k_am(t))+n_I(t)]^2+n_Q^2(t)}$$

$$=\frac{1}{\sqrt{2}}\sqrt{A_c^2(1+k_am(t))^2+2n_I(t)A_c(1+k_am(t))+|\tilde{n}(t)|^2}$$

$$\approx\frac{1}{\sqrt{2}}\sqrt{A_c^2(1+k_am(t))^2+2|\tilde{n}(t)|A_c(1+k_am(t))+|\tilde{n}(t)|^2}$$

$$=\frac{1}{\sqrt{2}}\sqrt{(A_c(1+k_am(t))+|\tilde{n}(t)|)^2}$$

$$\approx\frac{1}{\sqrt{2}}(A_c(1+k_am(t))+|\tilde{n}(t)|)$$

$$\overset{\text{block DC}}{=}\frac{1}{\sqrt{2}}[A_ck_am(t)+|\tilde{n}(t)|]$$

$$\sqrt{2}y(t)=\begin{cases}A_ck_am(t)+n_I(t), & A_c[1+k_am(t)]>>|\tilde{n}(t)|\\ A_ck_am(t)+|\tilde{n}(t)|, & A_c[1+k_am(t)]<<|\tilde{n}(t)|\end{cases}$$

$$\Rightarrow SNR_{O,\text{AM}}=\frac{E\left[A_c^2k_a^2m^2(t)\right]}{E\left[|\tilde{n}(t)|^2\right]}=\frac{A_c^2k_a^2P}{E[n_I^2(t)]+E[n_Q^2(t)]}=\frac{A_c^2k_a^2P}{4WN_0}$$

$$\Rightarrow\frac{SNR_{O,\text{AM}}}{SNR_{C,\text{AM}}}\approx\frac{A_c^2k_a^2P/(4WN_0)}{A_c^2(1+k_a^2P)/(2WN_o)}=\frac{k_a^2P}{2(1+k_a^2P)}<\frac{1}{2}$$

$$\begin{cases}\lim_{B\to 0}\left[\sqrt{(B+n_I)^2+n_Q^2}-(B+n_I)\right]=|\tilde{n}|-n_I\\ \lim_{B\to\infty}\left[\sqrt{(B+n_I)^2+n_Q^2}-(B+n_I)\right]=\lim_{B\to\infty}\frac{n_Q}{\left[\sqrt{(B+n_I)^2+n_Q^2}+(B+n_I)\right]}=0\end{cases}$$

$$\Rightarrow\sqrt{(B+n_I)^2+n_Q^2}\approx\begin{cases}(B+n_I)+|\tilde{n}|-n_I=B+|\tilde{n}|\\ B+n_I\end{cases}$$

$$\text{Also }\frac{d}{dB}\left[\sqrt{(B+n_I)^2+n_Q^2}-(B+n_I)\right]=\frac{B+n_I-\sqrt{(B+n_I)^2+n_Q^2}}{\sqrt{(B+n_I)^2+n_Q^2}}<0$$

2.12 Threshold Effect

$$\rho = \frac{A_c^2/2}{2WN_0} = \frac{A_c^2}{4WN_0}$$

- Threshold effect
  - For AM with envelope detection, there exists a *carrier-to-noise ratio* ρ (namely, the power ratio between unmodulated carrier $A_c\cos(2\pi f_c t)$ and the passband noise $n(t)$) below which the noise performance of a detector deteriorates rapidly.

$$\text{SNR}_{O,\text{AM}} = \begin{cases} \dfrac{A_c^2 k_a^2 P}{2WN_0} = 2k_a^2 P\rho, & \text{if } A_c[1+k_a m(t)] \gg |\tilde{n}(t)| \text{ i.e. } \rho \gg 1 \\ \dfrac{A_c^2 k_a^2 P}{4WN_0} = k_a^2 P\rho, & \text{if } A_c[1+k_a m(t)] \ll |\tilde{n}(t)| \text{ i.e. } \rho \ll 1 \end{cases}$$

2.12 General Formula for SNR_O in Envelope Detection

- For envelope detector, the noise is no longer additive; so the original definition of SNR_O (which is based on additive noise) may not be well-applied.
- A new definition should be given:
  - **Definition**. The (general) *output signal-to-noise ratio* for an output $y(t)$ due to a carrier input is defined as

$$SNR_O = \frac{s_o^2}{\text{Var}[y(t)]}$$

where $s_o = E[y(t)] - E[y_o(t)]$, and $y_o(t)$ is equal to $y(t)$ in the presense of noise alone.

$$\boxed{y(t) = s_o + y_o(t)}$$

2.12 General Formula for SNR_O in Envelope Detection

- s_o is named the *mean output signal*.
- Var[$y(t)$] is named the *mean output noise power*.
- **Example**. $y(t) = A + n_I(t)$, where $n_I(t)$ is zero mean..

$$\begin{cases} s_o = E[A + n_I(t)] - E[n_I(t)] = A \\ \mathrm{Var}[y(t)] = \mathrm{Var}[n_I(t)] = E[n_I^2(t)] \end{cases}$$

$$\Rightarrow SNR_O = \frac{A^2}{E[n_I^2(t)]}$$

This somehow shows the backward compatibility of the new definition.

2.12 General Formula for SNR_O in Envelope Detection

- Now for an envelope detector, the output due to a carrier input and additive Gaussian noise channel is given by:

$$y(t) = \sqrt{(A + n_I(t))^2 + n_Q^2(t)}$$

$I_0(\,)$ = modified Bessel function of the first kind of zero order.

$\Rightarrow y(t)$ is Rician distributed with pdf

$$f_{y(t)}(y) = \frac{y}{\sigma_N^2} \exp\left(-\frac{y^2 + A^2}{2\sigma_N^2}\right) I_0\left(\frac{Ay}{\sigma_N^2}\right) \quad \text{for } y \geq 0, \text{ where } \sigma_N^2 = E[\tilde{n}^2].$$

$\Rightarrow y_o(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$ is Rayleigh distributed with pdf

$$f_{y(t)}(y) = \frac{y}{\sigma_N^2} \exp\left(-\frac{y^2}{2\sigma_N^2}\right) \quad \text{for } y \geq 0, \text{ where } \sigma_N^2 = E[\tilde{n}^2].$$

$$
\begin{aligned}
E[y(t)] &= \int_0^\infty y f_{y(t)}(y)dy \\
&= \int_0^\infty \frac{y^2}{\sigma_N^2}\exp\left(-\frac{y^2+A^2}{2\sigma_N^2}\right) I_0\left(\frac{Ay}{\sigma_N^2}\right) dy \\
&= \frac{\sigma_N}{(2\rho)^{3/2}}\exp(-\rho)\int_0^\infty u^2\exp\left(-\frac{u^2}{4\rho}\right) I_0(u)du, \\
&\quad \text{by taking } u = Ay/\sigma_N^2 \text{ and } \rho = A^2/(2\sigma_N^2) = A^2/(4WN_0). \\
E[y_o(t)] &= \int_0^\infty y f_{y_o(t)}(y)dy = \int_0^\infty \frac{y^2}{\sigma_N^2}\exp\left(-\frac{y^2}{2\sigma_N^2}\right) dy \\
&= \sigma_N \int_0^\infty z^2 e^{-z^2/2}dz = \sigma_N \int_0^\infty z\cdot\left(ze^{-z^2/2}\right)dz \\
&= \sigma_N\left(z\cdot\left(-e^{-z^2/2}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-z^2/2}\right)dz\right) \\
&= \sigma_N \int_0^\infty e^{-z^2/2}dz = \sigma_N\sqrt{2\pi}\int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \sigma_N\sqrt{\frac{\pi}{2}}
\end{aligned}
$$

Appendix 4 Confluent Hypergeometric Functions

- The *Kummer confluent hypergeometric function* is a solution of *Kummer's equation*

$$x\frac{d^2y}{dx^2} + (b-x)\frac{dy}{dx} - ay = 0 \quad \text{for } a, b \text{ complex}$$

with boundary conditions $y(0) = 1$ and $y'(0) = a/b$.

- For $b \neq 0$, -1, -2, …., the *Kummer confluent hypergeometric function* is equal to ${}_1F_1(a;b;x)$.

Generalized hypergeometric function

$$ {}_pF_q(\vec{a};\vec{b};x) = \sum_{k=0}^\infty \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\cdot\frac{x^k}{k!}, \text{ where } \begin{cases}(a)_k = a(a+1)\cdots(a+k-1)\\ (a)_0 = 1\end{cases}.$$

A4.2 Properties of the Confluent Hypergeometric Function

1. ${}_1F_1(a;b;x) \approx 1 + \frac{a}{b}x \quad \text{as} \quad x \to 0.$
2. ${}_1F_1(-1;1;x) = 1 - x.$
3. $${}_1F_1(-1/2;1;-x) = \exp\left(-\frac{x}{2}\right) \times \left((1+x)I_0\left(\frac{x}{2}\right) + xI_2\left(\frac{x}{2}\right)\right)$$ $$\approx 2\sqrt{\frac{x}{\pi}} \text{ as } x \to \infty.$$
4. $$\int_0^\infty u^{m-1}\exp(-b^2u^2)I_0(u)du = \frac{\Gamma(m/2)}{2b^m}\left[{}_1F_1\left(\frac{m}{2};1;\frac{1}{4b^2}\right)\right]$$
5. $\exp(-u) \cdot {}_1F_1(\alpha;\beta;u) = {}_1F_1(\beta-\alpha;\beta;-u)$

2.12 General Formula for SNR_O in Envelope Detection

- Hence,

$$E[y(t)] = \frac{\sigma_N}{(2\rho)^{3/2}}\exp(-\rho)\int_0^\infty u^2\exp\left(-\frac{u^2}{4\rho}\right)I_0(u)du$$

$$= \frac{\sigma_N}{(2\rho)^{3/2}}\exp(-\rho)\frac{\Gamma(3/2)}{2(4\rho)^{-3/2}}\left[{}_1F_1\left(\frac{3}{2};1;\rho\right)\right] \quad \textit{By Property 4}$$

$$= \sqrt{\frac{\pi}{2}}\sigma_N\exp(-\rho)\left[{}_1F_1\left(\frac{3}{2};1;\rho\right)\right]$$

$$= \sqrt{\frac{\pi}{2}}\sigma_N\left[{}_1F_1\left(-\frac{1}{2};1;-\rho\right)\right] \quad \textit{By Property 5}$$

2.12 General Formula for SNR_O in Envelope Detection

- As a result,

$$s_o = E[y(t)] - E[y_o(t)] = \sqrt{\frac{\pi}{2}}\sigma_N \left[{}_1F_1\left(-\frac{1}{2};1;-\frac{A^2}{2\sigma_N^2}\right) - 1 \right].$$

- Similarly, we can obtain:

$$\text{Var}[y(t)] = 2\sigma_N^2 \left[{}_1F_1\left(-1;1;-\frac{A^2}{2\sigma_N^2}\right) - \frac{\pi}{4}\left[{}_1F_1\left(-\frac{1}{2};1;-\frac{A^2}{2\sigma_N^2}\right)\right]^2 \right]$$

$$= 2\sigma_N^2 \left[1 + \frac{A^2}{2\sigma_N^2} - \frac{\pi}{4}\left[{}_1F_1\left(-\frac{1}{2};1;-\frac{A^2}{2\sigma_N^2}\right)\right]^2 \right] \quad \textit{By Property 2}$$

2.12 General Formula for SNR_O in Envelope Detection

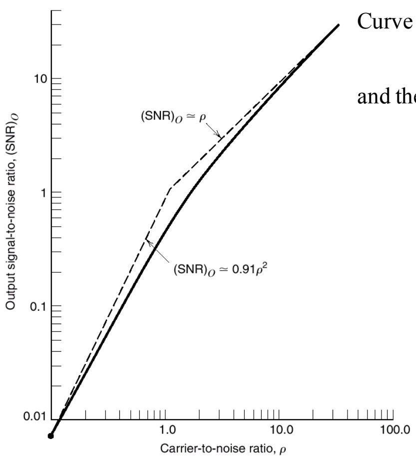
- This concludes to:

$$\begin{aligned}
\mathrm{SNR}_O &= \frac{[{}_1F_1(-1/2;1;-\rho) - 1]^2}{\frac{4}{\pi}(1+\rho) - [{}_1F_1(-1/2;1;-\rho)]^2}, \quad \text{where } \rho = \frac{A^2}{2\sigma_N^2} \\
&\approx \begin{cases} \dfrac{[2\sqrt{\rho/\pi} - 1]^2}{\frac{4}{\pi}(1+\rho) - [2\sqrt{\rho/\pi}]^2}, & \text{as } \rho \to \infty \\ \dfrac{[(1+\rho/2) - 1]^2}{\frac{4}{\pi}(1+\rho) - (1+\rho/2)^2}, & \text{as } \rho \to 0 \end{cases}
\end{aligned}$$

(Continue from the previous slide.)

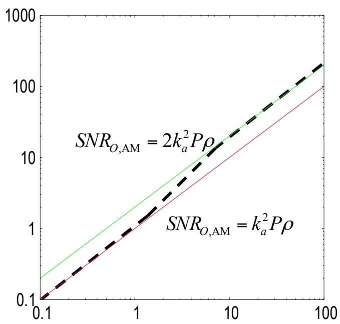
$$
\begin{aligned}
&= \begin{cases} \rho + \frac{\pi}{4} - \sqrt{\pi\rho}, & \text{as } \rho \to \infty \\ \dfrac{\pi\rho^2}{16(1+\rho) - \pi(2+\rho)^2}, & \text{as } \rho \to 0 \end{cases} \\
&\approx \begin{cases} \rho, & \text{as } \rho \to \infty \\ \frac{\pi\rho^2}{16-4\pi}, & \text{as } \rho \to 0 \end{cases} \\
&= \begin{cases} \rho, & \text{as } \rho \to \infty \\ 0.91\rho^2, & \text{as } \rho \to 0 \end{cases}
\end{aligned}
$$

2.12 General Formula for SNR_O in Envelope Detection



Curve of $SNR_O = \dfrac{\left[{}_1F_1(-1/2;1;-\rho)-1\right]^2}{\frac{4}{\pi}(1+\rho)-\left[{}_1F_1(-1/2;1;-\rho)\right]^2}$

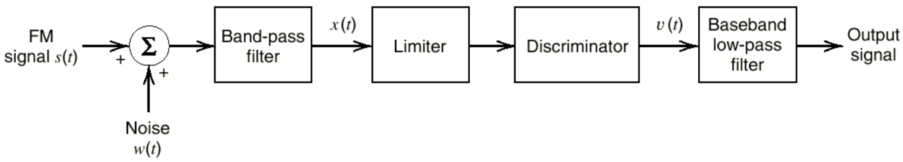
and the two limiting approximates.



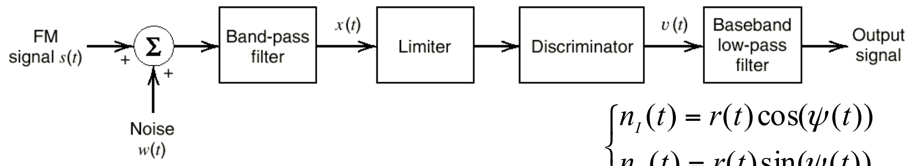
2.12 General Formula for SNR_O in Envelope Detection

- Remarks
  - For large *carrier-to-noise ratio* ρ, the envelope detector behaves like a coherent detector in the sense that the output SNR is proportional to ρ.
  - For small *carrier-to-noise ratio* ρ, the (newly defined) output signal-to-noise ratio of the envelope detector degrades faster than a linear function of ρ (decrease at a rate of ρ^2).
  - From "threshold effect" and "general formula for SNR_O," we can see that the envelope detector favors a strong signal. This is sometimes called "*weak signal suppression*."

2.13 Noise in FM Receivers



- To simplify the system analysis, we assume:
  - ideal band-pass filter (that is just wide enough to pass the modulated signal $s(t)$ without distortion),
  - ideal demodulator,
  - Gaussian distributed white Noise process.
- So the only source of imperfection is from the noise.

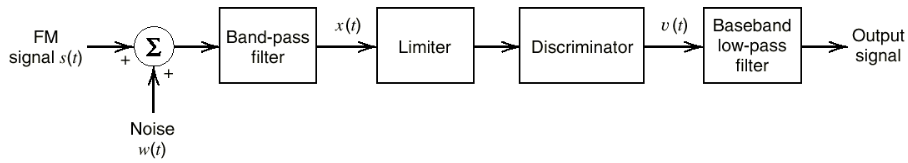


$$\begin{cases} n_I(t) = r(t)\cos(\psi(t)) \\ n_Q(t) = r(t)\sin(\psi(t)) \end{cases}$$

$$s(t) = A_c\cos[2\pi f_c t + \phi(t)], \text{ where } \phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau.$$

$$\begin{aligned} x(t) &= A_c\cos[2\pi f_c t + \phi(t)] + r(t)\cos[2\pi f_c t + \psi(t)] \\ &= A_c\cos[2\pi f_c t + \phi(t)] + r(t)\cos[2\pi f_c t + \phi(t) + \psi(t) - \phi(t)] \\ &= \left(A_c + r(t)\cos[\psi(t) - \phi(t)]\right)\cos[2\pi f_c t + \phi(t)] \\ &\quad - r(t)\sin[\psi(t) - \phi(t)]\sin[2\pi f_c t + \phi(t)] \\ &= \sqrt{\left(A_c + r(t)\cos[\psi(t) - \phi(t)]\right)^2 + r^2(t)\sin^2[\psi(t) - \phi(t)]}\cos[2\pi f_c t + \theta(t)] \end{aligned}$$

$$\text{where } \theta(t) = \phi(t) + \tan^{-1}\left\{\frac{r(t)\sin[\psi(t) - \phi(t)]}{A_c + r(t)\cos[\psi(t) - \phi(t)]}\right\}$$



$$x(t) = \sqrt{\left(A_c + r(t)\cos[\psi(t) - \phi(t)]\right)^2 + r^2(t)\sin^2[\psi(t) - \phi(t)]}\cos[2\pi f_c t + \theta(t)]$$

$$\xrightarrow{\text{Limiter}} A \cdot \cos[2\pi f_c t + \theta(t)]$$

Next, the signal will be passed through a Discriminator.

Recall on Slides 2-94 ~ 2-101, we have talked about the *Balanced Frequency Discriminator*, whose input and output satisfy:

$$\boxed{\text{Input } s(t) = A\cos(2\pi f_c t + \theta(t)) \quad \text{Output } \tilde{s}_o(t) = 2aA\theta'(t)}$$

Specifically, with $\tilde{s}(t) = A\exp(j\theta(t))$, we have:

$$\tilde{s}_1(t) = a\left[\frac{d\tilde{s}(t)}{dt} + j\pi B_T\tilde{s}(t)\right] = aAj[\theta'(t) + \pi B_T]\exp[j\theta(t)]$$

$$\tilde{s}_2(t) = -a\left[\frac{d\tilde{s}(t)}{dt} - j\pi B_T\tilde{s}(t)\right] = -aAj[\theta'(t) - \pi B_T]\exp[j\theta(t)]$$

$$\Rightarrow \tilde{s}_o(t) = |\tilde{s}_1(t)| - |\tilde{s}_2(t)| = 2aA\theta'(t)$$

Thus, after passing through the discriminator

$$v(t) = 2aA\theta'(t) = 2aA\frac{d\left(\phi(t) + \tan^{-1}\left\{\frac{r(t)\sin[\psi(t)-\phi(t)]}{A_c + r(t)\cos[\psi(t)-\phi(t)]}\right\}\right)}{dt}$$

The noise is a function of message signal $\phi(t)$.

Let $\alpha(t) = \psi(t) - \phi(t)$.

$$v(t) = 2aA\theta'(t) = 2aA\frac{d\left(\phi(t) + \tau{-1}\left\{\frac{r(t)\sin(\alpha(t))}{A_c + r(t)\cos(\alpha(t))}\right\}\right)}{dt}$$

Assumption 1: $\alpha(t) = \psi(t) - \phi(t)$ is uniformly distributed, and is independent of $m(t)$ and $r(t)$.

The above assumption is true for any deterministic (constant) $\phi(t)$! It has been shown that **Assumption 1** is justifiable for high carrier-to-noise ratio (or equivalently, under **Assumption 2**).

Assumption 2: $A_c \gg r(t)$ with high probability.

Thus $r(t)\cos(\alpha(t))$ and $r(t)\sin(\alpha(t))$ has the same distributions as $n_I(t)$ and $n_Q(t)$.

$$
\begin{aligned}
v(t) &= 2aA\theta'(t) = 2aA\frac{d}{dt}\left(\phi(t) + \tan^{-1}\left\{\frac{n_Q(t)}{A_c + n_I(t)}\right\}\right) \\
&= 2aA\left(\phi'(t) + \frac{1}{1+\left(\frac{n_Q(t)}{A_c+n_I(t)}\right)^2} \times \left(\frac{n'_Q(t)}{A_c + n_I(t)}\right.\right. \\
&\qquad \left.\left. - \frac{n_Q(t)n'_I(t)}{(A_c + n_I(t))^2}\right)\right)
\end{aligned}
$$

Assumption 2: $A_c \gg r(t)$ with high probability.

So that $A_c \gg |n_I(t)|$ and $A_c \gg |n_Q(t)|$ implies $A_c + n_I(t) \approx A_c$.

$$
v(t) \approx 2aA\left(\phi'(t) + \frac{1}{1+\left(\frac{n_Q(t)}{A_c}\right)^2} \times \left(\frac{n'_Q(t)}{A_c} - \frac{n_Q(t)}{A_c}\cdot\frac{n'_I(t)}{A_c}\right)\right)
$$

Assumption 2 implies $\frac{n_Q(t)}{A_c} \approx 0$.

$$
v(t) \approx 2aA\left(\phi'(t) + \frac{n'_Q(t)}{A_c}\right)
$$

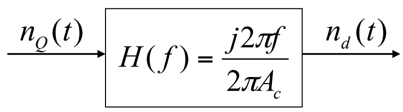
Assumption 3: $2aA = \frac{1}{2\pi}$.

$$
2\pi v(t) \approx \phi'(t) + \frac{n'_Q(t)}{A_c} = 2\pi k_f m(t) + 2\pi n_d(t),
$$

where $n_d(t) = \frac{n'_Q(t)}{2\pi A_c}$.

We then obtain the desired “additive” form.

Table 6.2 : 8. $\frac{d}{dt}g(t) \leftrightarrow j2\pi f G(f)$

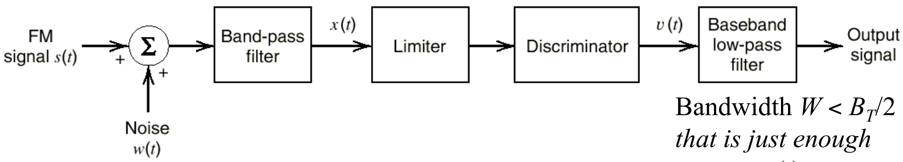


$$\Rightarrow S_{N_d}(f) = H(f)H(-f)S_{N_Q}(f) = \frac{jf}{A_c}\frac{j(-f)}{A_c}S_{N_Q}(f) = \frac{f^2}{A_c^2}S_{N_Q}(f)$$

$$\Rightarrow S_{N_d}(f) = \begin{cases} \frac{f^2}{A_c^2}N_0, & |f| \le \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

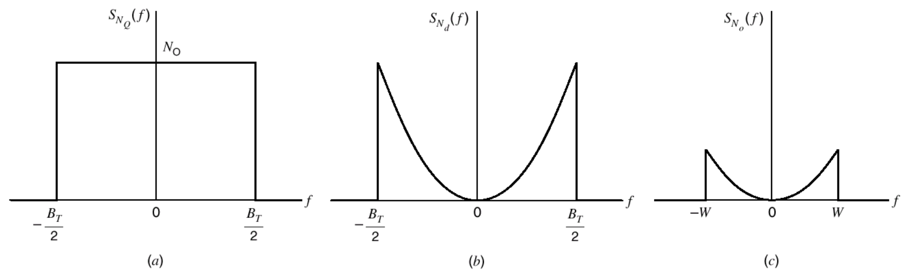
$$S_{N_Q}(f) = \begin{cases} S_N(f-f_c) + S_N(f+f_c), & \\ \quad \text{for } |f| < B_T/2 \\ 0, \text{ otherwise} \end{cases}$$

$v_o(t) \approx k_f m(t) + n_o(t)$



Bandwidth $W < B_T/2$ *that is just enough to pass* $m(t)$.

$$\Rightarrow S_{N_o}(f) = \begin{cases} \frac{f^2}{A_c^2}N_0, & |f| \le W \\ 0, & \text{otherwise} \end{cases}$$



$$\Rightarrow E[n_o^2(t)] = \frac{N_0}{A_c^2}\int_{-W}^{W} f^2 df = \frac{2N_0W^3}{3A_c^2}$$

Observation from the above formula:

In an FM system, increasing carrier power $A_c^2/2$

$\equiv$ Decreasing noise power. This is named the **noise quieting effect**.

As $v_o(t) \approx k_f m(t) + n_o(t)$,

$$\Rightarrow SNR_{O,\text{FM}} = \frac{k_f^2 E[m^2(t)]}{\frac{2N_0W^3}{3A_c^2}} = \frac{3A_c^2 k_f^2 P}{2N_0W^3}, \text{provided } A_c >> r(t).$$

We next turn to $SNR_{C,\text{FM}}$.

$s(t) = A_c \cos[2\pi f_c t + \phi(t)]$, where $\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau$.

$\Rightarrow$ average power in the modulated signal $s(t)$ is $A_c^2/2$.

Average noise power in the message bandwidth is $\int_{-W}^{W} \frac{N_0}{2} df = WN_0$.

$$\Rightarrow SNR_{C,\text{FM}} = \frac{A_c^2/2}{N_0W} = \frac{A_c^2}{2N_0W}.$$

$$\Rightarrow \frac{SNR_{O,\text{FM}}}{SNR_{C,\text{FM}}} = \frac{\frac{3A_c^2k_f^2P}{2N_0W^3}}{\frac{A_c^2}{2WN_0}} = \frac{3k_f^2P}{W^2}.$$

Remarks : $\boxed{\text{For fixed } W, \text{increasing } B_T \Leftrightarrow \text{increasing } \frac{SNR_{O,\text{FM}}}{SNR_{C,\text{FM}}}.}$

1. Deviation ratio $D = \frac{\Delta f}{W} \propto \frac{k_f P^{1/2}}{W}$. $\boxed{\Delta f = k_f \max |m(t)|}$

Hence, $\frac{SNR_{O,\text{FM}}}{SNR_{C,\text{FM}}} \propto D^2$.

2. $B_{T,\text{Carson}} = 2\Delta f\left(1 + \frac{1}{D}\right) = 2DW\left(1 + \frac{1}{D}\right) = 2W(D+1)$

2.13 Summary

- Specifically,
  - for high *carrier-to-noise ratio* ρ (equivalently to the assumption made in Assumption 1), an increase in transmission bandwidth B_T provides a corresponding quadratic increase in figure of merit of a FM system.
    - So, there is a tradeoff between B_T and figure of merit.
- Notably, figure of merit for an AM system is nothing to do with B_T.

Example 2.5 Single-Tone Modulation

- $m(t) = A_m \cos(2\pi f_m t)$
- Then we can represent the figure of merit in terms of modulation index (or deviation ratio) β as (cf. Slide 2-63):

$$\Rightarrow \frac{SNR_{O,\mathrm{FM}}}{SNR_{C,\mathrm{FM}}} = \frac{3k_f^2 P}{W^2} = \frac{3k_f^2 (A_m^2/2)}{W^2} = \frac{3}{2}\frac{\Delta f^2}{W^2} = \frac{3}{2}\beta^2.$$

- In order to make the figure of metric for an FM system to be superior to that for an AM system with 100% modulation, it requires:

$$\frac{3}{2}\beta^2 \geq \frac{1}{3} \Rightarrow \beta > \frac{\sqrt{2}}{3} = 0.471$$

2.13 Capture Effect

- Recall that in **Assumption 1**, we assume $A_c >> r(t)$.
- This somehow hints that the *noise suppression* of an FM modulation works well when the noise (*or other unwanted modulated signal that cannot be filtered out by the bandpass or lowpass filters*) is weaker than the desired FM signal.
- What if the unwanted FM signal is stronger than the desired FM signal.
  - The FM receiver will *capture* the unwanted FM signal!
- What if the unwanted FM signal has nearly equal strength as the desired FM signal.
  - The FM receiver will fluctuate back and forth between them!

2.13 FM Threshold Effect

- Recall that in **Assumption 1**, we assume $A_c >> r(t)$ (equivalently, a high *carrier-to-noise ratio*) to simplify $\theta(t)$ so that the next formula holds.

$$SNR_{O,\mathrm{FM}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}.$$

- However, a further decrease of carrier-to-noise ratio will break the FM receiver (from a clicking sound down to a crackling sound).
- As the same as the AM modulation, this is also named the *threshold effect*.

2.13 FM Threshold Effect

- Consider a simplified case with $m(t) = 0$ (no message signal).

$$2\pi\nu(t) = \theta'(t) = \frac{d\left(\tan^{-1}\left\{\frac{r(t)\sin[\psi(t)]}{A_c + r(t)\cos[\psi(t)]}\right\}\right)}{dt}$$

$$= \frac{A_c r(t)\psi'(t)\cos[\psi(t)] + A_c r'(t)\sin[\psi(t)] + r^2(t)\psi'(t)}{A_c^2 + 2A_c r(t)\cos[\psi(t)] + r^2(t)}$$

To facilitate the understanding of "clicking" sound effect, we let $r(t) = \lambda A_c$, a constant ratio of A_c.

2.13 FM Threshold Effect

$$\Rightarrow 2\pi\nu(t) = \theta'(t) = \lambda\psi'(t)\frac{\cos[\psi(t)] + \lambda}{1 + 2\lambda\cos[\psi(t)] + \lambda^2}$$

Then at the time, say, $\psi(t) \approx \pi$, and $\lambda > 1$ but $\lambda \approx 1$

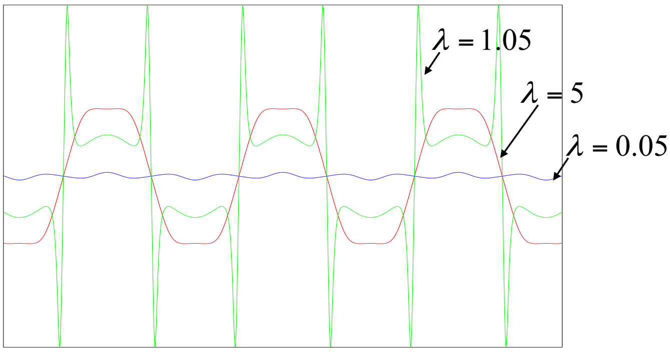
$$\Rightarrow 2\pi\nu(t) = \theta'(t) = \lambda\psi'(t)\frac{\cos[\psi(t)] + \lambda}{1 + 2\lambda\cos[\psi(t)] + \lambda^2} \approx \frac{\lambda}{\lambda - 1}\psi'(t)$$

Thus a sign change in $\psi'(t)$ will cause a spike!

Notably, when $\lambda = 0$ (no noise), the output equals $m(t) = 0$ as desired.

2.13 FM Threshold Effect

$$\psi(t) = \pi \sin(t) \Rightarrow 2\pi \nu(t) = \theta'(t) = \lambda\pi\cos(t)\frac{\cos[\pi\sin(t)] + \lambda}{1 + 2\lambda\cos[\pi\sin(t)] + \lambda^2}$$



2.13 Numerical Experiment

- Take $B_T/(2W) = 5$.
  - The average output signal power is calculated in the absence of noise.
  - The average output noise power is calculated when there is no signal present.

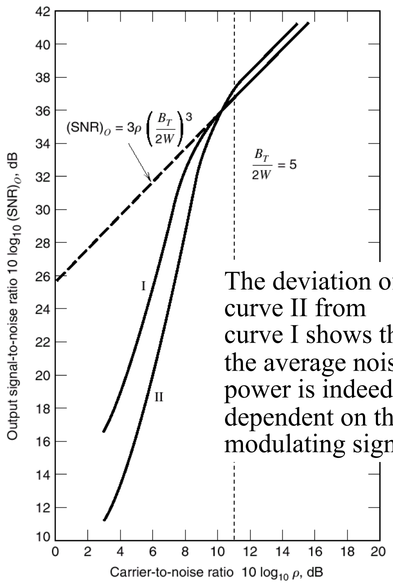
$$SNR_{O,\text{FM}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}, \text{ provided } A_c \gg r(t)$$

$$= \frac{3A_c^2 \Delta f^2}{4N_0 W^3}, \quad P = \frac{A_m^2}{2} \text{ and } \Delta f = k_f A_m$$

$$= 3\rho\left(\frac{B_T}{2W}\right)^3, \quad \Delta f = \frac{B_T}{2} \text{ and } \rho = \frac{A_c^2/2}{B_T N_0} \text{ is the carrier to noise ratio.}$$

2.13 Numerical of the formula in Slide 2-156

- Curve I: The average noise power is calculated assuming an unmodulated carrier.
- Curve II: The average noise power is calculated assuming a sinusoidally modulated carrier.
  - As text mentioned, for $\rho <$ 11 dB, the output signal-to-noise ratio deviates appreciably from the linear curve.
  - A true experiment found that occasional clicks are heard at ρ around 13 dB, only slightly larger than what theory indicates.



The deviation of curve II from curve I shows that the average noise power is indeed dependent on the modulating signal.

2.13 How to avoid "clicking" sound?

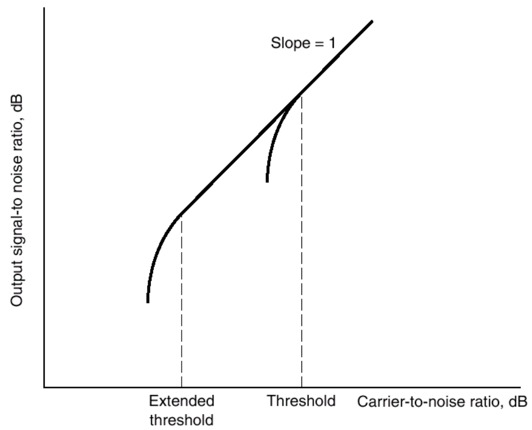
For given modulation index (or deviation ratio) β and message signal bandwidth W,

1. Determine B_T by Carson's rule or Universal curve.
2. For a specified noise level N_0, select A_c to satisfy:

$$10\log_{10}\left(\frac{A_c^2}{2B_TN_0}\right) \geq 13\,\text{dB} \quad \text{or equvalently, } \frac{A_c^2}{2B_TN_0} \geq 20.$$

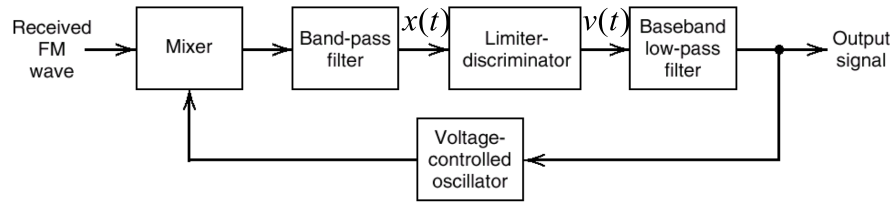
2.13 Threshold Reduction

- After our learning that FM modulation has threshold effect, the next question is naturally on "*how to reduce the threshold*?"



2.13 Threshold Reduction

- Threshold reduction in FM receivers may be achieved by
  1. negative feedback (commonly referred to as an *FMFB demodulator*), or
  2. *phase-locked loop demodulator*.
     - Why PLL can reduce threshold effect is not covered in this course. Notably, the PLL system analysis in Section 2.14 assumes noise-free transmission.



$s(t) = A_c \cos[2\pi f_c t + \phi(t)]$, where $\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau$.

$s_{vco}(t) = 2\cos[2\pi f_{vco} t + \phi_{vco}(t)]$, where $\phi_{vco}(t) = 2\pi\alpha k_f \int_0^t m(\tau)d\tau$.

$$s(t)s_{vco}(t) = 2A_c \cos[2\pi f_c t + \phi(t)]\cos[2\pi f_{vco} t + \phi_{vco}(t)]$$

$$\xrightarrow{\text{Bandpass}} A_c \cos[2\pi(f_c - f_{vco})t + (1-\alpha)\phi(t)]$$

The new frequency deviation $\Delta f_{new} = (1-\alpha)\Delta f_{original}$.

Thus, the bandpass filter can conceptually have a smaller passband as wide as $(1-\alpha)B_T$, centered at $(f_c - f_{vco})$.

$w(t)s_{vco}(t) = 2w(t)\cos[2\pi f_{vco} t + \phi_{vco}(t)]$

$\Rightarrow$ The noise at the Mixer output can be treated white with the same noise level as the input white noise.

$$\begin{aligned} x(t) &= A_c \cos[2\pi f_c' t + (1-\alpha)\phi(t)] + r(t)\cos[2\pi f_c' t + \psi(t)] \\ &\xrightarrow{\text{Limiter}} \cos[2\pi f_c' t + \theta(t)] \end{aligned}$$

where $E[n_I^2(t)] = E[n_Q^2(t)] = E[n^2(t)] = (1-\alpha)B_T N_0$,

and $\theta(t) = (1-\alpha)\phi(t) + \tan^{-1}\left\{\dfrac{r(t)\sin[\psi(t) - (1-\alpha)\phi(t)]}{A_c + r(t)\cos[\psi(t) - (1-\alpha)\phi(t)]}\right\}$.

Since $E[n_I^2(t)] = E[n_Q^2(t)]$ is smaller, and A_c remains the same,

the condition of $A_c >> r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$ holds with higer probability.

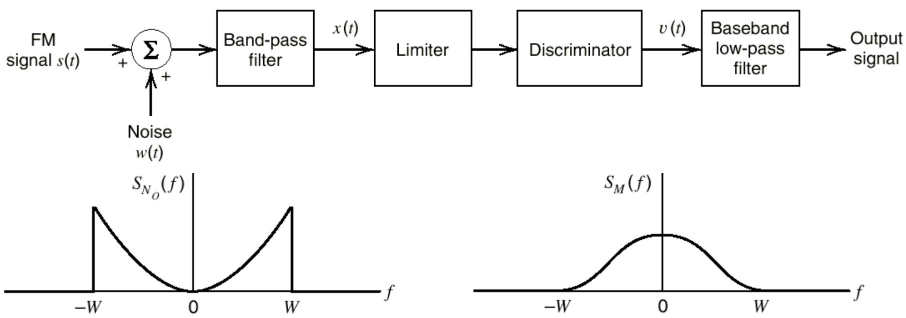
Experiments show that an FMFB receiver is capable of realizing a threshold extension on the order of 5~7 dB.

2.13 Threshold Reduction of an FMFB receiver

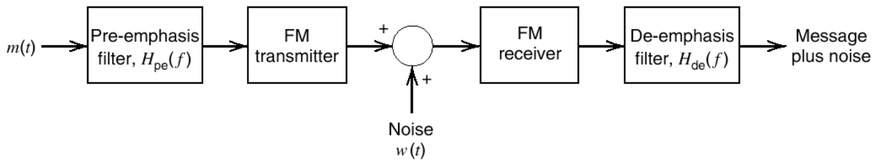
- To sum up:
  - An FMFB demodulator is essentially a *tracking filter* that can track only the *slowly varying frequency of a wideband FM signal (i.e., the message signal part)*, and consequently it responds only to a *narrowband of noise* centered about the *instantaneous carrier frequency*.

2.13 Pre-Emphasis and De-emphasis in FM

- Recall that the noise PSD at the output shapes like a bowel.
- If we can "equalize" the signal-to-noise power ratios over the entire message band, a better noise performance should result.



2.13 Pre-Emphasis and De-emphasis in FM

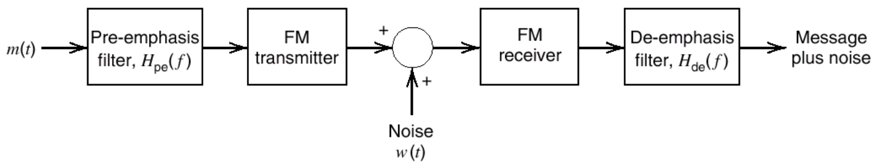


- In order to produce an *undistorted* version of the original message at the receiver output, we must have:

$$H_{pe}(f)H_{de}(f) = 1 \text{ for } -W \le f \le W.$$

- This relation guarantees the intactness of the message power.
- Next, we need to find $H_{de}(f)$ such that the noise power is optimally suppressed.

2.13 Pre-Emphasis and De-emphasis in FM



- Under the assumption of high carrier-to-noise ratio, the noise PSD at the de-emphasis filter output is given by:

$$|H_{de}(f)|^2 S_{N_o}(f) = \begin{cases} \dfrac{N_0 f^2}{A_c^2} |H_{de}(f)|^2, & |f| \le W \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \text{Average noise power} = \int_{-W}^{W} \frac{N_0 f^2}{A_c^2} |H_{de}(f)|^2 \, df$$

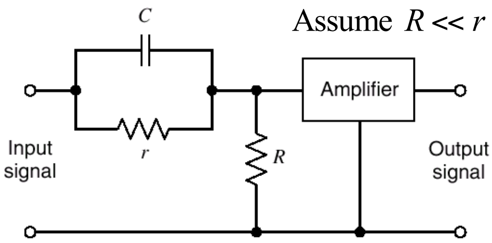
2.13 Pre-Emphasis and De-emphasis in FM

- Since the message power remains the same, the improvement factor of the output signal-to-noise ratio after and before pre/de-emphasis is:

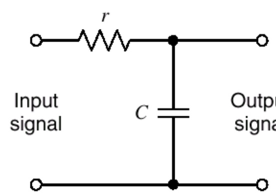
$$I = \frac{\int_{-W}^{W} \frac{N_0 f^2}{A_c^2} df}{\int_{-W}^{W} \frac{N_0 f^2}{A_c^2} |H_{de}(f)|^2 df} = \frac{\int_{-W}^{W} f^2 df}{\int_{-W}^{W} f^2 |H_{de}(f)|^2 df} = \frac{2W^3}{3\int_{-W}^{W} f^2 |H_{de}(f)|^2 df}$$

Example 2.6

Assume $R << r$ and $2\pi f C r << 1$ for $|f| \le W$.



Pre-emphasis filter



De-emphasis filter

$$H_{pe}(f) = 1 + j(f/f_0) \text{ and } H_{de}(f) = \frac{1}{1 + j(f/f_0)}, \text{ where } f_0 = 1/(2\pi C r).$$

Example 2.6

$$\Rightarrow I = \frac{2W^3}{3\int_{-W}^{W} f^2 |H_{de}(f)|^2 \, df} = \frac{2W^3}{3\int_{-W}^{W} \frac{f^2}{1+(f/f_0)^2} df} = \frac{(W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]}$$

With $f_0 = 2.1$ KHz and $W = 15$ KHz, we obtain $I = 22 = 13$ dB.
A significant noise performance improvement is therefore obtained.

2.13 Pre-Emphasis and De-emphasis in FM

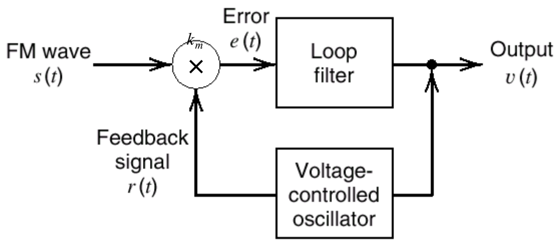
- Final remarks:
  - The previous example uses a linear pre-emphasis and de-emphasis filter to improve the noise.
  - A non-linear pre-emphasis and de-emphasis filters have been applied successfully to applications like tape recording. These techniques, known as Dolby-A, Dolby-B, and DBX systems, use a combination of filtering and dynamic range compression to reduce the effects of noise.

2.14 Computer Experiments: Phase-Locked Loop

- Phase-locked loop

$$s(t) = A_c \sin[2\pi f_c t + \phi_1(t)], \text{ where } \phi_1(t) = 2\pi k_f \int_0^t m(\tau)d\tau.$$

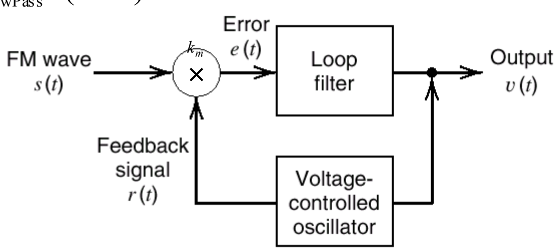
$$r(t) = A_v \cos[2\pi f_c t + \phi_2(t)], \text{ where } \phi_2(t) = 2\pi k_v \int_0^t v(\tau)d\tau.$$



Loop filter = low pass filter + filter $h(\tau)$.

$$\begin{aligned} e(t) &= k_m s(t) r(t) \\ &= k_m A_c \sin[2\pi f_c t + \phi_1(t)] \cdot A_v \cos[2\pi f_c t + \phi_2(t)] \\ &= \frac{k_m A_c A_v}{2}\left(\sin[4\pi f_c t + \phi_1(t) + \phi_2(t)] + \sin[\phi_1(t) - \phi_2(t)]\right) \\ &\xrightarrow{\text{Low Pass}} \frac{k_m A_c A_v}{2}\sin[\phi_e(t)], \quad \text{where } \phi_e(t) = \phi_1(t) - \phi_2(t). \end{aligned}$$

Also, $v(t) = \int_{-\infty}^{\infty} \{e(\tau)\}_{\text{LowPass}} h(t-\tau)d\tau.$



$$\phi_e(t) = \phi_1(t) - \phi_2(t) = \phi_1(t) - 2\pi k_v \int_0^t v(\tau)d\tau$$

$$\Rightarrow \quad \frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi k_v v(t)$$

$$= \frac{d\phi_1(t)}{dt} - 2\pi k_v \int_{-\infty}^{\infty} \{e(\tau)\}_{\text{LowPass}} h(t-\tau)d\tau$$

$$= \frac{d\phi_1(t)}{dt} - 2\pi k_0 \int_{-\infty}^{\infty} \sin[\phi_e(\tau)]h(t-\tau)d\tau$$

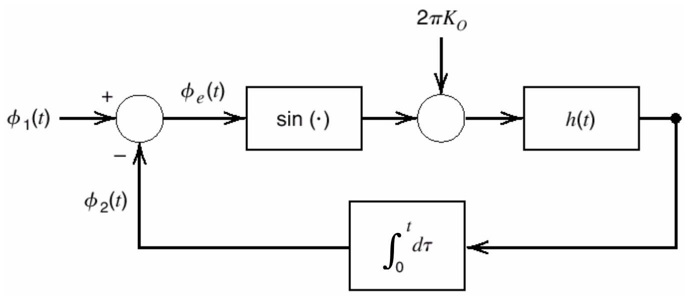
where $k_0 = k_m k_v A_c A_v / 2.$

$$\Rightarrow \quad \phi_e(t) = \int_0^t \frac{d\phi_e(u)}{du} du$$

$$= \int_0^t \frac{d\phi_1(u)}{du} du - 2\pi k_0 \int_0^t \left(\int_{-\infty}^{\infty} \sin[\phi_e(s)]h(u-s)ds \right) du$$

$$= \phi_1(t) - \int_0^t \left(2\pi k_0 \sin[\phi_e(u)] * h(u)\right) du$$

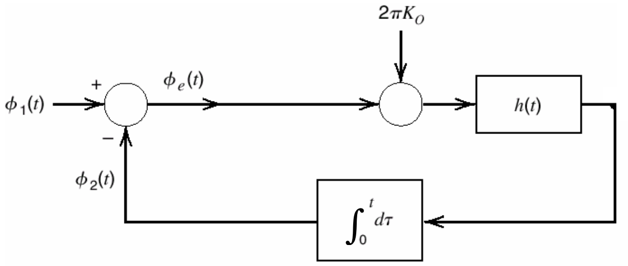
The previous formula suggests an equivalent analytical model for PLL.



When $\phi_e(t) = 0$, the system is said to be in *phase-lock*.
In this case, $\phi_1(t) = \phi_2(t)$ or equivalently, $k_v v(t) = k_f m(t)$.

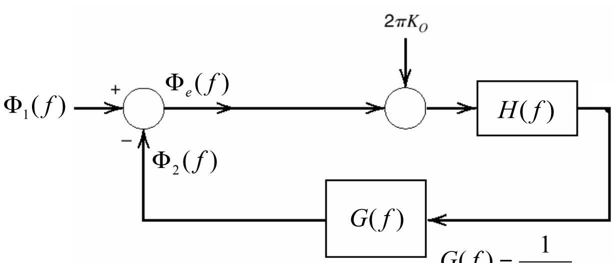
When $\phi_e(t)$ is small (< 0.5 radians), the system is said to be in *near phase-locked*.
In this case, we can approximate $\sin[\phi_e(t)]$ by $\phi_e(t)$; hence, a linear approximate model is resulted.

Linearization approximation model for PLL.



We can transform the above time-domain system to its equivalent frequency domain to facilitate its analysis.

Linearization approximation model for PLL.



$$G(f) = \frac{1}{j2\pi f}$$

$$\frac{\Phi_e(f)}{\Phi_1(f)} = \frac{\Phi_e(f)}{[\Phi_1(f) - \Phi_2(f)] + \Phi_2(f)}$$

$$= \frac{\Phi_e(f)}{\Phi_e(f) + 2\pi k_0 \Phi_e(f) H(f) G(f)}$$

$$= \frac{1}{1 + 2\pi k_0 H(f) G(f)} = \frac{jf}{jf + k_0 H(f)}$$

2.14 Experiment 0: First-Order PLL

$\boxed{H(f) = 1.}$

$$\frac{\Phi_e(f)}{\Phi_1(f)} = \frac{j(f/k_0)}{1 + j(f/k_0)}$$

A parameter k_0 controls both the loop gain and bandwidth of the filter. In other words, it is impossible to adjust the loop gain without changing the filter bandwidth.

2.14 Experiment 1: Second-Order Acquisition PLL

$H(f) = 1 + a/(jf)$ and using linear PLL model.

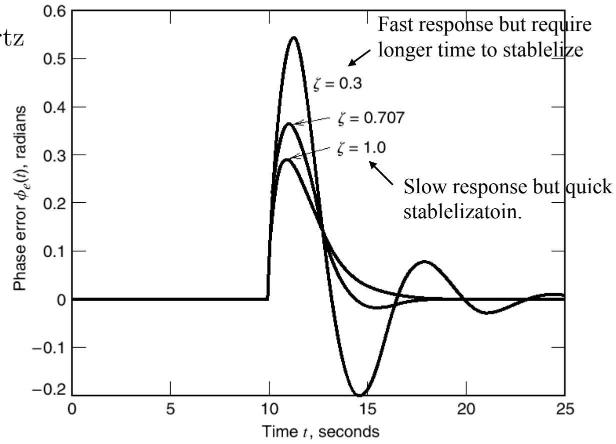
$$\frac{\Phi_e(f)}{\Phi_1(f)} = \frac{jf}{jf + k_0 H(f)} = \frac{jf}{jf + k_0(1 + a/(jf))} = \frac{(jf)^2}{(jf)^2 + k_0(jf) + k_0 a}$$

$$= \frac{(jf/f_n)^2}{1 + 2\zeta(jf/f_n) + (jf/f_n)^2}$$

where natural frequency $f_n = \sqrt{ak_0}$ and damping factor $\zeta = \sqrt{k_0/(4a)}$.

2.14 Experiment 1: Second-Order Acquisition PLL

Take $f_n = 1/(2\pi)$ Hertz



2.14 Experiment 2: Phase-Plain Portrait

$H(f) = 1 + a/(jf)$ and using nonlinear PLL model.

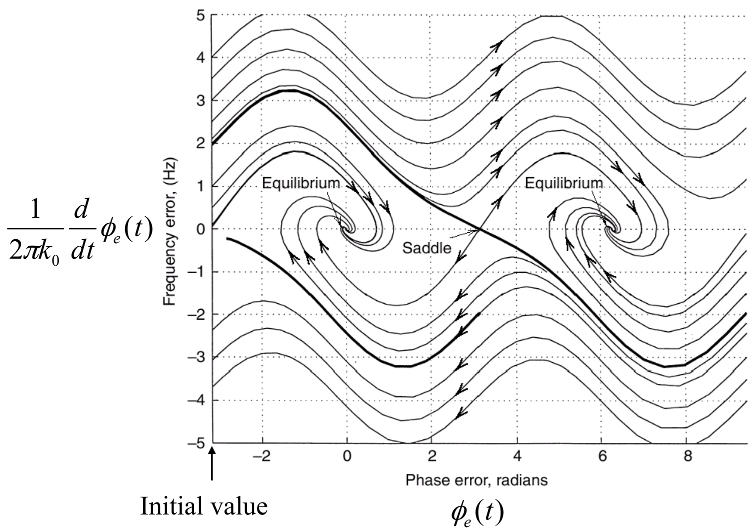
$$\text{Take} \begin{cases} f_n = \dfrac{50}{2\pi\sqrt{2}} \text{ Herz} \\ \zeta = \dfrac{1}{\sqrt{2}} \text{ (critical damping)} \end{cases}$$

Let $m(t) = A_m \cos(2\pi f_m t)$, where $f_m = \dfrac{50}{2\pi\sqrt{2\pi}}$ Hz.

$$\frac{1}{2\pi k_0}\frac{d\phi_e(t)}{dt} = \frac{1}{2\pi k_0}\frac{d\phi_1(t)}{dt} - \int_{-\infty}^{\infty} \sin[\phi_e(\tau)]h(t-\tau)d\tau$$

$$= \frac{k_f A_m}{k_0}\cos(2\pi f_m t) - \int_{-\infty}^{\infty} \sin[\phi_e(\tau)]h(t-\tau)d\tau$$

Phase-Plane Portrait



Remarks on the previous figure

- Each curve corresponds to different initial frequency error (i.e., boundary condition for the differential equation).
- The phase-plane portrait is periodic in $\phi_e(t)$ with period 2π, but the phase-plane portrait is not periodic in $d\phi_e(t)/dt$.
- There exists an initial frequency error (such as 2 in the previous figure) at which a *saddle line* appears (i.e., the solid line in the previous figure).
- In certain cases, the PLL will ultimately reach an equilibrium (stable) points at (0,0) or (2π,0).
- A slightest perturbation to the "saddle line" causes it to shift to the equilibrium points.

2.15 Summary and Discussion

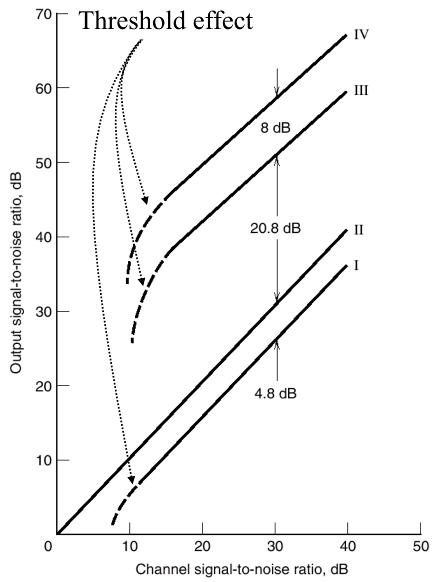
- Four types of AM modulations are introduced
  - (expensive) DSB-C transmitter + (inexpensive) envelope detector, which is good for applications like radio broadcasting.
  - (less expensive) DSB-SC transmitter + (more complex) coherent detector, which is good for applications like limited-transmitter-power point-to-point communication.
  - (less bandwidth) VSB transmitter + coherent detector, which is good for applications like television signals and high speed data.
  - (minimum transmission power/bandwidth) SSB transmitter + coherent detector, which is perhaps only good for applications whose message signals have an energy gap on zero frequency.

2.15 Summary and Discussion

- FM modulation, a representative of Angle Modulation
  - A nonlinear modulation process
  - Carson's rule and universal curve on transmission bandwidth

2.15 Summary and Discussion

- Noise performance
  - I. Full AM (DSB-C) with 100 percent modulation (See Slide 2-5)
  - II. Coherent DSB-SC & SSB
  - III. FM with $\beta = 2$ and 13 dB pre/de-emphasis improvement
  - IV. FM with $\beta = 5$ and 13 dB pre/de-emphasis improvement



2.15 Summary and Discussion

- Normalized transmission bandwidth in the previous figure

	DSB-C, DSB-SC	SSB	FM(β=2)	FM(β=5)
B_n	2	1	8	16

$$B_{n,\text{Universal Curve}} = \frac{B_T}{W} = \frac{B_T/\Delta f}{W/\Delta f} = \beta\frac{B_T}{\Delta f}$$

$= 2\times 4$ $\quad = 5\times 3.2$

(Refer to Slide 2-85.)

2.15 Summary and Discussion

- Observations from the figure
  1. SSB modulation is optimum in noise performance and bandwidth conservation in AM family.
  2. FM modulation improves the noise performance of AM family at the expense of an excessive transmission bandwidth.
  3. Curves III an IV indicate that FM modulation offers the tradeoff between transmission bandwidth and noise performance.