INFINITE SERIES

Sequence:

If a set of real numbers $u_1, u_2, \dots, u_n, \dots$ occur according to some definite rule, then it is called a sequence denoted by $\{u_n\}$.

Example (1): 1, 2, 3,… sequence of natural numbers

Example (2): 3, 6, 9, …. Sequence of multiples of 3

Series:

If $u_1, u_2, \dots, u_n, \dots$ is an infinite sequence of real numbers, then $u_1 + u_2 + \cdots + u_n + \cdots \infty$ is called an infinite series and is denoted by $\sum u_n$ and the sum of its first n terms is denoted by S_n

i.e., $S_n = u_1 + u_2 + \cdots + u_n$

Convergence, Divergence and Oscillation of a series:

Consider an infinite series, $\sum u_n = u_1 + u_2 + \cdots + u_n + \cdots \infty$

Let the sum of first n terms be $S_n = u_1 + u_2 + \cdots + u_n$

Clearly S_n is a function of n and as n increases indefinitely three possibilities arise:

(i) If $\lim_{n\to\infty} S_n = k, a\ finite\ quantity$, then the series $\sum u_n$ is said to be convergent.

(ii) If $\lim_{n\to\infty} S_n = \infty\ or -\infty$, then the series $\sum u_n$ is said to be divergent

(iii) If $\lim_{n\to\infty} S_n$ not unique, then the series $\sum u_n$ is said to be oscillatory

Example (1): $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}+\cdots$

Here $u_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$

$u_1=1-\frac{1}{2},\quad u_2=\frac{1}{2}-\frac{1}{3},\quad \ldots, u_n=\frac{1}{n}-\frac{1}{n+1}$

$$S_n=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]=1, \textit{finite}.$$

Hence the series is convergent

Example (2): $\sum u_n = 1 + 2 + 3 + \ldots$

$$S_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[\frac{n(n+1)}{2}\right] = \infty$$

Hence the series is divergent

Example (3): $\sum u_n = 1 - 1 + 1 - 1 + \ \dots$

$$Here, \quad S_n = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$\lim_{n\to\infty} S_n = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Hence $\sum u_n$ is oscillatory.

Geometric series:

The series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \ \dots + r^n + \dots \infty$

(i) Converges if $|r| < 1$, i.e., $-1 < r < 1$

(ii) Diverges if $r \geq 1$

(iii) Oscillates if $r \leq -1$

Properties of infinite series:

The convergence or divergence of an infinite series remains unaltered on multiplication of each term by $a\ non - zero\ constant$.

The convergence or divergence of an infinite series remains unaltered by addition or removal of a finite number of its terms.

Positive term series:

An infinite series in which all the terms after some particular term are positive is called a positive term series.

Ex: $-7 - 5 - 2 + 2 + 7 + 13 + 20 + \cdots$ is a positive term series.

If we consider the nth partial sum of this series, there exist only two possibilities,

(i) $\lim_{n \to \infty} S_n = k, finite$

(ii) $\lim_{n \to \infty} S_n = \infty$

According, a series of +ve terms either converges or diverges to $+\infty$.

Integral Test:

A positive term series $f(1) + f(2) + \dots\dots + f(n) + \dots,$ where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^\infty f(x)dx$ is finite or infinite.

p-series or Harmonic series test:

A positive term series $\sum u_n = \sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots\dots + \frac{1}{n^p} + \dots$ is

(i) Convergent if $p > 1$

(ii) Divergent if $p \leq 1$

Necessary condition for convergence:

If a positive term series $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$

Proof: Consider a +ve term series $\sum u_n = u_1 + u_2 + u_3 + \cdots \infty$, which is convergent.

If $\sum u_n$ is convergent then $\lim_{n\to\infty} S_n = k.$

Also, $\lim_{n\to\infty} S_{n-1} = k.$

Note: The limit of a convergent series is unique.

$$u_n = (u_1 + u_2 + \cdots \ldots + u_n) - (u_1 + u_2 + \cdots \ldots + u_{n-1})$$

$$= S_n - S_{n-1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = k - k = 0$$

Note: The Converse of the above result is not true.

i.e., Even if $\lim_{n\to\infty} u_n = 0$, then $\sum u_n$ need not be convergent.

Example 1: $\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Here, $\mathrm{u_n} = \frac{1}{\mathrm{n}}$, and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Given, $\sum u_n = \sum \frac{1}{n}$

Here $p = 1$, series is divergent. (using p-series test)

Hence $\lim_{n\to\infty} u_n = 0$ is a necessary condition but not a sufficient condition for convergence of $\sum u_n$.

Note: The above result leads to a simple test for divergence:

If $\lim_{n\to\infty} u_n \neq 0$, the series $\sum u_n$ must be divergent.

Problems:

1. Test the series for convergence, $\sum_{n=2}^{\infty} \frac{1}{nlogn}$

Ans: Consider $\int_2^{\infty} \frac{1}{nlogn} dn = [log(logn)]_2^{\infty} = \infty$

Therefore $\sum u_n$ is divergent by Integral test.

2. Test the series for convergence, $\sum_{n=1}^{\infty} ne^{-n^2}$

Put $x^2 = t$

$2x\,dx = dt$

Ans: Using integral test,

$$\int_1^{\infty} xe^{-x^2} dx = \int_1^{\infty} \frac{e^{-t}}{2} dt = \left[\frac{e^{-t}}{-2}\right]_1^{\infty} = \frac{1}{2e}$$

Therefore $\sum u_n$ is convergent.

Comparison test:

(1) Let $\sum u_n$ and $\sum v_n$ be two positive term series.

If $\sum v_n$ is convergent and $u_n \leq v_n, \ \forall\, n.$ Then $\sum u_n$ is also convergent.

(2) Let $\sum u_n$ and $\sum v_n$ be two positive term series.

If $\sum v_n$ is divergent and $u_n \geq v_n, \ \forall\, n$. Then $\sum u_n$ is also divergent.

(3) Limit test:

If $\sum u_n$ and $\sum v_n$ be two positive term series such that $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = k \ (\neq 0)$.

Then $\sum u_n$ and $\sum v_n$ behave alike.

$\text{i.e.}, \sum u_n$ and $\sum v_n$ converges or diverges together.

Problems:

1. Test the series for convergence, $\sum \frac{1}{2^n+1}$

Ans:

Let $u_n = \frac{1}{2^n+1}$ and $v_n = \frac{1}{2^n}$

$2^n + 1 > 2^n$ so that $\frac{1}{2^n+1} < \frac{1}{2^n}$

Take $v_n = \frac{1}{2^n}$ And $\sum v_n = \sum \left(\frac{1}{2}\right)^n$, *geometric series*

Since $r = \frac{1}{2} < 1$, the series $\sum v_n$ *converges.*

If $\sum v_n$ is convergent and $u_n \leq v_n, \forall n.$
Then $\sum u_n$ is also convergent.

We have, $\frac{1}{2^n+1} < \frac{1}{2^n}$, Therefore $\sum \frac{1}{2^n+1}$ is convergent by comparison test

(2) Test the series for convergence, $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots$

Ans: $u_n = \dfrac{2n-1}{n(n+1)(n+2)}$

$$= \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \qquad = \frac{1\left(2-\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Choose $v_n = \dfrac{1}{n^2}$ then $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2, finite\ (\neq 0)$

Therefore, $\sum u_n$ and $\sum v_n$ converges or diverges together.

A positive term series $\sum u_n = \sum 1/n^p$ Convergent if $p > 1$ AND Divergent if $p \leq 1$

But $\sum v_n = \sum \dfrac{1}{n^2}$ with $p = 2 > 1$.

Therefore $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Test the series for convergence, $\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)$

Ans: $u_n = \left(\sqrt{n^2+1}-n\right) \times \frac{\left(\sqrt{n^2+1}+n\right)}{\left(\sqrt{n^2+1}+n\right)} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$

$$= \frac{1}{n\left(\sqrt{1+1/n^2}+1\right)}$$

Let $\sum v_n = \sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2} \neq 0$$

But $\sum v_n = \sum \frac{1}{n}$, $p = 1$, by p –series test $\sum v_n$ is divergent.

By limit test $\sum u_n$ is also divergent.

Test the series for convergence $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\cdots$

Ans: $u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right]}{n^3\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}$

Let $\sum v_n = \sum \frac{1}{n^{5/2}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = 1\ (\neq 0)$$

But $\sum v_n = \sum \frac{1}{n^{5/2}}$, $p=\frac{5}{2}>1$, by p –series test, $\sum v_n$ convergent

Therefore , By limit test $\sum u_n$ is also convergent.

D’Alembert’s Ratio Test:

In a positive term series $\sum u_n$, if

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda,$$ a finite quantity, then the series

(i) Converges for $\lambda < 1$

(ii) Diverges for $\lambda > 1$

Note: Ratio test fails when $\lambda = 1$.

Raabe's Test:

In a positive term series $\sum u_n$, if

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = k,$$ a finite quantity, then the series

(i) Converges for $k > 1$

(ii) Diverges for $k < 1$

Note: Raabe's test fails when $k = 1$.

Cauchy's root Test:

In a positive term series $\sum u_n$, if

$$\lim_{n\to\infty} (u_n)^{1/n} = \lambda,$$ a finite quantity, then the series

(i) Converges for $\lambda < 1$

(ii) Diverges for $\lambda > 1$

Note: Cauchy's root test fails when $\lambda = 1$.

Test for convergence $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots \infty$

Ans: $u_n = \frac{n!}{n^n}$ $\qquad u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \frac{\cancel{(n+1)}\,\cancel{n}!}{(n+1)^n\cancel{(n+1)}} \times \frac{n^n}{\cancel{n}!}$$

$$\lim_{n\to\infty}\left(1 + {}^{1}/_{n}\right)^n = e$$

$$= \frac{n^n}{(n+1)^n} = \frac{n^n}{n^n\left(1 + \frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

By Ratio test, the series convergent.

Test for convergence $1 + \frac{1}{2} + \frac{1\cdot 2}{2\cdot 5} + \frac{1\cdot 2\cdot 3}{2\cdot 5\cdot 8} + \cdots \infty$

$$u_n = \frac{1\cdot 2\cdot 3\cdot \ \dots \ \cdot n}{2\cdot 5\cdot 8\cdot \ \dots \ \cdot (3n-1)}$$

$$u_{n+1} = \frac{1\cdot 2\cdot 3\cdot \ \dots \ \cdot n\cdot (n+1)}{2\cdot 5\cdot 8\cdot \ \dots \ \cdot (3n-1)(3n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{3n+2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{3} < 1$$

By D'Alembert's Ratio Test, $\sum u_n$ is convergent

Test for convergence $\frac{2}{3\cdot4}+\frac{2\cdot4}{3\cdot5\cdot6}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot8}+\cdots\infty$

$$u_n=\frac{2\cdot4\cdot6\cdot8\ldots2n}{3\cdot5\cdot7\cdot\ \ldots(2n+1)(2n+2)}$$

$$u_{n+1}=\frac{2\cdot4\cdot6\cdot8\ldots2n\,(2n+2)}{3\cdot5\cdot7\cdot\ \ldots(2n+1)(2n+3)(2n+4)}$$

$$\frac{u_{n+1}}{u_n}=\frac{2\cdot4\cdot6\cdot8\ldots2n\,(2n+2)}{3\cdot5\cdot7\cdot\ \ldots(2n+1)(2n+3)(2n+4)}\times\frac{3\cdot5\cdot7\cdot\ \ldots(2n+1)(2n+2)}{2\cdot4\cdot6\cdot8\ldots2n}$$

$$\frac{u_{n+1}}{u_n}=\frac{(2n+2)(2n+2)}{(2n+3)(2n+4)}\quad=\quad\frac{4n^2+8n+4}{4n^2+14n+12}$$

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{4n^2\left[1 + \frac{2}{n} + \frac{1}{n^2}\right]}{4n^2\left[1 + \frac{14}{4n} + \frac{3}{n^2}\right]} \quad = 1$$

Ratio test fails.

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{4n^2+14n+12}{4n^2+8n+4} - 1\right] = n\left[\frac{6n+8}{4n^2+8n+4}\right]$$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{6n+8}{4n^2+8n+4}\right] \quad = \frac{3}{2} > 1$$

By Raabe’s test, the series convergent.

Test for convergence $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}\dots\infty, \quad x>0$

$$u_n=\frac{x^{2n-2}}{(n+1)\sqrt{n}} \qquad u_{n+1}=\frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_{n+1}}{u_n}=\frac{x^{2n}}{(n+2)\sqrt{n+1}}\times\frac{(n+1)\sqrt{n}}{x^{2n-2}} \quad =\frac{(n+1)}{(n+2)}\times\frac{\sqrt{n}}{\sqrt{n+1}}\,x^2$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x^2$$

By Ratio test, the given series converges if $x^2<1$ and diverges if $x^2>1$ and test fails if $x^2=1$

If $\boldsymbol{x^2 = 1}$, $\quad u_n = \frac{1}{(n+1)\sqrt{n}} \quad = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$

Take $\boldsymbol{v_n = \frac{1}{n^{3/2}}}$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{1}{\left(1+\frac{1}{n}\right)} = 1 \ (\neq 0), a\ finite\ quantity.$$

But $\boldsymbol{\sum v_n = \sum \frac{1}{n^{3/2}}}$, $\mathrm{p} = \frac{3}{2} > 1$, by p − series test, $\sum v_n\ convergent.$

Hence $\sum u_n\ also\ convergent.$

$\therefore\ \boldsymbol{\sum u_n\ also\ converges\ if\ x^2 \leq 1}$ **and diverges if** $\boldsymbol{x^2 > 1}$

Test for convergence the series $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Ans: $u_n=\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$(u_n)^{1/n}=\left[\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}\right]^{\frac{1}{n}} \qquad =\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\lim_{n\to\infty}(u_n)^{1/n}=\frac{1}{e}$$

By Cauchy’s root test, the given series is convergent.

Test for convergence the series $\frac{1^3}{3}+\frac{2^3}{3^2}+1+\frac{4^3}{3^4}+\ \ldots$

Ans: $u_n=\frac{n^3}{3^n}$

By root test, $(u_n)^{\frac{1}{n}}=\left(\frac{n^3}{3^n}\right)^{\frac{1}{n}}$ $=\frac{\left((n)^{\frac{1}{n}}\right)^3}{3}$

$$\lim_{n\to\infty}(n)^{\frac{1}{n}}=1$$

$$\lim_{n\to\infty}(u_n)^{1/n}=\frac{1}{3}\lim_{n\to\infty}\left((n)^{\frac{1}{n}}\right)^3 =\frac{1}{3}<1$$

By Cauchy's root test, the given series $\sum u_n$ is convergent.

Consider an infinite series, $\sum u_n = u_1 + u_2 + \cdots + u_n + \cdots \infty$ and

Let the sum of first n terms be $S_n = u_1 + u_2 + \cdots + u_n$, then

$$\lim_{n\to\infty} S_n = finite \quad \to convergent$$

$$\lim_{n\to\infty} S_n = \infty\ 0r\ \ -\infty \quad \to divergent$$

$$\lim_{n\to\infty} S_n = not\ unique \quad \to oscillatory$$

The geometric series $\sum_{n=0}^{\infty} r^n$ (i) Converges if $|r| < 1$, i.e., $-1 < r < 1$

(ii) Diverges if $r \geq 1$

(iii) Oscillates if $r \leq -1$

- Integral test: $\int_1^{\infty} f(x)dx$ = finite → Convergent

  $\int_1^{\infty} f(x)dx$ = infinite → divergent

- Series $\sum \frac{1}{n^p}$ is Convergent if $p > 1$ **AND** Divergent if $p \leq 1$

- If a positive term series $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$

- If $\sum v_n$ is convergent and $u_n \leq v_n, \ \forall\, n$. Then $\sum u_n$ is also convergent.

- If $\sum v_n$ is divergent and $u_n \geq v_n, \ \forall\, n$. Then $\sum u_n$ is also divergent.

- **Limit test:**

If $\sum u_n$ and $\sum v_n$ be two positive term series such that $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = k\ (\neq 0)$.

➢ $\sum u_n$ and $\sum v_n$ converges or diverges together.

D'Alembert's Ratio Test:

In a positive term series $\sum u_n$, if

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$, a finite quantity,

then the series

(i) Converges for $\lambda < 1$

(ii) Diverges for $\lambda > 1$

Note: Ratio test fails when $\lambda = 1$.

Raabe's Test:

In a positive term series $\sum u_n$, if

$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = k$, a finite quantity,

then the series

(i) Converges for k > 1

(ii) Diverges for k < 1

Note: Raabe's test fails when k = 1.

Cauchy's root Test: In a positive term series $\sum u_n$, if $\lim_{n\to\infty} (u_n)^{1/n} = \lambda$, a finite quantity, then the series

(i) Converges for $\lambda < 1$

(ii) Diverges for $\lambda > 1$

Note: Cauchy's root test fails when $\lambda = 1$.