CSE1007: Logical Fundamentals of Programming

Part II - Predicate Logic

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1. Introduction to Quantification

Quantified Sentences

- Can we express the quantities of things in Propositional Logic

    - Every student who studies hard passes the examination.
    - There are some students who never study at home.

- Quantifiers in FOL
    - Universal Quantifiers: ∀
        - everything, all

    - Existential Quantifiers: ∃
        - something, at least one

Atomic Well-Formed Formulas (Atomic wffs)

Terms : expand with variables

- Simple terms
  - individual constants $+$ variable
  - **e.g.,** max, t, u, v, w, x, y, z, z_2
- Complex terms
  - formed by applying functional symbols to either simple terms or complex terms
  - **e.g.,** father(max), father(x)
    $(0 + 1) \times 1, (y + z) \times z$

Atomic wffs (well-formed formulas)

- Atomic sentences with variables(in place of individual constants)
  - **e.g.,** Home(x), Taller(max, y), Taller(father(z), z)
  - $\neq$ sentences
- Variables in atomic wffs are free(unbound)

Universal Quantifier : $\forall$

- Variable binding operator
  - always used in connection with a variables

- $\forall x$ reads:
  - "for every object x"
  - "for all x"

- Examples:
  - $\forall x$ Home(x)
    - everything is at home
  - $\forall x$ (Professor(x) $\rightarrow$ Smart(x))
    - every professor is smart
    - for every object x, if x is a professor, then x is smart
    - any object (x) is either not a professor or smart (or smart professor)

Existential Quantifier : ∃

- Variable binding operator
    - always used in connection with a variables

- ∃x reads:
    - "for some object x"
    - "for some x"

- Examples:
    - ∃x Home(x)
        - something is at home
    - ∃x (Professor(x) ∧ Smart(x))
        - some professor is smart
        - for some object x, x is both a professor and smart
        - there is at least one smart professor

Well-Formed Formulas (wffs)

1. If P is a wff, so is ¬P

2. If P_1, ..., P_n are wffs, so is $(P_1 \land ... \land P_n)$

3. If P_1, ..., P_n are wffs, so is $(P_1 \lor ... \lor P_n)$

4. If P and Q are wffs, so is $(P \rightarrow Q)$

5. If P and Q are wffs, so is $(P \leftrightarrow Q)$

6. If P is a wff and ν is a variable (i.e., one of t, u, v, w, x, ...), then $\forall\nu P$ is a wff

7. If P is a wff and ν is a variable (i.e., one of t, u, v, w, x, ...), then $\exists\nu P$ is a wff

e.g., $((Cube(x) \land Small(x)) \rightarrow \exists y\ LeftOf(x,y))$

Free and Bound Variable

1. Any variable in an atomic wffs is free or unbound

2. The free variables in P are also free in ¬P

3. The free variables in P_1, ..., P_n are also free in $(P_1 \wedge ... \wedge P_n)$

4. The free variables in P_1, ..., P_n are also free in $(P_1 \vee ... \vee P_n)$

5. The free variables in P and Q are also free in $(P \rightarrow Q)$

6. The free variables in P and Q are also free in $(P \leftrightarrow Q)$

7. All of the free variable in P are free in $\forall \nu P$, except for ν, and every occurrence of ν in P is said to be **bound**

8. All of the free variable in P are free in $\exists \nu P$, except for ν, and every occurrence of ν in P is said to be **bound**

e.g., $\forall x ((Cube(x) \wedge Small(x)) \rightarrow \exists y \, LeftOf(x,y))$

Sentence

- A **sentence** is a wff with NO free variables

- Scope of a quantifier
    - which variables fall under its influence and which don't

    $\exists x \ (\text{Professor}(x) \land \text{Smart}(x))$
    $\qquad\quad \downarrow \qquad\quad \downarrow$
    $\qquad\quad \text{bound} \qquad \text{bound}$

    $\exists x \ \text{Professor}(x) \land \text{Smart}(x)$
    $\qquad\quad \downarrow \qquad\quad \downarrow$
    $\qquad\quad \text{bound} \qquad\quad \text{free}$

- When you append either quantifier $\forall x$ or $\exists x$ to a wff P, we say that the quantifier binds all the free occurrences of x in P

Semantics for the Quantifiers

- Determine the truth value of quantified sentences
  - Propositional logic
    - the truth value of ¬P depends on the truth values of P
  - Predicate logic
    - cannot determine the truth value of ∃x Cube(x) depends on the truth values of Cube(x)

- Semantics: determine using auxiliary notion of **satisfaction**

  - ∃x S(x) is **true**
    iff there is at lease one object that satisfies the wff S(x)

  - ∀x S(x) is **true**
    iff every object satisfies the wff S(x)

Semantics for the Quantifiers

- Domain of Discourse (Domain of Quantification)
    - sentences with quantifiers are only true or false relative to some domain of discourse
    - specify domain explicitly unless it is clear from the context
    - In FOL, always assume that:
        - the domain of discourse contains at least one object
        - every individual constant in the language stands for an object in the domain

    A sentence of the form $\forall x\, S(x)$ is **true**   **iff**   the wff $S(x)$ is satisfied by every object in the domain of discourse

    A sentence of the form $\exists x\, S(x)$ is **true**   **iff**   the wff $S(x)$ is satisfied by some object in the domain of discourse

Semantics for the Quantifiers

- Notation
  - Propositional logic
    - use P or Q to stand for a possibly complex sentence
  - Predicate logic
    - use S(x) or P(y) to stand for a possibly complex wff

- P(y) = ∃x (LeftOf(x,y) ∨ RightOf(x,y))
  P(b) = ∃x (LeftOf(x,b) ∨ RightOf(x,b))

- S(x) = ∃x Professor(x) ∧ Smart(x)
  S(c) = ∃x Professor(x) ∧ Smart(c)

Game Rules for the Quantifiers in Tarski's World

Form	Your Commitment	Player to Move	Goal
P ∨ Q	TRUE FALSE	you Tarski's World	Choose one of P, Q that is true
P ∧ Q	TRUE FALSE	Tarski's World you	Choose one of P, Q that is false
∃x P(x)	TRUE FALSE	you Tarski's World	Choose some *b* that satisfies the wff P(x)
∀x P(x)	TRUE FALSE	Tarski's World you	Choose some *b* that does not satisfy the wff P(x)
¬P	either	—	Replace ¬P by P and switch commitment
P → Q	either	—	Replace P → Q by ¬P ∨ Q and keep commitment
P ↔ Q	either	—	Replace P ↔ Q by (P → Q) ∧ (Q → P) and keep commitment

Let's play in Tarski's World : Game World + Game Sentence

Aristotelian Forms

$$\text{All P's are Q's} \quad : \quad \forall x \ (P(x) \to Q(x))$$

$$\text{Some P's are Q's} \quad : \quad \exists x \ (P(x) \land Q(x))$$

$$\text{No P's are Q's} \quad : \quad \forall x \ (P(x) \to \neg Q(x)) \ \& \ \neg \exists x \ (P(x) \land Q(x))$$

$$\text{Some P's are not Q's} \quad : \quad \exists x \ (P(x) \land \neg Q(x))$$

Let's Play in Tarski's World!

1 How do we represent the following sentence in FOL

"There is a large cube!"

2 ∃x (Cube(x) → Large(x))

- build a world containing a single large cube and check if the above sentence is true or false
- change the large cube into a small tetrahedron and check if the above sentence is true or false

3 ∃x (Cube(x) ∧ Large(x))

- check if the above sentence is true or false

Translating Complex Noun Phrases

- How do we translate complex quantified noun phrases ?
  - employ conjunctions of atomic sentences
  - the order of an English sentence may not correspond to the order of FOL sentences (need a lot of practice)

- Existential Noun Phrases : use $\exists$ frequently together with $\wedge$
  - (subject) A small, happy dog is at home
    $\exists x[(Small(x) \wedge Happy(x) \wedge Dog(x)) \wedge Home(x)]$
  - (object) Max owns a small, happy dog
    $\exists x[(Small(x) \wedge Happy(x) \wedge Dog(x)) \wedge Owns(max, x)]$

- Universal Noun Phrases : use $\forall$ frequently together with $\rightarrow$
  - (subject) Every small dog at home is happy
    $\forall x[(Small(x) \wedge Dog(x) \wedge Home(x)) \rightarrow Happy(x)]$
  - (object) Max owns every small, happy dog
    $\forall x[(Small(x) \wedge Happy(x) \wedge Dog(x)) \rightarrow Owns(max, x)]$

Conversational Implicature

- "All P's are Q's" does not imply that there are some P's (though it may conversationally suggest)

    - Every 1st year students who took my class got an A
    - (fact) No 1st year students can take Logic course

- "Some P's are Q's" does not imply that not all P's are Q's (though it may conversationally suggest)

    - Some students got over 50% on the mid-term exam
    - (fact) All students got over 50% on the mid-term exam

Generalization

- **Vacuously True** Generalization
    - When there are no objects satisfying P(x), $\forall x\ (P(x) \rightarrow Q(x))$ is **vacuously true**
    - **e.g.** $\forall x\ (Tet(x) \rightarrow Small(x))$ is vacuously true when there are no tetrahedra at all. Why? We cannot find a counter example.

- **Inherently Vacuous** Generalization
    - $\forall x\ (P(x) \rightarrow Q(x))$ is **inherently vacuous** if the only worlds in which it is true are those in which $\forall x\ \neg P(x)$ is true.
    - **e.g.** $\forall y\ (Tet(y) \rightarrow Cube(y))$ is true when there are no tetrahedra, and is false when there are tetrahedra

- Let's play a game in Tarski's World
    - Peano's World + Dodgson's Sentences (sentence 1)
    - Peirce's World + Dodgson's Sentences (sentence 1, 4, 5)
    - find a sentence that is vacuously true in Peirce's World but non-vacuously true in Peano's World

2. The Logic of Quantifiers

Are these Arguments Logically Valid ?

$\forall x \ (Cube(x) \rightarrow Small(x))$
$\forall x \ Cube(x)$

$\forall x \ Small(x)$

valid

$\forall x \ Cube(x)$
$\forall x \ Small(x)$

$\forall x \ (Cube(x) \land Small(x))$

valid

$\exists x \ (Cube(x) \rightarrow Small(x))$
$\exists x \ Cube(x)$

$\exists x \ Small(x)$

invalid

counter-example :
  $(Cube(a) \land Large(a)) \land$
  $(Dodec(b) \land Large(b))$

$\exists x \ Cube(x)$
$\exists x \ Small(x)$

$\exists x \ (Cube(x) \land Small(x))$

invalid

counter-example :
  $(Cube(a) \land Large(a)) \land$
  $(Dodec(b) \land Small(b))$

Are these Sentences Logically True ?

Sentence	Logical Truth	Tautology
$\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$	Yes	No
$\forall x\ Cube(x) \lor \forall x\ \neg Cube(x)$	No	No
$\forall x\ Cube(x) \lor \neg \forall x\ Cube(x)$	Yes	Yes
$\exists x\ Cube(x) \lor \neg \exists x\ Cube(x)$	Yes	Yes

Method of Identifying the Tautologies of FOL

- Sentence with Universal/Existential Quantifiers
  $\Rightarrow$ Sentence in the truth-functional form

$(\exists y \, (P(y) \vee R(y)) \rightarrow \forall x \, (P(x) \wedge Q(x))) \rightarrow$
$(\neg \forall x \, (P(x) \wedge Q(x)) \rightarrow \neg \exists y \, (P(y) \vee R(y)))$

- replace $\exists y \, (P(y) \vee R(y))$ by A
- replace $\forall x \, (P(x) \wedge Q(x))$ by B

$(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

A	B	¬B	¬A	A → B	¬B → ¬A	(A → B) → (¬B → ¬A)
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

Truth Functional Form Algorithm

1. Start at the beginning of sentence S and proceed to the right
2. When you come across a quantifier ($\forall$ or $\exists$) or atomic sentence, underline the portion of the sentence
   - for a quantifier ($\forall$ or $\exists$), underline the quantifier and the entire formula that is applied to
   - for an atomic sentence, underline the atomic sentence
3. At the end of the underline, assign a sentence letter (A,B,C,...)
   - If the underlined part already appeared earlier, use the same sentence letter as before
   - If the underlined part appears for the first time, assign first sentence letter not yet used
4. Repeat the steps 2 and 3 until the end of the sentence S
5. Replace each underlined part with the sentence letter assigned The result is the truth-functional form of S

e.g., $\neg(\text{Tet}(d) \land \forall x \, \text{Small}(x)) \rightarrow (\neg\text{Tet}(d) \lor \neg\forall y \, \text{Small}(y))$

Ex : Truth-Functional Form Translation

First-Order (quantified) sentence	Truth-Functional form
$\forall x\ \text{Cube}(x) \lor \neg\forall x\ \text{Cube}(x)$	$A \lor \neg A$
$(\exists y\ \text{Tet}(y) \land \forall z\ \text{Small}(z)) \to \forall z\ \text{Small}(z)$	$(A \land B) \to B$
$\forall x\ \text{Cube}(x) \lor \exists y\ \text{Tet}(y)$	$A \lor B$
$\forall x\ \text{Cube}(x) \to \text{Cube}(a)$	$A \to B$
$\forall x\ (\text{Cube}(x) \lor \neg\text{Cube}(x))$	A
$\forall x\ (\text{Cube}(x) \to \text{Small}(x)) \lor \exists x\ \text{Dodec}(x)$	$A \lor B$

Tautologies of FOL

Definition : A quantified sentence of FOL is tautology
iff its truth-functional form is a tautology

- Every tautology is a logical truth

- But among quantified sentences there are many logical truth that are not tautologies

- Similarly, there are many logically valid arguments of FOL that are not tautologically valid

First-Order Logic

Propositional Logic	First-Order Logic	General Notation
Tautology	FO Validity	Logical Truth
Tautological Consequence	FO Consequence	Logical Consequence
Tautological Equivalence	FO Equivalence	Logical Equivalence

First-Order Validity (FO Validity)

- A **sentence of FOL** is a **FO validity** if it is a logical truth when you ignore the meanings of the names, function symbols, and predicates other than the identity symbol($=$)

- Which sentence is a FO validity?

$$\forall x\ SameSize(x,x)$$
$$\forall x\ Cube(x) \rightarrow Cube(b)$$
$$(Cube(b) \wedge b = c) \rightarrow Cube(c)$$
$$(Small(b) \wedge SameSize(b,c)) \rightarrow Small(c)$$

$$\forall x\ Outgrabe(x,x)$$
$$\forall x\ Tove(x) \rightarrow Tove(b)$$
$$(Tove(b) \wedge b = c) \rightarrow Tove(c)$$
$$(Slithy(b) \wedge Outgrabe(b,c)) \rightarrow Slithy(c)$$

Lewis Carroll's poem "Jabberwocky"

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First-Order Consequence (FO Consequence)

- A **sentence S** is a **FO consequence of premises $P_1, \ldots, P_n$**
  if S is a logical consequence of $P_1, \ldots, P_n$ when you ignore the
  meanings of the names, function symbols, and predicates
  other than identity($=$)

- In the following arguments,
  are the conclusion FO consequence of their premises?

  $\forall x\ (Tet(x) \rightarrow Large(x))$
  $\neg Large(b)$
  $\overline{\neg Tet(b)}$

  $\forall x\ (Borogove(x) \rightarrow Mimsy(x))$
  $\neg Mimsy(b)$
  $\overline{\neg Borogove(b))}$

First-Order Counterexample (FO Counterexample)

- In the following argument,
  is the conclusion FO consequence of its premises?

  ¬∃x Larger(x,a)   <u>Answer</u> : the conclusion could be
  ¬∃x Larger(b,x)   a logical consequence of its premises, but
  Larger(c,d)   **NOT** an FO consequence of its premises

  Larger(a,b)

                          FO Counterexample :
  ¬∃x R(x,a)   <u>Let R = Likes</u>, a = moriarty, b = scrooge,
  ¬∃x R(b,x)   c = romeo, d = juliet & Interpret this argument
  R(c,d)   with all your knowledge about the words :

  R(a,b)   premises are all true, but the conclusion is false
                          This is called a **FO Counterexample**!

Replacement Method

- Algorithm to check whether :
    - a sentence is a **FO validity**
    - the conclusion of an argument is a **FO consequence of its premises**

1. Replace all the predicates ($+$ functional symbols), except identity, with new and meaningless symbols

2. Check if a sentence S is a FO validity
    - find a counterexample in which S is false
    - if there is no such counterexample, S is a first-order validity

3. Check if a sentence S is a FO consequence of P_1, ..., P_n
    - find a counterexample in which S is false while P_1, ..., P_n is true
    - if there is no such counterexample, S is FO consequence of P_1,...,P_n

- Problem : Is it easier to find a counterexample?

    There are infinite number of cases to consider!

- Fitch - FO Con : pretty useful - it never gives wrong answer though sometimes it gets stuck (unable to give answer)

Tautology, FO Validity, and Logical Truth
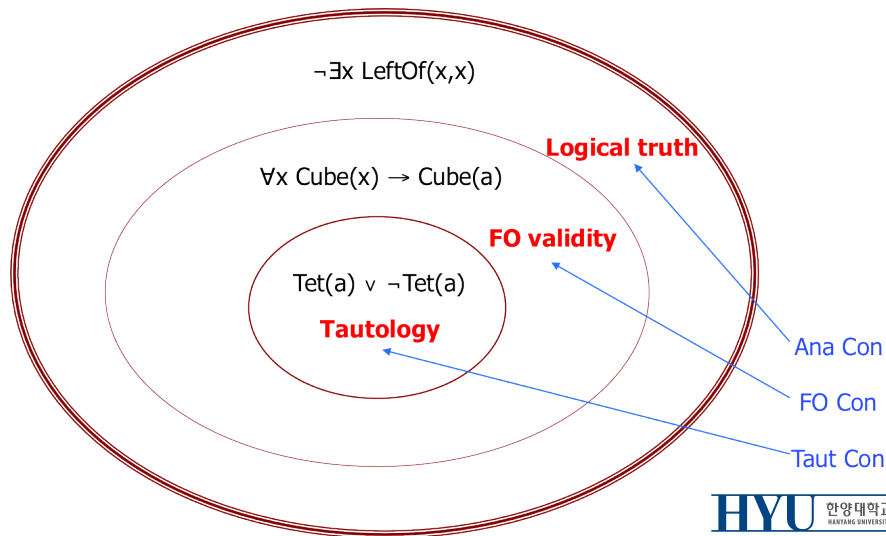
First-Order Equivalence (FO Equivalence)

$\neg(\exists x\ Cube(x) \land \forall y\ Dodec(y))$
&
$\neg\exists x\ Cube(x) \lor \neg\forall y\ Dodec(y)$
are tautologically equivalent

$\forall x\ (Cube(x) \to Small(x))$
&
$\forall x\ (\neg Small(x) \to \neg Cube(x))$
are FO equivalent
but not tautologically equivalent

In any possible circumstances, wffs $P(x) \to Q(x)$ and $\neg Q(x) \to \neg P(x)$ will be satisfied by exactly the same objects

Proof : (Indirect Proof)

Assume that in some circumstances, there is an object that satisfies one but not the other of these two formulas. Let's give this object a new name, say n_1. Consider the result of replacing x by n_1 in our formulas :

$$P(n_1) \to Q(n_1) \qquad \neg Q(n_1) \to \neg P(n_1)$$

Since x was the only free variable, these are sentences.
By contraposition, these two sentences are logically equivalent.
But by our assumption, one of them is true and the other is false.

- This is a contradiction!

First-Order Equivalence (FO Equivalence)

- Logically equivalent wffs
    - Two wffs with free variables are logically equivalent
        - if they are satisfied by the same objects in any circumstances
        - if, when replacing their free variables with new names,
            the resulting sentences are logically equivalent

- Substitution of equivalent wffs
    - Let P and Q be wffs, possibly containing free variables
    - Let S(P) be any sentence containing P as a component part
    - If P $\Leftrightarrow$ Q, then so is S(P) $\Leftrightarrow$ S(Q)

- Example :
    - $\forall x$ (Cube(x) $\rightarrow$ Small(x)) $\Leftrightarrow \forall x$ ($\neg$Cube(x) $\vee$ Small(x))
        $\Leftrightarrow \forall x$ ($\neg$Cube(x) $\vee$ $\neg\neg$Small(x))
        $\Leftrightarrow \forall x$ $\neg$(Cube(x) $\wedge$ $\neg$Small(x))

DeMorgan's Law

- For any wff P(x),
  1. $\neg \forall x \, P(x) \Leftrightarrow \exists x \, \neg P(x)$
  2. $\neg \exists x \, P(x) \Leftrightarrow \forall x \, \neg P(x)$

- Example : block world consisting of four named block a, b, c, d
  - $\forall x \, Small(x)$ is true **iff** $Small(a) \wedge Small(b) \wedge Small(c) \wedge Small(d)$ is true
  - $\exists x \, Small(x)$ is true **iff** $Small(a) \vee Small(b) \vee Small(c) \vee Small(d)$ is true

  $$\begin{aligned} \neg \forall x \, Small(x) &\Leftrightarrow \neg(Small(a) \wedge Small(b) \wedge Small(c) \wedge Small(d)) \\ &\Leftrightarrow \neg Small(a) \vee \neg Small(b) \vee \neg Small(c) \vee \neg Small(d)) \\ &\Leftrightarrow \exists x \, \neg Small(x) \end{aligned}$$

- Aristotelian sentences

  $$\begin{aligned} \neg \forall x \, (P(x) \rightarrow Q(x)) &\Leftrightarrow \neg \forall x \, (\neg P(x) \vee Q(x)) \\ &\Leftrightarrow \exists x \, \neg(\neg P(x) \vee Q(x)) \\ &\Leftrightarrow \exists x \, (\neg\neg P(x) \wedge \neg Q(x)) \\ &\Leftrightarrow \exists x \, (P(x) \wedge \neg Q(x)) \end{aligned}$$

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Other Quantifier Equivalences

- Pushing quantifiers past $\wedge$ and $\vee$
  - $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow \forall x\ P(x) \wedge \forall x\ Q(x)$
  - $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow \exists x\ P(x) \vee \exists x\ Q(x)$

- Null quantifiers (where x is not free in P)
  - $\forall x\ P \Leftrightarrow P$
  - $\exists x\ P \Leftrightarrow P$
  - $\forall x\ (P \vee Q(x)) \Leftrightarrow P \vee \forall x\ Q(x)$
  - $\exists x\ (P \wedge Q(x)) \Leftrightarrow P \wedge \exists x\ Q(x)$

- Replacing bound variables (where y does not occur in P(x))
  - $\forall x\ P(x) \Leftrightarrow \forall y\ P(y)$
  - $\exists x\ P(x) \Leftrightarrow \exists y\ P(y)$

Multiple Uses of a Single Quantifier

$\exists x \; \exists y \; [\text{Cube}(x) \wedge \text{Tet}(y) \wedge \text{LeftOf}(x,y)]$

$\qquad = \exists x \; [\text{Cube}(x) \wedge \exists y \; (\text{Tet}(y) \wedge \text{LeftOf}(x,y))]$

$\forall x \; \forall y \; [(\text{Cube}(x) \wedge \text{Tet}(y)) \rightarrow \text{LeftOf}(x,y)]$

$\qquad = \forall x \; [\text{Cube}(x) \rightarrow \forall y \; (\text{Tet}(y) \rightarrow \text{LeftOf}(x,y))]$

즉 분배 법칙

Let's play in Tarski's world : Cantor's World + Cantor's Sentence

$\forall x \; \forall y [(\text{Cube}(x) \wedge \text{Cube}(y)) \rightarrow (\text{LeftOf}(x,y) \vee \text{RightOf}(x,y))]$

$\forall x \; \forall y [(\text{Cube}(x) \wedge \text{Cube}(y) \wedge x \neq y) \rightarrow (\text{LeftOf}(x,y) \vee \text{RightOf}(x,y))]$

$\exists x \; \exists y [\text{Cube}(x) \wedge \text{Cube}(y)]$

$\exists x \; \exists y [\text{Cube}(x) \wedge \text{Cube}(y) \wedge x \neq y]$

Multiple Uses of a Single Quantifier

- To universally quantify two individual objects:

    $\forall x \ \forall y \ (x \neq y \rightarrow ...)$

- To existentially quantify two individual objects:

    $\exists x \ \exists y \ (x \neq y \wedge ...)$

- Two different variables do not guarantee that they represent different individual objects:

    - $\forall x \ \forall y \ P(x,y)$ logically implies $\forall x \ P(x,x)$

    - $\exists x \ P(x,x)$ logically implies $\exists x \ \exists y \ P(x,y)$

- The order is not important among the same quantifiers:

    - $\forall x \ \forall y \ \text{Likes}(x,y) \equiv \forall y \ \forall x \ \text{Likes}(x,y)$

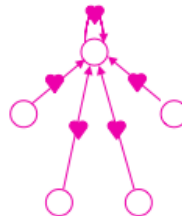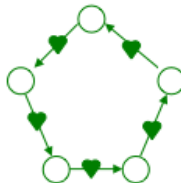   - $\exists x \ \exists y \ \text{Likes}(x,y) \equiv \exists y \ \exists x \ \text{Likes}(x,y)$

Mixed Quantifiers

$\forall x \ [\text{Cube}(x) \rightarrow \exists y \ (\text{Tet}(y) \land \text{LeftOf}(x,y))]$

$\quad = \forall x \ \exists y \ [\text{Cube}(x) \rightarrow (\text{Tet}(y) \land \text{LeftOf}(x,y))]$ ⟵ Prenex form

The order is important! $\quad \forall x \ \exists y \ \text{Likes}(x,y) \quad \neq \quad \exists y \ \forall x \ \text{Likes}(x,y)$

- **Let's play in Tarski's World** :
  - Mixed Sentences and Koenig's World

Mixed Quantifiers

- Order of variables

    $\forall x \, \exists y \, \text{Likes}(x,y)$

    $\forall x \, \exists y \, \text{Likes}(y,x)$

    $\exists y \, \forall x \, \text{Likes}(x,y)$

    $\exists y \, \forall x \, \text{Likes}(y,x)$

- At least two

    $\exists x \, \exists y \, (x \neq y \land \text{Cube}(x) \land \text{Cube}(y))$

- Exactly one

    $\exists x \, (\text{Cube}(x) \land \forall y \, (\text{Cube}(y) \rightarrow y = x))$

Step-by-step Method of Translation : English → FOL

noun phrase
with universal quantifier

noun phrase
with existential quantifier

↓   ↓

<u>Each Cube</u> is to the left of <u>a tetrahedron</u>.

⇓

$\forall x \, (\text{Cube}(x) \rightarrow$ x-is-to-the-left-of-a-tetrahedron$)$

⇓

$\exists y \, (\text{Tet}(y) \wedge \text{LeftOf}(x,y))$

Paraphrasing English

If a freshman takes a logic class, then he or she must be smart.

∃x (Freshman(x) ∧ ∃y (LogicClass(y) ∧ Takes(x,y))) → Smart(x)   X

⇓

Every freshman who takes a logic class must be smart.

∀x [(Freshman(x) ∧ ∃y (LogicClass(y) ∧ Takes(x,y))) → Smart(x)]   ✓

Donkey Sentences

Every farmer who owns a donkey beats it.

$\forall x\ (Farmer(x) \wedge \exists y\ (Donkey(y) \wedge Owns(x,y)) \rightarrow Beats(x,y))$ X

$\forall x\ (Farmer(x) \wedge \exists y\ (Donkey(y) \wedge Owns(x,y) \wedge Beats(x,y)))$ X

$\Downarrow$

Every donkey owned by any farmer is beaten by them.

$\forall x\ (Donkey(x) \rightarrow \forall y\ ((Farmer(y) \wedge Owns(y,x)) \rightarrow Beats(y,x)))$ √

Ambiguity and Context Sensitivity

Every minute a man is mugged in New York City.

$\forall x \, (Minute(x) \rightarrow \exists y \, (Man(y) \land MuggedDuring(y,x)))$ weak reading

$\Downarrow$

Every minute a man is mugged in New York City.
We are going to interview him tonight.

$\exists y \, (Man(y) \land \forall x \, (Minute(x) \rightarrow MuggedDuring(y,x)))$ strong reading

entails

Let's Play in Tarski's World

At least four medium dodecahedra are adjacent to a medium cube.

- Let's compare Anderson's First World & Second World
  - First world : Strong reading can also be True
  - Second world : only Weak reading can be True

Every medium dodecahedra are adjacent to a medium cube.

- **Weak reading** :
  - $\forall x \ ((\text{Medium}(x) \land \text{Dodec}(x)) \rightarrow$
    $\exists y \ (\text{Medium}(y) \land \text{Cube}(y) \land \text{Adjoins}(x,y)))$

- **Strong reading** :
  - $\exists y \ (\text{Medium}(y) \land \text{Cube}(y) \land$
    $\forall x \ ((\text{Medium}(x) \land \text{Dodec}(y)) \rightarrow \text{Adjoins}(x,y)))$

3. Informal Proofs

Proof Rules

Universal Quantifiers

- Introduction : Universal Generalization
- Elimination : Universal Instantiation

Existential Quantifiers

- Introduction : Existential Generalization
- Elimintation : Existential Instantiation

Universal Instantiation & Existential Generalization

Universal Instantiation

= Universal Elimination

- From $\forall x\ S(x)$, infer $S(c)$
- so long as **c** denotes an object in the domain of discourse

Existential Generalization

= Existential Introduction

- From $S(c)$, infer $\exists x\ S(x)$
- so long as **c** denotes an object in the domain of discourse

Example : Universal Instantiation & Existential Generalization

$\forall x \ [Cube(x) \rightarrow Large(x)]$
$\forall x \ [Large(x) \rightarrow LeftOf(x,b)]$
$Cube(d)$

$\exists x \ [Large(x) \land LeftOf(x,b)]$

Proof :

1. by universal instantiation: $Cube(d) \rightarrow Large(d)$

2. by universal instantiation: $Large(d) \rightarrow LeftOf(d,b)$

3. through Modus Ponens: $Large(d)$

4. through Modus Ponens: $LeftOf(d,b)$

5. then we have $Large(d) \land LeftOf(d,b)$

6. applying existential introduction: $\exists x \ [Large(x) \land LeftOf(x,b)]$

Existential Instantiation

Existential Instantiation

= Existential Elimination

- From $\exists x \, S(x)$, infer **S(c)**
- Since x that satisfies $S(x)$ is not known, select a temporary name c and assume that $S(c)$ is satisfied.
- Here, c must be selected with a **new name** which is not already in use.
- **e.g.** : (Math) shown that there is a prime number between *n* and *m*. Call it *p*. or Let *p* be such a prime number.

Example : Existential Instantiation

∀x [Cube(x) → Large(x)]
∀x [Large(x) → LeftOf(x,b)]
∃x Cube(x)

∃x [Large(x) ∧ LeftOf(x,b)]

Proof :

① The 3rd premise states that there is at least one cube.
Let "e" name one of these cube. [Existential Instantiation]

② by universal instantiation: Cube(e) → Large(e)

③ by universal instantiation: Large(e) → LeftOf(e,b)

④ through Modus Ponens: Large(e)

⑤ through Modus Ponens: LeftOf(e,b)

⑥ then we have Large(e) ∧ LeftOf(e,b)

⑦ applying existential introduction:
$$∃x [Large(x) ∧ LeftOf(x,b)]$$

General Conditional Proof

- the method of proof which chooses an arbitrary object and proves an argument in order to prove a universal claim

- Proof of $\forall x \ [P(x) \rightarrow Q(x)]$
- choose a new name c, assume P(c), and prove Q(c)

note make sure that Q does not contain any names introduced by existential instantiation after the assumption of P(c)

Example : General Conditional Proof

> Anyone who passes Logic course with an A is smart.
> Every CSE major has passed Logic course with an A.
>
> Every CSE major is smart.

Proof :

- Let "Jae" refer to any one of the CSE major.
- By the 2nd premise, Jae passed Logic course with an A.
- By the 1st premise, then, Jae is smart.
- But since Jae is an arbitrarily chosen CSE major, it follows that every CSE major is smart.

Correct Proof of Valid Argument
(General Conditional Proof)

$\exists y \; [\text{Girl}(y) \land \forall x \; (\text{Boy}(x) \rightarrow \text{Likes}(x,y))]$

$\forall x \; [\text{Boy}(x) \rightarrow \exists y \; (\text{Girl}(y) \land \text{Likes}(x,y))]$

Proof :

- Assuming the premise, at least one girl is liked by every boy.
- Let c be one of these popular girls,
  then $\forall x \; (\text{Boy}(x) \rightarrow \text{Likes}(x,c))$.
- Assume that d is any boy. Since every boy likes c, so d likes c.
- Thus, by existential generalization, d likes some girl.
- Since d was an arbitrarily chosen boy, by general conditional
  proof, every boy likes at least one girl.

Incorrect Proof of Invalid Argument
(General Conditional Proof)

$\forall$x [Boy(x) $\rightarrow$ $\exists$y (Girl(y) $\wedge$ Likes(x,y))]

$\exists$y [Girl(y) $\wedge$ $\forall$x (Boy(x) $\rightarrow$ Likes(x,y))]

Incorrect Proof :
- Assuming the premise, every boy likes at least one girl.
- Let e be any boy, then by the premise e likes some girl.
- **By existential instantiation, let f be some girl that e likes.**
- Since e was an arbitrarily chosen boy, by general conditional proof, **every boy likes f**.
- By existential generalization, at least one girl is liked by every boy.

Problem with Proof :
- a new name **f** (one of the girls that e likes) introduced by existential instantiation are contained in "**every boy likes f**"
  ($\forall$**x (Boy(x) $\rightarrow$ Likes(x,f)))**

Universal Generalization

Universal Generalization

= Universal Introduction

- From **S(c)**, infer $\forall$**x S(x)**

- introduce a new name c to stand for a completely arbitrary member of the domain of discourse and prove the sentence S(x), then can conclude $\forall$x S(x)

note make sure that S(c) does not contain any names introduced by existential instantiation after the introduction of c

Example : Universal Generalization

$\forall x \ (\text{Cube}(x) \rightarrow \text{Small}(x))$
$\forall x \ \text{Cube}(x)$

$\forall x \ \text{Small}(x)$

Proof :

① Choose a new name d to stand for any member of the domain of discourse.

② Applying universal instantiation to each premise,
$\text{Cube}(d) \rightarrow \text{Small}(d)$ and $\text{Cube}(d)$ is given.

③ By Modus Ponens, $\text{Small}(d)$ is concluded.

④ Since d denotes an arbitrary object in the domain,
$\forall x \ \text{Small}(x)$ can be concluded by universal generalization.

Example : Universal Generalization

$\forall x\ Cube(x)$
$\forall x\ Small(x)$

$\forall x\ (Cube(x) \land Small(x))$

Proof :

1. Choose a new name d to stand for any member of the domain of discourse.

2. Applying universal instantiation to each premise, Cube(d) and Small(d) is given.

3. Hence, Cube(d) ∧ Small(d).

4. Since d denotes an arbitrary object in the domain, $\forall x\ (Cube(x) \land Small(x))$ can be concluded by universal generalization.

Incorrect Proof of Invalid Argument
(Universal Generalization)

$\forall$x $\exists$y Adjoins(x,y)

$\exists$y $\forall$x Adjoins(x,y)

Incorrect Proof :

- Assume the premise and take c as a name for an arbitrary member of the domain. By universal instantiation, $\exists$y Adjoins(c,y) is given.
- Take a new name d and **apply existential instantiation, then Adjoins(c,d) is given**.
- Since c stands for an arbitrary object, **by universal generalization, $\forall$x Adjoins(x,d)**.
- Hence, by existential generalization, $\exists$y $\forall$x Adjoins(x,y)

Problem with Proof :

- a new name **d** introduced by existential instantiation are contained in $\forall$x **Adjoins(x,d)**

Famous Proof: Euclid's Theorem

$\forall x \; \exists y \; [y \geq x \land \textbf{Prime}(y)]$

any natural number has a prime number that is equal or larger than itself

Proof :

- Let n be an arbitrary natural number and try to prove that there exists a prime number at least as large as n.
- Let k be the product of all the prime numbers less than n.
  Thus each prime less than n divides k without remainder.
- Let m = k + 1, each prime less than n divides m with remainder 1.
  But we know that m can be factored into primes.
- Let p be on of these primes, then p must be greater than or equal to n (because number less than n cannot divide m without remainder).

- By existential generalization, there does indeed exist a prime number greater than or equal to n.
- Since n was arbitrary, this applies to all natural number as well.

Famous Proof: Barber's Paradox

∃z ∃x [BarberOf(x,z) ∧
        ∀y (ManOf(y,z) → (Shave(x,y) ↔ ¬Shave(y,y)))]

In a town, there was a barber who shaved all and only the men of the town who did not shave themselves.

Purported Proof :

- Assume that there is such a town. Let's call it Hoosierville and call its barber Fred. By assumption, Fred shaves all and only those men of the town who do not shave themselves. Fred shaves himself, or he doesn't. Show that either possibility leads to a contradiction.
    - **Fred does shave himself** : by assumption, Fred does not shaves any man of the town who shave himself. So he does not shave himself → contradiction!
    - **Fred does not shave himself** : by assumption, Fred shaves any man of the town who does not shave himself. So he must shave himself → contradiction!
- Contradiction follows from each possibilities, thus there is no such

4. Formal Proofs

Universal Quantifier Rules

- **Universal Elimination ($\forall$ Elim)** (= Universal Instantiation)

    $\forall x \, S(x)$
    :
  ▷ $S(c)$

- **Universal Conditional Proof ($\forall$ Intro)**

    ┌ $\boxed{c}$ $P(c)$
    │ :   the constant c does not occur outside
    │ $Q(c)$ the subproof where it is introduced
  ▷ $\forall x \, (P(x) \rightarrow Q(x))$

- **Universal Introduction ($\forall$ Intro)** (= Universal Generalization)

    ┌ $\boxed{c}$
    │ :   the constant c does not occur outside
    │ $P(c)$ the subproof where it is introduced
  ▷ $\forall x \, P(x)$

Universal Quantifier Rules

$\forall x\ (P(x) \to Q(x))$	Anyone who passes Logic course with an A is smart
$\forall z\ (Q(z) \to R(z))$	Every CSE major has passed Logic course with an A
$\forall x\ (P(x) \to R(x))$	Every CSE major is smart

- **Proof** : validity of the argument

  1. $\forall x\ (P(x) \to Q(x))$
  2. $\forall z\ (Q(z) \to R(z))$

     3. $\boxed{d}\ P(d)$

     4. $P(d) \to Q(d)$ $\forall$ Elim: 1
     5. $Q(d)$ $\to$ Elim: 3,4
     6. $Q(d) \to R(d)$ $\forall$ Elim: 2
     7. $R(d)$ $\to$ Elim: 5,6
  8. $\forall x\ (P(x) \to R(x))$ $\forall$ Intro: 3-7

- **Let's play in Fitch** : Universal 1 & 2

Existential Quantifier Rules

- **Existential Introduction (∃ Intro)**
  (= Existential Generalization)

  $\triangleright$ $\quad\left|\quad\begin{array}{l} S(c) \\ \vdots \\ \exists x\, S(x) \end{array}\right.$

- **Existential Elimination (∃ Elim)**
  (= Existential Instantiation)

  $\triangleright$ $\quad\left|\quad\begin{array}{ll} \exists x\, S(x) \\ \vdots \\ \quad\boxed{c}\ S(c) & \text{the constant c does not occur outside} \\ \quad\vdots \\ \quad Q & \text{the subproof where it is introduced} \\ \\ Q & \end{array}\right.$

  the constant c does not occur outside
  the subproof where it is introduced

  Here, Q should not contain the constant c

Existential Quantifier Rules

$\forall x \ [Cube(x) \rightarrow Large(x)]$
$\forall x \ [Large(x) \rightarrow LeftOf(x,b)]$
$\exists x \ Cube(x)$

$\exists x \ (Large(x) \wedge LeftOf(x,b))$

- **Proof** : validity of the argument
  1. $\forall x \ [Cube(x) \rightarrow Large(x)]$
  2. $\forall x \ [Large(x) \rightarrow LeftOf(x,b)]$
  3. $\exists x \ Cube(x)$

     4. $\boxed{e} \ Cube(e)$
        5. $Cube(e) \rightarrow Large(e)$ $\forall$ Elim: 1
        6. $Large(e)$ $\rightarrow$ Elim: 4,5
        7. $Large(e) \rightarrow LeftOf(e,b)$ $\forall$ Elim: 2
        8. $LeftOf(e,b)$ $\rightarrow$ Elim: 6,7
        9. $Large(e) \wedge LeftOf(e,b)$ $\wedge$ Intro: 6,8
        10. $\exists x \ (Large(x) \wedge LeftOf(x,b))$ $\exists$ Intro: 9
  11. $\exists x \ [Large(x) \wedge LeftOf(x,b)]$ $\exists$ Elim: 3, 4-10

- **Let's play in Fitch** : Existential 1

Strategy & Tactics

- Always be clear about the meaning of the sentences you are using

- A good strategy is to find an informal proof and then try to formalize it

- Working backwards can be very useful in proving universal claims especially those of the form $\forall x\,(P(x) \to Q(x))$

- Working backwards is not useful in proving an existential claim $\exists x\,S(x)$ unless you can think of a particular instance $S(c)$ of the claim that follows from the premises

- If you get stuck consider using proof by contradiction

Formal Proof Exercise

∃x [Tet(x) ∧ Small(x)]
∀x [Small(x) → LeftOf(x,b)]

∃x LeftOf(x,b)

- **Informal Proof** :

  We were told that there is a small tetrahedron.

  So, let's select it.

  But, we're also told that anything that is small is left of b.

  So if it is small it's got to be left of b, too!

  So, something's left of b, namely the small tetrahedron.

- **Formal Proof** : Let's try in Fitch!

Formal Proof Exercise : Proof by contradiction

$\neg \forall x\ P(x)$

$\exists x\ \neg P(x)$

- **Informal Proof** :

  To prove an existential sentence, our first thought would be to use existential introduction, say by proving $\neg P(c)$ for some c.

  But looking at the premise, we see there is no hope of proving of any particular thing that it satisfies $\neg P(x)$.

  From the fact that "not everything satisfies $P(x)$", we aren't going to prove of some specific c that $\neg P(c)$.

  So, this is surely a dead end!

  This leaves only one possible route to our desired conclusion :

  Proof by contradiction.

Formal Proof Exercise : Proof by contradiction

$\neg\forall x\ P(x)$

$\exists x\ \neg P(x)$

- **Informal Proof** : Proof by Contradiction

  So, negate our desired conclusion and try to obtain a contradiction.
  Assume $\neg\exists x\ \neg P(x)$. How can we hope to obtain a contradiction ?
  Since our only premise is $\neg\forall x\ P(x)$, the most promising approach
  would be to try for a proof of $\forall x\ P(x)$ using universal generalization.

  Thus, let c be an arbitrary individual in our domain of discourse.
  Our goal is to prove $P(c)$. How? Another proof by contradiction!
  If $P(c)$ were not the case, then we would have $\neg P(c)$, and hence
  $\exists x\ \neg P(x)$. This contradicts our assumption. Hence, $P(c)$ is the case.

  Since c were arbitrary, we get $\forall x\ P(x)$.

  But this contradicts our premise. Hence, $\exists x\ \neg P(x)$.

- **Formal Proof** : Let's try in Fitch - Quantifier Strategy 1