Chapter-2
Modelling in Frequency Domain

2.2 Laplace transform Review:

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$ ✓
$t^n u(t)$	$\frac{n!}{s^{n+1}}$.
$e^{-at}u(t)$	$\frac{1}{s+a}$ ✓
$\sin wt\, u(t)$	$\frac{w}{s^2+w^2}$
$\cos wt\, u(t)$	$\frac{s}{s^2+w^2}$.

Laplace transform theorem (P-37)

Example 2.2: (P-36)

Find inverse Laplace transform of $F_1(s) = \frac{1}{(s+3)^2}$.

We know,

$$\mathcal{L}[e^{-at}f(t)] = F(s+a). \quad \text{and} \quad \mathcal{L}[e^{-at}u(t)] = \frac{1}{(s+a)}.$$

Here $a = 3$

and $\mathcal{L}[tu(t)] = \frac{1}{s^2}$.

$$\therefore \mathcal{L}^{-1}[F_1(s)] = f_1(t) = e^{-3t}tu(t).$$

(Ans.)

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Partial Fraction Expansion:

Case 1: Roots of the Denominator $F(s)$ are real & distinct.
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$$\therefore F(s) = \frac{2}{(s+1)(s+2)}$$

$$\Rightarrow \frac{2}{(s+1)(s+2)} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)}.$$

$$\Rightarrow 2 = k_1(s+2) + k_2(s+1)$$

For $s=-2 \Rightarrow k_2 = -2$

and $s=-1 \Rightarrow k_1 = 2$

$$\therefore F(s) = \frac{2}{s+1} - \frac{2}{s+2}.$$

$$\Rightarrow \mathcal{L}^{-1}[F(s)] = (2e^{-t} - 2e^{-2t})u(t)$$

*Example-2.3 Given equ^n find $y(t)$. All initial conditions are zero.

$$\frac{d^2y}{dy^2} + 12\frac{dy}{dt} + 32y = 32u(t).$$

Sol^n:-

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}.$$

$$\Rightarrow Y(s)\left[s^2+12s+32\right] = \frac{32}{s}$$

$$\therefore Y(s) = \frac{32}{s(s^2+12s+32)}$$

$$= \frac{32}{s(s+4)(s+8)}$$

$$= \frac{k_1}{s} + \frac{k_2}{s+4} + \frac{k_3}{s+8}$$

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$k_1 = 1 \Big|_{s \to 0}$ $\quad k_2 = -2 \Big|_{s \to -4}$ $\quad k_3 = 1 \Big|_{s \to -8}$

$\therefore Y(s) = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$.

$\therefore y(t) = u(t) - 2e^{-4t}u(t) + e^{-8t}u(t)$.

(Ans.)

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Case 2: Roots are real and repeated.

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{k_1}{s+1} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)}$$

$\Rightarrow 2 = \cancel{k_1} + k_2(s+1) + k_3(s+1)(s+2) + k_1(s+2)^2$.

$\therefore k_1 = 2 \Big|_{s \to -1}$.

Again,

$$\frac{2}{(s+1)(s+2)^2} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)}$$

$\Rightarrow \frac{2}{s+1} = (s+2)^2 \frac{k_1}{(s+1)} + k_2 + (s+2)k_3$ [Mul. by $(s+2)^2$].

$\therefore k_2 = -2 \Big|_{s \to -2}$

Again,

$$\frac{2}{s+1} = (s+2)^2 \frac{k_1}{(s+1)} + k_2 + (s+2)k_3$$

$\Rightarrow \frac{-2}{(s+1)^2} = \frac{s(s+2)}{(s+1)} k_1 + k_3$ [differentiating both sides].

$\therefore k_3 = -2 \Big|_{s \to -2}$

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$$\therefore F(s) = \frac{2}{s+1} + \frac{-2}{(s+2)^2} + \frac{-2}{(s+2)}$$

$$\therefore f(t) = \left[2e^{-t} - 2te^{-2t} - 2e^{-2t}\right] u(t).$$

Ans.

<u>Case 3</u>: Roots are complex or Imaginary.

$$F(s) = \frac{3}{s(s^2+2s+5)}$$

$$= \frac{k_1}{s} + \frac{k_2 s + k_3}{s^2+2s+5}.$$

$$\Rightarrow 3 = k_1(s^2+2s+5) + (k_2 s + k_3)s \quad \text{-- (i)}$$

$$\therefore k_1 = \frac{3}{5}\Big|_{s\to 0}$$

Again from (i)

$$3 = k_1(s^2+2s+5) + (k_2 s + k_3)s$$

$$\Rightarrow 3 = \frac{3}{5}s^2 + \frac{6}{5}s + 3 + k_2 s^2 + k_3 s \quad \left[k_1 = \frac{3}{5}\right].$$

$$\Rightarrow 3 = \left(k_2 + \frac{3}{5}\right)s^2 + \left(k_3 + \frac{6}{5}\right)s + 3.$$

Balancing co-efficient we get,

$k_2 = -\frac{3}{5}$ and $k_3 = -\frac{6}{5}$.

$$\therefore F(s) = \frac{3/5}{s} - \frac{\frac{3}{5}s + \frac{6}{5}}{s^2+2s+5} = \frac{3/5}{s} - \frac{3}{5}\cdot\frac{(s+2)}{s^2+2s+5}.$$

$$= \frac{3/5}{s} - \frac{3}{5}\cdot\frac{(s+1) + (\frac{1}{2}).2}{(s+1)^2 + (2)^2}.$$

$$= \frac{3/5}{s} - \frac{3/5(s+1) + 3/10 \cdot 2}{(s+1)^2 + (2)^2}.$$

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We know,

$$\mathcal{L}\left[Ae^{-at}\cos\omega t\right] = \frac{A(s+a)}{(s+a)^2+\omega^2} \qquad \mathcal{L}\left[Be^{-at}\sin\omega t\right] = \frac{B\omega}{(s+a)^2+\omega^2}.$$

$$\mathcal{L}\left[Ae^{-at}\cos\omega t + Be^{-at}\sin\omega t\right] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2} \quad \text{--- (A)}$$

we have,

$$F(s) = \frac{3/5}{s} - \left[\frac{3}{5}\cdot\frac{(s+1)+(\frac{1}{2})(2)}{(s+1)^2+(2)^2}\right] \text{--- (B)}.$$

From (A) & (B);

$$A = \frac{3}{5} \; ; \; B = \cancel{\frac{6}{10}}\ \frac{3}{10} \; ; \; \omega = 2 \; ; \; a = 1.$$

$$\therefore f(t) = \frac{3}{5} - \left\{\frac{3}{5}e^{-t}\cos 2t + \cancel{\frac{6}{20}}\ \frac{3}{10}\, e^{-t}\sin 2t\right\}.$$

$$= \frac{3}{5} - \frac{3}{5}\left(e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t\right)$$

$$= \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right).$$

Ans.

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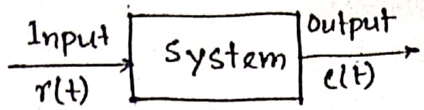
2.3 Transfer function:



n^{th} order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

Taking Laplace,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s)$$

$$\Rightarrow (a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

$$\Rightarrow \frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{\text{Output}}{\text{Input}}$$

The ratio $G(s)$ is called the transfer function.

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Ex-2.4 (P-45)

Find the transfer fun^c. Use the result to find out output for a unit step input. Assume zero initial condition.

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Sol^n:-

Taking laplace at both side.

$$sC(s) + 2C(s) = R(s)$$

$$\Rightarrow C(s)\,[s+2] = R(s)$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{1}{s+2} = G(s)$$ (Ans.)

Since, Given input $r(t) = u(t)$

$$\Rightarrow \mathcal{L}[r(t)] = R(s) = \frac{1}{s}.$$

$\therefore$ Output $C(s) = G(s)\,R(s)$

$$= \frac{1}{s(s+2)}$$

$$= \frac{1/2}{s} - \frac{1/2}{s+2}.$$

$$\therefore c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}.$$

Ans.

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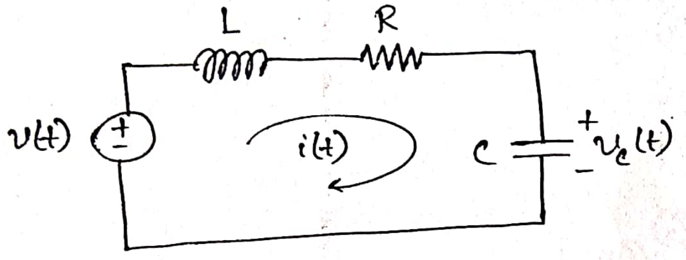
2.4 Electrical Network Transfer Function

Table 2.3 (P-48)

Example-2.6

Find the transfer function relating the capacitor voltage $V_c(s)$ to the i/p voltage $V(s)$. Assume zero initial condition.



Sol^n :- Always decide i/p & o/p.

Taking KVL,

$$L\frac{di(t)}{dt} + R\,i(t) + \frac{1}{C}\int_0^t i(t)dt = v(t) \quad \cdots \text{(i)}$$

Putting $i(t) = \frac{dq(t)}{dt}$ in (i) we get,

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \quad \cdots \text{(ii)}$$

Putting $q(t) = C\,v_c(t)$ in (ii) we get,

$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$$

$\Rightarrow LCs^2V_c(s) + RCsV_c(s) + V_c(s) = V(s)$ [Laplace transform]

$\Rightarrow (LCs^2 + RCs + 1)V_c(s) = V(s)$

$$\therefore \frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$ [Divided by $\frac{1}{LC}$].

(Ans)

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$$v(s) = \frac{1}{C} \cdot \frac{I(s)}{s}$$

$$\Rightarrow \frac{V(s)}{I(s)} = \frac{1}{Cs} = Z(s)$$

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Simple Circuits via Mesh Analysis

Transfer functions can be obtained using Kirchhoff's voltage law and summing voltages around loops or meshes.[3] We call this method *loop* or *mesh analysis* and demonstrate it in the following example.

Example 2.6

Transfer Function—Single Loop via the Differential Equation

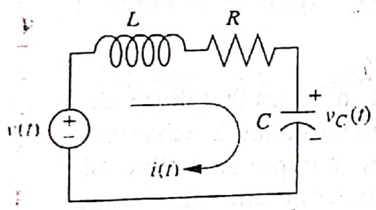

FIGURE 2.3 *RLC* network

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

SOLUTION: In any problem, the designer must first decide what the input and output should be. In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current. The problem statement, however, is clear in this case: We are to treat the capacitor voltage as the output and the applied voltage as the input.

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \tag{2.61}$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \tag{2.62}$$

From the voltage-charge relationship for a capacitor in Table 2.3,

$$q(t) = Cv_C(t) \tag{2.63}$$

Substituting Eq. (2.63) into Eq. (2.62) yields

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \tag{2.64}$$

[3] A particular loop that resembles the spaces in a screen or fence is called a *mesh*.

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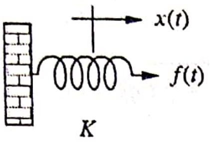
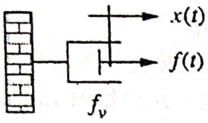
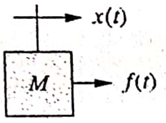
TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m(newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Mechanical systems parallel electrical networks to such an extent that there are analogies between electrical and mechanical components and variables. Mechanical systems, like electrical networks, have three passive, linear components. Two of them, the spring and the mass, are energy-storage elements; one of them, the viscous damper, dissipates energy. The two energy-storage elements are analogous to the two electrical energy-storage elements, the inductor and capacitor. The energy dissipator is analogous to electrical resistance. Let us take a look at these mechanical elements, which are shown in Table 2.4. In the table, K, f_v, and M are called *spring constant*, *coefficient of viscous friction*, and *mass*, respectively.

We now create analogies between electrical and mechanical systems by comparing Tables 2.3 and 2.4. Comparing the force-velocity column of Table 2.4 to the voltage-current column of Table 2.3, we see that mechanical force is analogous to electrical voltage and mechanical velocity is analogous to electrical current. Comparing the force-displacement column of Table 2.4 with the voltage-charge column of Table 2.3 leads to the analogy between the mechanical displacement and electrical charge. We also see that the spring is analogous to the capacitor, the viscous damper is analogous to the resistor, and the mass is analogous to the inductor. Thus, summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations. If the forces are written in terms of displacement, the resulting mechanical equations resemble, but are not analogous to, the mesh equations. We, however, will use this model for mechanical systems so that we can write equations directly in terms of displacement.

Another analogy can be drawn by comparing the force-velocity column of Table 2.4 to the current-voltage column of Table 2.3 in reverse order. Here the analogy is between force and current and between velocity and voltage. Also, the

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2.5 Translational Mechanical System. Transfer Function.

Table 2.4 (P-62)

Taking/Comparing Table 2.3 & 2.4

	Force - Velocity		Voltage - Current
Spring	$f(t) = k\int_0^t v(t)dt$	Capacitor	$v(t) = \frac{1}{C}\int_0^t i(t)dt$
Damper	$f(t) = f_v v(t)$	Resistor	$v(t) = R\,i(t)$
Mass	$f(t) = M\frac{dv(t)}{dt}$. $= M\frac{d^2x(t)}{dt^2}$	Inductor	$v(t) = L\frac{di(t)}{dt}$. $= L\frac{d^2q(t)}{dt^2}$.

Two energy storage elements

Spring ⇔ Capacitor

Mass ⇔ Inductor

One dissipating element

Viscous damper ⇔ Resistor

Units:

$f(t)$ = Newtons N

$x(t)$ = m

$v(t) = ms^{-1}$

K = spring constant (N/m)

f_v = co-efficient of viscous friction (N.s/m)

M = kg

Analogies: (Force-Voltage)

Mechanical force $f(t)$ ⇔ Electrical voltage $v(t)$

" velocity $v(t)$ ⇔ current $i(t)$

displacement $x(t)$ ⇔ charge $q(t)$

* Force - Current Analogy. (2.9. Electric ckt Analog P-84-87)

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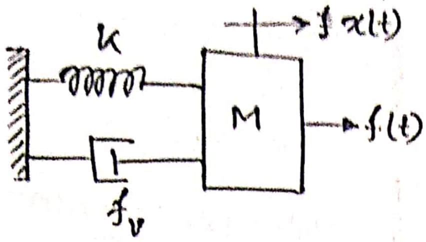
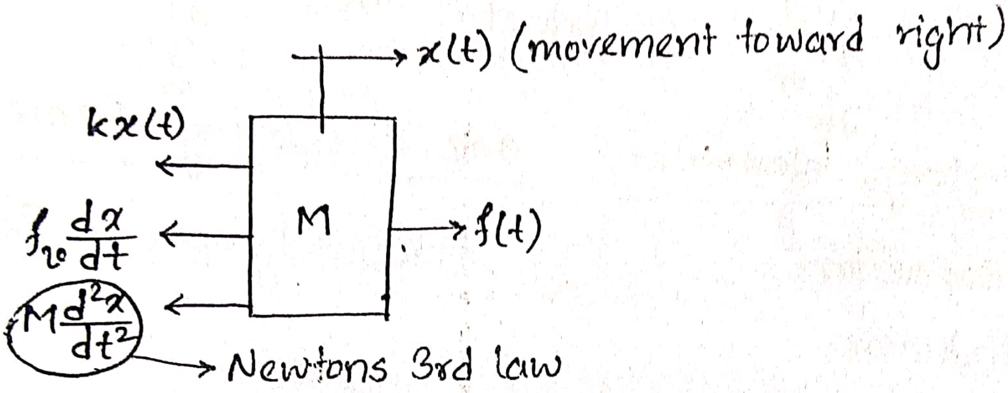
Example 2·16 (P-63)



Find the transfer function $X(s)/F(s)$

Soln:-

Assume movement toward right $x(t)$ as positive, draw the free body diagram.



Differential equation of motion using Newton's law we get.

$$M\frac{d^2x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + kx(t) = f(t)$$

$$\Rightarrow Ms^2X(s) + f_v sX(s) + kX(s) = F(s)$$

$$\Rightarrow X(s)\left[Ms^2 + f_v s + k\right] = F(s)$$

$$\therefore G(s) = \frac{X(s)}{F(s)} = \frac{\cancel{Ms^2+f_v s+k}}{} \quad \frac{1}{Ms^2+f_v s+k}$$

(Ans.)

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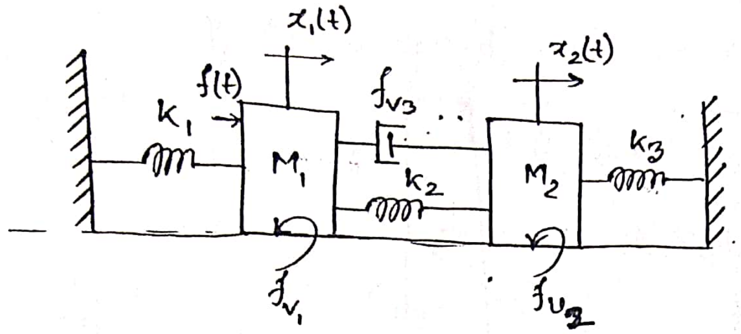
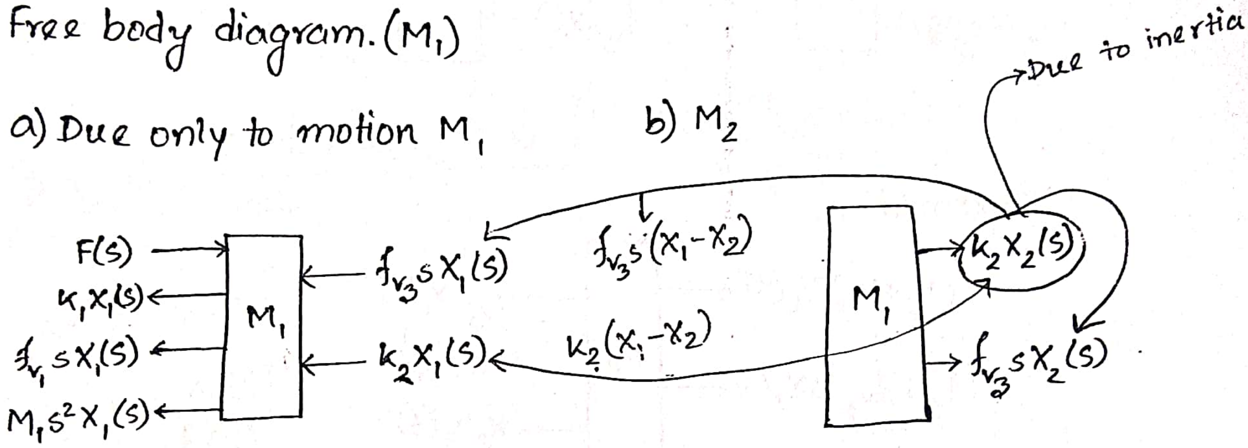
Ex-2.17 (P-65)

Find transfer func $X_2(s)/F(s)$

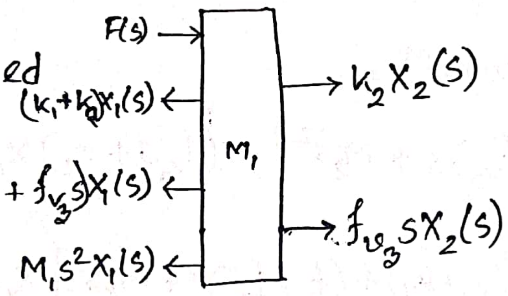


Free body diagram. (M_1)

a) Due only to motion M_1 b) M_2



c) combined



The equation of motion for M_1,

$$F(s) + K_2X_2(s) + f_{v_3}sX_2(s) - (K_1+K_2)X_1(s) - (f_{v_1}+f_{v_3})sX_1(s) - M_1s^2X_1(s) = 0$$

$$\Rightarrow F(s) - X_1(s)\left[(K_1+K_2) + (f_{v_1}+f_{v_3})s + M_1s^2\right] + X_2(s)\left[K_2 + f_{v_3}s\right] = 0$$

$$\therefore \left[M_1s^2 + (f_{v_1}+f_{v_3})s + (K_1+K_2)\right]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s) \quad \text{--- (i)}$$

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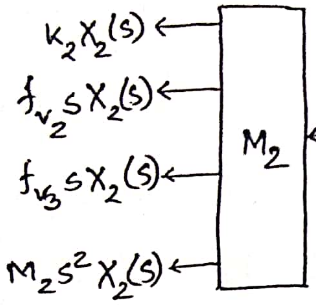
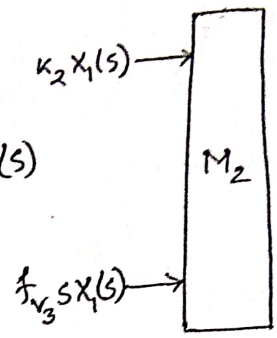
Free body diagram (M_2)

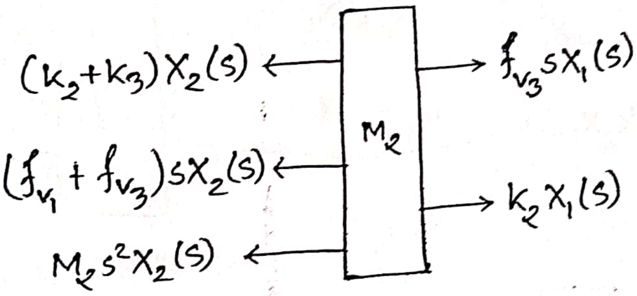
a) Due only to M_2



b) Due to M_1 only.



c) Combined



The equation of motion for M_2,

$$X_2(s)\left[(k_2+k_3)+(f_{v_1}+f_{v_3})s+M_2s^2\right]-(f_{v_3}s+k_2)X_1(s)=0$$

$$\Rightarrow -(f_{v_3}s+k_2)X_1(s)+\left[(k_2+k_3)+(f_{v_1}+f_{v_3})s+M_2s^2\right]X_2(s)=0 \quad \text{--- (ii)}$$

$$\left[M_1s^2+(f_{v_1}+f_{v_3})s+(k_1+k_2)\right]X_1(s)-(f_{v_3}s+k_2)X_2(s)=F(s) \quad \text{--- (i)}$$

~~From Cramer's~~ rule

$$\cancel{-(f_{v_3}s+k_2)}$$

$$\cancel{X_2(s)} = \frac{}{}$$

From Cramer's Rule.

$$X_2(s) = \frac{\begin{vmatrix} M_1s^2+(f_{v_1}+f_{v_3})s+(k_1+k_2) & F(s) \\ -(f_{v_3}s+k_2) & 0 \end{vmatrix}}{\begin{vmatrix} M_1s^2+(f_{v_1}+f_{v_3})s+(k_1+k_2) & -(f_{v_3}s+k_2) \\ -(f_{v_3}s+k_2) & [M_2s^2+(f_{v_2}+f_{v_3})s+(k_2+k_3)] \end{vmatrix}}$$

$$=\Delta$$

$$\Rightarrow X_2(s) = \frac{0-[-F(s)(f_{v_3}s+k_2)]}{\Delta} = \frac{F(s)(f_{v_3}s+k_2)}{\Delta}.$$

We know,

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{f_{v_3}s+k_2}{\Delta}.$$

(Ans).

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