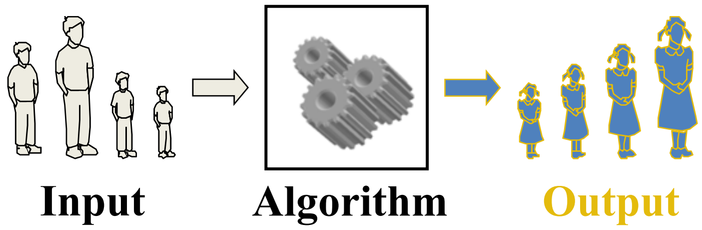
Lecture 2: Analysis of Algorithms



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- Experimental Analysis
- Theoretical Analysis
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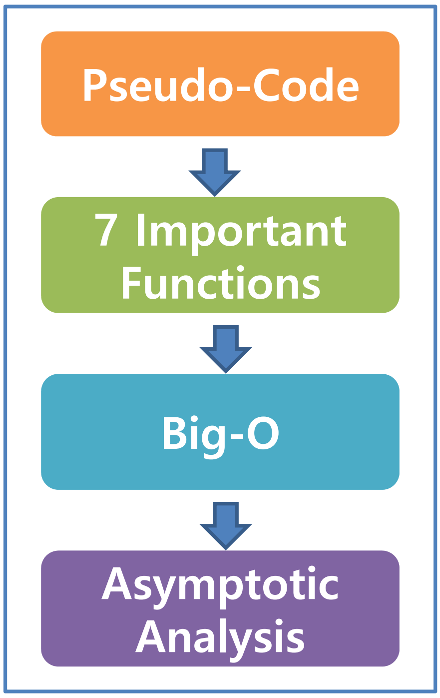
Lecture Objectives (cont'd)

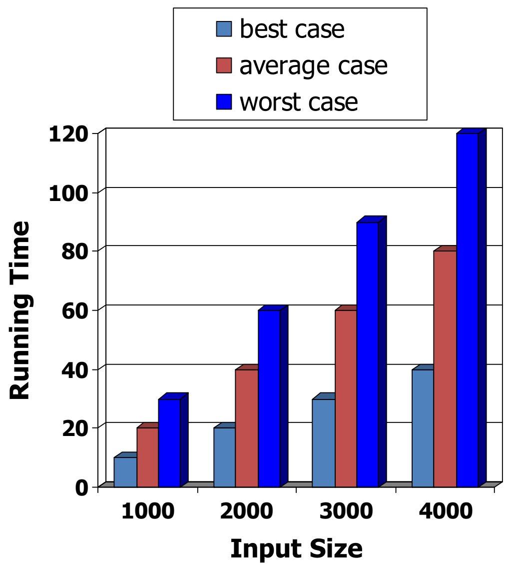
- Key Words

Algorithm Analysis



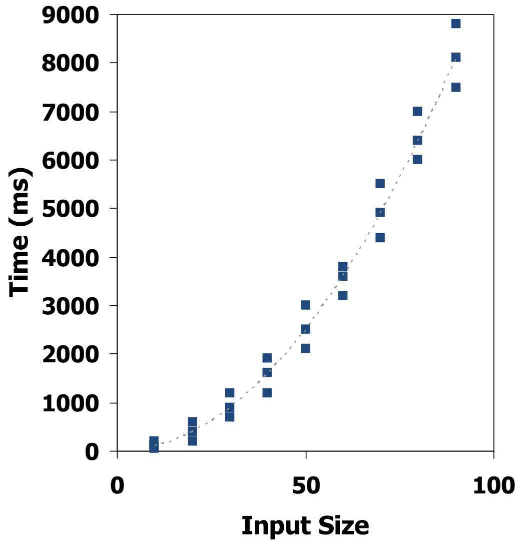
Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on **the worst case** running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time
- Plot the results



Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

```
Algorithm arrayMax(A, n)
  Input array A of n integers
  Output maximum element of A

  currentMax ← A[0]
  for i ← 1 to n − 1 do
    if A[i] > currentMax then
      currentMax ← A[i]
  return currentMax
```

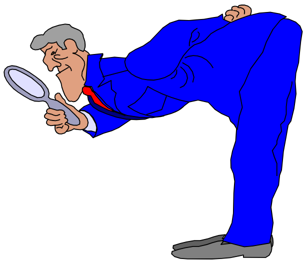
Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm ***method*** (*arg* [, *arg*…])

  Input …

  Output …



- Method call

  var.method (*arg* [, *arg*…])
- Return value

  return *expression*
- Expressions

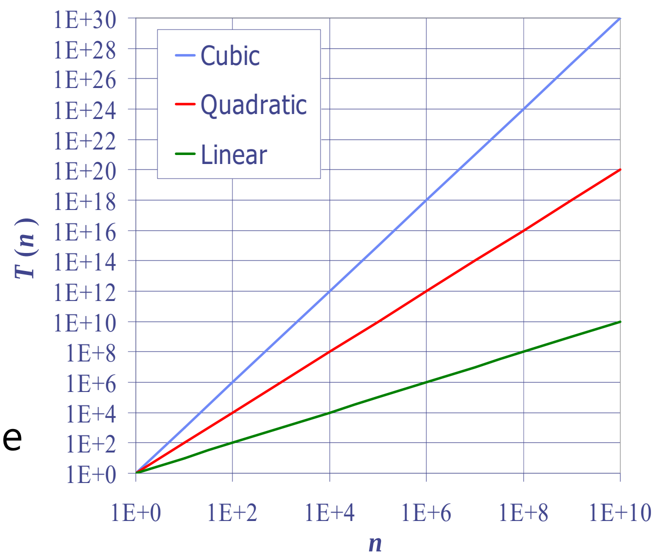
  ← Assignment (like = in Java)

  = Equality testing (like == in Java)

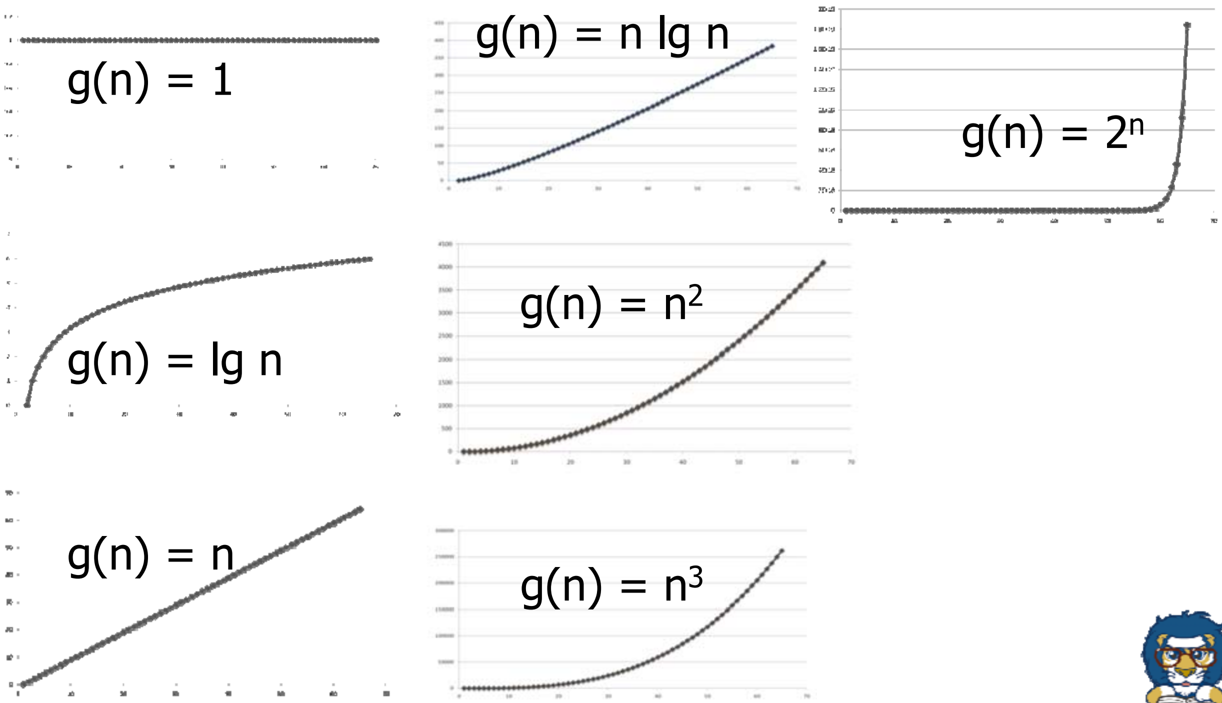
  n^2 Superscripts and other mathematical formatting allowed

Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ $\mathbf{1}$
  - Logarithmic ≈ log $\boldsymbol{n}$
  - Linear ≈ $\boldsymbol{n}$
  - N-Log-N ≈ $\boldsymbol{n}$ log $\boldsymbol{n}$
  - Quadratic ≈ $\boldsymbol{n}^2$
  - Cubic ≈ $\boldsymbol{n}^3$
  - Exponential ≈ $\mathbf{2}^n$
- In a log-log chart, the slope of the line corresponds to the growth rate



Functions Graphed Using “Normal” Scale



Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)

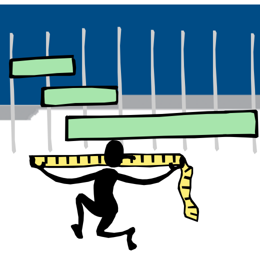
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm *arrayMax*(A, n)	# operations
currentMax $\leftarrow A[0]$	2
for $i \leftarrow 1$ **to** $n - 1$ **do**	$2\boldsymbol{n}$
if $A[i] >$ *currentMax* **then**	$2(\boldsymbol{n} - 1)$
currentMax $\leftarrow A[i]$	$2(\boldsymbol{n} - 1)$
{ increment counter $\boldsymbol{i}$ }	$2(\boldsymbol{n} - 1)$
return *currentMax*	2

Estimating Running Time



- Algorithm ***arrayMax*** executes $8\boldsymbol{n} - 2$ primitive operations in the worst case. Define:
  - $\boldsymbol{a}$ = Time taken by the fastest primitive operation
  - $\boldsymbol{b}$ = Time taken by the slowest primitive operation
- Let $\boldsymbol{T}(\boldsymbol{n})$ be worst-case time of ***arrayMax.*** Then
$$\boldsymbol{a}\,(8\boldsymbol{n} - 2) \le \boldsymbol{T}(\boldsymbol{n}) \le \boldsymbol{b}(8\boldsymbol{n} - 2)$$
- Hence, the running time $\boldsymbol{T}(\boldsymbol{n})$ is bounded by two linear functions

Growth Rate of Running Time

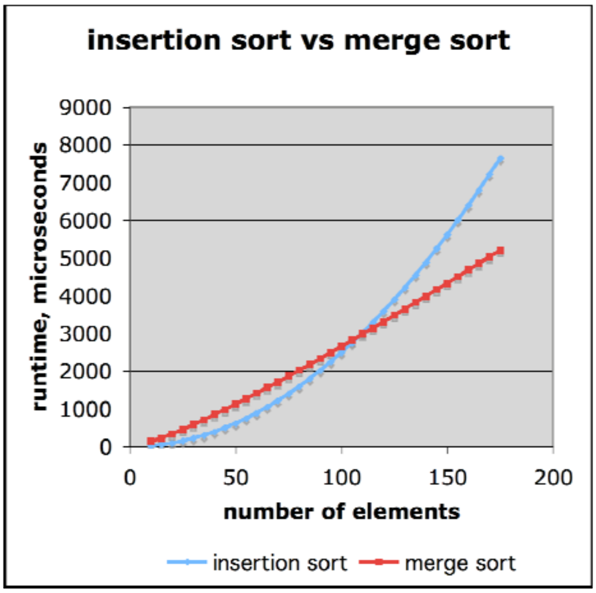
- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***

Why Growth Rate Matters

if runtime is...	time for n + 1	time for 2 n	time for 4 n
c lg n	c lg (n + 1)	c (lg n + 1)	c(lg n + 2)
c n	c (n + 1)	2c n	4c n
c n lg n	~ c n lg n + c n	2c n lg n + 2cn	4c n lg n + 4cn
c n^2	~ c n^2 + 2c n	**4c n^2**	16c n^2
c n^3	~ c n^3 + 3c n^2	8c n^3	64c n^3
c 2^n	c 2^{n+1}	c 2^{2n}	c 2^{4n}

runtime quadruples when problem size doubles

Comparison of Two Algorithms



insertion sort is

$n^2 / 4$

merge sort is

2 n lg n

sort a million items?

insertion sort takes roughly 70 hours

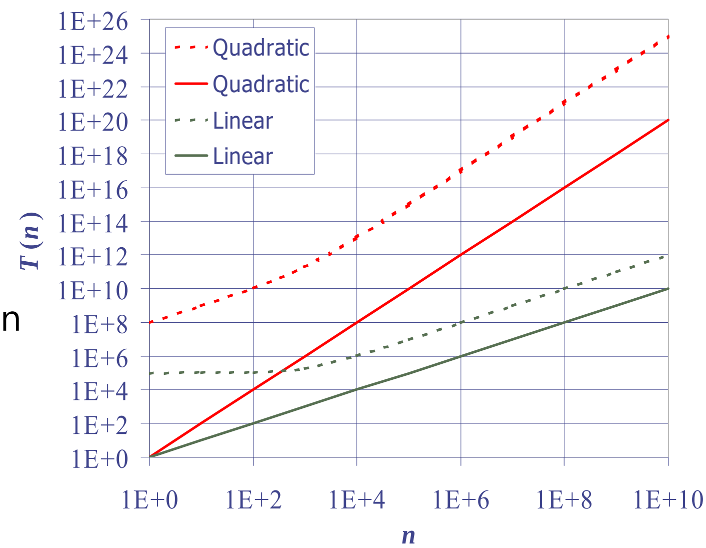
while

merge sort takes roughly 40 seconds

This is a slow machine, but if 100 x as fast then it's 40 minutes versus less than 0.5 seconds

Constant Factors

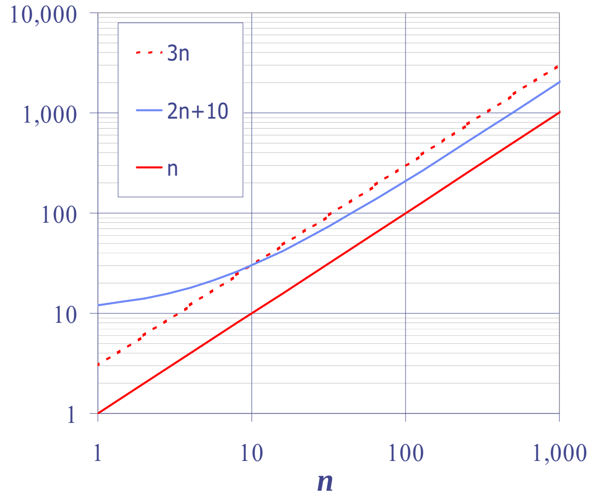
- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



Big-Oh Notation

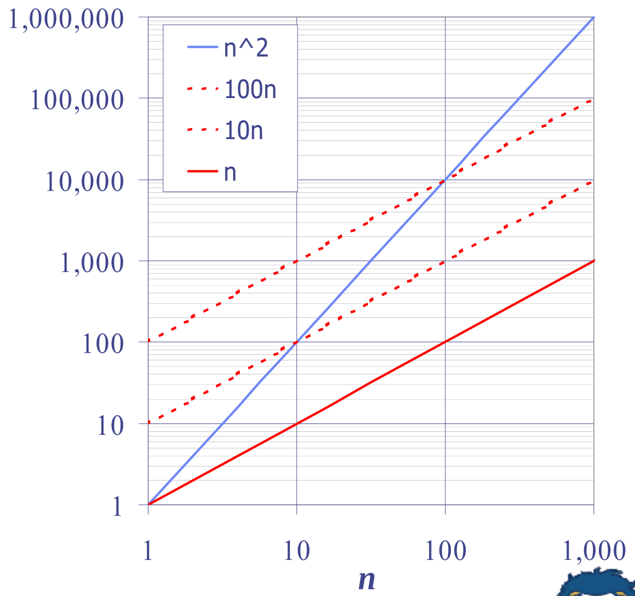
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

  $f(n) \le cg(n)$ for $n \ge n_0$
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \le cn$
  - $(c - 2)\, n \ge 10$
  - $n \ge 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \le cn$
  - $n \le c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples

- 7n-2

  7n-2 is O(n)

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \bullet n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \bullet n^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

- 3 log n + 5

  3 log n + 5 is O(log n)

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \bullet \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Rules

- If $f(n)$ is a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

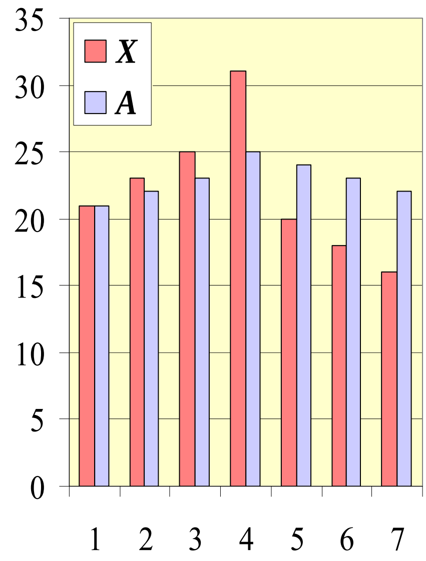
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $8n - 2$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The i-th prefix average of an array X is average of the first $(i+1)$ elements of X:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array A of prefix averages of another array X has applications to financial analysis



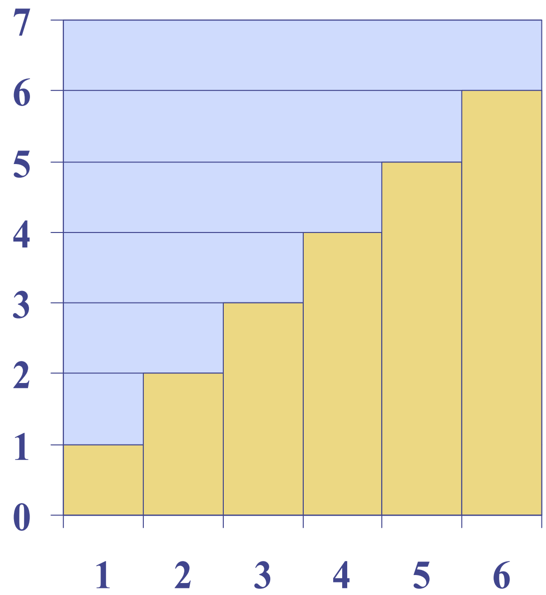
Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm *prefixAverages1*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n-1$ **do**	n
$s \leftarrow X[0]$	n
for $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n-1)$
$s \leftarrow s + X[j]$	$1 + 2 + \ldots + (n-1)$
$A[i] \leftarrow s / (i+1)$	n
return A	1

Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

Algorithm *prefixAverages2*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow s + X[i]$	n
$A[i] \leftarrow s / (i + 1)$	n
return A	1

- Algorithm ***prefixAverages2*** runs in $O(n)$ time

Math you need to Review

- Summations
- Linear Function
- Logarithms and Exponents
- Quadratic, Cubic, and other Polynomials

- Proof techniques
- Basic probability

- **properties of logarithms:**
  $\log_b(xy) = \log_b x + \log_b y$
  $\log_b (x/y) = \log_b x - \log_b y$
  $\log_b xa = a\log_b x$
  $\log_b a = \log_x a/\log_x b$
- **properties of exponentials:**
  $a^{(b+c)} = a^b a^c$
  $a^{bc} = (a^b)^c$
  $a^b / a^c = a^{(b-c)}$
  $b = a^{\log_a b}$
  $b^c = a^{c*\log_a b}$

Q & A

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