Propositional Logic Lecture 2

Hadjila Fethallah

Associate Professor at the Department of Computer Science

fethallah.hadjila@univ-tlemcen.dz

Proof Theory

- The purely semantic approach - based on model searching - is not practical
- To verify that A |= B, we must:
  - Find all models of A .
  - Check if these models are also models for B
- If A contains n atomic propositions then, it is necessary browse 2^n interpretations (the cost is exponential and not practical)
- Solution:
  - Possibility of using a syntactic approach
  - This means that it is only permitted to use inference rules and axioms

Formal system (proof system)

- A proof system (or axiomatic system) is a quadruplet (V, F, A, RI) such that:
- **V is a countable set of symbols**
- **F is a subset of V* called set of formulas**
- **A is a subset of F called a set of axioms**
- **RI is a subset of F called a set of inference rules**
- A rule of inference is an implication that is always true
- An axiom is a valid formula
- Examples :
- Axiom : $(p \rightarrow (q \rightarrow p))$
- modus ponens (RI): $\dfrac{p \, , \, p \rightarrow q}{q}$

Inference Rules (Examples)

Modus Ponens : $\{ P \rightarrow Q, \ P\} \vdash Q$

Modus Tollens : $\{ P \rightarrow Q , \ Not(Q)\} \vdash Not(P)$

Syllogism: $\{P \rightarrow Q , Q \rightarrow R\} \vdash P \rightarrow R$

Demonstration (proof)

- A demonstration in a formal system S, is a sequence of expressions A1,…An, such that:
- Each Ai is either:
  - An axiom of S
  - Or a consequence of the previous expressions, generated with one of the inference rules
- A theorem of S is the last expression of the demonstration
- We note it as : $\vdash$ An

Deductibility

- A given formula A is deductible from the set of hypotheses H, in a formal system S iff:
- There is a finite sequence of expressions A1,…An, such that , An=A, and for all $i \in \{1,\ldots,n\}$, Ai is created with one of the following scenarios:
- Ai is an axiom of S
- Ai is a consequence of the previous expressions, generated with one of the inference rules
- $Ai \in H$
- We note the relationship as : $H \vdash An$

Hilbert style system

- Classical propositional logic contains many proof systems, we cite the example of Łukasiewicz formal system:
- L1 (V={P $\cup$ {¬, → }},F, {A1,A2,A3}, MP)
- A1: (A →(B → A))
- A2: ((A → (B → C)) →((A → B) → (A → C));
- A3: ((¬B → ¬A) → (A → B))
- Inference rule:
- MP: A , (A → B) $\vdash$ B

Example of proof

- Is $P \rightarrow P$ provable ?

1.	$((P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P)))$	by Ax2
2.	$(P \to ((P \to P) \to P))$	by Ax1
3.	$((P \to (P \to P)) \to (P \to P))$	from 2, 1 by MP
4.	$(P \to (P \to P))$	Ax1
5.	$(P \to P)$	from 4, 3 by MP

Example of proof

- Can we prove P→ R from {P→Q,Q→ R} ?

Proof of: $(P \to Q), (Q \to R) \vdash_M (P \to R)$:

1.	$(P \to Q)$	premiss
2.	$(Q \to R)$	premiss
3.	$((Q \to R) \to (P \to (Q \to R)))$	by Ax1
4.	$(P \to (Q \to R))$	from 2, 3 by MP
5.	$((P \to (Q \to R)) \to ((P \to Q) \to (P \to R)))$	by Ax2
6.	$((P \to Q) \to (P \to R))$	from 4, 5 by MP
7.	$(P \to R)$	from 1, 6 by MP

Resolution method

- The resolution inference rule is invented by Robinson in 1965.
- All formulas of the resolution rule are under the conjunctive normal form CNF
- E.g., $(A \vee \neg B)$, $(B \vee \neg C \vee \neg D)$
- **Unitary resolution rule:**
- $l1 \vee \ldots \vee lk, \quad m$

- $L1 \vee \ldots \vee li\text{-}1 \vee li\text{+}1 \vee \ldots \vee lk$
- With li and m are the complementary (conflictual) literals.

Example:

$$\frac{P1.3 \vee \neg P2.2 \qquad P2.2}{P1.3}$$

Resolution rule

- We assume that li et mr are the conflictual literals
- The result is called resolvent
- $l1 \vee \ldots \vee l\,k, \quad m1 \vee \ldots \vee mi$

- $l1 \vee \ldots \vee li\text{-}1 \vee li{+}1 \vee \ldots \vee lk \vee m1 \vee \ldots \vee mr\text{-}1 \vee mr{+}1 \vee \ldots \vee mi$
- Example
- $C1 = (p \vee q \vee \neg r \vee s)$
- $C2 = (q \vee \neg p \vee t)$

Resolution over p and ¬p

- Resolvent :
- $(q \vee \neg r \vee s \vee q \vee t)$

Generalized Resolution rule

- It operates as the previous rule but it also removes (factorizes) the multiple copies of the same literal.
- Example **(factoring)**
- $C1 = (p \vee q \vee \neg r \vee s)$
- $C2 = (q \vee \neg p \vee t)$

Resolution over p and ¬p and factoring of q.

- Resolvent :
- $(\neg r \vee s \vee q \vee t)$
- This rule is **correct (sound)** and refutation **complete**.

Refutation principle

- To prove the clause A from a set of clauses H, it suffices to prove that H and A are unsatisfiable (inconsistent), and this means that $\square$ can be derived from H and $\neg A$
- To prove $H \vdash A$,it suffices to prove

  $H \cup \{\neg A\} \vdash \square$.

Resolution Algorithm

How do we prove H $\vdash$ A ?

Algorithm

1. H1 is first obtained by replacing the formulas of H by their CNF.
2. H2 = H1 $\cup$ {¬A}. (Where ¬ A is under CNF)
3. H3 is obtained by replacing the formulas of H2 with their clauses.
4. We iteratively apply (if possible) the resolution rule for any pair (Bj, Bi) where Bi,Bj $\in$ H3 and augment H3 with resolvent
5. we stop when we obtain the empty clause $\square$ (which is always unsatisfiable), in this case: H $\vdash$ A is confirmed. If the empty clause $\square$ cannot be obtained, then H $\nvdash$ A

Example (1)

- Can we demonstrate **a** from the set H ?
- H={
- $(b \wedge c) \rightarrow a$
- b
- $(d \wedge e) \rightarrow c$
- $e \vee f$
- $D \wedge \neg f$ }

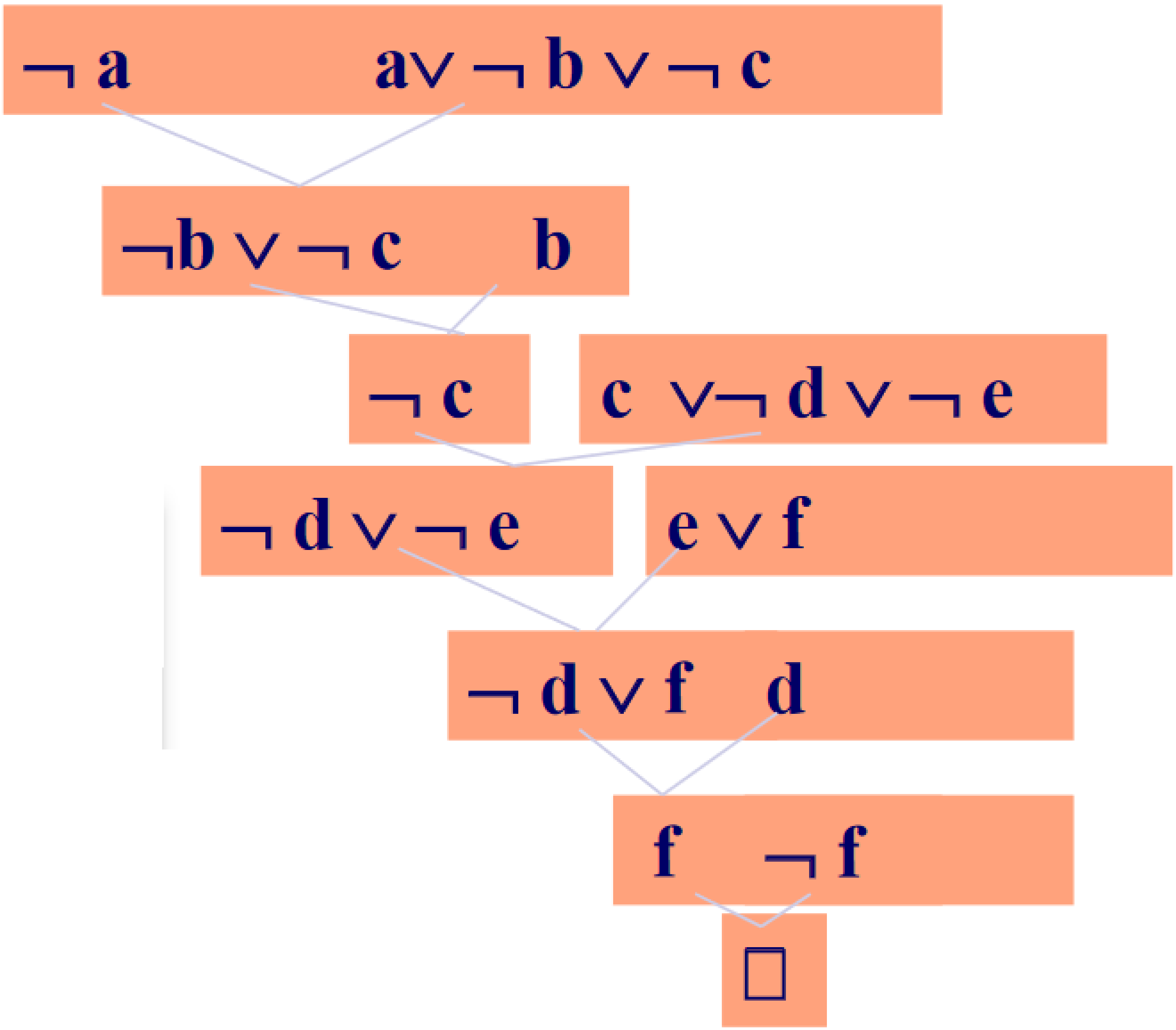
Example (2)

- CNF transformation of H
- $a \lor \neg b \lor \neg c$
- b
- $c \lor \neg d \lor \neg e$
- $e \lor f$
- d
- $\neg f$ }

Example (3)

- Application of refutation principle
- $a \vee \neg b \vee \neg c$
- b
- $c \vee \neg d \vee \neg e$
- $e \vee f$
- d
- $\neg f$
- $\neg a \}$

Example (4)



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