University of Tlemcen **Academic year 2023-2024**
Faculty of Sciences **(L1 ING-INF)** **Semester 2**
Department of Informatic **Algebra (First Year)**

Worksheet N°3/ "Vector Spaces"

<u>Exercice 01 :</u> *(Corrected in the course)*Determine whether a subset of a vector space is a subspace?

$$(1)\ A = \left\{(x,y) \in \mathbb{R}^2 / y = x^3\right\}$$

$$\text{For } \alpha = 2 \text{ and } (2,8) \text{ we have } 2 \times (2,8) = (4,16) \notin A.$$

wich implies that A is n't a subspace of $\mathbb{R}^2$.

$$(2)\ B = \left\{(x,y) \in \mathbb{R}^2 /\ x^2 + y^2 = 2\right\}.$$

$$0^2 + 0^2 = 0 \neq 2 \Rightarrow (0,0) \notin B,$$

wich implies that B is n't a subspace of $\mathbb{R}^2$.

$$(3)\ C = \left\{(x,y) \in \mathbb{R}^2 /\ x^2 + y^2 \leq 2\right\}.$$

$$\text{For } \alpha = 2 \text{ and } (1,1) \text{ we have } 2 \times (1,1) = (2,2) \notin A,$$

because $2^2 + 2^2 = 8 > 2$, wich implies that C is n't a subspace of $\mathbb{R}^2$.

$$\begin{aligned}(4)\ D &= \{(x,y,z)\,; 2x + 3y - 5z = 0\} \\ &= \left\{\left(\frac{-3y+5z}{2}, y, z\right); y, z \in \mathbb{R}\right\}\end{aligned}$$

is a subspace of $\mathbb{R}^3$, because:

$$a)\ (0,0,0) = \left(\frac{-3 \times 0 + 5 \times 0}{2}, 0, 0\right) \in D \Rightarrow D \neq \varnothing.$$

$$b)\ \forall u_1 \left(\frac{-3y_1 + 5z_1}{2}, y_1, z_1\right), u_2 \left(\frac{-3y_2 + 5z_2}{2}, y_2, z_2\right) \in D,$$

$$u_1 + u_2 =_1 \left(\frac{-3\,(y_1 + y_2) + 5\,(y_1 + y_2)}{2}, y_1, (z_1 + z_2)\right) \in D.$$

$$c)\ \forall \alpha \in \mathbb{R}, \forall u \left(\frac{-3y + 5z}{2}, y, z\right) \in D,$$

$$\alpha u =_1 \left(\frac{-3\,(\alpha y) + 5\,(\alpha z)}{2}, (\alpha y), (\alpha z)\right) \in D.$$

(5) $E = \{f \in E_1; f(0) = f'(0) = 0\}$.($E_1 = \mathcal{F}(\mathbb{R}, \mathbb{R})$ the set of derivable functions)

is a subspace of E_1, because:

$a)$ If we put the identity element $f_0(x) = 0, \forall x \in \mathbb{R}$ (null function)

$$\Rightarrow f_0(0) = f_0'(0) = 0 \Rightarrow f_0 \in E \Rightarrow E \neq \varnothing.$$

$$b)\ \forall f_1, f_2 \in E,$$
$$\left(f_1+f_2\right)(0) = \underbrace{f_1(0)}_{=0} + \underbrace{f_2(0)}_{=0} = 0,$$

and
$$\left(f_1'+f_2'\right)(0) = \underbrace{f_1'(0)}_{=0} + \underbrace{f_2'(0)}_{=0} = 0,$$

wich implies that $f_1 + f_2 \in E$.

$$c)\ \forall \alpha \in \mathbb{R}, \forall f \in E,$$
$$(\alpha f)(0) = \alpha \underbrace{f(0)}_{=0} = 0,$$

and
$$(\alpha f')(0) = \alpha \underbrace{f'(0)}_{=0} = 0,$$

wich implies that $\alpha f \in E$.

(6) $F = \{P \in \mathbb{R}_n[X] \,/\, P(1) = P'(1)\} .\mathbb{R}_n[X]:$ set polynomials of degree $\leq n$.

is a subspace of $\mathbb{R}_n[X]$, because:

$a)$ If we put the identity element $P_0(x) = 0, \forall x \in \mathbb{R}$ (null polynomial)

$$\Rightarrow P_0(1) = P_0'(1) = \ 0 \Rightarrow P_0 \in F \Rightarrow F \neq \varnothing.$$
$$b)\ \forall P_1, P_2 \in F,$$
$$\left(P_1+P_2\right)(1) = \underbrace{P_1(1)}_{\in F} + \underbrace{P_2(1)}_{\in F} = P_1'(1) + P_2'(1) = \left(P_1+P_2\right)'(1),$$

wich implies that $P_1 + P_2 \in F$.

$$c)\ \forall \alpha \in \mathbb{R}, \forall P \in F,$$
$$(\alpha P)(1) = \alpha \underbrace{P(1)}_{\in F} = \alpha P'(1) = (\alpha P)'(1),$$

wich implies that $\alpha P \in E$.

<u>Exercice 02 :</u>

(1) Are the following vectors linearly independent in E?

$$(1)\, v_1(1,2,-1), v_2(3,-2,2), v_3(-1,4,-1); E = \mathbb{R}^3.$$
$$(2)\, v_1(7,8,-1), v_2(9,-2,12), v_3(-1,4,-1), v_4(-5,5,-1); E = \mathbb{R}^3.$$
$$(3) P_1(X) = 2 - X, P_2(X) = 3 + X + X^2, P_3(X) = X - X^2 + 2X^3; E = \mathbb{R}_3[X].$$

$$\begin{aligned}(4) P_2(X) &= 4, P_2(X) = 2 - X, P_3(X) = 3 + X + X^2,\\ P_4(X) &= X^2 + 2X^3, P_5 = 5X^3; E = \mathbb{R}_3[X].\end{aligned}$$

(2) Write the vector $v = (7, -2, 4)$ as a linear combination of:

$$v_1 = (1, -1, 1), v_2 = (1, 2, -3) \text{ et } v_3 = (2, -5, 4).$$

(3) In $E = \mathbb{R}_2[X]$ the vector space of polynomials of degree less than or equal to 2 and with real coefficients, do the following vectors spans E?

$a)$ $P = X^2 - 5X + 1$ and $Q = X + 7$.
$b)$ $P = X^2 - 6,\ Q = 7X + 1$ and $R = 3X^2 + 8$.
$c)$ $P = 5, Q = 2X - 1$ and $R = X(X - 1)$.

Exercice 03 : Let $\mathbb{R}[X]$ the $\mathbb{R}$-vector space of polynomials with real coefficients and P_3 le the sub-set of $\mathbb{R}[X]$ such as $H_3 = \{P \in \mathbb{R}[X] / \deg P \leq 3\}$.

(1) Show that H_3 is a subspace of $\mathbb{R}[X]$.

(2) Let $Q_0 = 3, Q_1 = 1 - 2X, Q_2 = 3X - X^2$ and $Q_3 = 4X^2 - 5X^3$ be a polynomials.

a) verify that $B_2 = \{Q_0, Q_1, Q_2, Q_3\}$ is a basis of P_3.

b) Determine the coordinates of the polynomial $P = 1 + 6X - 4X^2 - 2X^3$ in B_2.

Exercice 04 : *(Corrected in the course)* Let:

$$\begin{aligned} E_1 &= \left\{(a, b, c) \in \mathbb{R}^3; a = 5c\right\}, \ E_2 = \left\{(a, b, c) \in \mathbb{R}^3; a + 3b - 7c = 0\right\} \\ \text{and } E_3 &= \{(0, 2c, c); c \in \mathbb{R}\}. \end{aligned}$$

(1) Show that E_i with $i = 1, 2, 3$ are a subspace of $\mathbb{R}^3$.

(2) Show that $\mathbb{R}^3 = E_1 + E_2, \mathbb{R}^3 = E_2 + E_3$ and $\mathbb{R}^3 = E_1 + E_3$.

(3) In which case is the sum is a direct sums?

Exercice 05 : *(Corrected in the course)* In $\mathbb{R}^3$ let the two subset:

$$E_1 = \left\{(3a - 5b, 2b - 4a, 3a) \in \mathbb{R}^3 /\ a, b \in \mathbb{R}\right\} \text{ and } E_2 = \left\{(4c, c, 6c) \in \mathbb{R}^3 /\ c \in \mathbb{R}\right\}.$$

(1) Show that E_1 and E_2 are a subspace of $\mathbb{R}^3$.

(2) determine a basis B_1 of E_1 and a basis B_2 of E_2.

(3) deduce $\dim E_1$ and $\dim E_2$.

(4) Show that: $\mathbb{R}^3 = E_1 + E_2$.

(5) deduce if the sum is a direct sum or not.

e 06 : (Additional) In $\mathbb{R}_8[X]$ the vector space of polynomials of degree less than or equal to 8 and with real coefficients,. We put: $E_0 = \{P \in \mathbb{R}_8[X] \ / \ P(0) = 0\}$,
$E_p = \{P \in \mathbb{R}_8[X] \ / \ \forall X \in \mathbb{R},\ P(X) = P(-X)\}$and $E_i = \{P \in \mathbb{R}_8[X] \ / \ P(X) = -P(-X)\}$.

(2) Show that: $\mathbb{R}_8[X] = E_0 + E_p$ and $\mathbb{R}_8[X] = E_i + E_p$.

(3) In which case the sum is a direct sum?