CSE1007: Logical Fundamentals of Programming

Part I - Propositional Logic

Scott Uk-Jin Lee

Department of Computer Science and Engineering
Hanyang University ERICA Campus

1^{st} Semester 2016

1. The Logic of Atomic Sentences

Objectives :

- learn **Forst-Order Logic (FOL)**
- understand concepts and methods for deducing logical results from the given facts

First-Order Logic (FOL)

- Logic :
  - "reasoning conducted or assessed according to strict principles of validity"
- Syntax
  - defines the structure of logical sentences
- Semantics
  - defines the meaning(intepretation) of logical sentences
  - either TRUE or FALSE

Individual Constants

- symbol/name that refer to an actually existing object

Natural Language	Ji-Sung Park	Hanyang University ERICA Campus	Three
individual constants of FOL (all small letters)	park	hyu-erica	3

- Restrictions
  - Every individual constant **must name** an actually existing object
  - **No** individual constant can name more than one object
    - Ji-Sung Park? Tae-Hwan Park? Chan Ho Park? ⇒ park X
  - A single object can have **more than one** name
    - Scott Lee ⇒ professor-lee, doctor-lee, mr-lee, scott, s √
  - An object can have **no name** at all

HYU 한양대학교 HANYANG UNIVERSITY

Terms

- Simple terms
  - individual constants

- Complex terms
  - formed using **functional symbol** with more than 1 (simple/complex) term as arguments
  - Hence, **functional symbol(terms)**

e.g,

Natural Language	Sarah's mother	Sarah's mother's mother
term of FOL (all small letters)	mother(sarah)	mother(mother(sarah))

  - used just like names(simple terms) → the same restrictions on individual constant applies to complex term
  - Hence, must refer to **one and only one** object

Predicate Symbols

- express **property** of objects or **relationship** between objects
- can be considered as a *verb* or *adjective*

Natural Language	Sarah is at Home	Sarah likes Tom's brother
predicate of FOL (starts with a Capital letter)	Home	Likes

- arity (number of arguments) is fixed

e.g., Home - predicate symbol with arity 1(unary)
Likes - predicate symbol with arity 2 (binary)
Sent - predicate symbol with arity 3 (ternary)

Atomic Sentences

- formed by **predicate** of arity *n* followed by *n* **terms** (predicate having terms as arguments)
- Hence, **predicate(term, ..., term)**

Natural Language	Sarah is at Home	Sarah likes Tom's brother
FOL	Home(sarah)	Likes(sarah, brother(tom))

- arity(no. of arguments) associated with a predicate is **fixed**

Atomic Sentence	Predicate	Arguments	Arity
Home(sarah)	Home	sarah	unary
Likes(sarah,brother(tom))	Likes	sarah, brother(tom)	binary
Sent(sarah, tom, email)	Sent	sarah, tom, email	ternary

한양대학교 HANYANG UNIVERSITY

Notation & Semantics of Atomic Sentences

Prefix/Infix Notation

- Prefix notation
  - **Predicate** comes first, then **arguments (terms)** surrounded with brackets comes after

e.g., Like(sarah, tom)

- Infix notation
  - **Predicate** comes in-between **arguments (terms)**

e.g., a = b

Semantics of Atomic Sentences

- Semantics: True or False

e.g., Teaches(professor-lee, logic) ⇒ True
Teaches(jaeho,logic) ⇒ False

Characteristics of Atomic Sentences

- The order of terms (arguments) for predicates are important!
  - change of order $\rightarrow$ change of meaning
  - Likes(sarah, tom)
  - Likes(tom, sarah)

- The meaning of an atomic sentence is always **determinate**!
  - always the same meaning! the meaning never changes!
  - Home(sarah)
  - Young(tom)
  - Loves(sarah, tom)

HYU 한양대학교 HANYANG UNIVERSITY

Differences between the below 3 Atomic Sentences?

- Syntax : TomGaveCandyTo(y)
- Semantic : Tom gave a candy to y

e.g., TomGaveCandyTo(sarah)

- Syntax : GaveCandyTo(x,y)
- Semantic : x gave a candy to y

e.g., GaveCandyTo(tom,sarah)

- Syntax : Gave(x,y,z)
- Semantic : x gave z to y

e.g., Gave(tom,sarah,candy)

Lets Play with Tarski's World!

- How to play with Tarski's World?
  - reference: LPL Software Manual
    - Chapter 3 Using Tarski's World
- Homework : get friendly with the Tarski's World
- Let's get into Tarski's World and interpret the meanings of atomic sentences
  - Let's play with the file, 'Wittgenstein's World'
  - Let's play with the file, 'Wittgenstein's Sentences'

Example of FOL Language 1 : FOL Language of Set Theory

Predicates

- $=$
  - predicate that checks whether two sets are the same (set equality)
  - infix & binary operator

e.g., atomic sentence: ansan $=$ haengdang, hanyang $=$ hanyang

- $\in$
  - predicate that checks whether the element is member of a set (set membership)
  - infix & binary operator

e.g., atomic sentence: let a be the name of 1 & b be the name of $\{1,3,5\}$, then the meaning of followings are?

- a $\in$ a
- a $\in$ b
- b $\in$ a
- b $\in$ b

Example of FOL Language 2 : FOL Language of Arithmetics

- Atomic Sentences
  - Names: 0, 1
  - Predicate: $=$, $<$ (infix, binary)
  - Functional symbols: $+$, x (infix, binary)
- Terms
  - Consideration: How many terms can we make? $\rightarrow$ too many
  - Definition (Inductive): term
    1. Then name 0 & 1 are terms
    2. if t_1 & t_2 are terms,
       then the expression $(t_1 + t_2)$ and $(t_1 \times t_2)$ are also terms
    3. Nothing is a term unless it can be obtained by
       repeated application of (1) and (2)

e.g., term : $(1+1)$ & $((1+1) \times (1+1))$
e.g., atomic sentence : $(1 \times 1) < (1+1)$ & $(1 \times 1) = (1+0)$

HYU 한양대학교 HANYANG UNIVERSITY

Arguments

- Arguments = a series of statements consisting of one or more **premises** & **conclusion**

statement 1	
statement 2	**premises**
...	
statement n	
statement	**conclusion**

- **Example**: All men are mortal. Socrates is a man. So, Socrates is mortal.
  - **premise 1**: All men are mortal.
  - **premise 2**: Socrates is a man.
  - **conclusion**: Socrates is mortal.

Logically Valid Arguments

- An argument is **valid** if :
  - the conclusion must be true
    in any circumstance in which the premises are true

- The conclusion of logically valid argument is a **logical consequence** of its premises.

Are these Arguments Logically Valid ?

- **Argument 1**:
  All men are mortal. Socrates is a man. So, Socrates is mortal.
  - **premise 1**: All men are mortal.
  - **premise 2**: Socrates is a man.
  - **conclusion**: Socrates is mortal.

  $\surd$ **Logically Valid**

- **Argument 2**:
  All smart students are mortal. All HYU students are smart students. So, all HYU students are mortal.
  - **premise 1**: All smart students are mortal.
  - **premise 2**: All HYU students are smart students.
  - **conclusion**: All HYU students are mortal.

  $\surd$ **Logically Valid**

Are these Arguments Logically Valid ?

- **Argument 3**:
  All smart students are mortal. All HYU students are mortal.
  So, all HYU students are smart students.
  - **premise 1**: All smart students are mortal.
  - **premise 2**: All HYU students are mortal.
  - **conclusion**: All HYU students are smart students.

X Logically Invalid

- How to prove logical invalidity of an argument ?
  - Find a counter-example
  - **Counter-example**: conclusion is false even if all of its premises are true.
  - counter-example of Argument 3
    - all smart students are mortal,
    - a HYU student Mr. 'Dumb' is mortal,
    - BUT Mr. 'Dumb' is not smart!

Logically Sound Arguments

- An argument is **sound** if :
  1. argument is valid
  2. its premises are all true

- Validity
  - guarantees the truth of the conclusion
  - assume premises to be true

- Soundness
  - guarantees the truth of the conclusion
  - all premises are true

  $\Rightarrow$ insure the truth of the conclusion

Are these Arguments Logically Sound ?

- **Argument 1**:
  All men are mortal. Socrates is a man. So, Socrates is mortal.
  - **premise 1**: All men are mortal.
  - **premise 2**: Socrates is a man.
  - **conclusion**: Socrates is mortal.

  $\surd$ **Logically Valid** & $\surd$ **Logically Sound**

- **Argument 2**:
  All women are mortal. Socrates is a woman. So, Socrates is mortal.
  - **premise 1**: All women are mortal.
  - **premise 2**: Socrates is a woman.
  - **conclusion**: Socrates is mortal.

  $\surd$ **Logically Valid** but **X Logically Unsound**

What is Important in the Study of Logic ?

- Answer: **Logical Validity**
- **Study of Logic**
  - focuses on proving the logical validity of an arguments
  - a valid argument are sound if all of its premises are true BUT finding out whether any premises is true or false is not part of the study

    - The study of logic is never interested in finding out whether Socrates was a man or a woman

Format for writing Arguments

Fitch* Format

All women are mortal. Socrates is a woman.	**premises**
Socrates is mortal.	**conclusion**

* Frederic Fitch: American Logician

Proof

- What is a **proof** ?
  - step-by-step demonstration which shows that
    - **statement (conclusion)** must be true
    - in any circumstances in which the **statement**$_1$**, statement**$_2$**, ... , statement**$_n$ **(premises)** are all true.

$$
\begin{array}{|l}
\text{statement}_1 \\
\text{statement}_2 \\
\ldots \\
\text{statement}_n \\
\hline
\text{statement}
\end{array}
$$

premises

conclusion

Proof Methods

- Informal Proof
  - using natural language
  - all proofs were done this way before Computer Science (most of the mathematical proof)
  - done by human
- Formal Proof
  - using symbolic logic
  - natural deduction system
  - done by computer (automatic theorem proving)
- Difference
  - both rigor
  - style: natural language $_{vs}$ symbolic logic

Informal Proofs

| Socrates is a man.
| All men are mortal.
| All mortals worry about dying sometimes.
|―――
| Socrates worries about dying sometimes.

- **Proof**
  - Socrates is a man and all men are mortal.
    So, Socrates is a mortal.
  - It is premised that all mortals worry about dying sometimes.
    So, Socrates worries about dying sometimes.

HYU 한양대학교 HANYANG UNIVERSITY

Fitch-style Proof

- using Deductive System designed by Frederic Fitch
- called "**System F**"

	P_1	
	...	
	P_n	
	S_1	justification 1
intermediate conclusion	...	...
	S_n	justification n
	S	justification n+1

Formal Proof Rules for Identity

- **Identity Introduction (= Intro)**
  - ▷ | $n = n$
- **Identity Elimination (= Elim)**

  | $P(n)$
  | ...
  | $n = m$
  | ...
  ▷ | $P(m)$

- **Reiteration (= Reit)**

  | P
  | ...
  ▷ | P

Example of Formal Proof

1.	Cube(c)
2.	c = b
3.	Cube(b)

- Symmetry of Identity

1.	a = b
	...
?.	b = a

Example of Formal Proof

1.	Cube(c)	
2.	c = b	
3.	Cube(b)	= Elim: 1,2

- Symmetry of Identity

1.	a = b	
2.	a = a	= Intro
3.	b = a	= Elim: 2,1

Another Example of Formal Proof

1. SameRow(a,a)
2. b = a

?. SameRow(b,a)

Another Example of Formal Proof

1.	SameRow(a,a)
2.	b = a
?.	a = b
?.	SameRow(b,a) = Elim: 1,?

Another Example of Formal Proof

1.	SameRow(a,a)	
2.	b = a	
3.	b = b	= Intro
4.	a = b	= Elim: 3,2
5.	SameRow(b,a)	= Elim: 1,4

Lets Play with Fitch!

- Fitch
  - is an implementation of the Proof System "System F"
  - is a software for constructing Formal Proofs
  - can automatically check constructed proofs

  - Let's construct previously covered proofs with Fitch

- Analytic Consequence (Ana Con)
  - Let's open 'Ana Con 1' file and play with it

Proof of invalidity

Proof of nonconsequences

- Find **counter-example**:
  - all premises (P_1, ... P_n) is true
  - BUT conclusion (S) is not true
  - $\Rightarrow$ proves that argument is NOT valid

| Kim Gu is a politician
| Most politicians are dishonest
|---
| Kim Gu is dishonest

2. The Boolean Logic

Boolean Logic Sentences

- Atomic sentences
- Complex sentences
  - atomic sentences connected with Boolean connectives (operators)
- Boolean connectives
  - Negation (not) : $\neg$
  - Conjunction (and) : $\wedge$
  - Disjunction (or) : $\vee$

Truth-Functional Connectives

- Boolean connectives ($\neg$, $\wedge$, $\vee$) are truth-functional connectives
- The truth value of a complex sentence (built up using connective) depends on nothing more than the truth values of the simpler sentences from which it is built
- Therefore, the semantics of a complex sentence built up using truth-functional connectives can be given by **truth table**
- Henkin-Hintikka game
  - one player claims a complex sentence is true & the other player claims it is false
  - reduce the complex sentence into a atomic sentence and justify their claims

Negation Symbol : $\neg$

- Syntax
  - if P is a sentence, then $\neg$P is also a sentence
  - prefix, unary
  - **literal** : if a sentence is an atomic sentence or the negation of an atomic sentence
  - abbreviation of negated identity claims: $\neq$
    e.g.1 $\neg(\mathsf{b} = \mathsf{c}) \to \mathsf{b} \neq \mathsf{c}$
- Semantics
  - $\neg$P is true **if and only if** P is false
  - Truth Table :

P	$\neg$P
true	false
false	true

  - Game: Let's play in Wittgenstein's World
    $\neg\neg\neg\neg\neg\neg$Between(e,d,f)

Conjunction Symbol : $\wedge$

- Syntax
  - if P and Q are sentences, then P$\wedge$Q is also a sentence
  - infix, binary
- Semantics
  - P$\wedge$Q is true **if and only if** both P and Q are true
  - Truth Table :

P	Q	P$\wedge$Q
true	true	true
true	false	false
false	true	false
false	false	false

  - Game: Let's play in Claire's World
    $\neg$Cube(a) $\wedge$ $\neg$Cube(b) $\wedge$ $\neg$Cube(c)

Disjunction Symbol : $\vee$

- Syntax
  - if P and Q are sentences, then $P \vee Q$ is also a sentence
  - infix, binary
- Semantics
  - $P \vee Q$ is true **if and only if** P is true or Q is true (or both P and Q are true)
  - Truth Table :

P	Q	$P \vee Q$
true	true	true
true	false	true
false	true	true
false	false	false

  - Game: Let's play in Ackermann's World

    $\text{Cube}(c) \vee \neg(\text{Cube}(a) \vee \text{Cube}(b))$

HYU 한양대학교 HANYANG UNIVERSITY

Ambiguity & Parentheses

- Ambiguous sentences

  e.g.1 : Home(max) $\lor$ Home(claire) $\land$ Happy(carl)
  e.g.2 : $\neg$Home(claire) $\land$ Home(max)

  - Are there sentences ambiguous? Why?

- How to avoid the ambiguity?
  - use parentheses

    e.g.1 (Home(max) $\lor$ Home(claire)) $\land$ Happy(carl)
    Home(max) $\lor$ (Home(claire) $\land$ Happy(carl))
    e.g.2 ($\neg$Home(claire)) $\land$ Home(max)
    $\neg$(Home(claire) $\land$ Home(max))

  - give different precedence

    $\neg \quad > \quad \land \quad > \quad \lor$

- ( ) must be used for $\land$ and $\lor$ when combined with other connectives

Useful Laws

- Double Negation

  $\neg\neg P \Leftrightarrow P$ for all P

- DeMorgan's Law

  $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$ for all P and Q

  $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$ for all P and Q

3 Concepts to Learn

- Tautologies and Logical Truth
- Logical and Tautological Equivalence
- Logical and Tautological Consequence

Interpret Boolean Logic Sentences with Truth Table

- Truth table can easily be generated with "Boole"
- Practice :
  1. $P \lor \neg P$
  2. $(A \land B) \lor \neg C$
  3. $\neg(A \land (\neg A \lor (B \land C))) \lor B$
  4. $P \land Q \land R$

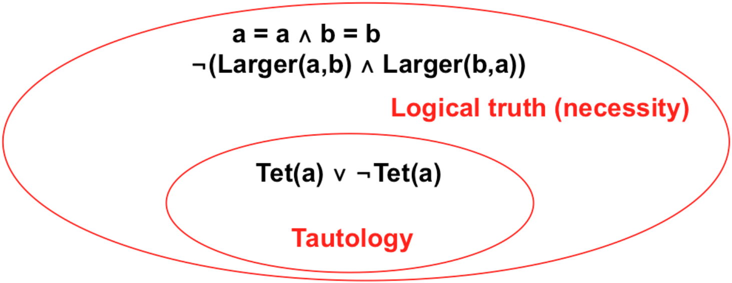
  * A, B, C, P, Q, R refers to atomic sentences

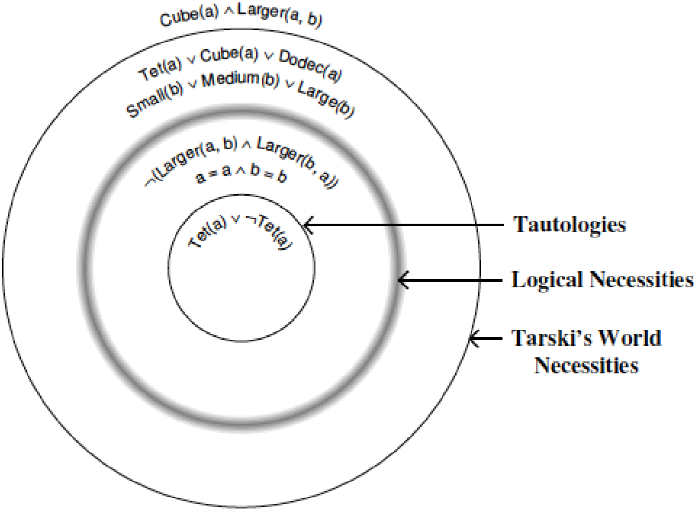
Tautologies and Logical Truth (Necessity)

- If a sentence is true in every logically possible situation, the sentence is called **logically true** or **logically necessary**
- a sentence that can only be true in any case is called **tautology**
  - a sentence is tautology **iff** every row of the truth table assigns TRUE to the sentence
  - is also called **TT-necessary**
    e.g. $P \lor \neg P$
  - If a sentence is true in every world in Tarski's World, then it is called **TW-necessary**

Tautologies and Logical Truth

- If a sentence is Tautology, then it is logical truth
- BUT if a sentence is logically true,it is not always tautology (some logical truths are not tautologies)




Cube(a) ∧ Larger(a, b)
Tet(a) ∨ Cube(a) ∨ Dodec(a)
Small(b) ∨ Medium(b) ∨ Large(b)
¬(Larger(a, b) ∧ Larger(b, a))
a = a ∧ b = b
Tet(a) ∨ ¬Tet(a)
Tautologies
Logical Necessities
Tarski's World Necessities

Logical Possibility

- If there exists a situation or a world where a sentence is true, then the sentence is **logically possible**
- If at least one row of the Truth Table assigns TRUE to a sentence, then the sentence is **TT-possible**
- If there is at least one world that is true in theTarski's World, then the sentence is called **TW-possible**
- logically possible vs. TW-possible
  - all TW-possible sentences are logically possible
  - BUT there exists a sentence that is logically possible but not TW-possible

    e.g. $\neg(\mathrm{Tet}(b) \lor \mathrm{Cube}(b) \lor \mathrm{Dedoc}(b))$

Tautological Equivalence

- S and S' are tautologically equivalent **iff** every column for S and S' in the joint Truth Table are the same

e.g. $\neg(A \wedge B)$ vs $\neg A \vee \neg B$

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$\neg A \vee \neg B$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\surd$ Tautologically Equivalent

Tautological Equivalence

e.g. $\neg((A \lor B) \land \neg C)$ <u>vs</u> $(\neg A \land \neg B) \lor C$

A	B	C	$A \lor B$	$\neg C$	$(A \lor B) \land \neg C$	$\neg((A \lor B) \land \neg C)$	$\neg A$	$\neg B$	$\neg A \land \neg B$	$\neg A \land \neg B) \lor C$
T	T	T	T	F	F	T	F	F	F	T
T	T	F	T	T	T	F	F	F	F	F
T	F	T	T	F	F	T	F	T	F	T
T	F	F	T	T	T	F	F	T	F	F
F	T	T	T	F	F	T	T	F	F	T
F	T	F	T	T	T	F	T	F	F	F
F	F	T	F	F	F	T	T	T	T	T
F	F	F	F	T	F	T	T	T	T	T

$\surd$ Tautologically Equivalent

Logical Equivalence

$a = b \land Cube(a)$ and $a = b \land Cube(b)$ are logically equivalent

Proof :

- Suppose that $a = b \land Cube(a)$ is true, then $a = b$ and $Cube(a)$ are both true. Since a and b are equal, $Cube(b)$ is also true (=Elim). Hence, $a = b \land Cube(b)$ is true.
  So the truth of $a = b \land Cube(a)$ logically implies the truth of $a = b \land Cube(b)$.
- Reverse :
  Suppose that $a = b \land Cube(b)$ is true, then $a = b$ is true and $Cube(b)$ is true. Then $b = a$ is also true (Symmetry of Identity). Since b and a are equal, $Cube(a)$ is also true (=Elim). Hence, $a = b \land Cube(a)$ is true.
  So the truth of $a = b \land Cube(b)$ logically implies the truth of $a = b \land Cube(a)$.
- Since both sentences can logically imply truth of each other in both directions, the two sentences are logically equivalent.

Tautological Equivalence vs. Logical Equivalence

- If S and S' are tautologically equivalent, then they are also logically equivalent.
- Some logically equivalent sentences are not tautologically equivalent.
  - e.g. the following two sentences are logically equivalent BUT not tautologically equivalent

    $\mathsf{a} = \mathsf{b} \wedge \mathsf{Cube(a)}$ vs $\mathsf{a} = \mathsf{b} \wedge \mathsf{Cube(b)}$
  - let's draw the joint Truth Table

Tautological Equivalence vs. Logical Equivalence

a=b ∧ Cube(a) and a=b ∧ Cube(b) are tautologically equivalent?

a=b	Cube(a)	Cube(b)	a=b ∧ Cube(a)	a=b ∧ Cube(b)
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

X NOT tautologically equivalent

Understand why?

Tautological & Logical Consequence

Let P_1, ..., P_n and Q be sentences & construct a joint Truth Table

- Tautological Consequence
  - Q is a tautological consequence of P_1, ..., P_n **iff** every row that assigns T to each of P_1, ..., P_n also assigns T to Q in the joint Truth Table
- Logical Consequence
  - Q is a logical consequence of P_1, ..., P_n **iff** Q is true when we assume all P_1, ..., P_n are true

Tautological & Logical Consequence

e.g. Determination of tautological consequences using Truth Table

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

$A \vee B$ is a tautological consequence of $A \wedge B$

= $A \vee B$ is a logical consequence of $A \wedge B$

= $A \wedge B$ logically implies $A \vee B$

$A \wedge B$ is NOT a tautological consequence of $A \vee B$

= $A \vee B$ does NOT logically implies $A \wedge B$

Logical Consequence

If Q is a tautological consequence of P_1, ..., P_n,
then Q is a logical consequence of P_1, ..., P_n

Proof :

- proves if Q is not a logical consequence of P_1, ..., P_n, then Q is not a tautological consequence of P_1, ..., P_n
- Suppose Q is not a logical consequence of P_1, ..., P_n.
  Then, by the definition of logical consequence, there must be a possible circumstance in which P is true but Q is false.
  This circumstance will determine the truth values of the atomic sentences in P and Q. Furthermore, the row for P and Q corresponding to these values exists in the joint Truth Table, then P will be assigned T and Q will be assigned F in that row.
  Hence, Q is not a tautological consequence of P_1, ..., P_n.

Logical but NOT Tautological Consequence

- a=c is a logical consequence of (a=b ∧ b=c)
- BUT a=c is not a tautological consequence of (a=b ∧ b=c)
- Let's draw a joint Truth Table and prove why

a=b	b=c	a=c	a=b ∧ b=c	a=c
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

More Examples

$$\begin{array}{|l} A \lor B \\ \neg A \\ \hline B \end{array} \qquad \begin{array}{|l} A \lor \neg B \\ B \lor C \\ \hline A \lor C \end{array}$$

- Create a Truth Table with 'Boole' and check whether the conclusion is a tautological consequence of premises.

HYU 한양대학교 HANYANG UNIVERSITY

Logical Consequence in Fitch

- **Taut Con** : **Taut**ological **Con**sequence
- **FO Con** : **F**irst-**O**rder **Con**sequence
  - understand not only tautological consequence but also $=$
- **Ana Con** : **Ana**lytic **Con**sequence
  - understand not only logical consequence but also predicates in Tarski's World
- Let's play with Fitch using the following files
  - Taut Con 1
  - Taut Con 2

Summary

Logical Consequence vs. Logical Equivalence / Logical Truth

- If P is a logical consequence of Q and Q is a logical consequence of P, then P and Q are logically equivalent.
- if a sentence is a logical consequence without premises, then it is logical truth.

3. Informal Proof

Problems & Limitations of Truth Table

1. only effective for showing validity of simple logical sentences that depends only on Boolean connectives
2. The table size gets extremely large as the number of atomic sentences goes up
   - number of atomic sentences = n
   - number of rows in the corresponding Truth Table = 2^n (exponential growth)
   - not very effective as too many calculations are required
3. cannot be easily extended
   - can only be used for sentences composed of Boolean connectives
   - can only detect tautological consequence

Informal Proof

- Overcomes the limitations of Truth Table
- The purposes of proofs
  - method of discovery:
    extract new information from information already obtained
  - method of communication:
    convey discoveries to others
- Proof must be written so that; the writer of the proof (proof writer) can convince the audience (proof reader)!
- There is a style in giving proofs
  - every proof writer has his/her own style

Informal Proof

- a "Good" proof is
  1. **correct**
  2. **easily understood**:
     proof readers should be able to follow the proof step without any complex reasoning of their own
  3. **significant**:
     the proof step should be informative, not a waste of the proof readers' time.

  - 2 and 3 are the opposite properties:
    more significant steps are harder to follow &
    easy to understand steps makes proof verbose

Hence a "good" proof requires a proper balance between the two.

Valid Inference Steps

- if Q is already known to be a logical consequence of sentence P_1, ..., P_n and each of P_1, ..., P_n has been proven from the premises, then Q can be asserted in the proof
- logical truths such as a= a or $P \vee \neg P$ can be asserted at any point in the proof
- The following are valid patterns of inference that generally go unmentioned in informal proofs
  1. **Conjunction Elimination**: From $P \wedge Q$, infer P
  2. **Conjunction Introduction**: From P and Q, infer $P \wedge Q$
  3. **Disjunction Introduction**: From P , infer $P \vee Q$

HYU 한양대학교 HANYANG UNIVERSITY

Informal Proof Methods

- Proof By Cases
- Indirect Proof
    - Proof By Contradiction

Proof by Cases

Disjunction Elimination in System F

- To prove S from $P_1 \vee \ldots \vee P_n$

  break disjunction into n cases
  prove S from P_1
  prove S from P_2
  ...
  prove S from P_n

- Pattern for proving S from $P \vee Q$
  - let's say that the desired goal is to prove S
  - assume that we already know $P \vee Q$
  - assume that P is true and prove if S is true
  - assume that Q is true and prove if S is true
  - since we know that either P or Q is true, S must be true

- cannot be proved with Truth Table

Proof by Cases: Example 1

Theorem: there exists irrational numbers b and c such that b^c is a rational number

Proof

- consider $\sqrt{2}^{\sqrt{2}}$, this number is either rational or irrational
- case 1: $\sqrt{2}^{\sqrt{2}}$ is a rational number
  - $\mathrm{b} = \mathrm{c} = \sqrt{2}$ exists
- case 2: $\sqrt{2}^{\sqrt{2}}$ is an irrational number
  - take $\mathrm{b} = \sqrt{2}^{\sqrt{2}}$ & $\mathrm{c} = \sqrt{2}$ and compute b^c
  - $\mathrm{b}^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}.\sqrt{2})} = \sqrt{2}^2 = 2$
  - in this case b^c is also a rational number

Proof by Cases: Example 2

Theorem: *Small(c)* is a logical consequences of

$$(Cube(c) \land Small(c)) \lor (Tet(c) \land Small(c))$$

Proof

- premise: (Cube(c) ∧ Small(c)) ∨ (Tet(c) ∧ Small(c))
- case 1: assume that (Cube(c) ∧ Small(c)) holds
  - then (by Conjunction Elimination) we have Small(c)
- case 2: assume that (Tet(c) ∧ Small(c)) holds
  - then (by Conjunction Elimination) we have Small(c)

Proof by Cases: Example 3

Theorem: assume that

$(Home(max) \wedge Happy(carl)) \vee (Home(claire) \wedge Happy(scruffy))$ is true

and prove that $Happy(carl) \vee Happy(scruffy)$ is true

Proof

- premise: (Home(max)∧Happy(carl)) ∨ (Home(claire)∧Happy(scruffy)) is in the form of disjunction
- case 1: assume that (Home(max) ∧ Happy(carl)) holds
  - then (by Conjunction Elimination) Happy(carl) is true
  - then (by Disjunction Introduction) Happy(carl) ∨ Happy(scruffy) is true
- case 2: assume that (Home(claire) ∧ Happy(scruffy)) holds
  - then (by Conjunction Elimination) Happy(scruffy) is true
  - then (by Disjunction Introduction) Happy(carl) ∨ Happy(scruffy) is true

HYU 한양대학교 HANYANG UNIVERSITY

Proof by Cases: Example 4 (Logician Couple's Story)

A logician and his wife recently realized that their parking meter had expired several hours earlier

Logician's Proof

- we've already gotten a ticket or we haven't
- case 1: if we've gotten a ticket, we won't get another one in the time it takes us to get to the car
- case 2: if we haven't got a ticket in the past several hours, it is extremely unlikely that we get one in the next few minutes
- In either event, there's no need to rush

Wife's Proof

- we are going to get a ticket in the next few minutes or we aren't
- case 1: if we are going to get a ticket, then rushing might prevent it
- case 2: if we aren't going to get a ticket, then it will still be a good exercise and will also show our respect for the law
- In either event, rushing back to the car is a good thing

Indirect Proof: Proof by Contradiction

= **Reductio ad absurdum**

= **Negation Introduction** in System F

- Mehod of Proof
  - let's say that the desired goal is to prove $\neg$S from some premises P_1, ..., P_n
  - temporarily assume that S is true and show a contradiction follows from this assumption
    (show that S, P_1, ..., P_n cannot be all true simultaneously)
  - can conclude that $\neg$S is a logical consequence of P_1, ..., P_n
    (if P_1, ..., P_n are true, then $\neg$S must be true)

- contradiction: $\perp \equiv P \wedge \neg P$ (can never be true)

Proof by Contradiction: Example 1

suppose Cube(c) $\vee$ Dodec(c) & Tet(b) are true, then $\neg$(b $=$ c) is true

Proof

- assume that b $=$ c are true & attempt to get a contradiction
- by the 1^{st} premise, c is either a Cube or a Dodec
- case where c is Cube: since b $=$ c, b is a Cube
  it contradicts the 2^{nd} premise, Tet(b)
- case where c is Dodec: since b $=$ c, b is a Dodec
  it contradicts the 2^{nd} premise, Tet(b)
- since neither case is possible, we have a contradiction
  Hence, b $=$ c must be false

Proof by Contradiction: Example 2

$\sqrt{2}$ is an irrational number

- hint 1: rational number can be expressed as a fraction p/q
  where at least one of p and q is an odd number
- hint 2: square of an odd number is always an odd number.
  So if n^2 is an even number, then n is also an even number
  Hence, if n^2 is even, it must be divisible by 4

Proof

- assume that $\sqrt{2}$ is a rational number & try to get contradiction
- $\sqrt{2}$ can be expressed as p/q and at least one of p and q is odd
- since $p/q = \sqrt{2}$, square both side to get $p^2/q^2 = 2$
- and then multiply both side with q^2 to get $p^2 = 2q^2$

HYU 한양대학교 HANYANG UNIVERSITY

Proof by Contradiction: Example 2

Proof (continues)

- so p^2 is an even number. Hence, p is even and p^2 is divisible by 4
- so $2q^2$ is divisible by 4, then q^2 is divisible by 2
- so q^2 is even, then q is even
- both p and q being even contradicts the fact that at least one of them is odd
- Hence, the assumption of $\sqrt{2}$ being rational leads to a contradiction
- $\sqrt{2}$ is an irrational number

Contradiction vs. Tautology

- contradiction $\equiv$ P $\wedge$ $\neg$P
- tautology $\equiv$ P $\vee$ $\neg$P
- S is a logical impossibility $\equiv$ $\neg$S is a logical truth (necessity)
- S is a tautology $\equiv$ $\neg$S is a contradiction
- P_1, P_2, ..., P_n are TT-contradictory
    - if every row has an F assigned to at least one of the sentences in the joint Truth Table
    - hence, P_1, P_2, ..., P_n cannot all be true at once

Proof by Contradiction: Example 3

Defense attorney presenting the following summary to the jury

- Prosecution claims that my client killed the owner of the KitKat Club
- Assume that prosecution is correct.
- You've heard their own experts testifying that the murder took place at 5:15pm
- According to the testimony of 5 co-workers, we also know the defendant was still at work at City Hall at 4:45pm.
- Then my client had to get from City Hall to the KitKat Club in 30 minutes or less.
- But to make that trip takes 35 minutes under the best of circumstances, and police records show that there was a massive traffic jam the day of the murder.
- Hence, my client is innocent.

Arguments with Inconsistent Premises

- if P_1, ..., P_n are contradictory, then they are called inconsistent
- any argument with an inconsistent set of premises is trivially valid BUT it is unsound

e.g.

$$
\begin{array}{|l}
\text{Home(max)} \lor \text{Home(claire)} \\
\neg\text{Home(max)} \\
\neg\text{Home(claire)} \\
\hline
\text{Home(max)} \land \text{Happy(carl)}
\end{array}
$$

4. Formal Proof

System F: Fitch-Style Natural Deduction System

- Proof rules
  - Conjunction rules
  - Disjunction rules
  - Negation rules
  - (Contradiction rules)
- Each rule consists of
  - an introduction rule
  - an elimination rule

Conjunction Rules

- **Conjunction Elimination Rule ($\wedge$ Elim)**

$$\begin{array}{r|l} & P_1 \wedge \ldots \wedge P_i \wedge \ldots \wedge P_n \\ & \vdots \\ \triangleright & P_i \end{array}$$

- **Conjunction Introduction Rule ($\wedge$ Intro)**

$$\begin{array}{r|l} & P_1 \\ & \Downarrow \\ & P_n \\ & \vdots \\ \triangleright & P_1 \wedge \ldots \wedge P_n \end{array}$$

Conjunction Rules

Example:

1.	$A \land B \land C$	
2.	B	$\land$ Elim: 1
3.	C	$\land$ Elim: 1
4.	$C \land B$	$\land$ Intro: 3,2

- **Let's construct a proof in Fitch**
  - Conjunction 1
  - Conjunction 2

Default and Generous Uses of the $\land$ Rules

17.	$Tet(a) \land Tet(b) \land Tet(c) \land Tet(d)$	
⋮		
26.	$Tet(d) \land Tet(b)$	$\land$ Elim: 17

13.	$Tet(a) \land Tet(b)$	
⋮		
21.	$Tet(b) \land Tet(a)$	$\land$ Elim: 13

- **Let's construct a proof in Fitch**
  - Conjunction 3
  - Conjunction 4

Disjunction Rules

- **Disjunction Introduction Rule ($\vee$ Intro)**

$$
\begin{array}{l|l}
 & P_i \\
 & \vdots \\
\triangleright & P_1 \vee \ldots \vee P_i \vee \ldots \vee P_n
\end{array}
$$

- **Disjunction Elimination Rule ($\vee$ Elim)**

$$
\begin{array}{l|l}
 & P_1 \vee \ldots \vee P_n \\
 & \vdots \\
 & \quad \begin{array}{|l} P_1 \\ \vdots \\ S \end{array} \\
 & \quad \Downarrow \\
 & \quad \begin{array}{|l} P_n \\ \vdots \\ S \end{array} \\
 & \vdots \\
\triangleright & S
\end{array}
$$

P_1, P_n : temporary assumptions

: subproofs

= Proof by Cases

Proof using Disjunction Rules: Example 1

1.	$(A \wedge B) \vee (C \wedge D)$	
2.	$A \wedge B$	
3.	B	$\wedge$ Elim: 2
4.	$B \vee D$	$\vee$ Intro: 3
5.	$C \wedge D$	
6.	D	$\wedge$ Elim: 5
7.	$B \vee D$	$\vee$ Intro: 6
8.	$B \vee D$	$\vee$ Elim: 1, 2-4, 5-7

한양대학교 HANYANG UNIVERSITY

Proof using Disjunction Rules: Example 2

1.	$(B \wedge A) \vee (A \wedge C)$	
2.	$B \wedge A$	
3.	A	$\wedge$ Elim: 2
4.	$A \wedge C$	
5.	A	$\wedge$ Elim: 4
6.	A	$\vee$ Elim: 1, 2-3, 4-5

Proof using Disjunction Rules: Example 3

1.	$(B \wedge A) \vee A$	
2.	$B \wedge A$	
3.	A	$\wedge$ Elim: 2
4.	A	
5.	A	Reit: 4
6.	A	$\vee$ Elim: 1, 2-3, 4-5

- **Let's construct a proof in Fitch**
  - Disjunction 1
  - Disjunction 2

Negation Rules

- **Negation Elimination Rule (¬ Elim)**

$$
\begin{array}{l|l}
 & \neg\neg P \\
 & \vdots \\
\triangleright & P
\end{array}
$$

- **Negation Introduction Rule (¬ Intro)**

$$
\begin{array}{l|l}
 & \begin{array}{|l} P \\ \hline \vdots \\ \bot \qquad\qquad\qquad\qquad (\equiv Q \wedge \neg Q) \end{array} \\
\triangleright & \neg P
\end{array}
$$

= Proof by Contradiction

Negation Rules

- **Contradiction Introduction Rule ($\perp$ Intro)**

$$\begin{array}{r|l} & P \\ & \vdots \\ & \neg P \\ & \vdots \\ \triangleright & \perp \end{array}$$

  - only used in subproof
  - can be used in the main proof when proving inconsistency

- **Let's construct a proof in Fitch**
  - Negation 1

Proving Inconsistency (between premises)

1.	$A \vee B$	
2.	$\neg A$	
3.	$\neg B$	
4.	A	
5.	$\perp$	$\perp$ Intro: 4,2
6.	B	
7.	$\perp$	$\perp$ Intro: 6,3
8.	$\perp$	$\vee$ Elim: 1, 4-5, 6-7

Introducing $\perp$ in Fitch

- **Taut Con**
  - can prove using $\wedge$, $\vee$, $\neg$, $\perp$ Intro/Elim
  - TT-contradiction:
    - $\neg(A \vee \neg A)$
    - $\neg A \vee \neg B$ and $A \wedge B$
- **FO Con**
  - can prove using $=$, $\wedge$, $\vee$, $\neg$, $\perp$ Intro/Elim
  - applying =Elim to Cube(b), $b = c$, $\neg$Cube(c) will give Cube(c), $\neg$Cube(c)
- **Ana Con**
  - understands predicates in Tarski's World (except Adjoins and Between)
  - Cube(b), Tet(b)
- **Let's construct a proof in Fitch** : Negation 2

Contradiction Rules

- **Contradiction Introduction Rule ($\perp$ Intro)**

$$
\begin{array}{r|l}
 & P \\
 & \vdots \\
 & \neg P \\
 & \vdots \\
\triangleright & \perp
\end{array}
$$

- **Contradiction Elimination Rule ($\perp$ Elim)**

$$
\begin{array}{r|l}
 & \perp \\
 & \vdots \\
\triangleright & P
\end{array}
$$

  - $\perp$ Elim rule is a shortened version of the below proof

17.	$\perp$	
18.	$\quad \neg P$	
19.	$\quad \perp$	Reit: 17
20.	$\neg\neg P$	$\neg$ Intro: 18-19
21.	P	$\neg$ Elim: 20

- **Let's construct a proof in Fitch**: Negation 3

Default and Generous Uses of the ¬ Rules

- ¬ Elim
  - Fitch allow you to remove any even number of negation signs (¬) in one step
- ¬ Intro
  - Fitch allow you to deduce unnegated sentence A from ¬A (instead of ¬¬A)
- **Let's construct a proof in Fitch**
  - Negation 4

Incorrect Proof

- if you are not careful,
  you may construct a proof that does not follow the premises

1.	$(B \wedge A) \vee (A \wedge C)$	
2.	$B \wedge A$	
3.	B	$\wedge$ Elim: 2
4.	A	$\wedge$ Elim: 2
5.	$A \wedge C$	
6.	A	$\wedge$ Elim: 5
7.	A	$\vee$ Elim: 1, 2-4, 5-6
8.	$A \wedge B$	$\wedge$ Intro: 7,3

- when the subproof ends, the subproof's assumption can no longer be used (assumption has been **discharged** / subproof has been **ended**)

Nested Subproofs

1.	$\neg(P \wedge Q)$	
2.	$\neg(\neg P \vee \neg Q)$	
3.	$\neg P$	
4.	$\neg P \vee \neg Q$	$\vee$ Intro: 3
5.	$\perp$	$\perp$ Intro: 4,2
6.	$\neg\neg P$	$\neg$ Intro: 3-5
7.	P	$\neg$ Elim: 6
8.	$\neg Q$	
9.	$\neg P \vee \neg Q$	$\vee$ Intro: 8
10.	$\perp$	$\perp$ Intro: 9,2
11.	$\neg\neg Q$	$\neg$ Intro: 8-10
12.	Q	$\neg$ Elim: 11
13.	$P \wedge Q$	$\wedge$ Intro: 7,12
14.	$\neg(P \wedge Q)$	Reit: 1
15.	$\perp$	$\perp$ Intro: 13,14
16.	$\neg\neg(\neg P \vee \neg Q)$	$\neg$ Intro: 2-15
17.	$\neg P \vee \neg Q$	$\neg$ Elim: 16

Proof Strategy and Tactics

1. Understand the **meaning of sentences**
2. Decide whether you think the conclusion follows from the premises (this can easily be checked using Taut Con in Fitch)
3. If you think it does not follow, or are not sure, try to find a counter-example
4. If you think it does follow, try to give **informal proof**
5. If a formal proof is called for, use the informal proof to guide you in finding one (there are always corresponding formal proof rules)
   - (informal) proof by contradiction $\Rightarrow$ (formal) $\neg$ Intro rule
   - (informal) proof by cases $\Rightarrow$ (formal) $\vee$ Elim rule
6. In giving consequence proofs, both formal and informal, don't forget the tactic of working backwards
7. In working backwards, though, always check that your intermediate goals are consequences of the available information

Proof Strategy & Tactics: try a bit more difficult proof

$\neg P \lor \neg Q$	This argument is Demorgan's Law
$\neg(P \land Q)$	$\rightarrow$ Hence, it is a valid argument

- Informal Proof: 1^{st} attempt
  - According to premises, we know that P is False or Q is False
  - Since either one of P and Q is False, $P \land Q$ is False
  - Hence, $\neg(P \land Q)$ is True
- It is not very easy to draw up a formal proof even after looking at this informal proof
- Let's work backwards!

$$
\begin{array}{|l}
\neg P \lor \neg Q \\
\hline
\neg(P \land Q)
\end{array}
$$

- Point 1: since conclusion is negation,
  we can assume $P \land Q$ then derive $\perp$
- New Proof Goal: set $P \land Q$ as a premises and derive $\perp$ as the conclusion
- Point 2: since premises are connected with Disjunction,
  we can derive $\perp$ by conducting a proof by cases
- Now, let's use this strategy and tactics to conduct formal proof with Fitch!

Proofs without Premises $\Rightarrow$ Prove Logical Truth

1.	$a = a$	$=$ Intro
2.	$b = b$	$=$ Intro
3.	$a = a \land b = b$	$\land$ Intro: 1,2

1.	$P \land \neg P$	
2.	P	$\land$ Elim: 1
3.	$\neg P$	$\land$ Elim: 1
4.	$\perp$	$\perp$ Intro: 2,3
5.	$\neg(P \land \neg P)$	$\neg$ Intro: 1-4

5. Conditionals

Truth-Functional Connectives

- **Truth-Functional Connective** :
  - **T** or **F** of a complex sentence (made with a connective) can be determined by the truth value of the connective's arguments
  - (can by captured by a Truth Table)
- Boolean connectives ($\neg$, $\wedge$, $\vee$) are truth-functional connectives
- FOL does not include connectives that are not truth-functional

Goal Let's include conditional connectives ($\rightarrow$, $\leftrightarrow$) in FOL

Conditional Symbol: $\rightarrow$

- Syntax
  - $P \rightarrow Q$ (P: antecedent, Q: consequent)
- Semantics
  - the sentence $P \rightarrow Q$ is true **iff** P is false or Q is true
  - $P \rightarrow Q \equiv \neg P \vee Q$
  - $P \rightarrow Q \approx$ if P then Q (various English expression)
    $=$ P only if Q
    $=$ Q provided P
    $=$ Q if P
- Truth Table :

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

Necessary Condition

- consequent Q in $P \rightarrow Q$
- A condition that must hold in order for something else to obtain
- English: "only if" (necessary condition comes after)
- Example:
  - You will pass the course **only if** you turn in all the homework
  - Turning in all the homework is necessary condition for passing
  - If you do not turn in all the homework, you will not pass
  - Does not guarantee that you will pass if you do turn in all the homework

  * It is impossible to pass and not turn in all the homework (exclude the case where $P = \mathbf{T}$ and $Q = \mathbf{F}$)

Sufficient Condition

- antecedent P in $P \to Q$
- A condition which guarantees that something else will obtain
- English: "if" (sufficient condition comes after)
- Example:
  - You will pass the course **if** you turn in all the homework
  - Turning in all the homework is sufficient condition for passing
  - If you do turn in all the homework, you will pass
  - Does guarantee that you will pass if you do turn in all the homework

  * It is impossible to turn in all the homework and fail (exclude the case where $P = \textbf{T}$ and $Q = \textbf{F}$)

Use of $\rightarrow$

- $\mathbf{Q}$ is logical consequence of $\mathbf{P}_1$, ..., $\mathbf{P}_n$

  if and only if

  the sentence $\mathbf{P}_1 \wedge \ldots \wedge \mathbf{P}_n \rightarrow \mathbf{Q}$ is a logical truth

- Unless P, Q

  = Q unless P

  = Q if not P

  $= \neg\mathbf{P} \rightarrow \mathbf{Q}$

Biconditional Symbol: $\leftrightarrow$

- Syntax
  - $P \leftrightarrow Q$: (if and only if $\equiv$ iff)
- Semantics
  - the sentence $P \leftrightarrow Q$ is true **iff** P and Q have the same truth value
  - $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
- Truth Table :

P	Q	P$\leftrightarrow$Q
T	T	T
T	F	F
F	T	F
F	F	T

HYU 한양대학교 HANYANG UNIVERSITY

$\Leftrightarrow$ vs. $\leftrightarrow$

- $\Leftrightarrow$: abbreviation of “is logically equivalent to”
  - NOT a truth-functional connective & NOT a FOL expression
  - $P \Leftrightarrow Q$: indicates that the sentence P is logically equivalent to the sentence Q
- $\leftrightarrow$: truth-functional connective to make a FOL sentence
  - $P \leftrightarrow Q$: is a FOL sentence if P and Q are FOL sentences

Truth-Functional Completeness

- Truth-functional connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
  - should we introduce more connectives?
  - is it possible to encounter a sentence that is truth-functional, but cannot express using the 5 truth-functional connectives?
  - do we need them all?
  - has the expressiveness been improved with $\rightarrow$ & $\leftrightarrow$?
- Definition
  - A set of connectives is truth-functionally complete if the connectives allow us to express every truth function.
- Theorem
  - Boolean connectives ($\neg$, $\wedge$, $\vee$) are truth-functionally complete
  - Let's prove it!

Proof: Binary Connectives

P	Q	P * Q
T	T	1
T	F	2
F	T	3
F	F	4

where

$C_1 = (P \wedge Q)$

$C_2 = (P \wedge \neg Q)$

$C_3 = (\neg P \wedge Q)$

$C_4 = (\neg P \wedge \neg Q)$

1	2	3	4	sentence with $\neg$, $\wedge$, $\vee$ only
F	F	F	F	$P \wedge \neg P \wedge Q \wedge \neg Q$
T	F	F	F	$C_1 = (P \wedge Q)$
F	T	F	F	C_2
F	F	T	F	C_3
F	F	F	T	C_4
T	T	F	F	$C_1 \vee C_2$
T	F	T	F	$C_1 \vee C_3$
T	F	F	T	$C_1 \vee C_4$
F	T	T	F	$C_2 \vee C_3$
F	T	F	T	$C_2 \vee C_4$
F	F	T	T	$C_3 \vee C_4$
T	T	T	F	$C_1 \vee C_2 \vee C_3 = (P \vee Q)$
T	T	F	T	$C_1 \vee C_2 \vee C_4$
T	F	T	T	$C_1 \vee C_3 \vee C_4$
F	T	T	T	$C_2 \vee C_3 \vee C_4$
T	T	T	T	$C_1 \vee C_2 \vee C_3 \vee C_4$

Proof: Unary Connectives

P	#P
T	1
F	2

1	2	sentence with $\neg$, $\wedge$, $\vee$ only
F	F	$P \wedge \neg P$
T	F	P
F	T	$\neg P$
T	T	$P \vee \neg P$

Proof: Ternary Connectives

P	Q	R	♣(P,Q,R)
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

where ♣(P,Q,R) = if P then Q else R

$\Longrightarrow$ sentence with $\neg$, $\wedge$, $\vee$ only:

$$
\begin{aligned}
&(P \wedge Q \wedge R) \\
&\vee (P \wedge Q \wedge \neg R) \\
&\vee (\neg P \wedge Q \wedge R) \\
&\vee (\neg P \wedge \neg Q \wedge R)
\end{aligned}
$$

n-ary Connectives

- Suppose that ♠ express an n-ary connective
- The algorithm that makes a sentence that is tautologically equivalent to ♠(P_1, ..., P_n)
  - Define the **conjunctions** $C_1, \ldots, C_{2n}$ that correspond to the 2^n rows of the truth table
  - Form a **disjunction** D that contains C_k as a disjunct iff the k^{th} row of the truth table has the value true
  - If all rows contain false, then we let $D \equiv P_1 \wedge \neg P_1$
  - Then this disjunction is tautologically equivalent to ♠(P_1, ..., P_n)

Theorems

- **Theorem**: The Boolean connectives $\neg$ and $\vee$ are truth-functionally complete
- **Theorem**: The Boolean connectives $\neg$ and $\wedge$ are truth-functionally complete
- **Theorem**: Let P↓Q express "neither P nor Q" then ↓ is truth-functionally complete
  - $\neg P = P \downarrow P$ = neither P nor P
  - $P \wedge Q = (P \downarrow P) \downarrow (Q \downarrow Q)$ = neither not P nor not Q

Disadvantages of Economizing on Connectives

- The fewer connectives we have
  - the harder it is to understand the sentences
  - the proofs become more complicated

Valid Steps of Informal Proof for Conditionals

- **Modus Ponens** (Conditional Elimination)
  - From $P \rightarrow Q$ and P, infer Q
- **Biconditional Elimination**
  - From P and either $P \leftrightarrow Q$ or $Q \leftrightarrow P$, infer Q
- **Contraposition**
  - $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
- **Logical Equivalences :**
  - $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
  - $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$
  - $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$
  - $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

Informal Proof: Method of Conditional Proof

Method

- to prove $P \rightarrow Q$, temporarily assume P and prove Q

Example

- Tet(a) $\rightarrow$ Tet(c) is logical consequence of Tet(a) $\rightarrow$ Tet(b) and Tet(b) $\rightarrow$ Tet(c). In other words, $\rightarrow$ is transitive.

Proof :

- As premises, Tet(a) $\rightarrow$ Tet(b) and Tet(b) $\rightarrow$ Tet(c) are given. We want to prove Tet(a) $\rightarrow$ Tet(c).
- To prove using conditional proof method, let's assume that Tet(a) is true. By applying Modus Ponens using the 1^{st} premise, we can conclude Tet(b). By applying Modus Ponens using the 2^{nd} premise, we can conclude Tet(c).
- Since we assumed Tet(a) and proved Tet(c), the rule of conditional proof assures Tet(a) $\rightarrow$ Tet(c).

Informal Proof: Method of Conditional Proof

Example

- $\text{Even}(n^2) \rightarrow \text{Even}(n)$

Proof : (conditional proof + proof by contradiction)

- We can assume $\text{Even}(n^2)$ and prove $\text{Even}(n)$.
- Assume n^2 is even and prove n is even.
- Assume n is odd. Then we can express n as $2m + 1$.
  Then
  $$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \end{aligned}$$
  This shows that n^2 is odd. It contradicts the premise.
- Thus, n is even.
- Hence, $\text{Even}(n^2) \rightarrow \text{Even}(n)$ is true.

Informal Proof: Method of Conditional Proof

Example

- $\text{Even}(n^2) \rightarrow \text{Even}(n)$

Proof : (contraposition)

- To prove $\text{Even}(n^2) \rightarrow \text{Even}(n)$, we can prove the contrapositive of this conditional, $\neg\text{Even}(n) \rightarrow \neg\text{Even}(n^2)$.
- Assume $\neg\text{Even}(n)$ which means assume n is odd.
- Then we can express n as $2m + 1$.

  Then
  $$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \end{aligned}$$
  This shows that n^2 is also odd. Therefore , $\neg\text{Even}(n^2)$ is true.
- Hence, $\neg\text{Even}(n) \rightarrow \neg\text{Even}(n^2)$ is true.

Informal Proof: Biconditionals

Method : (use conditional proof)

- arrange biconditionals into a cycle of conditionals

Example

- Q_1, Q_2, Q_3 are all equivalent.
  Hence, $Q_1 \leftrightarrow Q_2$, $Q_2 \leftrightarrow Q_3$, $Q_1 \leftrightarrow Q_3$.

Proof :

- Arrange the biconditionals as below and prove.

$$Q_1 \rightarrow Q_2 \rightarrow Q_3 \rightarrow Q_1$$

- Prove the following conditions are all equivalent.
  1. n is even
  2. n^2 is even
  3. n^2 is divisible by 4

  Proof : (3) $\rightarrow$ (2) $\rightarrow$ (1) $\rightarrow$ (3)

HYU 한양대학교 HANYANG UNIVERSITY

Formal Rules of Proof for $\rightarrow$

- **Conditional Elimination ($\rightarrow$ Elim)**

$$
\begin{array}{l|l}
 & P \rightarrow Q \\
 & \vdots \\
 & P \\
 & \vdots \\
\triangleright & Q
\end{array}
$$

Modus Ponens

- **Conditional Introduction ($\rightarrow$ Intro)**

$$
\begin{array}{l|l}
 & \begin{array}{|l} P \\ \hline \vdots \\ Q \end{array} \\
\triangleright & P \rightarrow Q
\end{array}
$$

Conditional Proof

- **Let's construct a proof in Fitch**
  - Conditional 1
  - Prove: $A \rightarrow \neg\neg A$
  - Conditional 2

HYU 한양대학교 HANYANG UNIVERSITY

Formal Rules of Proof for $\leftrightarrow$

- **Biconditional Elimination ($\leftrightarrow$ Elim)**

$$
\begin{array}{l|l}
 & P \leftrightarrow Q \text{ (or } Q \leftrightarrow P\text{)} \\
 & \vdots \\
 & P \\
 & \vdots \\
\triangleright & Q
\end{array}
$$

- **Biconditional Introduction ($\leftrightarrow$ Intro)**

$$
\begin{array}{l|l}
 & \begin{array}{|l} P \\ \hline \vdots \\ Q \end{array} \\
 & \\
 & \begin{array}{|l} Q \\ \hline \vdots \\ P \end{array} \\
\triangleright & P \leftrightarrow Q
\end{array}
$$

- **Let's construct a proof in Fitch**
  - Prove: $A \leftrightarrow \neg\neg A$ & Conditional 3

6. Soundness & Completeness of System F

Glossary

- F_T
  - portion of the deductive system F
  - contains $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ $\perp$ Intro/Elim rules
- $P_1, \ldots, P_n \quad \vdash_T \quad S$ ($\vdash$: provability)
  - there is a formal proof in F_T of S from premises $P_1, \ldots, P_n$.

- What we proven in F_T is genuinely valid?
- Are there any flaws in F_T?

Soundness Theorem :

- If $\quad P_1, \ldots, P_n \vdash_T S$
  then $\quad$ S is a tautological consequence of $P_1, \ldots, P_n$

Proof :

- Suppose that p is a proof constructed in the system F_T
- Any 'sentence' at any step (main + subproofs) in proof p is a tautological consequence of the 'assumptions' (including the main premises) in force at that step $\leftarrow$ need to prove
- Then, S appears at the main level of p and the only assumptions in force are the premises $P_1, \ldots, P_n$.
  Hence, S is a tautological consequence of $P_1, \ldots, P_n$

Soundness of F_T

Proof continues (Proof by Contradiction) :

- Suppose there is a step in proof p that is **not** a tautological consequence of the assumptions in force at that step (called invalid step)
- Then, look at the 1^{st} invalid step in p and show that none of the 12 rules of F_T could have justified that step ← need to prove (use 'proof by cases' and show that no matter which rule of F_T was applied at that invalid step, contradiction is derived)
- Thus, we can conclude that there can be no invalid steps in proofs in F_T

Soundness of F_T

Proof continues (Proof by Case) : →Intro

- Suppose the 1^{st} invalid step derives the sentence Q → R by applying →Intro to an earlier subproof with assumption Q and conclusion R
- Let A_1, ..., A_k be the assumptions in force at Q → R
- Then, assumptions in force at R are A_1, ..., A_k and Q
- Since step R is earlier than the first invalid step (Q → R), R must be tautological consequence of A_1, ..., A_k and Q
- By the assumption that Q → R is an invalid step, the following row exists in the joint Truth Table (Contradiction!)

```
:
| Q
|---
| :
| R
:
Q → R
:
```

A_1	...	A_k	Q	R	Q → R
T	T	T	T	T	F

R being T contradicts Q→R being F

Soundness of F_T

Proof continues (Proof by Case) : **$\rightarrow$Elim**

- Suppose the 1^{st} invalid step derives the sentence R by applying $\rightarrow$Elim to sentences $Q \rightarrow R$ and Q appearing earlier in the proof
- Let A_1, ..., A_k be all the assumptions in force at R
- If R is an invalid step,
  R is not a tautological consequence of A_1, ..., A_k
- Since steps $Q \rightarrow R$ and Q are earlier than the first invalid step (R), $Q \rightarrow R$ and Q are valid steps ($Q \rightarrow R$ and Q must be tautological consequence of the assumptions in force at that step)

Soundness of F_T

Proof continues (Proof by Case) : $\rightarrow$Elim

- [observation]: the assumptions in force at steps $Q \rightarrow R$ and Q are also in force at R
- By the assumption that R is an invalid step, the following row exists in the joint Truth Table (Contradiction!)

```
| A1
|---
| :
| Q → R
| :
| | A2
| |---
| | :
| | Q
| | :
| | | A3
| | |---
| | | :
| | | R
| | | :
```

A_1	...	A_k	$Q \rightarrow R$	Q	R
T	T	T	T	T	F

$Q \rightarrow R$ and Q being T contradicts R being F

Soundness of F_T

Proof continues (Proof by Case) : ⊥Elim

- Suppose the 1^{st} invalid step derives the sentence Q by applying ⊥Elim to ⊥
- [observation]: the assumptions in force at ⊥ are also in force at Q
- Let A_1, ..., A_k be the assumptions in force at ⊥ (and at Q)
- Since step ⊥ is earlier than the first invalid step Q, ⊥ must be tautological consequence of A_1, ..., A_k
- By the assumption that Q is an invalid step, the following row exists in the joint Truth Table (Contradiction!)

$$\begin{array}{|l} \vdots \\ \perp \\ Q \\ \vdots \end{array}$$

A_1	...	A_k	⊥	Q
TT-contradictory			T	F

A_1, ..., A_k cannot be all Ts !

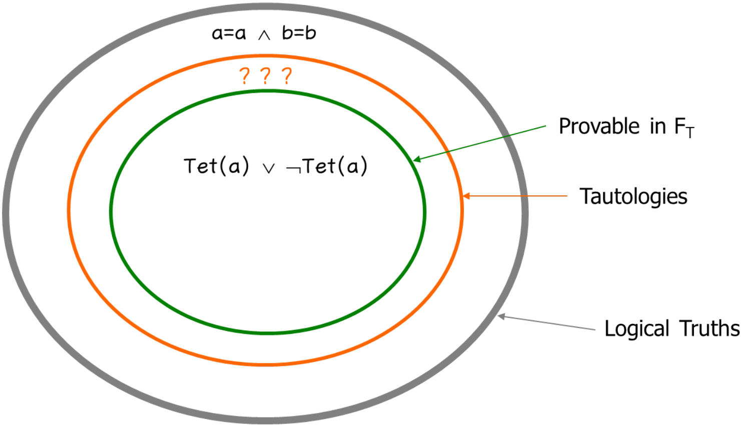
This also contradicts Q being F

Soundness of F_T

Proof continues (Proof by Case) : **9 remaining rules**

- Derive contradiction in all 12 cases as we have demonstrated before
- Conclude that
  "it is possible for a proof of F_T to contain an invalid step"
  must be false
- conclude the **Proof of Soundness**

Soundness of F_T



Soundness of F_T

- **Corollary**: If $\vdash_T$ S, then S is a tautology

Completeness of F_T

- Can F_T allow us to prove everything we should be able to prove?
- Given any premises P_1, ..., P_n and any tautological consequence S of these premises, can F_T allows us to construct a proof of S from P_1, ..., P_n?
- Could there be tautological consequence of some set of premises that cannot be proved using F_T?

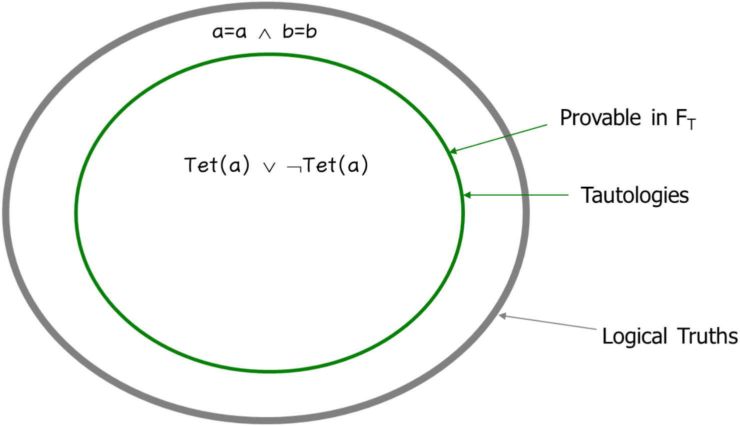
Completeness Theorem

- If a sentence S is a tautological consequence of P_1, ..., P_n, then $P_1, \ldots, P_n \vdash_T S$

Proof

- Refer to Chapter 17 of LPL textbook

Completeness and Soundness of F_T



Completeness and Soundness of F_T

- unprovable in F_T : $\text{Dodec}(b) \wedge b = c \quad \vdash_T \quad \text{Dodec}(c)$
  $\text{Larger}(b, c) \quad \vdash_T \quad \neg\text{Larger}(c, b)$

Practical Uses of Completeness Theorem

- Provides method for showing that an argument has a proof without actually having to find such proof
  - Just show that the conclusion is a tautological consequence of the premises

e.g. Since $A \rightarrow (B \rightarrow A)$ is a tautology, there must exists a proof

e.g. Since the sentence $B \wedge \neg D$ is a tautological consequence of $\neg((A \wedge B) \rightarrow (C \vee D))$, there must exists a proof

- How can we determine tautological consequence?
  - Draw a Truth Table in Boole or use Taut Con in Fitch

HYU 한양대학교 HANYANG UNIVERSITY

Practical Uses of Soundness Theorem

- Provides method for telling that an argument does not have a proof in F_T
  - Just show that the conclusion is not a tautological consequence of the premises

e.g. Since $A \to (A \to B)$ is not a tautology, it is impossible to construct a proof in F_T

e.g. Since the sentence $B \land \neg D$ is not a tautological consequence of $\neg((A \lor B) \to (C \land D))$, it is impossible to construct a proof in F_T

e.g. Since the sentence ¬Happy(carl) is not a tautological consequence of ¬(Happy(carl) ∧ Happy(scruffy)), it is impossible to construct a proof in F_T

Therefore...

- In Fitch,
- use Taut Con to determine :
  1. if a sentence is a tautological consequence of the cited sentence try to find a full proof of the sentence in F_T
  2. if a sentence is not a tautological consequence of the cited sentence there is no point in finding a proof in F_T

HYU 한양대학교 HANYANG UNIVERSITY