PARTIAL DIFFERENTIATION

OBJECTIVES

At the end of this lesson students should be able to

- understand the concept of partial differentiation.
- differentiate a function partially with respect to each of its variables in turn.
- apply Euler's theorem on homogeneous functions.
- find the partial derivatives of implicit and composite functions.
- apply the concept of partial differentiation in calculating the errors and approximations

1. INTRODUCTION

In many applications in science and engineering, a function of interest depends on multiple variables. For instance, the ideal gas law $p = \rho RT$ states that the pressure p is a function of both its density ρ, and its temperature, T. (The gas constant R is a material property and not a variable). Consider volume of a container in the shape of a right circular cone: $V = \frac{1}{3}\pi r^2 h$.

Here the volume V depends on the two quantities r and h, representing the base radius and altitude of the cone respectively. These are simple example of a function of more than one variable.

The process of taking the derivative, with respect to a single variable, and holding constant all of the other independent variables, is called finding (or, taking) a partial derivative. Partial derivatives are distinguished from ordinary derivatives by using a ∂ instead of a d.

2. FUNCTION OF TWO VARIABLES

Let D be a region in the xy – plane. Suppose that f is a function that associates every $(x, y) \in D$, with a unique real number z. We write $z = f(x, y)$ and we call z, a function two variables x and y. The variables x and y are called independent variables and z is called the dependent variable.

3. NEIGHBOURHOOD OF A POINT (a, b)

Let δ be any positive number. The points (x, y) such that $a - \delta \le x \le a + \delta$, $b - \delta \le y \le b + \delta$ determine a square bounded by the lines $x = a - \delta$, $x = a + \delta$; $y = b - \delta$, $y = b + \delta$. Its centre is at the point (a, b). This square is called a neighbourhood of the point (a, b). Thus the set $\{(x, y) : a-\delta \le x \le a+\delta,\ b-\delta \le y \le b+\delta\}$ is a neighbourhood of the point (a, b).

4. CONTINUITY OF A FUNCTION OF TWO VARIABLES.

We shall briefly introduce the concept of a continuous function in two variables. A real valued function w = f(x, y) defined on a region D of the xy-plane is said to be continuous at a point $(x_0, y_0) \in D$ if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$.

If f is continuous at every point in D, it is said to be continuous on D.

Example 1. Let $f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$

Consider the polar coordinates of the point (x, y), i.e. x = r cos θ and y = r sin θ. Then $\dfrac{xy}{\sqrt{x^2+y^2}}=r\cos\theta\sin\theta$. Since $r \to 0$ as $(x, y) \to (0, 0)$ and $|\sin\theta\cos\theta| \le 1$ for all θ, we have $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}=0=f(0,0)$. Therefore, the function is continuous at the origin.

Example 2. Let $f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$

Suppose that (x, y) approaches (0, 0) along the line y = x. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y)=\lim_{x\to 0} f(x,x)=\frac{1}{2}$$

Similarly, if (x, y) approaches (0, 0) along the line y = −x, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=-x}} f(x,y)=\lim_{x\to 0} f(x,-x)=-\frac{1}{2}$$

Therefore, the limit does not exist and the function is not continuous at the origin.

Remark: In order for the limit in $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$ to exist, f(x, y) must approach $f(x_0, y_0)$ for each and every path of approach of (x, y) to (x_0, y_0).

5. PARTIAL DERIVATIVES

Let z = f (x, y) be a function of two variables x and y. Then

$$\lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h},$$

if it exists, is said to be the first order partial derivative of f w.r.t x at (a, b) and is denoted by

$$\left(\frac{\partial z}{\partial x}\right)_{(a, b)} \text{ or } f_x(a, b)\cdot$$

Similarly,

$$\lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k},$$

if it exists, is said to be the first order partial derivative of f w.r.t y at (a, b) and is denoted by

$$\left(\frac{\partial z}{\partial y}\right)_{(a,b)} \text{ or } f_y(a,b)\cdot$$

If the partial derivatives of f exist at each point in the domain of definition of f, then they are given by

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = f_x$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = f_y$$

Example 3: Let $z = f(x, y) = x^3 + y^3 - 3xy$. Then

$$\frac{\partial z}{\partial x} = 3x^2 - 3y, \quad \frac{\partial z}{\partial y} = 3y^2 - 3x.$$

We see that f_x and f_y are functions of x and y again and hence may be differentiable wr.t x and w.r.t y. If the partial derivatives exist, they are called the second order partial derivatives ad are given as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}.$$

In a similar way one can define third and higher order partial derivatives.

Note: The two second partial derivatives f_{xy} and f_{yx} above, the ones with one partial derivative with respect to x and one with respect to y, are called mixed partial derivatives. If the partial order derivatives are continuous then the mixed partial derivatives are equal i.e., the order in which we differentiate f is immaterial. For instance

$f_{xy} = f_{yx}$; $f_{xxy} = f_{xyx} = f_{yxx}$, $f_{xyy} = f_{yyx} = f_{yxy}$.

Remark: Just because the order of partial differentiation doesn't (typically) matter as far as the final resulting higher-order partial derivative is concerned, that doesn't mean that calculating the partial derivatives in different orders is equally easy. For example, consider

$$f(x,y) = xe^{5y} + \frac{e^x \cos\left(x + \tan^{-1} x\right)}{\sqrt{1 - \log x}}$$

If you want to calculate the second partial derivative of f, once with respect to x and once with respect to y, it would be a **painful waste** of time to calculate ∂f/∂x first. If this isn't obvious to you, you should think about it until it's clear.

What you want to do is calculate the partial derivative with respect to y first, since, then, the entire right-hand ugly expression will disappear. Hence, we find that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(5xe^{5y}\right) = 5e^{5y}.$$

6. GEOMETRICAL REPRESENTATION OF PARTIAL DERIVATIVES OF THE FIRST ORDER.

The derivative of a one-variable function can be interpreted graphically as the slope of the tangent line. Is there also a way to interpret the partial derivatives graphically? Yes.

Geometrically the function z = f(x,y) represents a surface. Then $f_x(a,b)$ denotes the tangent of the angle which the tangent to the curve , in which the plane y = b parallel to the ZX plane cuts the surface at the point P(a,b, f(a,b)), makes with X-axis.

Similarly, $f_y(a, b)$ denotes the tangent of the angle which the tangent to the curve, in which the plane x = a parallel to the ZY plane cuts the surface at the point P(a,b, f(a,b)), makes with Y-axis.

Example 4: Find $\partial z/\partial x$ and $\partial z/\partial y$ if $xy + yz + zx = 1$.

Solution: $xy + yz + zx - 1 = 0$ (1)

Differentiating (1) partially w.r.to x, we get

$$y + y\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial z}{\partial x} = -\left(\frac{y+z}{x+y}\right)$$

Differentiating (1) partially w.r.to y, we get

$$x + y\frac{\partial z}{\partial y} + z + x\frac{\partial z}{\partial y} = 0$$

$$\therefore \frac{\partial z}{\partial y} = -\left(\frac{x+z}{x+y}\right)$$

Example 5: Verify $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ where $u = \sin^{-1}\dfrac{x}{y}$

Solution: $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y\sqrt{y^2 - x^2}}$, $\quad \dfrac{\partial^2 u}{\partial x \partial y} = -\dfrac{y}{\left(y^2 - x^2\right)^{\frac{3}{2}}}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}}, \quad \frac{\partial^2 u}{\partial y \partial x} = -\frac{y}{\left(y^2 - x^2\right)^{\frac{3}{2}}}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Example 6: If $z = e^{ax+by} f(ax - by)$ prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz.$

Solution: $\dfrac{\partial z}{\partial x} = a e^{ax+by} f(ax - by) + a e^{ax+by} f'(ax - by)$

$$\frac{\partial z}{\partial y} = b e^{ax+by} f(ax - by) - b e^{ax+by} f'(ax - by)$$

$$\therefore b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

Example 7 : If $u=f(r)$ where $r^2=x^2+y^2$ prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$

Solution : $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\times\frac{\partial r}{\partial x}=f'(r)\times\frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{r}f'(r)-\frac{x^2 f'(r)}{r^3}+\frac{x^2 f''(r)}{r^2}$$

similarly, $$\frac{\partial^2 u}{\partial y^2}=\frac{1}{r}f'(r)-\frac{y^2 f'(r)}{r^3}+\frac{y^2 f''(r)}{r^2}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$$

Example 8: If

$$u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right);\ xy\ \neq\ 0$$

prove that

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}.$$

Solution:

$$\frac{\partial u}{\partial y}=x^2\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}-2y\tan^{-1}\left(\frac{x}{y}\right)+y^2\frac{1}{1+\frac{x^2}{y^2}}\frac{x}{y^2}$$

$$=\frac{x^3}{x^2+y^2}+\frac{xy^2}{x^2+y^2}-2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$=x\ -\ 2y\tan^{-1}\left(\frac{x}{y}\right).$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$=1-2y\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y}=\frac{x^2-y^2}{x^2+y^2}.$$

PROBLEMS:

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

2. If $u=\left(1-2xy+y^2\right)^{-\frac{1}{2}}$, prove that $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=y^2u^3$

3. If $u=\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

4. Verify $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$ where $u=\log\left(\frac{x^2+y^2}{xy}\right)$

5. If $z(x+y)=x^2+y^2$ show that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$

7. HOMOGENEOUS FUNCTIONS:

A function z = f(x, y) is said to be a homogeneous function of degree n in x and y if we can write

$$z=x^n\phi\left(\frac{y}{x}\right)=y^n\psi\left(\frac{x}{y}\right).$$

Example 1:

Let $f(x, y)=\frac{x^3+y^3}{x-y}$. Then.

$$f(x,y)=\frac{x^3+y^3}{x-y}=\frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\left(\frac{y}{x}\right)\right]}=x^2\phi\left(\frac{y}{x}\right)$$

Thus f(x, y) is a homogeneous function of degree 2.

Example 2:

Let $f(x, y)=x^3+y^3\log y-y^3\log x+x^2y\sin\left(\frac{y}{x}\right)$. Then

$$f(x,y)=x^3+y^3\log\left(\frac{y}{x}\right)+x^2y\sin\left(\frac{y}{x}\right)$$

$$=\mathrm{x}^3\left[1+\left(\frac{y}{x}\right)^3\log\left(\frac{y}{x}\right)+\frac{y}{x}\sin\left(\frac{y}{x}\right)\right]$$

$$=\mathrm{x}^3\varphi\left(\frac{y}{x}\right).$$

8. EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS:

Statement: If z = f(x, y) is a homogeneous function of degree n in x and y then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz,\ \forall\ x,y \text{ in the domain of of f.}$$

Proof: Since z is a homogeneous function of degree n in x and y, we can express it in the form

$$z=x^n\phi\left(\frac{y}{x}\right).$$

$$\frac{\partial z}{\partial x}=x^n\phi'\left(\frac{y}{x}\right)\cdot y\left(\frac{-1}{x^2}\right)+nx^{n-1}\phi\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y}=x^n\phi'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right).$$

Hence,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nx^n\phi\left(\frac{y}{x}\right)+x^{n-1}\left[-y\phi'+y\phi'\right]=nz.$$

Corollary: If z is a homogeneous function of degree n, then

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z.$$

Proof:

Since z is a homogeneous function of degree n, we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \quad \text{...........................(1)}$$

Differentiate equation (1) partially with respect to *x*, we get

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}.$$

Multiplying by x, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+xy\frac{\partial^2 z}{\partial x\partial y}=(n-1)x\frac{\partial z}{\partial x} \quad ....(2)$$

Differentiate equation (1) partially with respect to y, we get

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=n\frac{\partial z}{\partial y}$$

Multiplying by y, we get

$$xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)y\frac{\partial z}{\partial y} \quad ....(3)$$

Adding equation (1) and (2), we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=n(n-1)z.$$

Example 1: Verify Euler's theorem for

$$z=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right).$$

Solution:

$$z=x^0\left[\sin^{-1}\left(\frac{1}{y/x}\right)+\tan^{-1}\left(\frac{y}{x}\right)\right]=x^0\phi\left(\frac{y}{x}\right).$$

Hence z is homogeneous function of degree n = 0.

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}.\frac{1}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{-y}{x^2}$$

$$=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}.$$

Similarly,

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{y^2-x^2}}\left(\frac{-x}{y}\right)+\frac{x}{x^2+y^2}$$

$$\therefore\ x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0=nz.$$

Hence verified.

Example 2: If

$$u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{2}\tan u.$$

Solution:

Let $z=\sin u$. Then

$$z=\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]=\frac{x\left[1+\frac{y}{x}\right]}{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}=x^{1/2}\phi\left(\frac{y}{x}\right).$$

Thus z is a homogeneous function of degree ½. Hence by Euler’s theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n\,z=\frac{1}{2}\sin u.$$

$$\Rightarrow x\frac{\partial(\sin u)}{\partial x}+y\frac{\partial(\sin u)}{\partial y}=\frac{1}{2}\sin u.$$

$$\Rightarrow \cos u\ x\frac{\partial u}{\partial x}+\cos u\ y\frac{\partial u}{\partial y}=\frac{1}{2}\sin u.$$

$$\Rightarrow x\frac{\partial u}{\partial x}+\ y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$$

Example 3: If

$$u=\operatorname{cosec}^{-1}\left[\frac{\sqrt{x}+\sqrt{y}}{\sqrt[3]{x}+\sqrt[3]{y}}\right]^{1/2}$$

prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right).$$

Solution: Let

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{12}z$$

$$-x\cos ecu\ \cot u\ \frac{\partial u}{\partial x}-y\cos ecu\ \cot u\frac{\partial u}{\partial y}=\frac{1}{12}\cos ecu$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\frac{\cos ecu}{\cos ecu\ \cot u}=-\frac{1}{12}\tan u \quad \text{................(1)}$$

Differentiate equation (1) partially with respect to *x*, We get

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(-1-\frac{1}{12}\sec^2 u\right)\cdot\frac{\partial u}{\partial x}$$

Multiplying by *x*, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=\left(-1-\frac{1}{12}\sec^2 u\right)\cdot x\frac{\partial u}{\partial x} \quad \text{..................(2)}$$

Differentiate equation (1) partially with respect to *y*, We get

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(-1-\frac{1}{12}\sec^2 u\right).\frac{\partial u}{\partial y}$$

Multiplying by *y*, we get

$$xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(-1-\frac{1}{12}\sec^2 u\right).y\frac{\partial u}{\partial y} \quad \text{..................(3)}$$

Adding equation (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left[-1-\frac{1}{12}\sec^2 u\right]\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=-\left[1+\frac{1}{12}\left(1+\tan^2 u\right)\right]\left(-\frac{1}{12}\tan u\right)$$

$$= \frac{1}{12}\tan u - \left[\frac{13}{12} + \frac{\tan^2 u}{12}\right]$$

Exercise :

1. Verify Euler's theorem for the following functions:

(i) $z = ax^2 + 2hxy + by^2$ (ii) $z = (x^2 + xy + y^2)^{-1}$.

2. If $u = \log\frac{x^2+y^2}{x+y}$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

3. If $u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

4. If $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$ then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(1-4\sin^2 u\right)\sin 2u$.

5. If $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$ then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\left[\tan^3 u - \tan u\right]$.

9. TOTAL DERIVATIVES:

Let z = f(x, y).Then the total differential dz is defined as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

10. COMPOSITE FUNCTIONS:

A function of a function is called a composite function.

Let z = f(x, y) where x = ϕ(t) and y = ψ(t).

Now, we can express z as s function of *t* alone by substituting the values of x and y in f(x, y).

Then z is a composite function of *t*.

Thus the ordinary derivative dz/dt which is called the **total derivative** of and is given by,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

COMPOSITE FUNCTIONS

Total partial derivative theorem for composite function:

If z = f(u, v) where u = ϕ(x, y) and v = ψ(x, y), then the partial derivatives of z are given by

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

Example 1:

Find $\frac{\mathrm{dz}}{\mathrm{dt}}$ given that $z=xy^2+x^2y,\ x=at^2,\ y=2at.$

Solution:

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$=\left(y^2+2xy\right)\left(2at\right)+\left(2xy+x^2\right)\left(2a\right)$$

$$=8a^3t^3+8a^3t^4+8a^3t^3+2a^3t^4$$

$$=16a^3t^3+10a^3t^4$$

Example 2:

If $\mathrm{z}=\mathrm{x}^2+y^2,\ x=\cos uv,\ y=\sin(u+v),$ find $\frac{\partial \mathrm{z}}{\partial \mathrm{u}}$ and $\frac{\partial \mathrm{z}}{\partial \mathrm{v}}$ interms of u and v.

Solution:

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$=2x\left[-v\sin uv\right]+2y\left[\cos(u+v)\right]$$

$$=-2v\sin uv\ \cos uv+2\sin(u+v)\cos(u+v)$$

$$=-v\sin 2uv+\sin 2(u+v).$$

Similarly,

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$=-u\sin 2uv+\sin 2(u+v).$$

Example 3:

If z = f(x, y), $x = e^u \sin v,\ y = e^u \cos v$ prove that $x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v} = \left(x^2 + y^2\right)\frac{\partial z}{\partial x}.$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\left[e^u \sin v\right] + \frac{\partial z}{\partial y}\left[e^u \cos v\right]$$

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left[e^u \cos v\right] + \frac{\partial z}{\partial y}\left[-e^u \cos v\right]$$

$$\therefore\ x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v} = x\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] + y\left[y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y}\right]$$

$$= \left(x^2 + y^2\right)\frac{\partial z}{\partial x}.$$

Example 4:

If $H = f\left(y - z,\ z - x,\ x - y\right)$, prove that $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0.$

Solution:

Let H = f(u, v, w) where $u = y - z,\ v = z - x$ and $w = x - y$.

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial x}.$$

$$= \frac{\partial H}{\partial u}(0) + \frac{\partial H}{\partial v}(-1) + \frac{\partial H}{\partial w}(1).$$

Similarly,

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}(1) + \frac{\partial H}{\partial v}(0) + \frac{\partial H}{\partial w}(-1).$$

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}(-1) + \frac{\partial H}{\partial v}(1) + \frac{\partial H}{\partial w}(0).$$

Hence

$$\frac{\partial H}{\partial x}+\frac{\partial H}{\partial y}+\frac{\partial H}{\partial z}=0.$$

Example 5:

If $z=f(x,y)$ and $x=r\cos\theta,\ y=r\sin\theta,$ prove that $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$

Solution:

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}.$$

$$=\frac{\partial z}{\partial x}\cdot\cos\theta+\frac{\partial z}{\partial y}\cdot\sin\theta.$$

Similarly,

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta).$$

Hence,

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\left[\cos^2\theta+\frac{1}{r^2}r^2\sin^2\theta\right]+\left(\frac{\partial z}{\partial y}\right)^2\left[\sin^2\theta+\frac{1}{r^2}r^2\cos^2\theta\right]$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Exercise

1. If $z=f(x,y)$ and $x=e^{u}+e^{-v},\ y=e^{-u}-e^{v},$ prove that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$

2. If $x=r\cos\theta,\ y=r\sin\theta,$ find (i) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}$ (ii) $\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}.$

3. If $z=e^{u-2v}$ and $u=\sin x,\ y=x^2+y^2,$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}.$

11. IMPLICIT DIFFERENTIATION:

When $f(x,y)$ is a function of two variables x and y, the equation $f(x,y)=c$ (where c is a constant) enables us to obtain values of y corresponding to values of x. Then we say that y

is an implicit function of x. We assume, for simplicity, that the above equation always defines a unique value of y for each value of x.

Since f is a function of x and y, and y is again a function of x, we can consider f a composite function of x. Then, its total derivative with respect to x is

$$\frac{df}{dx}=\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}.$$

But since $f(x,y)=c$, the total derivative of f must be identically 0. Thus

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0\Rightarrow \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial y/\partial y}=-\frac{f_x}{f_y} \text{ if } f_y\neq 0.$$

Differentiating again with respect to x, considering $\partial f/\partial x$ and $\partial f/\partial y$ as composite functions of x, we get

$$\frac{d^2y}{dx^2}=-\frac{\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y\partial x}\frac{dy}{dx}\right)\frac{\partial f}{\partial y}-\frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}\right)}{\left(\frac{\partial f}{\partial y}\right)^2}$$

$$=-\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2-2\frac{\partial^2 f}{\partial y\partial x}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3}$$

Thus,

$$\frac{dy}{dx}=-\frac{f_x}{f_y} \quad \text{and} \quad \frac{d^2y}{dx^2}=-\frac{f_{xx}\left(f_y\right)^2-2f_{yx}f_xf_y+f_{yy}\left(f_x\right)^2}{(f_y)^3}$$

Example 1: If $x^y=y^x$, find $\frac{dy}{dx}$.

Solution:

$x^y = y^x$

Taking log on both sides, we get

y logx = x logy

f(x, y) = y logx - x logy = 0

$$\therefore \ \frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{f_x}{f_y} = \frac{\frac{y}{x} - \log y}{\log x - \frac{x}{y}} = \frac{y(y - x\log y)}{x(x - y\log x)}.$$

Example 2: Prove that if $y^3 - 3ax^2 + x^3 = 0$, then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$.

Solution: Let

$$f(x,y) = y^3 - 3ax^2 + x^3 = 0.$$

We have

$$f_x = -6ax + 3x^2,\ f_y = 3y^2.$$

$$f_{xx} = -6a + 6x,\ f_{xy} = 0, f_{yy} = 6y$$

Therefore,

$$\frac{d^2y}{dx^2} = -\frac{6(x-a)9y^4 + (3x^2 - 6ax)^2 6y}{27y^6} = -2\frac{a^2x^2}{y^5}$$

EXERCISE

1. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(bx + by + f)^3}$$

2. If A, B, C are the angles of a triangle such that $\sin^2 A + \sin^2 B + \sin^2 C$ is constant, show that

$$\frac{dA}{dB} = \frac{\tan B - \tan C}{\tan C - \tan A}$$

12. ERRORS AND APPROXIMATIONS

Let $f(x,y)$ be a continuous function of x and y. If δx and δy are the increments of x and y, then the new value of $f(x,y)$ will be $f(x+\delta x, y+\delta y)$. Hence

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y).$$

Expanding $f(x+\delta x, y+\delta y)$ by Taylor's theorem and supposing δx, δy to be small enough that their products, squares, and higher powers can be neglected, we get

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y, \text{ approximately.}$$

The value δf is called the error in f due to the errors δx and δy in x and y. $\frac{\delta f}{f}$ is called the relative error in f and $\frac{\delta f}{f}\times 100$ is called the percentage error in f.

Example 1: If $PV^2 = K$ and if the relative errors in P is 0.05 and in V is 0.025 then show that error in K is 10%.

Solution:

$$\frac{\delta P}{P} = 0.05 \text{ and } \frac{\delta V}{V} = 0.025$$

$$PV^2 = K$$

Taking log on both sides of the equation,

$$\log P + 2\log V = \log K$$

$$\delta(\log P) + 2\delta(\log V) = \delta(\log K)$$

$$\frac{1}{P}\delta P + 2\frac{1}{V}\delta V = \frac{1}{K}\delta K \Rightarrow 0.05 + 2(0.025) = \frac{\delta K}{K}$$

$$\frac{\delta K}{K} = 0.1$$

Thus, error is

$$100\times\frac{\delta K}{K} = (0.1)\times 100 = 10\%$$

Example 2: The time T of a complete oscillation of a simple pendulum is given by the formula $T = 2\pi\sqrt{l/g}$. If g is a constant find the error in the calculated value of T due to an error of 3% in the value of l.

Solution:

$$T = 2\pi\sqrt{l/g}$$

Taking log on both sides,

$$\log T = \log 2\pi + \frac{1}{2}\log\frac{l}{g}$$

$$\delta(\log T) = \delta(\log 2\pi) + \delta\left(\frac{1}{2}(\log l - \log g)\right)$$

$$\frac{\delta T}{T} = 0 + \frac{1}{2}\frac{\delta l}{l} - 0$$

$$100\frac{\delta T}{T} = \frac{1}{2}\frac{\delta l}{l}\times 100 = \frac{1}{2}\times 3 = 1.5$$

Thus, the error in T is 1.5%

Example 3: If the sides and angles of a plane triangle ΔABC vary in such a way that its circum-radius remains constant, prove that $\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C}$, where δa, δb, and δc denote small increments in the sides a, b and c respectively.

Solution:

Let R be the circum-radius of the ΔABC, then

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

$$\Rightarrow 2R\sin A, b = 2R\sin B, c = 2R\sin C$$

Differentiating, we get

$$\delta a = 2R\cos A\,\delta A, \delta b = 2R\cos B\,\delta B, \delta c = 2R\cos C\,\delta C$$

$$\Rightarrow \frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2R(\delta A + \delta B + \delta C)$$

$$= 2R\delta(A + B + C) = 2R\delta(\pi) = 0.$$

MAXIMA AND MINIMA

Maxima and minim for a function of two variables:

Definitions: Let $f(x, y)$ be a function of 2 independent variables x and y. Then $f(x, y)$ is said to attain a maximum value at the point (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) in some neighbourhood of the point (a, b).

The above condition is equivalent to the condition $f(a, b) \geq f(a + h, b + k)$ for all small arbitrary values of h and k.

If $f(a, b) \leq f(x, y)$ for all (x, y) in some nbd of the point (a, b), then $f(a, b)$ is called a minimum value of $f(x, y)$. In other words if $f(a, b) \leq f(a + h, b + k)$ for all small arbitrary values of h, k then $f(a, b)$ is a minimum value.

An Extreme value is either a maximum or a minimum value.

Note: A function $f(x, y)$ can have many extreme values. These extreme values are called the local or relative extreme values of $f(x, y)$. If $f(a, b) \geq f(x, y)$ for all x and y, then $f(a, b)$ is called the global or absolute maximum value of $f(x, y)$.

Similarly if $f(a, b) \leq f(x, y)$ for all x and y, then $f(a, b)$ is called the global or absolute minimum of $f(x, y)$.

Note that the global extreme value of a function are unique.

Necessary conditions for a function to attain an extreme value:

Theorem: If $f(a, b)$ is an extreme value then $\frac{df(a,b)}{dx} = \frac{\partial f(a,b)}{\partial y} = 0$.

Proof: Let us assume that $f(x, y)$ possesses first order partial derivatives in a nbd of (a, b). Consider the function $g(x) = f(x, b)$, which is a function of single variable x, that attains an extreme value at $x = a$.

Then $\frac{\partial g(x)}{\partial x}|_{x=a} = 0 \ i.e \ \frac{\partial f(a,b)}{\partial x} = 0$

Similarly $h(y) = f(a, y)$ attains an extreme value at $y = b$.

$$\therefore \frac{d}{dy} h(y)|_{y=b} = 0 \ i.e \frac{\partial f(a, b)}{\partial y} = 0$$

Note 1: The conditions given above are only necessary but not sufficient i.e if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ then $f(a, b)$ need not be an extreme value.

For example, let $f(x, y)$ be defined by $f(x) = \begin{cases} 0, & if \ x = 0 \ or \ y = 0 \\ 1, & otherwise \end{cases}$

Then $\frac{\partial f(0,0)}{\partial x} = 0$, $\frac{\partial f(0,0)}{\partial y} = 0$. But $f(0,0)$ is not an extreme value.

A point (a, b) is called a stationary point of a function $f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Note that every extreme point is a stationary point but the converse is not true.

Sufficient conditions to attain extreme values:

Let $f(x, y)$ possess continuous second order partial derivatives in a nbd of a point (a, b).

If $f_x(a, b) = 0$, $f_y(a, b) = 0$ and $f_{x^2}(a, b) = A$, $f_{xy}(a, b) = B$, $f_{y^2}(a, b) = C$, then

i) $f(a, b)$ is a maximum value if $AC - B^2 > 0$ & $A < 0$.
ii) $f(a, b)$ is a minimum value if $AC - B^2 > 0$ & $A > 0$
iii) $f(a, b)$ is not an extreme value if $AC - B^2 < 0$
iv) The case is doubtful and needs further consideration if $AC - B^2 = 0$

Proof: Let us assume that $f(x, y)$ satisfies the conditions mentioned above. Then Taylor's expansion of $f(x, y)$ at (a, b) is

$$f(a+h, b+k) = f(a, b) + \left(hf_x(a, b) + kf_y(a, b)\right) + \frac{1}{2!}(h^2 f_x(a, b) + 2hkf_{xy}(a, b) + k^2 f_{y^2}(a, b) + \Delta$$

Where Δ contains 3rd and higher degree terms in h and k. Substituting the conditions given above we get

$$f(a+h, b+k) - f(a, b) = (h.0 + k.0) + \frac{1}{2}[h^2 A + 2hkB + k^2 C] + D \ldots\ldots\ldots\ldots\ldots(1)$$

For sufficiently smaller h and k, the sign of the LHS is the sign of $\frac{1}{2}(h^2 A + 2hkB + k^2 C)$.

Consider $\frac{1}{2}(h^2 A + 2hkB + k^2 C)$. Let $A \neq 0$. Then

$$= \frac{1}{2A}(h^2 A^2 + 2hkAB + k^2 AC)$$

$$= \frac{1}{2A}[(Ah + Bk)^2 + k^2(AC - B^2)]$$

If $AC - B^2 > 0$, then $[(Ah + Bk)^2 + k^2(AC - B^2)] \geq 0$ for all h and k.

$\therefore$ Sign of LHS of (1) is the same as the sign of A.

If $A < 0$, then $f(a+h, b+k) - f(a.b) \leq 0$

$=> f(a+h, b+k) \leq f(a,b)$ for all small h and k.

$=> f(a,b)$ is a maximum value.

If $A > 0$, then $f(a+h, b+k) \geq f(a,b)$ for all small h and k.

$=> f(a,b)$ is a minimum value.

If $AC - B^2 < 0$, let $A \neq 0$, then the sign of $\frac{1}{2A}[(Ah+Bk)^2 + k^2(AC-B^2)]$ varies as h & k varies.

$\therefore f(a,b)$ is not an extreme value.

If $A = 0$ & $C \neq 0$, then sign of $\frac{1}{2}(Ah^2 + 2hkB + k^2C) = \frac{1}{2C}[(hB+kC)^2 + k^2(AC-B^2)]$ varies as h & k varies.

$\therefore f(a,b)$ is not an extreme value.

If $A = 0, C = 0$, then sign of $\frac{1}{2}(Ah^2 + 2hkB + k^2C) = hkB$ varies as h & k varies.

$\therefore f(a,b)$ is not an extreme value.

If $AC - B^2 = 0$, suppose $A \neq 0$, then $\frac{1}{2}(Ah^2 + 2hkB + k^2C) = \frac{1}{2}[(Ah+Bk)^2 + k^2(AC-B^2)]$

$$= \frac{1}{2A}(Ah+Bk)^2$$

Which becomes zero if $Ah + Bk = 0$, hence the sign of LHS of (1) depends on the sign of Δ, the case is more complicated.

If $A = 0$, since $AC - B^2 = 0$, $B^2 = AC = 0$

$=> B = 0$

$\therefore Ah^2 + 2hkB + k^2C = k^2C$ which has value zero when $k = 0$ for any h. This case is doubtful.

Problems:

1. Find the extreme values of $f(x,y) = xy(a-x-y)$

We have $f_x = \frac{\partial}{\partial x}(axy - x^2y - xy^2)$

$$= ay - 2xy - y^2 = y(a-2x-y)$$

$f_y = ax - x^2 - 2xy = x(a-x-2y)$

$f_x = f_y = 0 \;\; => \;\; y(a - 2x - y) = 0, \qquad x(a - x - 2y) = 0$

The solutions are i) $x = 0, y = 0,$

ii) $y = 0, a - x - 2y = 0 \;\; => x = a,$

iii) $(a - 2x - y) = 0,\; x = 0 \;\; => \;\; x = 0, y = a,$

$iv)$ $a - 2x - y = 0$ and $a - x - 2y = 0 \;\; => \;\; x = y = \frac{a}{3}$

Stationary points are $(0,0), (a, 0), (0, a), (\frac{a}{3}, \frac{a}{3})$.

$A = \frac{\partial^2 f}{\partial x^2} = -2y \qquad , \; B = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y, \quad C = \frac{\partial^2 f}{\partial y^2} = -2x$

Stationary points	A	$AC - B^2$
$(0,0)$	0	$-a^2 < 0$
$(a, 0)$	0	$-a^2 < 0$
$(0, a)$	$-2a$	$-a^2 < 0$
$(\frac{a}{3}, \frac{a}{3})$	$-\frac{2a}{3}$	$\frac{a^2}{3} > 0$

$\therefore f\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{a^2}{2}$ is a maximum value if $a > 0$ or a minimum value if $a < 0$.

2. Find the extreme values of $f(x, y) = xy + 27(\frac{1}{x} + \frac{1}{y})$

$f_x = y - \frac{27}{x^2}, \quad f_y = x - \frac{27}{y^2}, \quad f_x = f_y = 0 \;\; => \;\; x = y = 3$

$A = f_{x^2} = \frac{54}{x^3}, \quad B = f_{xy} = 1, \quad C = f_{y^2} = \frac{54}{y^3}$

$AC - B^2 = \frac{54}{x^3} . \frac{54}{y^3} - 1$

At $(3,3), AC - B^2 = \frac{(54)^2}{3^3 . 3^3} - 1 = 4 - 1 = 3 > 0$

$A = \frac{54}{3^3} = 2 > 0$

$\therefore f$ has a minimum value at $x = 3, y = 3$ and minimum value is $f(3,3) = 27$

3. Find the extreme value of $f(x, y) = \sin x + \sin y + \sin(x + y),\; 0 \leq x,\; y \leq \frac{\pi}{2}$

$f_x = \cos x + \cos(x+y), \quad f_y = \cos y + \cos(x+y)$

$f_x = f_y = 0 \;\Rightarrow\; \cos x + \cos(x+y) = 0$

$\Rightarrow \cos y + \cos(x+y) = 0$

On subtracting, we get $\cos x - \cos y = 0$

$\therefore \cos x = \cos y \;\Rightarrow\; x = y$ or $x = 2\pi - y$

Substituting $x = y$, we get $\cos x + \cos 2x = 0$

$\Rightarrow 2\cos\frac{3x}{2}\cos\frac{x}{2} = 0 \quad \Rightarrow \quad \frac{3x}{2} = \pm\frac{\pi}{2},$ or $\frac{x}{2} = \pm\frac{\pi}{2}$

$\Rightarrow x = \pm\frac{\pi}{3}$ or $x = \pm\pi$

$\Rightarrow y = \pm\frac{\pi}{3}$ or $y = \pm\pi$

Now, $\quad A = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y), \quad B = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y), \quad C = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $AC - B^2 = 3 - \frac{1}{4} > 0, \; A = -\sqrt{3} < 0$

$\therefore f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = 2\sqrt{3}$ is a maximum value.

At $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, $AC - B^2 = 3 - \frac{1}{4} > 0, \quad A = \sqrt{3} > 0$

$\therefore f\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right) = 2\sqrt{3}$ is a minimum value.

Exercises: Find the extreme values:

1. $2(x-y)^2 - x^4 - y^4$
2. $2(x^2 - y^2) - x^4 + y^4$
3. $\frac{x^3 y^2}{6-x-y}$
4. $x^3 + y^3 - 3x - 12y + 20$
5. $x^2 y(x + 2y - 4)$
6. $2\sin(x+2y) + 3\cos(2x-y)$
7. $1 + \sin(x^2 + y^2)$

Lagrange's Method of Undetermined Multipliers

LAGRANGE METHOD OF UNDETERMINED MULTIPLIERS

Till now we have considered the method of optimizing a function of two variables without any conditions. But most of the optimizing problem we come across are of different type, where the function is optimized subject to some conditions.

Let us consider the problem of optimizing a function $z = f(x_1,\ x_2,\ x_3 \ldots . x_n)$ ……….(1)

subject to the conditions

$\emptyset_1(x_1, x_2, \ldots . . x_n) = c_1$

$\emptyset_2(x_1, x_2, \ldots . . x_n) = c_2$ ……………………….(2)

…………………………………….

$\emptyset_r(x_1, x_2, \ldots . . x_n) = c_r$

Where $c_1, c_2 \ldots . . c_r$ are all constant.

One of the way of solving this problem is to solve r, n-variables from conditions (2) and substituting in (1), which reduces to a function of $(n - r)$ variables, that can be solved by the direct methods.

Now let us consider different method, called the Lagrange's method, which gives the stationary points of the function f.

Consider the function $g(x_1, x_2 \ldots . . x_n) = f(x_1, x_2, \ldots . . x_n) + \sum_{i=1}^{r} \lambda_i \emptyset_i(x_1, x_2, \ldots x_n)$

Where λ_i , $i = 1,2, \ldots . . r$ are all constant called Lagrange's Mulitpliers.

If conditions defined by (2) are satisfied then $f(x_1, \ldots \ldots x_n)$ and $g(x_1, x_2 \ldots . . x_n)$ attain optimal value together.

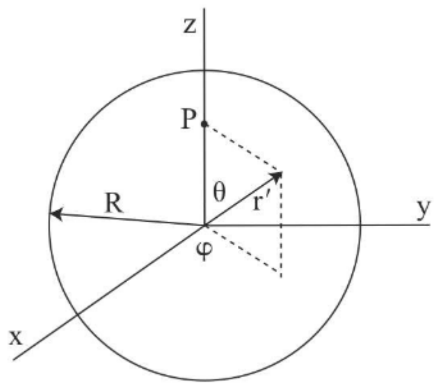
When optimal values are attained, then $\frac{\partial g}{\partial x_i} = 0 \quad i = 1,2, \ldots . n$

i.e, $\frac{\partial f}{\partial x_i} + \sum_{i=1}^{r} \lambda_i \frac{\partial \emptyset}{\partial x_i} = 0$

solving for $x_1, x_2, \ldots . x_n$ and $\lambda_1, \lambda_2, \ldots . . \lambda_r$ from (3) and (2), we get the stationary values of the defined by function $f(x_1, x_2, \ldots x_n)$ subject to the conditions (2). To determine the nature of these stationary points, we may have to use the physical nature of the function $f(x_1, x_2, \ldots x_n)$ subject to the conditions defined by (2).

Solved Problems

1. Find the points on the sphere $x^2 + y^2 + z^2 = a^2$ which are not minimum and maximum distance from the point (1,2,3). Let $P(x, y, z)$ be any point on the sphere $x^2 + y^2 + z^2 = a^2$.



Then the distance between $P(x, y, z)$ and $Q(1,2,3)$ is $PQ = r = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$ of r is optimal, then so is $r^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$.

Then the problem is to obtain the maximum and minimum value of $f(x, y, z) = r^2$ subject to the condition $x^2 + y^2 + z^2 = a^2$.

Let $g(x, y, z) = (xy)^2 + (y-2)^2 + (z-3)^2 + \lambda(x^2 + y^2 + z^2)$ when optimal value of $f = r^2$ are obtained subject to the condition $x^2 + y^2 + z^2 = a^2$, g is also optimal.

$$\therefore \frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0$$

$$=> 2(x-1) + \lambda(2x) = 0$$

$$2(y-2) + \lambda(2y) = 0$$

$$2(z-3) + \lambda(2z) = 0$$

$$=> \frac{x-1}{x} = \frac{y-2}{y} = \frac{z-3}{z} = -\lambda$$

$\frac{x-1}{x} = \frac{y-2}{y} => xy - y = xy - zx$ or $y = 2x$

$$\frac{x-1}{x} = \frac{z-3}{z} = z = 3x$$

Substituting in $x^2+y^2+z^2=a^2$, we get $x^2+4x^2+9x^2=a^2$

$\therefore x^2=\frac{a^2}{14}$ or $x=\pm\frac{a}{\sqrt{14}}$ $=> y=2x=\pm\frac{2a}{\sqrt{14}},\ z=3x=\pm\frac{3a}{\sqrt{14}}$

Stationary points are $A(\frac{a}{\sqrt{14}},\frac{2a}{\sqrt{14}},\frac{3a}{\sqrt{14}})$ and $B(-\frac{a}{\sqrt{14}},-\frac{2a}{\sqrt{14}},-\frac{3a}{\sqrt{14}})$

Since the points A and Q lies in the same octant, A is at a minimum distance from $Q(1,2,3)$.

Also B lies in the opposite octant of the point $Q(1,2,3)$, hence is at maximum distance from Q.

2. Find the minimum distance from the origin to the plane $lx+my+nz=p$.
   Let $P(x,y,z)$ be any point on the plane $lx+my+nz=p$.

Then the distance $OP=r=\sqrt{x^2+y^2+z^2}$.

When r is minimum, then so is $r^2=x^2+y^2+z^2$. Thus our objective is to minimize $f(x,y,z)=r^2=x^2+y^2+z^2$ subject to the conditions $lx+my+nz=p$.

Let $g(x,y,z)=x^2+y^2+z^2+\lambda(lx+my+nz)$.

When $lx+my+nz=p$ is satisfied, f and g assume minimum value together. Then $\frac{\partial g}{\partial x}=\frac{\partial g}{\partial y}=\frac{\partial g}{\partial z}=0$

i.e $2x+\lambda l=0=2y+\lambda m=2z+\lambda n$

$$\therefore \frac{x}{l}=\frac{y}{m}=\frac{z}{n}=-\frac{\lambda}{2}$$

$$=> y=\frac{mx}{l}, z=\frac{nx}{l}$$

Substituting in $lx+my+nz=p$, we get $lx+m.\frac{mx}{l}+n.\frac{nx}{l}=p$

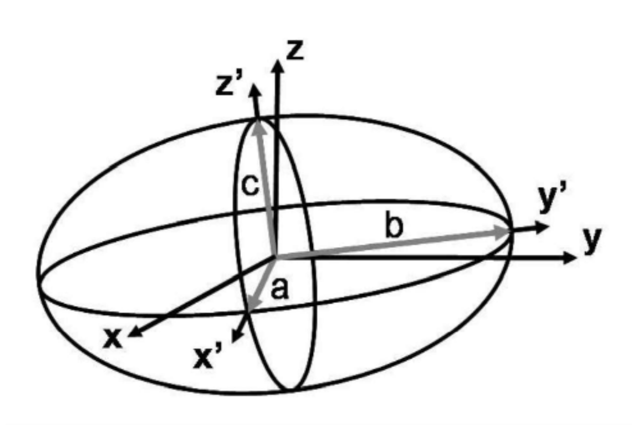
$$\therefore x=\frac{pl}{l^2+m^2+n^2} => y=\frac{pm}{l^2+m^2+n^2}, z=\frac{pn}{l^2+m^2+n^2}$$

$$\therefore r^2=x^2+y^2+z^2=\frac{(p^2l^2+m^2p^2+n^2p^2)}{(l^2+m^2+n^2)^2}=\frac{p^2}{l^2+m^2+n^2}$$

$$\therefore r=\sqrt{x^2+y^2+z^2}=\frac{p}{\sqrt{l^2+m^2+n^2}}$$

Which is the perpendicular distance from the origin to the plane $lx+my+nz=p$ and hence is the minimum distance.

3. Find the axes of the ellipse of the intersection of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $lx+my+nz=0$.



Let $P(x,y,z)$ be any point on the given ellipse.

Then the distance $OP = r = \sqrt{x^2+y^2+z^2}$.

The objective is to determine the maximum and minimum value of the distance $OP = r$ or $r^2 = u(x,y,z) = x^2+y^2+z^2$ subject to the condition that the point $P(x,y,z)$ lies on the ellipse or satisfies the conditions $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $lx+my+nz=0$.

**Since the semi-major and semi-minor axes of the ellipse are the maximum and the minimum distance from the origin to the point P i.e of the distance OP **

Let $g(x,y,z) = u + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) + \mu(lx+my+nz) = x^2+y^2+z^2+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\mu(lx+my+nz)$

Where λ and μ are constants.

When $g(x,y,z)$ is optimal, then so is $f(x,y,z)=u$ subject to the conditions defined above.

$$=> \frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0$$

i.e. $2x + \frac{\lambda 2x}{a^2} + \mu l = 0$ ………………..(1)

$2y + \frac{\lambda 2y}{l^2} + \mu m = 0$ ……………………..(2)

$2z + \frac{\lambda 2z}{c^2} + \mu n = 0$ ……………………..(3)

$\therefore$ (1)$*\, x$ + (2) $*\, y$ + (3) $*\, z$ gives

$$2(x^2 + y^2 + z^2) + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \mu(lx + my + nz) = 0$$

$\because \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $lx + my + nz = 0$, we get

$2x + 2\lambda = 0$ or $\lambda = -u$

Substituting in (1), (2) and (3), we get

$2x\left(1 - \frac{u}{a^2}\right) + \mu l = 0 \;\; => \;\; -\frac{2x}{\mu} = \frac{l}{1-\frac{u}{a^2}}$ …………(4)

$2y\left(1 - \frac{u}{b^2}\right) + \mu m = 0 \;\; => \;\; -\frac{2y}{\mu} = \frac{m}{1-\frac{u}{b^2}}$ ……….(5)

$2z\left(1 - \frac{u}{c^2}\right) + \mu n = 0 \;\; => \;\; -\frac{2z}{\mu} = \frac{n}{1-\frac{u}{c^2}}$ …………(6)

Now (4)$*\, l$ + (5)$*\, m$ + (6)$*\, n$ gives

$$\frac{l^2}{1-\frac{u}{a^2}} + \frac{m^2}{1-\frac{u}{b^2}} + \frac{n^2}{1-\frac{u}{c^2}} = -\frac{2}{\mu}(lx + my + nz) = 0$$

Or $l^2\left(1 - \frac{u}{b^2}\right)\left(1 - \frac{u}{c^2}\right) + m^2\left(1 - \frac{u}{a^2}\right)\left(1 - \frac{u}{c^2}\right) + n^2\left(1 - \frac{u}{a^2}\right)\left(1 - \frac{u}{b^2}\right) = 0$

Which is a quadratic equation in u and hence has two roots u_1 and u_2.

Let $u_1 > u_2$.

Then $\sqrt{u_1}$ is the semi-major axis and $\sqrt{u_2}$ is the semi-minor axis of the ellipse of the intersection of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane $lx + my + nz = 0$.

Problems

1. Find the extreme values of the function $f(x, y, z) = \sin x \sin y \sin z$ where x, y, z are the angles of a triangle.
2. Find the largest rectangular parallelepiped inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
3. Prove that of all rectangular parallelepiped of the same volume, the cube has the least surface.
4. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition

i) $yz + zx + xy = 3a^2$
ii) $x + y + z = 3a$
iii) $xyz = a^3$

5. Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.
6. A rectangular box open at the top have a volume of $32\ m^3$. Find the dimensions of the box requiring least material for construction.