INTRODUCTION TO INFINITE SERIES

INFINITE SERIES

Sequence:

If a set of real numbers $u_1, u_2, \dots\dots, u_n$ occur according to some definite rule, then it is called a sequence denoted by $\{S_n\} = \{u_1, u_2, \dots\dots, u_n\}$ if n is finite

Or $\{S_n\} = \{u_1, u_2, \dots\dots, u_n, \dots\dots.\}$ if n is infinite.

Series:

$u_1 + u_2 + \dots\dots + u_n$ is called a series and is denoted by $S_n = \sum_{k=1}^{n} u_k$

Infinite Series:

If the number of terms in the series is infinitely large, then it is called infinite series and is denoted by $\sum u_n = u_1 + u_2 + \dots\dots + u_n + \dots\dots$ and the sum of its first n terms be denoted by $S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \dots\dots + u_n$.

Convergence:

An infinite series $\sum u_n$ is said to be convergent if $\lim_{n\to\infty} S_n = k$, a definite unique number.

Example: $1 + \frac{1}{2} + \frac{1}{4} + \dots\dots..$

$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots\dots.. + \frac{1}{2^{n-1}} = \frac{\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)} = 2$, finite.

Therefore given series is convergent.

Divergence:

$\lim_{n\to\infty} S_n$ tends to either ∞ or $-\infty$ then the infinite series $\sum u_n$ is said to be divergent.

Example: $\sum u_n = 1 + 2 + 3 + \dots\dots..$

$$S_n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = \infty$$

Therefore $\sum u_n$ is divergent.

Oscillatory Series:

If $\lim_{n\to\infty} S_n$ tends to more than one limit either finite or infinite, then the infinite series $\sum u_n$ is said to be oscillatory series.

Example:1. $\sum u_n = 1 - 1 + 1 - 1 + \dots\dots..:$

$$S_n = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Therefore series is oscillatory.

2. $\sum u_n = 1 + (-3) + (-3)^2 + \dots\dots..$

$$S_n = \frac{1-(-1)^n 3^n}{1+3}$$

$$\lim_{n\to\infty} S_n = \begin{cases} \infty, & n \text{ is odd} \\ -\infty, & n \text{ is even} \end{cases}$$

Properties of infinite series:

1. The convergence or divergence of an infinite series remains unaltered on multiplication of each term by $c \neq o$.
2. The convergence or divergence of an infinite series remains unaltered by addition or removal of a finite number of its terms.

Positive term series:

An infinite series in which all the terms after some particular term are positive is called a positive term series.

Geometric Series test:

The series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \ldots\ldots + r^n + \ldots\ldots$

a. Converges if $|r| < 1$
b. Diverges if $r \geq 1$
c. Oscillates finitely if $r = -1$ and oscillates infinitely if $r < -1$

Proof:

Let S_n be the partial sum of $\sum_{n=0}^{\infty} r^n$.

$$S_n = 1 + r + r^2 + \ldots + r^{n-1}$$

Case 1: $|r| < 1$ i.e. $-1 < r < 1$

$$S_n = \frac{1-r^n}{1-r}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-r}$$

Therefore the series is convergent.

Case 2i: $r > 1$ i.e. $\lim_{n\to\infty} r^n = \infty$

$$S_n = \frac{r^n - 1}{r - 1}$$

$$\lim_{n\to\infty} S_n = \infty$$

Therefore the series is divergent.

Case 2ii: $r = 1$, $S_n = 1 + 1 + 1 + 1 + \cdots \ldots. +1 = n$
$\lim_{n\to\infty} S_n = \infty$. Therefore the series is divergent.

Case 3i: $r < -1$ i.e. $Let\ r = -m$

$$S_n = \frac{1-r^n}{1-r} = \frac{1-(-1)^n m^n}{1+m}$$

$$\lim_{n\to\infty} S_n = \begin{cases} \infty, & n\ is\ odd \\ -\infty, & n\ is\ even \end{cases}$$

Therefore the series is oscillatory.

Case 3ii: $r = -1$

i.e. $S_n = 1 - 1 + 1 - 1 + \ \ldots \ldots$

$$\lim_{n\to\infty} S_n = \begin{cases} 1, & n\ is\ odd \\ 0, & n\ is\ even \end{cases}$$

Therefore the series is oscillatory.

Note: If a series in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms are negative.

Integral Test:

A positive term series $f(1) + f(2) + \ \ldots \ldots . + f(n) + \ \ldots \ldots$ Where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^\infty f(x)dx$ is finite or infinite.

p-series or Harmonic series test:

A positive term series $\sum u_n = \sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ \ldots \ldots + \frac{1}{n^p} + \ \ldots \ldots$.is

i) Convergent if $p > 1$
ii) Divergent if $p \leq 1$

Proof:

Let $f(x) = \frac{1}{x^p}$

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^p} dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty, For\ p \neq 1$$

$$= \begin{cases} \infty, & if -p+1 > 0 \\ \frac{1}{p-1}, & if -p+1 < 0 \end{cases}$$

$$= \begin{cases} \infty, & if\ \ p < 1 \\ \frac{1}{p-1}, & if\ \ p > 1 \end{cases}$$

When $p = 1$, $\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x} dx = [logx]_1^\infty = \infty$

Thus $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem:

Let $\sum u_n$ be a positive term series. If $\sum u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$.

Proof:

If $\sum u_n$ is convergent then $\lim_{n\to\infty} S_n = k$.

$$u_n = (u_1 + u_2 + \cdots \ldots + u_n) - (u_1 + u_2 + \cdots \ldots + u_{n-1})$$

$$= S_n - S_{n-1}$$

$$\lim_{n\to\infty} S_{n-1} = k$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= k - k = 0$$

Note:

Converse need not be always true. i.e. Even if $\lim_{n\to\infty} u_n = 0$, then $\sum u_n$ need not be convergent.

Example 1: $\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \ldots \ldots$

$\sum u_n = \frac{1}{n}$ is divergent by integral test. But $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence $\lim_{n\to\infty} u_n = 0$ is a necessary condition but not a sufficient condition for convergence of $\sum u_n$.

Example 2

Test the series for convergence, $\sum_{n=2}^{\infty} \frac{1}{nlogn}$

Solution: Consider $\int_2^{\infty} \frac{1}{nlogn} dn = [log(logn)]_2^{\infty} = \infty$

Therefore $\sum u_n$ is divergent by Integral test.

Example 2

Test the series for convergence, $\sum ne^{-n^2}$

Solution: Let $x^2 = t$. Then $2x\, dx = dt$

$$\int_1^{\infty} xe^{-x^2} dx = \int_1^{\infty} \frac{e^{-t}}{2} dt = \left[\frac{e^{-t}}{-2}\right]_1^{\infty} = \frac{1}{2e}$$

Therefore $\sum u_n$ is convergent.

Comparison test:

1. Let $\sum u_n$ and $\sum v_n$ be two positive term series. If
    a. $\sum v_n$ is convergent
    b. $u_n \leq v_n, \ \forall\, n$

    Then $\sum u_n$ is also convergent.

    That is if a larger series converges then smaller also converge.

2. Let $\sum u_n$ and $\sum v_n$ be two positive term series. If
   c. $\sum v_n$ is divergent
   d. $u_n \geq v_n, \ \forall\, n$

   Then $\sum u_n$ is also divergent.

   That is if a smaller series diverges then larger also diverges.

Example 2

Test the series for convergence, $\sum_{n=2}^{\infty} \frac{1}{logn}$

Solution:

Let $u_n = \frac{1}{\log n}$ and $v_n = \frac{1}{n}$

$$\log n < n$$
$$\frac{1}{\log n} > \frac{1}{n}$$
$$u_n > v_n$$

But $\sum v_n = \sum \frac{1}{n}$ is a p-series with $p = 1$.

Therefore $\sum v_n$ is divergent.

By comparison test $\sum u_n$ is also divergent.

Example 2

Test the series for convergence, $\sum \frac{1}{2^n+1}$

Solution:

Let $u_n = \frac{1}{2^n+1}$ and $v_n = \frac{1}{2^n}$

$$2^n < 2^n + 1$$
$$\frac{1}{2^n} > \frac{1}{2^n + 1}$$
$$v_n > u_n$$

But $\sum v_n = \sum \frac{1}{2^n}$ is a geometric series with $r = \frac{1}{2} < 1$.

Therefore $\sum v_n$ is convergent.

By comparision test $\sum u_n$ is also convergent.

Another form of comparison test is

Limit test

Statement: If $\sum u_n$ and $\sum v_n$ be two positive term series such that $\lim_{n\to\infty}\frac{u_n}{v_n} = k\ (\neq 0)$. Then $\sum u_n$ and $\sum v_n$ behave alike.

That is if $\sum u_n$ converges then $\sum v_n$ also converge.

If $\sum u_n$ diverges then $\sum v_n$ also diverge.

Examples 3.

Test the series for convergence, $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots \ldots \ldots$

Solution:

$$u_n = \frac{2n-1}{n(n+1)(n+2)}$$

Choose $v_n = \frac{1}{n^2}$ then $\lim_{n\to\infty}\frac{u_n}{v_n} = 2$

But $\sum v_n = \sum \frac{1}{n^2}$ with $p = 2 > 1$.

Therefore $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Examples 4.

Test the series for convergence, $\sum_{n=1}^{\infty}\left(\sqrt{n^2+1} - n\right)$

Solution:

$$u_n = \left(\sqrt{n^2+1} - n\right)\frac{\left(\sqrt{n^2+1}+n\right)}{\left(\sqrt{n^2+1}+n\right)}$$

$$= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{n\left(\sqrt{1+n^2}+1\right)}$$

Let $\sum v_n = \sum \frac{1}{n}\ (p = 1)$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{2}$$

But $\sum v_n$ is divergent. By limit test $\sum u_n$ is also divergent.

Examples 5.

Test the series for convergence, $\sum \sqrt[3]{n^3+1} - n$

Solution:

$$u_n = \left(n^3+1\right)^{1/3} - n$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$a - b = \frac{a^3 - b^3}{a^2+ab+b^2}$$

$$u_n = \left(n^3+1\right)^{1/3} - n = \frac{n^3+1-n^3}{\left(n^3+1\right)^{2/3} + (n^3+1)^{1/3}n + n^2}$$

$$= \frac{1}{n^2\left[\left(1+\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^3}\right)^{1/3} + 1\right]}$$

Let $\sum v_n = \sum \frac{1}{n^2}$ with $p = 2 > 1$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$$

But $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Example 6.

Test the series for convergence, Solve $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots \ldots \ldots \ldots$

Solution:

$$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}\right)}{n^3\left(\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right)}$$

Let $\sum v_n = \sum \frac{1}{n^{5/2}}$ with $p = \frac{5}{2} > 1$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$$

But $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Example 7

Test the series for convergence, $\sum \frac{1}{n^3} \tan \frac{1}{n}$

Solution: $u_n = \frac{1}{n^3} \tan \frac{1}{n}$

We know that $\lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = 1$

Let $\sum v_n = \sum \frac{1}{n^4}$. Then $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$

But $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Example 8

Test the series for convergence, $\sum \frac{1}{n} - log\left(\frac{n+1}{n}\right)$

Solution: $u_n = \frac{1}{n} - log\left(1 + \frac{1}{n}\right)$

$$= \frac{1}{n} - \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} - \cdots \dots \dots\right]$$

$$= \left[\frac{1}{2n^2} - \frac{1}{6n^3} + \cdots \dots \dots\right]$$

Let $\sum v_n = \sum \frac{1}{n^2}$. Then $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$

But $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Exercises

Test for convergence of the series

1. $\sum_{n=0}^{\infty} \frac{2n^3+1}{4n^5+1}$

2. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots\dots \quad \dots\infty$

3. $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \dots\dots\dots \quad \dots\infty$

4. $\sum \sqrt{\frac{3^n-1}{2^n+1}}$

5. $\sum \frac{n^n}{(n+1)^{n+1}}$

6. $\frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \dots\dots \quad \dots\infty$

DIFFERENT TESTS OF CONVERGENCE

INFINITE SERIES

D'Alembert's Ratio Test: If $\sum u_n$ is a series of positive terms, and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} =$ $l(a\ finite\ value)$

then the series is convergent if $l < 1$, is divergent if $l > 1$ and the test fails if $l = 1$.

If the test fails, one should apply comparison test or the Raabe's test, as given below:

Raabe's Test: If $\sum u_n$ is a series of positive terms, and

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l(finite)$, then the series is convergent if $l > 1$, is divergent if $l < 1$ and the test fails if $l = 1$.

Remark: Ratio test can be applied when (i) v_n does not have the form $1/_{n^p}$

(ii) nth term has x^n, x^{2n} etc.

(iii) nth term has $n!$, $(n+1)!$, $(n!)^2$ ect.

(iv) the number of factors in numerator and denominator increase steadily, ex: $(\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \cdots)$

Example : Test for convergence the series

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots.$$

>> The given series is of the form $\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^3}{4!} + \ldots$ whose nth term is $u_n = \frac{n^2}{n!}$.

Therefore $u_{n+1} = \frac{(n+1)^2}{n+1!}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{n+1!}\,\frac{n!}{n^2} = \frac{(n+1)^2}{n^2}\cdot\frac{n!}{(n+1)(n!)} = \frac{n+1}{n^2}$$

Therefore $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n^2}\right) = \lim\limits_{n\to\infty} \left(\dfrac{1}{n}+\dfrac{1}{n^2}\right) = 0 < 1$

Therefore by ratio test, $\Sigma\, u_n$ is convergent.

Example : Discuss the nature of the series

$$\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \ldots.$$

>> $u_n = \dfrac{x^n}{n(n+1)}$

Therefore $u_{n+1} = \dfrac{x^{n+1}}{(n+1)(\overline{n+1}+1)} = \dfrac{x^{n+1}}{(n+1)(n+2)}$

Now $\dfrac{u_{n+1}}{u_n} = \dfrac{x^{n+1}}{(n+1)(n+2)} \cdot \dfrac{n(n+1)}{x^n} = \dfrac{n}{n+2}\,x$

Therefore $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \dfrac{n}{n+2}\,x = \lim\limits_{n\to\infty} \dfrac{1}{(1+2/n)}\,x = x$

Therefore by D'Alembert's ratio test $\Sigma\, u_n$ is $\begin{cases} \textit{convergent} \text{ if } x < 1 \\ \text{divergent} \quad \text{if } x > 1 \end{cases}$

And the test fails if $x = 1$

But when $x = 1$, $u_n = \dfrac{1^n}{n(n+1)} = \dfrac{1}{n(n+1)} = \dfrac{1}{n^2+n}$

u_n is of order $1/n^2$ ($p = 2 > 1$) and hence $\Sigma\, u_n$ is convergent (when $x = 1$). Hence we conclude that $\Sigma\, u_n$ is convergent $x \le 1$ and divergent if $x > 1$

Example : Find the nature of series $1 + \dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \ldots.$

>> Omitting the first term, the given series can be written in the form

$\frac{x^1}{1^2+1} + \frac{x^2}{2^2+1} + \frac{x^3}{3^2+1} + \ldots$ so that $u_n = \frac{x^n}{n^2+1}$

Therefore $u_{n+1} = \frac{x^{n+1}}{n^2+2n+2} \cdot \frac{n^2+1}{n^2+2n+2}$ x = $\lim_{n\to\infty} \frac{n^2(1+1/n^2)}{n^2(1+2/n+2/n^2)}$ .x

That is, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$

Hence by ratio test Σ u_n is $\begin{cases} \textit{convergent} \text{ if } x<1 \\ \text{divergent} \quad \text{if } x>1 \end{cases}$

and the test fails if x = 1.

But when x = 1, $u_n = \frac{1^n}{n^2+1} = \frac{1}{n^2+1}$ is of order $\frac{1}{n^2}$ (p = 2 > 1)

Therefore Σ u_n is convergent if x ≤ 1 and divergent if x > 1.

Example: Find the nature of the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \ldots$

>> omitting the first term, the general term of the series is given by $u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$

Therefore $u_{n+1} = \frac{x^2(n+1)}{(n+1+2)\sqrt{(n+1)+1}} = \frac{x^{2n+2}}{(n+3)\sqrt{n+2}}$

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n+2}}{(n+3)\sqrt{n+2}} \; \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \frac{n+2}{n+3}\sqrt{\frac{n+1}{n+2}}x^2 = \frac{\sqrt{(n+2)(n+1)}}{(n+3)}\; x^2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\sqrt{n(1+2/n)n(1+1/n)}}{n(1+3/n)} .\, x^2 = x^2$$

Hence by ratio test Σ u_n is $\begin{cases} \textit{convergent} \text{ if } x^2 < 1 \\ \text{divergent} \quad \text{if } x^2 > 1 \end{cases}$

and the fails if $x^2 = 1$.

When $x^2 = 1$, $u_n = \dfrac{(1)^n}{(n+2)\sqrt{n+1}} = \dfrac{1}{(n+2)\sqrt{n+1}}$

u_n is of order $1/n^{3/2}$ ($p = 3/2 > 1$) and hence Σ u_n is convergent.

Therefore Σ u_n is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

Example : Discus the convergence of the series

$$x + \frac{x^3}{2.3} + \frac{3}{2.4} + \frac{x^5}{5} + \frac{3.5}{2.4.6} \cdot \frac{x^7}{7} + \ldots \ (x > 0)$$

>> We shall write the given series in the form

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \ldots.$$

Now, omitting the first term we have

$$u_n = \frac{1.3.5...(2n-1)}{2.4.6...2n} \cdot \frac{x^{2n+1}}{2n+1}$$

$$u_{n+1} = \frac{1.3.5...[2(n+1)-1]}{2.4.6...2(n+1)} \cdot \frac{x^{2(n+1)+1}}{2(n+1)+1}$$

That is, $u_{n+1} = \dfrac{1.3.5...(2n+1)}{2.4.6....(2n+1)} \cdot \dfrac{x^{2n+3}}{2n+3}$

That is, $u_{n+1} = \dfrac{1.3.5...(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)} \cdot \dfrac{x^{2n+3}}{2n+3}$

Therefore $\dfrac{u_{n+1}}{u_n} = \dfrac{1.3.5...(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)} \cdot \dfrac{x^{2n+3}}{2n+3} \times \dfrac{2.4.6...2n}{1.3.5...(2n-1)} \cdot \dfrac{2n+1}{x^{2n+1}}$

That is, $\dfrac{u_{n+1}}{u_n} = \dfrac{(2n+1)(2n+1)x^2}{(2n+2)(2n+3)}$

Therefore $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n(2+1/n)n(2+1/n)x^2}{n(2+2/n)n(2+3/n)} = x^2$

Hence by ratio test, Σ u_n is $\begin{cases} \textit{convergent} \text{ if } x^2 < 1 \\ \text{divergent} \quad \text{if } x^2 > 1 \end{cases}$

And the test fails if $x^2 = 1$

When $x^2 = 1$, $\frac{u_{n+1}}{u_n} = \frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}$ and we shall apply Raabe's test.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(4n^2+10n+6)-(4n^2+4n+1)}{(2n+1)^2}\right]$$

$$= \lim_{n\to\infty} n\left(\frac{6n+5}{(2n+1)^2}\right) = \lim_{n\to\infty} \frac{n^2(6+5/n)}{n^2(2+1/n)^2} \; \frac{6}{4} = \frac{3}{2} > 1$$

Therefore Σ u_n is convergent (when $x^2 = 1$) by Rabbe's test.

Hence we conclude that, Σ u_n is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

Example : Examine the convergence of

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots + \frac{2^{n+1}-2}{2^{n+1}+1}x^n + \ldots.$$

\>> $u_n = \frac{2^{n+1}-2}{2^{n+1}+1}x^n$.

Therefore $u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1}x^{n+1}$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+2}-2}{2^{n+2}+1}x^{n+1} \times \frac{2^{n+1}+1}{2^{n+1}-2} \cdot \frac{1}{x^n}$$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+2}(1-2/2^{n+2})}{2^{n+2}(1+1/2^{n+2})} .x. \frac{2^{n+1}(1+1/2^{n+1})}{2^{n+1}(1-2/2^{n+1})}$$

$$= \frac{(1-1/2^{n+1})}{(1+1/2^{n+2})} .x . \frac{(1+1/2^{n+1})}{(1-1/2^{n})}$$

Therefore $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{(1-0)}{(1+0)} . x . \frac{(1+0)}{(1-0)} = x.$

Therefore by ratio test Σ u_n is $\begin{cases} convergent \text{ if } x<1 \\ \text{divergent} \quad \text{if } x>1 \end{cases}$ and the test fails if x = 1.

When x = 1, $u_n = \frac{2^{n+1}-2}{2^{n+1}+1}$

Therefore $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^{n+1}(1-1/2^{n})}{2^{n+1}(1+1/2^{n+1})} = 1$

Since $\lim_{n\to\infty} u_n = 1 \neq 0$, Σ u_n is divergent (when x = 1)

Hence Σ u_n is convergent if x < 1 and divergent if x $\geq$1.

Example : test for convergence of the infinite series

$$1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots$$

>> the first term of the given series can be written as $1!/1^1$ so that we have,

$u_n = \frac{n!}{n^n}$ and $u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)(n!)}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n}$

Therefore $\frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)^n} . \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \frac{n^n}{n^n(1+1/n)^n}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$$

Hence by ratio test Σ u_n is convergent.

Cauchy's Root Test: If $\sum u_n$ is a series of positive terms, and

$\boldsymbol{\lim_{n\to\infty}(u_n)^{1/n} = l\,(finite)},$

then, the series converges if $l < 1$, diverges if $l > 1$ and fails if $l = 1$.

Remark: Root test is useful when the terms of the series are of the form $u_n = [f(n)]^{g(n)}$.

We can note : (i) $\lim_{n\to\infty} n^{1/n} = 1$

(ii) $\lim_{n\to\infty}(1 + {}^{1}/{}_{n})^{1/n} = e$

(iii)) $\lim_{n\to\infty}(1 + {}^{x}/{}_{n})^{1/n} = e^x$

Example : Test for convergence $\sum_{n=1}^{\infty}\left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

>> $u_n = \left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

Therefore $(u_n)^{1/n} = \left\{\left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}\right\}^{1/n}$

$$= \left[1+\frac{1}{\sqrt{n}}\right]^{-n^{1/2}} = \left[1+\frac{1}{\sqrt{n}}\right]^{-\sqrt{n}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left[1+\frac{1}{\sqrt{n}}\right]^{-\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1.$$

Therefore as $n \to \infty$, $\sqrt{n}$ also $\to \infty$

Therefore by Cauchy's root test, Σu_n is convergent.

Example : Test for convergence $\sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^{n^2}$

>> $u_n = \left(1-\frac{3}{n}\right)^{n^2}$

Therefore $(u_n)^{1/n} = \left[\left(1-\frac{3}{n}\right)^{n^2}\right]^{1/n} = \left(1-\frac{3}{n}\right)^{n}$

$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(1+\frac{-3}{4}\right)^{n} = e^{-3}$.

Therefore $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$

That is, $\lim_{n\to\infty}(u_n)^{1/n} = \frac{1}{e^3} < 1$, therefore e = 2.7

Hence by Cauchy's root test, $\Sigma\, u_n$ is convergent.

Example : Find the nature of the series $\sum_{n=1}^{\infty}\left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

>> $u_n = \left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

Therefore $(u_n)^{1/n} = \left\{\left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}\right\}^{1/n}$

$$= \left[1+\frac{1}{\sqrt{n}}\right]^{-n^{1/2}} = \left[1+\frac{1}{\sqrt{n}}\right]^{-\sqrt{n}}$$

$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[1+\frac{1}{\sqrt{n}}\right]^{-\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1, \text{ since as n} \to \infty,\ \sqrt{n} \text{ also} \to \infty$$

Therefore by Cauchy's root rest, $\Sigma\, u_n$ is convergent.

Example : Test for convergence $\sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^{n^2}$

>> $u_n = \left(1-\frac{3}{n}\right)^{n^2}$

Therefore $(u_n)^{1/n} = \left[\left(1-\frac{3}{n}\right)^{n^2}\right]^{1/n} = \left(1-\frac{3}{n}\right)^{n}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(1+\frac{-3}{n}\right)^{n} = e^{-3}, \text{ since } \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{n} = e^{x}$$

That is, $\lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{e^3} < 1$, since $e = 2.7$.

Hence by Cauchy's root test, $\Sigma\, u_n$ is convergent.

ALTERNATING SERIES AND POWER SERIES

ALTERNATING SERIES

A series in which the terms are alternatively positive or negative is called an alternating series.

i.e., $u_1 - u_2 + u_3 - u_4 + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

LEBINITZ'S SERIES

An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges if

(i) each term is numerically less than its preceding term

(ii) $\lim_{n\to\infty} u_n = 0$

Note: If $\lim_{n\to\infty} u_n \neq 0$ then the given series is oscillatory.

Q Test the convergence of $\frac{1}{6}$ - $\frac{1}{13}$ $+\frac{1}{20}$ - $\frac{1}{27}$ $+ \ldots$

Solution: Here $u_n = \frac{1}{7n-1}$

then $u_{n+1} = \frac{1}{7(n=1)-1} = \frac{1}{7n+6}$

therefore, $u_n - u_{n+1} = \frac{1}{7n-1}$ - $\frac{1}{7n+6}$

$$= \frac{(7n+6)-(7n-1)}{(7n-1)(7n+6)} = \frac{7}{(7n-1)(7n+6)} > 0$$

That is, $u_n - u_{n+1} > 0, \Rightarrow u_n > u_{n+1}$

Also, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{7n-1} = \lim_{n\to\infty} \frac{1}{n} \frac{1}{(7-1/n)} = 0$

Therefore by Leibnitz test the given alternating series is convergent .

Q Find the nature of the series

$$\left(1-\frac{1}{\log 2}\right) - \left(1-\frac{1}{\log 3}\right) + \left(1-\frac{1}{\log 4}\right) - \left(1-\frac{1}{\log 5}\right) + \ldots$$

Solution: Here $u_n = 1 - \dfrac{1}{\log(n+1)}$ then $u_{n+1} = 1 - \dfrac{1}{\log(n+2)}$

Therefore, $u_n - u_{n+1} = \dfrac{1}{\log(n+2)} - \dfrac{1}{\log(n+1)}$

$= \dfrac{\log(n+1) - \log(n+2)}{\log(n+2)\log(n+1)} < 0.$

Since (n + 1) < (n + 2)

$u_n - u_{n+1} < 0 \Rightarrow u_n < u_{n+1}$

further $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} 1 - \left[\dfrac{1}{\log(n+1)}\right] = 1 - 0 = 1 \neq 0.$

Both the conditions of the Leibnitz test are not satisfied. So, we conclude that the series oscillates between $-\infty$ and $+\infty$.

Problems:

Test the convergence of the following series

$$(i)\, 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots.$$
$$(ii)\, \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots.$$
$$(iii)\, \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n+1}$$
$$(iv)\, \sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)} \quad for\ 0 < x < 1$$
$$(v)\, \sum \frac{1}{\sqrt{1+n^2}}$$

ABSOLUTELY & CONDITIONALLY CONVERGENT SERIES

An alternating series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$ is said to be absolutely convergent if the positive series $|a_1| + |a_2| + |a_3| + |a_4| + \ldots = \sum |a_n|$ is convergent.

An alternating series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$ is said to be conditionally convergent if

(i) $\sum |a_n|$ is divergent

(ii) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent

Theorem: An absolutely convergent series is convergent. The converse need not be true.

Proof: Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$ be an absolutely convergent series then $\sum |a_n|$ is convergent.

We know, $a_1 + a_2 + a_3 + a_4 + ... \le |a_1| + |a_2| + |a_3| + |a_4| + ...$

By comparison test, $\sum_{n=1}^{\infty} a_n$ is convergent.

Q. Show that each of the following series also converges absolutely

(i) $\Sigma\, a_n^2$; (ii) $\Sigma\, \dfrac{a_n^2}{1+a_n^2}$; (iii) $\Sigma\, \dfrac{a_n}{1+a_n}$

Solution: (i) Since $\Sigma\, a_n$ converges, we have $a_n \to 0$ as $n \to \infty$. Hence for some positive integer N, $|a_n| < 1$ for all $n \ge N$. This gives $a_n^2 \le |a_n|$ for all $n \ge N$. As $\Sigma\, |a_n|$ is convergent it follows $\Sigma\, a_n^2$ converges.

(as Σa_n^2 is a positive termed series, convergence and absolute convergence are identical).

(ii) As $1 + a_n^2 \ge 1$ for all n, we get $\dfrac{a_n^2}{1+a_n^2} \le a_n^2$

the convergence of $\Sigma\, a_n^2$ implies the convergence of $\Sigma\, \dfrac{a_n^2}{1+a_n^2}$.

(iii) $\left|\dfrac{a_n}{1+a_n}\right| = \dfrac{|a_n|}{|1+a_n|} < \dfrac{|a_n|}{1-|a_n|}$.

As $\Sigma\, |a_n|$ converges, $|a_n| \to 0$ as $n \to \infty$. Hence for some positive integer N, we have $|a_n| < \frac{1}{2}$ for all $n \ge N$.

This gives $\left|\dfrac{a_n}{1+a_n}\right| < 2|a_n|$ for all $n \ge N$.

Now, by comparison test, $\Sigma \left|\dfrac{a_n}{1+a_n}\right|$ converges.

That is, $\Sigma \dfrac{a_n}{1+a_n}$ converges absolutely.

Q. Test the convergence $\dfrac{1}{2^3}-\dfrac{1}{3^3}(1+2)+\dfrac{1}{4^3}(1+2+3)+\dfrac{1}{5^3}(1+2+3+4)+...\infty$

Solution: Here $a_n=(-1)^{n-1}\dfrac{(1+2+...+n)}{(n+1)^3}=(-1)^{n-1}\dfrac{n}{2(n+1)^2}=(-1)^{n-1}u_n$

then $u_n-u_{n-1}=\dfrac{1}{2}\dfrac{n^2+n-1}{(n+1)^2(n+2)^2}>0$

i.e., $u_{n+1}<u_n$ & $\lim\limits_{n\to\infty}u_n=0$

Thus by Lebinitz rule, $\sum a_n$ is convergent.

Also, $|a_n|=\dfrac{1}{2}\dfrac{n}{n^2+1}$. Take $v_n=\dfrac{1}{n}$

Then $\lim\limits_{n\to\infty}\dfrac{|a_n|}{v_n}=\dfrac{1}{2}\neq 0$

Since is $\sum v_n$ divergent, therefore $\sum|a_n|$ is also divergent.

Thus the given series is conditionally convergent.

POWER SERIES

A series of the form $a_0+a_1x+a_2x^2+...+a_nx^n+...-------------(i)$ where the a_i's are independent of x, is called a power series in x. Such a series may converge for some or all values of x.

INTERVAL OF CONVERGENCE

In the power series (i) we have $u_n=a_nx^n$

Therefore, $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)x$

If $\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)=l$, then by ratio test, the series (i) converges when $|x|<\dfrac{1}{l}$ and diverges for other values.

Thus the power series (i) has an interval $\frac{-1}{l} < x < \frac{1}{l}$ within which it converges and diverges for values of x outside the interval. Such interval is called the **interval of convergence** of the power series.

Q. Find the interval of convergence of the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - ....\infty$.

Solution: Here $u_n = (-1)^{n-1} \frac{x^n}{n}$ and $u_{n+1} = (-1)^n \frac{x^{n+1}}{n+1}$

Therefore, $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} x \right| = |x|$

By Ratio test the given series converges $|x| < 1$ for and diverges for $|x| > 1$.

When x=1 the series reduces to $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$, which is an alternating series and is convergent.

When x=-1 the series becomes $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...\right)$, which is divergent (by comparison with p-series when p=1)

Hence the interval of convergence is $-1 < x \le 1$.

Q. Show that the series $\sum_{1}^{\infty} (-1)^{n-1} \frac{x^n}{\sqrt{2n+1}}$ is absolutely convergent for | x | < 1, conditionally convergent for x = 1 and divergent for x = -1.

Solution. Here $u_n = (-1)^{n-1} \frac{x^n}{\sqrt{2n+1}}$

Therefore $u_{n+1} = \frac{(-1)^n x^{n+1}}{\sqrt{2n+3}}$

$$\lim_{n\to\infty} \left| \frac{u_{N+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n x^{n+1} \sqrt{2n+1}}{\sqrt{2n+3}(-1)^{n-1} x^n} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \sqrt{\frac{2n+1}{2n+3}} x \right|$$

$$= \lim_{n\to\infty} \left| (-1) \sqrt{\frac{n(2+1/n)}{n(2+3/n)}} x \right| = | x |$$

Therefore by generalized D' Alembert's test the series is absolutely convergent if | x | < 1, not convergent if | x | > 1 and the test fails if | x | = 1.

Now for |x | = 1, x can be +1 or – 1.

If x = 1 the given series becomes $\frac{1}{\sqrt{3}}$ - $\frac{1}{\sqrt{5}}$+ $\frac{1}{\sqrt{7}}$ - $\frac{1}{\sqrt{9}}$ + …

Here $u_n = \frac{1}{\sqrt{2n+1}}$, $u_{n+1} = \frac{1}{\sqrt{2n+3}}$

But 2n + 1< 2n + 3 $\Rightarrow u_n > u_{n+1}$

Also $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{2n+3}} = 0$

Therefore by Leibnitz test the series is convergence when x = 1.

But the absolute series $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \ldots$ whose general term is $u_n = \frac{1}{\sqrt{2n+1}}$ and is of order $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ and hence Σu_n is divergent

Since the alternating series is convergent and the absolute series is divergent when x = 1, the series is conditionally convergent when x = 1.

If x = -1, the series becomes $\frac{-1}{\sqrt{3}}$ - $\frac{1}{\sqrt{5}}$ - $\frac{1}{\sqrt{7}}$ - ….

$= - \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \ldots\right)$ where the series of positive terms is divergent as shown already.

Therefore the given series is divergent when x = -1.

Thus we have established all the results.

Problems:

1. Test the conditional convergence of $(i)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ $\quad (ii)\sum_{n=2}^{\infty}\frac{(-1)^{n-1}n}{n+1}$
2. Prove that $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots.$ is absolutely convergent
3. For what values of x the following series are convergent

   $(i)\, x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots.$

   $(ii)\, x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots.$

   $(iii)\, \frac{x}{1.2} - \frac{x^2}{2.3} + \frac{x^3}{3.4} - \frac{x^4}{4.5} + \ldots.$

   $(iv)\, 3x + 3^4x^4 + 3^9x^9 + \ldots. + 3^{n^2}x^{n^2} + \ldots$

4. Test the nature of convergence $\sum \frac{(-1)^{n-1}}{n\sqrt{n}}$

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