

INFINITE SERIES

Sequence:

If a set of real numbers u_1, u_2, \dots, u_n occur according to some definite rule, then it is called a sequence denoted by $\{S_n\} = \{u_1, u_2, \dots, u_n\}$ if n is finite

Or $\{S_n\} = \{u_1, u_2, \dots, u_n, \dots\}$ if n is infinite.

Series:

$u_1 + u_2 + \dots + u_n$ is called a series and is denoted by $S_n = \sum_{k=1}^n u_k$

Infinite Series:

If the number of terms in the series is infinitely large, then it is called infinite series and is denoted by $\sum u_n = u_1 + u_2 + \dots + u_n + \dots$ and the sum of its first n terms be denoted by $S_n = \sum_{k=1}^n u_k = u_1 + u_2 + \dots + u_n$.

Convergence:

An infinite series $\sum u_n$ is said to be convergent if $\lim_{n \rightarrow \infty} S_n = k$, a definite unique number.

Example: $1 + \frac{1}{2} + \frac{1}{4} + \dots$

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} = \frac{(1 - \frac{1}{2^n})}{(1 - \frac{1}{2})} = 2, \text{ finite.}$$

Therefore given series is convergent.

Divergence:

$\lim_{n \rightarrow \infty} S_n$ tends to either ∞ or $-\infty$ then the infinite series $\sum u_n$ is said to be divergent.

Example: $\sum u_n = 1 + 2 + 3 + \dots$

$$S_n = \frac{n(n+1)}{2}$$

$$\lim_{n \rightarrow \infty} S_n = \infty$$

Therefore $\sum u_n$ is divergent.

Oscillatory Series:

If $\lim_{n \rightarrow \infty} S_n$ tends to more than one limit either finite or infinite, then the infinite series $\sum u_n$ is said to be oscillatory series.

Example: 1. $\sum u_n = 1 - 1 + 1 - 1 + \dots$

$$S_n = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Therefore series is oscillatory.

2. $\sum u_n = 1 + (-3) + (-3)^2 + \dots$

$$S_n = \frac{1 - (-1)^n 3^n}{1 + 3}$$

$$\lim_{n \rightarrow \infty} S_n = \begin{cases} \infty, & n \text{ is odd} \\ -\infty, & n \text{ is even} \end{cases}$$

Properties of infinite series:

1. The convergence or divergence of an infinite series remains unaltered on multiplication of each term by $c \neq 0$.
2. The convergence or divergence of an infinite series remains unaltered by addition or removal of a finite number of its terms.

Positive term series:

An infinite series in which all the terms after some particular term are positive is called a positive term series.

Geometric Series test:

The series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots + r^n + \dots$

- a. Converges if $|r| < 1$
- b. Diverges if $r \geq 1$
- c. Oscillates finitely if $r = -1$ and oscillates infinitely if $r < -1$

Note: If a series in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms are negative.

Integral Test:

A positive term series $f(1) + f(2) + \dots + f(n) + \dots$ Where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^{\infty} f(x)dx$ is finite or infinite.

p-series or Harmonic series test:

A positive term series $\sum u_n = \sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$ is

- i) Convergent if $p > 1$
- ii) Divergent if $p \leq 1$

Theorem:

Let $\sum u_n$ be a positive term series. If $\sum u_n$ is convergent then $\lim_{n \rightarrow \infty} u_n = 0$.

Note:

Converse need not be always true. i.e. Even if $\lim_{n \rightarrow \infty} u_n = 0$, then $\sum u_n$ need not be convergent.

Example 1: $\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

$\sum u_n = \frac{1}{n}$ is divergent by integral test. But $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Hence $\lim_{n \rightarrow \infty} u_n = 0$ is a necessary condition but not a sufficient condition for convergence of $\sum u_n$.

Example 2

Test the series for convergence, $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

Solution: Consider $\int_2^{\infty} \frac{1}{n \log n} dn = [\log(\log n)]_2^{\infty} = \infty$

Therefore $\sum u_n$ is divergent by Integral test.

Example 3

Test the series for convergence, $\sum n e^{-n^2}$

Solution: Let $x^2 = t$. Then $2x dx = dt$

$$\int_1^{\infty} x e^{-x^2} dx = \int_1^{\infty} \frac{e^{-t}}{2} dt = \left[\frac{e^{-t}}{-2} \right]_1^{\infty} = \frac{1}{2e}$$

Therefore $\sum u_n$ is convergent.

Comparison test:

1. Let $\sum u_n$ and $\sum v_n$ be two positive term series. If
 - a. $\sum v_n$ is convergent
 - b. $u_n \leq v_n, \forall n$

Then $\sum u_n$ is also convergent.

That is if a larger series converges then smaller also converge.

2. Let $\sum u_n$ and $\sum v_n$ be two positive term series. If
 - c. $\sum v_n$ is divergent
 - d. $u_n \geq v_n, \forall n$

Then $\sum u_n$ is also divergent.

That is if a smaller series diverges then larger also diverges.

Example 2

Test the series for convergence, $\sum_{n=2}^{\infty} \frac{1}{\log n}$

Solution:

Let $u_n = \frac{1}{\log n}$ and $v_n = \frac{1}{n}$

$$\log n < n$$

$$\frac{1}{\log n} > \frac{1}{n}$$

$$u_n > v_n$$

But $\sum v_n = \sum \frac{1}{n}$ is a p-series with $p = 1$.

Therefore $\sum v_n$ is divergent.

By comparison test $\sum u_n$ is also divergent.

Example 2

Test the series for convergence, $\sum \frac{1}{2^{n+1}}$

Solution:

Let $u_n = \frac{1}{2^{n+1}}$ and $v_n = \frac{1}{2^n}$

$$2^n < 2^n + 1$$

$$\frac{1}{2^n} > \frac{1}{2^n + 1}$$

$$v_n > u_n$$

But $\sum v_n = \sum \frac{1}{2^n}$ is a geometric series with $r = \frac{1}{2} < 1$.

Therefore $\sum v_n$ is convergent.

By comparison test $\sum u_n$ is also convergent.

Another form of comparison test is

Limit test

Statement: If $\sum u_n$ and $\sum v_n$ be two positive term series such that $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = k (\neq 0)$. Then $\sum u_n$ and $\sum v_n$ behave alike.

That is if $\sum u_n$ converges then $\sum v_n$ also converge.

If $\sum u_n$ diverges then $\sum v_n$ also diverge.

Examples 3.

Test the series for convergence, $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \dots \dots \dots$

Solution:

$$u_n = \frac{2n - 1}{n(n + 1)(n + 2)}$$

Choose $v_n = \frac{1}{n^2}$ then $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 2$

But $\sum v_n = \sum \frac{1}{n^2}$ with $p = 2 > 1$.

Therefore $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Examples 4.

Test the series for convergence, $\sum_{n=1}^{\infty} (\sqrt{n^2 + 1} - n)$

Solution:

$$u_n = (\sqrt{n^2 + 1} - n) \frac{(\sqrt{n^2 + 1} + n)}{(\sqrt{n^2 + 1} + n)}$$

$$= \frac{n^2 + 1 - n^2}{\sqrt{n^2 + 1} + n}$$

$$= \frac{1}{n(\sqrt{1 + n^2} + 1)}$$

Let $\sum v_n = \sum \frac{1}{n} (p = 1)$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{2}$$

But $\sum v_n$ is divergent. By limit test $\sum u_n$ is also divergent.

Examples 5.

Test the series for convergence, $\sum \sqrt[3]{n^3 + 1} - n$

Solution:

$$u_n = (n^3 + 1)^{1/3} - n$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a - b = \frac{a^3 - b^3}{a^2 + ab + b^2}$$

$$u_n = (n^3 + 1)^{1/3} - n = \frac{n^3 + 1 - n^3}{(n^3 + 1)^{2/3} + (n^3 + 1)^{1/3}n + n^2}$$

$$= \frac{1}{n^2 \left[\left(1 + \frac{1}{n^3}\right)^{2/3} + \left(1 + \frac{1}{n^3}\right)^{1/3} + 1 \right]}$$

Let $\sum v_n = \sum \frac{1}{n^2}$ with $p = 2 > 1$.

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{3}$$

But $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Example 6.

Test the series for convergence, Solve $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \dots \dots \dots$

Solution:

$$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n} \left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}} \right)}{n^3 \left(\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3} \right)}$$

$$\text{Let } \sum v_n = \sum \frac{1}{n^{5/2}} \text{ with } p = \frac{5}{2} > 1.$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$$

But $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Example 7

Test the series for convergence, $\sum \frac{1}{n^3} \tan \frac{1}{n}$

Solution: $u_n = \frac{1}{n^3} \tan \frac{1}{n}$

We know that $\lim_{n \rightarrow \infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} = 1$

Let $\sum v_n = \sum \frac{1}{n^4}$. Then $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$

But $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Example 8

Test the series for convergence, $\sum \frac{1}{n} - \log \left(\frac{n+1}{n} \right)$

Solution: $u_n = \frac{1}{n} - \log \left(1 + \frac{1}{n} \right)$

$$= \frac{1}{n} - \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} - \cdots \cdots \cdots \right]$$

$$= \left[\frac{1}{2n^2} - \frac{1}{6n^3} + \cdots \cdots \cdots \right]$$

Let $\sum v_n = \sum \frac{1}{n^2}$. Then $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{2}$

But $\sum v_n$ is convergent. By limit test $\sum u_n$ is also convergent.

Exercises

Test for convergence of the series

1. $\sum_{n=0}^{\infty} \frac{2n^3+1}{4n^5+1}$

2. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots \dots \dots \infty$

3. $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \dots\dots\dots \dots\infty$

4. $\sum \sqrt{\frac{3^n - 1}{2^n + 1}}$

5. $\sum \frac{n^n}{(n+1)^{n+1}}$

6. $\frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \dots \dots \dots \infty$

D'Alembert's Ratio Test: If $\sum u_n$ is a series of positive terms, and $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} =$

 $l(\text{a finite value})$

then the series is convergent if $l < 1$, is divergent if $l > 1$ and the test fails if $l = 1$.

If the test fails, one should apply comparison test or the Raabe's test, as given below:

Raabe's Test: If $\sum u_n$ is a series of positive terms, and

$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = l(\text{finite})$, then the series is convergent if $l > 1$, is divergent if $l < 1$ and the test fails if $l = 1$.

Remark: Ratio test can be applied when (i) v_n does not have the form $1/n^p$

(ii) n^{th} term has x^n, x^{2n} etc.

(iii) n^{th} term has $n!$, $(n+1)!$, $(n!)^2$ ect.

(iv) the number of factors in numerator and denominator increase steadily, ex: $(\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \dots)$

Example : Test for convergence the series

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots$$

>> The given series is of the form $\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^3}{4!} + \dots$ whose n^{th} term is $u_n = \frac{n^2}{n!}$.

$$\text{Therefore } u_{n+1} = \frac{(n+1)^2}{n+1!}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{n+1!} \cdot \frac{n!}{n^2} = \frac{(n+1)^2}{n^2} \cdot \frac{n!}{(n+1)(n!)} = \frac{n+1}{n^2}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n^2} \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) = 0 < 1$$

Therefore by ratio test, $\sum u_n$ is convergent.

Example : Discuss the nature of the series

$$\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \dots$$

$$>> u_n = \frac{x^n}{n(n+1)}$$

$$\text{Therefore } u_{n+1} = \frac{x^{n+1}}{(n+1)(n+1+1)} = \frac{x^{n+1}}{(n+1)(n+2)}$$

$$\text{Now } \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^n} = \frac{n}{n+2} x$$

$$\text{Therefore } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{n}{n+2} x = \lim_{n \rightarrow \infty} \frac{1}{(1+2/n)} x = x$$

Therefore by D'Alembert's ratio test $\sum u_n$ is $\begin{cases} \text{convergent if } x < 1 \\ \text{divergent if } x > 1 \end{cases}$

And the test fails if $x = 1$

$$\text{But when } x = 1, u_n = \frac{1^n}{n(n+1)} = \frac{1}{n(n+1)} = \frac{1}{n^2 + n}$$

u_n is of order $1/n^2$ ($p = 2 > 1$) and hence $\sum u_n$ is convergent (when $x = 1$). Hence we conclude that $\sum u_n$ is convergent $x \leq 1$ and divergent if $x > 1$

Example : Find the nature of series $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots$

>> Omitting the first term, the given series can be written in the form

$$\frac{x^1}{1^2+1} + \frac{x^2}{2^2+1} + \frac{x^3}{3^2+1} + \dots \text{ so that } u_n = \frac{x^n}{n^2+1}$$

$$\text{Therefore } u_{n+1} = \frac{x^{n+1}}{n^2+2n+2} \cdot \frac{n^2+1}{n^2+2n+2} x = \lim_{n \rightarrow \infty} \frac{n^2(1+1/n^2)}{n^2(1+2/n+2/n^2)} \cdot x$$

$$\text{That is, } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x$$

Hence by ratio test $\sum u_n$ is $\begin{cases} \text{convergent if } x < 1 \\ \text{divergent if } x > 1 \end{cases}$

and the test fails if $x = 1$.

But when $x = 1$, $u_n = \frac{1^n}{n^2+1} = \frac{1}{n^2+1}$ is of order $\frac{1}{n^2}$ ($p = 2 > 1$)

Therefore $\sum u_n$ is convergent if $x \leq 1$ and divergent if $x > 1$.

Example: Find the nature of the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \dots$

>> omitting the first term, the general term of the series is given by $u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$\text{Therefore } u_{n+1} = \frac{x^2(n+1)}{(n+1+2)\sqrt{(n+1)+1}} = \frac{x^{2n+2}}{(n+3)\sqrt{n+2}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n+2}}{(n+3)\sqrt{n+2}} \cdot \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \frac{n+2}{n+3} \sqrt{\frac{n+1}{n+2}} x^2 = \frac{\sqrt{(n+2)(n+1)}}{(n+3)} x^2$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n(1+2/n)n(1+1/n)}}{n(1+3/n)} \cdot x^2 = x^2$$

Hence by ratio test $\sum u_n$ is $\begin{cases} \text{convergent if } x^2 < 1 \\ \text{divergent if } x^2 > 1 \end{cases}$

and the fails if $x^2 = 1$.

$$\text{When } x^2 = 1, u_n = \frac{(1)^n}{(n+2)\sqrt{n+1}} = \frac{1}{(n+2)\sqrt{n+1}}$$

u_n is of order $1/n^{3/2}$ ($p = 3/2 > 1$) and hence $\sum u_n$ is convergent.

Therefore $\sum u_n$ is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

Example : Discuss the convergence of the series

$$x + \frac{x^3}{2.3} + \frac{3}{2.4} + \frac{x^5}{5} + \frac{3.5}{2.4.6} \cdot \frac{x^7}{7} + \dots \quad (x > 0)$$

>> We shall write the given series in the form

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \dots$$

Now, omitting the first term we have

$$u_n = \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n} \cdot \frac{x^{2n+1}}{2n+1}$$

$$u_{n+1} = \frac{1.3.5 \dots [2(n+1)-1]}{2.4.6 \dots 2(n+1)} \cdot \frac{x^{2(n+1)+1}}{2(n+1)+1}$$

$$\text{That is, } u_{n+1} = \frac{1.3.5 \dots (2n+1)}{2.4.6 \dots (2n+1)} \cdot \frac{x^{2n+3}}{2n+3}$$

$$\text{That is, } u_{n+1} = \frac{1.3.5 \dots (2n-1)(2n+1)}{2.4.6 \dots (2n)(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

Therefore $\frac{u_{n+1}}{u_n} = \frac{1.3.5...(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)} \cdot \frac{x^{2n+3}}{2n+3} \times \frac{2.4.6...2n}{1.3.5...(2n-1)} \cdot \frac{2n+1}{x^{2n+1}}$

That is, $\frac{u_{n+1}}{u_n} = \frac{(2n+1)(2n+1)x^2}{(2n+2)(2n+3)}$

Therefore $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{n(2+1/n)n(2+1/n)x^2}{n(2+2/n)n(2+3/n)} = x^2$

Hence by ratio test, $\sum u_n$ is $\begin{cases} \text{convergent if } x^2 < 1 \\ \text{divergent if } x^2 > 1 \end{cases}$

And the test fails if $x^2 = 1$

When $x^2 = 1$, $\frac{u_{n+1}}{u_n} = \frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}$ and we shall apply Raabe's test.

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} n \left[\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1 \right]$$

$$= \lim_{n \rightarrow \infty} n \left[\frac{(4n^2 + 10n + 6) - (4n^2 + 4n + 1)}{(2n+1)^2} \right]$$

$$= \lim_{n \rightarrow \infty} n \left(\frac{6n+5}{(2n+1)^2} \right) = \lim_{n \rightarrow \infty} \frac{n^2(6+5/n)}{n^2(2+1/n)^2} \cdot \frac{6}{4} = \frac{3}{2} > 1$$

Therefore $\sum u_n$ is convergent (when $x^2 = 1$) by Raabe's test.

Hence we conclude that, $\sum u_n$ is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

Example : Examine the convergence of

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \dots + \frac{2^{n+1}-2}{2^{n+1}+1}x^n + \dots$$

$$>> u_n = \frac{2^{n+1}-2}{2^{n+1}+1} x^n.$$

Therefore $u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1} x^{n+1}$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+2} - 2}{2^{n+2} + 1} x^{n+1} \times \frac{2^{n+1} + 1}{2^{n+1} - 2} \cdot \frac{1}{x^n}$$

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{2^{n+2}(1 - 2/2^{n+2})}{2^{n+2}(1 + 1/2^{n+2})} \cdot x \cdot \frac{2^{n+1}(1 + 1/2^{n+1})}{2^{n+1}(1 - 2/2^{n+1})} \\ &= \frac{(1 - 1/2^{n+1})}{(1 + 1/2^{n+2})} \cdot x \cdot \frac{(1 + 1/2^{n+1})}{(1 - 1/2^n)} \end{aligned}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{(1 - 0)}{(1 + 0)} \cdot x \cdot \frac{(1 + 0)}{(1 - 0)} = x.$$

Therefore by ratio test $\sum u_n$ is $\begin{cases} \text{convergent} & \text{if } x < 1 \\ \text{divergent} & \text{if } x > 1 \end{cases}$ and the test fails if $x = 1$.

$$\text{When } x = 1, u_n = \frac{2^{n+1} - 2}{2^{n+1} + 1}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{2^{n+1}(1 - 1/2^n)}{2^{n+1}(1 + 1/2^{n+1})} = 1$$

Since $\lim_{n \rightarrow \infty} u_n = 1 \neq 0$, $\sum u_n$ is divergent (when $x = 1$)

Hence $\sum u_n$ is convergent if $x < 1$ and divergent if $x \geq 1$.

Example : test for convergence of the infinite series

$$1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots$$

>> the first term of the given series can be written as $1!/1^1$ so that we have,

$$u_n = \frac{n!}{n^n} \text{ and } u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)(n!)}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n}$$

$$\text{Therefore } \frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)^n} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \frac{n^n}{n^n(1 + 1/n)^n}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{1}{(1 + 1/n)^n} = \frac{1}{e} < 1$$

Hence by ratio test $\sum u_n$ is convergent.

Cauchy's Root Test: If $\sum u_n$ is a series of positive terms, and

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = l \text{ (finite),}$$

then, the series converges if $l < 1$, diverges if $l > 1$ and fails if $l = 1$.

Remark: Root test is useful when the terms of the series are of the form $u_n = [f(n)]^{g(n)}$.

We can note : (i) $\lim_{n \rightarrow \infty} n^{1/n} = 1$

$$(ii) \lim_{n \rightarrow \infty} (1 + 1/n)^{1/n} = e$$

$$(iii) \lim_{n \rightarrow \infty} (1 + x/n)^{1/n} = e^x$$

Example : Test for convergence $\sum_{n=1}^{\infty} \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

$$>> u_n = \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$$

$$\text{Therefore } (u_n)^{1/n} = \left\{ \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{3/2}} \right\}^{1/n}$$

$$= \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{1/2}} = \left[1 + \frac{1}{\sqrt{n}}\right]^{-\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{\sqrt{n}}\right]^{-\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1.$$

Therefore as $n \rightarrow \infty$, \sqrt{n} also $\rightarrow \infty$

Therefore by Cauchy's root test, $\sum u_n$ is convergent.

Example : Test for convergence $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^{n^2}$

$$>> u_n = \left(1 - \frac{3}{n}\right)^{n^2}$$

$$\text{Therefore } (u_n)^{1/n} = \left[\left(1 - \frac{3}{n}\right)^{n^2}\right]^{1/n} = \left(1 - \frac{3}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \left(1 + \frac{-3}{n}\right)^n = e^{-3}.$$

$$\text{Therefore } \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

$$\text{That is, } \lim_{n \rightarrow \infty} (u_n)^{1/n} = \frac{1}{e^3} < 1, \text{ therefore } e = 2.7$$

Hence by Cauchy's root test, $\sum u_n$ is convergent.

Example : Find the nature of the series $\sum_{n=1}^{\infty} \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

$$>> u_n = \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$$

$$\text{Therefore } (u_n)^{1/n} = \left\{ \left[1 + \frac{1}{\sqrt{n}} \right]^{-n^{3/2}} \right\}^{1/n}$$

$$= \left[1 + \frac{1}{\sqrt{n}} \right]^{-n^{1/2}} = \left[1 + \frac{1}{\sqrt{n}} \right]^{-\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} (\mathbf{u}_n)^{1/n} = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{\sqrt{n}} \right]^{-\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1, \text{ since as } n \rightarrow \infty, \sqrt{n} \text{ also } \rightarrow \infty$$

Therefore by Cauchy's root test, $\sum u_n$ is convergent.

Example : Test for convergence $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^{n^2}$

$$\gg u_n = \left(1 - \frac{3}{n}\right)^{n^2}$$

$$\text{Therefore } (u_n)^{1/n} = \left[\left(1 - \frac{3}{n} \right)^{n^2} \right]^{1/n} = \left(1 - \frac{3}{n} \right)^n$$

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \left(1 + \frac{-3}{n}\right)^n = e^{-3}, \text{ since } \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

That is, $\lim_{n \rightarrow \infty} (u_n)^{1/n} = \frac{1}{e^3} < 1$, since $e = 2.7$.

Hence by Cauchy's root test, $\sum u_n$ is convergent.

[illegible]