MEAN VALUE THEOREMS AND INDETERMINATE FORMS

Cauchy's Mean Value Theorem

Theorem: Suppose a two functions f(x) and g(x) satisfy the following conditions

1. f(x) and g(x) are continuous in closed interval [a, b]
2. f(x) and g(x) are differential in open interval (a, b)
3. $g'(x) \neq 0$

Then their exists at one point 'c' in the open interval (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Let us consider a new function $\phi(x)$ defined by

$$\phi(x) = f(x) - k\, g(x) \qquad \text{..............................(1)}$$

where k is constant to be chosen appropriately.

From the first two given conditions it follows that $\phi(x)$ is continuous in [a, b] and differentiable in (a, b). Also, we find that

$\phi(a) = f(a) - k\, g(a), \quad \phi(b) = f(b) - k\, g(b)$

Therefore, the condition $\phi(a) = \phi(b)$ holds if

$$f(a) - k\, g(a) = f(b) - k\, g(b)$$

$$k = \frac{f(b)-f(a)}{g(b)-g(a)} \qquad \text{.........................(2)}$$

Hence , if we chosen the constant 'k' as given by (2) , then $\phi(x)$ satisfies all the three conditions of Rolle's theorem , there exist at least one point 'c' in the open interval (a, b) such that $\phi'(c) = 0$

From (1) and (2), we find the result that $\phi'(c) = 0$ is same as $f'(c) = k\, g'(c)$

$$\frac{f'(c)}{g'(c)} = k = \frac{f(b)-f(a)}{g(b)-g(a)} \qquad \text{.. (3)}$$

This completes the proof.

Remarks

1) In the right band side of result (2), we have $g(a) \neq g(b)$. Because if, $g(a) = g(b)$,

Then g(x) satisfies all the conditions of Rolle's theorem and we get $g'(c) = 0$

For some 'c' in (a, b), which is not true. [It has been assumed that $g'(x) \neq 0$ for all x in (a, b)]

2) There may exist more than one c in (a, b)for which the result (3) holds.

3) If we take g(x) = x, then the result (3) becomes

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$ which is the Lagrange Mean Value Theorem.

Example 1. Verify Cauchy's Mean Value theorem for the functions f(x) = e^x and g(x) = e^{-x} in [a, b]

Solution : Here f(x) = e^x and g(x) = e^{-x} , which yield $f^1(x) = e^x$ and $g^1(x) = -e^{-x}$. Thus f(x) and g(x) are differentiable and therefore continuous, and $g^1(x)$ is not equal to zero for all x. Hence f(x) and g(x) satisfy all the three conditions of the Cauchy's Mean Value Theorem in every interval [a, b].

Now the result

$$\frac{f^1(c)}{g^1(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \qquad \text{......(1)}$$

$$\Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$e^{-2c} = \frac{e^b - e^a}{1/e^b - 1/e^a} = (e^b - e^a)\frac{e^{a+b}}{-(e^b - e^a)} = -e^{a+b}$$

$$\Rightarrow c = \frac{1}{2}(a+b)$$

Here $c = \frac{1}{2}(a+b)$, the result (1) holds, and 'c' lies between 'a' and 'b'

Thus, for f(x) = e^x and g(x) = e^{-x} defined in the interval [a, b], the Cauchy's Mean Value Theorem is verified.

Example 2. Verify Cauchy's Mean Value theorem for the functions f(x) = x^3 and g(x) = x^2 in [1, 2]

Solution: Here f(x) = x^3 and g(x) = x^2, which yield $f^1(x) = 3x^2$ and $g^1(x) = 2x$. Thus f(x) and g(x) are differentiable and therefore continuous, and $g^1(x)$ is not equal to zero for all x. Hence f(x) and g(x) satisfy all the three conditions of the Cauchy's Mean Value Theorem in every interval [a, b].

Now the result

$$\frac{f^1(c)}{g^1(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \qquad \text{......(1)}$$

$$\Rightarrow \frac{3c^2}{2c} = \frac{f(2)-f(10}{g(2)-g(1)} = \frac{8-1}{4-1} = \frac{7}{3}$$

$$\Rightarrow c = \frac{14}{9}$$

Here $c = \frac{14}{9} = 1.555$, the result (1) holds, and 'c' lies between 1 and 2

Thus, for f(x) = x^3 and g(x) = x^2 defined in the interval [1, 2], the Cauchy's Mean Value Theorem is verified.

Exercise

Verify Cauchy's Mean value theorem for the following functions. In each case, find an appropriate 'c'

1. $f(x) = x^2,\ g(x) = x \ in\ [1,\ 2]$
2. $f(x) = \frac{1}{x^2},\ g(x) = \frac{1}{x}\ in\ [a,\ b], b > a > 0$
3. $f(x) = \sin x,\ g(x) = \cos x\ in\ [a,\ b], 0 \le a < b \le \frac{\pi}{2}$
4. $f(x) = \sqrt{x},\ g(x) = \frac{1}{\sqrt{x}}\ in\ [\frac{1}{4},\ 1]$

Taylor's Theorem

Theorem: Suppose a function f(x) satisfies the following conditions

1. $f(x)$ and its n-1 derivatives are continuous in closed interval [a, b]
2. $f^{n-1}(x)$ is differentiable in (a, b)

Then there exist at least one point 'c' in the open interval (a, b) such that

$$f(b) = f(a) + (b-a)f^1(a) + \frac{(b-a)^2}{2!}f^{11}(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^n}{n!}f^n(c) \qquad (1)$$

Remark: For n =1, The Taylor's series reduces to the Lagrange's Mean Value Theorem.

For n = 2, 3, , . . ., we obtained second third, …. Mean Value theorems.

Aliter : Taking b - a = h and $c = a + \theta h$, where $\theta = (c-a)/(b-a)$, the result (1) may be put it in the following alternate form.

$$f(a+h) = f(a) + hf^{1}(a) + \frac{h^2}{2!} f^{11}(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^{n}(a+\theta h)$$

$$0 < \theta < 1 \qquad (2)$$

Note : The last term on the right hand side of the result (2) is known as reminder after n terms in the Taylor's theorem. This reminder is called the reminder in Lagrange form. The reminder can be rewritten in other forms also.

Taylor's Series

Taking $a + h = x$ in expression (2) we obtained

$$f(x) = f(a) + (x-a) f^{1}(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \ldots$$

$$\ldots \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^{n}[a + \theta(x-a)] \qquad (3)$$

This expression gives the expansion of f(x) in powers of (x - a), and the expansion contains n+1 terms. Let us denote the sum of the first n terms by $S_n(x)$ and the last term by $R_n(x)$ that is,

The expression (3) may be put it in the form

$$f(x) = S_n(x) + R_n(x) \qquad (6)$$

Now suppose that $f(x)$ possesses derivatives of all orders and that $R_n(x)$ tends to zero as $n \to \infty$. Then taking the limit as $n \to \infty$ on both sides of equation (6), we get

$$S_n(x) = f(a) + (x-a) f^{1}(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \ldots \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$= f(a) + \sum_{r=1}^{n-1} \frac{(x-a)^r}{r!} f^{r}(a) \qquad (4)$$

$$\text{and} \quad R_n(x) = \frac{(x-a)^n}{n!} f^{n}[a + \theta(x-a)] \qquad (5)$$

which is the reminder after n terms

(bearing in mind that the left hand side is independent of n)

$$f(x) = \lim_{n\to\infty} S_n(x)$$

$$= f(a) + \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{(x-a)^r}{r!} f^r(a)$$

$$= f(a) + \sum_{r=1}^{\infty} \frac{(x-a)^n}{n!} f^n(a) \qquad (7)$$

The right hand side of (7) is an infinite series in ascending powers of x – a. This series is called the Taylor's Series of the function f(x) about the point a. It is also referred to as the Taylor's Series expansion of f(x) in powers series about x = a.

Note: Changing x to x+h and a to x in (7), we obtain

$$f(x+h) = f(x) + \sum_{r=1}^{\infty} \frac{h^n}{n!} f^n(x) \qquad (8)$$

The right hand side of (8) is the Taylor's series expansion of f(x+h) in ascending powers of h about the point x.

Example.1 Expand $f(x) = \log x$ in a series of powers of (x-1) and hence evaluate log(1.1) correct to four decimal places.

Solution: The series expansion of $f(x)$ in powers of (x-1) is given by

$$f(x) = f(1) + \sum_{r=1}^{\infty} \frac{(x-1)^n}{n!} f^n(1) \qquad (i)$$

For the given $f(x)$, we have $f(1) = \log 1 = 0$, and

$$f^n(x) = \frac{d^n}{dx^n} \log x = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$f^n(1) = (-1)^{n-1}(n-1)!$$

So that , using these in equation (i), we get

$$\log x = 0 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n!} (-1)^{n-1}(n-1)! = \sum_{n=1}^{\infty} \frac{(x-1)^n}{n} (-1)^{n-1}(n-1)!$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - ....$$

This is the expansion of logx in powers of (x - 1)

Taking x = 1.1 in this expansion, we get

$$\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - .... \approx 0.0953$$

Example.2 Obtain the power series expansion of $f(x) = \cos x$ about $x = \pi/3$. Hence find the approximate value of $\cos 61^\circ$

Solution: We are required to expand $f(x) = \cos x$ in powers of $(x - \pi/3)$. This expansion is given by the formula

$$f(x) = f(\pi/3) + \sum_{r=1}^{\infty} \frac{\left(x - \pi/3\right)^n}{n!} f^n\left(\pi/3\right) \qquad (i)$$

We note that $f\left(\pi/3\right) = \cos\left(\pi/3\right) = 1/2$, and

$$f^n(x) = \frac{d^n}{dx^n} \cos x = \cos\left(x + n\pi/2\right)$$

Using this in equation (i), we get

$$\begin{aligned}
\cos x &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\left(x - \pi/3\right)^n}{n!} \cos\left(\frac{\pi}{3} + \frac{n\pi}{2}\right) \\
&= \frac{1}{2} - \left(x - \pi/3\right)\sin\left(\frac{\pi}{3}\right) - \frac{1}{2}\left(x - \pi/3\right)^2 \cos\left(\frac{\pi}{3}\right) + \frac{1}{3!}\left(x - \pi/3\right)^3 \sin\left(\frac{\pi}{3}\right) + \dots \\
&= \frac{1}{2} - \left(x - \pi/3\right)\sin\left(\frac{\pi}{3}\right) - \frac{1}{2}\left(x - \pi/3\right)^2 \cos\left(\frac{\pi}{3}\right) + \frac{1}{3!}\left(x - \pi/3\right)^3 \sin\left(\frac{\pi}{3}\right) + \dots \\
&= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \pi/3\right) - \frac{1}{4}\left(x - \pi/3\right)^2 + \frac{\sqrt{3}}{12}\left(x - \pi/3\right)^3 + \dots
\end{aligned}$$

This is the expansion of $f(x) = \cos x$ about the point $x = \dfrac{\pi}{3}$

To determine $\cos 61^\circ$, let us take $x = 61^\circ$ *so that* $x - \dfrac{\pi}{3} = 1^\circ = \dfrac{\pi}{180}$ radian. In this case The above equation become

$$\begin{aligned}
\cos 61^\circ &= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\pi}{180}\right)^3 + \dots \\
&= \frac{1}{2} - \frac{\sqrt{3}}{2}(0.01745) - \frac{1}{4}(0.01745)^2 + \frac{\sqrt{3}}{12}(0.01745)^3 + \dots \\
&\approx 0.4848
\end{aligned}$$

Example.3 Obtain the Taylor's series expansion of $f(x) = \log \cos x$ about $x = \pi/3$ up to fourth degree terms.

Solution: We are required to expand $f(x) = \log\cos x$ in powers of $(x - \pi/3)$ up to fourth degree terms. This expansion is given by the formula

$$f(x) = f(\pi/3) + \sum_{r=1}^{\infty} \frac{\left(x - \pi/3\right)^n}{n!} f^n\left(\pi/3\right) \qquad (i)$$

We note that $f\left(\pi/3\right) = \log\cos\left(\pi/3\right) = \log 1/2$, and

$$f^1(x) = -\frac{\sin x}{\cos x} = -\tan x, \; so \; that \; f^1(\pi/3) = -\sqrt{3}$$

$$f^{11}(x) = -\sec^2 x, \Rightarrow f^{11}\left(\pi/3\right) = -\sqrt{3}$$

$$f^{111}(x) = -2\sec^2 x \tan x, \Rightarrow f^{111}\left(\pi/3\right) = -8\sqrt{3}$$

$$f^{1v}(x) = -2\sec^2 x \tan^2 x - 2\sec^4 x, \Rightarrow f^{1v}\left(\pi/3\right) = -80$$

Using this in equation (i), we get

$$\log\cos x = \log\left(\frac{1}{2}\right) - \sqrt{3}\left(x - \frac{\pi}{3}\right) + \frac{(-4)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{(-8\sqrt{3})}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{(-80)}{4!}\left(x - \frac{\pi}{3}\right)^4$$

$$= \log\left(\frac{1}{2}\right) - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2 - \frac{4}{\sqrt{3}}\left(x - \frac{\pi}{3}\right)^3 - \frac{10}{3}\left(x - \frac{\pi}{3}\right)^4$$

This is the expansion of $f(x) = \log\cos x$ about the point $x = \frac{\pi}{3}$

Exercises

1. Expand $\tan^{-1} x$ in a power series of (x-1) upto the term containing the fourth degree.
2. Expand $\sin x$ in a series of powers of $\left(x - \frac{\pi}{2}\right)$ up to the term containing $\left(x - \frac{\pi}{2}\right)^4$.

   Hence find an approximate value of $\sin 91^\circ$
3. Using Taylor's series find the value of $\cos 62^\circ$ correct to four decimal places.

Maclaurin's Expansion

We know the Taylor's Expansion's of f(x) about x = a is given as follows

$$f(x) = f(a) + \sum_{r=1}^{n-1} \frac{(x-a)^r}{r!} f^r(a) + \frac{(x-a)^n}{n!} f^n\left(x + \theta(x-a)\right) \qquad (1)$$

For a = 0, this expansion becomes

$$f(x)=f(0)+\sum_{r=1}^{n-1}\frac{x^r}{r!}f^r(0)+\frac{x^n}{n!}f^n(\theta x) \qquad (2)$$

This result is known as ***Maclaurin' Theorem***

If we suppose that f(x) possesses of all orders derivatives of all orders and that the reminder after n terms $\frac{x^n}{n!}f^n(\theta x)$ tends to zero as $n\to\infty$, then the expression (2) becomes

$$f(x)=f(0)+\sum_{n=1}^{\infty}\frac{x^n}{n!}f^n(0)$$

$$=f(0)+x\,f^1(0)+\frac{x^2}{2!}f^{11}(0)+\frac{x^3}{3!}f^{111}(0)+\frac{x^4}{4!}f^{1v}(0)+\ldots \qquad (3)$$

This is the Taylor's series expansion of f(x) about the point x = 0. This expansion is known as the ***Maclaurin's series expansion of f(x)***

The right hand side of equation (3) , which is the power series in ascending powers of x
Suppose $f(x)=y\ \ so\ that\ \ f(0)=y(0)$

The expression (3) takes the form

$$y\ =y(0)+x\,y_1(0)+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+\frac{x^4}{4!}y_4(0)+\ldots \qquad (4)$$

Some standard Maclaurin's series

1. Exponential Series

Let us take $y=e^x, Then\ \ y(0)=e^0=1, Also,$

$$y_n=\frac{d^n}{dx^n}e^x=e^x,\ \ y_n(0)=e^0=1$$

Hence, the Maclaurin's series expansion of $y=e^x$

$$e^x\ =1+\sum_{n=1}^{\infty}\frac{x^n}{n!}$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

The right hand side of above series known as exponential series.

2. Logarithmic Series

Let us take $y=\log(1+x), Then\ \ y(0)=\log 1=0\ \ Also,$

$$y_n=\frac{d^n}{dx^n}\log(1+x)=\frac{(-1)^{n-1}(n-1)!}{(x+1)^n},$$

Hence, the Maclaurin's series expansion of $y = \log(1+x)$

$$\log(1+x) = 0 + \sum_{n=1}^{\infty} \frac{x^n}{n!}(-1)^{n-1}(n-1)!$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

The right hand side of above series known as logarithmic series.

3. Cosine Series

Let us take $y = \cos x, \textit{Then } y(0) = \cos 0 = 1, \textit{Also},$

$$y_n = \frac{d^n}{dx^n}\cos x = \cos\left(x + \frac{n\pi}{2}\right), \; y_n(0) = \cos\left(\frac{n\pi}{2}\right)$$

Hence, the Maclaurin's series expansion of $y = \cos x$

$$\cos x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}\cos\left(\frac{n\pi}{2}\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$$

The right hand side of above series known as cosine series.

Example.1 Expand tanx as ascending powers of x by using Maclaurin's series up to 5th degree terms.

Solution: $y = \tan x, \textit{Then } y(0) = \tan 0 = 0$

We find that

$$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2, \textit{Then } y_1(0) = 1$$

$$y_2 = 2yy_1, \textit{Then } y_2(0) = 0$$

$$y_3 = 2(y_1^2 + y\,y_2), \textit{Then } y_3(0) = 2$$

$$y_4 = 2(3y_1\,y_2 + yy_3), \textit{Then } y_4(0) = 0$$

$$y_5 = 2(3\,y_2^2 + 4y_1y_3 + y\,y_4), \textit{Then } y_5(0) = 16$$

Hence the Maclaurin's series expansion of $y = \tan x$, upto 5th degree is

$$\tan x = x + \frac{2x^2}{3!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$$

Example.1 Expand $y = e^{\tan^{-1} x}$ **as ascending powers of x by using Maclaurin's series up to 5th degree terms.**

Solution: $y = e^{\tan^{=1} x}, \textit{Then } y(0) = e^0 = 1$

We find that

$$y_1 = e^{\tan^{-1}x}\frac{1}{1+x^2} = \frac{y}{1+x^2} \Rightarrow y_1(0) = 1 \qquad (1)$$

Expression (1) *may bewritten as*

$$(1+x^2)y_2 + (2x-1)y_1 = 0 \qquad (2)$$

This gives $y_2(0) - y_1(0) = 0$ so that $y_2(0) = y_1(0) = 1$

Differentiating equation (2) with respect to x n times by using Leibnitz's theorem, we get

$$\left\{(1+x^2)y_{n+2} + (nC_1)(2x)y_{n+1} + (nC_2)(2)y_n\right\} + \left\{(2x-1)y_{n+1} + (nC_1)(2)y_n\right\} = 0$$

$$(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + 2\left(\frac{n(n-1)}{1.2} + n\right)y_n = 0$$

For x = 0, this becomes

$$y_{n+2}(0) - y_{n+1}(0) + [n^2+n]y_n(0) = 0$$

or

$$y_{n+2}(0) = y_{n+1}(0) - [n^2+n]y_n(0)$$

$\sin ce\ y(0) = y_1(0) = y_2(0) = 1,$ *this gives*

$y_3(0) = -1,\ y_4(0) = -7,\ \ y_5(0) = 5$

Hence the Maclaurin's series expansion yields

$$e^{\tan^{-1}x} = 0 + x + 1\frac{x^2}{2!} + (-1)\frac{x^3}{3!} + (-7)\frac{x^4}{4!} + 5\frac{x^5}{5!} + \ldots$$

$$= x + \frac{x^2}{2} - \frac{x^3}{3!} - 7\frac{x^4}{4!} + 5\frac{x^5}{5!} +$$

Example.3 Prove that for $x > 0,\quad \left(x - \frac{x^3}{6}\right) < \sin x < \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$

Solution : Recall the Maclaurin's Theorem reads

$$f(x) = f(0) + \sum_{r=1}^{n-1}\frac{x^r}{r!}f^r(0) + \frac{x^n}{n!}f^n(\theta x), \qquad 0 < \theta < 1 \qquad (1)$$

For $f(x) = \sin x$ we find $f(0) = 0$, and

$$f^r(x) = \frac{d^r}{dx^r}\sin x = \sin\left(x + \frac{\pi r}{2}\right)$$

The expression (1) becomes

For n =3 this yields

$$\sin x = \sum_{r=1}^{n-1} \frac{x^r}{r!}\sin\left(\frac{r\pi}{2}\right) + \frac{x^n}{n!}\sin\left(\theta x + \frac{n\pi}{2}\right), \qquad 0<\theta<1 \qquad (2)$$

$$\sin x = \sum_{r=1}^{2} \frac{x^r}{r!}\sin\left(\frac{r\pi}{2}\right) + \frac{x^3}{3!}\sin\left(\theta x + \frac{3\pi}{2}\right), \qquad 0<\theta_1<1$$

$$\sin x = x\sin\left(\frac{\pi}{2}\right) + \frac{x^2}{2!}\sin\left(\frac{2\pi}{2}\right) + \frac{x^3}{3!}\sin\left(\theta_1 x + \frac{3\pi}{2}\right), \qquad 0<\theta_1<1$$

$$= x - \frac{x^3}{3!}cos(\theta_1 x), \qquad 0<\theta_1<1 \qquad (3)$$

For n=5, expression (2) becomes

$$\sin x = \sum_{r=1}^{4} \frac{x^r}{r!}\sin\left(\frac{r\pi}{2}\right) + \frac{x^5}{5!}\sin\left(\theta x + \frac{5\pi}{2}\right), \qquad 0<\theta_2<1$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5}cos(\theta_2 x), \qquad 0<\theta_2<1 \qquad (4)$$

Since $\cos\theta \le 1$ *for* $\theta>0$, *we have*, *for* $x>0$

$$x - \frac{x^3}{3!} \le x - \frac{x^3}{3!}cos(\theta_1 x), \quad for \quad 0<\theta_1<1 \qquad (5)$$

$$and \quad x - \frac{x^3}{3!} + \frac{x^5}{5}cos(\theta_2 x) \le x - \frac{x^3}{3!} + \frac{x^5}{5} \;\; for \quad 0<\theta_2<1 \qquad (6)$$

In view of (3) and (4), these inequality yields

$$\left(x - \frac{x^3}{6}\right) < \sin x < \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

This is the required result

Exercises

I Obtain the Maclaurin's series expansions for the following functions

1. $\cos hx$
2. $\sec hx$
3. $e^x \cos x$
4. $\sqrt{1+\sin x}$
5. $e^{a\cos^{-1}x}$
6. $\log(\cos x)$
7. $\log(\sec x + \tan x)$
8. $\left(\sin^{-1}x\right)^2$
9. $e^{x\sec x}$
10. $\sin^{-1}x$

II Prove the following by Maclaurin's theorem

1. $\left(x-\frac{x^2}{2}\right) \le \sin x \le x$ 2. $1-\frac{x^2}{2} \le \cos x \le 1-\frac{x^2}{2}+\frac{x^2}{24}$

INDETERMINATE FORMS :

In connection with science and engineering we encounter the equations which are not determined for particular values of a parameter. For example $\frac{\sin\theta}{\theta}$ as $\theta \to 0$. which is an indeterminate form.

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist.

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ which do not have any definite value. Such an expression is called an indeterminate form.

The other indeterminate forms are $\frac{\infty}{\infty}$, $0 \bullet \infty$, $\infty - \infty$, 0^0, ∞^0 and 1^∞

<u>L' Hospital 's Rule</u>

If f(x) and g(x) are two functions such that

(i) $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

(ii) $f'(x)$ and $g'(x)$ exist and $g'(a) \neq 0$ and

(iii) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Note :

(i)

If $\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

We rewrite as $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} \right) = \frac{0}{0}$

(ii) Some times L' Hospital Rule used repeatedly to get the value of the limit. But at each step it has to be checked for indeterminate form before applying the rule.

(iii) While in the evaluation process some times we use

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \quad \text{and} \quad \lim_{x\to 0}\frac{\tan x}{x}=1 \quad \text{or} \quad \lim_{x\to 0}\frac{x}{\sin x}=1 \quad \text{and} \quad \lim_{x\to 0}\frac{x}{\tan x}=1$$

Examples : Evaluation of Indeterminate forms :

1. Evaluate: $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x}$

Solution : Let $L = \lim\limits_{x\to 0}\dfrac{a^x-b^x}{x} \quad \left[\dfrac{0}{0}\right]$

Applying L'Hospital's Rule we get

$$L=\lim_{x\to 0}\frac{a^x\log a-b^x\log b}{1}$$

$$=\log a-\log b$$

$$=\log\left(\frac{a}{b}\right).$$

2. Evaluate : $\lim\limits_{x\to 0}\dfrac{\sin x}{(e^x-1)^2}$

Solution : Let $L = \lim\limits_{x\to 0}\dfrac{\sin x}{(e^x-1)^2} \quad \left[\dfrac{0}{0}\right]$

Applying L'Hospital's Rule we get

$$L=\lim_{x\to 0}\frac{x\cos x+\sin x}{2(e^x-1)e^x} \quad \left[\frac{0}{0}\right]$$

Applying again L'Hospital's Rule we get

$$L=\lim_{x\to 0}\frac{-x\sin x+\cos x+\cos x}{2(2e^{2x}-e^x)}=\frac{0+2}{2(2-1)}=1$$

3. Evaluate : $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\log\cos x}{\tan x}$

Solution : Let $L = \lim\limits_{x\to\frac{\pi}{2}} \dfrac{\log\cos x}{\tan x} \quad \left[\dfrac{\infty}{\infty}\right]$

Applying L'Hospital's Rule we get

$$L = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\cos x}(-\sin x)}{\sec^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{\tan x}{\sec^2 x} \quad \left[\frac{\infty}{\infty}\right]$$

Applying again L'Hospital's Rule we get

$$L = \lim_{x\to\frac{\pi}{2}} -\frac{\sec^2 x}{2\sec x\ \sec x\ \tan x}$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{1}{2\tan x} = 0$$

4. Evaluate : $\lim\limits_{x\to\frac{1}{2}} \dfrac{\sec\pi x}{\tan 3\pi x}$

Solution : Let $L = \lim\limits_{x\to\frac{1}{2}} \dfrac{\sec\pi x}{\tan 3\pi x} \quad \left[\dfrac{\infty}{\infty}\right]$

$$= \lim_{x\to\frac{1}{2}} \frac{\sec\pi x}{\frac{\sin 3\pi x}{\cos 3\pi x}}$$

$$= \lim_{x\to\frac{1}{2}} \frac{1}{\sin 3\pi x} \frac{\sec\pi x}{\sec 3\pi x}$$

$$= \lim_{x\to\frac{1}{2}} \frac{1}{-1} \frac{\sec\pi x}{\sec 3\pi x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= -\lim_{x\to\frac{1}{2}} \frac{\cos 3\pi x}{\cos\pi x} \quad \left[\frac{0}{0}\right]$$

Applying L'Hospital's Rule we get

$$L = -\lim_{x\to\frac{1}{2}} \frac{-\sin 3\pi x.3\pi}{-\sin\pi x.\pi} = -1\,\frac{-3}{1} = 3$$

5. Evaluate : $\lim\limits_{x\to\infty} (a^{\frac{1}{x}} - 1)\, x$

Solution : Let $L = \lim\limits_{x\to\infty} (a^{\frac{1}{x}} - 1)\, x \quad [0 . \infty]$

$$= \lim_{x\to\infty} \frac{(a^{\frac{1}{x}} - 1)}{\frac{1}{x}} \quad \left[\frac{0}{0}\right]$$

put $\frac{1}{x} = t$ we get $t \to 0$ as $x \to \infty$

$$\therefore \quad L = \lim_{t\to 0} \frac{(a^t - 1)}{t} \quad \left[\frac{0}{0}\right]$$

Applying L'Hospital's Rule we get

$$L = \lim_{t\to 0} \frac{a^t \log a}{1} = \log a$$

6. Evaluate : $\lim\limits_{x\to 1} (x^2 - 1) \tan\frac{\pi x}{2}$

Solution : Let $L = \lim\limits_{x\to 1} (x^2 - 1) \tan\frac{\pi x}{2} \quad [0 . \infty]$

$$= \lim_{x\to 1} \frac{(x^2 - 1)}{\cot\frac{\pi x}{2}} \quad \left[\frac{0}{0}\right]$$

Applying L'Hospital's Rule we get

$$L = \lim_{x\to 1} \frac{2x}{-\operatorname{cosec}^2\frac{\pi x}{2} . \frac{\pi}{2}} = -\lim_{x\to 1} \frac{4}{\pi}\, x \sin^2\frac{\pi x}{2} = -\frac{4}{\pi}$$

7. Evaluate : $\lim_{x \to \frac{\pi}{2}} \left[\tan x - \frac{2x \sec x}{\pi} \right]$

Solution : Let $L = \lim_{x \to \frac{\pi}{2}} \left[\tan x - \frac{2x \sec x}{\pi} \right] \quad [\infty - \infty]$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{\sin x}{\cos x} - \frac{2x}{\pi \cos x} \right]$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{\pi \sin x - 2x}{\pi \cos x} \right] \quad \left[\frac{0}{0} \right]$$

Applying L'Hospital's Rule we get

$$L = \lim_{x \to \frac{\pi}{2}} \left[\frac{\pi \cos x - 2}{-\pi \sin x} \right] = \frac{0-2}{-\pi} = \frac{2}{\pi}$$

8. Evaluate : $\lim_{x\to 0} x^{\sin x}$

Solution : Let $L = \lim_{x\to 0} x^{\sin x} \quad \left[0^0\right]$

Taking logarithm on both sides, we get

$$\log L = \lim_{x\to 0} \sin x.\ \log x \quad [0.\infty]$$

$$= \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} \quad \left[\frac{\infty}{\infty}\right]$$

Applying L'Hospital's Rule we get

$$\log L = \lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x.\ \cot x}$$

$$= \lim_{x\to 0} -\frac{\sin x}{x} \tan x = 0$$

$$\log L = 0$$

$$\therefore \quad L = e^0 = 1.$$

9. Evaluate : $\lim_{x\to 0} \left(\frac{1}{x}\right)^{\tan x}$

Solution : Let $L = \lim_{x\to 0} \left(\frac{1}{x}\right)^{\tan x} \quad \left[\infty^0\right]$

Taking logarithm on both sides, we get

$$\log L = \lim_{x\to 0} \tan x.\ \log\left(\frac{1}{x}\right) \quad [0.\infty]$$

$$= \lim_{x\to 0} \frac{-\log x}{\cot x} \quad \left[\frac{\infty}{\infty}\right]$$

Applying L'Hospital's Rule we get

$$\log L = \lim_{x\to 0} \frac{-\frac{1}{x}}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \sin x = 0$$

$$\log L = 0$$

$$\therefore \quad L = e^0 = 1.$$

10. Evaluate : $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Solution : Let $L = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} \quad \left[1^{\infty}\right]$

Taking logarithm on both sides, we get

$$\log L = \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right).\ \log\left(2-\frac{x}{a}\right) \quad [\infty.0]$$

$$= \lim_{x\to a} \frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \left[\frac{0}{0}\right]$$

Applying L'Hospital's Rule we get

$$\log L = \lim_{x\to a} \frac{\frac{1}{2-\frac{x}{a}}.\left(-\frac{1}{a}\right)}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right).\left(\frac{\pi}{2a}\right)}$$

$$= \lim_{x\to a} \frac{2}{\pi}\frac{\sin^2\left(\frac{\pi x}{2a}\right)}{2-\frac{x}{a}} = \frac{2}{\pi}$$

$$\log L = \frac{2}{\pi}$$

$$\therefore \quad L = e^{\frac{2}{\pi}}.$$

Evaluate the following :

(i) $\underset{x \to 1}{\text{Lt}} \dfrac{1+\log x - x}{1-2x+x^2}$ (ii) $\underset{x \to 0}{\text{Lt}} \dfrac{x\cos x - \log(1+x)}{x^2}$

(iii) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$ (iv) $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x - x}{x^2 \tan x}$

(v) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \sin^{-1} x}{\sin^3 x}$ (vi) $\underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{x^2} - \cot^2 x\right)$

(vii) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(\sin x - \dfrac{1}{1-\sin x}\right)$

(viii) $\underset{x \to 0}{\text{Lt}} \log_{\tan x} \tan 2x$ (ix) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \dfrac{\tan x}{\tan 3x}$

(x) $\underset{x \to 0}{\text{Lt}} x\tan\left(\dfrac{\pi}{2} - x\right)$ (xi) $\underset{x \to 0}{\text{Lt}} x\log\tan x$

(xii) $\underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$ (xiii) $\underset{x \to 2}{\text{Lt}} \left(\dfrac{1}{x-2} - \dfrac{1}{\log(x-1)}\right)$

(xiv) $\underset{x \to 0}{\text{Lt}} \dfrac{\cot x}{\log x}$ (xv) $\underset{x \to 1}{\text{Lt}} x^{\left(\frac{1}{x-1}\right)}$.

SPHERE, CONE, CYLINDER

Sphere

Definition: A sphere is the locus of a point which remains at a constant distance from a fixed point. The fixed point is called the centre and the constant distance is the radius of the sphere.

Let $C(x_0, y_0, z_0)$ be the centre and r be the radius of a sphere S. Consider a point $P(x, y, z)$ on the sphere. Then the equation of the sphere is given by

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2 \qquad -(1)$$

If the center of the sphere is the origin the the equation is

$x^2+y^2+z^2=r^2$.

General equation:

Expanding (1), we get $x^2+y^2+z^2-2x_0x-2y_0y-2z_0z+x_0^{\,2}+y_0^{\,2}+z_0^{\,2}=r^2$

This equation is of the form $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ -(2)

where $u=-x_0, v=-y_0, w=-z_0, d=x_0^{\,2}+y_0^{\,2}+z_0^{\,2}-r^2$.

Since $C(x_0, y_0, z_0)$ is the centre and r is the radius of the sphere (2) represents the equation of the sphere whose center is $(-u,-v,-w)$ and radius is $\sqrt{u^2+v^2+w^2-d}$.

Example 1. Find the equation of the sphere whose centre is (3,-1,4) and which passes through the point (1,-2,0).

Solution. Since (3,-1,4) is the centre, equation is $(x-3)^2+(y+1)^2+(z-4)^2=r^2$

The sphere passes through the point (1,-2,0), we have

$(1-3)^2+(-2+1)^2+(0-4)^2=r^2$

$\Rightarrow\ r=\sqrt{21}$

The required equation is $(x-3)^2+(y+1)^2+(z-4)^2=21$.

2. Obtain the equation of the sphere which passes through the points (1,0,0), (0,1,0), (0,0,1) and which has its centre on the plane $x+y+z=6$.

Solution. Let $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ be the required equation. Since it passes through the given points, we have

$1+2u+d=0$

$1+2v+d=0$

$1+2w+d=0$

$\Rightarrow u = v = w = -\frac{1}{2}(d+1)$.

Its centre lies on the plane $x+y+z=6$.

$-u-v-w=6 \Rightarrow d=3$.

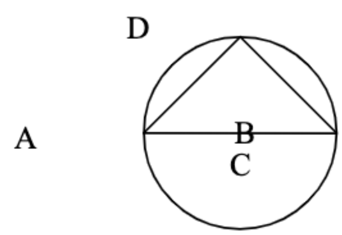
Required equation is

$x^2+y^2+z^2-4x-4y-4z+3=0$

3. a) Find the equation of the sphere which has (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter.

b) Find the equation of the sphere having the points (2,1,-3) and (1,-2,4) as the ends of a diameter Find its centre and radius.

Solution. Consider a point P(x,y,z) on the sphere S having the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as the extrimities of a diameter. Then AP and BP are at right angles. The direction ratios of AP are $(x-x_1, y-y_1, z-z_1)$ and those of BP are $(x-x_2, y-y_2, z-z_2)$.



$\therefore (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$ is the required equation of the sphere.

b) Given A=(2,1,-3), B= (1,-2,4)

Equation is $(x-2)(x-1)+(y-1)(y+2)+(z+3)(z-4)=0$.

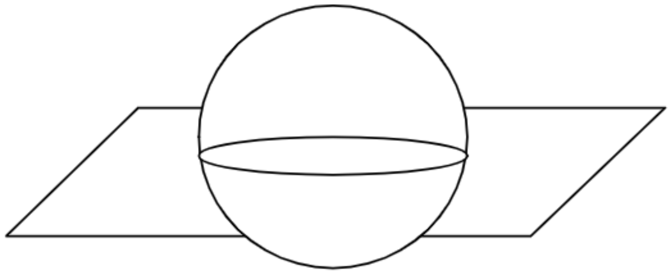
$\Rightarrow x^2+y^2+z^2-3x+y-z-12=0$

Centre$=(-u,-v,-w)=\left(\frac{3}{2}, \frac{-1}{2}, \frac{1}{2}\right)$.

And radius= $\sqrt{u^2+v^2+w^2-d} = \frac{\sqrt{59}}{2}$.

Intersection of a plane and a sphere:

Figure



Section of a sphere by a plane is a base circle and the section of a sphere by a plane through its centre is called a great circle.

The equations $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

And

$Ax+By+Cz+D=0$ taken together represents a circle. Having center M and the radius $MA=\sqrt{r^2-p^2}$.

Sphere through a circle of intersection

The equation of a sphere that passes through the circle of intersection of the sphere and the plane is given by S+kT=0 where S=0 is the equation of the sphere and T=0 is the equation of the plane.

Example 1. Find the centre, the radius and the area of the circle $x^2+y^2+z^2-2y-4z=11$**,** $x+2y+2z=15$**.**

Solution. Centre of the sphere, $C=(0,1,2)$

And radius, $r=\sqrt{1+4+11}$

Let $M(x_1,y_1,z_1)$ be the centre of the circle. Then the direction ratios of CM are (x_1,y_1-1,z_1-2)

Since CM is perpendicular to the plane

$$\frac{x_1}{1}=\frac{y_1-1}{2}=\frac{z_1-2}{2}=t$$

$$\Rightarrow \quad M=(x_1,y_1,z_1)=(t,2t+1,2t+2)$$

Since M lies on the given plane, we have

$$x_1+2y_1+2z_1=15$$

$$\Rightarrow t+4t+2+4t+4=15$$

$\Rightarrow t=1$

$\therefore M=(1,3,4)$ and $p=\sqrt{1+4+4}$

Radius of the circle= $\sqrt{4^2-3^2}=\sqrt{7}$

Area of the circle $= \pi M^2 = 7\pi$.

2. Find the equation of the sphere that passes through the circle $x^2+y^2+z^2-2x-3y+4z+8=0$, $x-2y+z-8=0$ and has its centre on the plane $4x-5y-z-3=0$.

Solution. Equation of a sphere is $x^2+y^2+z^2-2x-3y+4z+8+k(x-2y+z-8)=0$

Centre= $(-u,-v,-w)=\left(1-\frac{k}{2},\frac{3}{2}+k,-2-\frac{k}{2}\right)$

Since the centre lies on the plane $4x-5y-z-3=0$, we have

$$4\left(1-\frac{k}{2}\right)-5\left(\frac{3}{2}+k\right)-\left(-2-\frac{k}{2}\right)-3=0$$

$$\Rightarrow k=\frac{-9}{13}$$

Required equation of a sphere is $13(x^2+y^2+z^2-2x-3y+4z+8)-9(x-2y+z-8)=0$.

3. Find the equation of the sphere having the circle $x^2+y^2+z^2+10y-4z-8=0$, $x+y+z=3$ as a great circle.

Solution. The equation of the sphere passing through the given circle is

$$x^2+y^2+z^2+10y-4z-8)+k(x+y+z-3)=0$$

The given circle is a great circle of this sphere if the centre of the sphere and the centre of the circle coincide. This is possible if the centre of the sphere lies in the plane $x+y+z=3$ of the given circle.

Center of the sphere is $\left(\frac{-k}{2},-5-\frac{k}{2},2-\frac{k}{2}\right)$.

This centre lies on the plane if $\frac{-k}{2}-5-\frac{k}{2}+2-\frac{k}{2}=3$

$$\Rightarrow k=-4$$

Equation of the sphere is

$$x^2+y^2+z^2-4x+6y-8z+4=0$$

Exercise.

1.Show that the plane $x-2y-z=3$ cuts the sphere $x^2+y^2+z^2-x-z-2=0$ in a circle of radius unity. Also find the equation of the sphere which has this circle as a great circle.

2. Find the spheres passing through the circle $x^2+y^2+z^2-6x-2z+5=0$, $y=0$ and touching the plane $3y+4z+5=0$.

Orthogonal spheres:

Two spheres are said to be Orthogonal if the tangent planes at a point of intersection are at right angles. The radii of such spheres through their point of intersection P being perpendicular to the tangent planes at P are also at right angles.

Figure

Thus the spheres cut orthogonally, if the square of the distance between their centers equal to the sum of the squares of their radii.

$$(C_1C_2)^2 = r_1^2 + r_2^2$$

1 . Show that the condition for the spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ **and** $x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ **to cut orthogonally is** $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$.

Solution : The centers of the sphere are $C_1=(-u_1,-v_1,-w_1)$, $C_2=(-u_2,-v_2,-w_2)$ and $r_1=\sqrt{u_1^2+v_1^2+w_1^2-d_1}$, $r_2=\sqrt{u_2^2+v_2^2+w_2^2-d_2}$

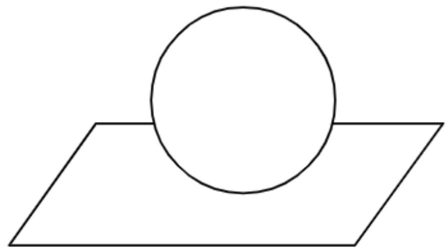
Spheres will cut orthogonally if $(C_1C_2)^2 = r_1^2 + r_2^2$.

$$\Rightarrow (u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2 = u_1^2+v_1^2+w_1^2-d_1+u_2^2+v_2^2+w_2^2-d_2$$

Simplifying, $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$ is the required condition.

Tangent plane to a sphere:

A plane q touches a sphere S if the perpendicular distance of the centre C of S from q is equal to the radius r of S. Then q is called a tangent plane to S.



Equation of the sphere S is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$, $C=(-u,-v,-w)$. Let $A=(x_1,y_1,z_1)$ be the point of contact between S and q.

Then the direction ratios of CA= (x_1+u, y_1+v, z_1+w). Consider any point P(x,y,z) in the tangent plane. The the direction ratios of $AP=(x-x_1, y-y_1, z-z_1)$.

Since the radius CA is perpendicular to q, we have

$(x-x_1)(x_1+u)+(y-y_1)(y_1+v)+(z-z_1)(z_1+w)=0$

$\Rightarrow xx_1+yy_1+zz_1+u(x-x_1)+v(y-y_1)+w(z-z_1)=x^2_1+y^2_1+z^2_1$

Since the point A lies on S, we have

$x^2_1+y^2_1+z^2_1=-2ux_1-2vy_1-2wz_1-d$

$\Rightarrow xx_1+yy_1+zz_1+u(x-x_1)+v(y-y_1)+w(z-z_1)=-2ux_1-2vy_1-2wz_1-d$

$\Rightarrow xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

This is the equation of the tangent plane q to S at the point $A=(x_1,y_1,z_1)$.

Example 1. Find the equation of the tangent plane at a point $(1,2,-2)$ to the sphere $x^2+y^2+z^2-2x-6y=0$.

Solution. $(u,v,w)=(-1,-3,0)$, d=0

Equation of the tangent plane is $x+2y-2z-(x+1)-3(y+2)=0$

$\Rightarrow y+2z+7=0$.

2. Find the tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$.

Solution. Any plane parallel to the given plane is $2x+2y-z+k=0$

This plane is a tangent plane to the given sphere S if the perpendicular distance p of the centre C of S is equal to the radius r of S.

$C=(2,-1,3), \quad r=\sqrt{4+1+9-5}=3$

$$p=\left|\frac{4-2-3+k}{\sqrt{4+4+1}}\right|=\frac{|k-1|}{3}$$

$$\therefore p=r \quad \text{if} \quad \frac{|k-1|}{3}=3$$

$\Rightarrow |k-1|=9, \; k=10$ or -8.

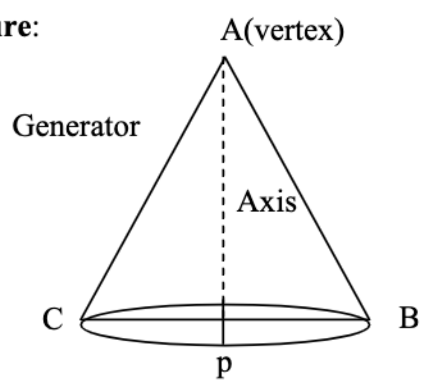
Equation of the tangent plane are $2x+2y-z+10=0$ and $2x+2y-z-8=0$.

Right Circular Cone :

Definition : A right circular cone is a surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line.

The constant angle θ is called the semi vertical angle, a fixed point is called a vertex and the fixed line AP is called the axis.

Figure:



Equation of a right circular cone:

Let (x_0, y_0, z_0) be the co-ordinates of the vertex A and (a, b, c) be the direction ratios of the axis. Consider any point $P(x, y, z)$ on the cone. Then the direction ratios of the generator AP are $(x-x_0, y-y_0, z-z_0)$ and

$$\cos\theta=\frac{a(x-x_0)+b(y-y_0)+c(z-z_0)}{\sqrt{\sum a^2}\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

Or $\left\{a(x-x_0)+b(y-y_0)+c(z-z_0)\right\}^2=(a^2+b^2+c^2)\left\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right\}\cos\theta$

This equation holds for any point P on the cone and hence is the equation of the right circular cone.

Example 1: Find the equation of the right circular cone whose vertex is the origin, whose axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and which has semi vertical angle of 30^0.

Solution. Given $(x_0,y_0,z_0)=(0,0,0)$. Direction ratios of the axis are $(1,2,3)$ and $\theta=30^0$.

Then the required equation is

$$(x+2y+3z)^2=(1^2+2^2+3^2)\left\{x^2+y^2+z^2\right\}\cos^2 30^0$$

$$\Rightarrow \quad x^2+4y^2+9z^2+4xy+12yz+6xz=14(x^2+y^2+z^2)\frac{3}{4}$$

$$\Rightarrow \quad 19x^2+13y^2+3z^2-8xy+24yz-12xz=0 \ .$$

Example 2: Find the equation of the right circular cone generated when the straight line $2y+3z=6,\ x=0$ revolves about z- axis.

Solution. Let $P(x,y,z)$ be any point on the cone. The vertex is the point of intersection of the line $2y+3z=6,\ x=0$ and the z- axis.

Therefore the vertex A is (0,0,2).

A generator of the cone is $\frac{x}{0}=\frac{y}{0}=\frac{z-2}{-2}$. Direction ratios of the generator are (0,3,2) and the axis are (0,0,1). The semi vertical angle θ is given by

$$\cos\theta=\frac{-2}{\sqrt{13}}$$

Let P(x,y,z) be any point on the cone so that the direction ratios of AP are (x,y,z-2). Since AP makes an angle θ with AZ, we have

$$\cos\theta=\frac{x.0+y.0+(z-2).1}{\sqrt{x^2+y^2+(z-2)^2}}$$

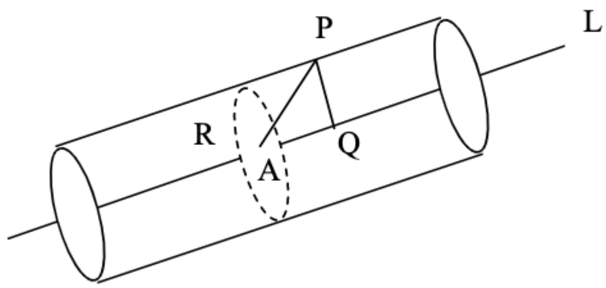
$$\Rightarrow \frac{4}{13}=\frac{(z-2)^2}{x^2+y^2+(z-2)^2}$$

$\Rightarrow 4x^2 + 4y^2 - 9z^2 + 36z - 36 = 0$, which is the required equation of the cone.

Right circular cylinder:

Definition: A right circular cylinder is a surface generated by a straight line which is parallel to a straight line and is at a constant distance from it. The constant distance is called the radius of the cylinder.

Equation of a right circular cylinder:



Let (l,m,n) be the direction cosines of the axis and $A(x_0, y_0, z_0)$ be a point on L. Consider an arbitrary point P(x,y,z) on the cylinder. If Q is the foot of the perpendicular from P onto L, then PQ=R, the radius of the cylinder. Also AQ is the projection of AP on L.

$$\therefore \; AQ = l(x - x_0) + m(y - y_0) + n(z - z_0)$$

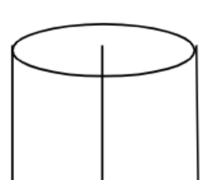
$$\text{Also} \quad AP^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

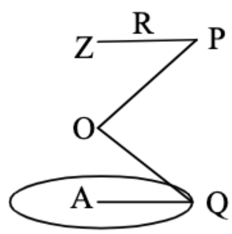
Then $AP^2 = AQ^2 + PQ^2$ is the required equation of the cylinder.

Example 1. The radius of a normal section of a right circular cylinder is 2 units. The axis lies along the straight line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$. Find its equation.

Solution.

Figure





Let P(x,y,z) be any point on the cylinder. Draw PN perpendicular to the axis AN. Then PN=2.

AN is the projection of AP on AN. Directio ratios of AN are (2,-1,5).

Direction cosines of AN= $\left(\frac{2}{\sqrt{30}},\frac{-1}{\sqrt{30}},\frac{5}{\sqrt{30}}\right)$

Then the required equation is

$$(x-1)^2+(y+3)^2+(z-2)^2-\left\{\frac{2}{\sqrt{30}}(x-1)-\frac{1}{\sqrt{30}}(y+3)+\frac{5}{\sqrt{30}}(z-2)\right\}^2=0.$$

Example 2: Find the equation of the right circular cylinder having the circle $x^2+y^2+z^2=9,\ x-y+z=3$ as base circle.

Solution.

The axis of the cylinder is the line through the centre of S and perpendicular to the plane q. We note that O(0,0,0) is the center of S and (1,-1,1) are the direction ratios of the normal to q.

The direction cosines of the axis are $\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$. The perpendicular distance from O on to the plane q is $OA=\left|\frac{-3}{\sqrt{1^2+1^2+1^2}}\right|=\sqrt{3}$

If Q is a point common to S and q, then OQ= radius of the sphere=3.

$\therefore AQ=\sqrt{OQ^2-OA^2}=\sqrt{9-3}=\sqrt{6}$

Let P(x,y,z) be any point on the cylinder. Then ZP= $\sqrt{6}$

The equation of the cylinder is

$$(x-0)^2+(y-0)^2+(z-0)^2-\left\{\frac{1}{\sqrt{3}}\left[(x-0)-(y-0)+(z-0)\right]\right\}^2=6$$

$$\Rightarrow x^2 + y^2 + z^2 + xy + yz - zx = 9.$$