بسم الله الرّحمن الرّحیم

دانشگاه صنعتی اصفهان
دانشکده برق و کامپیوتر

تجزیه و تحلیل سیگنال‌ها و سیستم‌ها

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جلسه بیست و دوم ـ بخش‌های 3.2 ، 3.6 و 3.7 کتاب

با سلام خدمت دانشجویان محترم

توابع ویژه (Eigenfunctions) و مقادیر ویژه (Eigenvalues) برای سیستم‌های LTI

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

نکته: اگر ورودی $x[n]$ را بتوان به صورت ترکیبی خطی از سیگنال‌های پایه (Basic) $x_i[n]$

به صورت $x[n] = \sum_i a_i x_i[n]$ نوشت و پاسخ سیستم LTI به $x_i[n]$ را $y_i[n]$

بیابیم، آنگاه $y[n] = \sum_i a_i y_i[n]$ است. درنتیجه $y_i[n]$ ها هم سیگنال‌های پایه

برای خروجی $y[n]$ هستند.

سؤال مهم: چه سیگنال‌های پایه $x_i[n]$ ای می‌توان یافت که:

اولاً : $x_i[n]$ ها فرم ساده‌ای داشته باشند.

ثانیاً : دستهٔ وسیعی از سیگنال‌های $x[n]$ را بتوان به صورت ترکیبی خطی از $x_i[n]$ ها نوشت.

ثالثاً : $y_i[n]$ یعنی پاسخ هر سیستم LTI به $x_i[n]$ ، شکل ساده‌ای داشته باشد به طوری که

بتوان پاسخ سیستم های LTI به دستهٔ وسیعی از سیگنال ها را به سادگی به دست آورد.

تعریف : $x_i[n]$ را تابع ویژهٔ یک سیستم می‌نامیم هرگاه پاسخ سیستم به آن ضریب ثابتی از ورودی $x_i[n]$ باشد.

(λ_i ثابت مختلط)

$$T\{x_i[n]\} = \lambda_i x_i[n]$$

به λ_i مقدار ویژهٔ متناظر با تابع ویژهٔ $x_i[n]$ گفته می‌شود.

قضیه: تابع نمایی مختلط $x[n]=z^n$ $(z=re^{j\omega})$ تابع ویژهٔ هر سیستم LTI است.

مقدار ویژهٔ متناظر آن ثابت $H(z)$ است که:
$$H(z)=\sum_{n=-\infty}^{\infty} h[n]z^{-n}=Z\{h[n]\}$$

اثبات:
$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$$
$$=\sum_{k=-\infty}^{\infty} h[k]z^{n-k}=z^n\cdot\sum_{k=-\infty}^{\infty} h[k]z^{-k}=z^n\cdot H(z)$$
$$=H(z).x[n]$$

$H(z)$ تبدیل Z پاسخ ضربهٔ سیستم یا همان تابع سیستم (System Function) است.

If the input to a discrete-time LTI system is represented as a linear combination of complex exponentials, that is, if $x[n] = \sum_k a_k z_k^n$, then the output will be

$$y[n] = \sum_k a_k H(z_k) z_k^n.$$

نمایش سری فوریهٔ سیگنال‌های متناوب زمان‌گسسته

A discrete-time signal $x[n]$ is periodic with period N if $x[n] = x[n + N]$.

The fundamental period is the smallest positive integer N for which the above equation holds, and $\omega_0 = 2\pi/N$ is the fundamental frequency.

For example, the complex exponential $e^{j(2\pi/N)n}$ is periodic with period N.

$$e^{j(\frac{2\pi}{N})(n+N)} = e^{j(\frac{2\pi}{N})n}$$

مجموعهٔ نمایی های مختلط و البته هارمونیکی

Furthermore, the set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots .$$

All of these signals have fundamental frequencies that are multiples of $2\pi/N$ and thus are harmonically related.

نکته مهم: وجود فقط N عضو متمایز در مجموعهٔ $\varphi_k[n]$

There are only N distinct signals in the set given by $\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}$,

This is a consequence of the fact that discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. Specifically, $\phi_0[n] = \phi_N[n], \phi_1[n] = \phi_{N+1}[n],.$

and, in general, $\phi_k[n] = \phi_{k+rN}[n]$.

$$\varphi_k[n] = e^{jk\omega_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\varphi_k[n+N] = e^{jk\left(\frac{2\pi}{N}\right)(n+N)} = e^{jk\left(\frac{2\pi}{N}\right)n} \cdot e^{jk2\pi} = e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$= \varphi_k[n] \quad , \quad \forall k$$

$$\varphi_{k+rN}[n] = e^{j(k+rN)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} \cdot e^{j2\pi rn}$$

$$= e^{jk\left(\frac{2\pi}{N}\right)n} = \varphi_k[n]$$

That is, when k is changed by any integer multiple of N, the identical sequence is generated

$$\left\{ \varphi_0[n], \varphi_1[n], \ldots, \varphi_{N-1}[n] \right\}$$: $\varphi_k[n]$ دنبالهٔ متناظر N

$0 \leq k \leq N-1$

نکته و یادآوری: در حالت زمان پیوسته، مجموعهٔ نمایی های مختلط والبته هارمونیکی $\varphi_k(t) = e^{jk\omega_0 t}$، یک مجموعهٔ نامتناهی است. امّا در حالت زمان گسسته مجموعهٔ $\varphi_k[n] = e^{jk\omega_0 n} = e^{jk(\frac{2\pi}{N})n}$، فقط N عضو دارد.

We now wish to consider the representation of more general periodic sequences in terms of linear combinations of the sequences $\phi_k[n]$

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}.$$

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

منظور از نماد $K = \langle N \rangle$، تغییرات K روی N عدد صحیح متوالی است.

فرم کلی نمایش سری فوریه زمان‌گسسته

A linear combination of harmonically related complex exponentials of the form

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

is referred to as the *discrete-time Fourier series* representation.

The coefficients a_k are the Fourier series coefficients.

نمایش (بسط) سری فوریه عبارتست از یک ترکیب خطی از نمایی های مختلط و البته هارمونیکی

$\varphi_k[n] = e^{jk(\frac{2\pi}{N})n}$ ، که حاصل آن یک دنباله متناوب $x[n]$ با دوره تناوب اصلی $N = \frac{2\pi}{\omega_0}$

است. ضرائب a_k $(k = \langle N \rangle)$ شکل دنباله $x[n]$ را تعیین می‌کند و بالعکس.

تعیین ضرائب نمایش بسط سری فوریه برای سیگنال‌های متناوب زمان‌گسسته

Suppose now that we are given a sequence $x[n]$ that is periodic with fundamental period N.

We would like to determine whether a representation of $x[n]$ in the form of

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

exists and, if so, what the values of the coefficients a_k are.

$$x[0] = \sum_{k=\langle N\rangle} a_k,$$

$$\Rightarrow \quad x[1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N}, \quad \Rightarrow$$

$$\vdots$$

$$x[N-1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}.$$

با فرض استقلال خطی N معادله (اثبات می‌شود)

می‌توان N مجهول $\{a_0, a_1, \dots, a_{N-1}\}$

را تعیین نمود.

روش دوم: تعیین فرم بسته ای برای تعیین ضرائب a_k و محاسبهٔ مستقیم

However, by following steps parallel to those used in continuous time, it is possible to obtain a closed-form expression for the coefficients a_k in terms of the values of the sequence $x[n]$.

مشابه روش تعیین ضرائب سری فوریه زمان پیوسته عمل می شود.

یک نکتهٔ کلیدی (در مسئله 3.54 کتاب اثبات می شود):

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}.$$

It states that the sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

Now consider the Fourier series representation of $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$.

Multiplying both sides by $e^{-jr(2\pi/N)n}$ and summing over N terms, we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n}.$$

r یک عدد صحیح معلوم از بین N عدد صحیح متوالی است.

Interchanging the order of summation on the right-hand side, we have

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N\rangle} a_k \underbrace{\sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}}_{}. \qquad r \in \langle N\rangle$$

$$\sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n} = \begin{cases} N \;, & k=r \\ 0 \;, & k\neq r \end{cases}$$

$$\Rightarrow \quad a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n}.$$

نتیجه‌گیری : زوج روابط سری فوریهٔ گسسته

This provides a closed-form expression for obtaining the Fourier series coefficients, and

we have the *discrete-time Fourier series pair*:

$$x[n] = \sum_{k=<N>} a_k\, e^{jk\omega_0 n} = \sum_{k=<N>} a_k\, e^{jk(\frac{2\pi}{N})n}$$

(معادلهٔ ترکیب) the *synthesis* equation

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

(معادلهٔ تحلیل) the *analysis* equation

توجه :

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \ldots + a_{N-1}\phi_{N-1}[n].$$

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \ldots + a_N\phi_N[n].$$

$$\Rightarrow \quad \boxed{a_k = a_{k+N}.}$$

That is, if we consider more than N sequential values of k, the values a_k repeat periodically with period N. It is important that this fact be interpreted carefully. In particular, since there are only N distinct complex exponentials that are periodic with period N, the discrete-time Fourier series representation is a finite series with N terms.

ضرائب سری فوریهٔ یک دنبالهٔ متناوب با دوره تناوب N ، خود یک دنبالهٔ متناوب

a_k با دورهٔ تناوب N است . از آنجا که فقط N دنبالهٔ متمایز $\varphi_k[n] = e^{jk(\frac{2\pi}{N})n}$

وجود دارد ، سری فوریهٔ زمان گسسته یک سری متناهی با N جمله است .

می توان a_k را یک دنبالهٔ نامتناهی متناوب در نظر داشت که در سری فوریه فقط N مقدار متوالی آن

مورد استفاده است .

مثال ۵) دنبالهٔ $x[n] = \sin(\omega_0 n)$ فقط درصورتی متناوب است که نسبت $\frac{2\pi}{\omega_0}$ یک عدد صحیح و یا گویا (نسبت دو عدد صحیح) باشد.

فرض : $x[n] = \sin(\frac{2\pi}{N} n) \Rightarrow x[n+N] = x[n]$

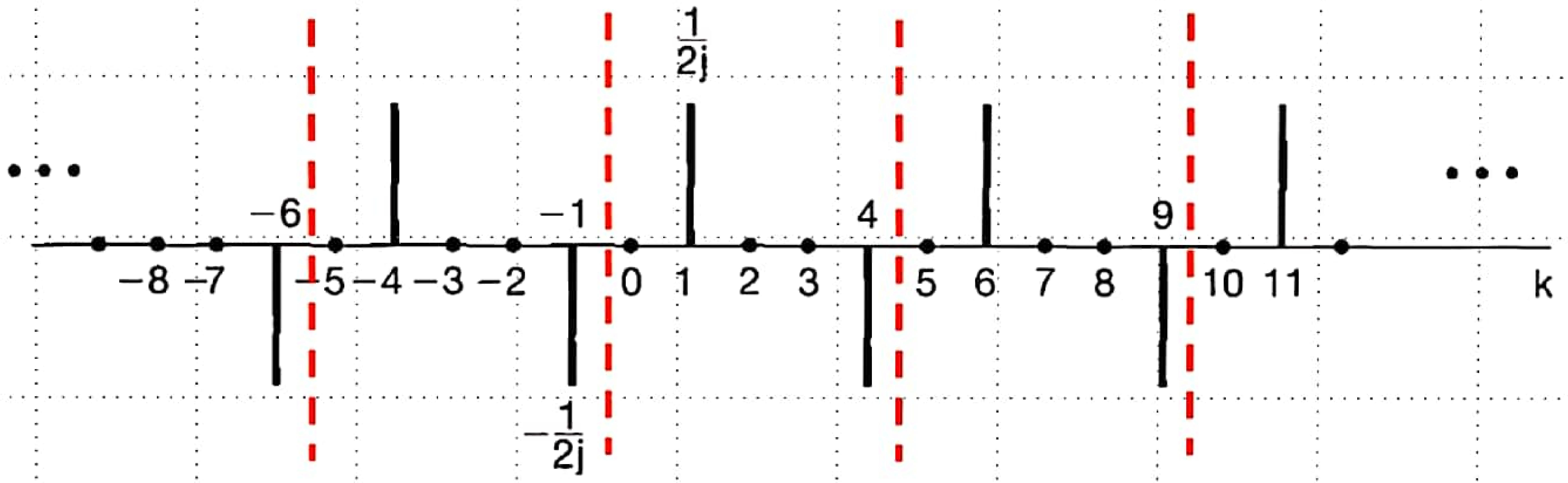
$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \Rightarrow a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j},$$

$$\Rightarrow \{ a_0 = 0 , a_1 = \frac{1}{2j} , a_2 = 0 , \dots , a_{N-1} = -\frac{1}{2j} \}$$

فرض : $N = 5$

$$\Rightarrow \{ a_0 = 0 , a_1 = \frac{1}{2j} , a_2 = 0 , a_3 = 0 , a_4 = -\frac{1}{2j} \}$$

و $a_{k+5} = a_k \; , \forall k$



Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

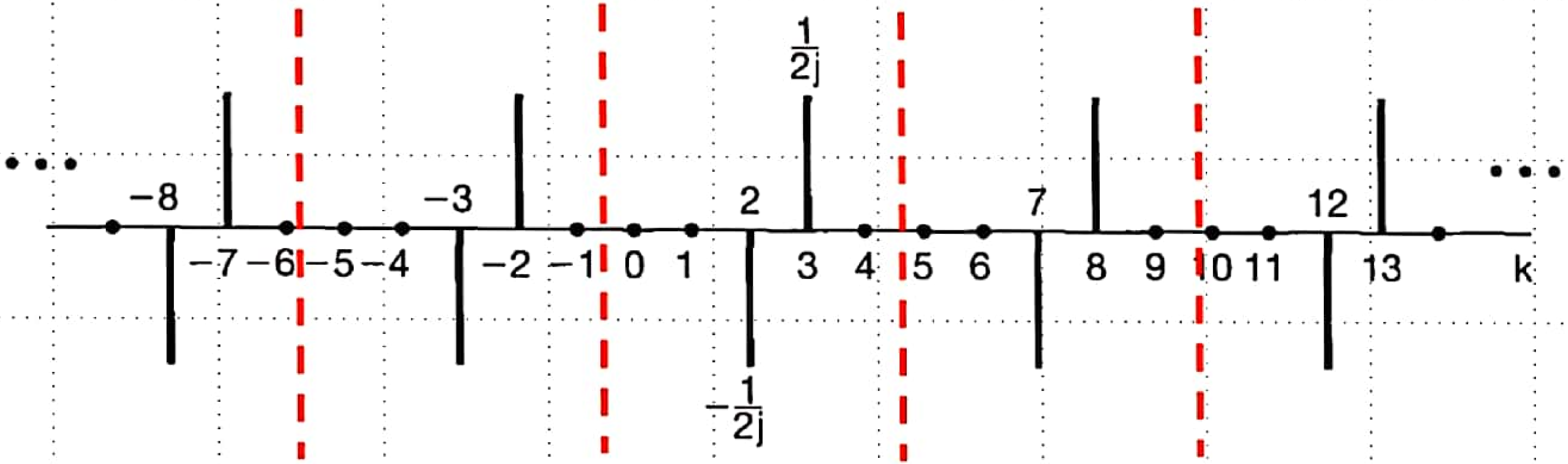
حال فرض کنیم $x[n] = Sin\left(\frac{2\pi M}{N} n\right)$ و $gcd(M, N) = 1$.

N دوره تناوب اصلی دنباله است. $\Rightarrow x[n+N] = x[n]$

$$x[n] = \frac{1}{2j} e^{jM(2\pi/N)n} - \frac{1}{2j} e^{-jM(2\pi/N)n}, \Rightarrow a_M = (1/2j), a_{-M} = (-1/2j),$$

فرض : $N = ۵$ و $M = ۳$

$$\Rightarrow \left\{ a_0 = 0 ,\ a_1 = 0 ,\ a_2 = a_{-3} = -\frac{1}{2j} ,\ a_3 = \frac{1}{2j} ,\ a_4 = 0 \right\}$$



Fourier coefficients for $x[n] = \sin 3(2\pi/5)n$.

Consider the signal (مثال ۵)

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

This signal is periodic with period N, and, we can expand $x[n]$ directly in terms of complex exponentials to obtain

$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}] + \frac{1}{2}[e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}].$$

$$\Rightarrow x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j(2\pi/N)n} + \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

Thus the Fourier series coefficients for this example are

$$a_0 = 1, \qquad a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j, \qquad a_2 = \frac{1}{2}j,$$

$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j, \qquad a_{-2} = -\frac{1}{2}j,$$

with $a_k = 0$ for other values of k in the interval of summation in the synthesis equation.

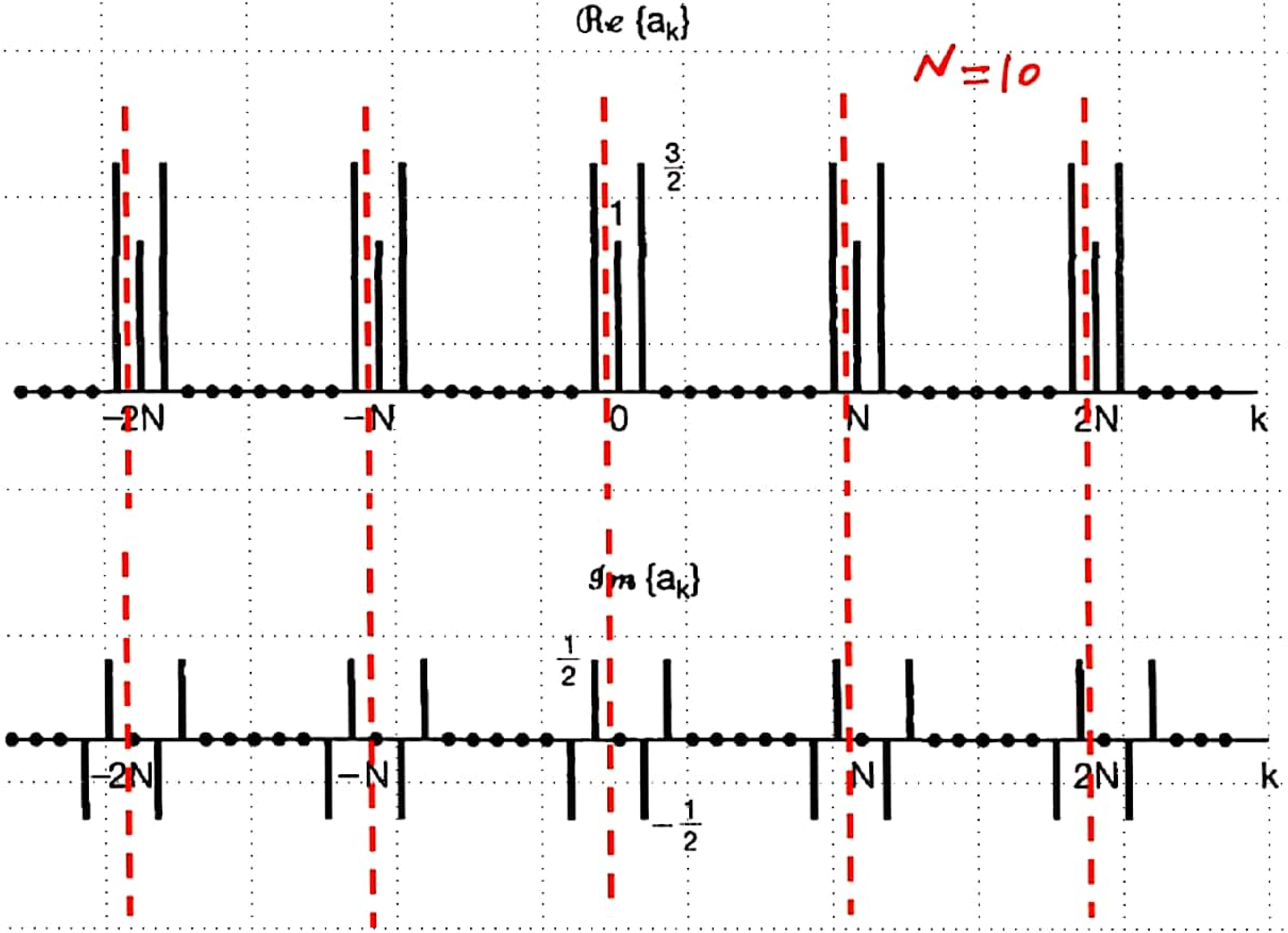
Again, the Fourier coefficients are periodic with period N, so, for example,

$$a_N = 1, a_{3N-1} = \tfrac{3}{2} + \tfrac{1}{2}j, \text{ and } a_{2-N} = \tfrac{1}{2}j.$$

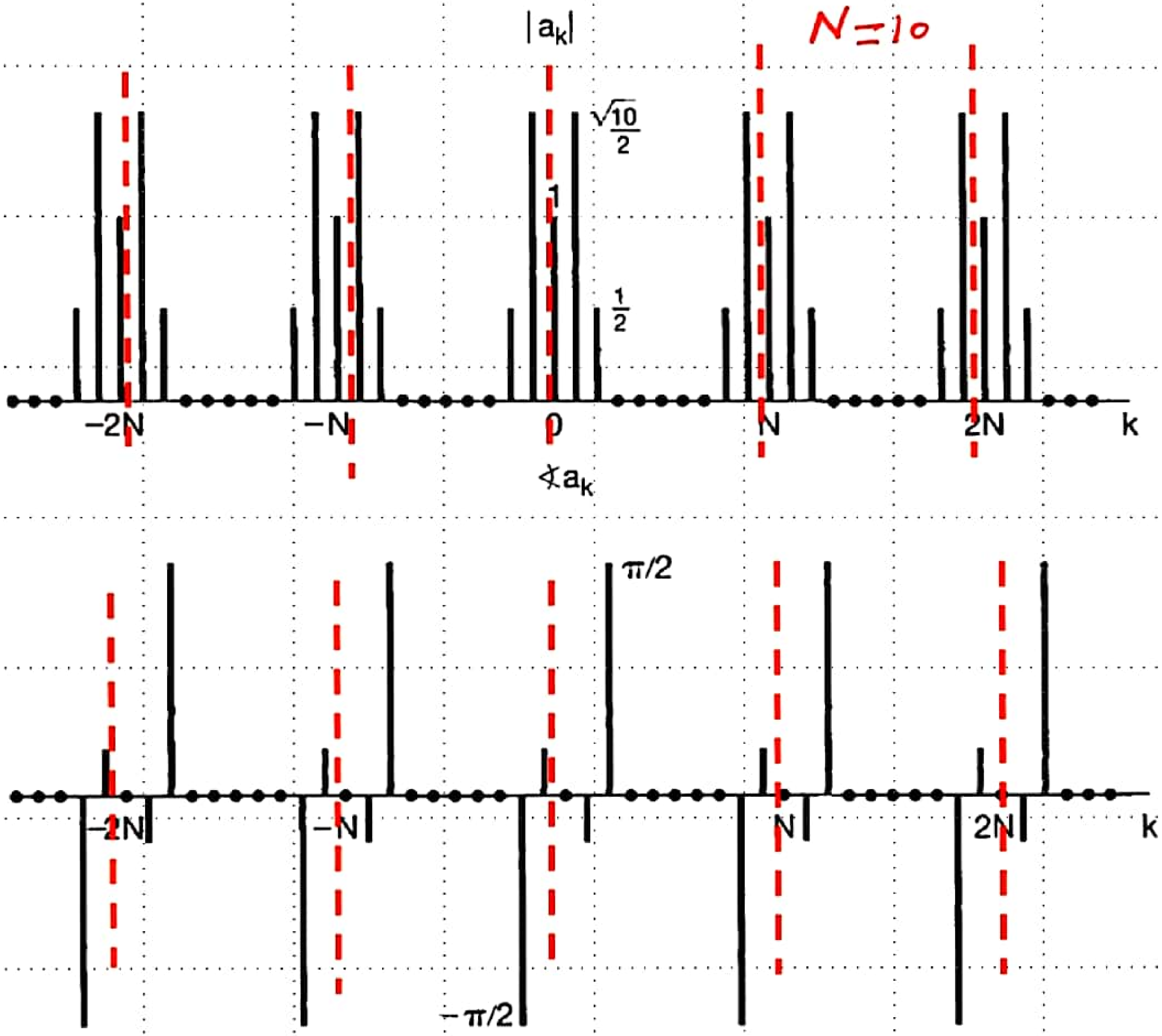
$$a_k = Re\{a_k\} + j\,Im\{a_k\}$$

$$a_k = |a_k|\, e^{j\measuredangle a_k}$$

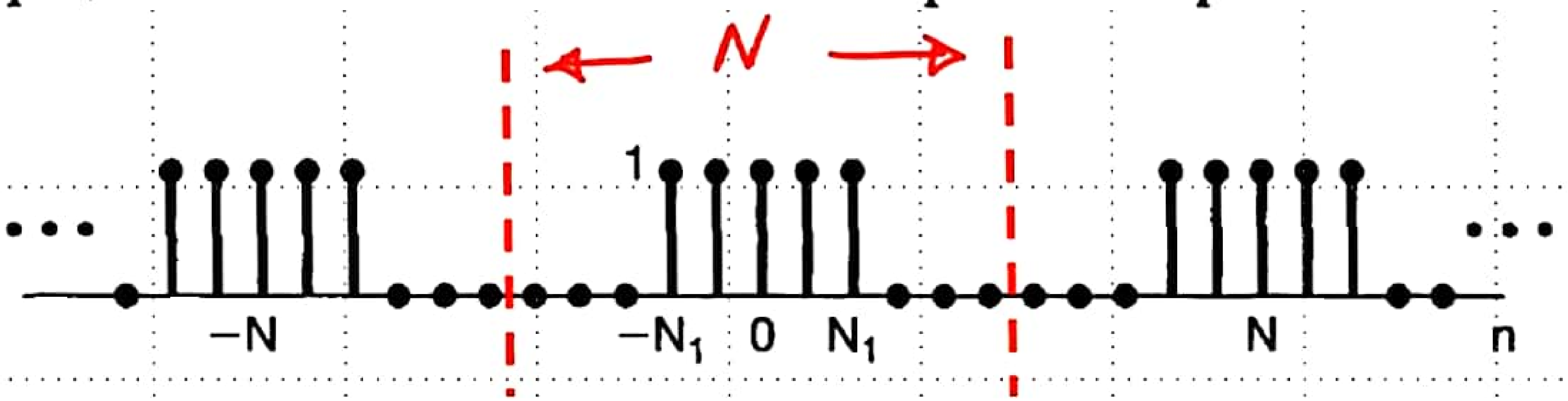
برای $N=10$ دنباله‌های
بخش حقیقی / موهومی و
اندازه / فاز a_k رسم شده است.



$\Re e\{a_k\}$
N=10
$\frac{3}{2}$
1
−2N
−N
0
N
2N
k
$\Im m\{a_k\}$
$\frac{1}{2}$
$-\frac{1}{2}$
−2N
−N
N
2N
k


$|a_k|$
$N=10$
$\frac{\sqrt{10}}{2}$
1
$\frac{1}{2}$
−2N
−N
0
N
2N
k
$\sphericalangle a_k$
π/2
−π/2
−2N
−N
N
2N
k

In this example, we consider the discrete-time periodic square wave shown in مثال)



Discrete-time periodic square wave.

$$x[n] = \begin{cases} 1 , & -N_1 \leq n \leq N_1 \\ 0 , & oth \end{cases} \quad , x[n+N] = x[n]$$

$$N > N_1$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}.$$

Letting $m = n + N_1$, we observe that

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} = \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}.$$

مجموع $2N_1+1$ جمله از یک تصاعد هندسی با قدر نسبت $e^{-jk(\frac{2\pi}{N})}$ و جملهٔ اول برابر ۱ .

$$\Rightarrow \quad a_k = \frac{1}{N} e^{jk(2\pi/N)N_1}\left(\frac{1-e^{-jk2\pi(2N_1+1)/N}}{1-e^{-jk(2\pi/N)}}\right)$$

$$= \frac{1}{N}\frac{e^{-jk(2\pi/2N)}\left[e^{jk2\pi(N_1+1/2)/N} - e^{-jk2\pi(N_1+1/2)/N}\right]}{e^{-jk(2\pi/2N)}\left[e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)}\right]}$$

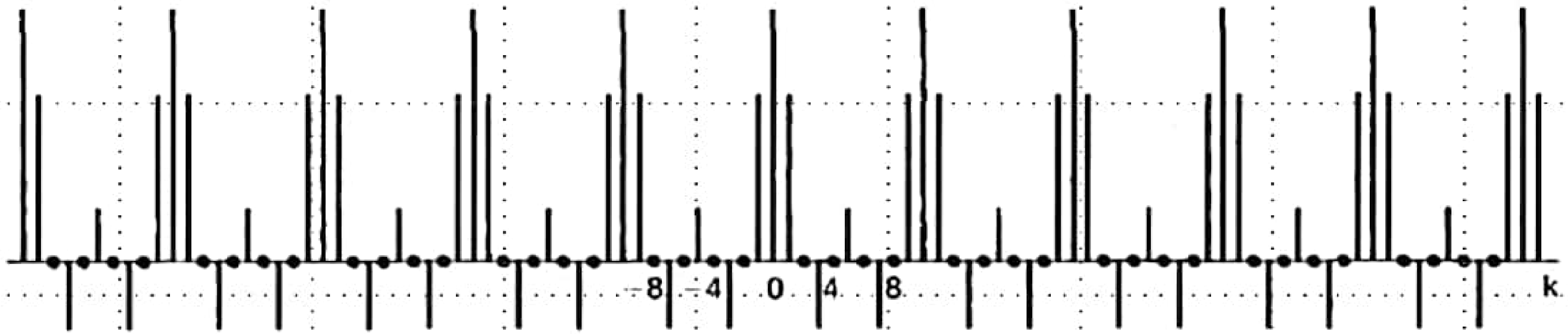
$$= \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}, \quad k \neq 0, \pm N, \pm 2N, \ldots$$

and $$a_k = \frac{2N_1+1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots$$

The coefficients a_k for $2N_1 + 1 = 5$ are sketched for $N = 10, 20,$ and 40 in

plots of Na_k for $2N_1 + 1 = 5$ and

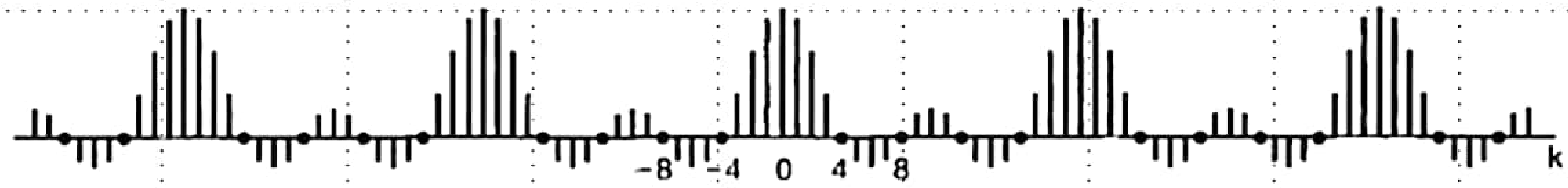
$N = 10$;



Fourier series coefficients for the periodic square wave

plots of Na_k for $2N_1 + 1 = 5$ and

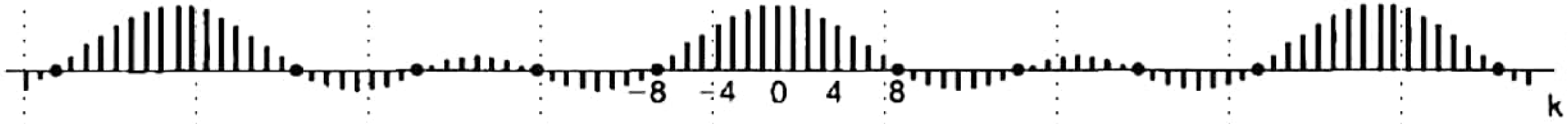
$N = 20$;



Fourier series coefficients for the periodic square wave

plots of Na_k for $2N_1 + 1 = 5$ and

$N = 40$.



Fourier series coefficients for the periodic square wave

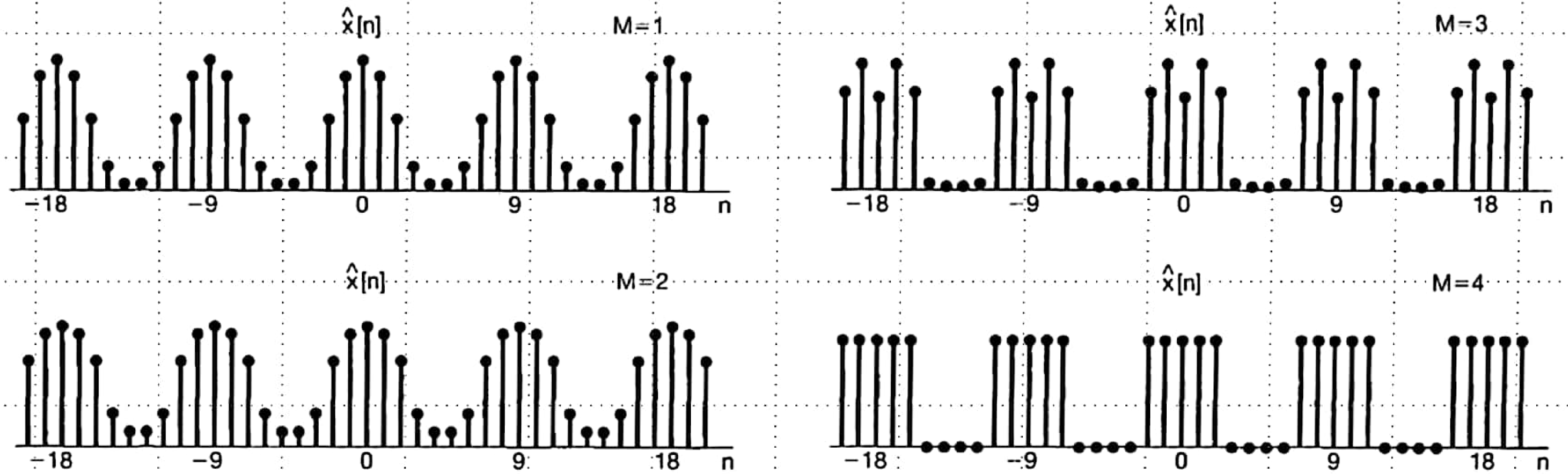
یادآوری : در نمایش سری فوریه برای سیگنال زمان پیوسته مربعی متقارن ، اثر افزایش تعداد جملات در سری فوریه در نمایش دقیق‌تر سیگنال و همچنین مشاهده اثر Gibbs در حوالی نقطه ناپیوستگی و جهش در پالس مربعی مورد بحث قرار گرفت .

سؤال : آیا در حالت زمان گسسته هم پدیده مشابهی مشاهده می‌شود ؟

فرض کنیم دوره تناوب دنباله پالس مربعی متناوب ، N (برای سهولت فرد فرض شود) و تعداد $2M+1 < N$ جمله در نمایش سری فوریه لحاظ شود . برخلاف حالت پیوسته به دلیل تعداد متناهی جملات سری ، مشکل واگرایی وجود ندارد و در حالت $2M+1 = N$ انطباق کامل است.

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n}$$

فرض : $N = 9$



نکته : اگر به تعداد مساوی با طول دوره تناوب، از ضرایب سری فوریه استفاده شود، $\hat{x}[n] = x[n]$

The reason for this stems from the fact that any discrete-time periodic sequence $x[n]$ is completely specified by a *finite* number N of parameters, namely, the values of the sequence over one period.

Thus, if N is odd and we take $M = (N-1)/2$ in eq. $\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n}$

if N is even and we take $M = N/2$, in eq. $\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk(2\pi/N)n}$,

$\Rightarrow \quad \hat{x}[n] = x[n].$

In contrast, a continuous-time periodic signal takes on a continuum of values over a single period, and an infinite number of Fourier coefficients are required to represent it.

خواص سری فوریهٔ سیگنال‌های متناوب زمان‌گسسته

$$x[n] = x[n+N], \qquad \omega_0 = 2\pi/N$$

دنبالهٔ متناوب اصلی $\quad x[n] \xleftrightarrow{\mathcal{FS}} a_k, \quad a_{k+N} = a_k \quad$ دنبالهٔ متناوب

ضرائب سری فوریه

$$\boxed{\begin{aligned} x[n] &= \sum_{k=<N>} a_k\, e^{jk\omega_0 n} = \sum_{k=<N>} a_k\, e^{jk\left(\frac{2\pi}{N}\right)n} \\ a_k &= \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n} \end{aligned}}$$

There are strong similarities between the properties of discrete-time and continuous-time Fourier series.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}

Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_lb_{k-l}$
First Difference	$x[n]-x[n-1]$	$(1-e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$

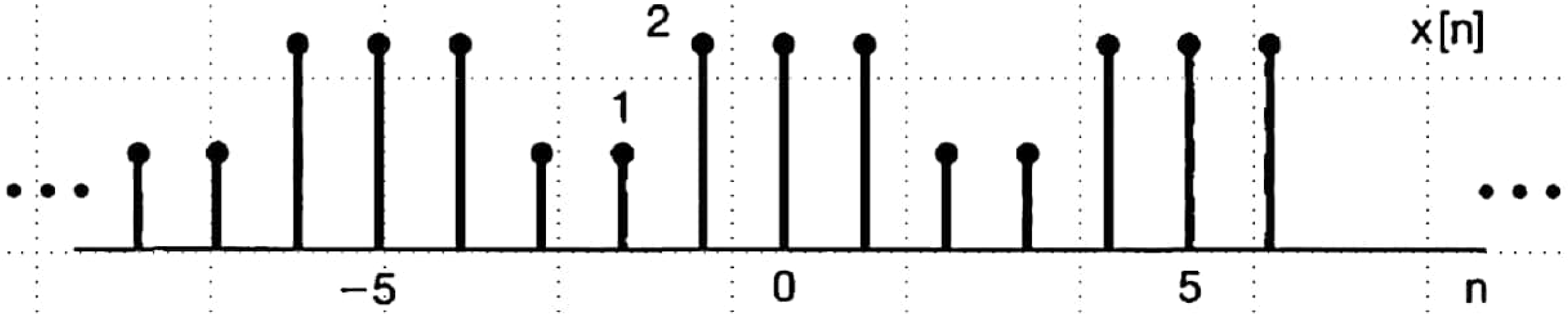
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} \quad [\mathrm{x[n]\ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} \quad [\mathrm{x[n]\ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

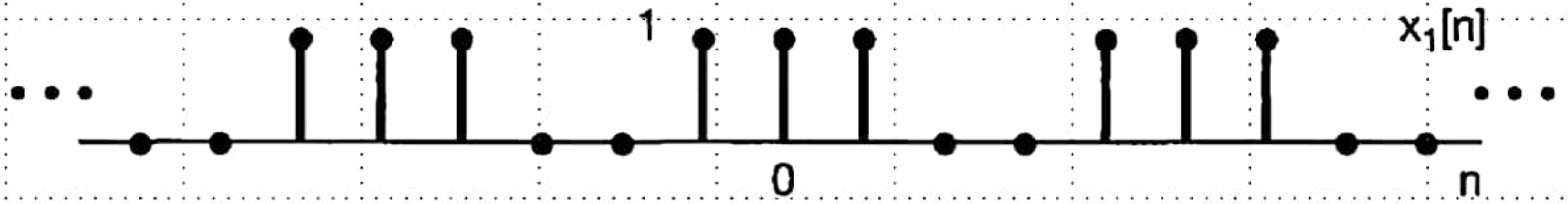
$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

مثال) استفاده از خاصیت خطی بودن

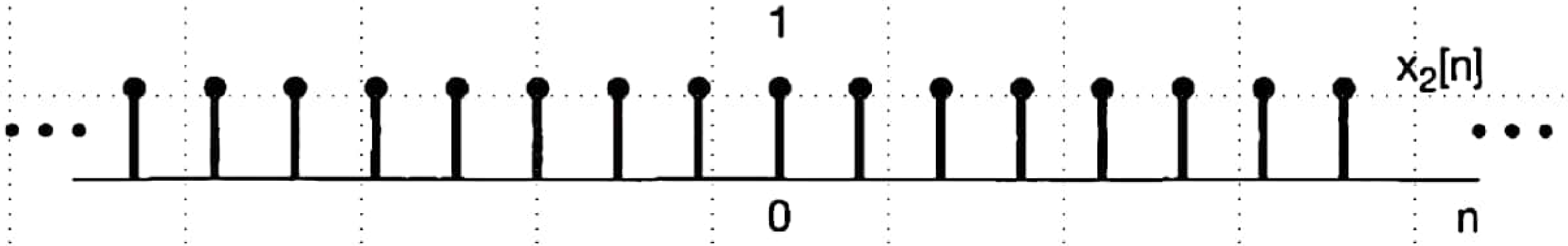
$$x[n] = x_1[n] + x_2[n]$$



$$x[n] \xleftrightarrow{\mathcal{FS}} a_k = ?$$



$$x_1[n] \xleftrightarrow{\mathcal{FS}} b_k \checkmark$$



$$x_2[n] \xleftrightarrow{\mathcal{FS}} c_k \checkmark$$

قبلاً ثابت شد :

$$b_k = \begin{cases} \dfrac{1}{N}\dfrac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\[2ex] \dfrac{2N_1 + 1}{N}, & k = 0, \pm N, \pm 2N, \ldots. \end{cases}$$

(with $N_1 = 1$ and $N = 5$)

$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\[2ex] \dfrac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

The sequence $x_2[n]$ has only a dc value, which is captured by its zeroth Fourier series coefficient:

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1.$$

Since the discrete-time Fourier series coefficients are periodic, it follows that $c_k = 1$ whenever k is an integer multiple of 5.

The remaining coefficients of $x_2[n]$ must be zero, because $x_2[n]$ contains only a dc component.

$$a_k = b_k + c_k \Rightarrow a_k = \begin{cases} b_k = \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\[2ex] \dfrac{8}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

مثال ۵)

Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.

2. $\sum_{n=0}^{5} x[n] = 2$.

3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.

4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

Let us determine the sequence $x[n]$. We denote the Fourier series coefficients of $x[n]$ by a_k.

From Fact 2, we conclude that $a_0 = 1/3$.

$$a_0 = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

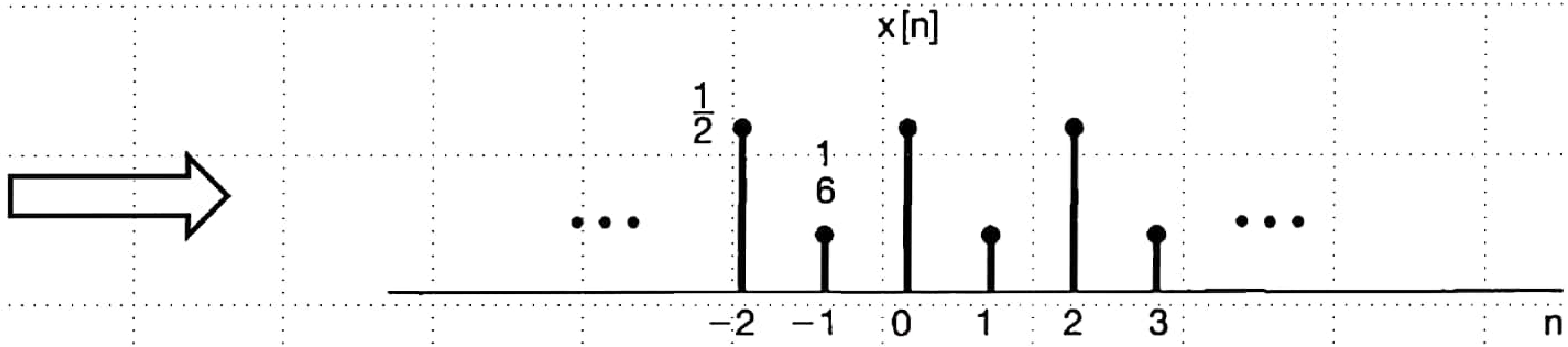
Noting that $(-1)^n = e^{-j\pi n} = e^{-j(2\pi/6)3n}$, we see from Fact 3 that $a_3 = 1/6$.

$$a_3 = \frac{1}{6} \sum_{n=\langle 6 \rangle} x[n] e^{-j3\left(\frac{2\pi}{6}\right)n} = \frac{1}{6} \sum_{n=\langle 6 \rangle} x[n](-1)^n = \frac{1}{6}$$

From Parseval's relation, the average power in $x[n]$ is $P = \sum_{k=0}^{5} |a_k|^2$.

بین همهٔ دنباله‌هایی که ۳ شرط اول را برآورده می‌کنند، $x[n]$ حداقل توان را در طول یک دورهٔ تناوب دارد.

Since each nonzero coefficient contributes a positive amount to P, and since the values of a_0 and a_3 are prespecified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a_5 = 0$. It then follows that $x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n$,



مثال)

In this example we determine and sketch a periodic sequence, given an algebraic expression for its Fourier series coefficients.

If $x[n]$ and $y[n]$ are periodic with period N, then the signal $w[n] = \sum_{r=\langle N \rangle} x[r]y[n-r]$ is also periodic with period N. Furthermore, the Fourier series coefficients of $w[n]$ are equal to Na_kb_k, where a_k and b_k are the Fourier coefficients of $x[n]$ and $y[n]$, respectively.

Suppose now that we are told that a signal $w[n]$ is periodic with a fundamental period of $N = 7$ and with Fourier series coefficients $c_k = \dfrac{\sin^2(3\pi k/7)}{7\sin^2(\pi k/7)}$.

We observe that $c_k = 7d_k^2$, where d_k denotes the sequence of Fourier series coefficients of a square wave $x[n]$, with $N_1 = 1$ and $N = 7$.

$$d_k = \begin{cases} \dfrac{1}{N}\dfrac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\[2em] \dfrac{2N_1 + 1}{N}, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$$

$N_1 = 1,\ N = \text{۷}$

$\Downarrow$

$$d_k = \frac{1}{\text{۷}} \frac{\sin\left(\frac{\text{۳}\pi k}{\text{۷}}\right)}{\sin\left(\frac{\pi k}{\text{۷}}\right)}$$

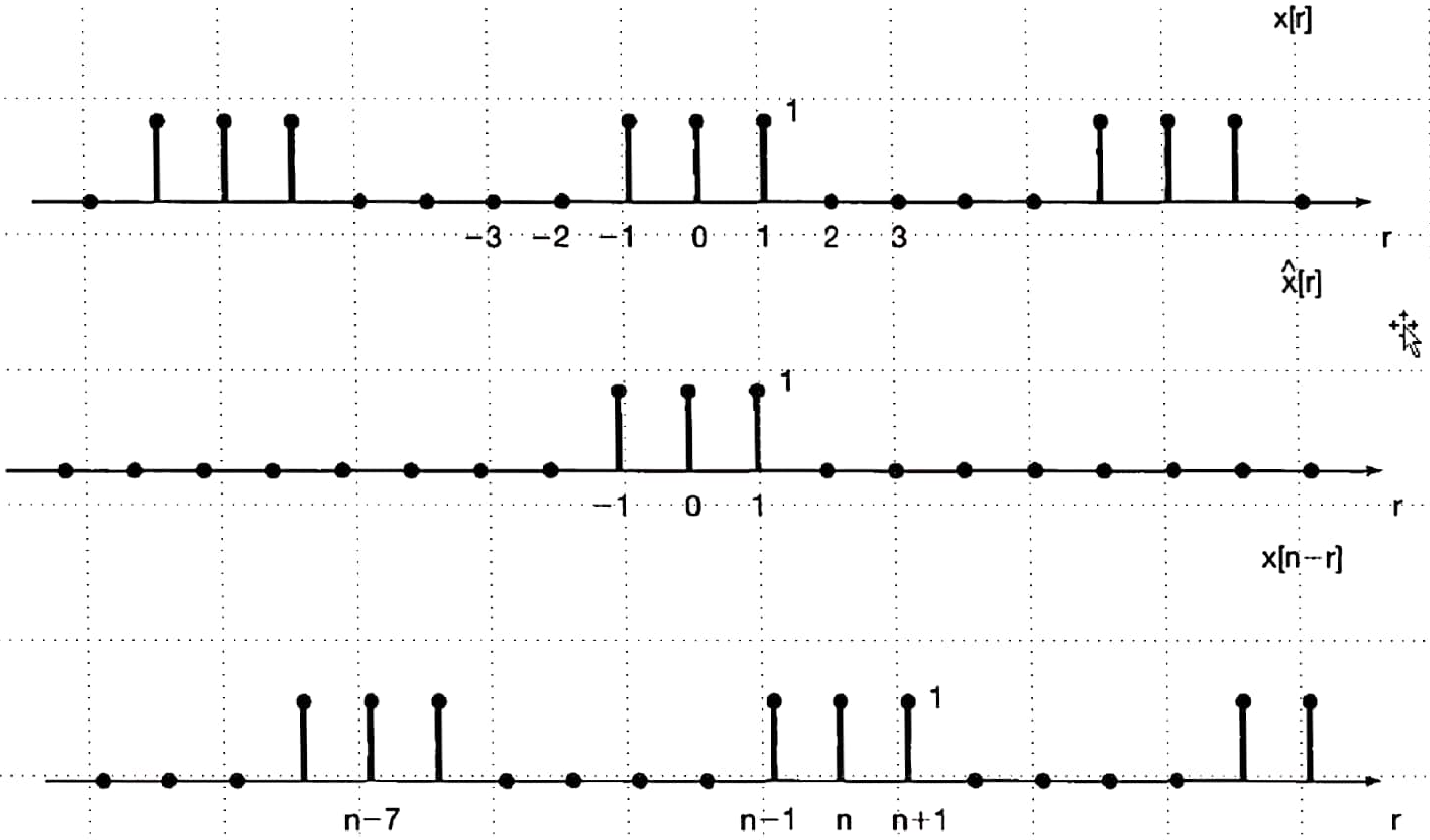
Using the periodic convolution property, we see that

$$w[n] = \sum_{r=\langle 7\rangle} x[r]x[n-r] = \sum_{r=-3}^{3} x[r]x[n-r],$$

where, in the last equality, we have chosen to sum over the interval $-3 \leq r \leq 3$.

we can convert the periodic convolution to an ordinary convolution by defining a signal $\hat{x}[n]$ that equals $x[n]$ for $-3 \leq n \leq 3$ and is zero otherwise.

Then,
$$w[n] = \sum_{r=-3}^{3} \hat{x}[r]x[n-r] = \sum_{r=-\infty}^{+\infty} \hat{x}[r]x[n-r].$$

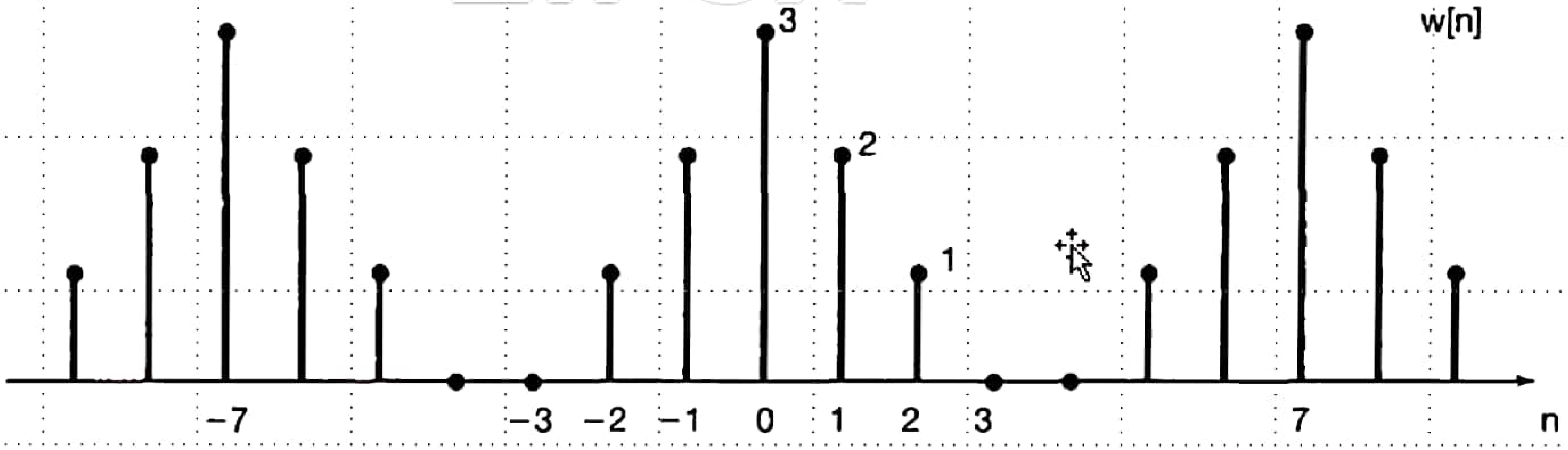


From the figure we can immediately calculate $w[n]$.

In particular we see that $w[0] = 3$; $w[-1] = w[1] = 2$; $w[-2] = w[2] = 1$;

and $w[-3] = w[3] = 0$.

Since $w[n]$ is periodic with period 7, we can then sketch $w[n]$ as shown in



بسم الله الرّحمن الرّحيم

دانشگاه صنعتی اصفهان
دانشکده برق و کامپیوتر

تجزیه و تحلیل سیگنال‌ها و سیستم‌ها

مدرس: مسعود عمومی

جلسه بیست و سوم ـ بخش‌های 3.8 ، 3.9 و 3.11 کتاب

با سلام خدمت دانشجویان محترم

یادآوری (توابع و مقادیر ویژهٔ سیستم‌های LTI)

Specifically, in continuous time, if $x(t) = e^{st}$ is the input to a continuous-time LTI system, then the output is given by $y(t) = H(s)e^{st}$, where, $H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$,

in which $h(t)$ is the impulse response of the LTI system.

Similarly, if $x[n] = z^n$ is the input to a discrete-time LTI system, then the output is given by $y[n] = H(z)z^n$, where, $H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$,

in which $h[n]$ is the impulse response of the LTI system.

یادآوری (خاصیت تناوب در ورودی و خروجی سیستم‌های LTI)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \quad y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$x(t) = x(t+T) \quad \Rightarrow \quad y(t) = y(t+T)$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow \quad y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$x[n] = x[n+N] \quad \Rightarrow \quad y[n] = y[n+N]$$

یادآوری (سری فوریه و سیستم‌های LTI زمان‌پیوسته و پایدار)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \qquad \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\Rightarrow \qquad y(t) = \sum_{k=-\infty}^{+\infty} \underline{a_k H(jk\omega_0)} e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} \underline{b_k} e^{jk\omega_0 t}$$

$$H(s) = \mathcal{L}\{h(t)\} = \int_{-\infty}^{+\infty} h(t) e^{-st} dt$$

System Function

$$s = j\omega \subset ROC \quad \Rightarrow \quad H(s) = H(j\omega)$$

Frequency Response

Stability Condition

سری فوریه و سیستم‌های LTI زمان‌گسسته و پایدار

$$x[n] = \sum_{k=<N>} a_k\, e^{jk\omega_0 n} = \sum_{k=<N>} a_k\, e^{jk\left(\frac{2\pi}{N}\right)n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$\Rightarrow \quad y[n] = \sum_{k=<N>} \underline{a_k\, H(e^{jk\omega_0})} e^{jk\omega_0 n} = \sum_{k=<N>} \underline{b_k}\, e^{jk\omega_0 n}$$

$$H(z) = z\{h[n]\} = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$$

System Function

$$z = e^{j\omega} \subset ROC \quad \Rightarrow \quad H(z) = H\left(e^{j\omega}\right)$$

Frequency Response

Stability Condition

مثال) ورودی یک سیستم LTI :

$$x[n] = \cos\left(\frac{2\pi n}{N}\right).$$

و پاسخ ضربهٔ آن (علّی و پایدار)

$$h[n] = \alpha^n u[n], -1 < \alpha < 1$$

مطلوبست نمایش سری فوریهٔ خروجی سیستم.

$x[n]$ can be written in Fourier series form as $x[n] = \frac{1}{2}e^{j(2\pi/N)n} + \frac{1}{2}e^{-j(2\pi/N)n}.$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n. \Longrightarrow H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}.$$

we then obtain the Fourier series for the output:

$$y[n] = \frac{1}{2}H\left(e^{j2\pi/N}\right)e^{j(2\pi/N)n} + \frac{1}{2}H\left(e^{-j2\pi/N}\right)e^{-j(2\pi/N)n}$$

$$y[n] = \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right)e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right)e^{-j(2\pi/N)n}.$$

If we write $\frac{1}{1-\alpha e^{-j2\pi/N}} = re^{j\theta}$, $\Rightarrow$ $y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right)$.

For example, if $N = 4$, $\frac{1}{1-\alpha e^{-j2\pi/4}} = \frac{1}{1+\alpha j} = \frac{1}{\sqrt{1+\alpha^2}}e^{j(-\tan^{-1}(\alpha))}$,

$$\Rightarrow \quad y[n] = \frac{1}{\sqrt{1+\alpha^2}}\cos\left(\frac{\pi n}{2} - \tan^{-1}(\alpha)\right).$$

فیلترینگ سیگنال‌های متناوب زمان‌پیوسته و زمان‌گسسته

$$x(t) = x(t+T) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t) = y(t+T)$$

$$-\infty < \omega < +\infty$$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$a_k \longleftrightarrow b_k = H(jk\omega_0).a_k$$

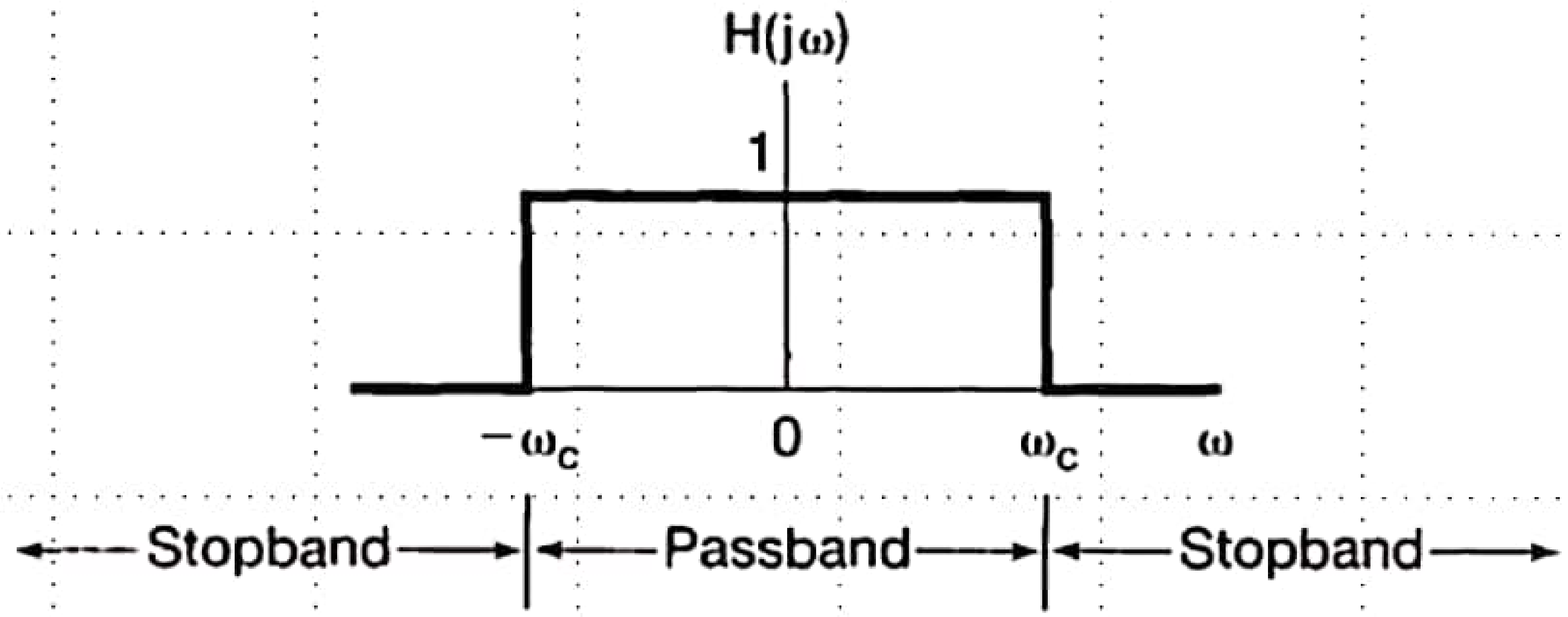
$$x[n] = x[n+N] \longrightarrow \boxed{H\left(e^{j\omega}\right)} \longrightarrow y[n] = y[n+N]$$

$$-\pi < \omega \leq +\pi$$

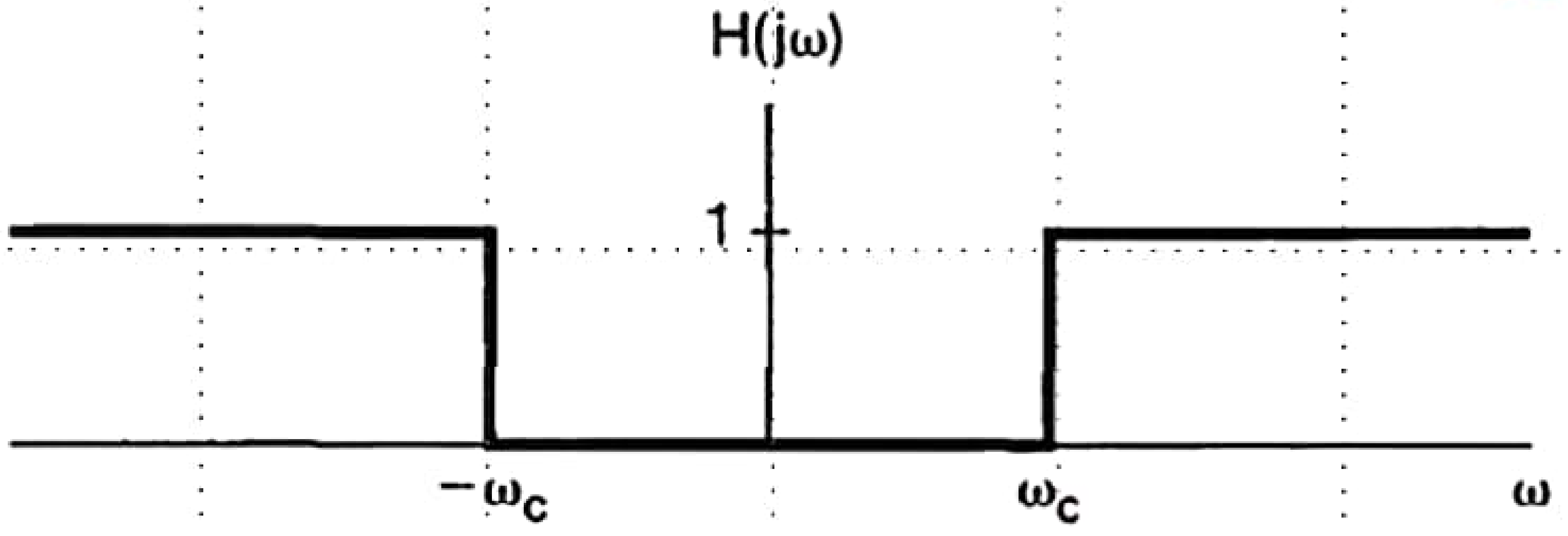
$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$a_k \longleftrightarrow b_k = H\left(e^{jk\omega_0}\right).a_k$$

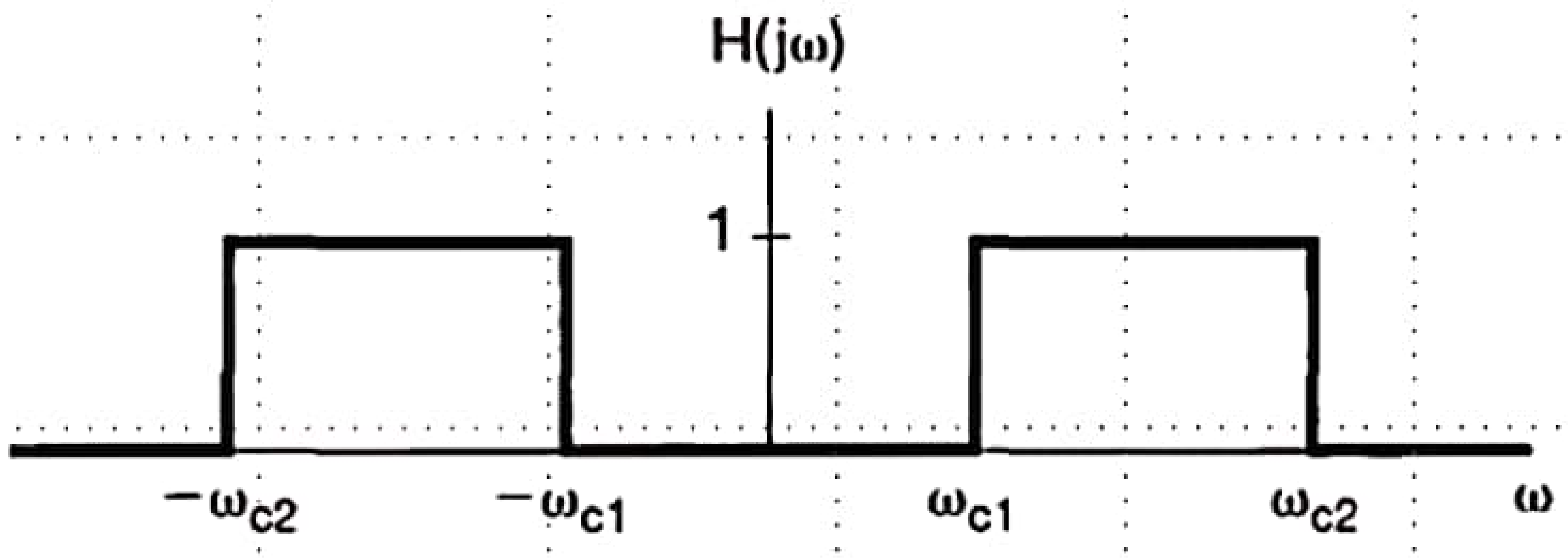
انواع فیلترهای ایده‌آل زمان‌پیوسته



Frequency response of
an ideal lowpass filter

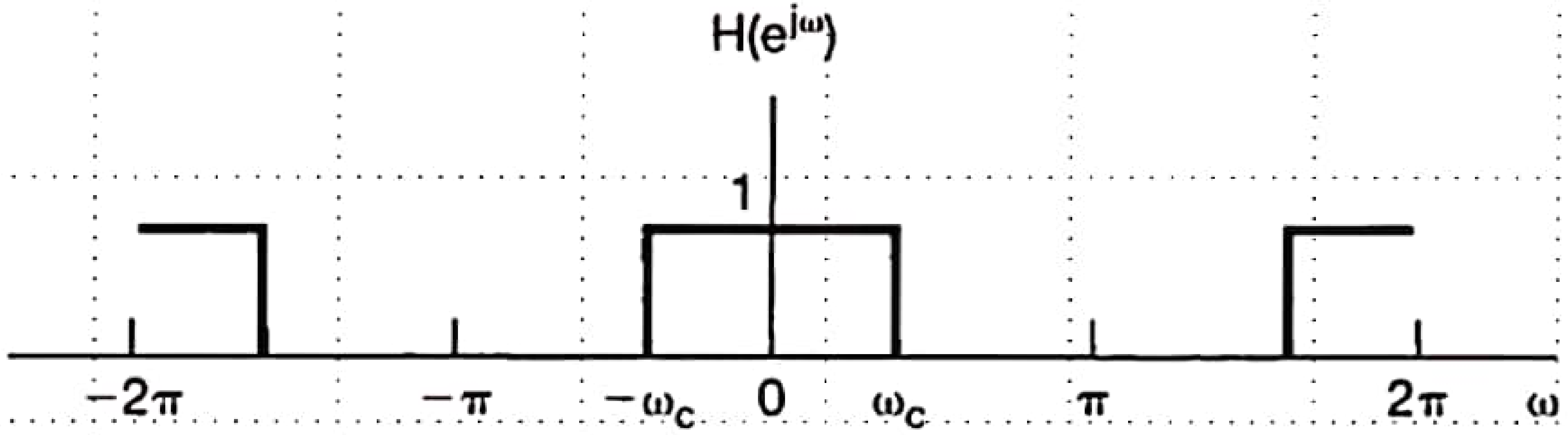


Frequency response of
an ideal highpass filter

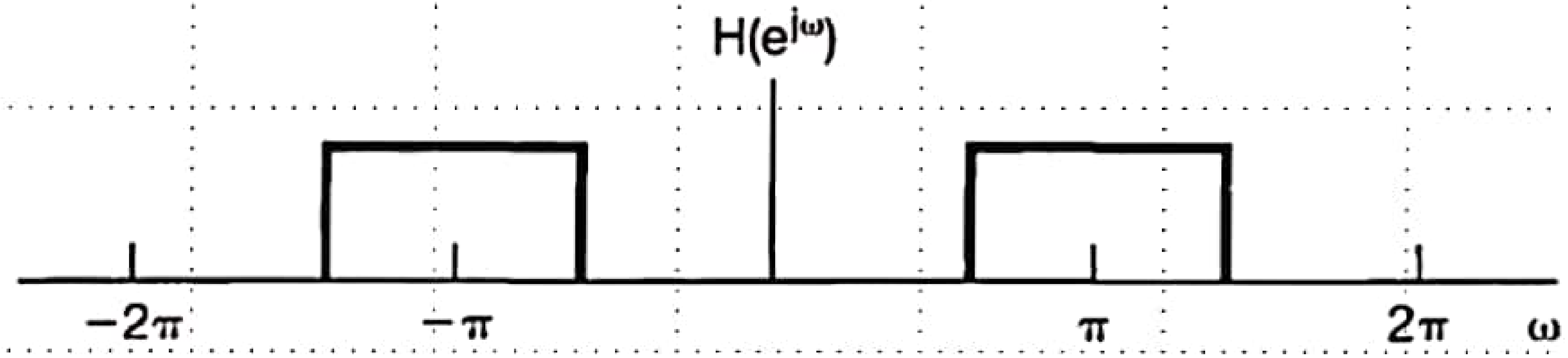


Frequency response of
an ideal bandpass filter

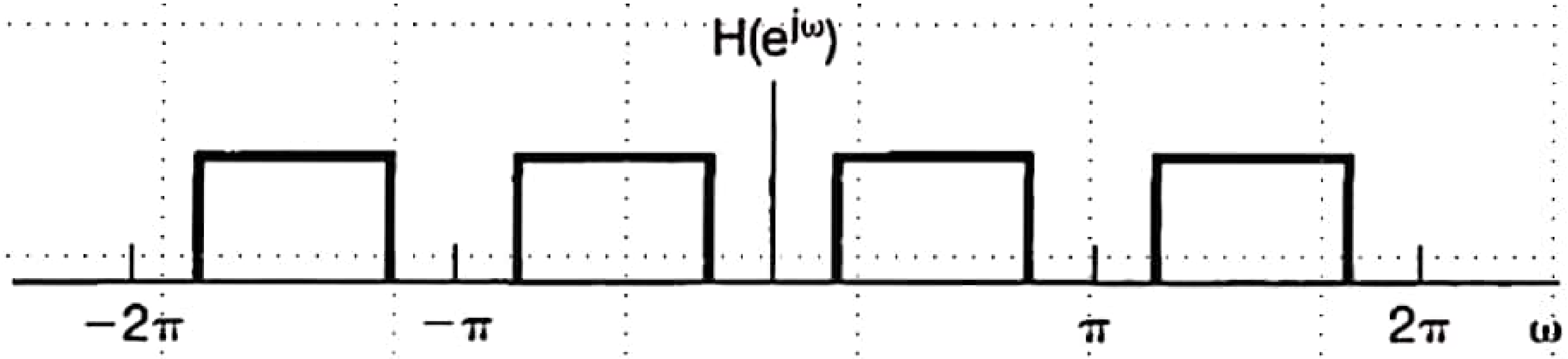
انواع فیلترهای ایده‌آل زمان گسسته



Frequency response of
an ideal lowpass filter



Frequency response of
an ideal highpass filter



Frequency response of
an ideal bandpass filter

مثال‌هایی از فیلترهای زمان‌گسسته توصیف‌شده توسط معادلات تفاضلی

مثال) فیلتر زمان‌گسسته بازگشتی مرتبه اول

The LTI system described by the first-order difference equation

$$y[n] - ay[n-1] = x[n].$$

تبدیل Z از طرفین $\Rightarrow (1 - az^{-1})Y(z) = X(z)$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}} \Rightarrow h[n] = a^n u[n]$$ پاسخ ضربه

$RoC: |z| > |a|$ (سیستم علّی و پایدار به شرط $|a| < 1$)

From the eigenfunction property of complex exponential signals, we know that if $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega})e^{j\omega n}$, where $H(e^{j\omega})$ is the frequency response of the system.

$$\Rightarrow H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}, \Rightarrow [1 - ae^{-j\omega}]H(e^{j\omega})e^{j\omega n} = e^{j\omega n},$$

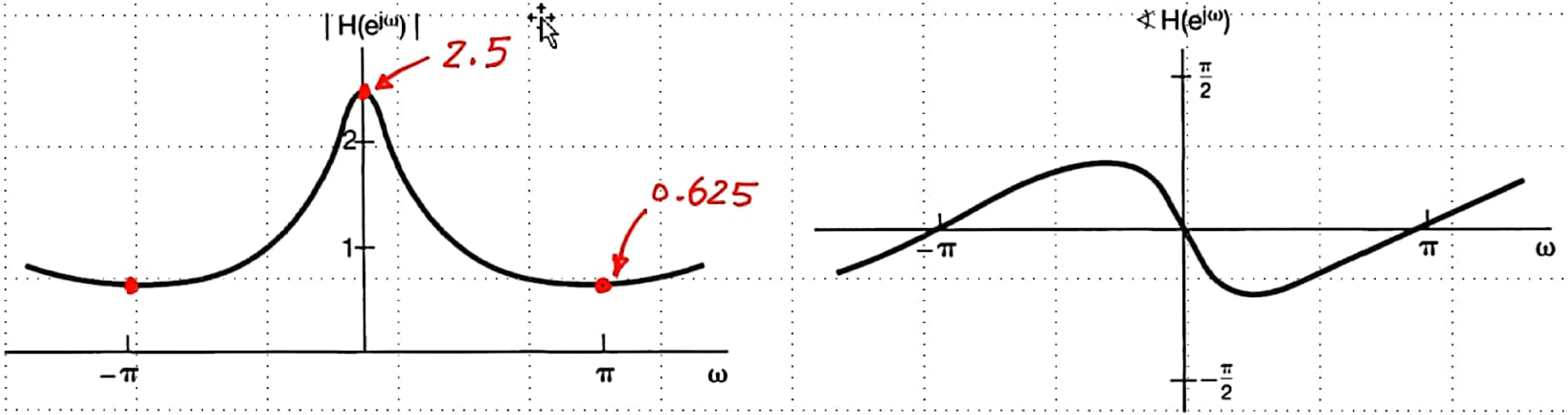
$$\Rightarrow \quad H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

پاسخ فرکانسی

$$H(e^{j\omega}) = \frac{1}{1 - a\cos\omega - ja\sin\omega} = |H(e^{j\omega})| e^{j \measuredangle H(e^{j\omega})}$$

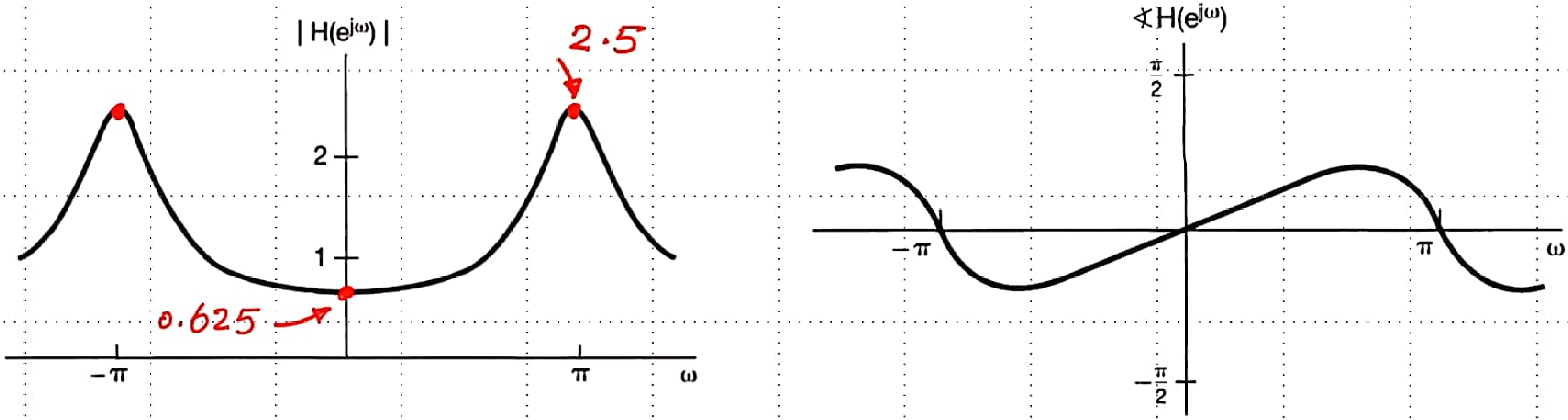
$$|H(e^{j\omega})| = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}} \quad \& \quad \measuredangle H(e^{j\omega}) = -\operatorname{tg}^{-1} \frac{a\sin\omega}{a\cos\omega - 1}$$

We observe that, for the positive value of a, the difference equation behaves like a lowpass filter with minimal attenuation of low frequencies near $\omega = 0$ and increasing attenuation as we increase ω toward $\omega = \pi$.



Frequency response of the first-order recursive discrete-time filter $a = 0.6$;

For the negative value of a, the system is a highpass filter, passing frequencies near $\omega = \pi$ and attenuating lower frequencies.



Frequency response of the first-order recursive discrete-time filter $a = -0.6$.

In fact, for any positive value of $a < 1$, the system approximates a lowpass filter, and for any negative value of $a > -1$, the system approximates a highpass filter, where $|a|$ controls the size of the filter passband, with broader passbands as $|a|$ is decreased.

$$y[n] - a\,y[n-1] = x[n]$$

$0 < a < 1$ LPF

$-1 < a < 0$ HPF

فیلترهای زمان‌گسسته غیربازگشتی

The general form of an FIR nonrecursive difference equation is

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k].$$

پایدار و غیرعلّی (به‌شرط $N>0$)

That is, the output $y[n]$ is a *weighted average* of the $(N + M + 1)$ values of $x[n]$ from $x[n-M]$ through $x[n+N]$, with the weights given by the coefficients b_k.

Systems of this form can be used to meet a broad array of filtering objectives, including frequency- selective filtering.

مثال) فیلتر LP متوسط متحرک

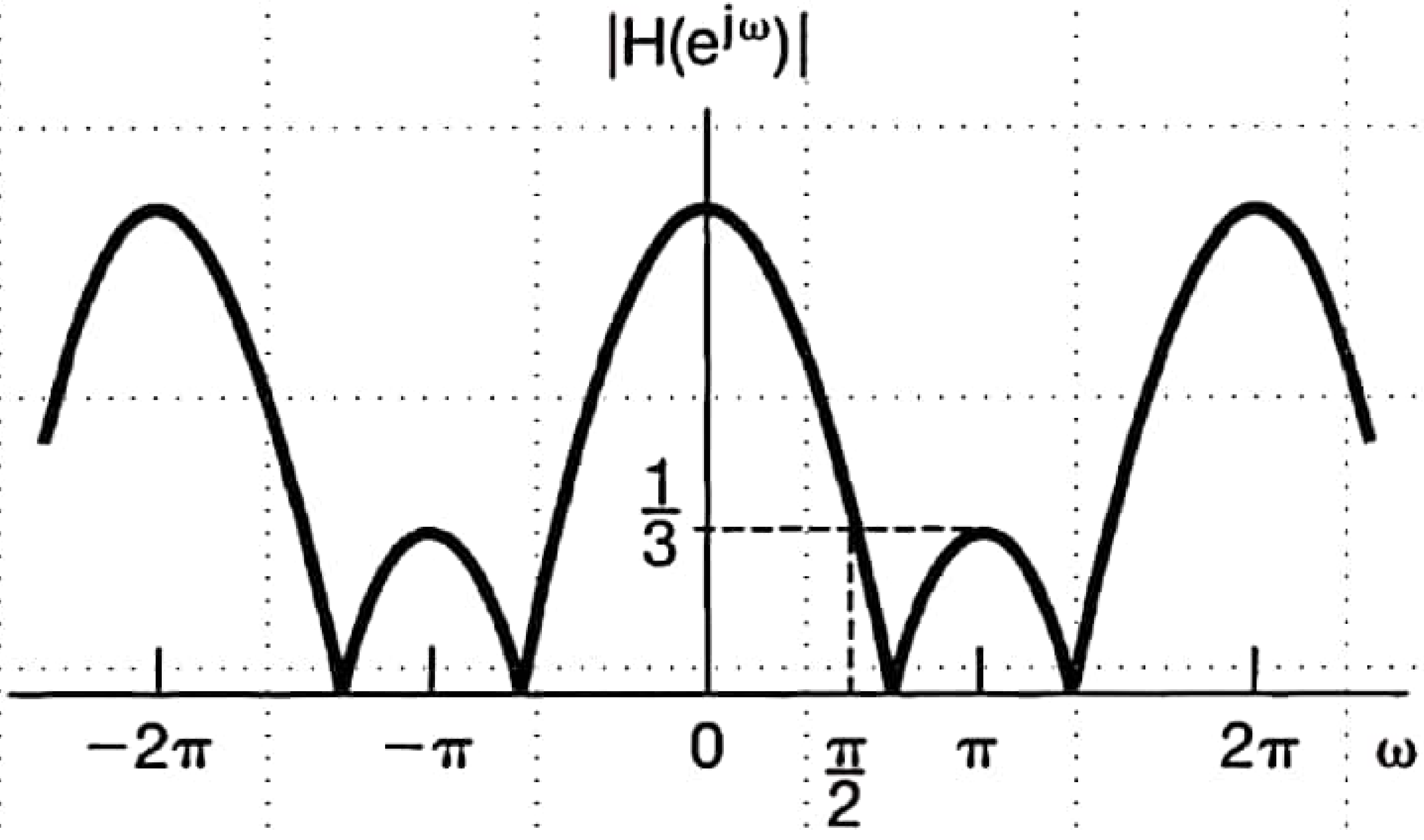
$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1]),$$

Moving Average (MA)

خروجی فیلتر متوسط سه نمونهٔ متوالی حال، گذشته و آینده‌ٔ ورودی آن است.

$$h[n] = \frac{1}{3}[\delta[n+1] + \delta[n] + \delta[n-1]],$$

$$H(e^{j\omega}) = \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3}(1 + 2\cos\omega).$$



Magnitude of the Frequency response of a three-point moving-average lowpass filter.

حالت کلی فیلتر پایین‌گذر متوسط متحرک (MA)

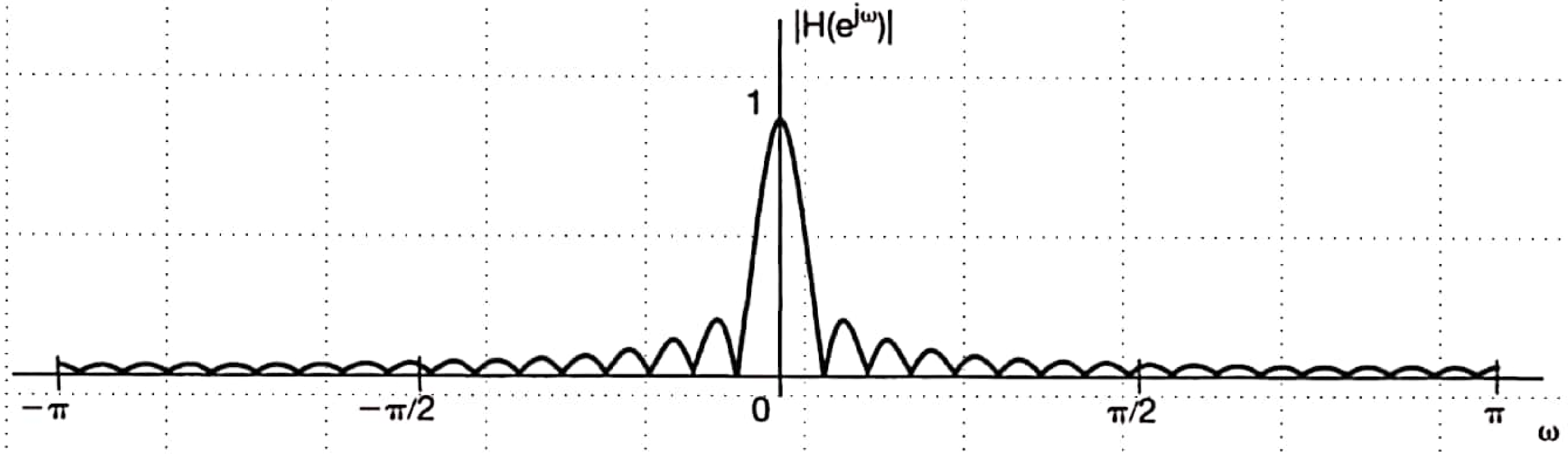
$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k].$$

The corresponding impulse response is a rectangular pulse

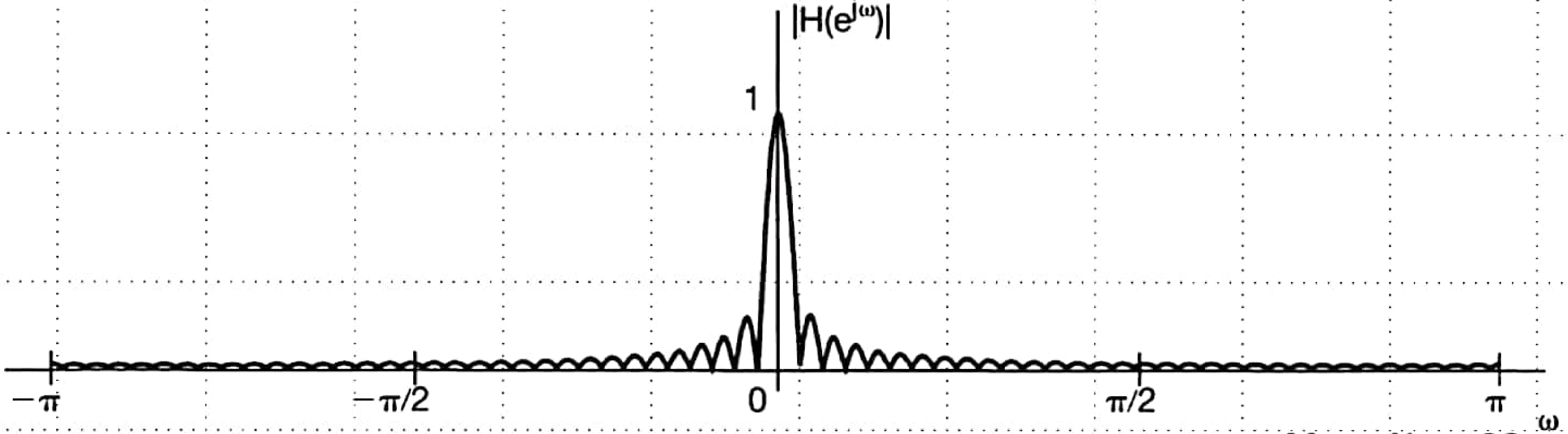
$$\Rightarrow \quad h[n] = 1/(N+M+1) \quad \text{for } -N \le n \le M, \text{ and } h[n] = 0 \text{ otherwise}$$

The filter's frequency response is $H(e^{j\omega}) = \dfrac{1}{N+M+1}\displaystyle\sum_{k=-N}^{M} e^{-j\omega k}.$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}.$$



Magnitude of the frequency response for the lowpass moving- average filter $M = N = 16$;



Magnitude of the frequency response for the lowpass moving- average filter $M = N = 32$.

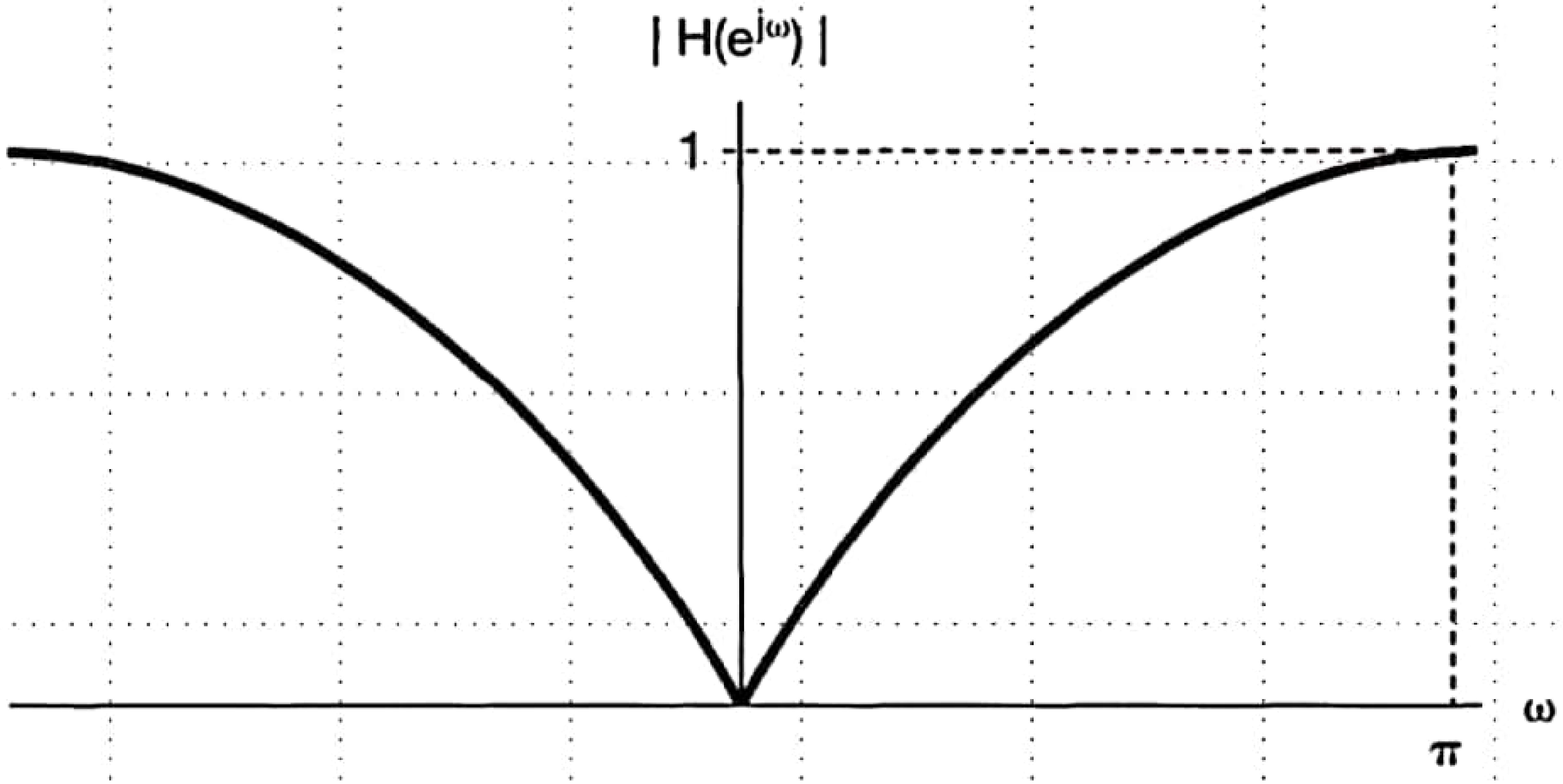
$$y[n] = \frac{x[n] - x[n-1]}{2}.$$

مثال) فیلتر HP غیربازگشتی

For input signals that are approximately constant, the value of $y[n]$ is close to zero. For input signals that vary greatly from sample to sample, the values of $y[n]$ can be expected to have larger amplitude. Thus, the system approximates a highpass filtering operation, attenuating slowly varying low-frequency components and passing rapidly varying higher frequency components with little attenuation.

$$\Longrightarrow h[n] = \tfrac{1}{2}\{\delta[n] - \delta[n-1]\} \Longrightarrow H(e^{j\omega}) = \frac{1}{2}[1 - e^{-j\omega}] = je^{j\omega/2}\sin(\omega/2).$$

we have plotted the magnitude of $H(e^{j\omega})$, showing that this simple system approximates a highpass filter, albeit one with a very gradual transition from pass- band to stopband.



Frequency response of a simple highpass filter.