# Guided Tutorials n°4

## Exercise 1

Formalize the given sentences using predicate logic:

1. Every even natural number greater than 2 is the sum of two prime numbers.
2. $\forall x \ [(nat(x) \wedge even(x) \wedge > (x, 2)) \Rightarrow \exists y \ \exists z \ (prime(y) \wedge prime(z) \wedge = (x, sum(y, z))]$
3. A student is a football-player if and only if he is athletic and disciplined.
4. $\forall x \ [(Student(x) \Rightarrow footballPlayer(x)) \Leftrightarrow (athletic(x) \wedge disciplined(x))]$
5. There is a smallest positive real number.
6. $\exists x \ [(real(x) \wedge positive(x)) \wedge \forall y \ ((real(y) \wedge positive(y)) \Rightarrow < (x, y)]$
7. There is no smallest positive real number.
8. $\neg[\exists x \ [(real(x) \wedge positive(x)) \wedge \forall y \ ((real(y) \wedge positive(y)) \Rightarrow < (x, y)]]$
9. Any student's absence of any exam implies his exclusion.
10. $\forall x \forall y \ [(student(x) \wedge exam(y)) \wedge absence(x, y)) \Rightarrow exclusion(x)]$
11. All non-athletic students are honest except one.
12. $\forall x \ [(\neg athletic(x) \wedge student(x) \wedge x \neq y)) \Rightarrow honest(x)]$

## Exercise 2

Find two interpretations $I_1$ and $I_2$ such that the formula below is true in $I_1$ and false in $I_2$.

$$F \equiv \forall x \ \forall y \ Q(g(x, y), g(y, y), z)$$

1)

- $I1 = (D = \{2, 3\}, IF1)$
- $IF1(Q) = \{(3, 3, 2), (3, 2, 2), (2, 3, 2), (2, 2, 2), (3, 3, 3), (3, 2, 3), (2, 3, 3), (2, 2, 3)\}$
- $IF1(g) = \{(3, 3, 2), (3, 2, 2), (2, 3, 2), (2, 2, 2)\}$
- $IF1(z) = 2$

Consequently, $IF1(F) = 1$

2)

- $I2 = (D = \{2, 3\}, IF2)$
- $IF2(Q) = \{\}$
- $IF2(g) = \{(3, 3, 2), (3, 2, 2), (2, 3, 2), (2, 2, 2)\}$
- $IF2(z) = 2$

Consequently, $IF2(F) = 0$

**Exercise 3**

The following three formulas express that the binary predicate $p$ is reflexive, symmetric, and transitive. Demonstrate that none of these formulas is a logical consequence of the two others.

1. $\forall x \; p(x, x)$
2. $\forall x \; \forall y \; (p(x, y) \Rightarrow p(y, x))$
3. $\forall x \; \forall y \; \forall z \; (p(x, y) \wedge p(y, z) \Rightarrow p(x, z))$

— **Solution :**

1. $F_1 \wedge F_2 \not\models F_3$
   — $\mathcal{D} = \{1, 2, 3\}$
   — $\mathcal{IF}(p) = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$
   En appliquant $\mathcal{I}$ sur les ingrédients de $F_1$, nous aurons :

   (a) $\mathcal{IF}(F_1) = 1$ car,

| $x$ | $p(x,x)$ | $F_1$ |
|---|---|---|
| 1 | 1 | vrai |
| 2 | 1 | vrai |
| 3 | 1 | vrai |

   (b) $\mathcal{IF}(F_2) = 1$ car,

| $x$ | $y$ | $p(x,y)$ | $p(y,x)$ | $F_2$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | vrai |
| 2 | 1 | 1 | 1 | vrai |
| 3 | 1 | 0 | 0 | vrai |
| 1 | 2 | 1 | 1 | vrai |
| 2 | 2 | 1 | 1 | vrai |
| 3 | 2 | 1 | 1 | vrai |
| 1 | 3 | 0 | 0 | vrai |
| 2 | 3 | 1 | 1 | vrai |
| 3 | 3 | 1 | 1 | vrai |

   (c) $\mathcal{IF}(F_3) = 0$ car, si $x = 1$, $y = 2$ et $z = 3$ on a :
   $\mathcal{IF}(p(1,2)) = 1$, $\mathcal{IF}(p(2,3)) = 1$ et $\mathcal{IF}(p(1,3)) = 0$
   l'implication est fausse $1 \wedge 1 \Rightarrow 0$

2. $F_1 \wedge F_3 \not\models F_2$
   — $\mathcal{D} = \{1, 2\}$
   — $\mathcal{IF}(p) = \{(1,1), (2,2), (1,2)\}$
   En appliquant $\mathcal{I}$ sur les ingrédients de $F_1$, nous aurons :

   (a) $\mathcal{IF}(F_1) = 1$ car,

| $x$ | $p(x,x)$ | $F_1$ |
|---|---|---|
| 1 | 1 | vrai |
| 2 | 1 | vrai |

(b) $\mathcal{IF}(F_3) = 1$ car,

| $x$ | $y$ | z | $p(x,y)$ | $p(y,z)$ | $p(x,z)$ | $F_2$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | vrai |
| 1 | 1 | 2 | 1 | 1 | 1 | vrai |
| 1 | 2 | 1 | 1 | 0 | 1 | vrai |
| 1 | 2 | 2 | 1 | 1 | 1 | vrai |
| 2 | 1 | 1 | 0 | 1 | 0 | vrai |
| 2 | 1 | 2 | 0 | 1 | 1 | vrai |
| 2 | 2 | 1 | 1 | 0 | 0 | vrai |
| 2 | 2 | 2 | 1 | 1 | 1 | vrai |

(c) $\mathcal{IF}(F_2) = 0$ car, si $x = 1$ et $y = 2$ on a :
$\mathcal{IF}(p(1,2)) = 1$, $\mathcal{IF}(p(2,1)) = 0$
l'implication est fausse $1 \Rightarrow 0$

3. $F_2 \wedge F_3 \not\models F_1$
— $\mathcal{D} = \{1, 2\}$
— $\mathcal{IF}(p) = \{(1, 1)\}$
En appliquant $\mathcal{I}$ sur les ingrédients de $F_2$, nous aurons :

(a) $\mathcal{IF}(F_2) = 1$ car,

| $x$ | $y$ | $p(x,y)$ | $p(y,x)$ | $F_2$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | vrai |
| 2 | 1 | 0 | 0 | vrai |
| 1 | 2 | 0 | 0 | vrai |
| 2 | 2 | 0 | 0 | vrai |

(b) $\mathcal{IF}(F_3) = 1$ car,

| $x$ | $y$ | z | $p(x,y)$ | $p(y,z)$ | $p(x,z)$ | $F_2$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | vrai |
| 1 | 1 | 2 | 1 | 0 | 0 | vrai |
| 1 | 2 | 1 | 0 | 0 | 1 | vrai |
| 1 | 2 | 2 | 0 | 0 | 0 | vrai |
| 2 | 1 | 1 | 0 | 1 | 0 | vrai |
| 2 | 1 | 2 | 0 | 0 | 0 | vrai |
| 2 | 2 | 1 | 0 | 0 | 0 | vrai |
| 2 | 2 | 2 | 0 | 0 | 0 | vrai |

(c) $\mathcal{IF}(F_1) = 0$ car, [redacted]

IF(F1)=0, car IF(p(2,2))=0 (puique (2,2) n'appartient pas à IF(p))

**Exercise 4**

Provide, when it exists, a unifier for each pair of atoms $(A_1, A_2)$.
— $A_1 = p(x, g(x), z)$ et $A_2 = p(f(y), g(f(b)), h(y))$
— $A_1 = p(x, h(b), h(x))$ et $A_2 = p(f(g(y)), y, h(f(g(h(a)))))$

— **Solution :**

$$\Rightarrow \begin{cases} A_1 = p(x, g(x), z) \\ A_2 = p(f(y), g(f(b)), h(y)) \end{cases} \Rightarrow \begin{cases} x = f(y) \\ g(x) = g(f(b)) \\ z = h(y) \end{cases} \Rightarrow \begin{cases} x = f(y) \\ g(f(y)) = g(f(b)) \\ z = h(y) \end{cases}$$

$$\Rightarrow \begin{cases} x = f(y) \\ f(y) = f(b) \\ z = h(y) \end{cases} \Rightarrow \begin{cases} x = f(y) \\ y = b \\ z = h(y) \end{cases} \Rightarrow \begin{cases} y = b \\ x = f(b) \\ z = h(b) \end{cases} \Rightarrow \{x/f(b), y/b, z/h(y)\}$$

— **Solution :**

$$\Rightarrow \begin{cases} A_1 = p(x, h(b), h(x)) \\ A_2 = p(f(g(y)), y, h(f(g(h(a))))) \end{cases} \Rightarrow \begin{cases} x = f(g(y)) \\ h(b) = y \\ h(x) = h(f(g(h(a)))) \end{cases} \Rightarrow \begin{cases} x = f(g(y)) \\ y = h(b) \\ h(f(g(y))) = h(f(g(h(a)))) \end{cases}$$

$$\Rightarrow \begin{cases} x = f(g(y)) \\ y = h(b) \\ h(f(g(h(b)))) = h(f(g(h(a)))) \end{cases} \Rightarrow \begin{cases} x = f(g(y)) \\ y = h(b) \\ b = a \end{cases} \quad \textit{echec}$$

## Exercise 5

Convert the following formulas into prenex form :

1. $(\forall x\ \exists y\ R(x, z, y)) \Rightarrow (\exists x\ \forall y\ \exists t\ S(x, z, t))$
2. $((\exists x\ \ A(x)\ \Rightarrow\ \exists y\ \ B(y))\ \Rightarrow\ \exists z\ \ C(z))\ \Rightarrow\ \exists t\ D(t)$
3. $(\forall x\ \exists y\ \forall t\ R(x, z, t)) \Rightarrow (\exists x\ \forall y\ \exists t\ S(x, z, t))$
4. $\exists x\ \forall y (P(x) \wedge Q(y)) \Rightarrow \forall x\ \forall y\ \neg R(x, y)$

Convert the following formulas into Skolem normal form :

1. $\forall x\ \forall z\ \exists y\ \exists w\ (\forall t\ P(x, y, z, t) \Rightarrow \exists t\ Q(w, t))$
2. $\exists x\ \exists y\ P(x, y) \wedge \forall x\ \neg P(x, x)$
3. $\forall x\ P(x) \wedge \forall x\ (P(x) \Rightarrow \exists R(x, y))$

— **Solution :**

1. $(\forall x\ \exists y\ R(x, z, y)) \Rightarrow (\exists x\ \forall y\ \exists t\ S(x, z, t))$
   $\neg(\forall x\ \exists y\ R(x, z, y)) \vee (\exists x\ \exists t\ S(x, z, t))$
   $(\exists x\ \forall y\ \neg R(x, z, y)) \vee (\exists x\ \exists t\ S(x, z, t))$
   $(\exists x\ \forall y\ \neg R(x, z, y)) \vee (\exists x_1\ \exists t\ S(x_1, z, t))$
   $\exists x\ \forall y\ \exists x_1\ \exists t\ (\neg R(x, z, y) \vee S(x_1, z, t))$