LAPLACE TRANSFORMS

Integral transform is a particular kind of mathematical operator which arises in the analysis of some boundary value and initial value problems of classical Physics. A function g defined by a relation of the form $g(y) = \int_a^b K(x,y) f(x) dx$ is called the integral transform of f(x). The function K(x, y) is called the kernel of the transform. The various types of integral transforms are Laplace transform, Fourier transform, Mellin transform, Hankel transform etc.

Laplace transform is a widely used integral transform in Mathematics with many applications in physics and engineering. It is named after Pierre-Simon Laplace, who introduced the transform in his work on probability theory. The Laplace transform is used for solving differential and integral equations. In physics and engineering it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems.

Definition

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$. The parameter s is a complex number.

L is called the Laplace transformation operator.

Transforms of elementary functions

1. $L\{1\} = \frac{1}{s}, \quad s > 0.$

Proof: $L\{1\} = \int_0^\infty e^{-st}.1 dt = \frac{1}{s}$ if $s>0$

2. $$L\{t^n\} = \begin{cases} \frac{n!}{s^{n+1}} \text{ when } n = 0,1,2,\ldots \\ \text{otherwise } \frac{\Gamma(n+1)}{s^{n+1}} \end{cases}$$

Proof:

$$L\{t^n\}=\int_0^\infty e^{-st}t^n dt=\left[t^n\frac{e^{-st}}{-s}\right]_0^\infty+\frac{n}{s}L\{t^{n-1}\}$$

$$=\frac{n}{s}\frac{(n-1)}{s}L\{t^{n-2}\}=...=\frac{n}{s}\frac{(n-1)}{s}...\frac{1}{s}L\{1\}$$

$$=\frac{n}{s}\frac{(n-1)}{s}...\frac{1}{s}\frac{1}{s}=\begin{cases}\dfrac{n!}{s^{n+1}} \text{ if } n \text{ is a positive integer}\\ \text{otherwise } \dfrac{\Gamma(n+1)}{s^{n+1}}\end{cases}$$

3. $L\{e^{at}\}=\dfrac{1}{s-a},\ \ s>a$

Proof:

$$L\{e^{at}\}=\int_0^\infty e^{-st}e^{at}dt=\int_0^\infty e^{-(s-a)t}dt$$

$$=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty=\frac{1}{s-a},\quad s>a$$

4. $L\{\text{sinat}\}=\dfrac{a}{s^2+a^2},\quad s>0$

Proof:

$$L\{\text{sinat}\}=\int_0^\infty e^{-st}\sin atdt=\frac{\left[e^{-st}\left(-s\sin at-a\cos at\right)\right]_0^\infty}{s^2+a^2}$$

$$=\frac{a}{s^2+a^2},\quad s>0$$

5. $L\{\text{cosat}\}=\dfrac{s}{s^2+a^2},\quad s>0$

Proof:

$$L\{\text{cosat}\}=\int_0^\infty e^{-st}\text{cosatdt}=\frac{\left[e^{-st}\left(\text{-scosat+asinat}\right)\right]_0^\infty}{s^2+a^2}$$

$$=\frac{s}{s^2+a^2},\quad s>0$$

6. $L\{\text{sinha t}\}=\dfrac{a}{s^2-a^2},\quad s>|a|$

Proof:

$$\mathrm{L}\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{a}{s^2-a^2}, \quad s>|a|$$

7. $\mathrm{L}\{\cosh at\} = \frac{\mathrm{s}}{\mathrm{s}^2\text{-}\mathrm{a}^2}, \quad s>|a|$

Proof:

$$\mathrm{L}\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{s}{s^2-a^2}, \quad s>|a|$$

Properties of Laplace transforms

1. Linearity property.

   If a, b, c be any constants and f, g, h be any functions of t then
   $\mathrm{L}\{\mathrm{af}(\mathrm{t})+\mathrm{bg}(\mathrm{t})-\mathrm{ch}(\mathrm{t})\} = \mathrm{aL}\{\mathrm{f}(\mathrm{t})\}+\mathrm{bL}\{\mathrm{g}(\mathrm{t})\}-\mathrm{cL}\{\mathrm{h}(\mathrm{t})\}$
   Because of the above property of L, it is called linear operator.

2. First shifting property.

   If $\mathrm{L}\{\mathrm{f}(\mathrm{t})\} = \mathrm{F(s)}$ then $\mathrm{L}\{\mathrm{e}^{\mathrm{at}}\mathrm{f}(\mathrm{t})\} = \mathrm{F}(\mathrm{s\text{-}a})$.

   Proof:

$$\mathrm{L}\{\mathrm{e}^{\mathrm{at}}\mathrm{f}(\mathrm{t})\} = \int_0^\infty \mathrm{e}^{-\mathrm{st}}\mathrm{e}^{\mathrm{at}}\mathrm{f}(\mathrm{t})\mathrm{dt} = \int_0^\infty \mathrm{e}^{-(\mathrm{s}-\mathrm{a})\mathrm{t}}\mathrm{f}(\mathrm{t})\mathrm{dt}$$

$$= \int_0^\infty \mathrm{e}^{-\mathrm{rt}}\mathrm{f}(\mathrm{t})\mathrm{dt} \quad \text{where } \mathrm{r}=\mathrm{s\text{-}a}.$$

$$= \mathrm{F(r)} = \mathrm{F}(\mathrm{s\text{-}a}).$$

   Application of this property leads to the following results.

1. $L\left\{e^{at}t^{n}\right\}=\begin{cases}\dfrac{n!}{(s\text{-}a)^{n+1}} & \text{when } n=0,1,2,\ldots \\ \text{otherwise} & \dfrac{\Gamma(n+1)}{(s\text{-}a)^{n+1}}\end{cases}$

2. $L\left\{e^{at}\sin bt\right\}=\dfrac{b}{(s\text{-}a)^{2}+b^{2}}$

3. $L\left\{e^{at}\cos bt\right\}=\dfrac{s\text{-}a}{(s\text{-}a)^{2}+b^{2}}$

4. $L\left\{e^{at}\sinh bt\right\}=\dfrac{b}{(s\text{-}a)^{2}-b^{2}}$

5. $L\left\{e^{at}\cosh bt\right\}=\dfrac{s\text{-}a}{(s\text{-}a)^{2}-b^{2}}$

   where in each case s > a.

Examples:

Find the Laplace transform of the following.

1. $\mathrm{Sin}^{3}2t$

Solution:

$$\sin 6t=3\sin 2t-4\sin^{3}2t$$

$$\sin^{3}2t=\frac{3}{4}\sin 2t-\frac{1}{4}\sin 6t$$

$$L\left\{\sin^{3}2t\right\}=\frac{3}{4}L\{\sin 2t\}-\frac{1}{4}L\{\sin 6t\}$$

$$=\frac{3}{4}.\frac{2}{s^{2}+4}-\frac{1}{4}.\frac{6}{s^{2}+36}=\frac{48}{\left(s^{2}+4\right)\left(s^{2}+36\right)}$$

2. $e^{4t}\sin 2t\cos t$

Solution:

$$L\left\{e^{4t}\sin 2t\cos t\right\}=\frac{1}{2}L\left\{e^{4t}(\sin 3t+\sin t)\right\}$$

$$=\frac{1}{2}\left[\frac{3}{(s-4)^{2}+9}+\frac{1}{(s-4)^{2}+1}\right]$$

3. $e^{2t}\cos^2 t$

Solution:

$$L\left\{e^{2t}\cos^2 t\right\}=\frac{1}{2}L\left\{e^{2t}\left(1+\cos 2t\right)\right\}$$
$$=\frac{1}{2}\left[\frac{1}{s-2}+\frac{(s-2)}{(s-2)^2+4}\right]$$

4. $f(t)=\begin{cases}1, & 0<t\le 1\\ t, & 1<t\le 2\\ 0, & t>2\end{cases}$

Solution:

$$\mathrm{L}\{\mathrm{f(t)}\}=\int_0^1 \mathrm{e}^{-st}.1dt+\int_1^2 \mathrm{e}^{-st}.\mathrm{tdt}+\int_2^\infty \mathrm{e}^{-st}.0\mathrm{dt}$$
$$=\frac{1}{s}-\frac{2e^{-2s}}{s}+\frac{e^{-s}}{s^2}-\frac{e^{-2s}}{s^2}$$

Exercises:

Find the Laplace transform of

1. $\sin^5 t$ 2. $\left(1+te^{-t}\right)^3$ 3. $\cosh at\sin at$

4. $f(t)=\begin{cases}\sin t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$ 5. $\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3$

Transforms of integrals

If $\mathrm{L}\{\mathrm{f(t)}\}=\mathrm{F(s)}$ then $\mathrm{L}\left\{\int_0^t \mathrm{f(u)du}\right\}=\frac{1}{\mathrm{s}}\mathrm{F(s)}$

Proof:

Let $\varphi(t)=\int_0^t f(u)du$, then $\varphi'(t)=f(t)$ and $\varphi(0)=0$.

\ $L\{\varphi'(t)\}=sL\{\varphi(t)\}-\varphi(0)$

Hence $L\{\varphi(t)\}=\frac{1}{s}L\{\varphi'(t)\}$

i.e. $L\left\{\int_0^t f(u)du\right\}=\frac{1}{s}F(s)$.

Multiplication by t^n

If $L\{f(t)\}=F(s)$

then $L\{t^n f(t)\}=(-1)^n \frac{d^n}{ds^n}F(s)$, $n=1,2,3,\ldots$

Proof:

We have $\int_0^\infty e^{-st}f(t)dt=F(s)$

Consider $\frac{d}{ds}\left\{\int_0^\infty e^{-st}f(t)dt\right\}=\frac{d}{ds}F(s)$

ByLeibnitz's rule for differentiation under the integral sign

$$\int_0^\infty \frac{\partial}{\partial s}e^{-st}f(t)dt=\frac{d}{ds}F(s)$$

$$\int_0^\infty e^{-st}\left[tf(t)\right]dt=-\frac{d}{ds}F(s)$$

which proves the theorem for n = 1.

Now assume the theorem to true for n = m(say), so that

$$\int_0^\infty e^{-st}\left[t^m f(t)\right]dt=(-1)^m \frac{d^m}{ds^m}F(s)$$

Then $\frac{d}{ds}\left[\int_0^\infty e^{-st}\left[t^m f(t)\right]dt\right]=(-1)^m \frac{d^{m+1}}{ds^{m+1}}F(s)$

By Leibnitz's rule,

$$\int_0^\infty e^{-st}\left[t^{m+1} f(t)\right]dt=(-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}F(s)$$

This shows that if the theorem is true for n = m, it is also true for n = m+1. But it is true for n = 1. Hence it is true for n = 1+1 = 2, n = 2+1 = 3 and so on.

Thus the theorem is true for all positive integral values of n.

Division by t

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty} F(s)ds$

Proof:

We have $F(s) = \int_0^{\infty} e^{-st} f(t)dt$

Consider $\int_s^{\infty} F(s)\,ds = \int_s^{\infty}\left[\int_0^{\infty} e^{-st} f(t)dt\right]ds$

$$= \int_0^{\infty}\int_s^{\infty} e^{-st} f(t)dsdt \qquad \text{(changing the order of integration)}$$

$$= \int_0^{\infty} f(t)\left(\int_s^{\infty} e^{-st} ds\right)dt \qquad (\because \text{t is independent of s})$$

$$= L\left\{\frac{1}{t} f(t)\right\}$$

Examples:

Find the Laplace transform of

1. tcosat

Solution:

$$L\{\text{cosat}\} = \frac{s}{s^2+a^2}$$

$$L\{\text{tcosat}\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2\text{-}a^2}{\left(s^2+a^2\right)^2}$$

2. $te^{-t}\sin 3t$

Solution:

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$\therefore L\{t\sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{\left(s^2+9\right)^2}$$

Now using first shifting property,we get

$$L\{e^{-t}t\sin 3t\} = \frac{6(s+1)}{\left((s+1)^2+9\right)^2} = \frac{6(s+1)}{\left(s^2+2s+10\right)^2}$$

3. $\dfrac{\text{cosat-cosbt}}{\text{t}}$

Solution:

$$\text{L}\{\text{cosat-cosbt}\} = \frac{\text{s}}{\text{s}^2+\text{a}^2} - \frac{\text{s}}{\text{s}^2+\text{b}^2}$$

$$\text{L}\left\{\frac{\text{cosat-cosbt}}{\text{t}}\right\} = \int_s^\infty \left(\frac{\text{s}}{\text{s}^2+\text{a}^2} - \frac{\text{s}}{\text{s}^2+\text{b}^2}\right)\text{ds}$$

$$= \log\left(\frac{\text{s}^2+\text{b}^2}{\text{s}^2+\text{a}^2}\right)^{1/2}$$

Exercises:

Find the Laplace transforms of the following functions.

1. tsin^2t 2. $\text{te}^{2\text{t}}\text{sin3t}$ 3. $\dfrac{\text{e}^{-\text{t}}\text{sint}}{\text{t}}$

4. $\dfrac{\text{e}^{\text{at}}-\text{cosbt}}{\text{t}}$ 5. $\dfrac{1-\cos 3\text{t}}{\text{t}}$

Periodic functions:

Suppose that the function $F(t)$ is periodic with period ω. Then its Laplace transform is given by

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt = \sum_{n=0}^{\infty}\int_{n\omega}^{(n+1)\omega} e^{-st}F(t)dt.$$

Let us put $t = n\omega + \beta$. Then $L\{F(t)\} = \sum_{n=0}^{\infty}\int_0^{\omega}\exp(-sn\omega - s\beta)F(\beta+n\omega)d\beta.$

But $F(\beta + n\omega) = F(\beta)$, we get

$$L\{F(t)\} = \sum_{n=0}^{\infty} \exp(-sn\omega) \int_0^{\omega} \exp(-s\beta) F(\beta) d\beta = \sum_{n=0}^{\infty} \left[\exp(-s\omega)\right]^n \int_0^{\omega} \exp(-s\beta) F(\beta) d\beta$$

$$= \frac{1}{1-e^{-sw}} \int_0^{\omega} \exp(-s\beta) F(\beta) d\beta.$$

Theorem

If $F(t)$ has a Laplace Transform and if $F(t+\omega) = F(t), L\{F(t)\} = \dfrac{\int_0^{\omega} e^{-s\beta} F(\beta) d\beta}{1-e^{-sw}}$.

Example (a)

Find the transform of the function $\psi(t,c) = \begin{cases} 1, & 0 < t < c \\ 0, & c < t < 2c \end{cases}$; $\psi(t+2c,c) = \psi(t,c)$.

$$L\{\psi(t,c)\} = \frac{\int_0^{c} e^{-s\beta} d\beta}{1-e^{-2cs}} = \frac{1}{s(1+e^{-cs})}.$$

Example (b)

Find the transform of the square-wave function $Q(t,c) = \begin{cases} 1, & 0 < t < c \\ -1, & c < t < 2c \end{cases}$; $Q(t+2c,c) = Q(t,c)$.

$$L\{Q(t,c)\} = \frac{\int_0^{c} e^{-s\beta} d\beta - \int_c^{2c} e^{-s\beta} d\beta}{1-e^{-2cs}} = \frac{(1-e^{-cs})}{s(1+e^{-cs})} = \frac{(1-e^{-cs})e^{cs/2}}{s(1+e^{-cs})e^{cs/2}} = \frac{1}{s}\tanh\frac{cs}{2}.$$

Exercise:

1. Define a triangular-wave function $T(t,c)$ by $T(t,c) = \begin{cases} t, & 0 \le t \le c \\ 2c-t, & c \le t \le 2c \end{cases}$; $T(t+2c,c) = T(t,c)$. Sketch $T(t,c)$ and find its Laplace Transform.

2. Define the function $G(t)$ by $G(t)=e^t, 0\le t<c;\ \mathrm{G}(t+c)=G(t), t\ge 0.$ Sketch the graph of $G(t)$ and find its Laplace Transform.
3. Define the function $S(t)$ by $S(t)=1-t, 0\le t<1;\ \mathrm{S}(t+1)=S(t), t\ge 0.$ Sketch the graph of $S(t)$ and find its Laplace Transform.
4. Sketch a half-wave rectification of the function $\sin\omega t$, as described below, and find its transform. $F(t)=\begin{cases}\sin\omega t, & 0\le t\le\dfrac{\pi}{\omega}\\ 0, & \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{w}\end{cases};\ \mathrm{F}\left(t+\dfrac{2\pi}{w}\right)=F(t).$
5. Find $L\{F(t)\}$ where $F(t)=t$ for $0<t<\omega$ and $F(t+\omega)=F(t)$.

A Step Function:

Applications frequently deal with situations that change abruptly at specified times. We need a notation for a function that will suppress a given term up to a certain value of t and insert the term for all larger t. The function we are about to introduce leads us to a powerful tool for constructing inverse transforms.

Let us define function $\alpha(t)$ by $\alpha(t)=\begin{cases}0, & t<0\\ 1, & t\ge 0\end{cases}.$

The definition says that $\alpha(t)$ is zero when the argument is negative and $\alpha(t)$ is unity when the argument is positive or is zero. It follows that

$$\alpha(t-c)=\begin{cases}0, & t<c\\ 1, & t\ge c\end{cases}.$$

The Laplace Transform of $\alpha(t-c)F(t-c)$ is

$$L\{\alpha(t-c)F(t-c)\}=\int_c^\infty e^{-st}F(t-c)dt.$$

Now put $t-c=v$ in the integral to obtain

$$L\{\alpha(t-c)F(t-c)\}=\int_0^\infty e^{-s(c+v)}F(v)dv=e^{-cs}L\{F(t)\}.$$

Theorem

If $L^{-1}\{f(s)\}=F(t)$, if $c\ge 0$, and if $F(t)$ be assigned values (no matter what ones) for $-c\le t<0,\ L^{-1}\{e^{-cs}f(s)\}=F(t-c)\alpha(t-c).$

Example (a):

Find $L\{y(t)\}$ where $y(t)=\begin{cases} t^2, & 0<t<2 \\ 6, & t>2 \end{cases}$.

$$y(t)=t^2\left[\alpha(t-0)-\alpha(t-2)\right]+6\left[\alpha(t-2)-\alpha(t-\infty)\right]=t^2+(6-t^2)\alpha(t-2)$$

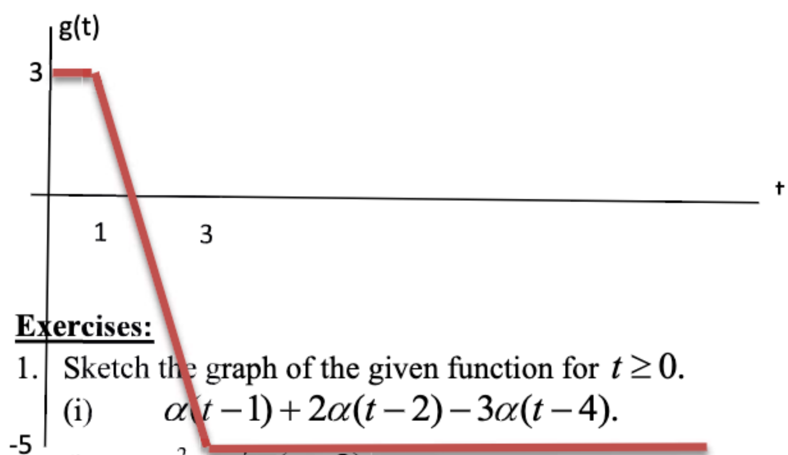
and $L\{y(t)\}=L\{t^2\}+e^{-2s}L\{6-(t+2)^2\}=\dfrac{2}{s^3}+e^{-2s}\left(\dfrac{2}{s}-\dfrac{4}{s^2}-\dfrac{2}{s^3}\right)$.

Example (b) :

Find and sketch a function $g(t)$ for which $g(t)=L^{-1}\left\{\dfrac{3}{s}-\dfrac{4e^{-s}}{s^2}-\dfrac{4e^{3s}}{s^2}\right\}$.

$$g(t)=3-4(t-1)\alpha(t-1)+4(t-3)\alpha(t-3)$$

$$=\begin{cases} 3, & 0\le t<1 \\ 7-4t, & 1\le t<3 \\ -5, & 3\le t \end{cases}$$



Exercises:

1. Sketch the graph of the given function for $t\ge 0$.
   (i) $\alpha(t-1)+2\alpha(t-2)-3\alpha(t-4)$.
   (ii) $t^2-t^2\alpha(t-2)$.
2. Express $F(t)$ in terms of the α function and find $L\{F(t)\}$.
   (i) $F(t)=\begin{cases} 4, & 0<t<2 \\ 2t-1, & t>2 \end{cases}$.
   (ii) $F(t)=\begin{cases} t^2, & 0<t<2 \\ t-1, & 2<t<3 \\ 7, & t>3 \end{cases}$.
   (iii) $F(t)=\begin{cases} \sin 3t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$.

INVERSE LAPLACE TRANSFORMATION

(continuation of previous exercise)

3. Find and sketch an inverse Laplace transform of $\frac{5e^{-3s}}{s} - \frac{e^{-s}}{s}$.

4. Evaluate $L^{-1}\left\{\frac{e^{-4s}}{(s+2)^3}\right\}$.

5. If $F(t)$ is to be continuous for $t \geq 0$ and $F(t) = L^{-1}\left\{\frac{(1-e^{2s})(1-3e^{-2s})}{s^2}\right\}$, evaluate $F(1), F(3), F(5)$.

INVERSE LAPLACE TRANSFORMS

Introduction:

Let L{f(t)} = F(s). Then f(t) is defined as the inverse Laplace transform of F(s) and is denoted by L^{-1} {F(s)}.

Thus L^{-1} F(s) = f(t) ……………..(1)

L^{-1} is known as the inverse laplace transform operator and is such that

$$LL^{-1} = L^{-1}L = 1$$

In the inverse problem (1), F(s) is given (known) and f(t) is to be determined.

Properties of Inverse Laplace transform

For each property on Laplace transform, there is a corresponding property on inverse Laplace transform, which readily follow from the definitions.

1) **Linearity Property**

Let L^{-1} {F(s)} = f(t) and L^{-1} {G(s)} = g(t) and a and b be any two constants. Then

$$L^{-1}[a\,F(s) \pm b\,G(s)] = a\,L^{-1}\{F(s)\} \pm b\,L^{-1}\{G(s)\}$$

2) **Shifting Property**

If L^{-1} {F(s)}=f(t) then L^{-1} [F(s-a)]= e^{at} L^{-1}{F(s)}

3) **Inverse transform of derivative**

If L^{-1}\{F(s)\}=f(t) then L^{-1}\{F^{n} (s)\}= $(-1)^n t^n L^{-1}\{F(s)\}$

4) **Division by s**

If $L^{-1}\{F(s)\}=f(t)$ then $L^{-1}\left\{\frac{F(s)}{s}\right\}=\int_0^t f(t)dt$

Table of Inverse Laplace Transforms of some standard functions

F(s)	$f(t)=L^{-1}F(s)$
$\frac{1}{s},\ s>0$	1
$\frac{1}{s-a},\ s>a$	e^{at}
$\frac{s}{s^2+a^2},\ s>0$	Cos at
$\frac{1}{s^2+a^2},\ s>0$	$\frac{\text{Sin at}}{\text{a}}$
$\frac{1}{s^2-a^2},\ s>\|a\|$	$\frac{\text{Sin h at}}{\text{a}}$
$\frac{s}{s^2-a^2},\ s>\|a\|$	Cos h at
$\frac{1}{s^{n+1}},\ s>0$ $n = 0, 1, 2, 3, \ldots$	$\frac{t^n}{n!}$
$\frac{1}{s^{n+1}},\ s>0$ $n > -1$	$\frac{t^n}{\Gamma(n+1)}$

Examples

1. Find the inverse Laplace transforms of the following:

$(i)\frac{1}{2s-5}$ $\quad (ii)\frac{s+b}{s^2+a^2}$ $\quad (iii)\frac{2s-5}{4s^2+25}+\frac{4s-9}{9-s^2}$

Solution:

$$(i)\ L^{-1}\frac{1}{2s-5}=\frac{1}{2}L^{-1}\frac{1}{s-5/2}=\frac{1}{2}e^{\frac{5t}{2}}$$

$$(ii)\ L^{-1}\frac{s+b}{s^2+a^2}=L^{-1}\frac{s}{s^2+a^2}+b\,L^{-1}\frac{1}{s^2+a^2}=\cos at+\frac{b}{a}\sin at$$

$$(iii)\ L^{-1}\left[\frac{2s-5}{4s^2+25}+\frac{4s-8}{9-s^2}\right]=\frac{2}{4}L^{-1}\frac{s-5/2}{s^2+\frac{25}{4}}-4L^{-1}\frac{s-9/2}{s^2-9}$$

$$=\frac{1}{2}\left[\cos\frac{5t}{2}-\sin\frac{5t}{2}\right]-4\left[\cos h3t-\frac{3}{2}\sin h3t\right]$$

Exercise:

Find the inverse Laplace transform of the following

$(i)\frac{s+2}{s^2+36}+\frac{4s-1}{s^2+25}$ (ii) $\frac{(s+2)^3}{s^6}$ (iii) $\frac{3s+5\sqrt{2}}{s^2+8}$ (iv) $\frac{1}{s\sqrt{s}}+\frac{3}{s^2\sqrt{s}}-\frac{8}{\sqrt{s}}$

Evaluation of L^{-1} F(s – a)

We have, if L{ f(t)} = F(s), then $L[e^{at} f(t)] = F(s - a)$, and so

$$L^{-1} F(s-a) = e^{at} f(t) = e^{at} L^{-1} F(s)$$

Examples

1. Evaluate: $L^{-1}\frac{3s+1}{(s+1)^4}$

Given $= L^{-1}\frac{3(s+1-1)+1}{(s+1)^4}=3\,L^{-1}\frac{1}{(s+1)^3}-2\,L^{-1}\frac{1}{(s+1)^4}$

$$=3e^{-t}\ L^{-1}\frac{1}{s^3}-2e^{-t}\ L^{-1}\frac{1}{s^4}$$

Using the formula

$$L^{-1}\frac{1}{s^{n+1}}=\frac{t^n}{n!}\quad \text{and taking } n=2 \text{ and } 3, \text{ we get}$$

$$Given = \frac{3e^{-t}t^2}{2} - \frac{e^{-t}t^3}{3}$$

$$2.\ \text{Evaluate}: L^{-1}\frac{s+2}{s^2-2s+5}$$

$$= L^{-1}\frac{s+2}{(s-1)^2+4} = L^{-1}\left[\frac{(s-1)+3}{(s-1)^2+4}\right] = L^{-1}\frac{s-1}{(s-1)^2+4} + 3L^{-1}\frac{1}{(s-1)^2+4}$$

$$= e^t\,L^{-1}\frac{s}{s^2+4} + 3\,e^t\,L^{-1}\frac{1}{s^2+4}$$

$$= e^t\ \cos 2t + \frac{3}{2}e^t \sin 2t$$

$$(3)\text{Evaluate}: L^{-1}\frac{2s+1}{s^2+3s+1}$$

$$= 2L^{-1}\frac{\left(s+\frac{3}{2}\right)-1}{\left(s+\frac{3}{2}\right)^2 - \frac{5}{4}} = 2\left[L^{-1}\frac{\left(s+\frac{3}{2}\right)}{\left(s+\frac{3}{2}\right)^2 - \frac{5}{4}} - L^{-1}\frac{1}{\left(s+\frac{3}{2}\right)^2 - \frac{5}{4}}\right]$$

$$= 2\left[e^{\frac{-3t}{2}}L^{-1}\frac{s}{s^2-\frac{5}{4}} - e^{\frac{-3t}{2}}L^{-1}\frac{1}{s^2-\frac{5}{4}}\right]$$

$$= 2e^{\frac{-3t}{2}}\left[\cos h\frac{\sqrt{5}}{2}t\frac{2}{\sqrt{5}}\sin h\frac{\sqrt{5}}{2}t\right]$$

Exercise

Find the inverse Laplace transforms of the following

(i) $\frac{s+5}{s^2-6s+13}$ **(ii)** $\frac{7s+4}{4s^2+4s+9}$ **(iii)** $\frac{e^{-4s}}{(s-2)^2}$ (iv) $\frac{(s+2)e^{-s}}{(s\ \ +1)^4}$

INVERSE LAPLACE TRANSFORM OF e^{-as} F(s)

Evaluation of $L^{-1}[e^{-as}\ F(s)]$

We have, if $L\{f(t)\} = F(s)$, then $L[f(t-a)\,H(t-a) = e^{-as}\ F(s)$, and so

$$L^{-1}[e^{-as} F(s)] = f(t-a)\,H(t-a)$$

Examples

$$(1) Evaluate : L^{-1}\frac{e^{-5s}}{(s-2)^4}$$

$$a = 5,\ F(s) = \frac{1}{(s-2)^4}$$

Here

$$Therefore\ \ f(t) = L^{-1}F(s) = L^{-1}\frac{1}{(s-2)^4} = e^{2t}L^{-1}\frac{1}{s^4} \quad = \frac{e^{2t}t^3}{6}$$

Thus

$$L^{-1}\frac{e^{-5s}}{(s-2)^4} = f(t-a)\,H(t-a) = \frac{e^{2(t-5)}(t-5)^3}{6}H(t-5)$$

$$(2)\ \ Evaluate : L^{-1}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+4}\right]$$

$$Given = f_1(t-\pi)H(t-\pi) + f_2(t-2\pi)H(t-2\pi) \qquad (1)$$

$$Here\ \ f_1(t) = L^{-1}\frac{1}{s^2+1} = \sin t$$

$$f_2(t) = L^{-1}\frac{s}{s^2+4} = \cos 2t$$

Now relation (1) *reads as*

$$Given\ = \sin(t-\pi)\,H(t-\pi) + \cos 2(t-2\pi)\,H(t-2\pi)$$

$$= -\cos t\ H(t-\pi) + \cos(2t)\,H(t-2\pi)$$

Exercise:
Find the inverse Laplace transform of the following

(i) $\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right]$ (ii) $\frac{\cosh 2s}{e^{3s}s^2}$

(iii) $\dfrac{1+e^{-3s}}{s^2}$ (iv) $\dfrac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

INVERSE LAPLACE TRANSFORM BY PARTIAL FRACTION AND LOGARITHMIC FUNCTION

Examples

(1) $\dfrac{1}{(s+1)(s-2)}$

Let F(s)= $\dfrac{1}{(s+1)(s-2)}$

By applying partial fraction we get

$$\frac{1}{(s+1)(s-2)}=\frac{A}{(s+1)}+\frac{B}{(s-2)}=\frac{A(s-2)+B(s+1)}{(s+1)(s-2)}$$

$$1=A(s-2)+B(s+1)$$

put s=2 $\Rightarrow 1=3B \Rightarrow B=\dfrac{1}{3}$

put s=-1 $\Rightarrow 1=-3A \Rightarrow A=-\dfrac{1}{3}$

Therefore

F(s)= $\dfrac{\frac{-1}{3}}{(s+1)}+\dfrac{\frac{1}{3}}{(s-2)}$

F(s)= $\dfrac{1}{3}\dfrac{1}{(s-2)}-\dfrac{1}{3}\dfrac{1}{(s+1)}$

Taking inverse laplace transform, we get

$$L^{-1}\{F(s)\}=\frac{1}{3}L^{-1}\frac{1}{(s-2)}-\frac{1}{3}L^{-1}\frac{1}{(s+1)}=\frac{1}{3}e^{2t}-\frac{1}{3}e^{-t}$$

2. Evaluate: $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$

we have

$$\frac{2s^2+5s-4}{s^3+s^2-2s}=\frac{2s^2+5s-4}{s(s^2+s-2)}=\frac{2s^2+5s-4}{s(s+2)(s-1)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s-1}$$

Then $2s^2+5s-4 = A(s+2)(s-1) + Bs(s-1) + Cs(s+2)$

For s = 0, we get A = 2, for s = 1, we get C = 1 and for s = -2, we get B = -1. Using these values in (1), we get

$$\frac{2s^2+5s-4}{s^3+s^2-2s} = \frac{2}{s} - \frac{1}{s+2} + \frac{1}{s-1}$$

$$L^{-1}\frac{2s^2+5s-4}{s^2+s^2-25} = 2 - e^{-2t} + e^{t}$$

3. Evaluate : $L^{-1}\dfrac{4s+5}{(s+1)^2+(s+2)}$

Consider

$$\frac{4s+5}{(s+1)^2+(s+2)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+2}$$

Then

$4s + 5 = A(s + 2) + B(s + 1)(s + 2) + C(s + 1)^2$

For s = -1, we get A = 1, for s = -2, we get C = -3

Comparing the coefficients of s^2, we get B + C = 0, so that B = 3. Using these values in (1), we get

$$\frac{4s+5}{(s+1)^2+(s+2)} = \frac{1}{(s+1)^2} + \frac{3}{(s+1)} - \frac{3}{s+2}$$

Hence

$$L^{-1}\frac{4s+5}{(s+1)^2+(s+2)} = e^{-t}L^{-1}\frac{1}{s^2} + 3e^{-t}L^{-1}\frac{1}{s} - 3e^{-2t}L^{-1}\frac{1}{s}$$

$$= te^{-t} + 3e^{-t} - 3e^{-2t}$$

4. Evaluate : $L^{-1}\dfrac{s^3}{s^4-a^4}$

Let

$$\frac{s^3}{s^4-a^4} = \frac{A}{s-a} + \frac{B}{s+a} + \frac{Cs+D}{s^2+a^2} \qquad (1)$$

Hence

$$s^3 = A(s + a)(s^2 + a^2) + B(s-a)(s^2+a^2)+(Cs + D)(s^2 - a^2)$$

For s = a, we get A = ¼; for s = -a, we get B = ¼; comparing the constant terms, we get

D = a(A-B) = 0; comparing the coefficients of s^3, we get

1 = A + B + C and so C = ½. Using these values in (1), we get

$$\frac{s^3}{s^4-a^4}=\frac{1}{4}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]+\frac{1}{2}\frac{s}{s^2+a^2}$$

Taking inverse transforms, we get

$$L^{-1}\frac{s^3}{s^4-a^4}=\frac{1}{4}\left[e^{at}+e^{-at}\right]+\frac{1}{2}\cos at$$

$$=\frac{1}{2}\left[\cos hat+\cos at\right]$$

5. Evaluate: $L^{-1}\dfrac{s}{s^4+s^2+1}$

Consider

$$\frac{s}{s^4+s^2+1}=\frac{s}{\left(s^2+s+1\right)+\left(s^2-s+1\right)}=\frac{1}{2}\left[\frac{2s}{\left(s^2+s+1\right)\left(s^2-s+1\right)}\right]$$

$$=\frac{1}{2}\left[\frac{\left(s^2+s+1\right)-\left(s^2-s+1\right)}{\left(s^2+s+1\right)\left(s^2-s+1\right)}\right]=\frac{1}{2}\left[\frac{1}{\left(s^2-s+1\right)}-\frac{1}{\left(s^2+s+1\right)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$L^{-1}\frac{s}{s^4+s^2+1}=\frac{1}{2}\left[e^{\frac{1}{2}t}L^{-1}\frac{1}{s^2+\frac{3}{4}}-e^{-\frac{1}{2}t}L^{-1}\frac{1}{s^2+\frac{3}{4}}\right]$$

$$=\frac{1}{2}\left[e^{\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}-e^{-\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right]$$

$$=\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\sin h\left(\frac{t}{2}\right)$$

Transform of logarithmic and inverse functions

We have, if $\mathrm{L}\{\mathrm{f(t)}\} = \mathrm{F(s)}$, then $\mathrm{L}[\mathrm{tf(t)}] = -\dfrac{d}{ds}F(s)$ *Hence.* $L^{-1}\left(-\dfrac{d}{ds}F(s)\right) = tf(t)$

Examples

(1) *Evaluate* : $L^{-1} \log\left(\dfrac{s+a}{s+b}\right)$

$$Let\ F(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$$

$$Then\ -\frac{d}{ds}F(s) = -\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$So\ that\ L^{-1}\left[-\frac{d}{ds}F(s)\right] = -\left[e^{-at} - e^{-bt}\right]$$

$$or\quad t\,f(t) = e^{-bt} - e^{-at}$$

$$Thus\quad f(t) = \frac{e^{-bt} - e^{-at}}{b}$$

(2) Evaluate $L^{-1} \tan^{-1}\left(\dfrac{a}{s}\right)$

$$Let\ F(s) = \tan^{-1}\left(\frac{a}{s}\right)$$

$$Then\ -\frac{d}{ds}F(s) = \left[\frac{a}{s^2 + a^2}\right]$$

$$or\quad L^{-1}\left[-\frac{d}{ds}F(s)\right] = \sin at \quad so\ that$$

$$or\quad t\,f(t) = \sin at$$

$$f(t) = \frac{\sin at}{a}$$

Inverse transform of $\left[\dfrac{\mathbf{F(s)}}{\mathbf{s}}\right]$

Since $L\int_0^t f(t)dt = \dfrac{F(s)}{s}$ we have $L^{-1}\left[\dfrac{F(s)}{s}\right] = \int_0^t f(t)dt$

Examples:

(1) *Evaluate :* $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Let us denote $F(s)=\frac{1}{s^2+a^2}$ *so that*

$f(t)=L^{-1}F(s)=\frac{\sin at}{a}$

Then $L^{-1}=\frac{1}{s(s^2+a^2)}=L^{-1}\frac{F(s)}{s}=\int_0^t\frac{\sin at}{a}dt$

$$=\frac{(1-\cos at)}{a^2}$$

(2) *Evaluate :* $L^{-1}\left[\frac{1}{s^2(s+a)^2}\right]$

we have $L^{-1}\frac{1}{(s+a)^2}=e^{-at}t$

Hence $L^{-1}\frac{1}{s(s+a)^2}=\int_0^t e^{-at}t\,dt$

$$=\frac{1}{a^2}\left[1-e^{-at}(1+at)\right],$$ on integration by parts.

Using this, we get

$$L^{-1}\frac{1}{s^2(s+a)^2}=\frac{1}{a^2}\int_0^t\left[1-e^{-at}(1+at)\right]dt$$

$$=\frac{1}{a^3}\left[at(1+e^{-at})+2(e^{-at}-1)\right]$$

CONVOLUTION THEOREM AND L.T. OF CONVOLUTION INTEGRAL

CONVOLUTION THEOREM AND APPLICATION OF LAPLACE TRANSFORMS

The convolution of two functions f(t) and g(t) denoted by f(t) $*$ g(t) is defined as

$$f(t) * g(t) = \int_0^t f(t-u)g(u)du$$

Property:

$$f(t) * g(t) = g(t) * f(t)$$

Proof :- By definition, we have

$$f(t) * g(t) = \int_0^t f(t-u)g(u)du$$

Setting t-u = x, we get

$$f(t) * g(t) = \int_t^0 f(x)g(t-x)(-dx)$$

$$= \int_0^t g(t-x)f(x)dx = g(t) * f(t)$$

This is the desired property. Note that the operation $*$ is commutative.

CONVOLUTION THEOREM:

$$L[f(t) * g(t)] = L\{f(t)\}.L\{g(t)\}$$

Proof: Let us denote

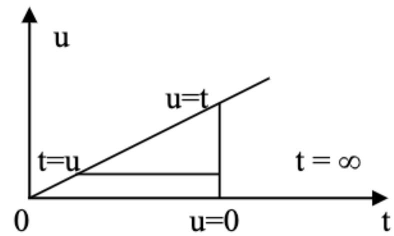
$$f(t) * g(t) = \phi(t) = \int_0^t f(t-u)g(u)du$$

Consider

$$L[\phi(t)] = \int_0^\infty e^{-st}[\int_0^t f(t-u)g(u)]dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(t-u)g(u)du \qquad (1)$$

We note that the region for this double integral is the entire area lying between the lines u =0 and u = t. On changing the order of integration, we find that t varies from u to ∞ and u varies from 0 to ∞.



Hence (1) becomes

$$L[\phi(t)] = \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(t-u)g(u)dtdu$$

$$= \int_0^{\infty} e^{-su} g(u)\left\{\int_u^{\infty} e^{-s(t-u)} f(t-u)dt\right\}du$$

$$= \int_0^{\infty} e^{-su} g(u)\left\{\int_0^{\infty} e^{-sv} f(v)dv\right\}du \quad , \quad \text{where } v = t-u$$

$$= \int_0^{\infty} e^{-su} g(u)du \int_0^{\infty} e^{-sv} f(v)dv$$

$= L\ g(t)\ .\ L\ f(t)$

Thus

$L\ f(t)\ .\ L\ g(t) = L[f(t) * g(t)]$

This is desired property.

Examples:

1. Verify Convolution theorem for the functions f(t) and g(t) in the following cases :

(i) $f(t) = t$, $g(t) = \sin t$ (ii) $f(t) = t$, $g(t) = e^t$

(i) Here,

$$f * g = \int_0^t f(u)g(t-u)du \quad = \int_0^t u\sin(t-u)du$$

Employing integration by parts, we get

$f * g = t - \sin t$

so that

$$L\,[f * g] = \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)} \qquad (1)$$

Next consider

$$L\,f(t)\,.\,L\,g(t) = \frac{1}{s^2}\cdot\frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)} \qquad (2)$$

From (1) and (2), we find that

$$L\,[f * g] = L\,f(t)\,.\,L\,g(t)$$

Thus convolution theorem is verified.

(ii) Here $\quad f * g = \int_0^t u e^{t-u}\,du$

Employing integration by parts, we get

$$f * g = e^t - t - 1$$

so that

$$L[f * g] = \frac{1}{s-1} - \frac{1}{s^2} - \frac{1}{s} = \frac{1}{s^2(s-1)} \qquad (3)$$

Next

$$L\,f(t)\,.\,L\,g(t) = \frac{1}{s^2}\cdot\frac{1}{s-1} = \frac{1}{s^2(s-1)} \qquad (4)$$

From (3) and (4) we find that

$$L[f * g] = L\,f(t)\,.\,L\,g(t)$$

Thus convolution theorem is verified.

2. By using the Convolution theorem, prove that

$$L\int_0^t f(t)dt = \frac{1}{s}Lf(t)$$

Let us define g(t) = 1, so that g(t-u) = 1

Then

$$L\int_0^t f(t)dt = L\int_0^t f(t)g(t-u)dt = L[f * g]$$

$$= L\ f(t)\ .\ L\ g(t) = L\ f(t)\ .\ \frac{1}{s}$$

Thus

$$L\int_0^t f(t)dt = \frac{1}{s}Lf(t)$$

This is the result as desired.

3. Using Convolution theorem, prove that

$$L\int_0^t e^{-u}\sin(t-u)du = \frac{1}{(s+1)(s^2+1)}$$

Let us denote, $f(t) = e^{-t}$ $g(t) = \sin t$, then

$$L\int_0^t e^{-u}\sin(t-u)du = L\int_0^t f(u)g(t-u)du \quad = L\ f(t)\ .\ L\ g(t)$$

$$= \frac{1}{(s+1)}\cdot\frac{1}{(s^2+1)} \qquad = \frac{1}{(s+1)(s^2+1)}$$

This is the result as desired.

4) Employ Laplace Transform method to solve the integral equation.

$$f(t) = 1 + \int_0^t f(u)\sin(t-u)du$$

Taking Laplace transform of the given equation, we get

$$L\,f(t) = \frac{1}{s} + L\int_0^t f(u)\sin(t-u)du$$

By using convolution theorem, here, we get

$$L\,f(t) = \frac{1}{s} + Lf(t)\cdot L\,\sin t = \frac{1}{s} + \frac{L\,f(t)}{s^2+1}$$

Thus

$$L\,f(t) = \frac{s^2+1}{s^3} \quad or \quad f(t) = L^{-1}\left(\frac{s^2+1}{s^3}\right) = 1 + \frac{t^2}{2}$$

This is the solution of the given integral equation.

Exercise:

Solve the following problems

1. Verify convolution theorem for the following pair of functions:

   (i) f(t) = cosat, g(t) = cosbt

   (ii) f(t) = t, g(t) = t e^{-t}

   (iii) f(t) = e^t g(t) = sint

2. Using the convolution theorem, prove the following:

   (i) $L\int_0^t (t-u)e^{u-1}\cos u du = \frac{s}{(s+1)^2(s^2+1)}$

   (ii) $L\int_0^t (t-u)ue^{-au}du = \frac{1}{s^2(s+a)^2}$

3) $f(t) = 1 + 2\int_0^t f(t-u)e^{-2u}\,du$

4) $f'(t) = t + \int_0^t f(t-u)\cos u\,du, \quad f(0) = 4$

Inverse transform of F(s) by using convolution theorem

We have, if $\mathrm{L(t) = F(s)}$ and $\mathrm{Lg(t) = G(s)}$, then

$$\mathrm{L[f(t) * g(t)]} = Lf(t)\cdot Lg(t) = F(s)\,G(s) \quad and\ so$$

$$L^{-1}[F(s)\,G(s)] = f(t) * g(t) = \int_0^t f(t-u)g(u)du$$

This expression is called the convolution theorem for inverse Laplace transform

Examples

Employ convolution theorem to evaluate the following :

(1) $L^{-1}\ \dfrac{1}{(s+a)(s+b)}$

Let us denote $\mathrm{F(s)} = \dfrac{1}{s+a},\ G(s) = \dfrac{1}{s+b}$

Taking the inverse, we get $f(t) = e^{-at},\ g(t) = e^{-bt}$

Therefore, by convolution theorem,

$$L^{-1}\frac{1}{(s+a)(s+b)} = \int_0^t e^{-a(t-u)}e^{-bu}du \quad = e^{-at}\int_0^t e^{(a-b)u}du$$

$$= e^{-at}\left[\frac{e^{(a-b)t}-1}{a-b}\right] \quad = \frac{e^{-bt}-e^{-at}}{a-b}$$

(2) $L^{-1}\ \dfrac{s}{\left(s^2+a^2\right)^2}$

Let us denote $F(s)=\dfrac{1}{s^2+a^2},\ G(s)=\dfrac{s}{s^2+a^2}$ Then

$f(t) = \dfrac{\sin at}{a},\ g(t) = \cos at$

Hence by convolution theorem,

$$L^{-1}\frac{s}{\left(s^2+a^2\right)^2}=\int_0^t \frac{1}{a}\sin a(t-u)\cos au\,du$$

$$=\frac{1}{a}\int_0^t \frac{\sin at+\sin(at-2au)}{2}du, \quad \text{by using compound angle formula}$$

$$=\frac{1}{2a}\left[u\sin at-\frac{\cos(at-2au)}{-2a}\right]_0^t=\frac{t\sin at}{2a}$$

(3) $L^{-1}\ \dfrac{s}{(s-1)\left(s^2+1\right)}$

Here

$$F(s)=\frac{1}{s-1},\ G(s)=\frac{s}{s^2+1}$$

Therefore

$f(t) = e^t,\ g(t) = \sin t$

By convolution theorem, we have

$$L^{-1}\frac{1}{(s-1)\left(s^2+1\right)}=\int e^{t-u}\sin u\,du \quad =e^t\left[\frac{e^{-u}}{2}(-\sin u-\cos u)\right]_0^t$$

$$=\frac{e^t}{2}\left[e^{-t}(-\sin t-\cos t)-(-1)\right]=\frac{1}{2}\left[e^t-\sin t-\cos t\right]$$

By employing convolution theorem, evaluate the following:

(1) $L^{-1}\dfrac{1}{(s+1)(s^2+1)}$ (4) $L^{-1}\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}, a \neq b$

(2) $L^{-1}\dfrac{s}{(s+1)^2(s^2+1)}$ (5) $L^{-1}\dfrac{1}{s^2(s+1)^2}$

(3) $L^{-1}\dfrac{1}{(s^2+a^2)^2}$ (6) $L^{-1}\dfrac{4s+5}{(s-1)^2(s+2)}$

Applications of Laplace transform

Laplace transform is very useful for solving linear differential equations with constant coefficients and with given initial conditions. We take the Laplace transforms of the differential equations and then make use of the initial conditions, which transforms the differential equations to an algebraic equation. Solve for Laplace transform from this algebraic equation and the required solution is obtained by taking the inverse of this transform.

Laplace transform of derivatives:

1. $L\{f'\} = sL(f) - f(0)$
2. $L\{f''\} = s^2 L(f) - s f(0) - f'(0)$
3. $L\{f'''\} = s^3 L(f) - s^2 f(0) - s f'(0) - f''(0)$ and so on.

Problems:

1. Solve $y'' + a^2 y = \cos at$, given $y = 0, y' = 0$ when $t = 0$.

   Solution: Taking Laplace transforms,

$$s^2 Y - s\,y(0) - y'(0) + a^2 Y = \frac{s}{s^2+a^2}$$

   On using the initial condition,

$$Y = \frac{s}{\left(s^2+a^2\right)^2}$$

   Taking inverse,

$$y=\frac{1}{2a}t\sin at$$

2. Solve $y''+5y'+6y=e^{-2t}$, given $y(0)=y'(0)=1$

Solution: Taking Laplace transforms,

$$s^2Y-sy(0)-y'(0)+5\left[sY-y(0)\right]+6Y=\frac{1}{s+2}$$

On using the initial condition,

$$Y=\frac{s^2+8s+13}{(s+2)\left(s^2+5s+6\right)}=\frac{s^2+8s+13}{(s+3)(s+2)^2}$$

Resolving into partial fractions,

$$Y=-\frac{2}{s+3}+\frac{3}{s+2}+\frac{1}{(s+2)^2}$$

Taking the inverse Laplace transforms,

$$y=-2e^{-3t}+3e^{-2t}+e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)=y=3e^{-2t}-2e^{-3t}+te^{-2t}$$

Exercises :-

Solve the following:

1. $y''-3y'+2y=1-e^{2t},\ y(0)=y'(0)=1$
2. $y'''+y''-4y'-4y=1+t,\ y(0)=y'(0)=y''(0)=0$

Electrical circuits:

Consider a simple circuit comprised of an inductance of magnitude L (henrys), a resistance of magnitude R (ohms), and capacitance of magnitude C (farads) connected in series.

If E is the emf (volts) applied to an LRC circuit, then the current i (amperes) in the circuit at time t is governed by the differential equation,

$$L\frac{di}{dt}+Ri+\frac{q}{C}=E$$

Here q is the charge(coulomb) is related to i through the relation $i = \frac{dq}{dt}$. If $q(0) = 0$ then the above equation can be rewritten as,

$$L\frac{di}{dx} + Ri + \frac{1}{C}\int_0^t i\,dt = E$$

Examples:

1. A LR circuit carries an emf of voltage $E = E_0 \sin \omega t$, where E_0 and ω are constants. Find the current i in the circuit if initially there is no current in the circuit.

   Solution: The differential equation governing the current i is,

   $$L\frac{di}{dt} + Ri = E_0 \sin \omega t$$

   or,
   $$\frac{di}{dt} + \frac{R}{L}i + = \frac{E_0}{L}\sin \omega t$$

   Taking Laplace transforms on both sides,

   $$sI - i(0) + aI = \frac{E_0}{L}\left(\frac{\omega}{s^2 + \omega^2}\right), \text{ where } a = \frac{R}{L}$$

   Applying the initial condition, $i(0) = 0$, we get,

   $$I = \frac{E_0 \omega}{L}\frac{1}{(s+a)(s^2+\omega^2)}$$

   By resolving into partial fractions,

   $$I = \frac{E_0}{L}\frac{1}{a^2+\omega^2}\left(\frac{\omega}{s+a} + a\frac{\omega}{s^2+\omega^2} - \omega\frac{s}{s^2+\omega^2}\right)$$

   Taking inverse Laplace transforms,

   $$i = \frac{E_0}{L}\frac{1}{a^2+\omega^2}\left(\omega e^{-at} + a\sin \omega t - \omega \cos \omega t\right)$$

2. A resistance R in series with an inductance L is connected with emf $E(t)$. The current i is given by, $L\frac{di}{dt} + Ri = E(t)$. The switch is connected at time $t = 0$ and disconnected at

time $t = a > 0$ Find the current i in terms of t, given that the emf is constant when the switch is on.

Solution: $$E(t) = \begin{cases} E & 0 < t \le a \\ 0 & t > a \end{cases}$$

Here consider E as a constant.

$$E(t) = E\left(H(t) - H(t-a)\right), t > 0$$

The governing equation becomes,

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\left(H(t) - H(t-a)\right)$$

Taking Laplace transforms and applying the initial condition, $i(0) = 0$,

$$sI + \alpha I = \frac{E}{L}\left(\frac{1}{s} - \frac{e^{-as}}{s}\right), \text{ where } \alpha = \frac{R}{L}$$

$$I = \frac{E}{L}\frac{1-e^{-as}}{s(s+\alpha)} = \frac{E}{R}\left(\frac{1}{s} - \frac{1}{s+\alpha}\right)\left(1-e^{-as}\right)$$

Taking inverse Laplace transforms,

$$i = \frac{E}{R}\left(\left(1-e^{-\alpha t}\right) - \left(1-e^{-\alpha(t-a)}\right)H(t-a)\right)$$

$$i = \begin{cases} \frac{E}{R}\left(1-e^{-\alpha t}\right) & 0 < t \le a \\ \frac{E}{R}e^{-\alpha t}\left(e^{\alpha a} - 1\right) & t > a \end{cases}$$

Exercises:

1. A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistance R. Show that current at time t is $\frac{E}{R-aL}\left(e^{-at} - e^{-Rt/L}\right)$.
2. A simple electrical circuit consists of resistance R and inductance L in a series with constant emf E. If the switch is closed when $t = 0$, find the current at any time t.

Mass spring systems

Consider a spring of length x, tied at one end to a support θ and the other end is tied to a fixed mass m which is free. If F is the force acting on the object, then from Newton's second law of motion,

$$m\frac{d^2x}{dt^2} = -kx$$

Suppose the medium through the setup is worked is resisting with a velocity of $x'(t)$ we have,

$$m\frac{d^2x}{dt^2} = -kx - cx'$$

$$mx''(t) + cx''(t) + kx(t) = 0$$

The auxiliary equation is given by, $mD^2 + cD + k = 0$

The roots of the quadratic equation are, $D = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

The value of $c^2 - 4mk$ determines the sensitivity of the medium.

1. $c^2 - 4mk < 0$ implies motion is under-damped.
2. $c^2 - 4mk = 0$ implies the motion is critically damped.
3. $c^2 - 4mk > 0$ implies the motion is over-damped.

Here k is the stiffness of the spring and can be given by the weight of the object per unit total length of the spring, $k = \frac{w}{b}$, where b is the length of the spring entirely and $w = mg$

1. A spring can extend $20\,cm$ when $0.5\,kg$ of mass is attached to it. It is suspended vertically from a support and set into vibration by pulling it down $10\,cm$ and imparting a velocity of $5\,cm/s$ vertically upwards. Find the displacement from its equilibrium.

   Solution: Let $x(t)$ be the displacement from its equilibrium.

   Here $b = 20\,cm, m = 500\,gm, x(0) = 10\,cm, x'(0) = -5cm/s$

   We can thus calculate the value of k, i.e. $k = 24500\,dy/cm$

   The equation of motion is , $500x''(t) + 24500x(t) = 0$

   $x''(t) + 49x(t) = 0$

   Taking Laplace transforms,

$$s^2X - sx(0) - x'(0) + 49X = 0$$

$$X = \frac{10s}{s^2+49} - \frac{5}{s^2+49}$$

Taking the inverse Laplace transform,

$$x(t) = \frac{10}{7}\cos 7t - \frac{5}{7}\sin 7t$$

2. A spring of stiffness k has a mass m attached to one end. It is acted upon by external force $A\sin\omega t$. Discuss its motion in general.

Solution: We know from the Newton's law of motion, $m\frac{d^2x}{dt^2} + kx = a\sin\omega t$,

$x(0) = x_0, x'(0) = v_0$

$$x''(t) + \frac{k}{m}x(t) = A\sin\omega t$$

Taking Laplace transforms,

$$s^2X - s\,x(0) - x'(0) + \frac{k}{m}X = \frac{A}{m}\frac{\omega}{\omega^2+s^2}$$

By applying the initial conditions,

$$X = \frac{A}{m}\frac{\omega}{\left(s^2+\omega^2\right)\left(s^2+\frac{k}{m}\right)} + x_0\frac{s}{s^2+\frac{k}{m}} + v_0\frac{1}{s^2+\frac{k}{m}}$$

Taking the inverse Laplace transforms,

$$x(t) = \frac{A\sin\omega t}{k-m\omega^2} + x_0\cos\sqrt{\frac{k}{m}}t + \frac{\sin\sqrt{\frac{k}{m}}t}{\sqrt{k/m}}\left(v_0 - \frac{A\omega}{k-m\omega^2}\right)$$

Exercise:

1. A spring is stretched 6 inches by a 12 pound weight. Let the weight be attached to the spring and pulled down 4 inches below the equilibrium point. If the weight id started with an upward velocity of 2 feet per second, describe the motion. No damping or impressed force is present.

2. A spring is such that 4 lb weight stretches it 6 inches. An impressed force $\frac{\cos 8t}{2}$ is acting on the spring. If the 4 pound weight is started from the equilibrium point with an upward velocity of 4 feet per second, describe the motion.