Ch-3

Modelling In Time Domain

*TF = Transfer function.

Disadvantage of frequency-domain (classial), TF based system

1. Only applicable for linear, time-invariant system.

Advantage of frequency domain

1. Rapidly provide stability & transient response information.

Advantages of time-domain, modern, state space approach

1. Can also be applied to nonlinear system.
2. Can handle system with non-zero initial condition.
3. Time-varying system can be represented.
4. Good for multiple-input multiple-output. (MIMO) system.

* Why Classical system not good for time-variant system.

$$x(t) \longrightarrow \boxed{\text{sys.}} \longrightarrow y(t) = x(-t)$$

$$\Rightarrow Y(s) = X(-s)$$

* $\frac{Y(s)}{X(s)}$ is not possible

* Not for Non-linear system

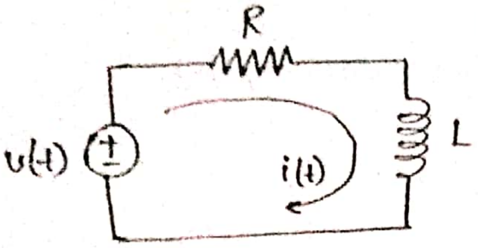
$$x(t) \longrightarrow \boxed{\text{sys}} \longrightarrow y(t) = x^2(t)$$

$$\Rightarrow Y(s) = X(s) \otimes X(s)$$

$\therefore \frac{Y(s)}{X(s)}$ not possible

3.2 Some Observation (P-119)

Consider an RL network shown below



We select i(t) and write loop equation,

$$L\frac{di}{dt} + Ri = v(t) \quad \text{----- (i)}$$

Laplace (i)

$$L\left[sI(s) - i(0)\right] + RI(s) = V(s) \quad \text{---- (ii)}$$

Assume the input v(t) to be a unit step u(t), $\therefore V(s) = \frac{1}{s}$.

We solve for I(s)

$$I(s) = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) + \frac{i(0)}{s+\frac{R}{L}} \quad \text{---- (iii)}$$

we get,

$$i(t) = \frac{1}{R}\left(1 - e^{-(R/L)t}\right) + i(0)e^{-(R/L)t} \quad \text{--- (iv)}$$

Here i(t) is a state variable and equ^n (i) is state equation.

Now we solve for other networks,

$$v_R(t) = Ri(t) \quad \text{--- (v)}$$

$$v_L(t) = v(t) - Ri(t) \quad \text{--- (vi)}$$

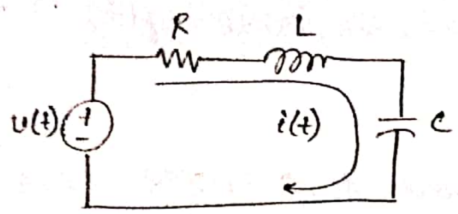
derivative of current

$$\frac{di(t)}{dt} = \frac{1}{L}\left[v(t) - Ri(t)\right] \quad \text{--- (vii)} \quad \left(\because v_L(t) = L\frac{di}{dt}\right)$$

Thus knowing state variables i(t) and input v(t) we can find the value or state of any network. Hence the equns. (v), (vi), (vii) are called output equations.

The combination of state equation (i) and the output equations (v)–(vii) is called state-space representation.

* Let us take a second-order system



The loop equ^n yields

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t) \quad --- (9)$$

Converting to charge using $i = \frac{dq}{dt}$;

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t) \quad ---- (10)$$

We convert (10) into two simultaneous, first-order differential equations in terms of i(t) and q(t), two state variables.

$$\frac{dq}{dt} = i \quad ----- (12a)$$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t)$$

$$\Rightarrow L\frac{di}{dt} + Ri + \frac{1}{C}q = v(t) \quad [\because \int i\,dt = q].$$

$$\Rightarrow L\frac{di}{dt} = -Ri - \frac{1}{C}q + v(t).$$

$$\therefore \frac{di}{dt} = -\frac{R}{L}i - \frac{1}{LC}q + \frac{1}{L}v(t) \quad ---- (12b)$$

Now here equ^n 12a and 12b are state equations and q(t) and i(t) state variable.

Thus if we want to get output across $v_L(t)$ the inductor we can write

$$v_L(t) = -v_C(t) - v_R(t) + v(t)$$

$$\therefore v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t) \quad \text{---- (13)}$$

Equ^n (13) is an output equation.

$v_L(t)$ is a linear combination of the state variables $q(t)$ & $i(t)$ and the input $v(t)$.

The combination of 12 upto 13 is called state space representation

We rearrange all the equations,

$$\left.\begin{aligned} \frac{dq}{dt} &= 0 + i + 0 \\ \frac{di}{dt} &= -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t) \end{aligned}\right\} \text{state equns.}$$

$$v_L(t) = -\frac{1}{C}q - Ri + v(t) \quad \} \text{ o/p equation.}$$

It can be written as. (only state equns.)

$$[\dot{x}] = [A][x] + [B][u] \quad \{\text{State equation}\}$$

where,

$$[\dot{x}] = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix} \qquad [B] = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$$

$$[A] = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \qquad [u] = [v(t)]$$

$$[x] = \begin{bmatrix} q \\ i \end{bmatrix}$$

The output equ^n can be written as, $[v_L(t) = -\frac{1}{C}q - Ri + v(t)]$

$$[y] = [C][x] + [D][u]$$

$$[y] = [v_L(t)]$$

$$[C] = \begin{bmatrix} -1/C & -R \end{bmatrix} \qquad [D] = [1]$$

$$[x] = \begin{bmatrix} q \\ i \end{bmatrix} \qquad [u] = [v(t)]$$

A system is represented in the state space by following equns.

$$[\dot{x}] = [A][x] + [B][u] \quad \text{state equation}$$

$$[y] = [C][x] + [D][u] \quad \text{output equation}$$

where,

$[x]$ = state vector

$[\dot{x}]$ = derivative of the state vector w.r. to time

$[y]$ = Output vector

$[u]$ = input or control vector

$[A]$ = System matrix

$[B]$ = Input matrix

$[C]$ = Output matrix

$[D]$ = Feedforward matrix

* The choice of state variables for a given system is not unique

Important Definitions (3.3, P-123)

3.5 Converting a Transfer Function to State Space

Let us begin by showing how to represent a general, n-th order, linear differential equⁿ with constant coefficient in state space in the phase variable form.

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots\cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u \quad \cdots\cdots (A)$$

choose output $y(t)$ upto its $(n-1)$ derivatives as state variables.

$$x_1 = y$$
$$x_2 = \frac{dy}{dt}$$
$$x_3 = \frac{d^2y}{dt^2}$$
$$\vdots$$
$$x_n = \frac{d^{n-1}y}{dt^{n-1}}.$$

To form [x] matrix

differenting yields

$$\dot{x}_1 = \frac{dy}{dt}$$
$$\dot{x}_2 = \frac{d^2y}{dt^2}$$
$$\dot{x}_3 = \frac{d^2y}{dt^3}$$
$$\vdots$$
$$\dot{x}_n = \frac{d^n y}{dt^n}.$$

To form [ẋ] matrix.

Substituting we get,

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}\right\} \quad (B1)$$

for $t \geq t_0$ and initial conditions, $\mathbf{x}(t_0)$, where

$\mathbf{x}$ = state vector
$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix

Equation (3.18) is called the *state equation*, and the vector $\mathbf{x}$, the *state vector*, contains the state variables. Equation (3.18) can be solved for the state variables, which we demonstrate in Chapter 4. Equation (3.19) is called the *output equation*. This equation is used to calculate any other system variables. This representation of a system provides complete knowledge of all variables of the system at any $t \geq t_0$.

As an example, for a linear, time-invariant, second-order system with a single input $v(t)$, the state equations could take on the following form:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1 v(t) \tag{3.20a}$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2 v(t) \tag{3.20b}$$

where x_1 and x_2 are the state variables. If there is a single output, the output equation could take on the following form:

$$y = c_1x_1 + c_2x_2 + d_1 v(t) \tag{3.21}$$

The choice of state variables for a given system is not unique. The requirement in choosing the state variables is that they be linearly independent and that a minimum number of them be chosen.

3.4 Applying the State-Space Representation

In this section, we apply the state-space formulation to the representation of more complicated physical systems. The first step in representing a system is to select the state vector, which must be chosen according to the following considerations:

1. A minimum number of state variables must be selected as components of the state vector. This minimum number of state variables is sufficient to describe completely the state of the system.
2. The components of the state vector (that is, this minimum number of state variables) must be linearly independent.

Let us review and clarify these statements.

Linearly Independent State Variables

The components of the state vector must be linearly independent. For example, following the definition of linear independence in Section 3.3, if x_1, x_2, and x_3 are chosen as state variables, but $x_3 = 5x_1 + 4x_2$, then x_3 is not linearly independent of x_1

and x_2, since knowledge of the values of x_1 and x_2 will yield the value of x_3. Variables and their successive derivatives are linearly independent. For example, the voltage across an inductor, v_L, is linearly independent of the current through the inductor, i_L, since $v_L = L di_L/dt$. Thus, v_L cannot be evaluated as a linear combination of the current, i_L.

Minimum Number of State Variables

How do we know the minimum number of state variables to select? Typically, the minimum number required equals the order of the differential equation describing the system. For example, if a third-order differential equation describes the system, then three simultaneous, first-order differential equations are required along with three state variables. From the perspective of the transfer function, the order of the differential equation is the order of the denominator of the transfer function after canceling common factors in the numerator and denominator.

In most cases, another way to determine the number of state variables is to count the number of independent energy-storage elements in the system.[5] The number of these energy-storage elements equals the order of the differential equation and the number of state variables. In Figure 3.2 there are two energy-storage elements, the capacitor and the inductor. Hence, two state variables and two state equations are required for the system.

If too few state variables are selected, it may be impossible to write particular output equations, since some system variables cannot be written as a linear combination of the reduced number of state variables. In many cases, it may be impossible even to complete the writing of the state equations, since the derivatives of the state variables cannot be expressed as linear combinations of the reduced number of state variables.

If you select the minimum number of state variables but they are not linearly independent, at best you may not be able to solve for all other system variables. At worst you may not be able to complete the writing of the state equations.

Often the state vector includes more than the minimum number of state variables required. Two possible cases exist. Often state variables are chosen to be physical variables of a system, such as position and velocity in a mechanical system. Cases arise where these variables, although linearly independent, are also *decoupled*. That is, some linearly independent variables are not required in order to solve for any of the other linearly independent variables or any other dependent system variable. Consider the case of a mass and viscous damper whose differential equation is $M dv/dt + Dv = f(t)$, where v is the velocity of the mass. Since this is a first-order equation, one state equation is all that is required to define this system in state space with velocity as the state variable. Also, since there is only one energy-storage element, mass, only one state variable is required to represent this system in state space. However, the mass also has an associated position, which is linearly independent of velocity. If we want to include position in the state vector along with velocity, then we add position as a state variable that is linearly independent of the other state variable, velocity. Figure 3.4 illustrates what is happening. The first block is the transfer

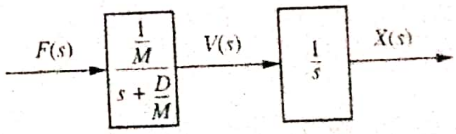


FIGURE 3.4 Block diagram of a mass and damper

[5] Sometimes it is not apparent in a schematic how many independent energy-storage elements there are. It is possible that more than the minimum number of energy-storage elements could be selected, leading to a state vector whose components number more than the minimum required and are not linearly independent. Selecting additional dependent energy-storage elements results in a system matrix of higher order and more complexity than required for the solution of the state equations.

function equivalent to $M\,dv(t)/dt + Dv(t) = f(t)$. The second block shows that we integrate the output velocity to yield output displacement (see Table 2.2, Item 10). Thus, if we want displacement as an output, the denominator, or characteristic equation, has increased in order to 2, the product of the two transfer functions. Many times, the writing of the state equations is simplified by including additional state variables.

Another case that increases the size of the state vector arises when the added variable is not linearly independent of the other members of the state vector. This usually occurs when a variable is selected as a state variable but its dependence on the other state variables is not immediately apparent. For example, energy-storage elements may be used to select the state variables, and the dependence of the variable associated with one energy-storage element on the variables of other energy-storage elements may not be recognized. Thus, the dimension of the system matrix is increased unnecessarily, and the solution for the state vector, which we cover in Chapter 4, is more difficult. Also, adding dependent state variables affects the designer's ability to use state-space methods for design.[6]

We saw in Section 3.2 that the state-space representation is not unique. The following example demonstrates one technique for selecting state variables and representing a system in state space. Our approach is to write the simple derivative equation for each energy-storage element and solve for each derivative term as a linear combination of any of the system variables and the input that are present in the equation. Next we select each differentiated variable as a state variable. Then we express all other system variables in the equations in terms of the state variables and the input. Finally, we write the output variables as linear combinations of the state variables and the input.

Example 3.1

Representing an Electrical Network

PROBLEM: Given the electrical network of Figure 3.5, find a state-space representation if the output is the current through the resistor.

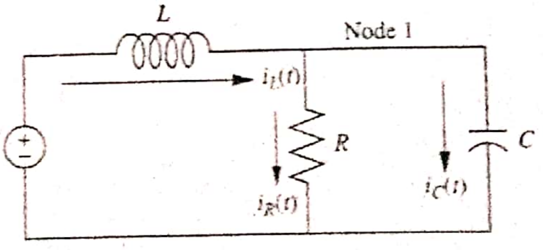


FIGURE 3.5 Electrical network for representation in state space

SOLUTION: The following steps will yield a viable representation of the network in state space.

Step 1 Label all of the branch currents in the network. These include i_L, i_R, and i_C, as shown in Figure 3.5.

[6] See Chapter 12 for state-space design techniques.

Step 2 Select the state variables by writing the derivative equation for all energy-storage elements, that is, the inductor and the capacitor. Thus,

$$C\frac{dv_C}{dt} = i_C \tag{3.22}$$

$$L\frac{di_L}{dt} = v_L \tag{3.23}$$

From Eqs. (3.22) and (3.23), choose the state variables as the quantities that are differentiated, namely v_C and i_L. Using Eq. (3.20) as a guide, we see that the state-space representation is complete if the right-hand sides of Eqs. (3.22) and (3.23) can be written as linear combinations of the state variables and the input.

Since i_C and v_L are not state variables, our next step is to express i_C and v_L as linear combinations of the state variables, v_C and i_L, and the input, $v(t)$.

Step 3 Apply network theory, such as Kirchhoff's voltage and current laws, to obtain i_C and v_L in terms of the state variables, v_C and i_L. At Node 1,

$$\begin{aligned} i_C &= -i_R + i_L \\ &= -\frac{1}{R}v_C + i_L \end{aligned} \tag{3.24}$$

which yields i_C in terms of the state variables, v_C and i_L.

Around the outer loop,

$$v_L = -v_C + v(t) \tag{3.25}$$

which yields v_L in terms of the state variable, v_C, and the source, $v(t)$.

Step 4 Substitute the results of Eqs. (3.24) and (3.25) into Eqs. (3.22) and (3.23) to obtain the following state equations:

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L \tag{3.26a}$$

$$L\frac{di_L}{dt} = -v_C + v(t) \tag{3.26b}$$

or

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L \tag{3.27a}$$

$$\frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v(t) \tag{3.27b}$$

Step 5 Find the output equation. Since the output is $i_R(t)$,

$$i_R = \frac{1}{R}v_C \tag{3.28}$$

The final result for the state-space representation is found by representing Eqs. (3.27) and (3.28) in vector-matrix form as follows:

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t) \tag{3.29a}$$

$$i_R = \begin{bmatrix} 1/R & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} \tag{3.29b}$$

where the dot indicates differentiation with respect to time.

From (A) we can write

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 \cdots\cdots - a_{n-1} x_n + b_0 u \quad \cdots (B2)$$

In vector-matrix form we get, (for B1 & B2)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u].$$

Since $y(t) = x_1$, we can write

$$y = [1 \ 0 \ 0 \cdots 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

* State Space model

$$[\dot{x}]_{n\times 1} = [A]_{n\times n} [x]_{n\times 1} + [B]_{n\times m} [u]_{m\times 1}.$$

$$[y]_{p\times 1} = [C]_{p\times n} [x]_{n\times 1} + [D]_{p\times m} [u]_{m\times 1}.$$

Example 3.4 (P-134)

Find state-space representation in phase variable form for the transfer function :

$$G(s) = \frac{C(s)}{R(s)} = \frac{24}{s^3+9s^2+26s+24} \quad \text{--- (i)}$$

Sol^n :- Cross-multiply --- (i)

$$(s^3+9s^2+26s+24)\,C(s) = 24\,R(s)$$

Inverse laplace gives (Assume zero initial condition)

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r \quad \text{---- (ii)}$$

* Select state variables to form $[x]$:

$x_1 = c \longrightarrow$ Output

$x_2 = \dot{c}$

$x_3 = \ddot{c}$

* Make derivative of state variables to form $[\dot{x}]$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \quad [\text{From (ii)}]$$

$$y = c = x_1$$

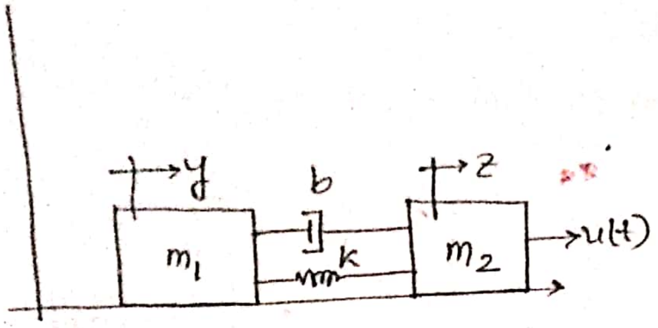
In vector-matrix form,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9 \end{bmatrix} \end{array} \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 24\end{bmatrix}[r].$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$

The 3rd row of system matrix has same coefficients as the denominator of the transfer function but negative and in reverse order.

State Space (Mechanical System)



Equations of motion:

$$m_1 \frac{d^2y}{dt^2} + b\left(\frac{dy}{dt} - \frac{dz}{dt}\right) + k(y-z) = 0 \quad \text{--- (i)}$$

$$m_2 \frac{d^2z}{dt^2} + b\left(\frac{dz}{dt} - \frac{dy}{dt}\right) + k(z-y) = u \quad \text{--- (ii)}$$

Energy Storage Element	State Variable
Spring	$x_1 = (y-z)$
mass 1	$x_2 = \dot{y}$
mass 2	$x_3 = \dot{z}$

* One choice of state variable
$x_1 = y$,
$x_2 = \dot{y}$,
$x_3 = z$,
$x_4 = \dot{z}$.

*mass stores energy in kinetic form $(=\frac{1}{2}mv^2)$ so velocities are taken.

Formation of $[\dot{x}]$ matrix

$$\dot{x}_1 = \dot{y} - \dot{z} = x_2 - x_3 \quad \text{--- (3a)}$$

$$\dot{x}_2 = \ddot{y} = \frac{1}{m_1}\left[-b(\dot{y}-\dot{z}) - k(y-z)\right] \text{ from (i)}$$

$$= \frac{1}{m_1}\left[-b(x_2-x_3) - kx_1\right] \quad \text{--- (3b)}$$

$$\dot{x}_3 = \ddot{z} = \frac{1}{m_2}\left[-b(\dot{z}-\dot{y}) - k(z-y) + u\right] \text{ - from (ii)}$$

$$= \frac{1}{m_2}\left[-b(x_3-x_2) + kx_1 + u\right] \quad \text{--- (3c)}$$

Matrix form, take coefficient of x_1, x_2, x_3 from equ^n 3a, 3b, 3c.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -\frac{k}{m_1} & -\frac{b}{m_1} & \frac{b}{m_1} \\ \frac{k}{m_2} & \frac{b}{m_2} & -\frac{b}{m_2} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} [u].$$

Suppose the output is $\dot{y} = x_2$

$$[y_{o/p}] = [0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0][u].$$

Step 4 Derive the output equation. Since the specified output variables are v_{R_2} and i_{R_2}, we note that around the mesh containing C, L, and R_2,

$$v_{R_2} = -v_C + v_L \tag{3.42a}$$

$$i_{R_2} = i_C + 4v_L \tag{3.42b}$$

Substituting Eqs. (3.38) and (3.39) into Eq. (3.42), v_{R_2} and i_{R_2} are obtained as linear combinations of the state variables, i_L and v_C. In vector-matrix form, the output equation is

$$\begin{bmatrix} v_{R_2} \\ i_{R_2} \end{bmatrix} = \begin{bmatrix} R_2/\Delta & -(1+1/\Delta) \\ 1/\Delta & (1-4R_1)/(\Delta R_1) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/\Delta \\ -1/\Delta \end{bmatrix} i(t) \tag{3.43}$$

In the next example, we find the state-space representation for a mechanical system. It is more convenient when working with mechanical systems to obtain the state equations directly from the equations of motion rather than from the energy-storage elements. For example, consider an energy-storage element such as a spring, where $F = Kx$. This relationship does not contain the derivative of a physical variable as in the case of electrical networks, where $i = C\,dv/dt$ for capacitors, and $v = L\,di/dt$ for inductors. Thus, in mechanical systems we change our selection of state variables to be the position and velocity of each point of linearly independent motion. In the example, we will see that although there are three energy-storage elements, there will be four state variables; an additional linearly independent state variable is included for the convenience of writing the state equations. It is left to the student to show that this system yields a fourth-order transfer function if we relate the displacement of either mass to the applied force, and a third-order transfer function if we relate the velocity of either mass to the applied force.

Read Carefully

Example 3.3

Representing a Translational Mechanical System

Class

PROBLEM: Find the state equations for the translational mechanical system shown in Figure 3.7.

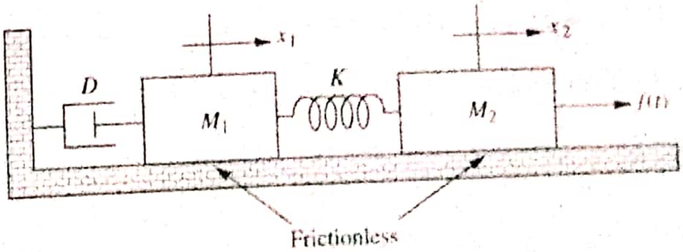


FIGURE 3.7 Translational mechanical system

SOLUTION: First write the differential equations for the network in Figure 3.7, using the methods of Chapter 2 to find the Laplace-transformed equations of motion. Next take the inverse Laplace transform of these equations, assuming zero initial conditions, and obtain

$$M_1 \frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0 \tag{3.44}$$

$$-Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t) \tag{3.45}$$

Now let $d^2x_1/dt^2 = dv_1/dt$, and $d^2x_2/dt^2 = dv_2/dt$, and then select x_1, v_1, x_2, and v_2 as state variables. Next form two of the state equations by solving Eq. (3.44) for dv_1/dt and Eq. (3.45) for dv_2/dt. Finally, add $dx_1/dt = v_1$ and $dx_2/dt = v_2$ to complete the set of state equations. Hence,

$$\frac{dx_1}{dt} = +v_1 \tag{3.46a}$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2 \tag{3.46b}$$

$$\frac{dx_2}{dt} = +v_2 \tag{3.46c}$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t) \tag{3.46d}$$

In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t) \tag{3.47}$$

where the dot indicates differentiation with respect to time. What is the output equation if the output is $x(t)$?

* There are infinite number of ways to form State Space Representations

* choices of state variable is not unique.

Skill-Assessment Exercise 3.1

PROBLEM: Find the state-space representation of the electrical network shown in Figure 3.8. The output is $v_o(t)$.

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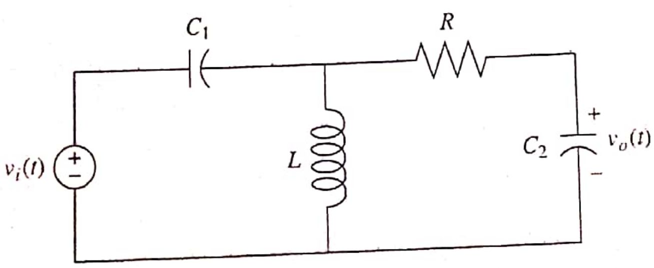

FIGURE 3.8 Electric circuit for Skill-Assessment Exercise 3.1

3.6 Converting from State Space to a Transfer Function

In Chapters 2 and 3, we have explored two methods of representing systems: the transfer function representation and the state-space representation. In the last section, we united the two representations by converting transfer functions into state-space representations. Now we move in the opposite direction and convert the state-space representation into a transfer function.

Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{3.68a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{3.68b}$$

take the Laplace transform assuming zero initial conditions:[8]

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{3.69a}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{3.69b}$$

Solving for $\mathbf{X}(s)$ in Eq. (3.69a),

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \tag{3.70}$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{3.71}$$

where $\mathbf{I}$ is the identity matrix.

Substituting Eq. (3.71) into Eq. (3.69b) yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \tag{3.72}$$

We call the matrix $[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]$ the transfer function matrix, since it relates the output vector, $\mathbf{Y}(s)$, to the input vector, $\mathbf{U}(s)$. However, if $\mathbf{U}(s) = U(s)$ and $\mathbf{Y}(s) = Y(s)$ are scalars, we can find the transfer function,

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}} \tag{3.73}$$

Let us look at an example.

[8] The Laplace transform of a vector is found by taking the Laplace transform of each component. Since $\dot{\mathbf{x}}$ consists of the derivatives of the state variables, the Laplace transform of $\dot{\mathbf{x}}$ with zero initial conditions yields each component with the form $sX_i(s)$, where $X_i(s)$ is the Laplace transform of the state variable. Factoring out the complex variable, s, in each component yields the Laplace transform of $\dot{\mathbf{x}}$ as $s\mathbf{X}(s)$, where $\mathbf{X}(s)$ is a column vector with components $X_i(s)$.

Example 3.6

State-Space Representation to Transfer Function

PROBLEM: Given the system defined by Eq. (3.74), find the transfer function, $T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u \tag{3.74a}$$

$$y = [1 \quad 0 \quad 0]\mathbf{x} \tag{3.74b}$$

SOLUTION: The solution revolves around finding the term $(s\mathbf{I} - \mathbf{A})^{-1}$ in Eq. (3.73).[9] All other terms are already defined. Hence, first find $(s\mathbf{I} - \mathbf{A})$:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} \tag{3.75}$$

Now form $(s\mathbf{I} - \mathbf{A})^{-1}$:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1} \tag{3.76}$$

Substituting $(s\mathbf{I} - \mathbf{A})^{-1}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ into Eq. (3.73), where

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0]$$

$$\mathbf{D} = 0$$

we obtain the final result for the transfer function:

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1} \tag{3.77}$$

MATLAB **ML**

Students who are using MATLAB should now run ch3p5 in Appendix B. You will learn how to convert a state-space representation to a transfer function using MATLAB. You can practice by writing a MATLAB program to solve Example 3.6.

Symbolic Math **SM**

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch3sp1 in Appendix F located at www.wiley.com/college/nise. You will learn how to use the Symbolic Math Toolbox to write matrices and vectors. You will see that the Symbolic Math Toolbox yields an alternative way to use MATLAB to solve Example 3.6.

[9] See Appendix G. It is located at www.wiley.com/college/nise and discusses the evaluation of the matrix inverse.