University of Tlemcen **Academic year 2023-2024**
Faculty of Sciences **(L1 ING-INF)**
Department of Informatic **Algebra (First Year)**

Worksheet N°2/ "Sets - Applications" Correction.

Exercise 01: Let $E = \{x, y, z\}$ and $F = \{-2, 2\}$.

(1) Identifie the power set (ensemble des parties): $\wp(E)$, $\wp(F)$ and $\wp(\wp(F))$.

$\{\}$ brace

The power set is the set of the sub-set of E.

$$\begin{aligned} \wp(E) &= \left\{ \emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, E \right\}.2^3 \\ G &= \{\emptyset\} \\ G &\subset \wp(E) \end{aligned}$$

$$\wp(F) = \left\{ \underbrace{\emptyset}_{a}, \underbrace{\{-2\}}_{b}, \underbrace{\{2\}}_{c}, \underbrace{F}_{d} \right\}.2^2; 2^4 = 16$$

$$\wp(\wp(F)) = \left\{ \begin{array}{c} \emptyset, \{\emptyset\}, \{\{-2\}\}, \{\{2\}\}, \{F\}, \{\emptyset, \{-2\}\}, \{\emptyset, \{2\}\}, \{\emptyset, F\}, \\ \{\{-2\}, F\}, \{\{2\}, F\}, \{\emptyset, \{-2\}, \{2\}\}, \{\emptyset, \{-2\}, E\}, \{\{-2\}, \{2\}\}, \\ \{\emptyset, \{2\}, F\}, \{\{-2\}, \{2\}, F\}, \wp(F) \end{array} \right\}$$

(2) Determine 3 examples of partition of the set E.

$$\begin{aligned} (1)P(E) &= \{\{x\}, \{y\}, \{z\}\}. \\ (2)P(E) &= \{\{x,y\}, \{z\}\}. \\ (3)P(E) &= \{\{x\}, \{y,z\}\}. \end{aligned}$$

(3) Add the following symbols: $\in, \notin, \subset$ or $\not\subset$.

1) element $\in$ or $\notin$ in a set.
2) set $\subset$ or $\not\subset$ in a set.

$E = \{x, y, z\}$ and $F = \{-2, 2\}$

$x \in E$	$\{y\} \subset E$	$3 \notin F$	$\{3\} \not\subset F$
$x \notin \wp(E)$	$\{y\} \in \wp(E)$	$\varnothing \in \wp(E)(\varnothing \subset \wp(E))$	$\{\varnothing\} \subset \wp(E)$
$\{3\} \notin \wp(F)$	$\{-2, 1\} \notin \wp(F)$	$\{-2\} \in \wp(F)$	$\{\{-2\}\} \subset \wp(F)$
$\{y\} \notin \wp(\wp(E))$	$\{\{y\}\} \in \wp(\wp(E))$	$\varnothing \in \wp(\wp(F))$ or $\subset$	$\{\varnothing\} \in \wp(\wp(F))$or $\subset$

$x \in E$	$\{y\} \subset E$	$3 \notin F$	$\{3\} \not\subset F$
$x \notin \wp(E)$	$\{y\} \in \wp(E)$	$\varnothing \in \wp(E)$	$\{\varnothing\} \subset \wp(E)$
$\{3\} \notin \wp(F)$	$\{-2, 1\} \notin \wp(F)$	$\{-2\} \in \wp(F)$	$\{\{-2\}\} \subset \wp(F)$
$\{y\} \notin \wp(\wp(E))$	$\{\{y\}\} \subset \wp(\wp(E))$	$\varnothing \in \wp(\wp(F))$	$\{\varnothing\} \in \wp(\wp(F))$

Exercise 02: Let E be a nonempty set, A, B and C three sub-sets of E.Prove that:

(1)
$$A \subset B \Leftrightarrow C_E^B \subset C_E^A \Leftrightarrow A \cup B = B.$$

Remark: To prove that:

$$H \Leftrightarrow K \Leftrightarrow P,$$

just prove:
$$H \Rightarrow K \Rightarrow P \Rightarrow H.$$

a) Prove that:
$$A \subset B \Rightarrow C_E^B \subset C_E^A ? C_E^B = \overline{B}.$$
$$\text{If } x \in C_E^B \text{ then } [x \in E \text{ and } x \notin B],$$

which implies that:

$$x \in E \text{ and } x \notin A \text{ because } A \subset B \Rightarrow x \in C_E^A.$$

Conclusion:
$$A \subset B \Rightarrow C_E^B \subset C_E^A.$$

b) Prove that:
$$C_E^B \subset C_E^A \Rightarrow A \cup B = B?$$

i) Prove that: $A \cup B \subset B$?

$$\begin{aligned} x &\in A \cup B \Rightarrow \begin{cases} x \in A \Rightarrow x \notin \overline{A} \Rightarrow x \notin \overline{B}(\text{Hyp}) \Rightarrow x \in B, \\ \text{or } x \in B, \end{cases} \\ &\Rightarrow x \in B. \text{ (in both cases)} \end{aligned}$$

ii) Let's prove that: $B \subset A \cup B$ (obvious in any case).
Conclusion:
$$A \cup B = B.$$

c) Prove that:
$$A \cup B = B \Rightarrow A \subset B?$$

Indeed:

$$\begin{aligned} x &\in A \Rightarrow x \in A \cup B \Rightarrow x \in B \text{ because } (A \cup B = B). \\ &\Rightarrow A \subset B. \end{aligned}$$

Conclusion:
$$A \subset B \Leftrightarrow C_E^B \subset C_E^A \Leftrightarrow A \cup B = B.$$

(2)

$$C_E^{A\cup B} = C_E^A \cap C_E^B.$$

<u>Remark:</u>

$$x \in A\cup B \Leftrightarrow x \in A \text{ or } x \in B.$$

$$\begin{aligned}
\text{If } x &\in C_E^{A\cup B} \Leftrightarrow x \in E \text{ and } x \notin A\cup B\\
&\Leftrightarrow x \in E \text{ and } [x \notin A \text{ and } x \notin B]\\
&\Leftrightarrow [x \in E \text{ and } x \notin A\ ] \text{ and } [x \in E \text{ and } x \notin B] \text{ (distributive property)}\\
&\Leftrightarrow \left(x \in C_E^A\right) \text{ and } \left(x \in C_E^B\right) \Leftrightarrow x \in C_E^A \cap C_E^B.
\end{aligned}$$

(3)

$$A\cap B = A\cap C \Leftrightarrow A\cap C_E^B = A\cap C_E^C$$

So we prove that:

$$\begin{aligned}
A\cap B &= A\cap C \Rightarrow A\cap C_E^B = A\cap C_E^C?\\
&\text{and}\\
A\cap C_E^B &= A\cap C_E^C \Rightarrow A\cap B = A\cap C?
\end{aligned}$$

" $\Rightarrow$ " hypothesis: $A\cap B = A\cap C$.
Problem: $A\cap C_E^B = A\cap C_E^C$?
a) prove that: $A\cap C_E^B \subset A\cap C_E^C$.
If,

$$\begin{aligned}
x &\in A\cap C_E^B \Rightarrow x\in A \text{ and } x \in C_E^B\\
&\Rightarrow x\in A \text{ and } [x\in E \text{ and } x\notin B] \ (P_1 \text{ and } P_2 \text{ and } P_3)\\
&\Rightarrow x\in E \text{ and } [x\in A \text{ and } x\notin B] (\ P_2\text{and } P_1 \text{ and } P_3)\\
&\Rightarrow x\in E \text{ and } [x\notin A\cap B]\\
&\Rightarrow x\in E \text{ and } [x\notin A\cap C] \text{ because } A\cap B = A\cap C \text{ (hypothesis).}\\
&\Rightarrow x\in E \text{ and } [x\in A \text{ and } x\notin C] \text{ because } x\in A,\\
&\Rightarrow x\in A \text{ and } [x\in E \text{ and } x\notin C]\\
&\Rightarrow x\in A \text{ and } x\in C_E^C \Rightarrow x\in A\cap C_E^C.
\end{aligned}$$

b) Prove that: $A\cap C_E^C \subset A\cap C_E^B$?
If,

$$\begin{aligned}
x &\in A\cap C_E^C \Rightarrow x\in A \text{ and } x \in C_E^C\\
&\Rightarrow x\in A \text{ and } [x\in E \text{ and } x\notin C]\\
&\Rightarrow x\in E \text{ and } [x\in A \text{ and } x\notin C]\\
&\Rightarrow x\in E \text{ and } [x\notin A\cap C]\\
&\Rightarrow x\in E \text{ and } [x\notin A\cap B] \text{ because } A\cap B = A\cap C.\text{(hypothesis)}\\
&\Rightarrow x\in E \text{ and } [x\in A \text{ and } x\notin B] \text{ because } x\in A\\
&\Rightarrow x\in A \text{ and } [x\in E \text{ et } x\notin B]\\
&\Rightarrow x\in A \text{ and } x\in C_E^B \Rightarrow x\in A\cap C_E^B.
\end{aligned}$$

" $\Leftarrow$ " hypothesis: $A\cap C_E^B = A\cap C_E^C$.

<u>First method:</u>

Problem: $A\cap B = A\cap C$?

a) Prove that: $A\cap B\subset A\cap C$?

If,

$$\begin{aligned} x &\in A\cap B \Rightarrow x\in A \text{ and } x\in B \\ &\Rightarrow x\in A \text{ and } x\notin C_E^B \Rightarrow x\notin A\cap C_E^B \\ &\Rightarrow x\notin A\cap C_E^C \Rightarrow x\notin C_E^C \text{ because } x\in A. \\ &\Rightarrow x\in A \text{ and } x\in C \Rightarrow x\in A\cap C \end{aligned}$$

b) Prove that: $A\cap C\subset A\cap B$?

If,

$$\begin{aligned} x &\in A\cap C \Rightarrow x\in A \text{ and } x\in C \\ &\Rightarrow x\in A \text{ and } x\notin C_E^C \Rightarrow x\notin A\cap C_E^C \\ &\Rightarrow x\notin A\cap C_E^B \Rightarrow x\notin C_E^B \text{ because } x\in A. \\ &\Rightarrow x\in A \text{ and } x\in B \Rightarrow x\in A\cap B. \end{aligned}$$

Conclusion:

$$A\cap B = A\cap C \Leftrightarrow A\cap C_E^B = A\cap C_E^C.$$

<u>2nd method:</u> We have: $\forall A,B,C$:

$$\forall a\in\mathbb{R}, a^2+a+1>0.$$

$$A\cap B = A\cap C \Rightarrow A\cap\overline{B} = A\cap\overline{C}...(1)$$

$\forall A,G,L$:

$$A\cap G = A\cap L \Rightarrow A\cap\overline{G} = A\cap\overline{L}...(2)$$

We put:

$$G=\overline{B} \text{ and } L=\overline{C}.$$

So (2) implies: $\forall A,B,C$:

$$A\cap\overline{B} = A\cap\overline{C} \Rightarrow A\cap B = A\cap C.$$

(b) Give two assertions equivalent to the following assertion:

$$x\in(A-B)\Rightarrow x\notin A\cap B.$$

$$[K\Rightarrow L]\Leftrightarrow\left[\overline{L}\Rightarrow\overline{K}\right] \text{ and } [K\Rightarrow L]\Leftrightarrow\overline{\overline{[K\Rightarrow L]}}.$$

First: (Contraposition)

$$x\in A\cap B\Rightarrow x\notin(A-B)\,.\text{(True)}$$

Second: (Double negation)

$$\overline{K \Rightarrow L} \Leftrightarrow K \wedge \overline{L}.$$

$$\begin{aligned} \overline{\overline{[x \in (A-B) \Rightarrow x \notin A \cap B]}} &\Leftrightarrow \overline{[x \in (A-B)] \wedge [x \in A \cap B]} \\ &\Leftrightarrow [x \notin (A-B)] \vee x \notin (A \cap B). \end{aligned}$$

- Say if this implication is true or false? The proposition is True because the contraposition is true.

Exercise 03 : Let f and g be two defined applications of $\mathbb{R}$ in $\mathbb{R}$ such as:

$$f(x) = x^2 - 3x + 3 \text{ and } g(x) = \frac{x^2-5}{x^2+2}.$$

(1) f and g are they injective? surjective? (Method of definitions)
For the function:

$$f(x) = x^2 - 3x + 3.$$

a) For injective,
<u>**1st method:**</u>

$$\begin{aligned} \forall x_1, x_2 &\in \mathbb{R}, f(x_1) = f(x_2) \Rightarrow x_1^2 - 3x_1 + 3 = x_2^2 - 3x_2 + 3 \\ &\Rightarrow x_1^2 - x_2^2 - 3(x_1 - x_2) = 0 \\ &\Rightarrow (x_1 - x_2)[(x_1 + x_2 - 3)] = 0, \end{aligned}$$

then just choose x_1 and x_2 that check $x_1 + x_2 - 3 = 0$, for example:

$$x_1 = 1 \text{ and } x_2 = 2, \text{ so:}$$

$$x_1 \neq x_2 \text{ but } f(x_1) = f(x_2) = 1.$$

Conclusion: f is not injective.
<u>**2nd method:**</u>

$$f(x) = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4}(\text{canonical form}).$$

For example we put:

$$x - \frac{3}{2} = 2 \text{ or } x - \frac{3}{2} = -2, (\text{we have a square})$$

wich implies that: $x_1 = \frac{7}{2}$ or $x_2 = -\frac{1}{2}$, so:

$$x_1 \neq x_2 \text{ but } f(x_1) = f(x_2) = \frac{19}{4}.$$

Conclusion: f is not injective.
b) For surjective,
<u>1st method:</u> The problem is to show if we have:

$$\forall y\in\mathbb{R},\exists x\in\mathbb{R},f\left(x\right)=y?$$

Indeed,

$$\begin{aligned} f\left(x\right) &= y\Leftrightarrow x^{2}-3x+3=y\\ &\Leftrightarrow x^{2}-3x+3-y=0,....(1)\end{aligned}$$

$$\triangle=9-4\left(3-y\right)=4y-3,$$

so the equation (1) admits solutions if $\triangle\geq 0$, otherwise for example if $y=\frac{1}{2}$, then in this case $\triangle<0$ so,

$$\forall x\in\mathbb{R},f\left(x\right)\neq y,$$

wich implies that f is n't surjective.
<u>2nd method:</u>

$$f\left(x\right)=\left(x-\frac{3}{2}\right)^{2}+\frac{3}{4}(\text{canonical form}).$$

So, $f\left(x\right)\geq\frac{3}{4}$ then for $y<\frac{3}{4}$,

$$\forall x\in\mathbb{R},f\left(x\right)\neq y,$$

wich implies that f is n't surjective.
For the function:

$$g\left(x\right)=\frac{x^{2}-5}{x^{2}+2}=\frac{x^{2}+2-2-5}{x^{2}+2}=1-\frac{7}{x^{2}+2}.$$

a) For injective:

$$\text{For } x_{1}=1 \text{ and } x_{2}=-1,x_{1}\neq x_{2} \text{ but } g\left(x_{1}\right)=g\left(x_{2}\right)=\frac{-4}{3}.$$

Conclusion: g is not injective.
b) For surjective,
The problem is to show if we have:

$$\forall y\in\mathbb{R},\exists x\in\mathbb{R},g\left(x\right)=y?$$

Indeed,

$$g\left(x\right)=y\Rightarrow 1-\frac{7}{x^{2}+2}=y\Rightarrow\underbrace{1-y}_{>0\text{ and }\leq 0}=\underbrace{\frac{7}{x^{2}+2}}_{>0},$$

so, for example if $y=1$,

$$\forall x\in\mathbb{R},g\left(x\right)\neq y,$$

wich implies that g is n't surjective.

Reminder: 1) If $f: E \to F$ is bijective so f is injective and surjective and the inverse application exists noted f^{-1} with:

$$\begin{array}{rcl} f^{-1} & : & F \to E \\ y & \mapsto & f^{-1}(y) = x. \end{array}$$

In this we have: $f^{-1}(y), \forall y \in F$.

2) If $f: E \to F$ be an application and $B \subset F$, so $f^{-1}(B)$ exists with:

$$f^{-1}(B) = \{x \in E, \exists y \in B / f(x) = y\}.$$

Example 1

$$\begin{array}{rcl} f & : & \mathbb{R} \to \mathbb{R} \\ x & \mapsto & f(x) = \sin x. \end{array}$$

$f^{-1}(0)$ *does n't exist because* f *is n't bijective.*

But:

$$f^{-1}\{0\} = \{k\pi, k \in \mathbb{Z}\}.$$

because: $\sin k\pi = 0, \forall k \in \mathbb{Z}$.

(2) Say if the following propositions are true or false?

$$f(x) = x^2 - 3x + 3 = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} \text{ and } g(x) = \frac{x^2 - 5}{x^2 + 2}.$$

The table of variation for the function f is:(Ici c'est $\frac{3}{2}$)

x	$-\infty$		$\frac{2}{3}$		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$\frac{3}{4}$	$\nearrow$	$+\infty$

The table of variation for the function g is:

x	$-\infty$		0		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+1$	$\searrow$	$\frac{-5}{2}$	$\nearrow$	$+1$

(a)$f(\{0\}) = 3$ is false because $f(\{0\}) = \{3\}$.
(b) $0 \in f^{-1}(\{3\})$ is true because $f(0) = 3$.
(c) $\{0\} = f^{-1}(\{3\})$ is false because $f(0) = 3$ but $\exists x_2 \neq 0$ and $f(x_2) = 3$.
(d) $g^{-1}(0) = \sqrt{5}$ is false because g is n't bijective.
(e) $g^{-1}(\{0\}) = \left\{\sqrt{5}\right\}$ is false because $g^{-1}(\{0\}) = \left\{\sqrt{5}, -\sqrt{5}\right\}$.
(f) $g^{-1}(\{10\}) = \emptyset$ is true because $g(\mathbb{R}) = \left[-\frac{5}{2}; 1\right]$.

Remainder :

$$f(A \cup B) = f(A) \cup f(B) \text{ and } f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

(3) Find

$$\begin{aligned} f([0;1]) &= f\left(\left[0;\frac{2}{3}\right] \cup \left[\frac{2}{3};1\right]\right) = f\left(\left[0;\frac{2}{3}\right]\right) \cup f\left(\left[\frac{2}{3};1\right]\right) \\ &= \left(\left[f\left(\frac{2}{3}\right);f(0)\right]\right) \cup \left(\left[f\left(\frac{2}{3}\right);f(1)\right]\right) \\ &= \left(\left[\frac{3}{4};3\right]\right) \cup \left(\left[\frac{3}{4};1\right]\right) = \left[\frac{3}{4};3\right]. \end{aligned}$$

$$\begin{aligned} f^{-1}([0;1]) &= f^{-1}\left(\left[0;\frac{3}{4}\right[\cup \left[\frac{3}{4};1\right]\right) \\ &= f^{-1}\left(\left[0;\frac{3}{4}\right[\right) \cup f^{-1}\left(\left[\frac{3}{4};1\right]\right) \\ &= \varnothing \cup [1;2] \text{ because } f^{-1}(\{1\}) = \{1,2\} \\ &= [1;2]. \end{aligned}$$

$x^2 - 3x + 3 = 1 \Rightarrow x^2 - 3x + 2 = 0, \triangle = 1, x_1 = 1$ and $x_2 = 2$.

$$\begin{aligned} f(\mathbb{R}) &= \left[\frac{3}{4};+\infty\right[. \\ g([0;1]) &= [g(0);g(1)] = \left[-\frac{5}{2};-\frac{4}{3}\right]. \end{aligned}$$

<u>Exercise 04:</u> Let f defined from E in F by : $f(x) = \frac{3x}{x^2+x-2}$.

(1) Find E for f to be an application.
f is an application if and only if: $E = D_f$.

$$D_f = \left\{x \in \mathbb{R}/x^2 + x - 2 \neq 0\right\} = \mathbb{R} - \{-2, 1\}.$$

$$\underrightarrow{-\infty \qquad + \qquad -2 \qquad - \qquad 1 \qquad + \qquad +\infty}$$

(2) Study the application f and draw its table of variations.

$$
\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{3}{x} = 0.\\
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{3}{x} = 0.\\
\lim_{\substack{x\to-2\\ <}} f(x) &= \lim_{\substack{x\to-2\\ <}} \frac{3x}{x^2+x-2} = \frac{-6}{0^+} = -\infty.\\
\lim_{\substack{x\to-2\\ >}} f(x) &= \lim_{\substack{x\to-2\\ >}} \frac{3x}{x^2+x-2} = \frac{-6}{0^-} = +\infty.
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\substack{x\to1\\ <}} f(x) &= \lim_{\substack{x\to1\\ <}} \frac{3x}{x^2+x-2} = \frac{3}{0^-} = -\infty.\\
\lim_{\substack{x\to1\\ >}} f(x) &= \lim_{\substack{x\to1\\ >}} \frac{3x}{x^2+x-2} = \frac{3}{0^+} = +\infty.
\end{aligned}
$$

The derivative of:

$$f(x) = \frac{3x}{x^2+x-2}.$$

$$
\begin{aligned}
f'(x) &= \frac{3\left(x^2+x-2\right) - (2x+1)\,3x}{\left(x^2+x-2\right)^2}\\
&= \frac{-3x^2-6}{\left(x^2+x-2\right)^2} = -3\frac{\left(x^2+2\right)}{\left(x^2+x-2\right)^2} < 0.
\end{aligned}
$$

The table of variation for the function f is:

x	$-\infty$		-2		1		$+\infty$
$f'(x)$		$-$	‖	$-$	‖	$-$	
$f(x)$	0	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	0

(3) Say if f is injective and if it is surjective from E in $\mathbb{R}$? (Don't forget to write the definitions and the rationale for your answer).
a) For injective,

$$\forall x_1, x_2 \in D_f, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

f is n't injective because: (We use the table)

$\exists x_1 \in \left]-\infty, ; -2\right[, x_2 \in \left]-2; 1\right[, x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$ (negative value).

b) For surjective,

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = y \text{ or } f(E) = \mathbb{R}.$$

We use the table, we have: $f(E) = \mathbb{R}$, which implies that f is surjective.

(4) Otherwise give examples from the table of variations where f is bijective. A few cases we use the table, f is bijective for example if:

$$\begin{aligned} (1) f &: \left]-\infty;-2\right[\to \left]-\infty;0\right[. \\ (2) f &: \left]-2;1\right[\to \left]-\infty;+\infty\right[. \\ (3) f &: \left]0;+\infty\right[\to \left]0;+\infty\right[. \end{aligned}$$

(5) (Additional) Answer the same questions to: $g(x) = \frac{2}{\sqrt{(x+4)^2+1}} - 3$.

$$D_g = \mathbb{R}.$$

Limits:

$$\begin{aligned} \lim_{x\to-\infty} g(x) &= -3. \\ \lim_{x\to+\infty} g(x) &= -3. \end{aligned}$$

The derivative is:

$$g'(x) = \frac{-\frac{2(x+4)\times 2}{2\sqrt{(x+4)^2+1}}}{\left(\sqrt{(x+4)^2+1}\right)^2} = -2\frac{(x+4)}{\left((x+4)^2+1\right)\sqrt{(x+4)^2+1}}.$$

The sign of the derivative:

$$\underrightarrow{-\infty \quad + \quad -4 \quad - \quad +\infty}$$

The table of variation for the function g is:

x	$-\infty$		-4		$+\infty$
$g'(x)$		$+$	0	$-$	
$g(x)$	-3	$\nearrow$	-1	$\searrow$	-3

(6) (Additional) Answer the same questions to: $h(x) = \frac{2}{\sqrt{x^2-1}}$.

$$D_h = \left]-\infty;-1\right[\cup \left]1;+\infty\right[.$$

Limits:

$$\begin{aligned}
\lim_{x\to-\infty} h(x) &= 0. \\
\lim_{x\to+\infty} h(x) &= 0. \\
\lim_{\substack{x\to-1 \\ <}} h(x) &= +\infty. \\
\lim_{\substack{x\to 1 \\ >}} h(x) &= +\infty.
\end{aligned}$$

The derivative is:

$$h'(x) = \frac{-\frac{2x\times 2}{2\sqrt{x^2-1}}}{\left(\sqrt{x^2-1}\right)^2} = -2\frac{x}{(x^2-1)\sqrt{x^2-1}}.$$

The sign of the derivative:

$$\overrightarrow{-\infty \qquad + \qquad -1 * * * * * 0 * * * * * 1 \qquad - \qquad +\infty}$$

The table of variation for the function h is:

x	$-\infty$ -1		1 $+\infty$
$h'(x)$	$+$	//////	$-$
$h(x)$	$0 \nearrow +\infty$	//////	$+\infty \searrow 0$

Exercise 05: (Additional) Let $a, b, c\ d \in \mathbb{R}^*$and let f be defined as follows:

$$\begin{aligned}
f \quad &: \quad A \to B \\
x \quad &\mapsto \quad f(x) = \frac{ax+c}{bx+d}.
\end{aligned}$$

How should the greatest unknowns A and B and other constants be selected so that f is:

(1) f is an application if: $\forall x \in A, \exists y \in B$ with: $y = f(x)$.

We remark that $f(x)$ exists for $x \neq -\frac{d}{b}$, which implies that:

$$A = \mathbb{R} - \left\{-\frac{d}{b}\right\}.$$

(2) f is injective if:

$$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

$$
\begin{aligned}
f(x_1) &= f(x_2) \Rightarrow \frac{ax_1+c}{bx_1+d} = \frac{ax_2+c}{bx_2+d} \\
&\Rightarrow (ax_1+c)(bx_2+d) = (bx_1+d)(ax_2+c), \\
&\Rightarrow abx_1x_2 + adx_1 + cbx_2 + cd = abx_1x_2 + bcx_1 + dax_2 + cd, \\
&\Rightarrow adx_1 + cbx_2 = bcx_1 + dax_2, \\
&\Rightarrow (ad-bc)\,x_1 = (ad-bc)\,x_2,
\end{aligned}
$$

so to have $x_1 = x_2$ it's sufficient that: $(ad-bc) \neq 0$.
Conclusion: for f to be injective:

$$
(ad-bc) \neq 0 \text{ and } \quad \underset{\text{(the condition of application)}}{A = \mathbb{R} - \left\{-\frac{d}{b}\right\}}.
$$

(3) f is surjective if:

$$
\forall y \in B, \exists x \in A \text{ such as: } f(x) = y.
$$

$$
\begin{aligned}
f(x) &= y \Leftrightarrow \frac{ax+c}{bx+d} = y \Leftrightarrow ax+c = (bx+d)\,y \\
&\Leftrightarrow \quad x = \frac{dy-c}{a-by} \text{ that exists if } y \neq \frac{a}{b}.
\end{aligned}
$$

So f is surjective if:

$$
y \neq \frac{a}{b} \Leftrightarrow B = \mathbb{R} - \left\{\frac{a}{b}\right\} \text{ and } \quad \underset{\text{(the condition of application)}}{A = \mathbb{R} - \left\{-\frac{d}{b}\right\}}.
$$

(4) f is a bijective application if and only if: f is injective and surjective so:

$$
A = \mathbb{R} - \left\{-\frac{d}{b}\right\}, (ad-bc) \neq 0 \text{ and } B = \mathbb{R} - \left\{\frac{a}{b}\right\}.
$$

Sincere wishes you success (MESSIRDI BACHIR)