Beta and Gamma Functions

If (i) the interval [a, b] is finite

(ii) the function f(x) is bounded in [a, b], that is, f(x) does not become infinite at any point in the interval, and

(iii) $\frac{d}{dx}[\phi(x)] = f(x)$, then $\int_a^b f(x)dx$ = ϕ(b) - ϕ(a) is called a proper integral.

In condition (i) is not satisfied (that is, a, b or both are infinite), the integral $\int_a^b f(x)dx$ is called an improper integral of first kind.

Definition: If m and n are positive, then $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called a **Beta function** and denoted by B(m, n) or β(m, n).

Note: If m and n are both greater than or equal to 1, then the above integral is a proper integral. On the other hand if either m or n is less than 1, then the integral becomes improper but may be convergent.

Properties of Beta Functions:

1. Symmetry: B(m, n) = B(n, m)

Verification: By definition, B(m, n) = $\int_0^1 x^{m-1}(1-x)^{n-1}dx$

= $\int_0^1 (1-x)^{m-1}[(1-(1-x))]^{n-1}dx$ (since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$)

= $\int_0^1 (1-x)^{m-1}x^{n-1}dx$

= $\int_0^1 x^{n-1}(1-x)^{m-1}dx$ = B(n, m).

2. If p > -1, q > -1 then $\int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta\ d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$.

Verification: $\int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta\ d\theta = \int_0^{\frac{\pi}{2}} (sin^2\theta)^{(p-1)/2}\ (cos^2\theta)^{(q-1)/2} \sin\theta \cos\theta\ d\theta$

Put $\sin^2\theta = x$. Then $\cos^2\theta\ d\theta = 1 - x$.

When θ = 0, we have x = 0 and θ = $\frac{\pi}{2}$, we have x = 1.

Therefore $\int_0^{\frac{\pi}{2}} sin^p\theta \; cos^q\theta \; d\theta$ = $\int_0^1 x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}.\frac{1}{2}dx$

= $\frac{1}{2}\int_0^1 x^{\left(\frac{p+1}{2}\right)-1}(1-x)^{\left(\frac{q+1}{2}\right)-1}dx$

= $\frac{1}{2}$B$\left(\frac{p+1}{2},\frac{q+1}{2}\right)$.

Note: By taking $\frac{p+1}{2}$ = m and $\frac{q+1}{2}$ = n, we get p = 2m-1 and q = 2n-1.

Therefore B(m, n) = 2 $\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta \; cos^{2n-1}\theta \; d\theta.$

3. B(m, n + 1) + B(m + 1, n) = B(m, n).

4. $\frac{B(m,n+1)}{n} = \frac{B(m+1,n)}{m} = \frac{B(m,n)}{m+n}$

Verification: Left to the student's practice

Definition: If n is positive then $\int_0^\infty e^{-x}\, x^{n-1}dx$ is called a Gamma function denoted by Γ(n).

Properties of Gamma function:

Property 1: Γ(1) = 1

Verification: By definition Γ(n) = $\int_0^\infty e^{-x}\, x^{n-1}dx.$

Therefore Γ(1) = $\int_0^\infty e^{-x}\, x^0 dx$

= $\int_0^\infty e^{-x}dx$ = $\left[\frac{e^{-x}}{-1}\right]_0^\infty$

= $-\left\{\lim_{x\to\infty} e^{-x} - e^0\right\}$

= -(0-1) = 1.

Property 2: Reduction formula: Γ(n + 1) = n Γ(n).

Verification: By definition, Γ(n + 1) = $\int_0^\infty e^{-x}\, x^n dx$. Applying integration by parts, Γ(n + 1) = $\left[x^n\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty$ - $\int_0^\infty \frac{e^{-x}}{-1} nx^{n-1}dx = -\left\{\lim_{x\to\infty}\frac{x^n}{e^x} - 0\right\} + n\int_0^\infty e^{-x}\, x^{n-1}dx$

= -(0-0) + nΓ(n) = nΓ (n).

Property 3: If n is a positive integer then Γ(n) = (n-1)!

Verification: Γ(n + 1) = n · Γ(n). Changing n to n-1, we get Γ(n) = (n-1)Γ(n-1). Similarly

Γ(n -1) = (n -2)Γ(n-2).

By substitution Γ(n) = (n-1) (n-2)Γ(n-2).

Similarly, Γ(n - 2) = (n -3)Γ(n-3), and again by substitution, Γ(n) = (n-1)(n-2)(n-3)Γ(n-3). This process can be continued successively, and it ends with Γ(1), since n is a positive integer.

Therefore Γ(n) = (n-1) (n-2)(n-3) … 1Γ(1). But Γ(1) = 1.

Hence Γ(n) = (n-1) (n-2)(n-3) … 1.

Property 4: Relation between beta and gamma function:

$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Verification: By definition, $\int_0^\infty e^{-t}\, t^{n-1} dt$, using t as the variable.

Put t = x^2. Then dt = 2x dx.

When t = 0, we have x = 0 and when t = ∞, we have x = ∞.

Therefore Γ(n) = $\int_0^\infty e^{-x^2}\, (x^2)^{n-1} 2x dx$ = 2 $\int_0^\infty e^{-x^2}\, x^{2n-1} dx$.

Similarly Γ(m) = 2 $\int_0^\infty e^{-y^2}\, y^{2n-1} dy$.

Multiplying, Γ(m) · Γ(n) = 2 $\int_0^\infty e^{-x^2}\, x^{2n-1} dx$ · 2 $\int_0^\infty e^{-y^2}\, y^{2n-1} dy$

= 4 $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, x^{2n-1} y^{2n-1} dx dy$ ·

The range of this integral is the entire first quadrant of the XOY plane. Converting to polar coordinates, by writing x = r cos θ, y = r sin θ so that dx dy = r dθ dr.

The limits being θ from 0 to $\frac{\pi}{2}$ and r from θ to ∞.

Therefore Γ(m) · Γ(n) = 4 $\int_0^{\frac{\pi}{2}} \int_{r=0}^\infty e^{-r^2}\, (r\cos\theta)^{2n-1} r\, d\theta\, dr$ ·

$$= 4\left(\int_0^{\frac{\pi}{2}}\int_{r=0}^{\infty} sin^{2m-1}\theta\ cos^{2n-1}\theta\ \ d\theta\right)\cdot\left(\int_{r=0}^{\infty} e^{-r^2}\ \ (r)^{2(m+n)-1}\ \ dr\right)$$

$$= 4\left(\frac{1}{2}B(m,n)\right)\ \left(\frac{1}{2}\ \Gamma(m+n)\right)$$

$$= \mathrm{B(m, n)}\ \Gamma(\mathrm{m + n}).$$

Property 5: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$; $\Gamma\left(\frac{-1}{2}\right) = -2\sqrt{\pi}$.

Verification: Use $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, by taking $m = n = \frac{1}{2}$, we get

$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}$. But $\Gamma(1) = 1$.

Therefore $B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} sin^{2\left(\frac{1}{2}\right)-1}\theta cos^{2\left(\frac{1}{2}\right)-1}\theta d\theta$

$= 2\int_0^{\frac{\pi}{2}}(\sin\theta)^0\ (\cos\theta)^0\ d\theta$

$= 2\left(\frac{\pi}{2} - 0\right) = \pi$.

Therefore $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Now $\Gamma(\mathrm{n}) = (\mathrm{n}-1)\Gamma(\mathrm{n}-1) \Rightarrow \Gamma(\mathrm{n}-1) = \frac{1}{n-1}\Gamma(n)$.

Put $n = \frac{1}{2}$, we get $\Gamma\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}}\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$.

Property 6: Duplication formula: $\Gamma(\mathrm{m})\ \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

Verification: $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Therefore $B(m,m) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)} = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$ _______(equation 1)

But $B(m,m) = 2\int_0^{\pi/2} sin^{2m-1}\theta\ cos^{2m-1}\theta d\theta =$

$\frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}(2sin\theta cos\theta)^{2m-1}d\theta$

$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}(sin2\theta)^{2m-1}d\theta$. Put $2\theta = \phi$.

Then dθ = $\frac{1}{2}\,d\theta$. When θ = 0, we have ϕ = 0 and when θ = $\frac{\pi}{2}$ we get ϕ = π.

Therefore B(m, m) = $\frac{2}{2^{2m-1}}\int_0^{\pi}(sin\phi)^{2m-1}\frac{1}{2}\ d\phi$

$= \frac{1}{2^{2m-1}}\,2\int_0^{\frac{\pi}{2}}(sin\phi)^{2m-1}d\phi$

$= \frac{1}{2^{2m-1}}\,2\int_0^{\frac{\pi}{2}} sin^{2m-1}\phi cos^{2\left(\frac{1}{2}\right)-1}\,\phi\ d\phi$

$= \frac{1}{2^{2m-1}}B(m,\frac{1}{2}) = \frac{1}{2^{2m-1}}\frac{\Gamma(m).\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+\frac{1}{2})}$ ________(equation 2).

Hence from (1) and (2) we get

$\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\frac{\Gamma(m).\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+\frac{1}{2})}$. Cancelling Γ(m) both sides and write $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ we get

$\Gamma(m)\ \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$.

Property 7: $\Gamma(n) = \int_0^1\left(log\frac{1}{y}\right)^{n-1}dy$

Hint: Use $\Gamma(n) = \int_0^{\infty} e^{-x}\,x^{n-1}dx$. Put e^{-x} = y so that e^{x} = y or x = log $\frac{1}{y}$.

Example: Evaluate $\int_0^1 x^4\,(1-x)^3dx$

Solution: By definition $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ = B(m, n).

Take m = p + 1 and n = q + 1.

We get $\int_0^1 x^p(1-x)^q dx$ = B(p+1, q+1).

Therefore $\int_0^1 x^4\,(1-x)^3dx$ = B(5, 4) = $\frac{\Gamma(5)\Gamma(4)}{\Gamma(5+4)} = \frac{4!3!}{8!} = \frac{1}{280}$.

Example: Evaluate $\int_0^{\frac{\pi}{2}} sin^5\theta\ cos^7\theta\ d\theta$

Solution: Use $\int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta\ d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$.

$\int_0^{\frac{\pi}{2}} sin^5\theta\ cos^7\theta\ d\theta = \frac{1}{2}B\left(\frac{6}{2},\frac{8}{2}\right) = \frac{1}{2}\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{1}{2}\frac{2!3!}{6!} = \frac{1}{120}$.

Example: Evaluate $\int_0^1(x-a)^p(b-x)^q dx$

Solution: Put x = a + (b - a)t.

Then dx = (b - a)dt, x – a = (b - a)t, b – x = (b - a)(1- t).

When x = a, we have t = 0 and when x = b we have t = 1.

Therefore $\int_0^1 (x-a)^p (b-x)^q dx$

$$= \int_0^1 [(b-a)t]^p \, [(b-a)(1-t)]^q (b-a)\cdot$$

$$= (b-a)^{p+q+1} \int_0^1 t^p \, [(1-t)]^q \cdot$$

$$= (b-a)^{p+q+1} B(p+1, q+1).$$

Example: Show that $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\ d\theta \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}}\ d\theta = \pi$.

Solution: Use $\int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta\ d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$, taking p = ½ and q = 0, we get

$$\int_0^{\frac{\pi}{2}} sin^{\frac{1}{2}}\theta\ cos^0\theta\ d\theta = \frac{1}{2}B\left(\frac{\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\ d\theta = \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{2}\right) \_\_\_\_\_\_\_\_ \text{(equation 1)}$$

Taking p = -1/2, q = 0, we get $\int_0^{\frac{\pi}{2}} sin^{\frac{-1}{2}}\theta\ cos^0\theta\ d\theta = \frac{1}{2}B\left(\frac{\frac{-1}{2}+1}{2}, \frac{0+1}{2}\right)$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}}\ d\theta = \frac{1}{2}B\left(\frac{1}{4}, \frac{1}{2}\right) \_\_\_\_\_\_\_\_ \text{(equation 2).}$$

Multiplying (1) and (2)

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\ d\theta \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}}\ d\theta = \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{2}\right) \frac{1}{2}B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2 . \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{1}{4} \frac{[\sqrt{\pi}]^2 . \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{1}{4} \frac{\pi . \Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi.$$

Example: Show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Solution: By definition of gamma function Γ(n) = $\int_0^\infty e^{-t}\, t^{n-1} dt$. Put t = x^2.

Then dt = 2x dx. When t = 0, we get x = 0; when t = ∞ we get x = ∞.

Therefore Γ(n) = $\int_0^\infty e^{-x^2}\, (x^2)^{n-1} 2x dx = \int_0^\infty e^{-x^2}\, x^{2n-1} dx$.

Taking n = 1/2, we get

$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-x^2}\, x^{2\left(\frac{1}{2}\right)-1} dx \Rightarrow \sqrt{\pi} = 2\int_0^\infty e^{-x^2}\, x^0 dx = 2\int_0^\infty e^{-x^2} dx.$

Therefore $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Example: Evaluate $\int_0^a x^4\sqrt{a^2 - x^2}$ dx

Solution: Put $x^2 = a^2 t$. Then dx = $a \cdot \frac{1}{2\sqrt{t}} dt$.

When x = 0, we have t = 0; when x = a, we have t = 1.

Therefore $\int_0^a x^4\sqrt{a^2 - x^2}$ dx

$= \int_0^1 a^4 r^2 \sqrt{a^2 - a^2 t}\ \frac{a}{2\sqrt{t}} dt$

$= \frac{a^6}{2} \int_0^1 t^{\frac{3}{2}} (1-t)^{\frac{1}{2}}\ dt$

$= \frac{a^6}{2}\ B\left(\frac{3}{2} + 1, \frac{1}{2} + 1\right)$

$= \frac{a^6}{2}\ B\left(\frac{5}{2}, \frac{3}{2}\right)$

$= \frac{a^6}{2} \frac{\Gamma\left(\frac{5}{2}\Gamma\left(\frac{3}{2}\right)\right)}{\Gamma(4)}$

$= \frac{a^6}{2} \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{3!}$

$= \frac{\pi a^6}{32}$.

Example: Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \frac{\pi}{\sqrt{2}}$

Solution: $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\sin\theta}{\cos\theta}}$ dθ

$= \int_0^{\frac{\pi}{2}} sin^{\frac{1}{2}}\theta\ cos^{\frac{-1}{2}}\theta d\theta$

$= \frac{1}{2}\ B\left(\frac{\frac{1}{2}+1}{2}, \frac{\frac{-1}{2}+1}{2}\right)$

$= \frac{1}{2}\mathrm{B}\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$.

Using Duplication formula with m = $\frac{1}{4}$, we get

$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{\frac{1}{2}-1}}\Gamma\left(\frac{1}{2}\right) = \sqrt{2}\ \pi$.

Therefore $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \frac{1}{2}\sqrt{2}\,\pi = \frac{\pi}{\sqrt{2}}$.

Example: Express $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$ in terms of gamma function.

Solution: Put $x^2 = \sin\theta$. Then $dx = \frac{1}{2} sin^{\frac{-1}{2}}\theta \cos\theta\ d\theta$.

Therefore $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} =$

$\int_0^{\frac{\pi}{2}} \quad \frac{1}{2} \quad \frac{sin^{\frac{-1}{2}}\theta\cos\theta\ d\theta}{\sqrt{(1-sin^2\theta)}}$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^{\frac{-1}{2}}\theta\ d\theta$

$= \frac{1}{2}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{\frac{-1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{-1}{2}+2}{2}\right)}$

$= \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$.

Exercises:

1. Compute $\Gamma(3.5)$, $\Gamma(4.5)$, $\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4})$
2. Express $\int_0^\infty x^n e^{-a^2x^2}$ in terms of gamma function.
3. Prove that $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} B(m+1, n+1)$.
4. Evaluate $\int_0^1 x^5 (1-x^3)^{10} dx$.
5. Prove that $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$.