بسم الله الرحمن الرحیم

ساختمان‌های داده

جلسه ۲۵

مجتبی خلیلی
دانشکده برق و کامپیوتر
دانشگاه صنعتی اصفهان

Heaps

Recall Priority Queue ADT

- A priority queue stores a collection of entries
- Typically, an entry is a pair (key, value), where the key indicates the priority
- Main methods of the Priority Queue ADT
  - insert(e) inserts an entry e
  - removeMin()
    removes the entry with smallest key
- Additional methods
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), empty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Recall PQ Sorting

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- We use a priority queue
  - Insert the elements with a series of insert operations
  - Remove the elements in sorted order with a series of removeMin operations
- The running time depends on the priority queue implementation:
  - Unsorted sequence gives selection-sort: $O(n^2)$ time
  - Sorted sequence gives insertion-sort: $O(n^2)$ time
- Can we do better? Balancing the above

Algorithm PriorityQueueSort(L, P):

Input: An STL list L of n elements and a priority queue, P, that compares elements using a total order relation

Output: The sorted list L

while !L.empty() **do**
    $e \leftarrow L$.front
    L.pop_front() {remove an element e from the list}
    P.insert(e) {... and it to the priority queue}
while !P.empty() **do**
    $e \leftarrow P$.min()
    P.removeMin() {remove the smallest element e from the queue}
    L.push_back(e) {... and append it to the back of L}

We will have these results soon …

List-based

Operation	*Unsorted List*	*Sorted List*
size, empty	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$
min, removeMin	$O(n)$	$O(1)$

Heap-based

Operation	*Time*
size, empty	$O(1)$
min	$O(1)$
insert	$O(\log n)$
removeMin	$O(\log n)$

Heap: Overview

- A heap is a binary tree storing keys at its nodes and satisfying the following properties:

1. Heap-order property

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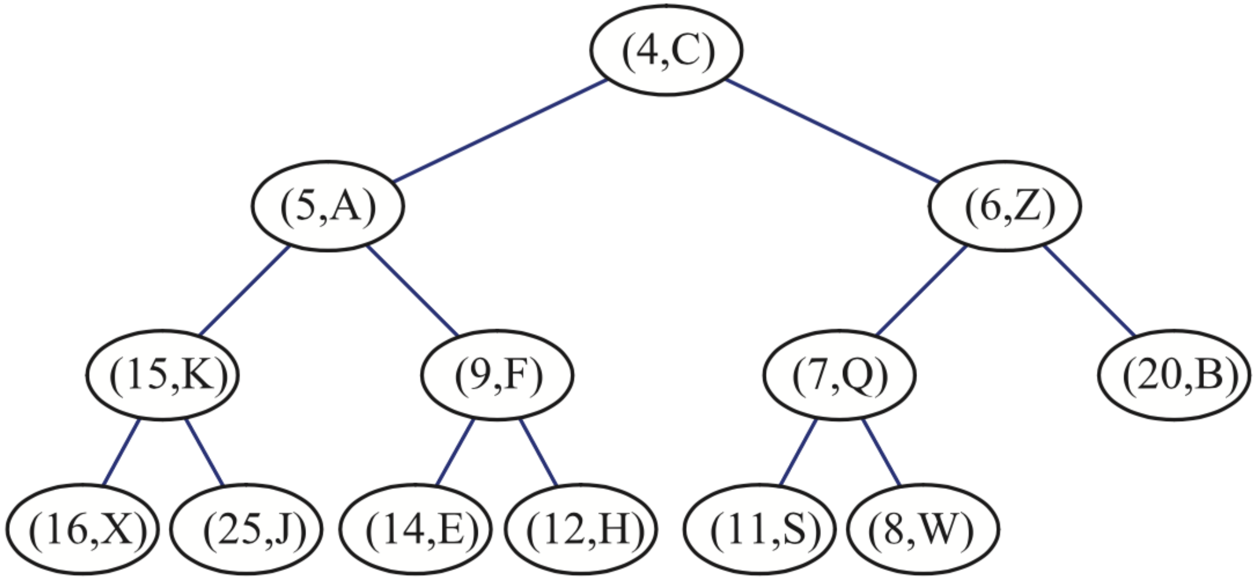
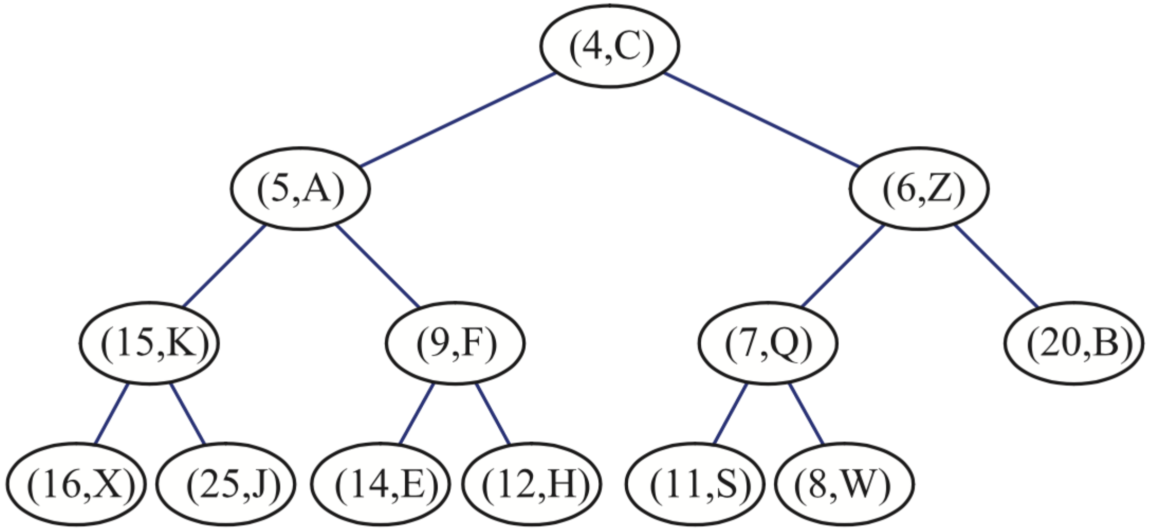
- 1. Heap-Order: for every internal node v other than the root,
  key(***v***) $\geq$ ***key***(***parent***(***v***))
  - The keys encountered on a path from the root to a leaf T are nondecreasing
  - A minimum key: always at the root



2. Complete binary tree property

- Complete Binary Tree
  - Roughly speaking, every level, except for the last level, is completely filled, and all nodes in the last level are as far left as possible.
- let $\boldsymbol{h}$ be the height of the heap
  - for $\boldsymbol{i} = 0, \ldots, \boldsymbol{h} - 1$, there are 2^i nodes of depth $\boldsymbol{i}$
  - at depth $\boldsymbol{h} - 1$, the internal nodes are to the left of the external nodes



Heap: Overview

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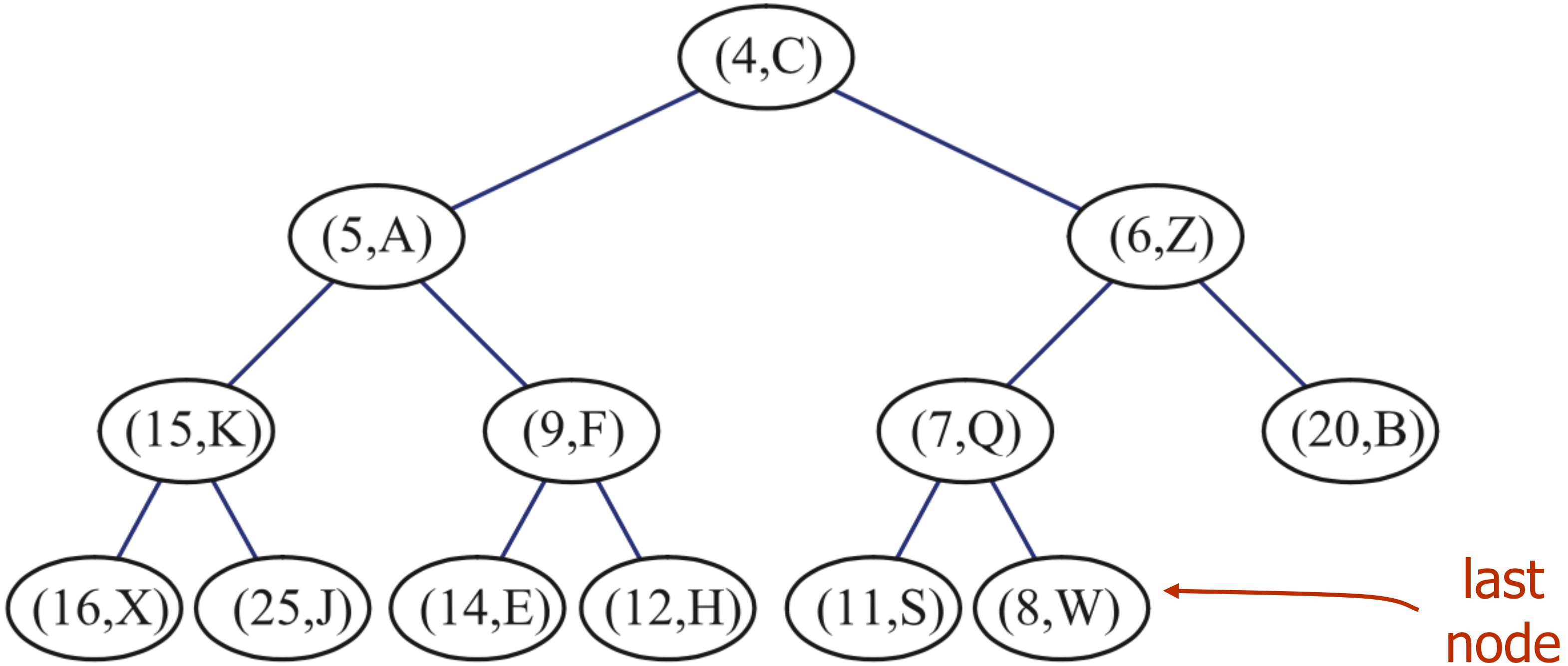
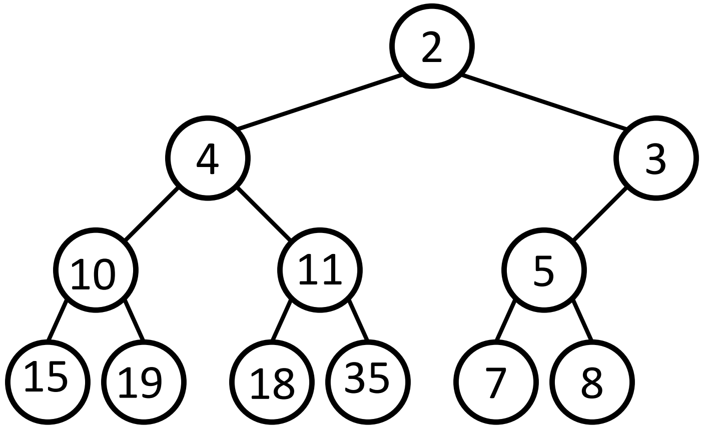
- A heap is a binary tree storing keys at its nodes and satisfying the following properties:
  - 1. Heap-order property
  - 2. Complete binary tree property
- The last node of a heap is the rightmost node of maximum depth



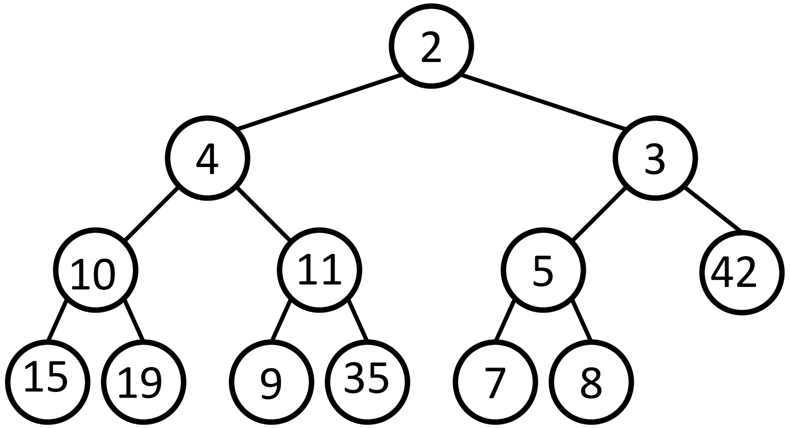
Example

- Min-Heap?



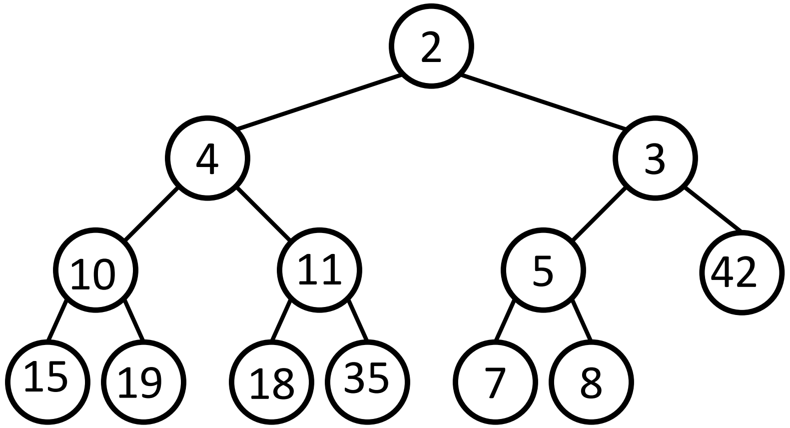
Example

- Min-Heap?



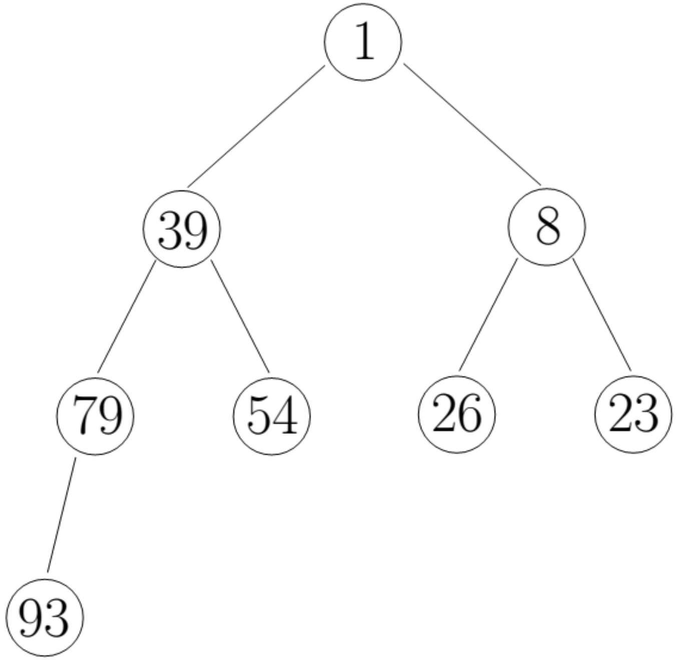
Example

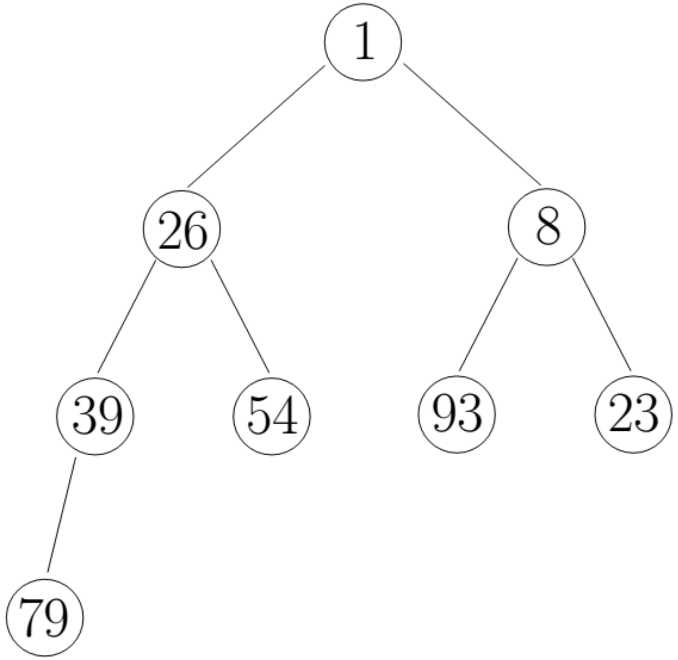
- Min-Heap?



Example

$$S = \{93, 39, 1, 26, 8, 23, \ 79, 54\}$$





Height of a Heap of *n* elements

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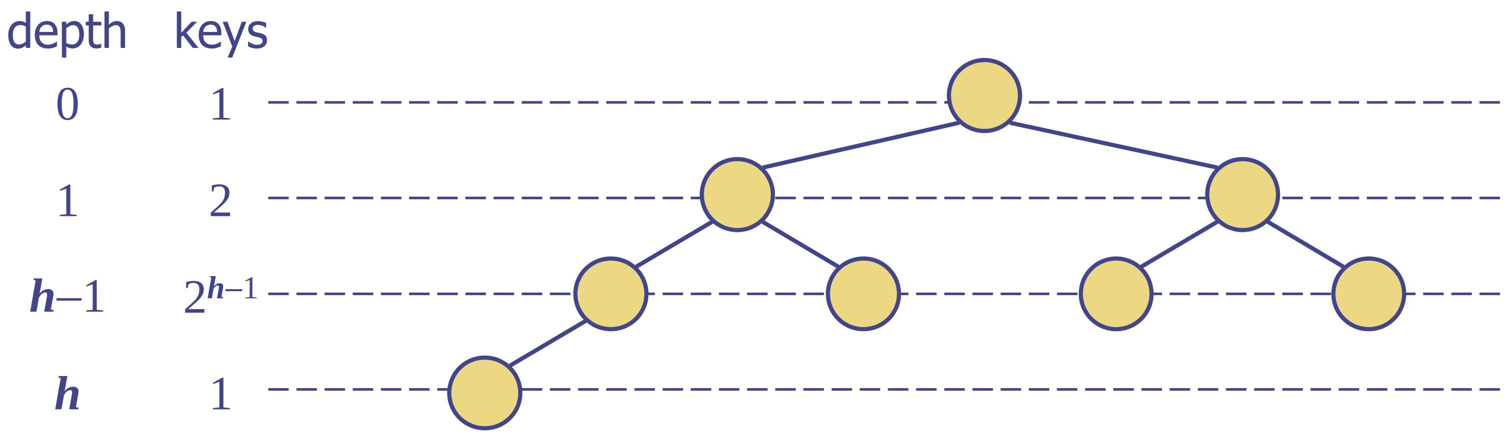
- Theorem: A heap storing $\boldsymbol{n}$ keys has height $\boldsymbol{O}(\log \boldsymbol{n})$

  Proof: (we apply the complete binary tree property)
  - Let $\boldsymbol{h}$ be the height of a heap storing $\boldsymbol{n}$ keys
  - Since there are 2^{i} keys at depth $\boldsymbol{i} = 0, \ldots, \boldsymbol{h} - 1$ and at least one key at depth $\boldsymbol{h}$, we have $\boldsymbol{n} \geq 1 + 2 + 4 + \ldots + 2^{\boldsymbol{h}-1} + 1$
  - Thus, $\boldsymbol{n} \geq 2^{\boldsymbol{h}}$, i.e., $\boldsymbol{h} \leq \log \boldsymbol{n}$

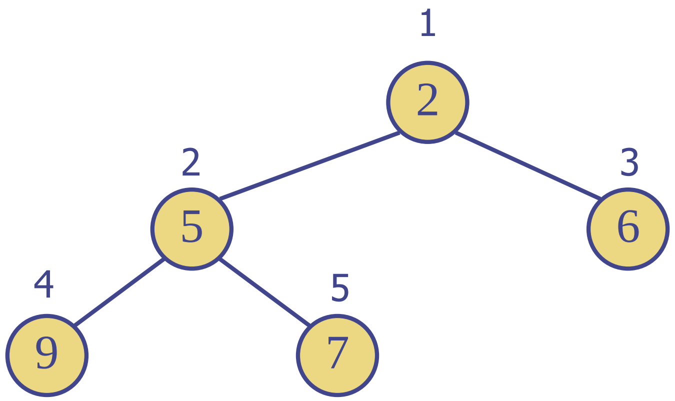


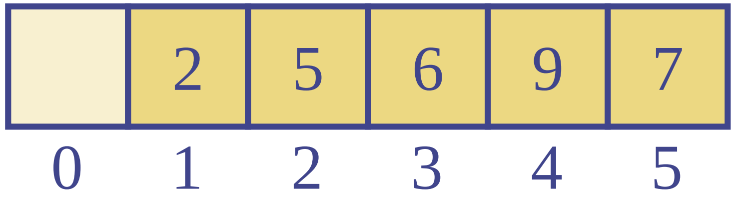
Heap

- دیدیم که heap به صورت درخت باینری تقریبا کامل است:
- پیاده سازی ساده با آرایه
- حفظ اوردر لگاریتمی

Vector-based Heap Implementation

- We can represent a heap with $\boldsymbol{n}$ keys by means of a vector of length $\boldsymbol{n} + 1$
- For the node at rank $\boldsymbol{i}$
  - the left child is at rank $2\boldsymbol{i}$
  - the right child is at rank $2\boldsymbol{i} + 1$
- Links between nodes are not explicitly stored
- The cell of at rank 0 is not used
- Last node at rank n





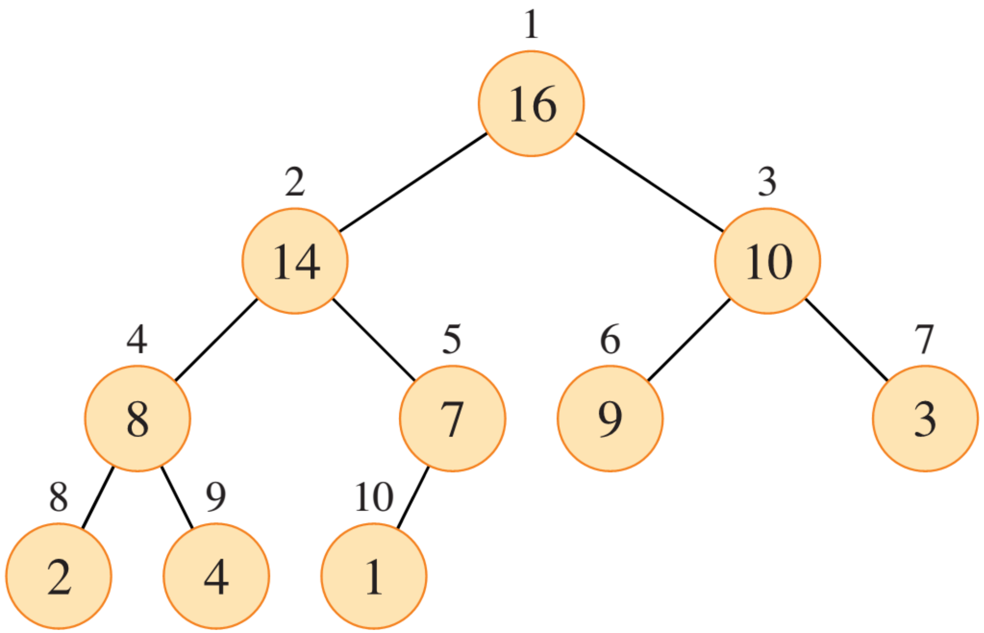
Example

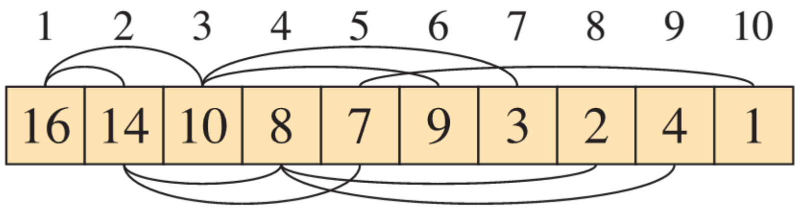
- Min heap?

[1,8,5,9,12,11,7]

Example

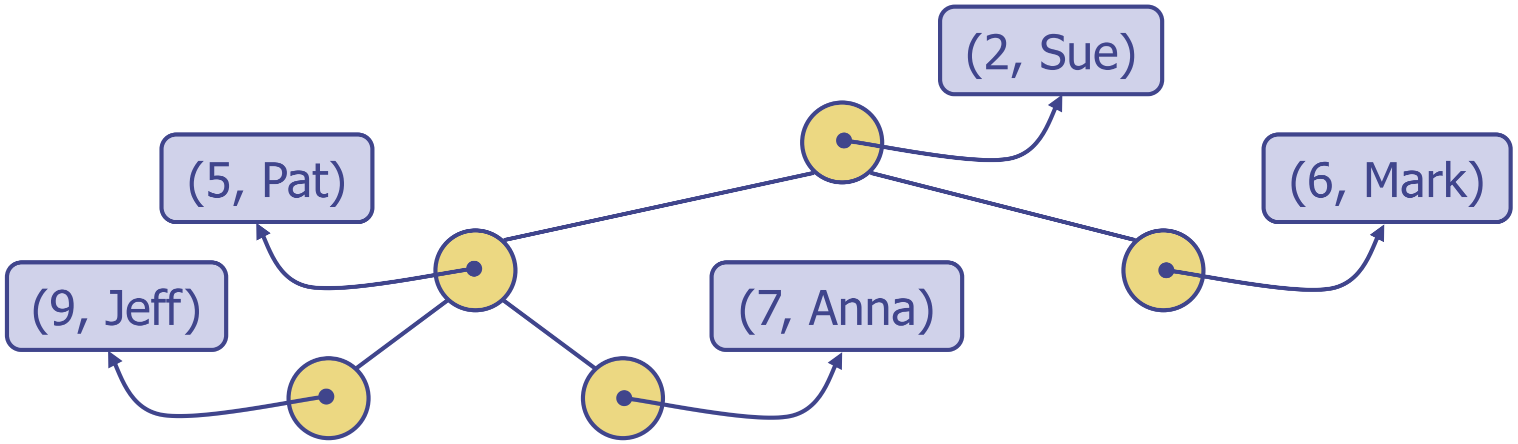
- Max heap?





Heaps and Priority Queues

- We can use a heap to implement a priority queue
  - We say “heap-based PQ implementation”
- We store a (key, element) item at each internal node
- We keep track of the position of the last node
  - I am able to know who is the last node in O(1) time
  - Easy



Insertion into a Heap

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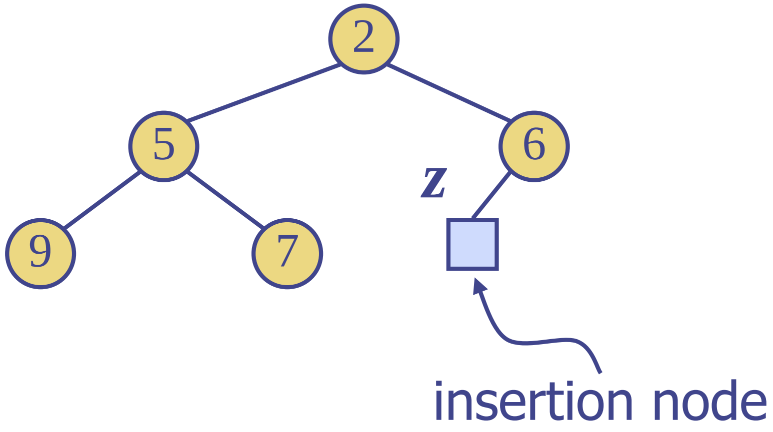
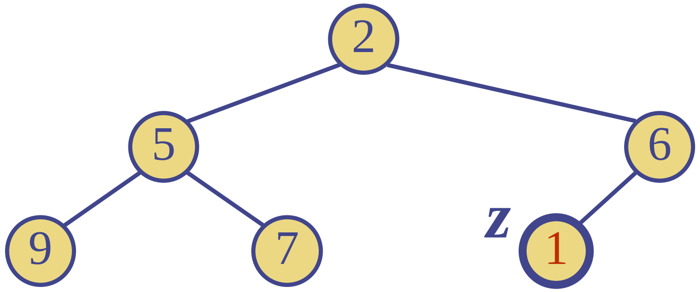
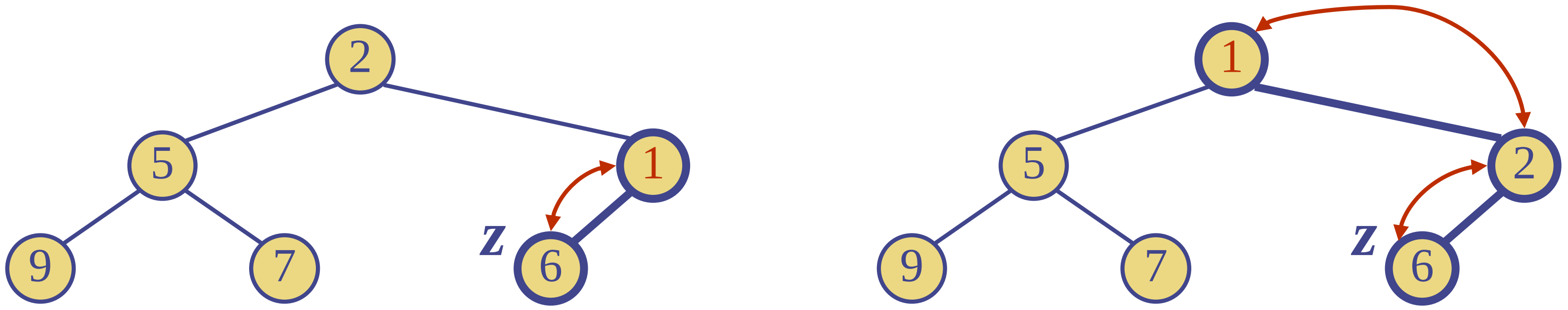
- Method **insert** of the priority queue ADT corresponds to the insertion of a key k to the heap
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
    - How? discussed later
  - Store k at z
  - Restore the heap-order property (discussed next)





Upheap

- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time



Insert: (2,T)

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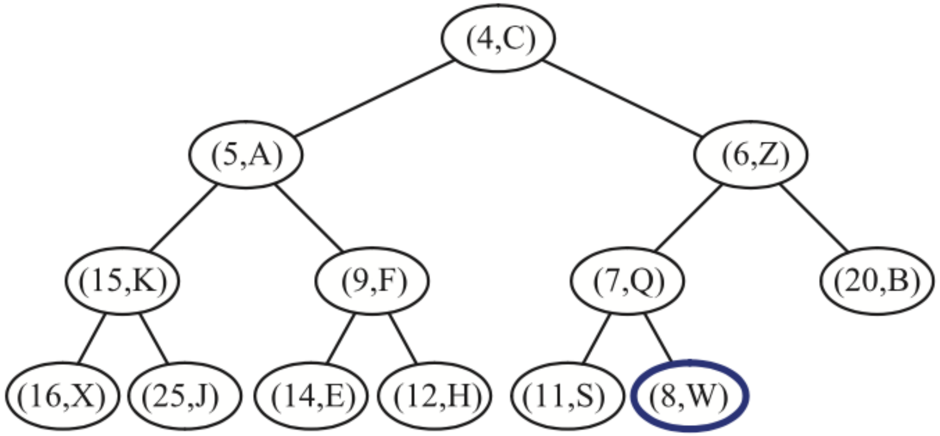
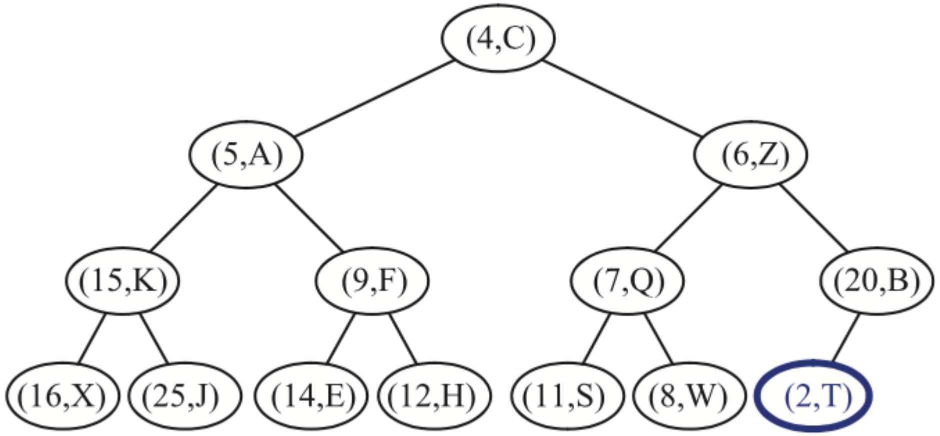
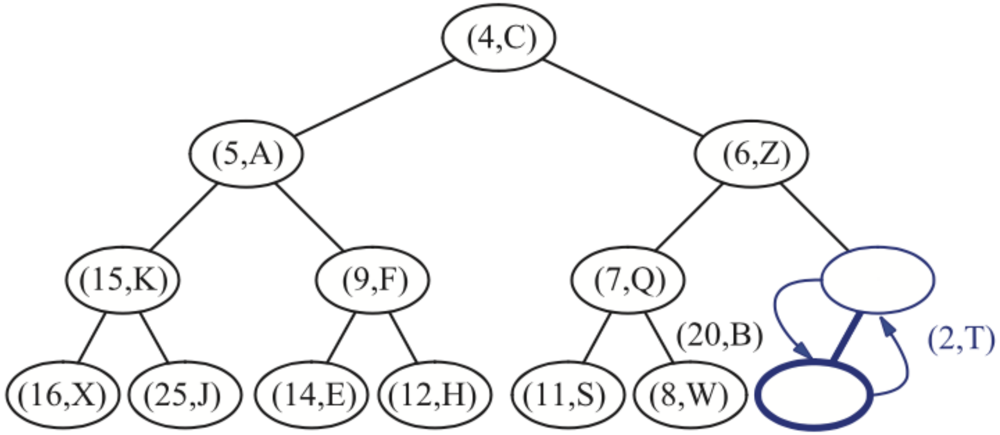


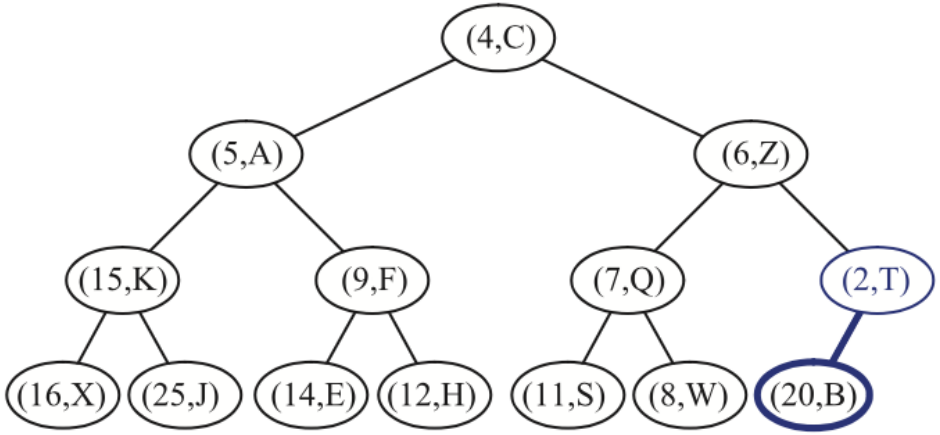
(a)



(b)



(c)



(d)

Insert: (2,T)

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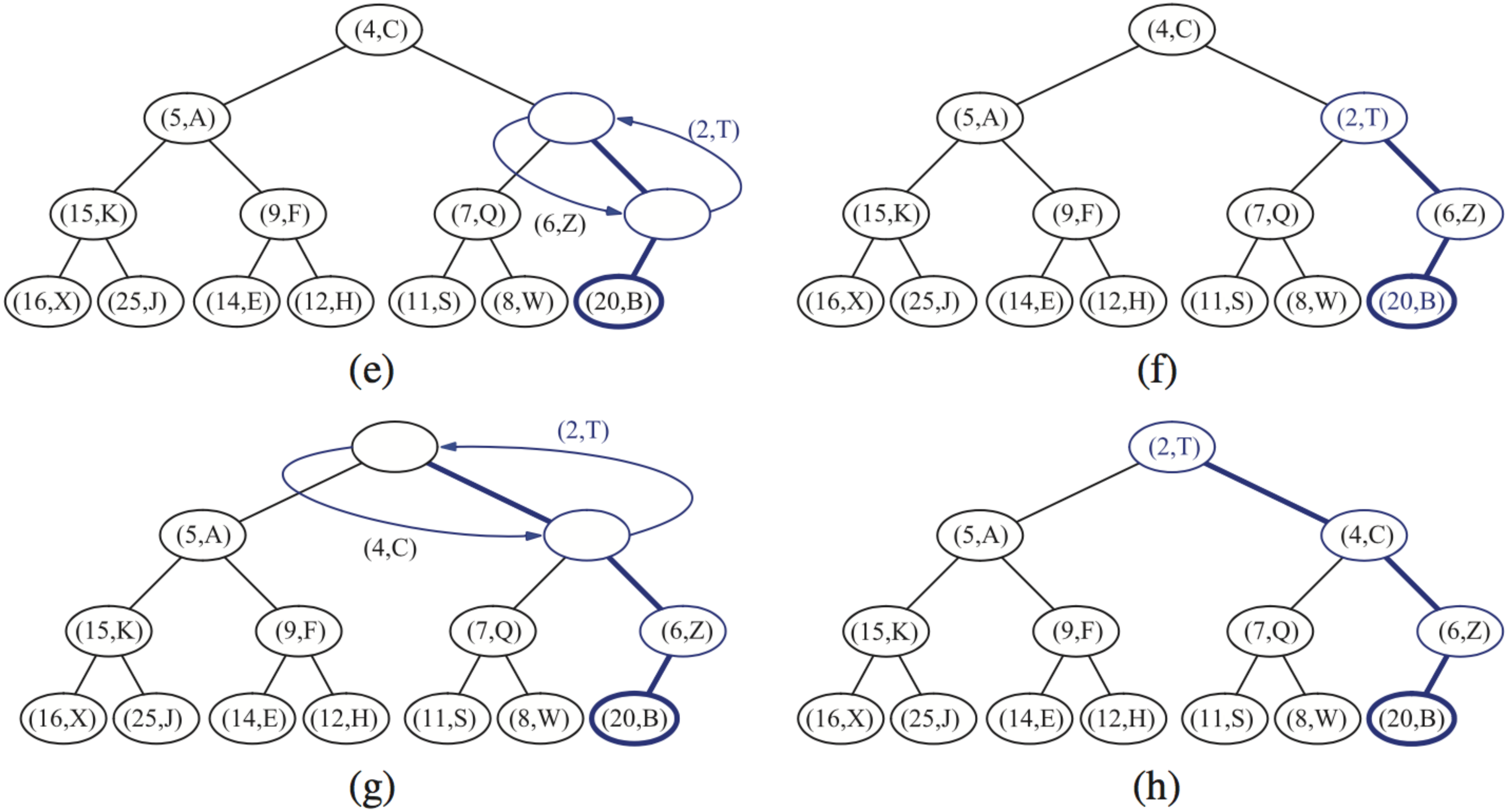
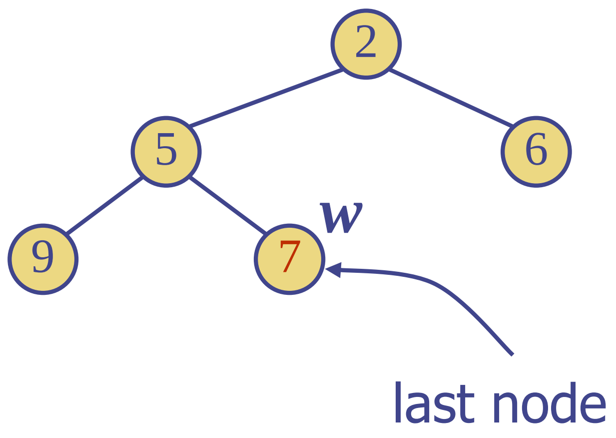
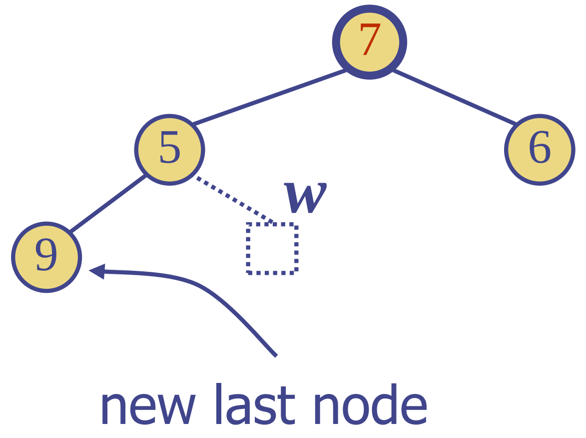


Removal from a Heap

- Method **removeMin** of the priority queue ADT corresponds to the removal of the root key from the heap

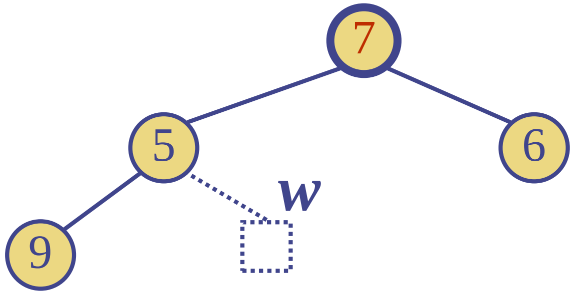
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node w
  - Remove w
  - Restore the heap-order property (discussed next)

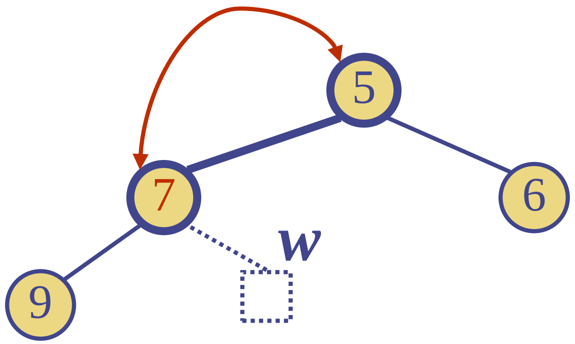




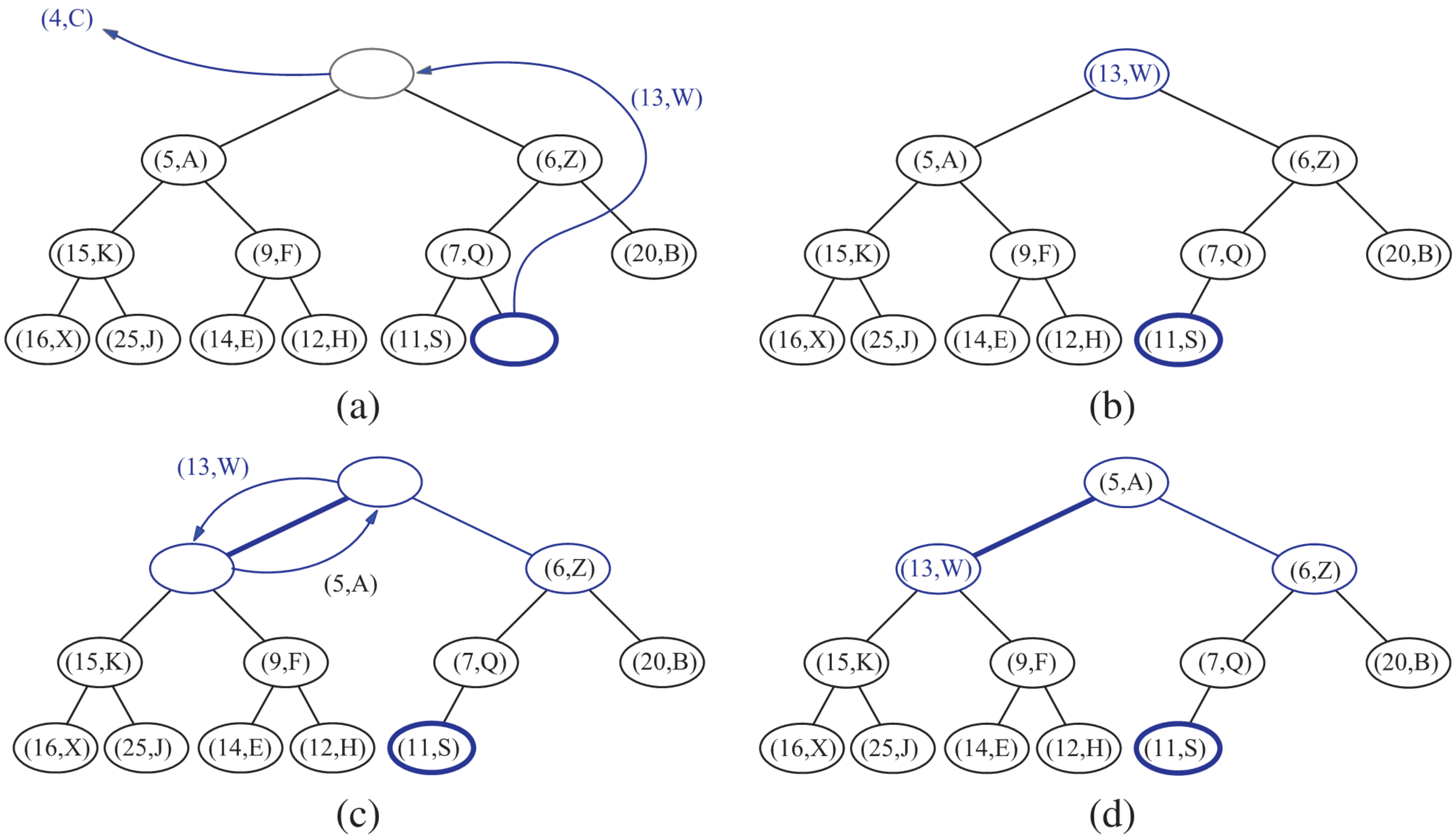
Downheap

- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root (but which path?)
- Upheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time





removeMin


(4,C)
(13,W)
(5,A)
(6,Z)
(15,K)
(9,F)
(7,Q)
(20,B)
(16,X)
(25,J)
(14,E)
(12,H)
(11,S)
(a)
(13,W)
(5,A)
(6,Z)
(15,K)
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(7,Q)
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(11,S)
(c)
(5,A)
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(15,K)
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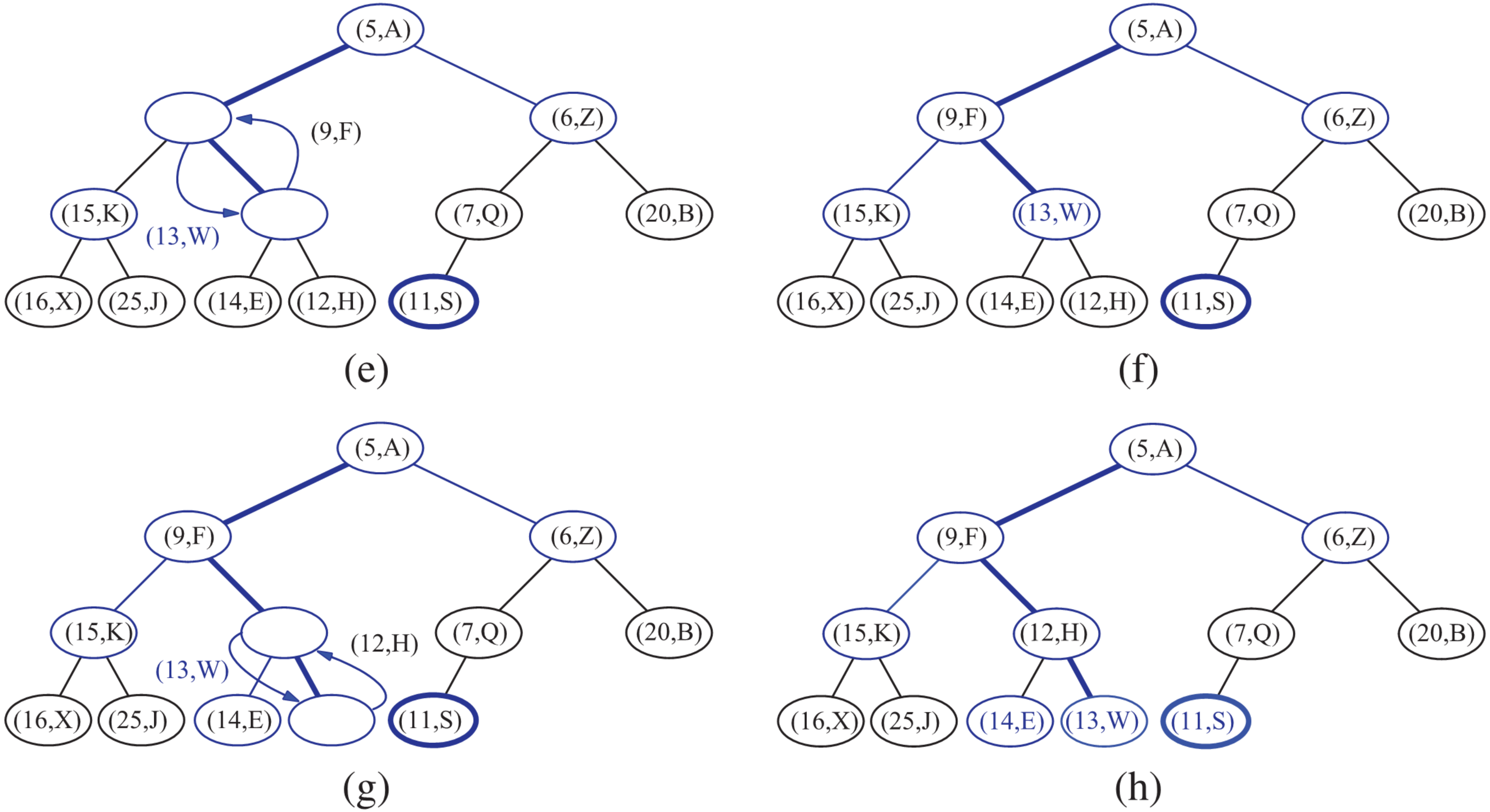
(14,E)
(12,H)
(11,S)
(d)

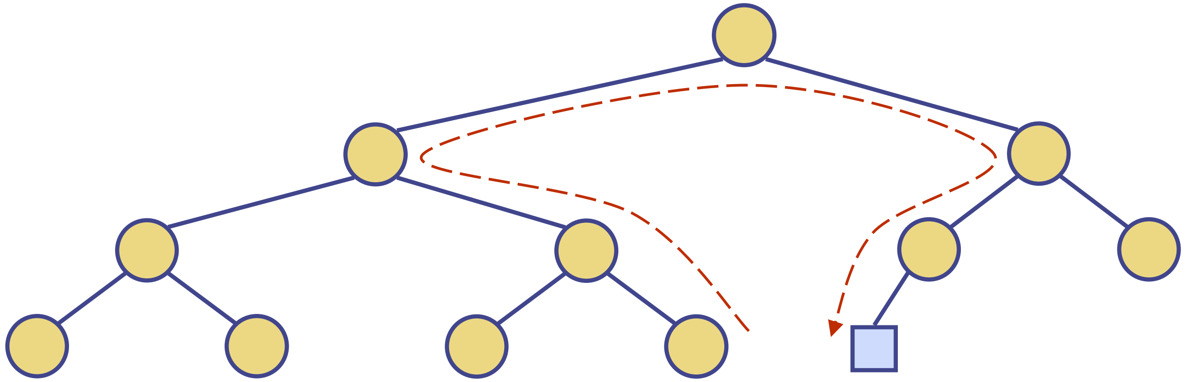
removeMin



Updating the Last Node

- How can we find the insertion node (a new last node)?
  - The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - (1) Go up until a left child or the root is reached
  - (2) If a left child is reached, go to the right child
  - (3) Go down left until a leaf is reached

- Similar algorithm for updating the last node after a removal



Vector-based Heap Implementation

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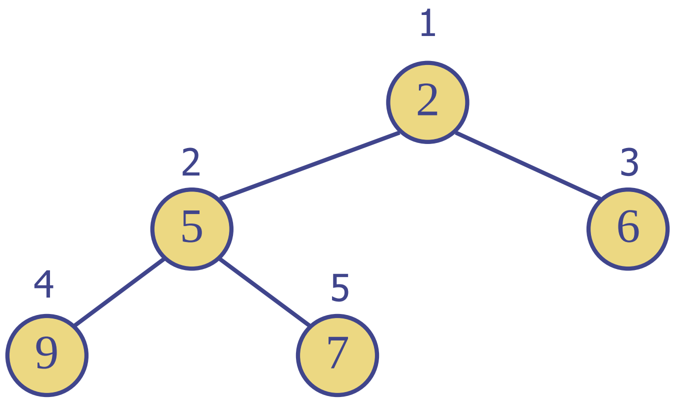
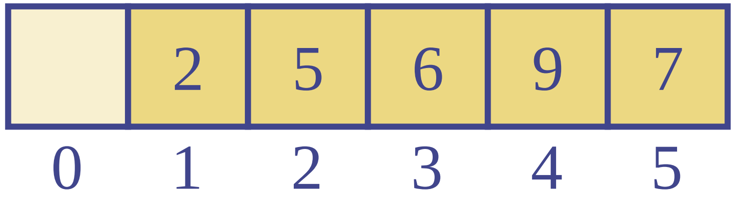
- We can represent a heap with n keys by means of a vector of length $n + 1$
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i + 1$
- Links between nodes are not explicitly stored
- The cell of at rank 0 is not used
- Operation insert corresponds to inserting at rank $n + 1$
- Operation removeMin corresponds to removing at rank n





List-based vs. Heap-based

List-based

Operation	*Unsorted List*	*Sorted List*
size, empty	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$
min, removeMin	$O(n)$	$O(1)$

Heap-based

Operation	*Time*
size, empty	$O(1)$
min	$O(1)$
insert	$O(\log n)$
removeMin	$O(\log n)$