Algebra 1 Course: Chapter 1, **Logic Concepts**

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Chapter 1

Logic Concepts

1.1 Truth table

1.1.1 Statement

Definition 1. *A* statement *is a text (one or more sentences) of which one can say whether it is true or false and not both at the same time (i.e. without ambiguity).*
This statement is generally noted by a letter : P, Q, R, S, T, ...

Example 1.

- *"$4 > 0$" is a sentence that can be said to be true, so it is indeed a statement (true).*
- *"The sun is blue" is a sentence that can be said to be false, so it is indeed a statement (false).*

☐ *"Tomorrow, it will be nice" is not a statement because it cannot be said to be true or false.*

☐ *"I am a liar" is not a statement, because if this sentence is true, then it is false (because of the meaning), and if it is false then it is true (because of the meaning), so there is ambiguity.*

☐ *"This sentence is false" is not a statement, because if this sentence is true, then it is false (because of the meaning), and if it is false then it is true (because of the meaning), so there is ambiguity.*

1.1.2 Truth table

Definition 2. *The following table is called the **truth table** of a statement P :*

P
1
0

This table indicates the truth values*, which are **True** or **False**, that the statement P can take.*
*In this table, the number **1** represents the value **True** and the number **0** the value **False**. We can also put the letters **V** (French writing) or **T** for True and **F** for False.*

1.1.3 Negation of a statement

Definition 3. *Let P be a statement, we call the **negation** of P the statement denoted $\overline{P}$(ou $\neg P$), which is true if P is false and which is false if P is true. The truth table of $\overline{P}$ is as follows :*

P	$\overline{P}$
1	0
0	1

1.1.4 logical connectors

Given two statements P, Q, we are then faced with 4 possible cases: Both statements are true, both statements are false, statement P is true and statements Q is false and the last one, the statement P is false and statements Q is true. The following truth table summarizes all this:

P	Q
1	1
1	0
0	1
0	0

If we have three statements, then we have 8 possible cases.

P	Q	R
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

In general, with n statements we are faced with 2^n possible cases.

To two statements P, Q, we associate a third statement, which is the result of a *logic conncection* between the two statements P, Q. What are, then, the *truth values* of the resulting statement of this *logic connection* of P and Q. Well, this depends on the **logical connector** applied.

I. Connector Conjunction "and"

Given two statements P, Q, we call the **conjunction** of these two statements the resulting statement (P and Q) denoted $P \wedge Q$, **which is true only if the two statements P and Q are true at the same time** and is false in the other cases.

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Example 2. *Consider the following propositions P, Q :*
P: The board is green. (true statement)
Q: The board is circular. (false statement)
Let us apply the conjunction connector to these two statements :
$P \wedge Q$: The board is green ***and*** *The board is circular. Written correctly, this sentence becomes:*
$P \wedge Q$: The board is green and circular. According to the truth table of the conjunction, this statement is therefore false, which can be easily verified with the meaning of the sentence P.

II. Connector Disjunction "or"

Given two statements P, Q, we call the **disjunction** of these two statements the resulting statement (P or Q) denoted $P \vee Q$, **which is false only if the two statements P and Q are false at the**

same time and is true in the other cases.

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Example 3. *Let's consider the same propositions P, Q as before:*
P: The board is green. (true statement)
Q: The board is circular. (false statement)
Let us apply the disjunction connector to these two statements :
$P \vee Q$: The board is green ***or*** *The board is circular. Written correctly, this sentence becomes:*
$P \vee Q$: The board is green or circular. According to the truth board of the disjunction, this statement is therefore true, which can be easily verified with the meaning of the sentence.

III. Connector Implication

Given two statements P, Q, the resulting statement is (P implies Q) denoted $P \Longrightarrow Q$, **which is false only if the first statement P is true and the second one Q is false** and is true in the other cases.

P	Q	$P \Longrightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

Remark 1.

1. *$P \Longrightarrow Q$ is also defined by the statement $\overline{P} \vee Q$.*
2. *Defining the order of the statements is essential when using the implication connector :*

   $P \Longrightarrow Q$ is an implication with P the first statement and Q the second statement
   $Q \Longrightarrow P$ is an implication with Q the first statement and P the second statement.

   This order is not necessary for the other connectors defined here.
3. *If the statement $P \Longrightarrow Q$ is true, then it reads :* **If** P is true **then** Q is true.
4. *If the statement $P \Longrightarrow Q$ is true, then the statement Q is said to be* ***the necessary condition****, On the other hand, the proposition P is said to be the* ***sufficient condition****.*
5. *In exercises, where you have to prove that an implication is true, the first statement is supposed to be true this is the* ***hypothesis*** *and you have to show that the second one (the* ***conclusion****) is true too.*

Example 4. *Consider the following true proposition :*

If 2 is a natural number then 2 is a relative number.

Is there a logical connector in this statement? Yes, the keywords ***If*** *and* ***then*** *allow us to determine an implication whose first statement is :*
2 is a natural number *(true statement),*
and the second is :
2 is a relative number *(true statement)*
According to the truth table of implication, the initial statement is then true.

Example 5. *Let the following true statement be :*

$$P : (X-1)(X-2)+2 \text{ is a polynomial of } X \text{ of degree } 2,$$

and let the following false statement be:

$$Q : (X-1)(X-2)+2 \text{ has two real roots.}$$

$P \implies Q$ is therefore false according to the truth table. (Q is false because $(X-1)(X-2)+2$ has no roots. Take a look at its discriminant which is $\Delta = -7 < 0$ and therefore no real roots for our polynomial)

IV. Connector Equivalence

Given two statements P, Q, the resulting statement is (P *is equavalent to* Q) denoted by $P \Leftrightarrow Q$, **which is true if the two statements P and Q are both true or both false** and is false otherwise.

P	Q	$P \Leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Remark 2.

1. *$P \Leftrightarrow Q$ is also defined by the statement $(P \implies Q) \wedge (Q \implies P)$.*
2. *$P \Leftrightarrow Q$ is also read :*
   - *P is true if and only if Q is also true.*
   - *A necessary and sufficient condition for P to be true is that Q is true*

V. Connector XOR "exclusive or"

The **XOR**, denoted $P \oplus Q$, is an *exclusive disjunction*, that is to say that the case where both propositions are true at the same time is excluded from the cases that give a true value. In other words, the **XOR is true if one of the two statements is true but not both at the same time**.

P	Q	$P \oplus Q$
1	1	0
1	0	1
0	1	1
0	0	0

$P \oplus Q$ is also defined by $(P \wedge \overline{Q}) \vee (\overline{P} \wedge Q)$ and also by $(P \vee Q) \wedge (\overline{P \wedge Q})$

1.1.5 Negations of statements with connectors

I. Negation of conjunction (NAND)

$$\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}$$

This means that the negation of a statement resulting from the conjunction of two other statements is nothing other than the disjunction of the negations of the two statements. The demonstration is done by the use of the truth table:

P	Q	$P \wedge Q$	$\overline{P \wedge Q}$	$\overline{P}$	$\overline{Q}$	$\overline{P} \vee \overline{Q}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1
			①			②

Columns numbered ① and ② are identical (there are the same truth values in each row), so the equivalence is demonstrated.

Example 6. *Let us find the negation of the following statement :*

The integer 12 *is even* ***and*** *is divisible by* 4.

The negation, according to the previous equivalence, is therefore :

The integer 12 *is not even* ***or*** *is not divisible by* 4.

Of course, the first statement is true because the two statements that compose it, by conjunction, are both true
The negation is then false because it is the negation of a true statement. We can also say that it is false because the two statements that compose it, by disjunction, are both false. This can be easily deduced just by reading the two statements.

II. Négation de la disjonction (NOR)

$$\overline{P \vee Q} \Leftrightarrow \overline{P} \wedge \overline{Q}$$

For proof, see the exercises in the Tutorials.

III. Négation de l'implication

$$\overline{P \implies Q} \Leftrightarrow P \wedge \overline{Q}$$

For proof, see the exercises in the Tutorials.

IV. Négation de l'équivalence

$$\overline{P \Leftrightarrow Q} \Leftrightarrow \overline{(P \implies Q) \wedge (Q \implies P)} \ldots$$

Find the equivalent in the tutorial session.

V. Négation de XOR

$$\overline{P \oplus Q} \Leftrightarrow \overline{P} \oplus \overline{Q}$$

For proof, see the exercises in the Tutorials.

1.1.6 Properties

Let P, Q, R be three statements, then we have the following properties :

• **Commutativity**

1. $(P \wedge Q) \Leftrightarrow (Q \wedge P)$.

2. $(P \vee Q) \Leftrightarrow (Q \vee P)$.

3. $(P \oplus Q) \Leftrightarrow (Q \oplus P)$.

• **Associativity**

4. $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$.

5. $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$.

6. $(P \oplus Q) \oplus R \Leftrightarrow P \oplus (Q \oplus R)$.

• **Transitivity**

7. $[(P \implies Q) \wedge (Q \implies R)] \implies (P \implies R)$.

8. $[(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R)] \implies (P \Leftrightarrow R)$.

• **Distributivity**

9. $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$.

10. $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$.

• Useful en cryptography

11. $(P \oplus Q) \oplus Q \Leftrightarrow P$.

• Let the implication $P \implies Q$. We have the following two properties :

12. $Q \implies P$ is called **Reciprocal Implication** which has nothing to do with the implication $P \implies Q$.

13. $\overline{Q} \implies \overline{P}$ is called **Contrapositive Implication** and verifies
$$P \implies Q \Leftrightarrow \overline{Q} \implies \overline{P}.$$
Not to be confused with the negation of implication seen above.

• The double ngation

14. $\overline{\overline{P}} \Leftrightarrow P$

• A *Tautology* is a statement that is always true.

15. $P \vee \overline{P}$ called *The excluded middle* is a tautology.

16. $\overline{P \wedge \overline{P}}$ called *The non-contradiction* is also a tautology.

1.1.7 Statement dependent on variables (Predicates)

A statement may depend on one or more variables. We denote by $P(x)$ the statement that depends on the variable x where $x \in \mathbb{E}$ and $\mathbb{E}$ a given set, for example $\mathbb{N}, \mathbb{R} \ldots$ (see the next chapter for the definition of a set).

Example 7. *For $x \in \mathbb{R}$, let the following predicate $P(x)$ be*

$$x \text{ is an odd integer.}$$

$P(3)$ is then true and $P(\pi)$ is false.

1.2 Quantifiers

1.2.1 Statement with a single quantifier

Let $P(x)$ be a statement depending on a variable $x \in \mathbb{E}$. The following cases may arise :

1. The statement $P(x)$ **is verified for all** $x \in \mathbb{E}$, we then write :

$$\forall x \in \mathbb{E},\ P(x)$$

   we can also say : **Whatever** $x \in \mathbb{E}$ we have $P(x)$ (means that $P(x)$ is true). or **for every**..., or **for any** ...

   Example 8. $\forall x \in \mathbb{R},\ x^2 \geq 0.$

2. The statement $P(x)$ **is verified for at least one** $x \in \mathbb{E}$, we then write :

$$\exists x \in \mathbb{E},\ P(x)$$

   we can also say : **There exists at least one** $x \in \mathbb{E}$ such that $P(x)$ (means that $P(x)$ is true).

   Example 9. $\exists n \in \mathbb{N},\ n \leq \pi.$

3. **For any** $x \in \mathbb{E}$, the statement $P(x)$ **is not verified**, we then write :

$$\forall x \in \mathbb{E},\ \overline{P(x)}.$$

   Example 10. *For any x of $\mathbb{R}$, $e^x \leq 0$ is not true , we then write : $\forall x \in \mathbb{R}, e^x > 0$, because $e^x > 0$ is the negation of $e^x \leq 0$.*

4. The statement $P(x)$ **is verified for a unique** $x \in \mathbb{E}$, we then write:

$$\exists! x \in \mathbb{E},\ P(x).$$

   we read, **there exists a unique** $x \in \mathbb{E}$ such that $P(x)$.

   Example 11. $\exists! x \in \mathbb{R}_+,\ \ln x = 0.$

   These four cases are also statements.
   The symbols $\forall$ and $\exists$ are, respectively, called **universal quantifier** and **existential quantifier**, because they are used to specify the quantity of elements that we are considering.

1.2.2 Statement with multiple quantifiers

The following two statements are equivalent: :

$$\begin{aligned} &\forall x \in \mathbb{E}, \quad \forall y \in \mathbb{F}, \quad P(x,y),\\ &\forall y \in \mathbb{F}, \quad \forall x \in \mathbb{E}, \quad P(x,y), \end{aligned}$$

these two too :

$$\begin{aligned} &\exists x \in \mathbb{E}, \quad \exists y \in \mathbb{F}, \quad P(x,y),\\ &\exists y \in \mathbb{F}, \quad \exists x \in \mathbb{E}, \quad P(x,y), \end{aligned}$$

Example 12.

$$\forall x \in \mathbb{R}_+, \forall y \in \mathbb{R}_-,\ x^2 + y^2 \geq 0 \Longleftrightarrow \forall y \in \mathbb{R}_-, \forall x \in \mathbb{R}_+,\ x^2 + y^2 \geq 0$$

$$\exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}_-,\ (m^2 - 9)(n + 4) > 0 \Longleftrightarrow \exists n \in \mathbb{Z}_-, \exists m \in \mathbb{Z},\ (m^2 - 9)(n + 4) > 0$$

because, in a statement, one can swap between quantifiers of the same type. This is not always possible if the quantifiers are of different types.

Example 13.

1. • *The following statement :*

$$\forall x \in \mathbb{R}, \quad \exists n \in \mathbb{N}, \quad x \leq n$$

*is a true statement, because it means that for **any** real number x ($\forall x \in \mathbb{R}$), **there exists at least** one natural number n ($\exists n \in \mathbb{N}$) such that $x \leq n$, this is obvious :*

Let's take an example. For the real number $x = 3.1$, I can subsequently choose a natural number $n = 4$ so that $x \leq n$ is verified. What must be noted in this example, which is not a demonstration, is the choice of the natural number n which was performed after having fixed the real number x and it means that this choice is not free, but is linked to the fixed real number x; that is to say that I could not have chosen as a natural number $n = 2$, because $x \leq n$ would not be verified

• *On the other hand, the following statement :*

$$\exists n \in \mathbb{N}, \quad \forall x \in \mathbb{R}, \quad x \leq n$$

*is a false statement, because it means that **there exist** a natural number n ($\exists n \in \mathbb{N}$), such that **for every** real number x ($\forall x \in \mathbb{R}$), we have $x \leq n$, in other words, there exists an integer n greater than all the real numbers! This is absolutely absurd.*
Cconclusion : we cannot interchange the quantifiers of this proposition.

2. *The two following statements :*

$$\forall x \in \mathbb{R}_-, \exists n \in \mathbb{N}, x \leq n,$$
$$\exists n \in \mathbb{N}, \forall x \in \mathbb{R}_-, x \leq n,$$

are both true, despite the permutation of the quantifiers, nevertheless, the two propositions do not have the same meaning. In the second, the n that verifies the inequality must be the same for all x, on the other hand in the first statement the n can be chosen differently for each x (it can also be the same for all x in this example).

1.2.3 Negation of statement with quantifiers

The rules for negating statements with quantifiers are as follows :

$$\overline{\forall x \in \mathbb{E}, P(x)} \Leftrightarrow \exists x \in \mathbb{E}, \overline{P(x)},$$
$$\overline{\exists x \in \mathbb{E}, P(x)} \Leftrightarrow \forall x \in \mathbb{E}, \overline{P(x)}.$$

When there are several quantifiers with a statement $P(x)$, we apply the rules above for each quantifier, without permuting them of course, and we negate $P(x)$.

Example 14. *Let us determine the negation of the following statement $P(x)$:*

$$P(x) : \forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x^2 < 10^n$$

Solution.

$$\begin{aligned}\overline{P(x)} &\Leftrightarrow \overline{\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x^2 < 10^n} \\ &\Leftrightarrow \overline{\forall x \in \mathbb{R}},\ \overline{\exists n \in \mathbb{N}},\ \overline{x^2 < 10^n} \\ &\Leftrightarrow \exists x \in \mathbb{R},\ \forall n \in \mathbb{N},\ x^2 \geq 10^n\end{aligned}$$

Note that since the statement $P(x)$ is true, then we immediately deduce that its negation $\overline{P(x)}$ is false.

1.3 Reasoning

We will present some methods, called **Reasoning Methods**, that are used to demonstrate the truth of a given statement. To demonstrate that $P \implies Q$ (in order not to make the notations cumbersome, P and Q can also depend on variables) is true, here are the methods that can be used :

1.3.1 Direct reasoning

This is the first method that comes to mind for the demonstration. We take the hypothesis P which we assume to be true and then with the help of deductions and logical reasoning we try to show that Q is also true.

Example 15. *Show that if $x \geq 2$ alors $x^2 - 3x + 2 \geq 0$.*

Solution. *Let's make a direct reasoning.*
Suppose that $x \geq 2$, then $x - 2 \geq 0$.
We note that 2 is a root of the polynomial $x^2 - 3x + 2$.

$$\begin{aligned} 2 \textit{ is a root of } x^2 - 3x + 2 &\implies x^2 - 3x + 2 = (x-2)(x-\alpha) \\ &\implies x^2 - 3x + 2 = (x-2)(x-1) \\ &(\textit{we find } \alpha = 1 \textit{ by simple identification}) \end{aligned}$$

but $x \geq 2$, then $x - 1 \geq 0$ too. Hence :

$$x^2 - 3x + 2 = (x-2)(x-1) \geq 0.$$

1.3.2 Reasoning by contraposition

In this method we show that the contrapositive implication $\overline{Q} \implies \overline{P}$, which gave its name to this mode of resonance, is true, because we have the property :

$$P \implies Q \Leftrightarrow \overline{Q} \implies \overline{P}$$

Example 16. *Show that : $mn \neq 0 \implies m \neq 0 \wedge n \neq 0$*

Solution. *Let us reason by contraposition.*
Let us show that : $m = 0 \vee n = 0 \implies mn = 0$.
It is clear that if $m = 0$ or $n = 0$ then the product mn is zero, that is $mn = 0$.
We have therefore shown by contraposition that if $mn \neq 0$ then $m \neq 0$ and $n \neq 0$.

1.3.3 Reasoning by absurdity (contradiction)

In this mode of reasoning, to show that $P \implies Q$ is true, we assume that its negation is true ($\overline{P \implies Q}$ true) and that this leads to an absurdity or a contradiction, this will mean that $\overline{P \implies Q}$ is false, hence $P \implies Q$ is true.
Knowing that $\overline{P \implies Q} \Leftrightarrow P \wedge \overline{Q}$, we therefore assume at the beginning of the demonstration that P and $\overline{Q}$ are true.

Example 17. *Let $n, m \in \mathbb{N}^*$. Show that if $n^2 + m^2$ is odd, then n and m have not the same parity.*

Solution. *Let's reason by contradiction.*
So let's assume that $n^2 + m^2$ is odd and that n and m have the same parity, that is to say they are both even or both odd.

- *Let us first assume that n^2+m^2 is odd and that n and m are even.*

$$\begin{aligned} n, m \text{ are even} &\implies \exists k, l \in \mathbb{N}^*, n = 2k, m = 2l \\ &\implies n^2+m^2 = 4k^2+4l^2 \\ &\implies n^2+m^2 = 2(2k^2+2l^2) \end{aligned}$$

that is to say n^2+m^2 is even and this is a contradiction with our hypothesis.

- *Now suppose that n^2+m^2 is odd and that n and m are odd.*

$$\begin{aligned} n, m \text{ are odd} &\implies \exists k, l \in \mathbb{N}^*, n = 2k+1, m = 2l+1 \\ &\implies n^2+m^2 = (2k+1)^2+(2l+1)^2 \\ &\implies n^2+m^2 = 4k^2+4k+1+4l^2+4l+1 \\ &\implies n^2+m^2 = 2(2k^2+2k+2l^2+2l+1) \end{aligned}$$

that is to say n^2+m^2 is even and this is a contradiction with our hypothesis.

Conclusion, if n^2+m^2 is odd, then n and m are not of the same parity.

1.3.4 Reasoning by induction

In a demonstration by induction, we show that a statement $P(n)$ depending on a natural integer n is true for all $n \in \mathbb{N}$. It is done in three steps :

1. **Initialization :** In this step, we show that $P(0)$ is true.
2. **Heredity :** Here, we start by expressing the **Induction Hypothesis** which is formulated as follows : *suppose that for a given $n \in \mathbb{N}$, $P(n)$ is true.*
   We then show the **heredity**, that is to say that $P(n+1)$ is true
3. **Conclusion :** Deduce that the statement $P(n)$ is true for all integers n.

Example 18. *Show that for all $n \in \mathbb{N}, 2^n \geq n+1$.*

Solution. *Let's use a reasoning by induction.*
Let :

$$P(n) : 2^n \geq n+1$$

1. ***Initialization :*** *Let us show that $P(0)$ is true, that is to say that $2^0 \geq 0+1$ is true :*
   $2^0 = 1$ and $0+1=1$, so we get $2^0 \geq 0+1$ because $1 \geq 1$, that is, $P(0)$ is true.
2. ***Heredity :*** *l'**Induction Hypothesis** : Suppose for a fixed $n \in \mathbb{N}$, that $P(n)$ is true, that is to say that $2^n \geq n+1$ is true for a certain $n \in \mathbb{N}$.*
   Now let us show that $P(n+1)$ is also true, that is, that $2^{n+1} \geq n+1+1$ is true.

   By the induction hypothesis, we have $2^n \geq n+1$.

$$\begin{aligned} 2^n \geq n+1 &\implies 2\ 2^n \geq 2n+2, \\ &\implies 2\ 2^n \geq n+2 \ \text{ because } n \geq 0 \implies 2n \geq n, \\ &\implies 2^{n+1} \geq n+1+1, \end{aligned}$$

   that is $P(n+1)$ is true too.
3. ***Conclusion :*** *For all $n \in \mathbb{N}, P(n)$ is true, that is $2^n \geq n+1$.*

1.3.5 Raisonneing by counter example

To show the falsity of a statement with a universal quantifier, that is to say in the form :

$$\forall x \in \mathbb{E}, P(x)$$

we just need to find a **counter example**, that is, find $x \in \mathbb{E}$ such that $P(x)$ is false.

Example 19. *Is the following statement: :*

$$\textit{For any real } x, \textit{ such that } x < 1, \textit{ we have } |x| < 1,$$

true?

Solution. *The given statement is false, it suffices to take $x = -2$, we then have $x = -2 < 1$, but $|x| = |-2| = 2 > 1$.*

Remark 3. *To show that a statement of the type $\forall x \in \mathbb{E}, P(x)$, is true,* ***it is not enough*** *to find one, or even several, $x \in \mathbb{E}$ such that $P(x)$ is true. It is necessary to show that $P(x)$ is true for all x in $\mathbb{E}$. This is not always possible, because often the set $\mathbb{E}$ is infinite, it is then* ***sufficient*** *to show that $P(x)$ is true* ***for any*** *x , that is to say without giving x a particular value.*