Course Algebra 1 : Chapter 2, **Sets and Applications**

1^{st} year Licence LMD **Informatique**

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Chapter 2

Sets and Applications

2.1 Sets : Definitions, properties et examples

2.1.1 Definitions

Definition 1.

1. *A* ***set*** *is a* collection *or* gathering *of distinct objects or elements. It is usually denoted by a capital letter* $E, F, A, B, \ldots$ *or by enclosing its constituent elements in braces* $\{\ldots\}$.

   Example 1. $\mathbb{R}$ *: Set of real numbers,* $\{5, 6, 7, 8, 9, 10\}$ *: Set of natural numbers between 5 and 10,* E *: Set of vectors of the plane,* $\mathbf{K}[X]$ *: Set of polynomials with one variable* $X, \ldots$.

2. *If a set is determined by* enumerating *its elements, we say that the set is defined in* ***Roster method*** *(in french :* ***en extension****).*

   Example 2. $\{0, 1, 2, 3, 4, 5\}$ *is the set of natural numbers less or equal to 5.*

3. *If a set is determined by a characteristic property of its elements, we say that the set is defined by* ***abstraction method*** *(in french :* ***en compréhension****).*

   Example 3. $E = \{n \in \mathbb{N}, n \text{ is divisible by } 2\}$ *designates the set of numbers* n *belonging to* $\mathbb{N}$ *(natural numbers) that are divisible by* 2*, i.e. even numbers.*

4. *If a set* E *contains a finite number* n *of elements, then* E *is said to be a* ***finite set*** *and* n *is its* ***cardinal****, which is denoted by* $CardE = n$.

   Example 4. The set of natural numbers between 5 and 10, *from the previous example has the cardinal* $Card\{5, 6, 7, 8, 9, 10\} = 6$.

Definition 2.

1. *If* a *is one of the elements making up the set* E*, we say that* a belongs to E *or that* a *is* an element of E*, and we denote* $a \in E$*. Otherwise,* a *is said* not to belong to E*, and we denote* $a \notin E$.

   Example 5. $2 \in \mathbb{N}$ *because* 2 *is an element of* $\mathbb{N}$. $\pi \notin \mathbb{Q}$*, because* π *is an irrational number.*

2. *If the set* E *contains only one element* a*, i .e.* $E = \{a\}$*, it is called a* ***singleton****.*

3. *If* E *contains no elements, it is called an* ***empty set*** *and is denoted* $\varnothing$ *or* $\{\}$.

Definition 3. *Given two sets E and F, if all the elements of E also belong to F, and we write :*

$$\forall x \in E, x \in F$$

*we then say that E **is included in** F or that E is **a subset** of F, or that E is **a part** of F, or that F **contains** E. This inclusion is denoted by*

$$E \subset F$$

The negation of the inclusion is :

$$E \not\subset F$$

therefore expressed as :

$$\exists x \in E, x \notin F.$$

Example 6. $\mathbb{N} \subset \mathbb{Z}$. $\{-1, \sqrt{2}, 3\} \not\subset \mathbb{Z}$.

Properties 1.

1. $\forall E$, $\varnothing \subset E$: *the empty set* $\varnothing$ *is included in all sets.*
2. $\forall E$, $E \subset E$: *any set is included in itself.*
3. $E = F \Leftrightarrow (E \subset F \wedge F \subset E)$.
4. $(E \subset F \wedge F \subset G) \implies E \subset G$: *the inclusion is* transitive.

Remark 1. *Belonging, whose symbol is $\in$, of an element in a set should not be confused with inclusion, whose symbol is $\subset$, of a set in another set.*

Example 7.

$$2 \in \mathbb{N} \qquad \text{and} \qquad \{2\} \subset \mathbb{N}$$

Definition 4. *Given a set E, the **set of parts of set** E, denoted $\mathcal{P}(E)$, is defined as the set containing all the subsets of E. $card\mathcal{P}(E) = 2^n$ where $n = card(E)$.*

Example 8. *Let E be the set defined by :*

$$E = \{-1, 0, 2\}$$

The subsets that can be constructed from E are then :

$$\{-1\}, \{0\}, \{2\}, \{-1, 0\}, \{-1, 2\}, \{0, 2\} \text{ but also } \{-1, 0, 2\} = E, \varnothing$$

So

$$\mathcal{P}(E) = \{\{-1\}, \{0\}, \{2\}, \{-1, 0\}, \{-1, 2\}, \{0, 2\}, E, \varnothing\}$$

Clearly $card\mathcal{P}(E) = 2^3 = 8$.

Remark 2. *$\mathcal{P}(E)$ is a set whose elements are also sets. We deduce that a given subset of E **belongs** ($\in$) to $\mathcal{P}(E)$. We therefore have the following relations:*

$$a \in E \Leftrightarrow \{a\} \subset E \Leftrightarrow \{a\} \in \mathcal{P}(E)$$

2.1.2 Sets Operations

Diagrams representing sets can be used to make certain concepts easier to understand.
Let E be a set and A, B two subsets of E. The following operations are defined :

I. Union

The union of the two sets A and B is the set C of elements belonging to A or B, written as :

$$C = A \cup B$$

we read: C equals A union B. So we write :

$$C = A \cup B = \{x \in E, x \in A \text{ or } x \in B\}$$

C is also a subset of E.
We have the equivalence :

$$x \in A \cup B \Leftrightarrow (x \in A \text{ or } x \in B)$$

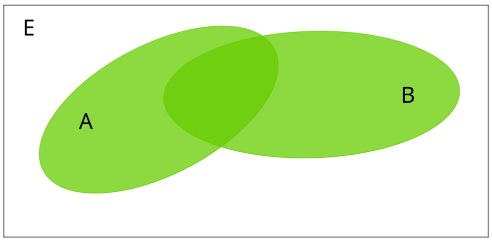


Figure 2.1: Diagram of Union. The set $C = A \cup B$ is the green part.

Example 9.

Let be $E = \{-3, -1, 0, 2\}$ *and* $F = \{-3, 0, 5, 10, 25\}$, *then* $E \cup F = \{-3, -1, 0, 2, 5, 10, 25\}$.

II. Intersection

The intersection of the two sets A and B is the set D of the elements belonging to A and B, we write :

$$D = A \cap B$$

we read : D equals A inter B. We therefore write :

$$D = A \cap B = \{x \in E, x \in A \text{ and } x \in B\}$$

D is also a subset of E.
The equivalence is :

$$x \in A \cap B \Leftrightarrow (x \in A \text{ and } x \in B)$$

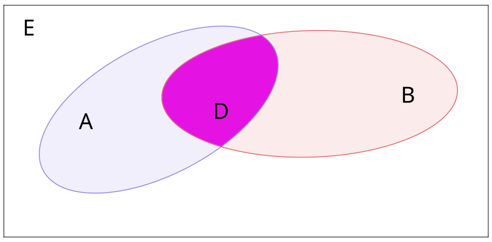


Figure 2.2: Diagram of the intersection. The set $D = A \cap B$ is the purple part.

Example 10.
Let be $E =]-\infty, -1.5[$ *and* $F = [-3, 7]$, *then* $E \cap F = [-3, -1.5[$.

III. Set difference

The set difference A minus B is the set F of elements that **belong** to A and **do not belong** to B, written as :

$$F = A - B \text{ or } A \backslash B$$

We read : F equals A minus B. So we write :

$$F = A - B = \{x \in E, x \in A \text{ and } x \notin B\}$$

F is also a subset of E.
The equivalence is :

$$x \in A - B \Leftrightarrow (x \in A \text{ and } x \notin B)$$

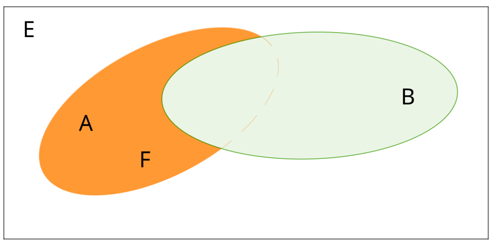


Figure 2.3: Diagram of the set difference $A - B$. The set $F = A - B$ is the orange part (which is not A).

Example 11.
Let be $E = \mathbb{Z}$ *and* $F = \{0\}$, *then* $E \backslash F = \mathbb{Z}^*$.

IV. Symmetrical difference

The symmetrical difference of A and B is the set G of elements belonging to one and only one of the sets A and B, written as :

$$G = A\Delta B$$

We read : G equals the symmetrical difference between A and B. We therefore write :

$$G = A\Delta B = \{x \in E, x \in A \text{ or } x \in B \text{ and } x \notin A \cap B\}$$

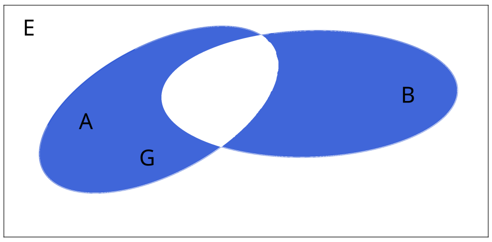


Figure 2.4: Diagram of the Symmetric Difference $A\Delta B$. the set $G = A\Delta B$ is the blue part.

G is also a subset of E.
We have the equivalence :

$$x \in A\Delta B \Leftrightarrow (x \in A \text{ or } x \in B \text{ and } x \notin A \cap B)$$

Example 12.
Let be $E = [-5, 2]$ *and* $F =]0, 10]$*, then* $E\Delta F = [-5, 0[\cup]2, 10]$.

2.1.3 The Complement

The two subsets A and B of a set E are called complementary, if their union is the set E and their intersection the empty set $\varnothing$, we write :

$$B = \complement_E A \text{ or } A^c \text{ or } \overline{A}$$

We read : B equals the complement of A in E. So we write

$$B = \complement_E A = \{x \in E, x \notin A\}$$

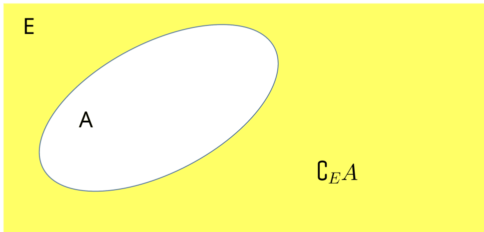


Figure 2.5: Diagram of the complement $\complement_E A$. The set $B = \complement_E A$ is the part in yallow colour.

$\complement_E A$ is also a subset of E.
We have the equivalence :

$$x \in \complement_E A \Leftrightarrow x \notin A$$

Example 13. $\complement_{\mathbb{Z}} \mathbb{Z}_- = \mathbb{Z}^*_+$.

2.1.4 Union and intersection of parts of a set

Definition 5. *Let* I *be a part of* $\mathbb{N}$ *and* $(E_i)_{i\in I}$ *a family of parts of* E*. We Note :*

$$\underset{i\in I}{\cup} E_i = \{x \in E, \exists i \in I, x \in E_i\}$$

$$\underset{i\in I}{\cap} E_i = \{x \in E, \forall i \in I, x \in E_i\}$$

Example 14. *Let be the set :* $E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ *and* $I = \{0, 1, 2\}$ *such that :*

$$E_0 = \{0, 1, 3\},\ E_1 = \{3, 5, 8, 10\},\ E_2 = \{3, 6\}$$

then $(E_i)_{i\in I} = \{E_0, E_1, E_2\}$ *is a family of parts of* E*. We have then :*

$$\underset{i\in I}{\cup} E_i = E_0 \cup E_1 \cup E_2 = \{0, 1, 3, 5, 6, 8, 10\}$$

we can see that $0, 1, 3 \in E_0$*,* $5, 8, 10 \in E_1$ *and* $6 \in E_2$*.*
We hae too :

$$\underset{i\in I}{\cap} E_i = E_0 \cap E_1 \cap E_2 = \{3\}$$

we can see that $3 \in E_0$*,* $3 \in E_1$ *and* $3 \in E_2$*.*

Remark 3. *We have then, for all x of E :*

$$x \in \bigcup_{i\in I} E_i \Longleftrightarrow \exists i \in I,\ x \in E_i$$

$$x \in \bigcap_{i\in I} E_i \Longleftrightarrow \forall i \in I,\ x \in E_i$$

Definition 6. *Let I be a part of $\mathbb{N}$ and $(E_i)_{i\in I}$ a family of non-empty parts of E. We say that this family forms a* ***parition*** *of E if :*

$$\begin{cases} \bigcup\limits_{i\in I} E_i = E \\ \forall (i,j) \in I^2, i \neq j \Longrightarrow E_i \cap E_j = \varnothing \end{cases}$$

Example 15.

1. *Let be the set :*

$$E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

   - *The family : $\{\{0\}, \{1, 2, 3\}, \{4, 5, 6, 7, 8, 9\}\}$ forms a partition of E.*
   - *The family : $\{\{0, 1\}, \{2, 3, 4, 5\}, \{6, 7, 8, 9\}\}$ also forms a partition of E.*

2. *The part made up of even integers and the part made up of odd integers form a partition of $\mathbb{Z}$.*

2.1.5 Some properties

Let A, B and C be subsets of E, then we have the following properties :

1. **Commutativity** $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
2. **Associativity** $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
3. **Distributivity** $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ and $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.
4. **Idempotency** $A \cup A = A$ et $A \cap A = A$.
5. **De Morgan's laws** $\complement_E(A \cup B) = \complement_E A \cap \complement_E B$ and $\complement_E(A \cap B) = \complement_E A \cup \complement_E B$.
6. $A \cap E = A$ and $A \cup E = E$.
7. $A \cap B = B \Longleftrightarrow B \subset A$ and $A \cup B = A \Longleftrightarrow B \subset A$
8. $A \subset A \cup K$ for any K subset of E.

2.1.6 Examples : Famous sets

The usual sets are denoted by a capital letter in double bar (Blackboard Bold) $\mathbb{E}$. If a star is added to the power $\mathbb{E}^*$ this denotes the same set $\mathbb{E}$ but without the null element. If the symbol $+$ (respectively $-$) is added to the subscript, this denotes the subset of $\mathbb{E}$ whose elements are of positive (resp. negative) sign or equal to zero :

$\mathbb{N}$: The set of natural numbers : $0, 1, 2, 3 \ldots$

$\mathbb{Z}$: The set of integers : $\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, \ldots$

$\mathbb{Q}$: The set of rational numbers : $\left(\frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z}^*\right)$.

$\mathbb{R}$: The set of real numbers : $\ldots, -3.1, \ldots, -1, \ldots, 0, \ldots, \pi, \ldots, 10, \ldots$

$\mathbb{C}$: The set of complex numbers : $(a + i \cdot b,\ a, b \in \mathbb{R})$.

2.1.7 Cartesian product

Definition 7. *Let A and B be two sets. The set of pairs (a, b) where $a \in A$ and $b \in B$, is called the* ***Cartesian product*** *of sets A and B. It is denoted by :*

$$A \times B$$

If $A \neq B$, then $A \times B \neq B \times A$.
If $A = B$ then, $A \times B = A \times A = A^2$.
If A and B are finite sets, then the cardinal is : $card(A \times B) = cardA \times cardB$.

Example 16.

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

is the set of pairs (x, y) where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.
Similarly

$$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^n$$

is the set of families with n elements $(x_1, x_2, \ldots, x_n)$ where $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}$.

2.2 Applications

Definition 8. *Let E and F be two sets.*

An ***application*** *from E to F is any correspondence f associating* ***each*** *element x of E a* ***single*** *element y of F.*

E is the ***starting set****. An element of E, usually x, is an* ***antecedent*** *or a* ***pre-image****.*

F is the ***arrival set****. An element of F, usually y, which is associated with x by the application f is the* ***image of*** *x* ***by*** *f. All this is denoted by :*

$$\begin{aligned} f \,:\, E &\to F \\ x &\mapsto y = f(x) \end{aligned}$$

The set of all applications from E into F is denoted by : F^E.

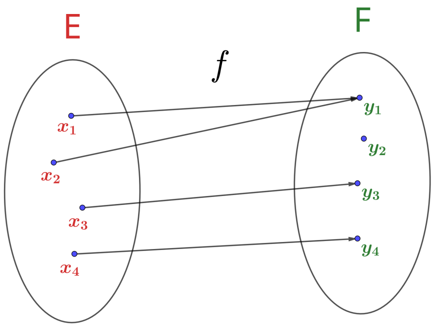


Figure 2.6: Schema of an application f from a set E to a set F

Remark 4.

1. *An application is a* ***function*** *whose domain of definition is the entire starting set.*

2. *The* ***graph*** *of an application $f : E \longrightarrow F$ is the set : $\Gamma_f = \{(x, f(x)) \in E \times F, x \in E\}$*

Example 17.

1. *Let be the application f and its schema :*

$$f : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto y = f(n) = 2n$$

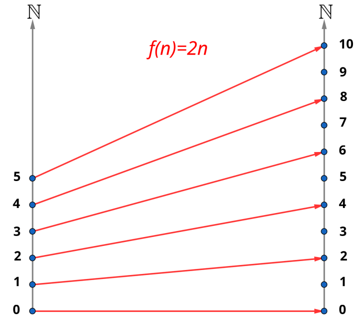


Figure 2.7: Schema of the apllication f

2. *The application :*

$$f : E \to E$$
$$x \mapsto y = f(x) = x$$

is called the ***identity application*** *in E and is denoted : Id_E.*

3. *Let $A \subset E$. The appllication :*

$$f : E \to \{0, 1\}$$
$$x \mapsto \begin{cases} 1 & si\ x \in A \\ 0 & si\ x \in \complement_E A \end{cases}$$

is called the ***indicator*** *or* ***characteristic application*** *of part A and is denoted by : 1_A or χ_A.*

Definition 9. *Let E and F be two sets and f an application from E to F.*

(a) *For any part A of E , the* ***direct image*** *of A by f, denoted $f(A)$, is defined by :*

$$f(A) = \{f(x) \in F, x \in \ A\}$$

i.e.: the images, of all elements x of A, which belong to F. We have $f(A) \subset F$.

(b) *For any part B of F, we define the* ***reciprocal image*** *of B by f, denoted $f^{-1}(B)$, by :*

$$f^{-1}(B) = \{x \in E, f(x) \in B\}$$

i.e.: The elements of E (not necessarily all of them) whose images $f(x)$ belong to B. We have $f^{-1}(B) \subset E$.

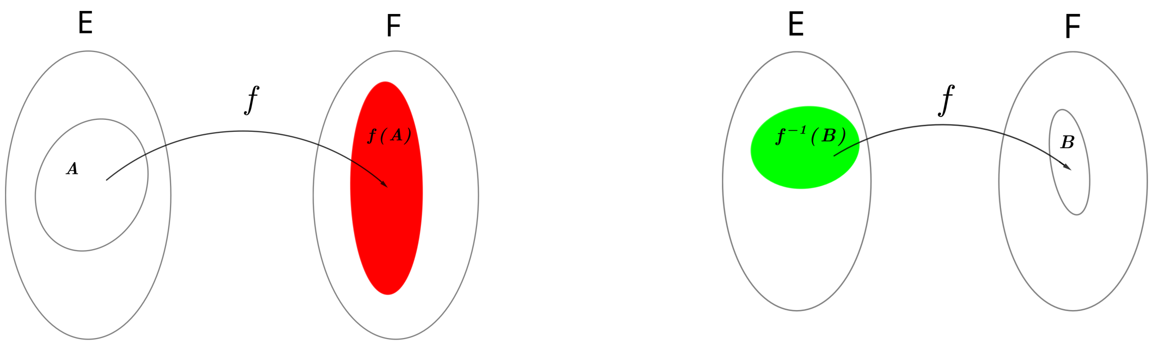
Example 18. *Let f be the following application :*

$$f :]-\sqrt{3}, +\infty[\to \mathbb{R}_+$$
$$x \mapsto f(x) = x^2 - 3$$

1. *Determine the direct image of the set $A = \{-\sqrt{2}, 3, 0.5, \pi, 8\}$ by the application f.*

2. *Determine the reciprocal image of the set $B = [0, 1]$ by the application f.*

Solution.



(a) Direct image $f(A)$ (b) Reciprocal image $f^{-1}(B)$

Figure 2.8: Direct and reciprocal images

1. $f(A) = \{f(x) \in \mathbb{R}_+, x \in A\}$.
The elements x of A being : $-\sqrt{2}, 3, 0.5, \pi, 8$ let's calculate their images by the application f : $f(-\sqrt{2}) = -1,\ f(3) = 6,\ f(0.5) = -2.75,\ f(\pi) = \pi - 3,\ f(8) = 61$ *but* $f(-\sqrt{2}) = -1 \notin \mathbb{R}_+$ *and* $f(0.5) = -2.75 \notin \mathbb{R}_+$ *and all the other images belong to* $\mathbb{R}_+$*. So :*

$$\begin{aligned} f(A) &= \{f(3), f(\pi), f(8)\} \\ &= \{6, \pi - 3, 61\} \end{aligned}$$

2. $f^{-1}(B) = \{x \in]-\sqrt{3}, +\infty[, f(x) \in B\}$.
Let $x \in]-\sqrt{3}, +\infty[$,

$$\begin{aligned} f(x) \in B &\iff f(x) \in [0, 1] \\ &\iff 0 \leq f(x) \leq 1 \\ &\iff 0 \leq x^2 - 3 \leq 1 \\ &\iff 3 \leq x^2 \leq 4 \\ &\iff -\sqrt{3} \leq x \leq -2 \ \textit{ou} \ 2 \leq x \leq \sqrt{3} \\ &\iff x \in [-\sqrt{3}, -2] \cup [2, \sqrt{3}] \end{aligned}$$

but $-\sqrt{3} \notin]-\sqrt{3}, +\infty[$ *Therefore*

$$f^{-1}(B) =]-\sqrt{3}, -2] \cup [2, \sqrt{3}]$$

2.2.1 Injection

Let f be an application from E to F. If for each two (distinct) elements of E there correspond, through f , two distinct elements of F, we say that f realizes an **injection** from E into F or that f is an **injective application** (also called a **one-to-one application**), then we have :

$$\forall x_1, x_2 \in E \qquad x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$$

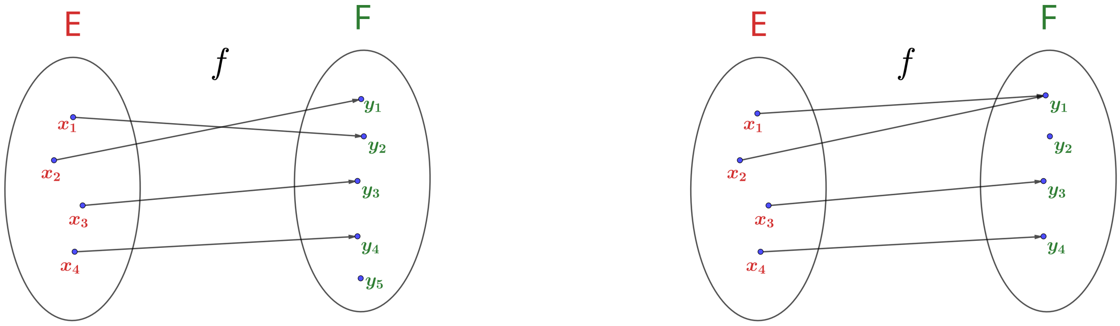
or what amounts to the same thing :

$$\forall x_1, x_2 \in E \qquad f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$$

The application f is **non-injective** if the negation of the previous statement is verified i.e.

$$\exists x_1, x_2 \in E \qquad f(x_1) = f(x_2) \text{ and } x_1 \neq x_2$$

Example 19.



(a) Injective application (b) Non-injective application

Figure 2.9: In figure (b), the two elements x_1, x_2 have the same image y_1 , which makes the application f non-injective.

1. *Let be the following application :*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\mapsto f(x) = x + 1 \end{aligned}$$

Show that f is an injective application.

Let x_1, x_2 be any two elements of $\mathbb{R}$.

$$\begin{aligned} f(x_1) = f(x_2) &\Longrightarrow x_1 + 1 = x_2 + 1 \\ &\Longrightarrow x_1 = x_2 \end{aligned}$$

then f is injective.

2. *Let be the application :*

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\mapsto g(x) = x^2 \end{aligned}$$

Is g injective?

Method 1 : Note that since 2 and -2 are both elements of $\mathbb{R}$, we can calculate $g(2) = 2^2 = 4$ and $g(-2) = (-2)^2 = 4$, hence $g(2) = g(-2)$ and $2 \neq -2$ and this is a negation of the definition, which is $\exists x_1, x_2 \in \mathbb{R},\ g(x_1) = g(x_2)$ and $x_1 \neq x_2$, hence g is not injective.

Method 2 : Try to show that g is injective, using the definition :
Let $x_1, x_2 \in \mathbb{R}, g(x_1) = g(x_2)$.

$$\begin{aligned} g(x_1) = g(x_2) &\Longleftrightarrow x_1^2 = x_2^2 \\ &\Longrightarrow x_1 = x_2 \text{ or } x_1 = -x_2 \end{aligned}$$

and so g is not injective, because $x_1 = -x_2$ means that there are $x_1, x_2 \in \mathbb{R}$ which are not equal and which have the same image by g.

2.2.2 Surjection

Let f be an application from E to F. If any element of F is the image by the application f of **at least** one element of E, then we say that f is a **surjection** of E onto F or that f is a **surjective application**. Then we have :

$$\forall y \in F,\ \exists x \in E,\ y = f(x)$$

The application f is **non-surjective** if we have the negation of the previous statement :

$$\exists y \in F,\ \forall x \in E,\ y \neq f(x)$$

(a) Surjective application (b) Non-surjective application

Figure 2.10: In figure (b), the element y_2 has no antecedent, which makes the application f non-superjective.

Example 20.

1. *Let be the application :*

$$f : [-1, 0] \longrightarrow [0, 1]$$
$$x \mapsto f(x) = \sqrt{x+1}$$

Is f surjective?

Let $y \in [0, 1]$, check if there is an $x \in [-1, 0]$ such that y is its image by f. Let's assume that this x exists. y is the image of x by f, which means : $y = f(x)$

$$\begin{aligned} y = f(x) &\Longleftrightarrow y = \sqrt{x+1} \\ &\Longrightarrow y^2 = x + 1 \textit{ because } y \geq 0 \\ &\Longrightarrow x = y^2 - 1 \in [-1, 0] \textit{ because } y \in [0, 1] \textit{ and then } y^2 \in [0, 1] \end{aligned}$$

hence f is surjective.

2. *Let be the application :*

$$g : \mathbb{R}^* \longrightarrow \mathbb{R}$$
$$x \mapsto \frac{1}{x}$$

Is g surjective?

Method 1 : Note that :

$$\forall x \in \mathbb{R}^*, \;\; g(x) = \frac{1}{x} \neq 0$$

but the $0 \in \mathbb{R}$, this means that the 0has no antecedent in $\mathbb{R}^*$. So g is not surjective.

Method 2 : Try to show that g is surjective.

Let $y \in \mathbb{R}$, check if there is an $x \in \mathbb{R}^*$ such that y is its image by g. Let's assume that this x exists. y is the image of x by g, which means : $y = g(x)$

$$\begin{aligned} y = g(x) &\Longleftrightarrow y = \frac{1}{x} \\ &\Longrightarrow x = \frac{1}{y} \quad \textit{for } y \neq 0 \end{aligned}$$

If $y = 0$ then we cannot find an $x = \frac{1}{y}$, which means that $y = 0$ has no antecedent, hence g is not surjective.

2.2.3 Bijection

If f is both **injective** and **surjective**, we say that f makes a **bijection** from E to F, or that f is a **bijective application**. This is expressed as :

$$\text{Any element } y \in F \underset{\text{f surjective}}{\text{ is the image}} \text{ of } \underset{\text{f injective}}{\text{a unique}} \text{ element } x \in E.$$

$$\forall y \in F, \; \exists! x \in E, \; y = f(x)$$

Since each element $y \in F$ can be associated with a single element $x \in E$, we can then define an application from F to E, called the **inverse** or **reciprocal application** of f and denoted f^{-1}. This application is characterised by :

$$\forall x \in E, \; \forall y \in F, \; y = f(x) \Leftrightarrow x = f^{-1}(y)$$

Remark 5. *The reciprocal application f^{-1}, which only exists if f is bijective, should not be confused with the reciprocal image of a set B, $f^{-1}(B)$, which is a set and always exists.*

(a) Bijective application

(b) Bijective application and its inverse

Example 21.

Let f be the following application :

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}_+^* \\ x &\mapsto f(x) = \mathrm{e}^x \end{aligned}$$

Is f bijective? If so, determine its reciprocal.

Let $y \in \mathbb{R}_+^*$, check if there is a unique $x \in \mathbb{R}$ such that y is its image by f. Assume that x exists. y is the image of x by f, which means that $y = f(x)$.

$$\begin{aligned} y = f(x) &\Longleftrightarrow y = \mathrm{e}^x \\ &\Longleftrightarrow \ln y = \ln \mathrm{e}^x \;\; \textit{since } y > 0 \\ &\Longleftrightarrow x = \ln y \in \mathbb{R} \end{aligned}$$

hence f is bijective.

Since the application f is bijective, we can define its reciprocal application as follows :

$$\begin{aligned} f^{-1} : \mathbb{R}_+^* &\longrightarrow \mathbb{R} \\ y &\mapsto f^{-1}(y) = \ln y. \end{aligned}$$

Remark 6. *Let $f : E \longrightarrow F$ be an application*

1. *f is surjective if and only if $f(E) = F$.*

2. *If f is strictly monotonic on E then it is injective.*

2.2.4 Applications composition

Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be two applications.
We define the**composite application** :

$$\begin{aligned} g \circ f : E &\longrightarrow G \\ x &\mapsto y = (g \circ f)(x) = g(f(x)) \end{aligned}$$

$$\begin{tikzcd} \boldsymbol{E} \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \boldsymbol{F} \arrow[r, "g"] & \boldsymbol{G} \end{tikzcd}$$

Figure 2.12: Applications composition

Example 22. *Let be the following two applications :*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}_+ \\ x &\longrightarrow \mathrm{e}^x \end{aligned} \qquad\qquad \begin{aligned} g : \mathbb{R}_+ &\longrightarrow \mathbb{R}_+ \\ x &\longrightarrow \sqrt{x} \end{aligned}$$

then the composite application $g \circ f$ is defined as follows :

$$\begin{aligned} g \circ f : \mathbb{R} &\longrightarrow \mathbb{R}_+ \\ x &\mapsto (g \circ f)(x) = g(f(x)) = \sqrt{\mathrm{e}^x}. \end{aligned}$$

Properties 2.

1. *For all applications : $f : E \longrightarrow F$, $g : F \longrightarrow G$ and $h : G \longrightarrow H$, we have :*

$$(h \circ g) \circ f = h \circ (g \circ f)$$

2. *The composite of two injective (resp. surjective) applications is an injective (resp. surjective) application.*

3. *The composite of two bijective applications is a bijective application, and we have :*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

4. *For $f : E \longrightarrow F$ bijective, we have :*

$$f^{-1} \circ f = Id_E \quad \text{ and } \quad f \circ f^{-1} = Id_F$$

and $(f^{-1})^{-1} = f$.

Definition 10. *The applications $f : E \longrightarrow F$, and $g : G \longrightarrow H$ are said to be equal if :*

$$E = G, \ F = H, \ \text{ and } \ \forall x \in E = G, \ f(x) = g(x).$$

Definition 11. *Let $f : E \longrightarrow F$ ba an application :*

1. *Let $A \subset E$ and $g : A \longrightarrow F$ an application. If for all $x \in A$, $g(x) = f(x)$, then the application g : is the* ***restriction*** *of f to A and is denoted $f_{|A}$.*

2. *Let G be a set such that $E \subset G$ and $h : G \longrightarrow F$ an application. If for all $x \in G$, $h(x) = f(x)$ then the application h is called: the **extension** of f to G.*

Example 23. *Let be the application :*

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R}$$
$$x \mapsto f(x) = x^2.$$

The application :

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto f(x) = x^2.$$

is the extension of the application f to $\mathbb{R}$,
and the application

$$h : \mathbb{R}_+^* \longrightarrow \mathbb{R}$$
$$x \mapsto f(x) = x^2.$$

is the restriction of the application f to $\mathbb{R}_+^*$.

Definition 12. *A part A of a set E is said to be **stable** by an application $f : E \longrightarrow E$ if and only if :*

$$\forall x \in A,\ f(x) \in A.$$

Properties 3. *Let $f : E \longrightarrow F$ be an application*

1. *For all parts A, B of E, we have :*
   (a) $A \subset B \Longrightarrow f(A) \subset f(B)$
   (b) $f(A \cup B) = f(A) \cup f(B)$
   (c) $f(A \cap B) \subset f(A) \cap f(B)$
2. *For all parts C, D of F, we have :*
   (a) $C \subset D \Longrightarrow f^{-1}(C) \subset f^{-1}(D)$
   (b) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
   (c) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
   (d) $f^{-1}(\complement_F C) = \complement_E f^{-1}(C)$

Proof.

1. *Let's show the property $A \subset B \Longrightarrow f(A) \subset f(B)$*
   Assume that $A \subset B$ and let's show that $f(A) \subset f(B)$. Let $y \in\ f(A)$.

$$\begin{aligned} y \in f(A) &\Longrightarrow \exists x \in A, y = f(x) \text{ from the definition of the direct image } f(A). \\ &\Longrightarrow \exists x \in B, y = f(x) \text{ because } A \subset B \\ &\Longrightarrow y \in f(B) \text{ from the definition of the direct image } f(B). \end{aligned}$$

   so $f(A) \subset f(B)$.

2. *Let us show the property* $f(A \cap B) \subset f(A) \cap f(B)$.

   Let $y \in f(A \cap B)$.

$$\begin{aligned} y \in f(A \cap B) &\Longrightarrow \exists x \in (A \cap B), y = f(x) \\ &\Longrightarrow (\exists x \in A, y = f(x)) \text{ and } (\exists x \in B, y = f(x)) \\ &\Longrightarrow (y \in f(A)) \text{ and } (y \in f(B)) \\ &\Longrightarrow y \in (f(A) \cap f(B)) \end{aligned}$$

   then $f(A \cap B) \subset f(A) \cap f(B)$.

   Example 24. *Let the application :* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by :* $\forall x \in \mathbb{R}, f(x) = x^2$ *and the subsets of* $\mathbb{R}$, $A = [0, 1]$ *and* $B = [-1, 0]$. *We then have :*

   $f(A) = f(B) = [0, 1]$ *hence* $f(A) \cap f(B) = [0, 1]$, *but* $A \cap B = \{0\}$, *hence* $f(A \cap B) = f(\{0\}) = \{0\}$

   so $f(A \cap B) \subset f(A) \cap f(B)$ *only without equality.*

   Let us show that the inverse inclusion occurs if f *is injective.*

   Let $y \in f(A) \cap f(B)$.

$$\begin{aligned} y \in f(A) \cap f(B) &\Longrightarrow y \in f(A) \text{ and } y \in f(B) \\ &\Longrightarrow (\exists x_1 \in A, y = f(x_1)) \text{ and } (\exists x_2 \in B, y = f(x_2)), \text{ if } f \text{ is injective then } x_1 = x_2 \\ &\Longrightarrow x_1 \in B \text{ too and et } x_2 \in A \text{ too.} \\ &\Longrightarrow x_1 = x_2 \in A \cap B \\ &\Longrightarrow y = f(x_1) = f(x_2) \in f(A \cap B) \end{aligned}$$

   so $f(A) \cap f(B) \subset f(A \cap B)$ *if* f *is injective.*
   Conclusion : *If f is injective, then* $f(A \cap B) = f(A) \cap f(B)$.

3. *Let us show the property :* $C \subset D \Longrightarrow f^{-1}(C) \subset f^{-1}(D)$.

   Let $x \in f^{-1}(C)$.

$$\begin{aligned} x \in f^{-1}(C) &\Longrightarrow f(x) \in C \\ &\Longrightarrow f(x) \in D \text{ because } C \subset D \\ &\Longrightarrow x \in f^{-1}(D) \end{aligned}$$

   so $f^{-1}(C) \subset f^{-1}(D)$.