Maxima and minim for a function of two variables:

Definitions: Let $f(x, y)$ be a function of 2 independent variables x and y. Then $f(x, y)$ is said to attain a maximum value at the point (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) in some neighbourhood of the point (a, b).

The above condition is equivalent to the condition $f(a, b) \geq f(a + h, b + k)$ for all small arbitrary values of h and k.

If $f(a, b) \leq f(x, y)$ for all (x, y) in some nbd of the point (a, b), then $f(a, b)$ is called a minimum value of $f(x, y)$. In other words if $f(a, b) \leq f(a + h, b + k)$ for all small arbitrary values of h, k then $f(a, b)$ is a minimum value.

An Extreme value is either a maximum or a minimum value.

Note: A function $f(x, y)$ can have many extreme values. These extreme values are called the local or relative extreme values of $f(x, y)$. If $f(a, b) \geq f(x, y)$ for all x and y, then $f(a, b)$ is called the global or absolute maximum value of $f(x, y)$.

Similarly if $f(a, b) \leq f(x, y)$ for all x and y, then $f(a, b)$ is called the global or absolute minimum of $f(x, y)$.

Note that the global extreme value of a function are unique.

Necessary conditions for a function to attain an extreme value:

Theorem: If $f(a, b)$ is an extreme value then $\frac{df(a,b)}{dx} = \frac{\partial f(a,b)}{\partial y} = 0$.

Proof: Let us assume that $f(x, y)$ possesses first order partial derivatives in a nbd of (a, b). Consider the function $g(x) = f(x, b)$, which is a function of single variable x, that attains an extreme value at $x = a$.

Then $\frac{\partial g(x)}{\partial x}|_{x=a} = 0 \;\; i.e \; \frac{\partial f(a,b)}{\partial x} = 0$

Similarly $h(y) = f(a, y)$ attains an extreme value at $y = b$.

$$\therefore \frac{d}{dy} h(y)|_{y=b} = 0 \;\; i.e \frac{\partial f(a, b)}{\partial y} = 0$$

Note 1: The conditions given above are only necessary but not sufficient i.e if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ then $f(a, b)$ need not be an extreme value.

For example, let $f(x, y)$ be defined by $f(x) = \begin{cases} 0, & if\ x = 0\ or\ y = 0 \\ 1, & otherwise \end{cases}$

Then $\frac{\partial f(0,0)}{\partial x} = 0$, $\frac{\partial f(0,0)}{\partial y} = 0$. But $f(0,0)$ is not an extreme value.

A point (a, b) is called a stationary point of a function $f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Note that every extreme point is a stationary point but the converse is not true.

Sufficient conditions to attain extreme values:

Let $f(x, y)$ possess continuous second order partial derivatives in a nbd of a point (a, b).

If $f_x(a, b) = 0$, $f_y(a, b) = 0$ and $f_{x^2}(a, b) = A$, $f_{xy}(a, b) = B$, $f_{y^2}(a, b) = C$, then

i) $f(a, b)$ is a maximum value if $AC - B^2 > 0$ & $A < 0$.
ii) $f(a, b)$ is a minimum value if $AC - B^2 > 0$ & $A > 0$
iii) $f(a, b)$ is not an extreme value if $AC - B^2 < 0$
iv) The case is doubtful and needs further consideration if $AC - B^2 = 0$

Proof: Let us assume that $f(x, y)$ satisfies the conditions mentioned above. Then Taylor's expansion of $f(x, y)$ at (a, b) is

$$f(a + h, b + k) = f(a, b) + \left(hf_x(a, b) + kf_y(a, b)\right) + \frac{1}{2!}(h^2 f_x(a, b) + 2hkf_{xy}(a, b) + k^2 f_{y^2}(a, b) + \Delta$$

Where Δ contains 3rd and higher degree terms in h and k. Substituting the conditions given above we get

$$f(a + h, b + k) - f(a, b) = (h.0 + k.0) + \frac{1}{2}[h^2 A + 2hkB + k^2 C] + D \ldots\ldots\ldots\ldots\ldots(1)$$

For sufficiently smaller h and k, the sign of the LHS is the sign of $\frac{1}{2}(h^2 A + 2hkB + k^2 C)$.

Consider $\frac{1}{2}(h^2 A + 2hkB + k^2 C)$. Let $A \neq 0$. Then

$$= \frac{1}{2A}(h^2 A^2 + 2hkAB + k^2 AC)$$

$$= \frac{1}{2A}[(Ah + Bk)^2 + k^2(AC - B^2)]$$

If $AC - B^2 > 0$, then $[(Ah + Bk)^2 + k^2(AC - B^2)] \geq 0$ for all h and k.

$\therefore$ Sign of LHS of (1) is the same as the sign of A.

If $A < 0$, then $f(a + h, b + k) - f(a.b) \leq 0$

=> $f(a + h, b + k) \leq f(a, b)$ for all small h and k.

=> $f(a, b)$ is a maximum value.

If $A > 0$, then $f(a + h, b + k) \geq f(a, b)$ for all small h and k.

=> $f(a, b)$ is a minimum value.

If $AC - B^2 < 0$, let $A \neq 0$, then the sign of $\frac{1}{2A}[(Ah + Bk)^2 + k^2(AC - B^2)]$ varies as h & k varies.

$\therefore f(a,b)$ is not an extreme value.

If $A = 0$ & $C \neq 0$, then sign of $\frac{1}{2}(Ah^2 + 2hkB + k^2C) = \frac{1}{2C}[(hB + kC)^2 + k^2(AC - B^2)]$ varies as h & k varies.

$\therefore f(a,b)$ is not an extreme value.

If $A = 0, C = 0$, then sign of $\frac{1}{2}(Ah^2 + 2hkB + k^2C) = hkB$ varies as h & k varies.

$\therefore f(a,b)$ is not an extreme value.

If $AC - B^2 = 0$, suppose $A \neq 0$, then $\frac{1}{2}(Ah^2 + 2hkB + k^2C) = \frac{1}{2}[(Ah + Bk)^2 + k^2(AC - B^2)]$

$$= \frac{1}{2A}(Ah + Bk)^2$$

Which becomes zero if $Ah + Bk = 0$, hence the sign of LHS of (1) depends on the sign of Δ, the case is more complicated.

If $A = 0$, since $AC - B^2 = 0$, $B^2 = AC = 0$

$=> B = 0$

$\therefore Ah^2 + 2hkB + k^2C = k^2C$ which has value zero when $k = 0$ for any h. This case is doubtful.

<u>Problems:</u>

1. Find the extreme values of $f(x,y) = xy(a - x - y)$

We have $f_x = \frac{\partial}{\partial x}(axy - x^2y - xy^2)$

$= ay - 2xy - y^2 = y(a - 2x - y)$

$f_y = ax - x^2 - 2xy = x(a - x - 2y)$

$f_x = f_y = 0 \;\; => \;\; y(a - 2x - y) = 0, \qquad x(a - x - 2y) = 0$

The solutions are i) $x = 0, y = 0,$

ii) $y = 0, a - x - 2y = 0 \;\; => x = a,$

iii) $(a - 2x - y) = 0,\; x = 0 \;\; => \;\; x = 0, y = a,$

$iv)$ $a - 2x - y = 0$ and $a - x - 2y = 0 \;\; => \;\; x = y = \frac{a}{3}$

Stationary points are $(0,0), (a,0), (0,a), (\frac{a}{3}, \frac{a}{3})$.

$A = \frac{\partial^2 f}{\partial x^2} = -2y, \quad B = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y, \quad C = \frac{\partial^2 f}{\partial y^2} = -2x$

Stationary points	A	$AC - B^2$
$(0,0)$	0	$-a^2 < 0$
$(a,0)$	0	$-a^2 < 0$
$(0,a)$	$-2a$	$-a^2 < 0$
$(\frac{a}{3},\frac{a}{3})$	$-\frac{2a}{3}$	$\frac{a^2}{3} > 0$

$\therefore f\left(\frac{a}{3},\frac{a}{3}\right) = \frac{a^2}{2}$ is a maximum value if $a > 0$ or a minimum value if $a < 0$.

2. Find the extreme values of $f(x,y) = xy + 27(\frac{1}{x} + \frac{1}{y})$

$$f_x = y - \frac{27}{x^2}, \quad f_y = x - \frac{27}{y^2}, \quad f_x = f_y = 0 \ => \quad x = y = 3$$

$$A = f_{x^2} = \frac{54}{x^3}, \quad B = f_{xy} = 1, \quad C = f_{y^2} = \frac{54}{y^3}$$

$AC - B^2 = \frac{54}{x^3}.\frac{54}{y^3} - 1$

At (3,3), $AC - B^2 = \frac{(54)^2}{3^3.3^3} - 1 = 4 - 1 = 3 > 0$

$$A = \frac{54}{3^3} = 2 > 0$$

$\therefore f$ has a minimum value at $x = 3, y = 3$ and minimum value is $f(3,3) = 27$

3. Find the extreme value of $f(x,y) = \sin x + \sin y + \sin(x+y)$, $0 \leq x,\ y \leq \frac{\pi}{2}$

$f_x = \cos x + \cos(x+y), \quad f_y = \cos y + \cos(x+y)$

$f_x = f_y = 0 \ => \cos x + \cos(x+y) = 0$

$=> \cos y + \cos(x+y) = 0$

On subtracting, we get $\cos x - \cos y = 0$

$\therefore \cos x = \cos y \ => \ x = y$ or $x = 2\pi - y$

Substituting $x = y$, we get $\cos x + \cos 2x = 0$

$=> 2\cos\frac{3x}{2}\cos\frac{x}{2} = 0 \quad => \quad \frac{3x}{2} = \pm\frac{\pi}{2}$, or $\frac{x}{2} = \pm\frac{\pi}{2}$

$=> x = \pm\frac{\pi}{3}$ or $x = \pm\pi$

$=> y = \pm\frac{\pi}{3}$ or $y = \pm\pi$

Now, $A = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$, $B = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$, $C = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $AC - B^2 = 3 - \frac{1}{4} > 0$, $A = -\sqrt{3} < 0$

$\therefore f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = 2\sqrt{3}$ is a maximum value.

At $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, $AC - B^2 = 3 - \frac{1}{4} > 0$, $A = \sqrt{3} > 0$

$\therefore f\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right) = 2\sqrt{3}$ is a minimum value.

Exercises: Find the extreme values:

1. $2(x-y)^2 - x^4 - y^4$
2. $2(x^2 - y^2) - x^4 + y^4$
3. $\frac{x^3 y^2}{6-x-y}$
4. $x^3 + y^3 - 3x - 12y + 20$
5. $x^2 y(x + 2y - 4)$
6. $2\sin(x+2y) + 3\cos(2x - y)$
7. $1 + \sin(x^2 + y^2)$

Lagrange's Method of Undetermined Multipliers

Till now we have considered the method of optimizing a function of two variables without any conditions. But most of the optimizing problem we come across are of different type, where the function is optimized subject to some conditions.

Let us consider the problem of optimizing a function $z = f(x_1,\ x_2,\ x_3 \dots . x_n)$ ……….(1)

subject to the conditions

$\emptyset_1(x_1, x_2, \dots \dots x_n) = c_1$

$\emptyset_2(x_1, x_2, \dots \dots x_n) = c_2$ ………………………(2)

…………………………………

$\emptyset_r(x_1, x_2, \dots \dots x_n) = c_r$

Where $c_1, c_2 \dots \dots c_r$ are all constant.

One of the way of solving this problem is to solve r, n-variables from conditions (2) and substituting in (1), which reduces to a function of $(n-r)$ variables, that can be solved by the direct methods.

Now let us consider different method, called the Lagrange's method, which gives the stationary points of the function f.

Consider the function $g(x_1, x_2 \ldots\ldots x_n) = f(x_1, x_2, \ldots\ldots x_n) + \sum_{i=1}^{r} \lambda_i \emptyset_i(x_1, x_2, \ldots x_n)$

Where λ_i , $i = 1,2, \ldots\ldots r$ are all constant called Lagrange's Mulitpliers.

If conditions defined by (2) are satisfied then $f(x_1, \ldots \ldots x_n)$ and $g(x_1, x_2 \ldots\ldots x_n)$ attain optimal value together.

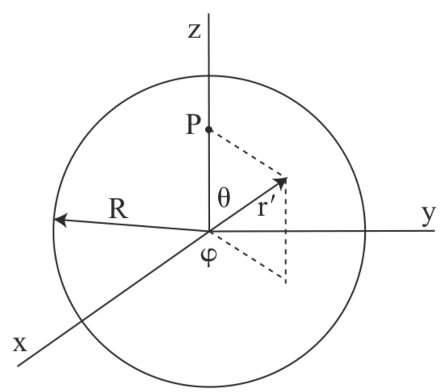
When optimal values are attained, then $\frac{\partial g}{\partial x_i} = 0 \quad i = 1,2, \ldots . n$

i.e, $\frac{\partial f}{\partial x_i} + \sum_{i=1}^{r} \lambda_i \frac{\partial \emptyset}{\partial x_i} = 0$

solving for $x_1, x_2, \ldots . x_n$ and $\lambda_1, \lambda_2, \ldots\ldots \lambda_r$ from (3) and (2), we get the stationary values of the defined by function $f(x_1, x_2, \ldots x_n)$ subject to the conditions (2). To determine the nature of these stationary points, we may have to use the physical nature of the function $f(x_1, x_2, \ldots x_n)$ subject to the conditions defined by (2).

Solved Problems

1. Find the points on the sphere $x^2 + y^2 + z^2 = a^2$ which are not minimum and maximum distance from the point (1,2,3). Let $P(x, y, z)$ be any point on the sphere $x^2 + y^2 + z^2 = a^2$.



Then the distance between $P(x, y, z)$ and $Q(1,2,3)$ is $PQ = r = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$ of r is optimal, then so is $r^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$.

Then the problem is to obtain the maximum and minimum value of $f(x, y, z) = r^2$ subject to the condition $x^2 + y^2 + z^2 = a^2$.

Let $g(x,y,z) = (xy)^2 + (y-2)^2 + (z-3)^2 + \lambda(x^2+y^2+z^2)$ when optimal value of $f = r^2$ are obtained subject to the condition $x^2+y^2+z^2 = a^2$, g is also optimal.

$$\therefore \frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0$$

$=> 2(x-1) + \lambda(2x) = 0$

$2(y-2) + \lambda(2y) = 0$

$2(z-3) + \lambda(2z) = 0$

$=> \dfrac{x-1}{x} = \dfrac{y-2}{y} = \dfrac{z-3}{z} = -\lambda$

$\frac{x-1}{x} = \frac{y-2}{y} => xy - y = xy - zx$ or $y = 2x$

$\dfrac{x-1}{x} = \dfrac{z-3}{z} = z = 3x$

Substituting in $x^2+y^2+z^2 = a^2$, we get $x^2 + 4x^2 + 9x^2 = a^2$

$\therefore x^2 = \frac{a^2}{14}$ or $x = \pm\frac{a}{\sqrt{14}}$ $=> y = 2x = \pm\frac{2a}{\sqrt{14}}$, $z = 3x = \pm\frac{3a}{\sqrt{14}}$

Stationary points are $A(\frac{a}{\sqrt{14}}, \frac{2a}{\sqrt{14}}, \frac{3a}{\sqrt{14}})$ and $B(-\frac{a}{\sqrt{14}}, -\frac{2a}{\sqrt{14}}, -\frac{3a}{\sqrt{14}})$

Since the points A and Q lies in the same octant, A is at a minimum distance from $Q(1,2,3)$.

Also B lies in the opposite octant of the point $Q(1,2,3)$, hence is at maximum distance from Q.

2. Find the minimum distance from the origin to the plane $lx + my + nz = p$.
   Let $P(x,y,z)$ be any point on the plane $lx + my + nz = p$.

Then the distance $OP = r = \sqrt{x^2+y^2+z^2}$.

When r is minimum, then so is $r^2 = x^2+y^2+z^2$. Thus our objective is to minimize $f(x,y,z) = r^2 = x^2+y^2+z^2$ subject to the conditions $lx + my + nz = p$.

Let $g(x,y,z) = x^2+y^2+z^2 + \lambda(lx + my + nz)$.

When $lx + my + nz = p$ is satisfied, f and g assume minimum value together. Then $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0$

i.e $2x + \lambda l = 0 = 2y + \lambda m = 2z + \lambda n$

$$\therefore \frac{x}{l} = \frac{y}{m} = \frac{z}{n} = -\frac{\lambda}{2}$$

$$=> y = \frac{mx}{l}, z = \frac{nx}{l}$$

Substituting in $lx + my + nz = p$, we get $lx + m.\frac{mx}{l} + n.\frac{nx}{l} = p$

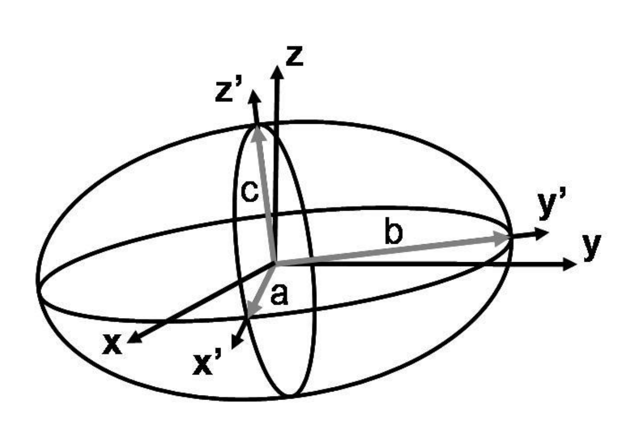
$$\therefore x = \frac{pl}{l^2 + m^2 + n^2} => y = \frac{pm}{l^2 + m^2 + n^2}, z = \frac{pn}{l^2 + m^2 + n^2}$$

$$\therefore r^2 = x^2 + y^2 + z^2 = \frac{(p^2 l^2 + m^2 p^2 + n^2 p^2)}{(l^2 + m^2 + n^2)^2} = \frac{p^2}{l^2 + m^2 + n^2}$$

$$\therefore r = \sqrt{x^2 + y^2 + z^2} = \frac{p}{\sqrt{l^2 + m^2 + n^2}}$$

Which is the perpendicular distance from the origin to the plane $lx + my + nz = p$ and hence is the minimum distance.

3. Find the axes of the ellipse of the intersection of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = 0$.



Let $P(x, y, z)$ be any point on the given ellipse.

Then the distance $OP = r = \sqrt{x^2 + y^2 + z^2}$.

The objective is to determine the maximum and minimum value of the distance $OP = r$ or $r^2 = u(x, y, z) = x^2 + y^2 + z^2$ subject to the condition that the point $P(x, y, z)$ lies on the ellipse or satisfies the conditions $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $lx + my + nz = 0$.

**Since the semi-major and semi-minor axes of the ellipse are the maximum and the minimum distance from the origin to the point P i.e of the distance OP **

Let $g(x, y, z) = u + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \mu(lx + my + nz) = x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \mu(lx + my + nz)$

Where λ and μ are constants.

When $g(x, y, z)$ is optimal, then so is $f(x, y, z) = u$ subject to the conditions defined above.

$$=> \frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0$$

i.e. $2x + \frac{\lambda 2x}{a^2} + \mu l = 0$ ....(1)

$2y + \frac{\lambda 2y}{l^2} + \mu m = 0$ ....(2)

$2z + \frac{\lambda 2z}{c^2} + \mu n = 0$ ....(3)

$\therefore$ (1)$*\, x +$ (2) $*\, y +$ (3) $*\, z$ gives

$$2(x^2 + y^2 + z^2) + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \mu(lx + my + nz) = 0$$

$\because \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $lx + my + nz = 0$, we get

$2x + 2\lambda = 0$ or $\lambda = -u$

Substituting in (1), (2) and (3), we get

$2x\left(1 - \frac{u}{a^2}\right) + \mu l = 0 \;\; => \;\; -\frac{2x}{\mu} = \frac{l}{1-\frac{u}{a^2}}$ ....(4)

$2y\left(1 - \frac{u}{b^2}\right) + \mu m = 0 \;\; => \;\; -\frac{2y}{\mu} = \frac{m}{1-\frac{u}{b^2}}$ ....(5)

$2z\left(1 - \frac{u}{c^2}\right) + \mu n = 0 \;\; => \;\; -\frac{2z}{\mu} = \frac{n}{1-\frac{u}{c^2}}$ ....(6)

Now (4)$*\, l +$ (5)$*\, m +$ (6)$*\, n$ gives

$$\frac{l^2}{1-\frac{u}{a^2}} + \frac{m^2}{1-\frac{u}{b^2}} + \frac{n^2}{1-\frac{u}{c^2}} = -\frac{2}{\mu}(lx + my + nz) = 0$$

Or $l^2\left(1 - \frac{u}{b^2}\right)\left(1 - \frac{u}{c^2}\right) + m^2\left(1 - \frac{u}{a^2}\right)\left(1 - \frac{u}{c^2}\right) + n^2\left(1 - \frac{u}{a^2}\right)\left(1 - \frac{u}{b^2}\right) = 0$

Which is a quadratic equation in u and hence has two roots u_1 and u_2.

Let $u_1 > u_2$.

Then $\sqrt{u_1}$ is the semi-major axis and $\sqrt{u_2}$ is the semi-minor axis of the ellipse of the intersection of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane $lx + my + nz = 0$.

Problems

1. Find the extreme values of the function $f(x, y, z) = \sin x \sin y \sin z$ where x, y, z are the angles of a triangle.
2. Find the largest rectangular parallelepiped inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
3. Prove that of all rectangular parallelepiped of the same volume, the cube has the least surface.
4. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition
   i) $yz + zx + xy = 3a^2$
   ii) $x + y + z = 3a$
   iii) $xyz = a^3$
5. Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.
6. A rectangular box open at the top have a volume of $32\ m^3$. Find the dimensions of the box requiring least material for construction.