PARTIAL DIFFERENTIATION

OBJECTIVES

At the end of this lesson students should be able to

- understand the concept of partial differentiation.
- differentiate a function partially with respect to each of its variables in turn.
- apply Euler's theorem on homogeneous functions.
- find the partial derivatives of implicit and composite functions.
- apply the concept of partial differentiation in calculating the errors and approximations

1. INTRODUCTION

In many applications in science and engineering, a function of interest depends on multiple variables. For instance, the ideal gas law $p = \rho RT$ states that the pressure p is a function of both its density ρ, and its temperature, T. (The gas constant R is a material property and not a variable). Consider volume of a container in the shape of a right circular cone: $V = \frac{1}{3}\pi r^2 h.$

Here the volume V depends on the two quantities r and h, representing the base radius and altitude of the cone respectively. These are simple example of a function of more than one variable.

The process of taking the derivative, with respect to a single variable, and holding constant all of the other independent variables, is called finding (or, taking) a partial derivative. Partial derivatives are distinguished from ordinary derivatives by using a ∂ instead of a d.

2. FUNCTION OF TWO VARIABLES

Let D be a region in the xy – plane. Suppose that f is a function that associates every $(x, y) \in D$, with a unique real number z. We write z = f(x, y) and we call z, a function two variables x and y. The variables x and y are called independent variables and z is called the dependent variable.

3. NEIGHBOURHOOD OF A POINT (a, b)

Let δ be any positive number. The points (x, y) such that $a-\delta \le x \le a+\delta$, $b-\delta \le y \le b+\delta$ determine a square bounded by the lines $x = a-\delta$, $x = a+\delta$; $y = b-\delta$, $y = b+\delta$. Its centre is at the point (a, b). This square is called a neighbourhood of the point (a, b). Thus the set $\{(x, y) : a-\delta \le x \le a+\delta,\ b-\delta \le y \le b+\delta\}$ is a neighbourhood of the point (a, b).

4. CONTINUITY OF A FUNCTION OF TWO VARIABLES.

We shall briefly introduce the concept of a continuous function in two variables. A real valued function w = f(x, y) defined on a region D of the xy-plane is said to be continuous at a point $(x_0, y_0) \in D$ if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

If f is continuous at every point in D, it is said to be continuous on D.

Example 1. Let $f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$

Consider the polar coordinates of the point (x, y), i.e. x = r cos θ and y = r sin θ. Then $\dfrac{xy}{\sqrt{x^2+y^2}} = r\cos\theta\sin\theta$. Since r → 0 as (x, y) → (0, 0) and |sin θ cos θ| ≤ 1 for all θ, we have

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$. Therefore, the function is continuous at the origin.

Example 2. Let $f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$

Suppose that (x, y) approaches (0, 0) along the line y = x. Then

$$\lim_{\substack{(x,y)\to(0,0) \\ y=x}} f(x,y) = \lim_{x\to 0} f(x,x) = \frac{1}{2}$$

Similarly, if (x, y) approaches (0, 0) along the line y = −x, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ y=-x}} f(x,y) = \lim_{x\to 0} f(x,-x) = -\frac{1}{2}$$

Therefore, the limit does not exist and the function is not continuous at the origin.

Remark: In order for the limit in $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ to exist, f(x, y) must approach $f(x_0, y_0)$ for each and every path of approach of (x, y) to (x_0, y_0).

5. PARTIAL DERIVATIVES

Let z = f (x, y) be a function of two variables x and y. Then

$$\lim_{h \to 0} \frac{f(a+h, b)-f(a, b)}{h},$$

if it exists, is said to be the first order partial derivative of f w.r.t x at (a, b) and is denoted by

$$\left(\frac{\partial z}{\partial x}\right)_{(a, b)} \text{ or } f_x(a, b)\cdot$$

Similarly,

$$\lim_{k \to 0} \frac{f(a, b+k)-f(a, b)}{k},$$

if it exists, is said to be the first order partial derivative of f w.r.t y at (a, b) and is denoted by

$$\left(\frac{\partial z}{\partial y}\right)_{(a,b)} \text{ or } f_y(a,b)\cdot$$

If the partial derivatives of f exist at each point in the domain of definition of f, then they are given by

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y)-f(x, y)}{\Delta x} = f_x$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y)-f(x, y)}{\Delta y} = f_y$$

Example 3: Let z = f(x, y) = x^3+ y^3- 3xy. Then

$$\frac{\partial z}{\partial x} = 3x^2 - 3y, \quad \frac{\partial z}{\partial y} = 3y^2 - 3x.$$

We see that f_x and f_y are functions of x and y again and hence may be differentiable wr.t x and w.r.t y. If the partial derivatives exist, they are called the second order partial derivatives ad are given as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}.$$

In a similar way one can define third and higher order partial derivatives.

Note: The two second partial derivatives f_{xy} and f_{yx} above, the ones with one partial derivative with respect to x and one with respect to y, are called mixed partial derivatives. If the partial order derivatives are continuous then the mixed partial derivatives are equal i.e., the order in which we differentiate f is immaterial. For instance

$f_{xy} = f_{yx}$; $f_{xxy} = f_{xyx} = f_{yxx}$, $f_{xyy} = f_{yyx} = f_{yxy}$.

Remark: Just because the order of partial differentiation doesn't (typically) matter as far as the final resulting higher-order partial derivative is concerned, that doesn't mean that calculating the partial derivatives in different orders is equally easy. For example, consider

$$f(x,y) = xe^{5y} + \frac{e^x \cos\left(x + \tan^{-1} x\right)}{\sqrt{1 - \log x}}$$

If you want to calculate the second partial derivative of f, once with respect to x and once with respect to y, it would be a **painful waste** of time to calculate $\partial f/\partial x$ first. If this isn't obvious to you, you should think about it until it's clear.

What you want to do is calculate the partial derivative with respect to y first, since, then, the entire right-hand ugly expression will disappear. Hence, we find that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(5xe^{5y}\right) = 5e^{5y}.$$

6. GEOMETRICAL REPRESENTATION OF PARTIAL DERIVATIVES OF THE FIRST ORDER.

The derivative of a one-variable function can be interpreted graphically as the slope of the tangent line. Is there also a way to interpret the partial derivatives graphically? Yes.

Geometrically the function z = f(x,y) represents a surface. Then $f_x(a,b)$ denotes the tangent of the angle which the tangent to the curve , in which the plane y = b parallel to the ZX plane cuts the surface at the point P(a,b, f(a,b)), makes with X-axis.

Similarly, $f_y(a, b)$ denotes the tangent of the angle which the tangent to the curve, in which the plane x = a parallel to the ZY plane cuts the surface at the point P(a,b, f(a,b)), makes with Y-axis.

Example 4: Find ∂z/∂x and ∂z /∂y if xy + yz + zx = 1.

Solution: xy + yz + zx – 1= 0 (1)

Differentiating (1) partially w.r.to x, we get

$$y + y\,\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial z}{\partial x} = -\left(\frac{y+z}{x+y}\right)$$

Differentiating (1) partially w.r.to y, we get

$$x + y\,\frac{\partial z}{\partial y} + z + x\frac{\partial z}{\partial y} = 0$$

$$\therefore \frac{\partial z}{\partial y} = -\left(\frac{x+z}{x+y}\right)$$

Example 5 : Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ where $u = \sin^{-1}\frac{x}{y}$

Solution : $\frac{\partial u}{\partial y} = -\frac{x}{y\sqrt{y^2 - x^2}}$, $\quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{y}{\left(y^2 - x^2\right)^{\frac{3}{2}}}$

$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}}$, $\quad \frac{\partial^2 u}{\partial y \partial x} = -\frac{y}{\left(y^2 - x^2\right)^{\frac{3}{2}}}$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Example 6 : If $z = e^{ax+by} f\left(ax - by\right)$ prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$

Solution : $\frac{\partial z}{\partial x} = a e^{ax+by} f\left(ax - by\right) + a e^{ax+by} f'\left(ax - by\right)$

$$\frac{\partial z}{\partial y} = b e^{ax+by} f\left(ax - by\right) - b e^{ax+by} f'\left(ax - by\right)$$

$$\therefore\ b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

Example 7 : If $u=f(r)$ where $r^2=x^2+y^2$ prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$

Solution : $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\times\frac{\partial r}{\partial x}=f'(r)\times\frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{r}f'(r)-\frac{x^2 f'(r)}{r^3}+\frac{x^2 f''(r)}{r^2}$$

similarly, $$\frac{\partial^2 u}{\partial y^2}=\frac{1}{r}f'(r)-\frac{y^2 f'(r)}{r^3}+\frac{y^2 f''(r)}{r^2}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$$

Example 8: If

$$u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right);\ xy\neq 0$$

prove that

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}.$$

Solution:

$$\frac{\partial u}{\partial y}=x^2\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}-2y\tan^{-1}\left(\frac{x}{y}\right)+y^2\frac{1}{1+\frac{x^2}{y^2}}\frac{x}{y^2}$$

$$=\frac{x^3}{x^2+y^2}+\frac{xy^2}{x^2+y^2}-2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$=x-2y\tan^{-1}\left(\frac{x}{y}\right).$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$=1-2y\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y}=\frac{x^2-y^2}{x^2+y^2}.$$

PROBLEMS:

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

2. If $u=\left(1-2xy+y^2\right)^{-\frac{1}{2}}$, prove that $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=y^2u^3$

3. If $u=\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

4. Verify $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$ where $u=\log\left(\frac{x^2+y^2}{xy}\right)$

5. If $z\left(x+y\right)=x^2+y^2$ show that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$

7. HOMOGENEOUS FUNCTIONS:

A function z = f(x, y) is said to be a homogeneous function of degree n in x and y if we can write

$$z=x^n\phi\left(\frac{y}{x}\right)=y^n\psi\left(\frac{x}{y}\right).$$

Example 1:

Let $f(x, y)=\frac{x^3+y^3}{x-y}$. Then.

$$f(x,y)=\frac{x^3+y^3}{x-y}=\frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\left(\frac{y}{x}\right)\right]}=x^2\phi\left(\frac{y}{x}\right)$$

Thus f(x, y) is a homogeneous function of degree 2.

Example 2:

Let f(x, y) = $x^3+y^3\log y-y^3\log x+x^2y\sin\left(\frac{y}{x}\right)$. Then

$$f(x,y) = x^3 + y^3 \log\left(\frac{y}{x}\right) + x^2 y \sin\left(\frac{y}{x}\right)$$

$$= \mathrm{x}^3\left[1+\left(\frac{y}{x}\right)^3 \log\left(\frac{y}{x}\right)+\frac{y}{x}\sin\left(\frac{y}{x}\right)\right]$$

$$= \mathrm{x}^3\varphi\left(\frac{y}{x}\right).$$

8. EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS:

Statement: If z = f(x, y) is a homogeneous function of degree n in x and y then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz,\ \forall\ x, y \text{ in the domain of of f.}$$

Proof: Since z is a homogeneous function of degree n in x and y, we can express it in the form

$$z = x^n \phi\left(\frac{y}{x}\right).$$

$$\frac{\partial z}{\partial x} = x^n \phi'\left(\frac{y}{x}\right)\cdot y\left(\frac{-1}{x^2}\right) + nx^{n-1}\phi\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right).$$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) + x^{n-1}\left[-y\phi' + y\phi'\right] = nz\,.$$

Corollary: If z is a homogeneous function of degree n, then

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

Proof:

Since z is a homogeneous function of degree n, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad ....(1)$$

Differentiate equation (1) partially with respect to *x*, we get

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x}.$$

Multiplying by *x*, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+xy\frac{\partial^2 z}{\partial x\partial y}=(n-1)x\frac{\partial z}{\partial x}\quad ....(2)$$

Differentiate equation (1) partially with respect to *y*, we get

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=n\frac{\partial z}{\partial y}$$

Multiplying by *y*, we get

$$xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)y\frac{\partial z}{\partial y}\quad ....(3)$$

Adding equation (1) and (2), we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=n(n-1)z.$$

Example 1: Verify Euler's theorem for

$$z=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right).$$

Solution:

$$z=x^0\left[\sin^{-1}\left(\frac{1}{y/x}\right)+\tan^{-1}\left(\frac{y}{x}\right)\right]=x^0\phi\left(\frac{y}{x}\right).$$

Hence z is homogeneous function of degree n = 0.

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}.\frac{1}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{-y}{x^2}$$

$$=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}.$$

Similarly,

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{y^2-x^2}}\left(\frac{-x}{y}\right)+\frac{x}{x^2+y^2}$$

$$\therefore\ x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0=nz.$$

Hence verified.

Example 2: If

$$u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{2}\tan u.$$

Solution:

Let $z=\sin u$. Then

$$\mathrm{z}=\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]=\frac{x\left[1+\frac{y}{x}\right]}{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}=x^{1/2}\phi\left(\frac{y}{x}\right).$$

Thus z is a homogeneous function of degree ½. Hence by Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n\,z=\frac{1}{2}\sin u.$$

$$\Rightarrow x\frac{\partial(\sin u)}{\partial x}+y\frac{\partial(\sin u)}{\partial y}=\frac{1}{2}\sin u.$$

$$\Rightarrow \cos u\ x\frac{\partial u}{\partial x}+\cos u\ y\frac{\partial u}{\partial y}=\frac{1}{2}\sin u.$$

$$\Rightarrow x\frac{\partial u}{\partial x}+\ y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$$

Example 3: If

$$u=\operatorname{cosec}^{-1}\left[\frac{\sqrt{x}+\sqrt{y}}{\sqrt[3]{x}+\sqrt[3]{y}}\right]^{1/2}$$

prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right).$$

Solution: Let

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{12}z$$

$$-x\cos ecu\ \ \cot u\ \ \frac{\partial u}{\partial x}-y\cos ecu\ \ \cot u\frac{\partial u}{\partial y}=\frac{1}{12}\cos ecu$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\frac{\cos ecu}{\cos ecu\ \ \cot u}=-\frac{1}{12}\tan u \qquad ....(1)$$

Differentiate equation (1) partially with respect to *x*, We get

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(-1-\frac{1}{12}\sec^2 u\right)\cdot\frac{\partial u}{\partial x}$$

Multiplying by *x*, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=\left(-1-\frac{1}{12}\sec^2 u\right)\cdot x\frac{\partial u}{\partial x} \qquad ....(2)$$

Differentiate equation (1) partially with respect to *y*, We get

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(-1-\frac{1}{12}\sec^2 u\right).\frac{\partial u}{\partial y}$$

Multiplying by *y*, we get

$$xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(-1-\frac{1}{12}\sec^2 u\right).y\frac{\partial u}{\partial y} \qquad ....(3)$$

Adding equation (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left[-1-\frac{1}{12}\sec^2 u\right]\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=-\left[1+\frac{1}{12}\left(1+\tan^2 u\right)\right]\left(-\frac{1}{12}\tan u\right)$$

$$= \frac{1}{12}\tan u - \left[\frac{13}{12} + \frac{\tan^2 u}{12}\right]$$

Exercise :

1. Verify Euler's theorem for the following functions:

 (i) $z = ax^2 + 2hxy + by^2$ (ii) $z = (x^2 + xy + y^2)^{-1}$.

2. If $u = \log\dfrac{x^2 + y^2}{x + y}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.$

3. If $u = \tan^{-1}\left[\dfrac{x^3 + y^3}{x + y}\right]$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u.$

4. If $u = \tan^{-1}\left[\dfrac{x^3 + y^3}{x - y}\right]$ then show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \left(1 - 4\sin^2 u\right)\sin 2u.$

5. If $u = \sin^{-1}\left[\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right]$ then show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{4}\left[\tan^3 u - \tan u\right].$

9. TOTAL DERIVATIVES:

Let z = f(x, y).Then the total differential dz is defined as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

10. COMPOSITE FUNCTIONS:

A function of a function is called a composite function.

Let z = f(x, y) where x = $\phi(t)$ and y = $\psi(t)$.

Now, we can express z as s function of *t* alone by substituting the values of x and y in f(x, y).

Then z is a composite function of *t*.

Thus the ordinary derivative dz/dt which is called the **total derivative** of and is given by,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Total partial derivative theorem for composite function:

If z = f(u, v) where u = ϕ(x, y) and v = ψ(x, y), then the partial derivatives of z are given by

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

Example 1:

Find $\frac{\mathrm{dz}}{\mathrm{dt}}$ given that $z = xy^2 + x^2 y,\ x = at^2,\ y = 2at.$

Solution:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \left(y^2 + 2xy\right)\left(2at\right) + \left(2xy + x^2\right)\left(2a\right)$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$= 16a^3t^3 + 10a^3t^4$$

Example 2:

If z = $x^2 + y^2,\ x = \cos uv,\ y = \sin(u+v),$ find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ interms of u and v.

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= 2x\left[-v\sin uv\right] + 2y\left[\cos(u+v)\right]$$

$$= -2v\sin uv\ \cos uv + 2\sin(u+v)\cos(u+v)$$

$$= -v\sin 2uv + \sin 2(u+v).$$

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= -u\sin 2uv + \sin 2(u+v).$$

Example 3:

If z = f(x, y), $x = e^u \sin v,\ y = e^u \cos v$ prove that $x\dfrac{\partial z}{\partial u} + y\dfrac{\partial z}{\partial v} = \left(x^2 + y^2\right)\dfrac{\partial z}{\partial x}.$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\left[e^u \sin v\right] + \frac{\partial z}{\partial y}\left[e^u \cos v\right]$$

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left[e^u \cos v\right] + \frac{\partial z}{\partial y}\left[-e^u \cos v\right]$$

$$\therefore\ x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v} = x\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] + y\left[y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y}\right]$$

$$= \left(x^2 + y^2\right)\frac{\partial z}{\partial x}.$$

Example 4:

If $H = f\left(y - z,\ z - x,\ x - y\right),$ prove that $\dfrac{\partial H}{\partial x} + \dfrac{\partial H}{\partial y} + \dfrac{\partial H}{\partial z} = 0.$

Solution:

Let H = f(u, v, w) where $u = y - z,\ v = z - x$ and $w = x - y.$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial x}.$$

$$= \frac{\partial H}{\partial u}(0) + \frac{\partial H}{\partial v}(-1) + \frac{\partial H}{\partial w}(1).$$

Similarly,

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}(1) + \frac{\partial H}{\partial v}(0) + \frac{\partial H}{\partial w}(-1).$$

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}(-1) + \frac{\partial H}{\partial v}(1) + \frac{\partial H}{\partial w}(0).$$

Hence

$$\frac{\partial H}{\partial x}+\frac{\partial H}{\partial y}+\frac{\partial H}{\partial z}=0.$$

Example 5:

If $z=f(x,y)$ and $x=r\cos\theta,\ y=r\sin\theta,$ prove that $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$

Solution:

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}.$$

$$=\frac{\partial z}{\partial x}\cdot\cos\theta+\frac{\partial z}{\partial y}\cdot\sin\theta.$$

Similarly,

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta).$$

Hence,

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\left[\cos^2\theta+\frac{1}{r^2}r^2\sin^2\theta\right]+\left(\frac{\partial z}{\partial y}\right)^2\left[\sin^2\theta+\frac{1}{r^2}r^2\cos^2\theta\right]$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Exercise

1. If $z=f(x,y)$ and $x=e^{u}+e^{-v},\ y=e^{-u}-e^{v},$ prove that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$

2. If $x=r\cos\theta,\ y=r\sin\theta,$ find (i) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}$ (ii) $\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}.$

3. If $z=e^{u-2v}$ and $u=\sin x,\ y=x^2+y^2,$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}.$

11. IMPLICIT DIFFERENTIATION:

When $f(x,y)$ is a function of two variables x and y, the equation $f(x,y)=c$ (where c is a constant) enables us to obtain values of y corresponding to values of x. Then we say that y

is an implicit function of x. We assume, for simplicity, that the above equation always defines a unique value of y for each value of x.

Since f is a function of x and y, and y is again a function of x, we can consider f a composite function of x. Then, its total derivative with respect to x is

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

But since $f(x, y) = c$, the total derivative of f must be identically 0. Thus

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial y / \partial y} = -\frac{f_x}{f_y} \text{ if } f_y \neq 0.$$

Differentiating again with respect to x, considering $\partial f / \partial x$ and $\partial f / \partial y$ as composite functions of x, we get

$$\frac{d^2 y}{dx^2} = -\frac{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dx}\right)\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}\right)}{\left(\frac{\partial f}{\partial y}\right)^2}$$

$$= -\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial^2 f}{\partial y \partial x}\ \frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3}$$

Thus,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \quad \text{and} \quad \frac{d^2 y}{dx^2} = -\frac{f_{xx}\left(f_y\right)^2 - 2 f_{yx} f_x f_y + f_{yy}\left(f_x\right)^2}{(f_y)^3}$$

Example 1: If $x^y = y^x$, find $\frac{dy}{dx}$.

Solution:

$x^y = y^x$
Taking log on both sides, we get
y logx = x logy
f(x, y) = y logx - x logy = 0

$$\therefore \ \frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{f_x}{f_y} = \frac{\frac{y}{x} - \log y}{\log x - \frac{x}{y}} = \frac{y\left(y - x\log y\right)}{x\left(x - y\log x\right)}.$$

Example 2: Prove that if $y^3 - 3ax^2 + x^3 = 0$, then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$.

Solution: Let

$$f(x, y) = y^3 - 3ax^2 + x^3 = 0.$$

We have

$$f_x = -6ax + 3x^2, \ f_y = 3y^2.$$

$$f_{xx} = -6a + 6x, \ f_{xy} = 0, \ f_{yy} = 6y$$

Therefore,

$$\frac{d^2y}{dx^2} = -\frac{6(x-a)9y^4 + (3x^2 - 6ax)^2 6y}{27y^6} = -2\frac{a^2x^2}{y^5}$$

EXERCISE

1. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(bx + by + f)^3}$$

2. If A, B, C are the angles of a triangle such that $\sin^2 A + \sin^2 B + \sin^2 C$ is constant, show that

$$\frac{dA}{dB} = \frac{\tan B - \tan C}{\tan C - \tan A}$$

12. ERRORS AND APPROXIMATIONS

Let $f(x, y)$ be a continuous function of x and y. If δx and δy are the increments of x and y, then the new value of $f(x, y)$ will be $f(x + \delta x, y + \delta y)$. Hence

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y).$$

Expanding $f(x+\delta x, y+\delta y)$ by Taylor's theorem and supposing δx, δy to be small enough that their products, squares, and higher powers can be neglected, we get

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \text{, approximately.}$$

The value δf is called the error in f due to the errors δx and δy in x and y. $\frac{\delta f}{f}$ is called the relative error in f and $\frac{\delta f}{f}\times 100$ is called the percentage error in f.

Example 1: If $PV^2 = K$ and if the relative errors in P is 0.05 and in V is 0.025 then show that error in K is 10%.

Solution:

$$\frac{\delta P}{P} = 0.05 \text{ and } \frac{\delta V}{V} = 0.025$$

$$PV^2 = K$$

Taking log on both sides of the equation,

$$\log P + 2\log V = \log K$$

$$\delta(\log P) + 2\delta(\log V) = \delta(\log K)$$

$$\frac{1}{P}\delta P + 2\frac{1}{V}\delta V = \frac{1}{K}\delta K \Rightarrow 0.05 + 2(0.025) = \frac{\delta K}{K}$$

$$\frac{\delta K}{K} = 0.1$$

Thus, error is

$$100\times\frac{\delta K}{K} = (0.1)\times 100 = 10\%$$

Example 2: The time T of a complete oscillation of a simple pendulum is given by the formula $T = 2\pi\sqrt{l/g}$. If g is a constant find the error in the calculated value of T due to an error of 3% in the value of l.

Solution:

$$T = 2\pi\sqrt{l/g}$$

Taking log on both sides,

$$\log T = \log 2\pi + \frac{1}{2}\log\frac{l}{g}$$

$$\delta(\log T) = \delta(\log 2\pi) + \delta\left(\frac{1}{2}(\log l - \log g)\right)$$

$$\frac{\delta T}{T} = 0 + \frac{1}{2}\frac{\delta l}{l} - 0$$

$$100\frac{\delta T}{T} = \frac{1}{2}\frac{\delta l}{l}\times 100 = \frac{1}{2}\times 3 = 1.5$$

Thus, the error in T is 1.5%

Example 3: If the sides and angles of a plane triangle ΔABC vary in such a way that its circum-radius remains constant, prove that $\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C}$, where δa, δb, and δc denote small increments in the sides a, b and c respectively.

Solution:

Let R be the circum-radius of the ΔABC, then

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

$$\Rightarrow 2R\sin A, b = 2R\sin B, c = 2R\sin C$$

Differentiating, we get

$$\delta a = 2R\cos A\,\delta A, \delta b = 2R\cos B\,\delta B, \delta c = 2R\cos C\,\delta C$$

$$\Rightarrow \frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2R(\delta A + \delta B + \delta C)$$

$$= 2R\delta(A + B + C) = 2R\delta(\pi) = 0.$$