

CSE1007: Logical Fundamentals of Programming

Part I - Propositional Logic

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1. The Logic of Atomic Sentences

Objectives :

- learn **Forst-Order Logic (FOL)**
- understand concepts and methods for deducing logical results from the given facts

First-Order Logic (FOL)

- Logic :
 - "reasoning conducted or assessed according to strict principles of validity"
- Syntax
 - defines the structure of logical sentences
- Semantics
 - defines the meaning(intepretation) of logical sentences
 - either TRUE or FALSE

Individual Constants

- symbol/name that refer to an actually existing object

Natural Language	Ji-Sung Park	Hanyang University ERICA Campus	Three
individual constants of FOL (all small letters)	park	hyu-erica	3

- Restrictions
 - Every individual constant **must name** an actually existing object
 - **No** individual constant can name more than one object
 - Ji-Sung Park? Tae-Hwan Park? Chan Ho Park? \Rightarrow park X
 - A single object can have **more than one** name
 - Scott Lee \Rightarrow professor-lee, doctor-lee, mr-lee, scott, s ✓
 - An object can have **no name** at all

Terms

- Simple terms
 - individual constants
- Complex terms
 - formed using **functional symbol** with more than 1 (simple/complex) term as arguments
 - Hence, **functional symbol(terms)**

e.g.,

Natural Language	Sarah's mother	Sarah's mother's mother
term of FOL (all small letters)	<code>mother(sarah)</code>	<code>mother(mother(sarah))</code>

- used just like names(simple terms) → the same restrictions on individual constant applies to complex term
- Hence, must refer to **one and only one** object

Predicate Symbols

- express **property** of objects or **relationship** between objects
- can be considered as a *verb* or *adjective*

Natural Language	Sarah is at Home	Sarah likes Tom's brother
predicate of FOL (starts with a Capital letter)	Home	Likes

- arity (number of arguments) is fixed
 - e.g., Home - predicate symbol with arity 1(unary)
 - Likes - predicate symbol with arity 2 (binary)
 - Sent - predicate symbol with arity 3 (ternary)

Atomic Sentences

- formed by **predicate** of arity n followed by n **terms** (predicate having terms as arguments)
- Hence, **predicate(term, ..., term)**

Natural Language	Sarah is at Home	Sarah likes Tom's brother
FOL	<code>Home(sarah)</code>	<code>Likes(sarah, brother(tom))</code>

- arity(no. of arguments) associated with a predicate is **fixed**

Atomic Sentence	Predicate	Arguments	Arity
<code>Home(sarah)</code>	<code>Home</code>	<code>sarah</code>	unary
<code>Likes(sarah, brother(tom))</code>	<code>Likes</code>	<code>sarah, brother(tom)</code>	binary
<code>Sent(sarah, tom, email)</code>	<code>Sent</code>	<code>sarah, tom, email</code>	ternary

Notation & Semantics of Atomic Sentences

Prefix/Infix Notation

- Prefix notation
 - **Predicate** comes first, then **arguments (terms)** surrounded with brackets comes after
e.g., Like(sarah, tom)
- Infix notation
 - **Predicate** comes in-between **arguments (terms)**
e.g., $a = b$

Semantics of Atomic Sentences

- Semantics: True or False
e.g., Teaches(professor-lee, logic) \Rightarrow True
Teaches(jaeho, logic) \Rightarrow False

Characteristics of Atomic Sentences

- The **order** of terms (arguments) for predicates are important!
 - change of order → change of meaning
 - Likes(sarah, tom)
 - Likes(tom, sarah)
- The meaning of an atomic sentence is always **determinate**!
 - always the same meaning! the meaning never changes!
 - Home(sarah)
 - Young(tom)
 - Loves(sarah, tom)

Differences between the below 3 Atomic Sentences?

- Syntax : TomGaveCandyTo(**y**)
- Semantic : Tom gave a candy to **y**
e.g., TomGaveCandyTo(sarah)
- Syntax : GaveCandyTo(**x,y**)
- Semantic : **x** gave a candy to **y**
e.g., GaveCandyTo(tom,sarah)
- Syntax : Gave(**x,y,z**)
- Semantic : **x** gave **z** to **y**
e.g., Gave(tom,sarah,candy)

Lets Play with Tarski's World!

- How to play with Tarski's World?
 - reference: LPL Software Manual
 - Chapter 3 Using Tarski's World
- Homework : get friendly with the Tarski's World
- Let's get into Tarski's World and interpret the meanings of atomic sentences
 - Let's play with the file, 'Wittgenstein's World'
 - Let's play with the file, 'Wittgenstein's Sentences'

Example of FOL Language 1 :

FOL Language of Set Theory

Predicates

• $=$

- predicate that checks whether two sets are the same (set equality)
- infix & binary operator

e.g., atomic sentence: $\text{ansan} = \text{haengdang}$, $\text{hanyang} = \text{hanyang}$

• \in

- predicate that checks whether the element is member of a set (set membership)
- infix & binary operator

e.g., atomic sentence: let a be the name of 1 & b be the name of $\{1,3,5\}$, then the meaning of followings are?

- $a \in a$
- $a \in b$
- $b \in a$
- $b \in b$

Example of FOL Language 2 :

FOL Language of Arithmetics

- Atomic Sentences

- Names: 0, 1
- Predicate: =, < (infix, binary)
- Functional symbols: +, × (infix, binary)

- Terms

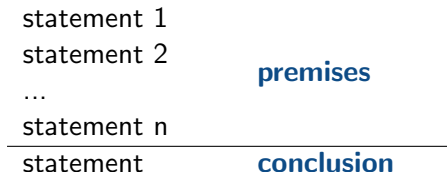
- Consideration: How many terms can we make? → **too many**
- Definition (Inductive): term
 1. Then name 0 & 1 are terms
 2. if t_1 & t_2 are terms,
then the expression $(t_1 + t_2)$ and $(t_1 \times t_2)$ are also terms
 3. Nothing is a term unless it can be obtained by
repeated application of (1) and (2)

e.g., term : $(1+1)$ & $((1+1) \times (1+1))$

e.g., atomic sentence : $(1 \times 1) < (1+1)$ & $(1 \times 1) = (1+0)$

Arguments

- Arguments = a series of statements consisting of one or more **premises** & **conclusion**



- Example:** All men are mortal. Socrates is a man. So, Socrates is mortal.
 - premise 1:** All men are mortal.
 - premise 2:** Socrates is a man.
 - conclusion:** Socrates is mortal.

Logically Valid Arguments

- An argument is **valid** if :
 - the conclusion must be true in any circumstance in which the premises are true
- The conclusion of logically valid argument is a **logical consequence** of its premises.

Logically Sound Arguments

- An argument is **sound** if :
 - ① argument is valid
 - ② its premises are all true
- Validity
 - guarantees the truth of the conclusion
 - assume premises to be true
- Soundness
 - guarantees the truth of the conclusion
 - all premises are true
 - ⇒ insure the truth of the conclusion

Are these Arguments Logically Sound ?

- **Argument 1:**

All men are mortal. Socrates is a man. So, Socrates is mortal.

- **premise 1**: All men are mortal.
- **premise 2**: Socrates is a man.
- **conclusion**: Socrates is mortal.

✓ Logically Valid & ✓ Logically Sound

- **Argument 2:**

All women are mortal. Socrates is a woman. So, Socrates is mortal.

- **premise 1**: All women are mortal.
- **premise 2**: Socrates is a woman.
- **conclusion**: Socrates is mortal.

✓ Logically Valid but X Logically Unsound

What is Important in the Study of Logic ?

- Answer: **Logical Validity**
- **Study of Logic**
 - focuses on proving the logical validity of an arguments
 - a valid argument are sound if all of its premises are true
BUT finding out whether any premises is true or false is not part of the study
 - The study of logic is never interested in finding out whether Socrates was a man or a woman

Format for writing Arguments

Fitch* Format

All women are mortal.

Socrates is a woman.

premises

Socrates is mortal.

conclusion

* Frederic Fitch: American Logician

Proof

- What is a **proof** ?
 - step-by-step demonstration which shows that
 - **statement (conclusion)** must be true
 - in any circumstances in which the **statement₁, statement₂, ... , statement_n (premises)** are all true.

conclusion

Proof Methods

- Informal Proof
 - using natural language
 - all proofs were done this way before Computer Science (most of the mathematical proof)
 - done by human
- Formal Proof
 - using symbolic logic
 - natural deduction system
 - done by computer (automatic theorem proving)
- Difference
 - both rigor
 - style: natural language *vs* symbolic logic

Informal Proofs

Socrates is a man.

All men are mortal.

All mortals worry about dying sometimes.

Socrates worries about dying sometimes.

• Proof

- Socrates is a man and all men are mortal.
So, Socrates is a mortal.
- It is premised that all mortals worry about dying sometimes.
So, Socrates worries about dying sometimes.

Formal Proofs

Fitch-style Proof

- using Deductive System designed by Frederic Fitch
- called “**System F**”

intermediate conclusion

P_1	
\dots	
P_n	
S_1	justification 1
\dots	\dots
S_n	justification n
S	justification n+1

Formal Proof Rules for Identity

- Identity Introduction (= Intro)

$$\triangleright \mid n = n$$

- Identity Elimination (= Elim)

$$\triangleright \begin{array}{|l} P(n) \\ \dots \\ n = m \\ \dots \\ P(m) \end{array}$$

- Reiteration (= Reit)

▷	P
	...
	P

Example of Formal Proof

- | | |
|----|---------|
| 1. | Cube(c) |
| 2. | $c = b$ |
| 3. | Cube(b) |

- Symmetry of Identity

- | | |
|----|---------|
| 1. | $a = b$ |
| | ... |
| ? | $b = a$ |

Example of Formal Proof

1.	$\text{Cube}(c)$	
2.	$c = b$	
<hr/>		
3.	$\text{Cube}(b)$	$= \text{Elim: } 1,2$

- Symmetry of Identity

1.	$a = b$	
<hr/>		
2.	$a = a$	$= \text{Intro}$
3.	$b = a$	$= \text{Elim: } 2,1$

Another Example of Formal Proof

1. SameRow(a,a)

2. $b = a$

? SameRow(b,a)

Another Example of Formal Proof

1. SameRow(a,a)

2. $b = a$

? $a = b$

? SameRow(b,a) = Elim: 1,?

Another Example of Formal Proof

- | | | |
|-------|--------------|-------------|
| 1. | SameRow(a,a) | |
| 2. | $b = a$ | |
| <hr/> | | |
| 3. | $b = b$ | = Intro |
| 4. | $a = b$ | = Elim: 3,2 |
| 5. | SameRow(b,a) | = Elim: 1,4 |

Lets Play with Fitch!

- Fitch
 - is an implementation of the Proof System “System F”
 - is a software for constructing Formal Proofs
 - can automatically check constructed proofs
- Let's construct previously covered proofs with Fitch
- Analytic Consequence (Ana Con)
 - Let's open 'Ana Con 1' file and play with it

Proof of invalidity

Proof of nonconsequences

- Find **counter-example**:
 - all premises (P_1, \dots, P_n) is true
 - BUT conclusion (S) is not true \Rightarrow proves that argument is NOT valid

Kim Gu is a politician

Most politicians are dishonest

Kim Gu is dishonest

2. The Boolean Logic

Boolean Logic Sentences

- Atomic sentences
- Complex sentences
 - atomic sentences connected with Boolean connectives (operators)
- Boolean connectives
 - Negation (not) : \neg
 - Conjunction (and) : \wedge
 - Disjunction (or) : \vee

Truth-Functional Connectives

- Boolean connectives (\neg , \wedge , \vee) are truth-functional connectives
- The truth value of a complex sentence (built up using connective) depends on nothing more than the truth values of the simpler sentences from which it is built
- Therefore, the semantics of a complex sentence built up using truth-functional connectives can be given by **truth table**
- Henkin-Hintikka game
 - one player claims a complex sentence is true & the other player claims it is false
 - reduce the complex sentence into a atomic sentence and justify their claims

Negation Symbol : \neg

- Syntax

- if P is a sentence, then $\neg P$ is also a sentence
- prefix, unary
- **literal** : if a sentence is an atomic sentence or the negation of an atomic sentence
- abbreviation of negated identity claims: \neq
e.g.1 $\neg(b = c) \rightarrow b \neq c$

- Semantics

- $\neg P$ is true **if and only if** P is false
- Truth Table :

P	$\neg P$
true	false
false	true

- Game: Let's play in Wittgenstein's World
 $\neg\neg\neg\neg$ Between(e,d,f)

Conjunction Symbol : \wedge

- Syntax

- if P and Q are sentences, then $P \wedge Q$ is also a sentence
- infix, binary

- Semantics

- $P \wedge Q$ is true **if and only if** both P and Q are true
- Truth Table :

P	Q	$P \wedge Q$
true	true	true
true	false	false
false	true	false
false	false	false

- Game: Let's play in Claire's World
 $\neg \text{Cube}(a) \wedge \neg \text{Cube}(b) \wedge \neg \text{Cube}(c)$

Disjunction Symbol : \vee

- Syntax

- if P and Q are sentences, then $P \vee Q$ is also a sentence
- infix, binary

- Semantics

- $P \vee Q$ is true **if and only if** P is true or Q is true (or both P and Q are true)
- Truth Table :

P	Q	$P \vee Q$
true	true	true
true	false	true
false	true	true
false	false	false

- Game: Let's play in Ackermann's World
 $\text{Cube}(c) \vee \neg(\text{Cube}(a) \vee \text{Cube}(b))$

Ambiguity & Parentheses

- Ambiguous sentences

e.g.1 : $\text{Home}(\text{max}) \vee \text{Home}(\text{claire}) \wedge \text{Happy}(\text{carl})$

e.g.2 : $\neg \text{Home}(\text{claire}) \wedge \text{Home}(\text{max})$

- Are there sentences ambiguous? Why?

- How to avoid the ambiguity?

- use parentheses

e.g.1 $(\text{Home}(\text{max}) \vee \text{Home}(\text{claire})) \wedge \text{Happy}(\text{carl})$
 $\text{Home}(\text{max}) \vee (\text{Home}(\text{claire}) \wedge \text{Happy}(\text{carl}))$

e.g.2 $(\neg \text{Home}(\text{claire})) \wedge \text{Home}(\text{max})$
 $\neg (\text{Home}(\text{claire}) \wedge \text{Home}(\text{max}))$

- give different precedence

$\neg > \wedge > \vee$

- $()$ must be used for \wedge and \vee when combined with other connectives

Useful Laws

- Double Negation

$$\neg\neg P \Leftrightarrow P$$

for all P

- DeMorgan's Law

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

for all P and Q

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

for all P and Q

3 Concepts to Learn

- Tautologies and Logical Truth
- Logical and Tautological Equivalence
- Logical and Tautological Consequence

Interpret Boolean Logic Sentences with Truth Table

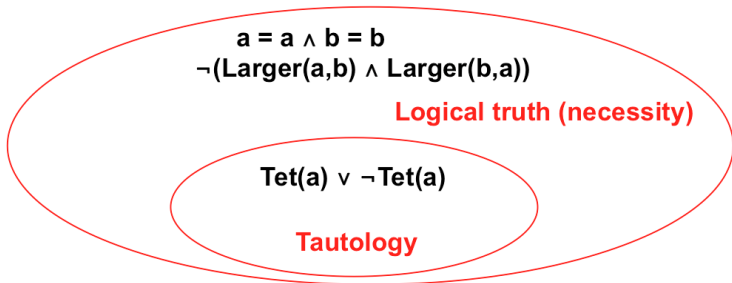
- Truth table can easily be generated with “Boole”
 - Practice :
 1. $P \vee \neg P$
 2. $(A \wedge B) \vee \neg C$
 3. $\neg(A \wedge (\neg A \vee (B \wedge C))) \vee B$
 4. $P \wedge Q \wedge R$
- * A, B, C, P, Q, R refers to atomic sentences

Tautologies and Logical Truth (Necessity)

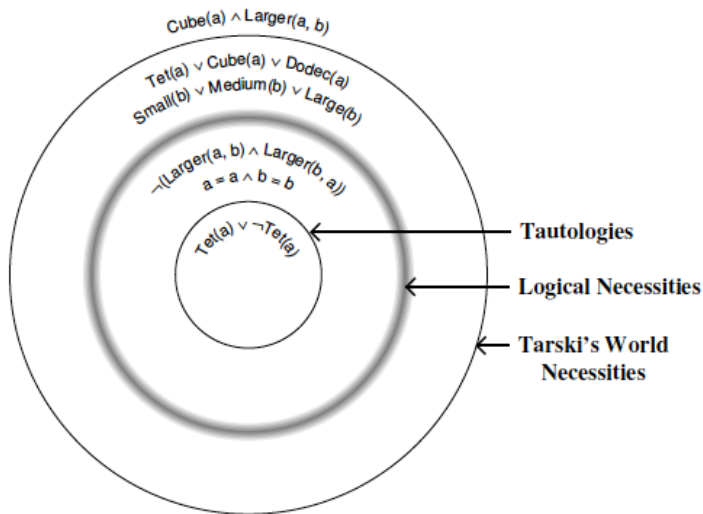
- If a sentence is true in every logically possible situation, the sentence is called **logically true** or **logically necessary**
- a sentence that can only be true in any case is called **tautology**
 - a sentence is tautology **iff** every row of the truth table assigns TRUE to the sentence
 - is also called **TT-necessary**
e.g. $P \vee \neg P$
 - If a sentence is true in every world in Tarski's World, then it is called **TW-necessary**

Tautologies and Logical Truth

- If a sentence is Tautology, then it is logical truth
- BUT if a sentence is logically true, it is not always tautology (some logical truths are not tautologies)



Tautology and Logical Truth



Logical Possibility

- If there exists a situation or a world where a sentence is true, then the sentence is **logically possible**
 - If at least one row of the Truth Table assigns TRUE to a sentence, then the sentence is **TT-possible**
 - If there is at least one world that is true in the Tarski's World, then the sentence is called **TW-possible**
 - logically possible vs. TW-possible
 - all TW-possible sentences are logically possible
 - BUT there exists a sentence that is logically possible but not TW-possible
- e.g. $\neg(\text{Tet}(b) \vee \text{Cube}(b) \vee \text{Dedoc}(b))$

Tautological Equivalence

- S and S' are tautologically equivalent **iff** every column for S and S' in the joint Truth Table are the same

e.g. $\neg(A \wedge B)$ vs $\neg A \vee \neg B$

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$\neg A \vee \neg B$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

✓ Tautologically Equivalent

Tautological Equivalence

e.g. $\neg((A \vee B) \wedge \neg C)$ vs $(\neg A \wedge \neg B) \vee C$

A	B	C	$A \vee B$	$\neg C$	$(A \vee B) \wedge \neg C$	$\neg((A \vee B) \wedge \neg C)$	$\neg A$	$\neg B$	$\neg A \wedge \neg B$	$(\neg A \wedge \neg B) \vee C$
T	T	T	T	F	F	T	F	F	F	T
T	T	F	T	T	T	F	F	F	F	F
T	F	T	T	F	F	T	F	T	F	T
T	F	F	T	T	T	F	F	T	F	F
F	T	T	T	F	F	T	T	F	F	T
F	T	F	T	T	T	F	T	F	F	F
F	F	T	F	F	F	T	T	T	T	T
F	F	F	F	T	F	T	T	T	T	T

✓ Tautologically Equivalent

Logical Equivalence

$a = b \wedge \text{Cube}(a)$ and $a = b \wedge \text{Cube}(b)$ are logically equivalent

Proof :

- Suppose that $a = b \wedge \text{Cube}(a)$ is true, then $a = b$ and $\text{Cube}(a)$ are both true. Since a and b are equal, $\text{Cube}(b)$ is also true ($=\text{Elim}$). Hence, $a = b \wedge \text{Cube}(b)$ is true.
So the truth of $a = b \wedge \text{Cube}(a)$ logically implies the truth of $a = b \wedge \text{Cube}(b)$.
- Reverse :
Suppose that $a = b \wedge \text{Cube}(b)$ is true, then $a = b$ is true and $\text{Cube}(b)$ is true. Then $b = a$ is also true ($\text{Symmetry of Identity}$). Since b and a are equal, $\text{Cube}(a)$ is also true ($=\text{Elim}$). Hence, $a = b \wedge \text{Cube}(a)$ is true.
So the truth of $a = b \wedge \text{Cube}(b)$ logically implies the truth of $a = b \wedge \text{Cube}(a)$.
- Since both sentences can logically imply truth of each other in both directions, the two sentences are logically equivalent.

Tautological Equivalence vs. Logical Equivalence

- If S and S' are tautologically equivalent, then they are also logically equivalent.
- Some logically equivalent sentences are not tautologically equivalent.
 - e.g. the following two sentences are logically equivalent
BUT not tautologically equivalent
 $a = b \wedge \text{Cube}(a)$ vs $a = b \wedge \text{Cube}(b)$
 - let's draw the joint Truth Table

Tautological Equivalence vs. Logical Equivalence

$a=b \wedge \text{Cube}(a)$ and $a=b \wedge \text{Cube}(b)$ are tautologically equivalent?

$a=b$	$\text{Cube}(a)$	$\text{Cube}(b)$	$a=b \wedge \text{Cube}(a)$	$a=b \wedge \text{Cube}(b)$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

X NOT tautologically equivalent

Understand why?

Tautological & Logical Consequence

Let P_1, \dots, P_n and Q be sentences & construct a joint Truth Table

- Tautological Consequence
 - Q is a tautological consequence of P_1, \dots, P_n **iff**
every row that assigns **T** to each of P_1, \dots, P_n
also assigns **T** to Q in the joint Truth Table
- Logical Consequence
 - Q is a logical consequence of P_1, \dots, P_n **iff**
 Q is true when we assume all P_1, \dots, P_n are true

Tautological & Logical Consequence

e.g. Determination of tautological consequences using Truth Table

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

$A \vee B$ is a tautological consequence of $A \wedge B$

= $A \vee B$ is a logical consequence of $A \wedge B$

= $A \wedge B$ logically implies $A \vee B$

$A \wedge B$ is NOT a tautological consequence of $A \vee B$

= $A \vee B$ does NOT logically implies $A \wedge B$

Logical Consequence

If Q is a tautological consequence of P_1, \dots, P_n ,
then Q is a **logical consequence** of P_1, \dots, P_n

Proof :

- proves if Q is not a logical consequence of P_1, \dots, P_n , then Q is not a tautological consequence of P_1, \dots, P_n
- Suppose Q is not a logical consequence of P_1, \dots, P_n .
Then, by the definition of logical consequence, there must be a possible circumstance in which P is true but Q is false.
This circumstance will determine the truth values of the atomic sentences in P and Q . Furthermore, the row for P and Q corresponding to these values exists in the joint Truth Table, then P will be assigned T and Q will be assigned F in that row.
Hence, Q is not a tautological consequence of P_1, \dots, P_n .

Logical but NOT Tautological Consequence

- $a=c$ is a logical consequence of $(a=b \wedge b=c)$
- BUT $a=c$ is not a tautological consequence of $(a=b \wedge b=c)$
- Let's draw a joint Truth Table and prove why

$a=b$	$b=c$	$a=c$	$a=b \wedge b=c$	$a=c$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

More Examples

$$\begin{array}{|l} A \vee B \\ \neg A \\ \hline B \end{array}$$

$$\begin{array}{|l} A \vee \neg B \\ B \vee C \\ \hline A \vee C \end{array}$$

- Create a Truth Table with 'Boole' and check whether the conclusion is a tautological consequence of premises.

Logical Consequence in Fitch

- **Taut Con** : **Taut**ological **Con**sequence
- **FO Con** : **F**irst-**O**rd**er** **Con**sequence
 - understand not only tautological consequence but also =
- **Ana Con** : **Ana**lytic **Con**sequence
 - understand not only logical consequence but also predicates in Tarski's World
- Let's play with Fitch using the following files
 - Taut Con 1
 - Taut Con 2

Logical Consequence vs. Logical Equivalence / Logical Truth

- If P is a logical consequence of Q and Q is a logical consequence of P , then P and Q are logically equivalent.
- if a sentence is a logical consequence without premises, then it is logical truth.

3. Informal Proof

Problems & Limitations of Truth Table

- 1 only effective for showing validity of simple logical sentences that depends only on Boolean connectives
- 2 The table size gets extremely large as the number of atomic sentences goes up
 - number of atomic sentences = n
 - number of rows in the corresponding Truth Table = 2^n (exponential growth)
 - not very effective as too many calculations are required
- 3 cannot be easily extended
 - can only be used for sentences composed of Boolean connectives
 - can only detect tautological consequence

Informal Proof

- Overcomes the limitations of Truth Table
- The purposes of proofs
 - method of discovery:
extract new information from information already obtained
 - method of communication:
convey discoveries to others
- Proof must be written so that; the writer of the proof (proof writer) can convince the audience (proof reader)!
- There is a style in giving proofs
 - every proof writer has his/her own style

Informal Proof

- a “Good” proof is
 - ① **correct**
 - ② **easily understood:**

proof readers should be able to follow the proof step without any complex reasoning of their own
 - ③ **significant:**

the proof step should be informative, not a waste of the proof readers’ time.
 - 2 and 3 are the opposite properties:

more significant steps are harder to follow &
easy to understand steps makes proof verbose
- Hence** a “good” proof requires a proper balance between the two.

Valid Inference Steps

- if Q is already known to be a logical consequence of sentence P_1, \dots, P_n and each of P_1, \dots, P_n has been proven from the premises, then Q can be asserted in the proof
- logical truths such as $a = a$ or $P \vee \neg P$ can be asserted at any point in the proof
- The following are valid patterns of inference that generally go unmentioned in informal proofs
 - ① **Conjunction Elimination:** From $P \wedge Q$, infer P
 - ② **Conjunction Introduction:** From P and Q , infer $P \wedge Q$
 - ③ **Disjunction Introduction:** From P , infer $P \vee Q$

Informal Proof Methods

- Proof By Cases
- Indirect Proof
 - Proof By Contradiction

Proof by Cases

= **Disjunction Elimination** in System F

- To prove S from $P_1 \vee \dots \vee P_n$
 - break disjunction into n cases
 - prove S from P_1
 - prove S from P_2
 - ...
 - prove S from P_n
- Pattern for proving S from $P \vee Q$
 - let's say that the desired goal is to prove S
 - assume that we already know $P \vee Q$
 - assume that P is true and prove if S is true
 - assume that Q is true and prove if S is true
 - since we know that either P or Q is true, S must be true
- cannot be proved with Truth Table

Proof by Cases: Example 1

Theorem: there exists irrational numbers b and c such that b^c is a rational number

Proof

- consider $\sqrt{2}^{\sqrt{2}}$, this number is either rational or irrational
- case 1: $\sqrt{2}^{\sqrt{2}}$ is a rational number
 - $b = c = \sqrt{2}$ exists
- case 2: $\sqrt{2}^{\sqrt{2}}$ is an irrational number
 - take $b = \sqrt{2}^{\sqrt{2}}$ & $c = \sqrt{2}$ and compute b^c
 - $b^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2$
 - in this case b^c is also a rational number

Proof by Cases: Example 2

Theorem: $Small(c)$ is a logical consequences of $(Cube(c) \wedge Small(c)) \vee (Tet(c) \wedge Small(c))$

Proof

- **premise:** $(\text{Cube}(c) \wedge \text{Small}(c)) \vee (\text{Tet}(c) \wedge \text{Small}(c))$
- **case 1:** assume that $(\text{Cube}(c) \wedge \text{Small}(c))$ holds
 - then (by Conjunction Elimination) we have $\text{Small}(c)$
- **case 2:** assume that $(\text{Tet}(c) \wedge \text{Small}(c))$ holds
 - then (by Conjunction Elimination) we have $\text{Small}(c)$

Proof by Cases: Example 3

Theorem: assume that

$(\text{Home}(\text{max}) \wedge \text{Happy}(\text{carl})) \vee (\text{Home}(\text{claire}) \wedge \text{Happy}(\text{scruffy}))$ is true
and prove that $\text{Happy}(\text{carl}) \vee \text{Happy}(\text{scruffy})$ is true

Proof

- **premise:** $(\text{Home}(\text{max}) \wedge \text{Happy}(\text{carl})) \vee (\text{Home}(\text{claire}) \wedge \text{Happy}(\text{scruffy}))$ is in the form of disjunction
- **case 1:** assume that $(\text{Home}(\text{max}) \wedge \text{Happy}(\text{carl}))$ holds
 - then (by Conjunction Elimination) $\text{Happy}(\text{carl})$ is true
 - then (by Disjunction Introduction) $\text{Happy}(\text{carl}) \vee \text{Happy}(\text{scruffy})$ is true
- **case 2:** assume that $(\text{Home}(\text{claire}) \wedge \text{Happy}(\text{scruffy}))$ holds
 - then (by Conjunction Elimination) $\text{Happy}(\text{scruffy})$ is true
 - then (by Disjunction Introduction) $\text{Happy}(\text{carl}) \vee \text{Happy}(\text{scruffy})$ is true

Proof by Cases: Example 4 (Logician Couple's Story)

A logician and his wife recently realized that their parking meter had expired several hours earlier

Logician's Proof

- we've already gotten a ticket or we haven't
- **case 1**: if we've gotten a ticket, we won't get another one in the time it takes us to get to the car
- **case 2**: if we haven't got a ticket in the past several hours, it is extremely unlikely that we get one in the next few minutes
- In either event, there's no need to rush

Wife's Proof

- we are going to get a ticket in the next few minutes or we aren't
- **case 1**: if we are going to get a ticket, then rushing might prevent it
- **case 2**: if we aren't going to get a ticket, then it will still be a good exercise and will also show our respect for the law
- In either event, rushing back to the car is a good thing

Indirect Proof: Proof by Contradiction

모순유도 증명

= **Reductio ad absurdum**

= **Negation Introduction** in System F

- Method of Proof

- let's say that the desired goal is to prove $\neg S$ from some premises P_1, \dots, P_n
- temporarily assume that S is true and show a contradiction follows from this assumption
(show that S, P_1, \dots, P_n cannot be all true simultaneously)
- can conclude that $\neg S$ is a logical consequence of P_1, \dots, P_n
(if P_1, \dots, P_n are true, then $\neg S$ must be true)

- contradiction: $\perp \equiv P \wedge \neg P$ (can never be true)

Proof by Contradiction: Example 2

$\sqrt{2}$ is an irrational number

- hint 1: rational number can be expressed as a fraction p/q
where at least one of p and q is an odd number
- hint 2: square of an odd number is always an odd number.
So if n^2 is an even number, then n is also an even number
Hence, if n^2 is even, it must be divisible by 4

Proof

- assume that $\sqrt{2}$ is a rational number & try to get contradiction
- $\sqrt{2}$ can be expressed as p/q and **at least one of p and q is odd**
- since $p/q = \sqrt{2}$, square both side to get $p^2/q^2 = 2$
- and then multiply both side with q^2 to get $p^2 = 2q^2$

Proof by Contradiction: Example 2

Proof (continues)

- so p^2 is an even number. Hence, p is even and p^2 is divisible by 4
- so $2q^2$ is divisible by 4, then q^2 is divisible by 2
- so q^2 is even, then q is even
- both p and q being even contradicts the fact that at least one of them is odd
- Hence, the assumption of $\sqrt{2}$ being rational leads to a contradiction
- $\sqrt{2}$ is an irrational number

Contradiction vs. Tautology

- contradiction $\equiv P \wedge \neg P$
- tautology $\equiv P \vee \neg P$
- S is a logical impossibility $\equiv \neg S$ is a logical truth (necessity)
논리적으로 거짓이면
- S is a tautology $\equiv \neg S$ is a contradiction
- P_1, P_2, \dots, P_n are TT-contradictory
 - if every row has an **F** assigned to at least one of the sentences in the joint Truth Table \rightarrow 모두 참이 되는 열이 없다
 - hence, P_1, P_2, \dots, P_n cannot all be true at once

Proof by Contradiction: Example 3

Defense attorney presenting the following summary to the jury

- Prosecution claims that my client killed the owner of the KitKat Club
- Assume that prosecution is correct.
- You've heard their own experts testifying that the murder took place at 5:15pm
- According to the testimony of 5 co-workers, we also know the defendant was still at work at City Hall at 4:45pm.
- Then my client had to get from City Hall to the KitKat Club in 30 minutes or less.
- But to make that trip takes 35 minutes under the best of circumstances, and police records show that there was a massive traffic jam the day of the murder.
- Hence, my client is innocent.

Arguments with Inconsistent Premises

- if P_1, \dots, P_n are contradictory, then they are called **inconsistent** 전제가 모순이면 일관성이 없다(증명할 필요 없음)
- any argument with an inconsistent set of premises is trivially valid BUT it is **unsound**
일관성없는 전제들을 가진 어떤 인자는 평범하게 유효하지만 옳지 않다()

e.g.

Home(max) \vee Home(claire)

\neg Home(max)

\neg Home(claire)

Home(max) \wedge Happy(carl)

いみ ないです。

4. Formal Proof

System F: Fitch-Style Natural Deduction System

- Proof rules
 - Conjunction rules
 - Disjunction rules
 - Negation rules
 - (Contradiction rules)
- Each rule consists of
 - an introduction rule
 - an elimination rule

Conjunction Rules

- **Conjunction Elimination Rule (\wedge Elim)**

$$\triangleright \left| \begin{array}{l} P_1 \wedge \dots \wedge P_i \wedge \dots \wedge P_n \\ \vdots \\ P_i \end{array} \right. \text{아무거나 빼도 참}$$

- **Conjunction Introduction Rule (\wedge Intro)**

	P_1	
	\Downarrow	
	P_n	
	:	전제가 참이라는 전제가 있기 때문에 and로 조인해도 무관
\triangleright	$P_1 \wedge \dots \wedge P_n$	

Conjunction Rules

Example:

1.	$A \wedge B \wedge C$	
2.	B	\wedge Elim: 1
3.	C	\wedge Elim: 1
4.	$C \wedge B$	\wedge Intro: 3,2

- **Let's construct a proof in Fitch**

- Conjunction 1
- Conjunction 2

Default and Generous Uses of the \wedge Rules

Generous Uses 시험안냄

17. $\text{Tet}(a) \wedge \text{Tet}(b) \wedge \text{Tet}(c) \wedge \text{Tet}(d)$

:

26. $\text{Tet}(d) \wedge \text{Tet}(b)$ \wedge Elim: 17

13. $\text{Tet}(a) \wedge \text{Tet}(b)$

:

21. $\text{Tet}(b) \wedge \text{Tet}(a)$ \wedge Elim: 13

- Let's construct a proof in Fitch

- Conjunction 3
- Conjunction 4

Disjunction Rules

- Disjunction Introduction Rule (\vee Intro)

\triangleright $\begin{array}{|l} P_i \\ : \\ P_1 \vee \dots \vee P_i \vee \dots \vee P_n \end{array}$

- Disjunction Elimination Rule (\vee Elim)

\triangleright $\begin{array}{|l} P_1 \vee \dots \vee P_n \\ : \\ \begin{array}{|l} P_1 \\ : \\ S \end{array} \\ \Downarrow \\ \begin{array}{|l} P_n \\ : \\ S \end{array} \\ : \\ S \end{array}$

P_1, P_n : temporary assumptions
[] : subproofs

= Proof by Cases

Proof using Disjunction Rules: Example 1

1.	$(A \wedge B) \vee (C \wedge D)$	
2.	$A \wedge B$	
3.	B	\wedge Elim: 2
4.	$B \vee D$	\vee Intro: 3
5.	$C \wedge D$	
6.	D	\wedge Elim: 5
7.	$B \vee D$	\vee Intro: 6
8.	$B \vee D$	\vee Elim: 1, 2-4, 5-7

Proof using Disjunction Rules: Example 2

1.	$(B \wedge A) \vee (A \wedge C)$	
2.	$B \wedge A$	
3.	A	\wedge Elim: 2
4.	$A \wedge C$	
5.	A	\wedge Elim: 4
6.	A	\vee Elim: 1, 2-3, 4-5

Proof using Disjunction Rules: Example 3

1.	$(B \wedge A) \vee A$	
2.	$B \wedge A$	
3.	A	\wedge Elim: 2
4.	A	
5.	A	Reit: 4
6.	A	\vee Elim: 1, 2-3, 4-5

- Let's construct a proof in Fitch

- Disjunction 1
- Disjunction 2

Negation Rules

- Negation Elimination Rule (\neg Elim)

\triangleright $\begin{array}{|l} \neg\neg P \\ : \\ P \end{array}$

- Negation Introduction Rule (\neg Intro)

\triangleright $\begin{array}{|l} \begin{array}{|l} P \\ : \\ \perp \end{array} \\ \neg P \end{array}$

$(\equiv Q \wedge \neg Q)$

= Proof by Contradiction

Negation Rules

- **Contradiction Introduction Rule (\perp Intro)**

\triangleright $\begin{array}{|l} P \\ : \\ \neg P \\ : \\ \perp \end{array}$

- only used in subproof
- can be used in the main proof when proving inconsistency

- **Let's construct a proof in Fitch**

- Negation 1

Proving Inconsistency (between premises)

1.	$A \vee B$	
2.	$\neg A$	
3.	$\neg B$	
<hr/>		
4.	A	
5.	\perp	\perp Intro: 4,2
6.	B	
7.	\perp	\perp Intro: 6,3
8.	\perp	\vee Elim: 1, 4-5, 6-7

Introducing \perp in Fitch

- **Taut Con**

- can prove using $\wedge, \vee, \neg, \perp$ Intro/Elim
- TT-contradiction:
 - $\neg(A \vee \neg A)$
 - $\neg A \vee \neg B$ and $A \wedge B$

- **FO Con**

- can prove using $=, \wedge, \vee, \neg, \perp$ Intro/Elim
- applying $=$ Elim to $\text{Cube}(b)$, $b = c$, $\neg \text{Cube}(c)$ will give $\text{Cube}(c)$, $\neg \text{Cube}(c)$

- **Ana Con**

- understands predicates in Tarski's World (except Adjoins and Between)
- $\text{Cube}(b)$, $\text{Tet}(b)$

- **Let's construct a proof in Fitch** : Negation 2

Contradiction Rules

- Contradiction Introduction Rule (\perp Intro)

\triangleright $\begin{array}{|l} P \\ : \\ \neg P \\ : \\ \perp \end{array}$

- Contradiction Elimination Rule (\perp Elim)

\triangleright $\begin{array}{|l} \perp \\ : \\ P \end{array}$

- \perp Elim rule is a shortened version of the below proof

17.	\perp	
	18.	$\neg P$
	19.	\perp
20.	$\neg\neg P$	Reit: 17
21.	P	\neg Intro: 18-19
		\neg Elim: 20

- Let's construct a proof in Fitch: Negation 3

Default and Generous Uses of the \neg Rules

- \neg Elim
 - Fitch allow you to remove any even number of negation signs (\neg) in one step
- \neg Intro
 - Fitch allow you to deduce unnegated sentence A from $\neg A$ (instead of $\neg\neg A$)
- **Let's construct a proof in Fitch**
 - Negation 4

Incorrect Proof

- if you are not careful,
you may construct a proof that does not follow the premises

1.	$(B \wedge A) \vee (A \wedge C)$	
2.	$B \wedge A$	
3.	B	\wedge Elim: 2
4.	A	\wedge Elim: 2
5.	$A \wedge C$	
6.	A	\wedge Elim: 5
7.	A	\vee Elim: 1, 2-4, 5-6
8.	$A \wedge B$	\wedge Intro: 7,3

- when the subproof ends, the subproof's assumption can no longer be used (assumption has been **discharged** / subproof has been **ended**)

Nested Subproofs

1.	$\neg(P \wedge Q)$	
2.	$\neg(\neg P \vee \neg Q)$	
3.	$\neg P$	
4.	$\neg P \vee \neg Q$	\vee Intro: 3
5.	\perp	\perp Intro: 4,2
6.	$\neg\neg P$	\neg Intro: 3-5
7.	P	\neg Elim: 6
8.	$\neg Q$	
9.	$\neg P \vee \neg Q$	\vee Intro: 8
10.	\perp	\perp Intro: 9,2
11.	$\neg\neg Q$	\neg Intro: 8-10
12.	Q	\neg Elim: 11
13.	$P \wedge Q$	\wedge Intro: 7,12
14.	$\neg(P \wedge Q)$	Reit: 1
15.	\perp	\perp Intro: 13,14
16.	$\neg\neg(\neg P \vee \neg Q)$	\neg Intro: 2-15
17.	$\neg P \vee \neg Q$	\neg Elim: 16

Proof Strategy and Tactics

- 1 Understand the **meaning of sentences**
- 2 Decide whether you think the conclusion follows from the premises (this can easily be checked using Taut Con in Fitch)
- 3 If you think it does not follow, or are not sure, try to find a counter-example
- 4 If you think it does follow, try to give **informal proof**
- 5 If a formal proof is called for, use the informal proof to guide you in finding one (there are always corresponding formal proof rules)
 - (informal) proof by contradiction \Rightarrow (formal) \neg Intro rule
 - (informal) proof by cases \Rightarrow (formal) \vee Elim rule
- 6 In giving consequence proofs, both formal and informal, don't forget the tactic of working backwards
- 7 In working backwards, though, always check that your intermediate goals are consequences of the available information

Proof Strategy & Tactics: try a bit more difficult proof

$$\begin{array}{|l} \neg P \vee \neg Q \\ \hline \neg(P \wedge Q) \end{array}$$

This argument is Demorgan's Law
 \rightarrow Hence, it is a valid argument

- Informal Proof: 1st attempt
 - According to premises, we know that P is False or Q is False
 - Since either one of P and Q is False, P \wedge Q is False
 - Hence, $\neg(P \wedge Q)$ is True
- It is not very easy to draw up a formal proof even after looking at this informal proof
- Let's work backwards!

Proof Strategy & Tactics: try a bit more difficult proof

$$\frac{\neg P \vee \neg Q}{\neg(P \wedge Q)}$$

- Point 1: since conclusion is negation,
we can assume $P \wedge Q$ then derive \perp
- New Proof Goal: set $P \wedge Q$ as a premises and derive \perp as the conclusion
- Point 2: since premises are connected with Disjunction,
we can derive \perp by conducting a proof by cases
- Now, let's use this strategy and tactics to conduct formal proof with Fitch!

Proofs without Premises \Rightarrow Prove Logical Truth

	1. $a = a$	= Intro
	2. $b = b$	= Intro
	3. $a = a \wedge b = b$	\wedge Intro: 1,2

	1. $P \wedge \neg P$	
		2. P
		\wedge Elim: 1
		3. $\neg P$
		\wedge Elim: 1
		4. \perp
		\perp Intro: 2,3
	5. $\neg(P \wedge \neg P)$	\neg Intro: 1-4

5. Conditionals

Truth-Functional Connectives

- **Truth-Functional Connective :**

- **T** or **F** of a complex sentence (made with a connective) can be determined by the truth value of the connective's arguments
 - (can be captured by a Truth Table)

- Boolean connectives (\neg , \wedge , \vee) are truth-functional connectives
- FOL does not include connectives that are not truth-functional

Goal Let's include conditional connectives (\rightarrow , \leftrightarrow) in FOL

Conditional Symbol: \rightarrow

- Syntax

- $P \rightarrow Q$ (P: antecedent, Q: consequent)

- Semantics

- the sentence $P \rightarrow Q$ is true **iff** P is false or Q is true
 - $P \rightarrow Q \equiv \neg P \vee Q$
 - $P \rightarrow Q \approx$ if P then Q (various English expression)
 - = P only if Q
 - = Q provided P
 - = Q if P

- Truth Table :

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Necessary Condition

- consequent Q in $P \rightarrow Q$
 - A condition that must hold in order for something else to obtain
 - English: “only if” (necessary condition comes after)
 - Example:
 - You will pass the course only if you turn in all the homework
- 보충 조건
- Turning in all the homework is necessary condition for passing
 - If you do not turn in all the homework, you will not pass
 - Does not guarantee that you will pass if you do turn in all the homework
 - * It is impossible to pass and not turn in all the homework (exclude the case where $P = \text{T}$ and $Q = \text{F}$)

Sufficient Condition

- antecedent P in $P \rightarrow Q$
- A condition which guarantees that something else will obtain
- English: “if” (sufficient condition comes after)
- Example:
 - You will pass the course if you turn in all the homework
 - Turning in all the homework is sufficient condition for passing
 - If you do turn in all the homework, you will pass
 - Does guarantee that you will pass if you do turn in all the homework
 - * It is impossible to turn in all the homework and fail
(exclude the case where $P = \text{True}$ and $Q = \text{False}$)

Use of \rightarrow

- **Q** is logical consequence of **P**₁, ..., **P**_n

if and only if

the sentence $\mathbf{P}_1 \wedge \dots \wedge \mathbf{P}_n \rightarrow \mathbf{Q}$ is a logical truth

- Unless P, Q
 $= Q \text{ unless } P$
 $= Q \text{ if not } P$
 $= \neg P \rightarrow Q$

Biconditional Symbol: \leftrightarrow

- Syntax

- $P \leftrightarrow Q$: (if and only if \equiv iff)

- Semantics

- the sentence $P \leftrightarrow Q$ is true **iff** P and Q have the same truth value
 - $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

- Truth Table :

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

- \Leftrightarrow : abbreviation of “is logically equivalent to”
 - NOT a truth-functional connective & NOT a FOL expression
 - $P \Leftrightarrow Q$: indicates that the sentence P is logically equivalent to the sentence Q
- \leftrightarrow : truth-functional connective to make a FOL sentence
 - $P \leftrightarrow Q$: is a FOL sentence if P and Q are FOL sentences

Truth-Functional Completeness

- Truth-functional connectives: \neg , \wedge , \vee , \rightarrow , \leftrightarrow
 - should we introduce more connectives?
 - is it possible to encounter a sentence that is truth-functional, but cannot express using the 5 truth-functional connectives?
 - do we need them all?
 - has the expressiveness been improved with \rightarrow & \leftrightarrow ?

더 많이 표현할 수 있지만 더 많이 만들수는 없음

- Definition
 - A set of connectives is truth-functionally complete if the connectives allow us to express every truth function.
- Theorem
 - Boolean connectives (\neg , \wedge , \vee) are truth-functionally complete
 - Let's prove it!

Proof: Binary Connectives

P	Q	$P * Q$
T	T	1
T	F	2
F	T	3
F	F	4

where

$$C_1 = (P \wedge Q)$$

$$C_2 = (P \wedge \neg Q)$$

$$C_3 = (\neg P \wedge Q)$$

$$C_4 = (\neg P \wedge \neg Q)$$

1	2	3	4	sentence with \neg, \wedge, \vee only
F	F	F	F	$P \wedge \neg P \wedge Q \wedge \neg Q$
T	F	F	F	$C_1 = (P \wedge Q)$
F	T	F	F	C_2
F	F	T	F	C_3
F	F	F	T	C_4
T	T	F	F	$C_1 \vee C_2$
T	F	T	F	$C_1 \vee C_3$
T	F	F	T	$C_1 \vee C_4$
F	T	T	F	$C_2 \vee C_3$
F	T	F	T	$C_2 \vee C_4$
F	F	T	T	$C_3 \vee C_4$
T	T	T	F	$C_1 \vee C_2 \vee C_3 = (P \vee Q)$
T	T	F	T	$C_1 \vee C_2 \vee C_4$
T	F	T	T	$C_1 \vee C_3 \vee C_4$
F	T	T	T	$C_2 \vee C_3 \vee C_4$
T	T	T	T	$C_1 \vee C_2 \vee C_3 \vee C_4$

Proof: Unary Connectives

P	#P
T	1
F	2

1	2	sentence with \neg , \wedge , \vee only
F	F	$P \wedge \neg P$
T	F	P
F	T	$\neg P$
T	T	$P \vee \neg P$

Proof: Ternary Connectives

P	Q	R	$\clubsuit(P,Q,R)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

where $\clubsuit(P,Q,R) = \text{if } P \text{ then } Q \text{ else } R$

\Rightarrow sentence with \neg, \wedge, \vee only:

$$\begin{aligned} & (P \wedge Q \wedge R) \\ & \vee (P \wedge Q \wedge \neg R) \\ & \vee (\neg P \wedge Q \wedge R) \\ & \vee (\neg P \wedge \neg Q \wedge R) \end{aligned}$$

n-ary Connectives

- Suppose that \spadesuit express an n-ary connective
- The algorithm that makes a sentence that is tautologically equivalent to $\spadesuit(P_1, \dots, P_n)$
 - Define the **conjunctions** C_1, \dots, C_{2^n} that correspond to the 2^n rows of the truth table
 - Form a **disjunction** D that contains C_k as a disjunct iff the k^{th} row of the truth table has the value true
 - If all rows contain false, then we let $D \equiv P_1 \wedge \neg P_1$
 - Then this disjunction is tautologically equivalent to $\spadesuit(P_1, \dots, P_n)$

Theorems

- **Theorem:** The Boolean connectives \neg and \vee are truth-functionally complete
- **Theorem:** The Boolean connectives \neg and \wedge are truth-functionally complete
- **Theorem:** Let $P \downarrow Q$ express “neither P nor Q ” then \downarrow is truth-functionally complete
 - $\neg P = P \downarrow P = \text{neither } P \text{ nor } P$
 - $P \wedge Q = (P \downarrow P) \downarrow (Q \downarrow Q) = \text{neither not } P \text{ nor not } Q$

Disadvantages of Economizing on Connectives

- The fewer connectives we have
 - the harder it is to understand the sentences
 - the proofs become more complicated

Valid Steps of Informal Proof for Conditionals

- **Modus Ponens** (Conditional Elimination)
 - From $P \rightarrow Q$ and P , infer Q
- **Biconditional Elimination**
 - From P and either $P \leftrightarrow Q$ or $Q \leftrightarrow P$, infer Q
- **Contraposition**
 - $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
- **Logical Equivalences :**
 - $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
 - $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$
 - $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$
 - $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

Informal Proof: Method of Conditional Proof

Method

- to prove $P \rightarrow Q$, temporarily assume P and prove Q

Example

- $\text{Tet}(a) \rightarrow \text{Tet}(c)$ is logical consequence of $\text{Tet}(a) \rightarrow \text{Tet}(b)$ and $\text{Tet}(b) \rightarrow \text{Tet}(c)$. In other words, \rightarrow is transitive.

Proof :

- As premises, $\text{Tet}(a) \rightarrow \text{Tet}(b)$ and $\text{Tet}(b) \rightarrow \text{Tet}(c)$ are given. We want to prove $\text{Tet}(a) \rightarrow \text{Tet}(c)$.
- To prove using conditional proof method, let's assume that $\text{Tet}(a)$ is true. By applying Modus Ponens using the 1st premise, we can conclude $\text{Tet}(b)$. By applying Modus Ponens using the 2nd premise, we can conclude $\text{Tet}(c)$.
- Since we assumed $\text{Tet}(a)$ and proved $\text{Tet}(c)$, the rule of conditional proof assures $\text{Tet}(a) \rightarrow \text{Tet}(c)$.

Informal Proof: Method of Conditional Proof

Example

• $\text{Even}(n^2) \rightarrow \text{Even}(n)$

Proof : (conditional proof + proof by contradiction)

- We can assume $\text{Even}(n^2)$ and prove $\text{Even}(n)$.
- Assume n^2 is even and prove n is even.
- Assume n is odd. Then we can express n as $2m + 1$.

$$\begin{aligned}\text{Then } n^2 &= (2m + 1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1\end{aligned}$$

This shows that n^2 is odd. It contradicts the premise.

- Thus, n is even.
- Hence, $\text{Even}(n^2) \rightarrow \text{Even}(n)$ is true.

Informal Proof: Method of Conditional Proof

Example

• $\text{Even}(n^2) \rightarrow \text{Even}(n)$

Proof : (contraposition)

- To prove $\text{Even}(n^2) \rightarrow \text{Even}(n)$, we can prove the contrapositive of this conditional, $\neg \text{Even}(n) \rightarrow \neg \text{Even}(n^2)$.
- Assume $\neg \text{Even}(n)$ which means assume n is odd.

- Then we can express n as $2m + 1$.

$$\begin{aligned}\text{Then } n^2 &= (2m + 1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1\end{aligned}$$

This shows that n^2 is also odd. Therefore, $\neg \text{Even}(n^2)$ is true.

- Hence, $\neg \text{Even}(n) \rightarrow \neg \text{Even}(n^2)$ is true.

Informal Proof: Biconditionals

Method : (use conditional proof)

- arrange biconditionals into a cycle of conditionals

Example

- Q_1, Q_2, Q_3 are all equivalent.
Hence, $Q_1 \leftrightarrow Q_2, Q_2 \leftrightarrow Q_3, Q_1 \leftrightarrow Q_3$.

Proof :

- Arrange the biconditionals as below and prove.

$$Q_1 \rightarrow Q_2 \rightarrow Q_3 \rightarrow Q_1$$

- Prove the following conditions are all equivalent.

1. n is even
2. n^2 is even
3. n^2 is divisible by 4

Proof : (3) \rightarrow (2) \rightarrow (1) \rightarrow (3)

Formal Rules of Proof for \rightarrow

- Conditional Elimination (\rightarrow Elim)

\triangleright $\begin{array}{|l} P \rightarrow Q \\ : \\ P \\ : \\ Q \end{array}$

Modus Ponens

- Conditional Introduction (\rightarrow Intro)

\triangleright $\begin{array}{|l} \begin{array}{|l} P \\ \hline : \\ Q \end{array} \\ P \rightarrow Q \end{array}$

Conditional Proof

- Let's construct a proof in Fitch

- Conditional 1
- Prove: $A \rightarrow \neg\neg A$
- Conditional 2

Formal Rules of Proof for \leftrightarrow

- **Biconditional Elimination (\leftrightarrow Elim)**

P	↔ Q (or Q ↔ P)
⋮	
P	
⋮	
Q	

- **Biconditional Introduction (\leftrightarrow Intro)**

P	P	P
:	:	:
Q	Q	Q
Q	Q	Q
:	:	:
P	P	P

$\triangleright \quad P \leftrightarrow Q$

- Let's construct a proof in Fitch

- Prove: $A \Leftrightarrow \neg\neg A$ & Conditional 3

6. Soundness & Completeness of System F

Glossary

- F_T
 - portion of the deductive system F
 - contains $\neg, \wedge, \vee, \rightarrow, \leftrightarrow \perp$ Intro/Elim rules
- $P_1, \dots, P_n \vdash_T S$ (\vdash : provability)
 - there is a formal proof in F_T of S from premises P_1, \dots, P_n .

Soundness of F_T

- What we proven in F_T is genuinely valid?
- Are there any flaws in F_T ?

Soundness Theorem :

- If $P_1, \dots, P_n \vdash_T S$
then S is a tautological consequence of P_1, \dots, P_n

Proof :

- Suppose that p is a proof constructed in the system F_T
- Any 'sentence' at any step (main + subproofs) in proof p is a tautological consequence of the 'assumptions' (including the main premises) in force at that step ← need to prove
- Then, S appears at the main level of p and the only assumptions in force are the premises P_1, \dots, P_n .

Hence, S is a tautological consequence of P_1, \dots, P_n

Soundness of F_T

Proof continues (Proof by Contradiction) :

- Suppose there is a step in proof p that is **not** a tautological consequence of the assumptions in force at that step (called **invalid step**)
- Then, look at the 1st invalid step in p and show that none of the 12 rules of F_T could have justified that step \leftarrow **need to prove** (use 'proof by cases' and show that no matter which rule of F_T was applied at that invalid step, contradiction is derived)
- Thus, we can conclude that there can be no invalid steps in proofs in F_T

Soundness of F_T

Proof continues (Proof by Case) : \rightarrow Intro

- Suppose the 1st invalid step derives the sentence $Q \rightarrow R$ by applying \rightarrow Intro to an earlier subproof with assumption Q and conclusion R
- Let A_1, \dots, A_k be the assumptions in force at $Q \rightarrow R$
- Then, assumptions in force at R are A_1, \dots, A_k and Q
- Since step R is earlier than the first invalid step ($Q \rightarrow R$), R must be tautological consequence of A_1, \dots, A_k and Q
- By the assumption that $Q \rightarrow R$ is an invalid step, the following row exists in the joint Truth Table (**Contradiction!**)

\vdots
 $\frac{Q}{\vdots}$
 R

\vdots
 $Q \rightarrow R$
 \vdots

A_1	\dots	A_k	Q	R	$Q \rightarrow R$
T	T	T	T	T	F

R being T contradicts $Q \rightarrow R$ being F

Soundness of F_T

Proof continues (Proof by Case) : \rightarrow Elim

- Suppose the 1st invalid step derives the sentence R by applying \rightarrow Elim to sentences $Q \rightarrow R$ and Q appearing earlier in the proof
- Let A_1, \dots, A_k be all the assumptions in force at R
- If R is an invalid step,
R is not a tautological consequence of A_1, \dots, A_k
- Since steps $Q \rightarrow R$ and Q are earlier than the first invalid step (R),
 $Q \rightarrow R$ and Q are valid steps ($Q \rightarrow R$ and Q must be tautological consequence of the assumptions in force at that step)

Soundness of F_T

Proof continues (Proof by Case) : \rightarrow Elim

- [observation]: the assumptions in force at steps $Q \rightarrow R$ and Q are also in force at R
- By the assumption that R is an invalid step, the following row exists in the joint Truth Table (**Contradiction!**)

A_1
:
$Q \rightarrow R$
:
A_2
:
Q
:
A_3
:
R
:

A_1	...	A_k	$Q \rightarrow R$	Q	R
T	T	T	T	T	F

$Q \rightarrow R$ and Q being T
contradicts R being F

Soundness of F_T

Proof continues (Proof by Case) : \perp Elim

- Suppose the 1st invalid step derives the sentence Q by applying \perp Elim to \perp
- [observation]: the assumptions in force at \perp are also in force at Q
- Let A_1, \dots, A_k be the assumptions in force at \perp (and at Q)
- Since step \perp is earlier than the first invalid step Q, \perp must be tautological consequence of A_1, \dots, A_k
- By the assumption that Q is an invalid step, the following row exists in the joint Truth Table (**Contradiction!**)

:	A_1	...	A_k	\perp	Q
\perp	TT-contradictory			T	F
Q					
:					

A_1, \dots, A_k cannot be all Ts !

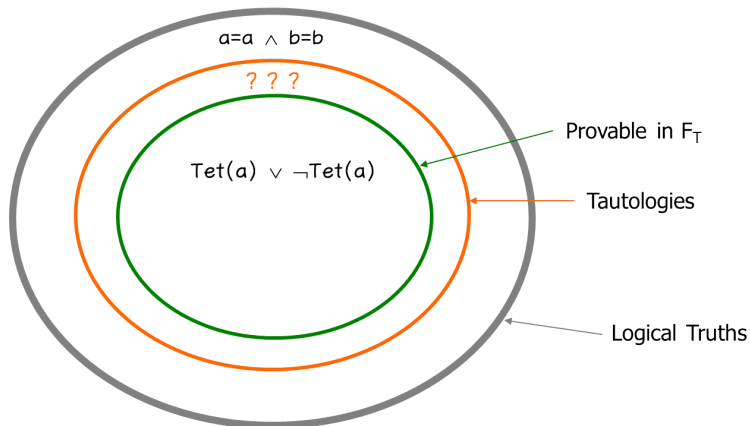
This also contradicts Q being F

Soundness of F_T

Proof continues (Proof by Case) : 9 remaining rules

- Derive contradiction in all 12 cases as we have demonstrated before
- Conclude that
“it is possible for a proof of F_T to contain an invalid step”
must be false
- conclude the **Proof of Soundness**

Soundness of F_T



Soundness of F_T

- **Corollary:** If $\vdash_T S$, then S is a tautology

Completeness of F_T

- Can F_T allow us to prove everything we should be able to prove?
- Given any premises P_1, \dots, P_n and any tautological consequence S of these premises, can F_T allow us to construct a proof of S from P_1, \dots, P_n ?
- Could there be tautological consequence of some set of premises that cannot be proved using F_T ?

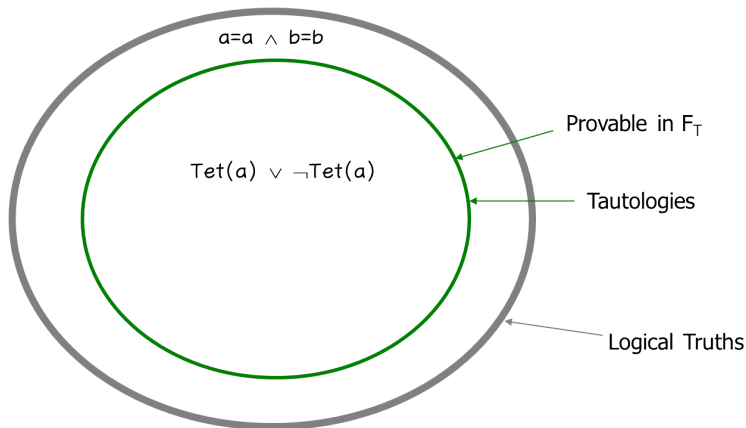
Completeness Theorem

- If a sentence S is a tautological consequence of P_1, \dots, P_n , then $P_1, \dots, P_n \vdash_T S$

Proof

- Refer to Chapter 17 of LPL textbook

Completeness and Soundness of F_T



Completeness and Soundness of F_T

- unprovable in F_T : $\text{Dodec}(b) \wedge b = c \vdash_T \text{Dodec}(c)$
 $\text{Larger}(b, c) \vdash_T \neg \text{Larger}(c, b)$

Practical Uses of Completeness Theorem

- Provides method for showing that an argument has a proof without actually having to find such proof
 - Just show that the conclusion is a tautological consequence of the premises

e.g. Since $A \rightarrow (B \rightarrow A)$ is a tautology, there must exist a proof

e.g. Since the sentence $B \wedge \neg D$ is a tautological consequence of $\neg((A \wedge B) \rightarrow (C \vee D))$, there must exist a proof

- How can we determine tautological consequence?
 - Draw a Truth Table in Boole or use Taut Con in Fitch

Practical Uses of Soundness Theorem

- Provides method for telling that an argument does not have a proof in F_T
 - Just show that the conclusion is not a tautological consequence of the premises

e.g. Since $A \rightarrow (A \rightarrow B)$ is not a tautology, it is impossible to construct a proof in F_T

e.g. Since the sentence $B \wedge \neg D$ is not a tautological consequence of $\neg((A \vee B) \rightarrow (C \wedge D))$, it is impossible to construct a proof in F_T

e.g. Since the sentence $\neg \text{Happy}(\text{carl})$ is not a tautological consequence of $\neg(\text{Happy}(\text{carl}) \wedge \text{Happy}(\text{scruffy}))$, it is impossible to construct a proof in F_T

Therefore...

- In Fitch,
- use Taut Con to determine :
 - ① if a sentence is a tautological consequence of the cited sentence try to find a full proof of the sentence in F_T
 - ② if a sentence is not a tautological consequence of the cited sentence there is no point in finding a proof in F_T