National Higher School of Mathematics

Textbook Analysis 1

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Version : 2022/2023

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english

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CHAPTER

1

REAL NUMBER SYSTEM

1.1 Algebraic and Order axioms

The real number system consists of the real numbers, together with the two operations, addition (denoted by $+$) and multiplication (denoted by $\times$) and the less than relation (denoted by $<$). One also singles out two particular real numbers, zero or 0 and one or 1. If a and b are real numbers, then so are $a+b$ and $a \times b$. We say that the real numbers are closed under addition and multiplication. We usually write

$$ab \text{ for } a \times b.$$

For any two real numbers a and b, the statement $a < b$ is either true or false. We will soon see that one can define subtraction and division in terms of $+$ and $\times$; and $\leq, >$, etc. can be defined from $<$. There are three categories of properties of the real number system: the algebraic properties, the order properties and the completeness property. We will discuss the completeness property in a later section of this chapter. Here we begin with certain basic algebraic and order properties, usually called the algebraic and order axioms, from which we can prove all the other algebraic and order properties of the real numbers. For all real numbers a, b and c :

1. $(a+b)+c = a+(b+c)$ (associative axiom for addition)
2. $a+0 = 0+a = a$ (additive identity axiom)
3. there is a real number, denoted $-a$, such that $a+(-a) = (-a)+a = 0$ (additive inverse axiom)
4. $a+b = b+a$ (commutative axiom for addition)
5. $(a \times b) \times c = a \times (b \times c)$ (associative axiom for multiplication)
6. $a \times 1 = 1 \times a = a$, moreover $0 \neq 1$ (multiplicative identity axiom)
7. if $a \neq 0$ then there is a real number, denoted a^{-1}, such that $a \times a^{-1} = a^{-1} \times a = 1$ (multiplicative inverse axiom)
8. $a \times (b+c) = a \times b + a \times c$ (distributive axiom)
9. $a \times b = b \times a$ (commutative axiom for multiplication)
10. exactly one of the following holds: $a < b$, $a = b$ or $b < a$ (trichotomy axiom)
11. if $a < b$ and $b < c$, then $a < c$ (transitivity axiom)
12. if $a < b$ then $a + c < b + c$ (addition and order axiom)
13. if $a < b$ and $0 < c$, then $a \times c < b \times c$ (multiplication and order axiom)

1.1.1 Remarks

- For equality, denoted by the symbol "=", we mean "the same thing as", or equivalently, "is the same real number as". We take "=" to be a logical notion and do not write axioms for it. [2] Instead, we use any property of "=" which follow from its logical meaning. For example $a = a$; if $a = b$ then $b = a$; if $a = b$ and $b = c$ then $a = c$; if $a = b$ and something is true of a then it's also true of b (since a and b denote the same real number!).
- When we write $a \neq b$, we just mean that a is different real number as b.
- The assertion $0 \neq 1$ in axiom 6 may seem silly. But it doesn't follow from the other axioms, since all the other axioms hold for the set containing just the number 0 .

- Some of the axioms are redundant. For example, from axiom 4 and the property $a + 0 = a$ it follows that $0 + a = a$. Similar comments apply to axiom 3 ; and because of axiom 6 to axioms 8 and 9.

1.1.2 Algebraic Structures

- Axioms $1, 2, 3, 4$ allow $\mathbb{R}$ the structure of an additive Abelian group.
- Axioms $5, 6, 7$ allow $\mathbb{R}$ the structure of a multiplicative group.
- Axioms $1, ..., 8$ allow us to justify that $\mathbb{R}$ is a field.
- Commutativity of the multiplication operation $\times$ makes $\mathbb{R}$ a commutative field.

From axiom 8, we can write

$$a\left(b + (-b)\right) = 0 = ab + a\left(-b\right),$$

which means

$$-\left(ab\right) = a\left(-b\right)$$

same as above we obtain

$$-\left(ab\right) = \left(-a\right)b.$$

1.1.3 Algebraic consequences

Certain not so obvious "rules", such as "the product of minus times minus is plus" and the rule for adding two fractions, follow from the axioms. If we want the properties given by axioms 1-9 to be true for the real numbers (and we do), then there is no choice other than to have $(-a)(-b) = ab$ and $(a/c) + (b/d) = (ad + bc)/cd$ (see the following theorem). We won't emphasise the idea of making deductions from the axioms. Nonetheless, you should have some appreciation of the ideas involved, and thus you should work through a couple of proofs.

Theorem 1.1. *If a, b, c, d are real numbers and $c \neq 0, d \neq 0$ then*

1. *$ac = bc$ implies $a = b$.*

2. $a0 = 0$

3. $-(-a) = a$

4. $(c^{-1})^{-1} = c$

5. $(-1)a = -a$

6. $a(-b) = -(ab) = (-a)b$

7. $(-a) + (-b) = -(a + b)$

8. $(-a)(-b) = ab$

9. $(a/c)(b/d) = (ab)/(cd)$

10. $(a/c) + (b/d) = (ad + bc)/cd$

1.2 $\mathbb{R}$ is a totally ordered field

Order consequences

All the standard properties of inequalities for the real numbers follow from axioms 1-13.
<u>More definitions</u>:
One defines " $>$ ", " $\leq$ " and " $\geq$ " in terms of $<$ as

$$a > b \text{ if } b < a,$$
$$a \leq b \text{ if } (a < b \text{ or } a = b),$$
$$a \geq b \text{ if } (a > b \text{ or } a = b).$$

(Note that the statement $1 \leq 2$, although it isn't one we are likely to make, is indeed true. Why?)
We define $\sqrt{b}$, for $b \geq 0$, to be the number $c \geq 0$ such that $c^2 = b$. Similarly, if n is a natural number, then $\sqrt[n]{b}$ is the number $c \geq 0$ such that $c^n = b$. To prove such a number c always exists requires the "completeness axiom" (see later). To prove the uniqueness of such a number requires the "order axioms". If $0 < a$, we say a is positive and if $a < 0$, we say a is negative.

Some properties of inequalities:

The following are consequences of the axioms which are provided without proofs.

Theorem 1.2. *If a, b and c are real numbers then*

1. $a < b$ *and* $c < 0$ *implies* $ac > bc$
2. $0 < 1$ *and* $-1 < 0$
3. $a > 0$ *implies* $1/a > 0$
4. $0 < a < b$ *implies* $0 < 1/b < 1/a$
5. $|a + b| \leq |a| + |b|$ *(triangle inequality)*
6. $||a| - |b|| \leq |a - b|$ *(a consequence of the triangle inequality)*

1.2.1 Order relation

We defined in $\mathbb{R}$ an order relation $\leqslant$ by $a \leqslant b$ or $b \geqslant a$. We recall the axioms:

$$(A_1) : \forall a \in \mathbb{R} : a \leqslant a \;\; (Reflexive)\,.$$

$$(A_2) : a \leqslant b \;\; and \;\; a \leqslant b \Longleftrightarrow a = b \, (Antisymmetric)\,.$$

$$(A_3) : a \leqslant b \;\; and \;\; b \leqslant c \Longrightarrow a \leqslant c \, (transitive)\,.$$

We can show that all elements of $\mathbb{R}$ are comparable with respect to the order relation $\leqslant$ and as such we imply a totally ordered set (total order relation)

1.2.2 Example

One can show the utility of such order relation

1. Let a be a real number such that $|a| < \varepsilon, \forall \varepsilon > 0$, we have $a = 0$ (elsewhere, if we choose $\varepsilon = \dfrac{|a|}{2}$, contradiction.

2. Let a and b be two real numbers such that $a < b + \varepsilon, \forall \varepsilon > 0$, then $a \leqslant b$ (else, if we take $\varepsilon = \dfrac{b-a}{2}$, we get a contradiction).

1.3 Natural Numbers and Induction

Definition 1.1. *Mathematical induction is a mathematical proof technique requiring essentially that a statement $P(n)$ holds for every natural number $n = 0, 1, 2, 3, ...$; that is, the overall statement is a sequence of infinitely many cases*
$P(0), P(1), P(2), P(3), ...$

A proof by induction consists of two steps: first step, the base, proves the statement for $n = 0$ without assuming any knowledge of other cases. Second step, the induction, proves that if a statement holds for any given case $n = k$, then it must also hold for the next case $n = k + 1$. These two steps establish that the statement holds for every natural number n. The base case doesn't necessarily begin with $n = 0$, but often with $n = 1$ and possibly with any fixed natural number $n = N$, establishing the truth of statement for all natural numbers $n \geqslant N$.

Example 1.1. *Prove the following statements:*

a- $1 + 3 + 5 + ... + (2n - 1) = n^2$

b- $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

c- $1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 1.2. *Prove that for all numbers $x$different from 1:* $(1 + x)(1 + x^2)(1 + x^4)...(1 + x^2n) = \frac{1-x^2n+1}{1-x}$

1.4 Absolute value

The absolute value (or modulus) for any real number a, denoted by $|a|$, is defined as

$$a \longrightarrow |a| \in \mathbb{R}^+ = \begin{cases} a, \text{ si } a \geqslant 0 \\ -a, \text{ si } a < 0 \end{cases},$$

$$|a| = \sup(a, -a).$$

The absolute value has the following fundamental properties:

(1) $|a| = 0 \Longleftrightarrow a = 0.$

(2) $|a| = |-a|.$

(3) $|a.b| = |a| \, . \, |b|$.

(4) si $a > 0 : |x| \leqslant a \Longleftrightarrow -a \leqslant x \leqslant a.$

Indeed, if $x \geqslant 0$ we have $x \geqslant -a$ and

$$|x| \leqslant a \Longleftrightarrow x \leqslant a.$$

if $x \leqslant 0$ we have $x \leqslant a$ and

$$|x| \leqslant a \Longleftrightarrow -x \leqslant a \text{ soit } x \geqslant -a.$$

(5) $|a+b| \leqslant |a| + |b|$.

Indeed, if a and b have the same sign, then the inequality is true. If $a \leqslant 0 \leqslant b$, then $a+b \leqslant b \leqslant b+|a|$ (because $a \leqslant 0 \leqslant |a|$), $|a| = -a$. i.e. $a+b \leqslant |a|+|b|$. also $b \geqslant 0 \geqslant -|b|$, $a+b \geqslant a \geqslant a-|b|$. which means that $a+b \geqslant -|a|-|b|$, and using (4), we get

$$|a+b| \leqslant |a| + |b|.$$

We can show, by induction that

$$|a_1 + a_2 + ... + a_n| \leqslant |a_1| + |a_2| + ... + |a_n|.$$

(6) $||a| - |b|| \leqslant |a-b|$.

$|a| = |b + (a-b)|$ and $|b| = |a + (b-a)|$, then, by (5), we obtain

$$|a| \leqslant |b| + |a-b|$$

$$|b| \leqslant |a| + |b-a|$$

$$-|a-b| \leqslant |a| - |b| \leqslant |a-b|,$$

by (4), we claim

$$||a| - |b|| \leqslant |a-b|.$$

1.5 Intervals

In mathematics, a (real) interval is a set that contains all real numbers lying between any two numbers. more precisely

Definition 1.2. *Let a and b bet two real numbers such that $b > a$. The set $\{x : a < x < b\}$ is called open interval and it noted by $]a, b[$. The set $[a, b] = \{x : a \leqslant x \leqslant b\}$ is called closed interval (compact interval). The sets $[a, b[= \{x : a \leqslant x < b\},]a, b] = \{x : a < x \leqslant b\}$, are called (respectively right and left) half-open intervals.*

For all intervals, the points a and b are called endpoints. If $a = b$, we put by definition $[a, a] = \{a\}$ (*degenerate closed interval*) and $]a, a[= \phi$. The length of the interval (closed, open, or half-open) is given by the real number $b - a$.

Examples: 1- The set $\{x : x \leqslant a\}$ is a left unbounded closed interval, noted $]-\infty, a]$. 2- The set $\{x : x < a\}$ is a left unbounded open interval, noted $]-\infty, a[$. 3- The set $\{x : x \geqslant a\}$ is a right unbounded closed interval, noted $[a, +\infty[$. 4- The set $\{x : x > a\}$ is a right unbounded open interval, noted $]a, +\infty[$. 5- The set $\mathbb{R}$ is also noted $]-\infty, +\infty[$. $-\infty$ and $+\infty$ represent infinity numbers.

1.6 Archimedean property

This property does not follow from the algebraic and order axioms alone. It states, informally, that there are no real numbers beyond all the natural numbers.

1.6.1 Archimedean axiom

For every real number a there is a natural number n such that $a < n$. Equivalently, the set $\mathbb{N}$ is not bounded above. We say that $\mathbb{R}$ is Archimidean.

Corollary 1.1. *For all real numbers a and b such that $a > 0$, there exists $n \in \mathbb{N}$ such that $na > b$.*

Proof. Just replace in the axiom a by $\frac{b}{a}$. $\square$

Remark 1.1. *This property seems trivial, actually it's very important, and it allows us to define the famous definition of the integer part of a real number.*

Proposition 1.1. *Let $x \in \mathbb{R}$, there exists a unique integer (called integer part) denoted by $E(x)$, or $[x]$, such that:*

$$E(x) \leqslant x \leqslant E(x) + 1.$$

Example 1.3. $E(e) = 2$, $E(-e) = -3$, $E(1,45632) = 1$.

1.7 The completeness Axiom

In this section we give the completeness Axiom for $\mathbb{R}$. This Axiom will guarantee that $\mathbb{R}$ has no "gaps".

Definition 1.3. *Let S be a nonempty subset of $\mathbb{R}$.*

a. *If S contains the largest element s_0 [that is, s_0 belongs to S and $s \leqslant s_0$ for all $s \in S$, then we call s_0 the maximum of S and write $s_0 = \max S$.*

b. *If S contains the smallest element then we call the smallest element the minimum of S and write $\min S$.*

Example 1.4. *The set $\mathbb{R}$ has no maximum and minimum.*

Example 1.5. *The interval $]a, b[$ has no maximum nor minimum.*

Example 1.6. $\mathbb{N} = \{0, 1, ...\}$, $\min \mathbb{N} = 0$, $\mathbb{N}$ *has no maximum.*

Example 1.7. $\min\, [a, b[= a$, $\max\, [a, b[$ *does not exist.*

Example 1.8. *Let A be a subset of $\subset \mathbb{R}$, defined by $A = \{x \in \mathbb{R} : 0 \leqslant \ln x < 1\}$ Check the min and the max for A. Indeed, one has $0 \leqslant \ln x < 1$, which is equivalent to*

$$1 \leqslant x < e.$$

So $A = [1, e[$, we get, $\min A = 1$ and $\max A$ does not exist.

Definition 1.4. *Let S be a nonempty subset of $\mathbb{R}$.*

a. *If a real number M satisfies $s \leqslant M$ for all $s \in S$, then M is called an upper bound of S and the set S is said to be bounded above.*

b. *If a real number m satisfies $m \leqslant s$ for all $s \in S$, then m is called a lower bound of S and the set S is said to be bounded below.*

c. *The set S is said to be bounded if it is bounded above and bellow. Thus S is bounded if there exist real numbers m and M such that $S \in [m, M]$.*

Example 1.9. *The set $A = \{\sin n, \; n \in \mathbb{N}\}$ is bounded, because*

$$\forall n \in \mathbb{N} : -1 \leqslant \sin n \leqslant 1.$$

Example 1.10. *The set $A = [1, 3]$ is bounded, we can easily check that*

$$\forall x \in A : 1 \leqslant x \leqslant 3.$$

Example 1.11. *Let $A = \left\{\frac{1}{n+1}, \; n \geqslant 1\right\}$. A given real $M \geqslant \frac{1}{2}$ is an upper bound for A, and a given real $m \leqslant 0$ is a lower bound for A.*

Definition 1.5. *Let S be a nonempty subset of $\mathbb{R}$*

a. *If S is bounded above and S has a least upper bound, then we will call it the supremum of S and denote it by $\sup S$.*

b. *If S is bounded below and S has a greatest lower bound, then we will call it the infimum of S and denote it by $\inf S$.*

Example 1.12. *Let* $A =]1, 2[$*, we have*

$$\forall x \in A : 1 \leqslant x \leqslant 2,$$

hence $\inf A = 1$, $\sup A = 2$.

Example 1.13. *For the set* $A = \left\{ \frac{n-1}{n},\ n \geqslant 1 \right\}$*, one has* $\inf A = 0$, $\sup A = 1$ *because*

$$\forall n \geqslant 1 : 0 \leqslant \frac{n-1}{n} \leqslant 1.$$

Example 1.14. *Let* $A = \left\{ \frac{1}{n}, n = 1, 2, 3, 4 \right\}$*, we have* $\inf A = \min A = \frac{1}{4}$ *and* $\max A = \sup A = 1$.

Theorem 1.3. *Every nonempty subset* S *of* $\mathbb{R}$ *that is bounded above has a least upper bound. In other words,* $\sup S$ *exists and is a real number.*

Corollary 1.2. *Every nonempty subset* S *of* $\mathbb{R}$ *that is bounded below has a greatest lower bound* $\inf S$*.*

Exercise 1.1. *Let* A *and* B *be nonempty subsets of real numbers such that* $A \subset B$*, prove that*

1. *If* B *is upper bounded, the* $\sup B$ *exists and* $\sup A \leq \sup B$*.*

2. *If* B *is lower bounded, the* $\inf B$ *exists and* $\inf B \leq \inf A$*.*

Exercise 1.2. *Let* A *and* B *be nonempty subsets of real numbers, show that*

1. $\sup(A \cup B) = \max \{\sup A, \sup B\}$.

2. $\inf(A \cup B) = \min \{\inf A, \inf B\}$.

3. *If* $A \cap B \neq= \emptyset$*, show that* $\sup(A \cap B) \leq \min \{\sup A, \sup B\}$ *and* $\inf(A \cap B) \geq \max \{\inf A, \inf B\}$*.*

1.8 Characterization of supremum and infimum

Theorem 1.4. *Let X be a subset of $\mathbb{R}$. The real M is the supremum for X, if and only if the following hold :*

a. $\forall x \in X$, *x satisfies* $x \leqslant M$.

b. $\forall \varepsilon > 0, \exists\ x_\varepsilon \in X$, *satisfying* $M - \varepsilon < x_\varepsilon \leq M$.

Remark 1.2. *(a) means that M is an upper bound for X. (b) indicates that $M - \varepsilon$ is not an upper bound for all $\varepsilon > 0$.*

Example 1.15. *Let be* $A = \left\{1 - \dfrac{1}{n}, n \geqslant 1\right\}$. *Prove that* $\sup A = 1$. *Using the upper bound characterization, one can see that 1)* $\forall x = 1 - \dfrac{1}{n} \in A : x = 1 - \dfrac{1}{n} \leqslant 1$. *2) We claim that :* $\forall \varepsilon > 0,\ \exists x_\varepsilon \in A : x_\varepsilon > 1 - \varepsilon$

$$\begin{aligned} x_\varepsilon &> 1 - \varepsilon \Leftrightarrow 1 - \frac{1}{n} > 1 - \varepsilon \\ &\Leftrightarrow \frac{1}{n} < \varepsilon \\ &\Leftrightarrow n > \frac{1}{\varepsilon}. \end{aligned}$$

So, let $\varepsilon > 0$ and $n \in N$ such that $n > \frac{1}{\varepsilon}$ (n exists because $\mathbb{R}$ is Archimedean set, then

$$x_\varepsilon = 1 - \frac{1}{n} > 1 - \varepsilon.$$

Thus $\forall \varepsilon > 0,\ \exists x_\varepsilon \in X : x_\varepsilon > 1 - \varepsilon$ *which means* $\sup A = 1$.

Theorem 1.5. *Let X be a subset of $\mathbb{R}$. The real m is the infimum for X, if and only if the following hold :*

a. $\forall x \in X$, *x satisfies* $x \geqslant m$.

b. $\forall \varepsilon > 0, \exists\ x_\varepsilon \in X$, *satisfying* $x_\varepsilon < m + \varepsilon$.

Example 1.16. *Let be* $A=\left\{1+\dfrac{1}{n}, n\geqslant 1\right\}$. *Prove that* $\inf A=1$.

We have 1) $\forall x\in A: x=1+\dfrac{1}{n}\geqslant 1$. *2) We prove that* $\forall\varepsilon>0$, $\exists x_\varepsilon\in A: m\leq x_\varepsilon<1+\varepsilon$.*, indeed*

$$\begin{aligned} x_\varepsilon &< 1+\varepsilon \Leftrightarrow 1+\frac{1}{n}<1+\varepsilon \\ &\Leftrightarrow \frac{1}{n}<\varepsilon \\ &\Leftrightarrow n>\frac{1}{\varepsilon}. \end{aligned}$$

So, let be $\varepsilon>0$ and $n\in N$ such that $n>\dfrac{1}{\varepsilon}$. Then $x_\varepsilon<1+\varepsilon$. Thus $\inf A=1$.

Theorem 1.6. *Let* A, B *be two nonempty subsets of* $\mathbb{R}$. *Define*

$$A+B:=\{x+y : x\in A \ \ \textit{and} \ \ y\in B\},$$

and

$$A-B:=\{x-y : x\in A \ \ \textit{and} \ \ y\in B\}.$$

we have

$$\sup(A+B)=\sup A+\sup B \quad \textit{and} \quad \sup(A-B)=\sup A-\inf B.$$

Establish similar formulas for $\inf(A+B)$ *and* $\inf(A-B)$.

1.9 Extended real number line

Definition 1.6. *The extended real number line is obtained from the real number line* $\mathbb{R}$ *by adding two infinity elements* $+\infty$ *and* $-\infty$ *endowed by the totally order relation extended from that of* $\mathbb{R}$ *to* $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$, *where* $\overline{\mathbb{R}}$ *denotes the extended real number line.*

Operations on $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$ are defined by

$$\begin{aligned} x+(+\infty) &= +\infty+x=+\infty, \forall x\in\mathbb{R}, \\ x+(-\infty) &= -\infty+x=-\infty, \forall x\in\mathbb{R}, \end{aligned}$$

$$x\,(\pm\infty) = (\pm\infty)\,x = \begin{cases} \pm\infty \text{ si } x > 0 \\ \mp\infty \text{ si } x < 0 \end{cases},$$

$$\begin{aligned} (+\infty) + (+\infty) &= +\infty, \\ (-\infty) + (-\infty) &= -\infty, \\ (\pm\infty)\,(\pm\infty) &= +\infty, \\ (\pm\infty)\,(\mp\infty) &= -\infty. \end{aligned}$$

As the sum $(+\infty) + (-\infty)$ and the product $0\,(\pm\infty) = +\infty$ are not well defined, so $\overline{\mathbb{R}}$ does not have any algebraic structures.

1.10 Topology of the line $\mathbb{R}$

1.10.1 Open sets, closed sets, neighbourhood

Definition 1.7. *A subset A of $\mathbb{R}$ is said to be open if it is empty or if for every $x \in A$ there exists an open interval containing x and contained in A.*

In other words an open set in $\mathbb{R}$ is a set which is the union of open intervals. The following assertions are an almost immediate consequence of this definition.

$\mathbf{O}_1$ Every union (finite or infinite) of open sets is open;

$\mathbf{O}_2$ Every finite intersection of open sets is open;

$\mathbf{O}_3$ The line $\mathbb{R}$ and the empty set $\emptyset$ are open sets.

Property O_1 results from the fact that every union of sets, each of which is a union of open intervals , is itself a union of open intervals. To prove the property O_2, it is sufficient to prove it for the intersection of two open sets A, B : By hypothesis

$$A = \cup_i A_i, B = \cup_j B_j$$

where A_i and B_j are open intervals. Therefore

$$A \cap B = (\cup_i A_i) \cap (\cup_j B_j) = \cup_{i,j}(A_i \cap B_j).$$

Since each of the sets $A_i \cap B_j$ is either empty or an open interval, $A \cap B$ is open. Finally, property O_3 is obvious.

Example 1.17. *Every open interval is an open set.*

Example 1.18. *The union of the open interval* $]n, n+1[$ *where* $n \in \mathbb{Z}$, *is an open set.*

The intersection of infinite number of open sets is not always open. For example $\cap_{n\in\mathbb{N}^*}(\left]\frac{-1}{n}, \frac{1}{n}\right[) = \{0\}$.

Definition 1.8. *A subset* A *of* $\mathbb{R}$ *is said to be closed when its complement* $C_{\mathbb{R}}^{A}$ *is open.*

Each of the properties O_1, O_2, O_3 at one implies a dual property for closed sets.

Example 1.19. *Every closed interval* $[a, b]$ *(where* $a \leq b$*) is a closed set. Indeed, the complement of* $[a, b]$ *is the union of the two open intervals* $]-\infty, a[$ *and* $]b, +\infty[$, *and is therefore an open set.*

It should be observed that a set can be neither open nor closed.

Neighbourhood (or neighborhood): Let $x \in \mathbb{R}$ and $\varepsilon > 0$. A neighbourhood of x is a subset of $\mathbb{R}$ which contains an open interval $V_{(x,\varepsilon)} =]x-\varepsilon, x+\varepsilon[$, containing x.

Example 1.20. *The interval* $]-\varepsilon, \varepsilon[$ $(\varepsilon > 0)$ *is a neighbourhood of* 0. *The interval* $\left]-\frac{1}{n}, \frac{1}{n}\right[$ $(n > 0)$ *is a neighbourhood of* 0.

Properties

1) The intersection of finite neighbourhoods of a point x is also its neighbourhood

2) If x and y are two distinct real numbers of $\mathbb{R}$, there exist two neighbourhoods V of x and W of y such that $V \cap W = \emptyset$. ($\mathbb{R}$ is a separated space (or Hausdorff).

Proposition 1.2. *A subset* S *is open if and only if* S *a neighbourhood of all the points of* S.

Example 1.21. *If* $a, b \in \mathbb{R}$ $(a < b)$, *the intervals* $]a, b[$, $]-\infty, a[$, $]a, +\infty[$ *are neighbourhoods for all their points.*

Remark 1.3. $\mathbb{R}$ *is both open and closed subset, elsewhere* $[a, b[$ $(a < b)$ *is neither open nor closed.*

The meaning which we have just given to the word "neighbourhood" appears different from the one defined in ordinary usage, since for us a point x of $\mathbb{R}$ has many neighbourhoods, and one of them is the space $\mathbb{R}$ itself.

Accumulation points of a set

Definition 1.9. *If A is a subset of $\mathbb{R}$, a point x of $\mathbb{R}$ is called an accumulation point of A if, in every neighbourhood of x, there exists at least one point of A different from x. In other words, if A is a subset of $\mathbb{R}$, a point x of $\mathbb{R}$ is called an accumulation point of A if, $\forall \varepsilon > 0$, $A \cap \left]x-\varepsilon, x+\varepsilon\right[/ \{x\} \neq \emptyset$.*

The set of accumulation points is denoted by A' (it can be empty).

Example 1.22. $A = [1, 2]$, *then* $A' = [1, 2]$.

Example 1.23. $A = \left]0, 1\right[\cup \{2\}$, *so* $A' = [0, 1]$.

Example 1.24. $A = \{1, 2, 3, 4\}$, *therefore* $A' = \phi$.

Example 1.25. $A = \left\{\frac{1}{n}, n \geqslant 1\right\}$, *thus* $A' = \{0\}$.

Remark 1.4. *An accumulation point of a set does not necessarily belong to the set. For example, the point 0 is an accumulation point of the set of point $A = \left\{\frac{1}{n}, n \geqslant 1\right\}$, but does not belong to this set. Again 0 and 1 are accumulation points of $\left]0, 1\right[$ without belonging to this interval.*

Proposition 1.3. *Every closed set contains its accumulation points. Conversely, every set which contains its accumulation points is closed.*

Proof. Let A be a closed set; if $x \in C_{\mathbb{R}}^{A}$, then the open set $C_{\mathbb{R}}^{A}$ is a neighbourhood of x and does not contain any point of A nor does it contain any point of A. Thus x cannot be an accumulation point of A. Conversely, if A is such that no point of $C_{\mathbb{R}}^{A}$ are an accumulation point of A, then there exists for each $x \in C_{\mathbb{R}}^{A}$ a neighbourhood of x not containing any point of A, and therefore contained in $C_{\mathbb{R}}^{A}$; the set $C_{\mathbb{R}}^{A}$ is thus a neighbourhood of each of its points, i.e., it is open; in other words, A is closed. $\square$

Isolated points

Definition 1.10. *An isolated point of a set A is a point x of A which is not an accumulation point of A. In other words, it is a point x of A which has a neighbourhood V such that $A \cap V = x$. ($\exists \varepsilon > 0,\ A \cap \,]x - \varepsilon, x + \varepsilon[\, = \{x\}$.*

Example 1.26. *Let $A = [0, 1] \cup \mathbb{N}$, then the isolated points of A are $\{2, 3, ..., n, ...\}$.*

CHAPTER

2

COMPLEX NUMBERS

2.1 Algebraic properties

Let (x, y) and (x', y') be two elements of $\mathbb{R}^2$. We define two operations on $\mathbb{R}^2$, by setting $(x, y) \times (x', y') = (xx' - yy', xy' + yx')$ et $(x, y) + (x', y') = (x + x', y' + y')$. This two composition operations define a new field, which is the complex commutative field denoted by $\mathbb{C}$. The additive neutral element is given by $0 = (0, 0)$ and the multiplicative neutral element is $(1, 0)$, the multiplicative inverse of $(x, y) \neq (0, 0)$ is $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. By identifying $(x, 0) \in \mathbb{R}^2$ with $x \in \mathbb{R}$, and by setting $i = (0, 1)$.

$$\mathbb{C} = \left\{ z/\ z = x + iy \text{ with } x, y \in \mathbb{R} \text{ and } i^2 = -1 \right\}.$$

So we do calculus with complex numbers as what we do with real numbers taking into account that $i^2 = -1$.

Example 2.1. *For all $n \in \mathbb{N}$, we have*

$$1 + i + i^2 + ... + i^n = \frac{1 - i^{n+1}}{1 - i} = \begin{cases} 1 & \text{if } n = 4k \\ 1 + i & \text{if } n = 4k + 1 \\ i & \text{if } n = 4k + 2 \\ 0 & \text{if } n = 4k + 3 \end{cases}$$

Definition 2.1. *A complex number is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit. The notations $a + ib$ and $a + bi$ are used interchangeably. The real number a in $z = a + ib$ is called the real part of $z = a + ib$; the real number b is called the imaginary part of $z = a + ib$. The real and imaginary parts of a complex number $z = a + ib$ are abbreviated $Re(z)$, and $Im(z)$, respectively.*

Definition 2.2. *Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$.*

Conjugate and modulus of a complex number

Definition 2.3. *Let $z = x + iy$ be a complex number, we define the conjugate of $\overline{z} = x + iy$ by $\overline{z} = x - iy$. The positive number $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ denotes its modulus.*

The modulus $|z|$ of a complex number z is also called the absolute value of z. We shall use both words modulus and absolute value throughout this text.

Example 2.2. *If $z = 2 - 3i$, then we find the modulus of the number to be $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. If $z = -9i$, then $|z| = |-9i| = \sqrt{(-9)^2} = 9$.*

The following properties hold

1) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$,

2) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$,

3) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

4) $|z| \geqslant 0$ et $|z| = 0 \Leftrightarrow z = 0$,

5) $|z_1 z_2| = |z_1| |z_2|$

6) $||z_1| - |z_2|| \leqslant |z_1 - z_2| \leqslant |z_1| + |z_2|$

7) $|Rez| \leqslant |z|$, $|Imz| \leqslant |z|$.

8) $\frac{1}{z} = \frac{\overline{z}}{|z|}$.

2.2 Complex plane

A complex number $z = x + iy$ is uniquely determined by an ordered pair of real numbers (x, y). The first and second entries of the ordered pairs correspond, in turn, with the real and imaginary parts of the complex number. For example, the ordered pair $(2, -3)$ corresponds to the complex number $z = 2 - 3i$. Conversely, $z = 2 - 3i$ determines the ordered pair $(2, -3)$. The numbers $7, i$, and $-5i$ are equivalent to $(7, 0), (0, 1), (0, -5)$, respectively. In this manner we are able to associate a complex number $z = x + iy$ with a point (x, y) in a coordinate plane.

Complex plane

Because of the correspondence between a complex number $z = x + iy$ and one and only one point (x, y) in a coordinate plane, we shall use the terms complex number and point interchangeably. The coordinate plane is called the complex plane or simply the z -plane. The horizontal or $x - axis$ is called the real axis because each point on that axis represents a real number. The vertical or $y - axis$ is called the imaginary axis because a point on that axis represents a pure imaginary number.

Vector

A complex number $z = x + iy$ can also be viewed as a two dimensional position vector, that is, a vector whose initial point is the origin and whose terminal point is the point (x, y). This vector interpretation prompts us to define the length of the vector z as the distance $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ from the origin to the point (x, y) . This length is the modulus.

2.3 Polar Form of Complex Numbers

Recall that a point P in the plane whose rectangular coordinates are (x, y) can also be described in terms of polar coordinates. The polar coordinate system, invented by Isaac Newton, consists of point O called the pole and the horizontal half-line emanating from the pole called the polar axis. If r is a directed distance from the pole to P and θ is an angle of inclination (in radians) measured from the polar axis to the line OP, then the point can be described by the ordered pair (r, θ), called the polar coordinates of P.

Polar form

Suppose, that a polar coordinate system is superimposed on the complex plane with the polar axis coinciding with the positive x-axis and the pole O at the origin. Then x, y, r and θ are related by $x = rcos\theta, y = rsin\theta$. These equations enable us to express a nonzero complex number $z = x + iy$ as $z = (rcos\theta) + i(rsin\theta)$. We say that $z = r(cos\theta + isin\theta)$ is the polar form or polar representation of the complex number z. Again, the coordinate r can be interpreted as the distance from the origin to the point (x, y) . In other words, we shall adopt the convention that r is never negative so that we can take r to be the modulus of z, that is, $r = |z|$. The angle θ of inclination of the vector z, which will always be measured in radians from the positive real axis, is positive when measured counterclockwise and negative when measured clockwise. The angle θ is called an argument of z and is denoted by $\theta = arg(z)$. An argument θ of a complex number must satisfy the equations $cos\theta = \frac{x}{r}$ and $sin\theta = \frac{y}{r}$. An argument of a complex number z is not unique since $cos\theta$ and sin $sin\theta$ are $2\pi - periodic$. In practice we use $tan\theta = \frac{y}{x}$ to find θ. However, because $tan\theta$ is $\pi - periodic$, some care must be exercised in using the last equation. The following example illustrates how this is done.

Example 2.3. *Express $-\sqrt{3} - i$ in polar form.*
Solution
With $x = -\sqrt{3}$ and $y = -1$ we obtain $r = |z| = 2$. Now $\frac{y}{x} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$, and so a $\left(tan\frac{1}{\sqrt{3}}\right)^{-1} = \pi/6$, which is an angle whose terminal side is in the first quadrant. But since the point $(-\sqrt{3}, -1)$ lies in the third quadrant, we take the solution of $tan\frac{1}{\sqrt{3}}$ to be $\theta = arg(z) = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$. It follows that a polar form of the number is $z = 2(cos\frac{7\pi}{6} + isin\frac{7\pi}{6})$.

Principal Argument

The symbol $arg(z)$ actually represents a set of values, but the argument θ of a complex number that lies in the interval $-\pi < \theta < \pi$ is called the principal value of $arg(z)$ or the principal argument of z. The principal argument of z is unique and is represented by the symbol $Arg(z)$, that is, $-\pi < Arg(z) < \pi$. For example, if $z = i$, we have some values of $arg(i)$ as $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{-3\pi}{2}$, and so on, but $Arg(i) = \frac{\pi}{2}$. Similarly, we can verify that the principal argument of $-\sqrt{3} - i$ is $Arg(z) = \frac{\pi}{6} - \pi = \frac{-5\pi}{6}$. Using $Arg(z)$ we can express the complex number $-\sqrt{3} - i$ in the alternative polar form $z = 2\left(cos\frac{-5\pi}{6} + isin\frac{-5\pi}{6}\right)$.

Moivre's formula

If $z \neq 0$ we have $z = r(\cos\theta + i\sin\theta)$ where θ is the principal argument. By definition, $-\pi \leq Arg(z) \leq \pi$. The polar form of a complex number is especially convenient when multiplying or dividing two complex numbers. We can verify that

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right).$$

Thus $|z_1 z_2| = |z_1| |z_2|$ and $\arg(z_1 z_2) = (\arg(z_1) + \arg(z_2))\, mod 2\pi$, and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right).$$

continuing in this manner, we obtain a formula for the-nth power of z.

$$\left(r\cos\theta + ir\sin\theta\right)^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right).$$

Euler's formula

Let $z \neq 0$, and $z = r\left(\cos\varphi + i\sin\varphi\right)$, if $r = 1$, then $z = \cos\varphi + i\sin\varphi$. Put

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which is called Euler's formula.

Example 2.4. $e^{2\pi i} = 1$, $e^{\pi i} = -1$, $e^{-\frac{\pi}{2}i} = -i$, $e^{\frac{\pi}{2}i} = i$.
Replacing θ by $(-\theta)$ in $e^{i\theta} = \cos\theta + i\sin\theta$, we obtain

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

From the last two formulas above, we deduce Euler's formulas:

$$\cos\varphi = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}.$$

Properties

1) $e^{i\theta}.e^{-i\theta'} = e^{i(\theta+\theta')}$
2) $\dfrac{e^{i\theta}}{e^{i\theta'}} = e^{i(\theta-\theta')}$
3) $\left(e^{i\theta}\right)^n = e^{in\theta}$, $n \in \mathbb{N}$.

Definition 2.4. *Let $z \neq 0$. Then $z = r\left(\cos\theta + i\sin\theta\right) = re^{i\theta}$ is called the exponential form of z.*

The nth root of a complex number

All nonzero complex number $z = re^{i\theta}$ admits n roots $n - iémes$ w_k where, $w_k = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}}$ with $k \in \{0, 1, ..., n-1\}$, $n > 1$.

Example 2.5. *Solve the following equation $z^4 = 1 + i$.*

The solutions are the fourth roots of $1 + i$, so we have $w_k = \sqrt[4]{2}e^{\frac{i}{4}\left(\frac{\pi}{4}+2\pi k\right)}$, $k = 0, 1, 2, 3$.

CHAPTER

3

SEQUENCES OF REAL NUMBERS

3.0.1 General definitions

Definition 3.1. *A real sequence or a sequence of real numbers is defined as a function from* $\mathbb{N}$*, the set of natural number to* $\mathbb{R}$*, the set of real numbers. In other words* $\mathbb{N} : n \in \mathbb{N} \longmapsto u_n \in \mathbb{R}$, $u_n = f(n)$*. It is customary to denote a sequence by a letter such as u and to denote its value at* n *as* $(u_n)_{n\in\mathbb{N}}$ *or more clearly* (u_n), $u_n = (u_0, u_1, ...)$.

The real numbers $u_0, u_1, ...$ are called elements or terms of the sequence u_n. The number u_n is called the nth term of rank n of the sequence or general term.

Examples:
$$\left(\frac{1}{n+1}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, ...\right),$$

$$\left(2^{(-1)^n}\right)_{n\in\mathbb{N}\cup 0} = \left(2, \frac{1}{2}, 2, \frac{1}{2}, ...\right)$$

Definition 3.2. *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a real sequence, it is said to be :*

- constant if there exists $a \in \mathbb{R}$ such that: $\forall n \in \mathbb{N} : u_n = a$.
- stationary if there exists $n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0 : u_n = u_{n_0}$.
- truncated, if its general term can only be defined after a certain value of n, let $n \geqslant N_0$. If, necessary we can complete by putting $u_n = 0$ pour $n < N_0$.

Example 3.1. $\left(sin\left(2n + \frac{1}{2}\right)\pi\right) = (1, 1, 1, ...)$ *is a constant sequence.*

$\left(sinn!\frac{\pi}{5}\right)_{n\in\mathbb{N}} = \left(\sin\frac{\pi}{5}, \sin\frac{\pi}{5}, \sin 2\frac{\pi}{5}, \sin\left(6\frac{\pi}{5}\right), \sin\left(24\frac{\pi}{5}\right), 0, 0, ...\right)$ *is a stationary sequence.*

$\left(\frac{1}{n(n-3)}\right)$ *is a truncated sequence, it is defined after $n = 4$.*

3.0.2 Bounded sequences, convergent sequences

Definition 3.3. *A sequence (u_n) is said to be upper bounded (resp. lower bonded), if there exists $M \in \mathbb{R}$ such that $\forall n \in \mathbb{N} : u_n \leqslant M$ (resp. $u_n \geqslant M$).*
A sequence is said to be bounded if it is upper and lower bounded.

in other terms (u_n) is said to be bounded if there exists $M > 0, M \in \mathbb{R}$, such that $\forall n \in \mathbb{N} : |u_n| \leqslant M$.

Example 3.2. *The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is bounded, because, $\forall n \geqslant 1 : 0 < u_n = \frac{1}{n} \leqslant 1$.*

The sequence $((-1)^n)_{n\in\mathbb{N}}$ is bounded because $\forall n \in \mathbb{N} : |(-1)^n| = 1$.

The sequence $(n)_{n\in\mathbb{N}}$ is lower bounded, but the sequence $((-1)^n n)$ is neither upper bounded nor lower bounded.

Definition 3.4. *We say that the sequence (u_n) converges to the limit l as n approaches infinity, and write $\lim\limits_{n\to+\infty} u_n = l$ or $u_n \to l$, $n \to +\infty$, if*

$$\forall \varepsilon > 0, \exists N_0 = N_0(\varepsilon) \in \mathbb{N}, \forall n > N_0(\varepsilon) \Rightarrow |u_n - l| < \varepsilon. \tag{3.1}$$

A sequence that does not converge to some real number is said to diverge.
Resume

$$\lim_{n\to+\infty} u_n = l \Leftrightarrow \forall \varepsilon > 0, \exists N_0(\varepsilon) \in \mathbb{N}, \forall n > N_0(\varepsilon) \Rightarrow |u_n - l| < \varepsilon.$$

Theorem 3.1. *The limit of a convergent sequence is unique.*

Proof. By contradiction technique, we suppose that we have two limits l_1, l_2, we must show that $l_1 = l_2$, indeed by the definition of limit there must exist N_1 so that

$$\forall n > N_1(\varepsilon) \Rightarrow |u_n - l_1| < \frac{\varepsilon}{2},$$

and must exist N_2 so that

$$\forall n > N_2(\varepsilon) \Rightarrow |u_n - l_2| < \frac{\varepsilon}{2}.$$

For $n > max\{N_1, N_2\}$, the triangle shows that

$$|l_1 - l_2| = |l_1 - l_2 + u_n - u_n| < |u_n - l_1| + |u_n - l_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

This shows that $|l_1 - l_2| < \varepsilon$ for all positive ε. It follows that $|l_1 - l_2| = 0$. □

Example 3.3. *The following sequence* $((-1)^n)_{n\in\mathbb{N}} = (1, -1, 1, -1, ...)$ *does not converge to any real number.*

3.0.3 Convergent sequences properties

Theorem 3.2. *Convergent sequences are bounded.*

Proof. Let (u_n) be a convergent sequence, and let $\lim\limits_{n\to+\infty} u_n = l$. Applying Definition 3.4 with $\varepsilon = 1$, we obtain $N \in \mathbb{N}$ so that

$$\forall n > N(\varepsilon) \Rightarrow |u_n - l| < 1,$$

From the triangle inequality we see that $N \in \mathbb{N}$ implies

$$\forall n > N \Rightarrow |u_n| < 1 + |l|,$$

define $M = max\{1 + |l|, |u_1|, |u_2|, |u_3|, ..., |u_N|\}$. Then we have $|u_n| \leqslant M$, for all $n \in \mathbb{N}$, so (u_n) is a bounded sequence. □

Remark 3.1. *The boundness of a sequence of real numbers is necessary for convergence but not sufficient.*

Example 3.4. *La suite* $((-1)^n n)_{n\in\mathbb{N}} = (1, -1, 1, -1, ...)$ *is unbounded, so divergent.*

Theorem 3.3. *If* $\lim_{n\to+\infty} u_n = a$, $\lim_{n\to+\infty} v_n = b$, *and* $u_n \leqslant v_n$, $(\forall n \in \mathbb{N})$. *Then* $a \leqslant b$.

Theorem 3.4. *Let be* $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, *two sequences which converge to* a *and (resp.* b*), despite that* $(\forall n \in \mathbb{N})$ $(u_n < v_n)$, *we obtain* $a \leqslant b$.

Example 3.5. *Let be the following sequences* $\left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$, *and* $\left(\frac{1}{n+2}\right)_{n\in\mathbb{N}}$,, *we have* $\forall n \in \mathbb{N}$ $\frac{1}{n+2} < \frac{1}{n+1}$, *but the two limits are equal to* 0.

Theorem 3.5. *(Squeeze Theorem) Suppose that* $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, *and* $\{w_n\}_{n\in\mathbb{N}}$, *are sequences of real numbers such that 1)* $\forall n \in \mathbb{N} : u_n \leqslant w_n \leqslant v_n$. *2)* $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = a$. *Then* $\lim_{n\to+\infty} w_n = a$.

Example 3.6. *Find* $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{n^2+k}$.

We have $\forall k \in \{1, 2, ..., n\}$ $\frac{n}{n^2+n} \leqslant \frac{n}{n^2+k} \leqslant \frac{n}{n^2+1}$

where $\frac{n^2}{n^2+n} \leqslant \sum_{k=1}^{n} \frac{n}{n^2+k} \leqslant \frac{n^2}{n^2+1}$. *Since* $\lim_{n\to+\infty} \frac{n^2}{n^2+n} = \lim_{n\to+\infty} \frac{n^2}{n^2+1} = 1$, *then by the squeeze technique, we obtain,*

$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{n^2+k} = 1$.

Theorem 3.6. *If* $\lim_{n\to+\infty} u_n = l$, *then* $\lim_{n\to+\infty} |u_n| = |l|$.

$$\lim_{n\to+\infty} u_n = l \Rightarrow \lim_{n\to+\infty} |u_n| = |l|.$$

Proof. It follows from the inequality $||x| - |y|| \leqslant |x - y|$. □

Remark 3.2. *The converse is in general wrong.*

Example 3.7. *Consider the sequence defined by* $((-1)^n)_{n\in\mathbb{N}}$. *One has* $\lim_{n\to+\infty} |u_n| = \lim_{n\to+\infty} |(-1)^n| = 1$, *despite that* $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} (-1)^n$ *does not exist.*

Remark 3.3. $\lim_{n\to+\infty} u_n = 0 \Longleftrightarrow \lim_{n\to+\infty} |u_n| = 0$.

3.0.4 Combination Rules for convergent sequences

Theorem 3.7. *Suppose that the following sequences are convergent* $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$. *Set* $\lim_{n\to+\infty} u_n = a$, $\lim_{n\to+\infty} v_n = b$, *Then*

1) $\forall c \in \mathbb{R}$, $\lim_{n\to+\infty} cu_n = c \lim_{n\to+\infty} u_n = ca$.

2) $\lim_{n\to+\infty} u_n \pm v_n = a \pm b$

3) $\lim_{n\to+\infty} u_n v_n = ab$

4) If $(\forall n \in \mathbb{N})$ $(u_n \neq 0)$, *if* $b \neq 0$, *then* $\lim_{n\to+\infty} \frac{u_n}{v_n} = \frac{a}{b}$.

Proof. 1) We assume $c \neq 0$, since this result is trivial for $c = 0$, Let $\varepsilon > 0$ and note that we need to show that $|cu_n - ca| < \varepsilon$ for large n. Since $\lim\limits_{n\to+\infty} u_n = a$, there exists N such that

$$\forall n > N \Rightarrow |u_n - a| < \frac{\varepsilon}{|c|}.$$

Then

$$\forall n > N \Rightarrow |cu_n - ca| < \varepsilon.$$

2) Let $\varepsilon > 0$; we need to show that $|u_n \pm v_n - (a \pm b)| < \varepsilon$, for large n, we note that $|u_n \pm v_n - (a \pm b)| < |u_n - a| + |v_n - b)|$. Since there exists N_1 such that

$$\forall n > N_1 \Rightarrow |u_n - a| < \frac{\varepsilon}{2}.$$

Likewise, there exists N_2 such that

$$\forall n > N_2 \Rightarrow |u_n - a| < \frac{\varepsilon}{2}.$$

Let $N = max\{N_1, N_2\}$. Then clearly

$$\forall n > N \Rightarrow |u_n \pm v_n - (a \pm b)| < \varepsilon.$$

3) The trick here is to look at the inequality

$$|u_n v_n - ab| = |u_n v_n + u_n b - u_n b - ab| < |u_n v_n - u_n b| + |u_n b - ab| = |u_n|\,|v_n - b| + |b|\,|u_n - a|$$

4) To prove (4) it suffices to show that $\lim\limits_{n\to+\infty} \dfrac{1}{v_n} = \dfrac{1}{b}$.
The result (4) then follows from (3). Since $b \neq 0$, and $\lim\limits_{n\to+\infty} v_n = b$,, there exists a positive integer N_0 such that $|v_n - b| < \frac{1}{2}|b|$, for all $n \geqslant N_0$.
Also, since $|b| < |v_n - b| + |b| < \frac{1}{2}|b| + |v_n|$, for $n \geqslant N_0$, we have $|v_n| \geqslant \frac{1}{2}|b|$ for all $n \geqslant N_0$.
Therefore,
$\left|\frac{1}{v_n} - \frac{1}{b}\right| = \frac{|b - v_n|}{|bv_n|} \leqslant \frac{2}{|b|^2}|v_n - b|$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to+\infty} v_n = b$, we can choose an integer $N1 \geqslant N_0$ so that $|v_n - b| \leqslant \varepsilon \frac{|b|^2}{2}$ for all $n \geqslant N1$.
Therefore
$\left|\frac{1}{v_n} - \frac{1}{b}\right| \leqslant \varepsilon$, for all $n \geqslant N1$. $\square$

Sequences which tend to infinity

$$\lim_{n\to+\infty} u_n = +\infty \Leftrightarrow \forall A > 0, \exists N \in \mathbb{N}, \forall n > N \Rightarrow u_n > A.$$

$$\lim_{n\to+\infty} u_n = -\infty \Leftrightarrow \forall A > 0, \exists N \in \mathbb{N}, \forall n > N \Rightarrow u_n < -A.$$

Example 3.8. *Let* $a \in \mathbb{R}$, $a > 1$. *Show that* $\lim_{n\to+\infty} a^n = +\infty$. *Let* $A > 0$. *Then* $a^n > A \Leftrightarrow n > \dfrac{\ln A}{\ln a}$.

So $\forall A > 0, \exists N = E\left(\dfrac{\ln A}{\ln a}\right)$, $\forall n > N = E\left(\dfrac{\ln A}{\ln a}\right) \Rightarrow a^n > A \Leftrightarrow \lim_{n\to+\infty} a^n = +\infty$ $(a > 1)$.

3.0.5 Monotone sequences

Definition 3.5. *Let* (u_n) *be a sequence of real numbers. We say that* (u_n) *is nondecreasing (resp. nonincreasing) if it satisfies the inequality* $\forall n \in \mathbb{N} : u_n \leqslant u_{n+1}$ *(resp.* $u_n \geqslant u_{n+1}$*).*

If $u_n < u_{n+1}$ *(resp.* $u_n > u_{n+1}$*), we say that* (u_n) *is increasing (resp. decreasing). We say that* (u_n) *is monotone if it is either nonincreasing or nondecreasing. We say that* (u_n) *is strictly monotone if it is either increasing or decreasing.*

Corollary 3.1. *An nondecreasing sequence is lower bounded, and the nonincreasing one is upper bounded.*

Example 3.9. *Let* $\left(\dfrac{n-1}{n}\right)_{n\in\mathbb{N}}$ *be a real sequence. Since* $u_{n+1} - u_n = \dfrac{n}{n+1} - \dfrac{n-1}{n} = \dfrac{1}{n(n+1)} > 0$, $\forall n \in \mathbb{N}$, *then* $u_{n+1} > u_n$ $(n \in \mathbb{N})$. *So the real sequence* $\left(\dfrac{n-1}{n}\right)_{n\in\mathbb{N}}$ *is increasing.*

Example 3.10. *The sequence* $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ *is decreasing because,* $\forall n \in \mathbb{N} : u_{n+1} = \dfrac{1}{n+1} < \dfrac{1}{n} = u_n$.

Remark 3.4. *We check the monotony by evaluating the sign of* $u_{n+1} - u_n$. *If the sequence is positive, we can just compare* $\dfrac{u_{n+1}}{u_n}$ *with* 1.

Theorem 3.8. *All bounded monotone sequences converge.*

Proof. Let (u_n) be a bounded nondecreasing sequence. Let S denote the set $\{u_n : n \in \mathbb{N}\}$, and let $u = supS$. Since S is bounded, $u = supS$ represents a real number. We show that $limu_n = u$. Let $\varepsilon > 0$. Since $u - \varepsilon$ is not an upper bound for S, there exists N such that $u_N > u - \varepsilon$. Since (u_n) is nondecreasing, we have $u_N \leqslant u_n$ for all $n \geqslant N$. Of course, $u_n \leqslant u$

for all n, $n \geqslant N$ so n > N implies $u - \varepsilon < u_n \leqslant u$, which implies $|u_n - u| < \varepsilon$. This shows that $lim u_n = u$. The proof for bounded nonincreasing sequences is left as exercise. $\square$

3.0.6 Adjacent sequences

Definition 3.6. *Let (u_n) (v_n) be two real sequences. We say that the two sequences are adjacent if the first is nondecreasing, the second is nonincreasing, and their difference converges to* 0*. In other words*

1) *The sequence (u_n) is nondecreasing and the sequence (v_n) is nonincreasing,*
2) *The difference $(v_n - u_n)$ converges to 0, when n approaches ∞.*

Example 3.11. $u_n = 1 + \frac{1}{n^2}$ *and* $v_n = 1 - \frac{1}{n^2}$

Proposition 3.1. *Two adjacent sequences converge, and converge to the same limit.*

Proof. Let (u_n) and (v_n) be two real sequences such as that (u_n) is nondecreasing, the sequence (v_n) is nonincreasing, and $(v_n - u_n)$ converges to 0, when n approaches ∞. We first show that $v_n > u_n$. Put $W_n = v_n - u_n$. We check the sign of $W_{n+1} - W_n = v_{n+1} - u_{n+1} - v_n + u_n = v_{n+1} - v_n - (u_{n+1} - u_n) < 0$, because (u_n) is nondecreasing, and (v_n) is nonincreasing. So (W_n) is nonicreasing and converges to 0, thus (W_n) is positive and $v_n > u_n$. Now we show that (u_n) and (v_n) converge. Indeed, one has $u_n < v_n < v_0$, so u_n is nonincreasing and bounded above by v_0, all bounded monotone sequences converge. Likewise $u_0 < u_n < v_n$, we do and write the same things. v_n converges. The two adjacent sequences admit the same limit. Since $lim u_n = l_1, lim v_n = l_2$, we have $lim W_n = lim(v_n - u_n) = 0 \Leftrightarrow l_1 = l_2$. $\square$

3.0.7 Subsequences

Definition 3.7. *A subsequence of a sequence (u_n) is a sequence formed by deleting elements of the u_n to produce a new u_n. This subsequence is usually written as $v_n = u_{\rho(n)}$, $n \in \mathbb{N}$, where $\rho : \mathbb{N} \to \mathbb{N}$ is an increasing sequence of positive integers.*

Example 3.12. *Let consider the real sequence defined by $\left((-1)^n\right)_{n \in \mathbb{N}}$. The sequences defined by :*

$$v_n = u_{2n} = (-1)^{2n} = 1 \, (n \in \mathbb{N})$$

$$w_n = u_{2n+1} = (-1)^{2n+1} = -1 \, (n \in \mathbb{N})$$

are subsequences of the sequence $((-1)^n)_{n\in\mathbb{N}}$.

Remark 3.5. *If* $\rho : \mathbb{N} \to \mathbb{N}$ *is an increasing application, then* $\forall n \in \mathbb{N} : \rho(n) \geqslant n$.

Corollary 3.2. *Let* (u_n) *be a real sequence, if* u_n *converges to* l. *Then all subsequences* $\left(v_n = u_{\rho(n)}\right)$ *of* (u_n) *converges to* l.

Remark 3.6. *The converse is in general wrong.*

Example 3.13. *The divergent sequence* $((-1)^n)_{n\in\mathbb{N}}$ *admits the following convergent subsequences* $u_{2n} = 1\ (n \in \mathbb{N})$, $u_{2n+1} = -1\ (n \in \mathbb{N})$.

Theorem 3.9. *(Bolzano-Weierstrass) Every bounded sequence admits a convergent subsequence.*

Proof. Let $(u_n$ be a real sequence and $m \in \mathbb{N}$. we say that m is a peak of the sequence $(u_n$ if : $n > m \Rightarrow u_n < u_m$. Suppose that (u_n) has an infinite numbers of peaks. $k_0 < k_1 < k_2 < k_3 < k_4 < k_5 < ... < k_n < ...$ and consider the subsequence (u_{k_n}). Then (u_{k_n}) is decreasing since $k_n > k_m \Rightarrow u_{k_n} < u_{k_m}$ and thus (u_{k_n}) is monotone. Suppose that (u_{k_n}) has a finite number of peaks and let N be the last (greatest) peak. Then $k_0 = N+1$ is not a peak and so there exists k_1 such that $u_{k_1} > u_{k_0}$. Having defined k_n such that $k_n > k_{n-1} > N$, then there exists $k_{n+1} > k_n$ such that $u_{k_{n+1}} > u_{k_n}$. The subsequence u_{k_n} is obviously increasing and so it is monotone. Now if u_n is in addition bounded, so is u_{k_n} and applying the Monotone Convergence Theorem yields that the subsequence has a finite limit. $\square$

3.0.8 Cauchy sequences

Definition 3.8. *A sequence* (u_n) *of real numbers is called a Cauchy sequence if*

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}, \forall n, m > N(\varepsilon) \Rightarrow |u_n - u_m| < \varepsilon.$$

Example 3.14. *The real sequence* $u_n = \dfrac{\sin 1}{2} + \dfrac{\sin 2}{2^2} + ... + \dfrac{\sin n}{2^n}$, $n \geqslant 1$ *is a Cauchy sequence.*

Indeed, for all $(n, m) \in \mathbb{N}^2, n > m$, *we have*

$$
\begin{aligned}
|u_n - u_m| &= \left|\frac{\sin(m+1)}{2^{m+1}} + \frac{\sin(m+2)}{2^{m+2}} + ... + \frac{\sin n}{2^n}\right| \\
&\leqslant \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + ... + \frac{1}{2^n} \\
&< \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + ... + \frac{1}{2^n} + ... \\
&= \frac{1}{2^{m+1}}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{2^m}
\end{aligned}
$$

Since $\frac{1}{2^m} \to 0, m \to \infty,$ *then*

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}, \forall n > m > N(\varepsilon) \Rightarrow |u_n - u_m| < \frac{1}{2^m} < \varepsilon.$$

Example 3.15. *The sequence* $((-1)^n)_{n \in \mathbb{N}}$ *is not Cauchy sequence. Remark that for* $\varepsilon = 1$

$$\forall N \in \mathbb{N}, \exists n = N+1, m = N+2, n > m > N$$

and

$$|u_{N+2} - u_{N+1}| = \left|(-1)^{N+2} - (-1)^{N+1}\right| = 2 > 1.$$

3.1 The Cauchy Criterion

Our difficulty in proving " $u_n \to \ell$ " is this: What is ℓ ? Cauchy saw that it was enough to show that if the terms of the sequence got sufficiently close to each other. then completeness will guarantee convergence.

Theorem 3.10. *Every Cauchy sequence is bounded* $[\mathbb{R}$ *or* $\mathbb{C}]$.

Proof. $1 > 0$ so there exists N such that $m, n \geqslant N \Longrightarrow |u_m - u_n| < 1$. So for $m \geqslant N$, $|u_m| \leqslant 1 + |u_N|$ by the Δ law. So for all m

$$|u_m| \leqslant 1 + |u_1| + |u_2| + \cdots + |u_N|.$$

□

Theorem 3.11. *Every convergent sequence is Cauchy.*

Proof. Let $u_n \to l$ and let $\varepsilon > 0$. Then there exists N such that

$$k \geqslant N \implies |u_k - l| < \varepsilon/2$$

For $m, n \geqslant N$ we have

$$|u_m - l| < \varepsilon/2$$
$$|u_n - l| < \varepsilon/2$$

So

$$\begin{aligned} |u_m - u_n| &\leqslant |u_m - l| + |u_n - l| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned} \qquad \text{by the } \Delta \text{ law}$$

□

Theorem 3.12. *Every real Cauchy sequence is convergent.*

Proof. Let the sequence be (u_n). By the above, (u_n) is bounded. By Bolzano-Weierstrass (u_n) has a convergent subsequence $(u_{n_k}) \to l$, say. So let $\varepsilon > 0$. Then

$$\exists N_1 \text{ such that } \quad r \geqslant N_1 \implies |u_{n_r} - l| < \varepsilon/2$$

$$\exists N_2 \text{ such that } m, n \geqslant N_2 \implies |u_m - u_n| < \varepsilon/2$$

Put $s := \min\{r \mid n_r \geqslant N_2\}$ and put $N = n_s$. Then

$$m, n \geqslant N \implies |u_n - l| = |u_n - u_{n_s} + u_{n_s} - l| \leqslant |u_n - u_{n_s}| + |u_{n_s} - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

□

Theorem 3.13. *Every complex Cauchy sequence is convergent.*

Proof. Put $z_n = x_n + \mathrm{i}y_n$. Then x_n is Cauchy: $|x_x - x_m| \leqslant |z_n - z_m|$ (as $|Re(w)| \leqslant |w|$). So $x_n \to x, y_n \to y$ and so $z_n \to x + \mathrm{i}y$. □

Example 3.16. *Using Cauchy criterion, show that the sequence*

$$u_n = 1 + \frac{1}{2} + ... + \frac{1}{n}\,(n \geqslant 1)$$

diverges.

Let $\varepsilon \in \left]0, \frac{1}{2}\right[$. *Then for* $n \geqslant 1$:

$$\begin{aligned} |u_{2n} - u_n| &= \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} \\ &\geqslant \frac{1}{2n} + \frac{1}{2n} + ... + \frac{1}{2n} = n\frac{1}{2n} = \frac{1}{2} > \varepsilon \end{aligned}$$

So the Cauchy criterion is not satisfied.

3.2 Limit supremum and limit infimum

Let $(s_n)_n$ be a sequence of real numbers and define the sequences

$$u_k = \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\} = \sup_{n \geq k} s_n$$
$$l_k = \inf\{s_k, s_{k+1}, s_{k+2}, \ldots\} = \inf_{n \geq k} s_n$$

For a simple example, consider the sequence $s_n = 1/n$. Then for each index k

$$u_k = \sup\left\{\frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2}, \ldots\right\} = \frac{1}{k}$$

because $1/k$ has the smallest denominator of all the fractions inside the braces, thus it must be the largest fraction. On the other hand,

$$l_k = \inf\left\{\frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2}, \ldots\right\} = 0$$

because the fractions inside the braces get smaller and smaller, approaching 0 .

For another example, consider the alternating sequence $s_n = (-1)^n$. In this case,

$$u_k = \sup\left\{(-1)^k, (-1)^{k+1}, (-1)^{k+2}, \ldots\right\} = 1$$

because the numbers inside the braces are always either -1 or 1 . Similarly,

$$l_k = \inf\left\{(-1)^k, (-1)^{k+1}, (-1)^{k+2}, \ldots\right\} = -1$$

Here is one more example that in a sense combines the previous two: $s_n = (-1)^n/n$. For every index k

$$u_k = \sup\left\{\frac{(-1)^k}{k}, \frac{(-1)^{k+1}}{k+1}, \frac{(-1)^{k+2}}{k+2}, \ldots\right\} = \begin{cases} 1/k & \text{if } k \text{ is even} \\ 1/(k+1) & \text{if } k \text{ is odd} \end{cases}$$

while

$$l_k = \inf\left\{\frac{(-1)^k}{k}, \frac{(-1)^{k+1}}{k+1}, \frac{(-1)^{k+2}}{k+2}, \ldots\right\} = \begin{cases} -1/(k+1) & \text{if } k \text{ is even} \\ -1/k & \text{if } k \text{ is odd} \end{cases}$$

There are also sequences for which (u_k) or (l_k) may be equal to plus or minus infinity. Consider the simple sequence $s_n = 2n$ for which we get

$$u_k = \sup\{2k, 2k+2, 2k+4, \ldots\}$$

The supremum equals plus infinity because the set in braces has no finite upper bounds. On the other hand,

$$l_k = \inf\{2k, 2k+2, 2k+4, \ldots\} = 2k$$

is well-defined for all k.

The following lists the basic properties of the two sequences (u_k) and (l_k).

Lemma 3.14. *Let (s_n) be a given sequence of real numbers.*

(a) The sequences (u_k) and (l_k) bound the sequence (s_n) in the following sense:

$$l_k \leq s_k \leq u_k$$

(b) (u_k) is a nonincreasing sequence, and (l_k) is a nondecreasing sequence.

Proof. (a) is clear from the definition of supremum and infimum of sets.

(b), we show that for every index k

$$u_k \geq u_{k+1} \quad \text{and} \quad l_k \leq l_{k+1}$$

For the first inequality, recall that for every k,

$$u_k = \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$$
$$u_{k+1} = \sup\{s_{k+1}, s_{k+2}, s_{k+3}, \ldots\}$$

The only difference between the two quantities is that the second set doesn't contain (s_k). If (s_k) is less than or equal to one of the other numbers $s_{k+1}, s_{k+2}, \ldots$ inside the braces, then the supremum isn't affected by dropping it, and we have $u_{k+1} = u_k$. But if s_k is greater than all the other numbers inside the braces, then dropping it will reduce the supremum: $u_{k+1} < u_k$. $\square$

The previous Lemma shows that the bounding sequences u_k and l_k are monotone sequences. As such, each can either have a real number for a limit or diverge to ∞ or $-\infty$. Because u_k is nonincreasing, if its limit is a real number, then it must be the greatest lower bound or infimum of the sequence u_k and can thus be represented as

$$\lim_{k\to\infty} u_k = \inf_{k\geq 1} u_k = \inf_{k\geq 1} \sup \{s_k, s_{k+1}, s_{k+2}, \ldots\} = \inf_{k\geq 1} \sup_{n\geq k} s_n$$

Similarly, for l_k, which is nondecreasing, we can write

$$\lim_{k\to\infty} l_k = \sup_{k\geq 1} l_k = \sup_{k\geq 1} \inf \{s_k, s_{k+1}, s_{k+2}, \ldots\} = \sup_{k\geq 1} \inf_{n\geq k} s_n$$

If the limits are $\pm\infty$ instead of real numbers, then we use those symbols to indicate the limits. With this in mind, we have the following definition.

Let s_n be a given sequence of real numbers. If the sequence u_k converges to a real number, then its limit is the limit supremum (or limit superior) of s_n and denoted by

$$\limsup_{n\to\infty} s_n = \inf_{n\geq 1} \sup \{s_n, s_{n+1}, s_{n+2}, \ldots\} = \overline{lim}_{k\to\infty} u_k$$

If u_k diverges to ∞ or $-\infty$, then we use these symbols to denote the limit supremum. Similarly, the limit infimum (or limit inferior) of s_n is

$$\liminf_{n\to\infty} s_n = \sup_{n\geq 1} \inf \{s_n, s_{n+1}, s_{n+2}, \ldots\} = \underline{lim}_{k\to\infty} l_k$$

or ∞ or $-\infty$ as appropriate.

For example, referring to the example we discussed earlier, we have for $s_n = 1/n$

$$\limsup_{n\to\infty} \frac{1}{n} = \overline{\lim}_{k\to\infty} \frac{1}{k} = 0, \quad \liminf_{n\to\infty} \frac{1}{n} = \underline{lim}_{k\to\infty} 0 = 0$$

Similarly, for $s_n = (-1)^n$

$$\limsup_{n\to\infty}(-1)^n = \overline{\lim}_{k\to\infty} 1 = 1, \quad \liminf_{n\to\infty}(-1)^n = \underline{\lim}_{k\to\infty}(-1) = -1$$

And for $s_n = 2n$

$$\limsup_{n\to\infty}(2n) = \infty, \quad \liminf_{n\to\infty}(2n) = \lim_{k\to\infty} 2k = \infty$$

Notice that of the above three sequences, only $1/n$ converges to a real number, and it has the property that its limit supremum and limit infimum are equal real numbers.

Theorem 3.15. *A sequence (s_n) converges to a real number s if and only if*

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$$

Proof. First, we assume that $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$ is given and prove that s_n must converge to the number s. By

$$l_n \leq s_n \leq u_n$$

Since $\lim_{n\to\infty} l_n = s$ and also $\lim_{n\to\infty} u_n = s$, the squeeze theorem implies that $\lim_{n\to\infty} s_n$ exists and equals s.

Conversely, assume that s_n converges to a number s. Then by the definition of convergence, for every $\varepsilon > 0$, we can find a positive integer N such that

$$|s_n - s| < \varepsilon \quad \text{for all } n \geq N$$

or equivalently,

$$s - \varepsilon < s_n < s + \varepsilon \quad \text{for all } n \geq N$$

In particular, $s_{n+j} < s + \varepsilon$ for all $j = 1, 2, 3, \ldots$ and $n \geq N$, and this implies that

$$\sup\{s_n, s_{n+1}, s_{n+2}, \ldots\} \leq s + \varepsilon \quad \text{for all } n \geq N$$

because $s + \varepsilon$ is an upper bound of the set $\{s_n, s_{n+1}, s_{n+2}, \ldots\}$, while the supremum is its least upper bound.

Similarly, because $s_{n+j} > s - \varepsilon$ for all j, we infer that $s - \varepsilon$ is a lower bound for

$\{s_n, s_{n+1}, s_{n+2}, \ldots\}$, and therefore, the greatest lower bound of this set satisfies

$$\inf\{s_n, s_{n+1}, s_{n+2}, \ldots\} \geq s - \varepsilon \quad \text{for all } n \geq N$$

From above, we conclude that

$$s - \varepsilon \leq l_n \leq u_n \leq s + \varepsilon \quad \text{for all } n \geq N$$

These inequalities imply the following:

$$|u_n - s| \leq \varepsilon, \quad |l_n - s| \leq \varepsilon \quad \text{for all } n \geq N$$

Since these inequalities hold for all $\varepsilon > 0$, we conclude that

$$\lim_{n\to\infty} u_n = s \text{ and } \lim_{n\to\infty} l_n = s.$$

□

An immediate consequence of the above theorem is the following:

Corollary 3.3. *If $\limsup_{n\to\infty} s_n \neq \liminf_{n\to\infty} s_n$ for a sequence (s_n), then (s_n) diverges.*

Notice that the above corollary includes the cases where limit supremum or limit infimum are ∞ or $-\infty$. For example, the previous Corollary implies the divergence of both of the sequences $(-1)^n$ and $2n$ that we discussed earlier. A sequence (s_n) can have a limit s only when the upper and lower bounding sequences meet:

$$l_1 \leq l_2 \leq \cdots \leq l_k \leq \cdots \to s \leftarrow \cdots \leq u_k \leq \cdots \leq u_2 \leq u_1$$

If the lower sequence does not meet the upper one, then there is a nonempty open interval of numbers (l, u) between them:

$$l_1 \leq l_2 \leq \cdots \to \lim_{n\to\infty} l_n = l < u = \lim_{n\to\infty} u_n \leftarrow \cdots \leq u_2 \leq u_1$$

The sequence s_n cannot converge to a limit if the interval (l, u) is not empty, i.e., $u - l > 0$ because not matter how large the index N we choose, there are terms $s_k \leq l$ and other terms $s_m \geq u$ with $k, m \geq N$; so if we choose, say, $\varepsilon = (u - l)/2$, then no valid threshold index N can be found to match such values of ε.

Every sequence (s_n) has a limit supremum u and a limit infimum l (they could be ∞ or $-\infty$ if (s_n) is unbounded). Although u and l are limits of monotone sequences (u_k) and (l_k) that are derived from (s_n), these bounding sequences may or may not contain terms of s_n; in fact, there are sequences where $u_k \neq s_n$ and $l_k \neq s_n$ for every k and every n. On the other hand, the definitions of the bounding sequences suggest that the numbers (u_k) and (l_k) are increasingly aligned with the terms of (s_n) as k and n get larger. This raises a natural question: are there subsequences of (s_n) that converge to the limits u and l ?

To answer this question, let (s_n) be a given sequence and consider its upper bounding sequence

$$u_k = \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$$

If u is the limit supremum of s_n, then because u_k is a nonincreasing sequence,

$$u = \inf_{k \geq n} u_k = \lim_{k\to\infty} u_k$$

If we pick any number $\varepsilon > 0$, then there is a positive integer N such that

$$u_k - u = |u_k - u| < \varepsilon \quad \text{for all } k \geq N$$

Further, $u_k \geq s_n$ for all $n \geq k$ so that

$$s_n \leq u_k < u + \varepsilon \quad \text{for all } n \geq N$$

Next, since u_k is the least upper bound of the set $\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ for each k and further, $u \leq u_k$ for all k, it follows that $u - \varepsilon$ is not an upper bound of this set. This means that there is an integer $m \geq 0$ such that $u - \varepsilon < s_{k+m}$. If $k \geq N$, then

$$u - \varepsilon < s_{k+m} < u + \varepsilon \quad k \geq N$$

These inequalities help us identify a subsequence of s_n that converges to u.

Theorem 3.16. *The following statements are equivalent*
(a) If A has a supremum or least upper bound, then it is unique. Also, a greatest lower bound is unique, if it exists. (b) There is a sequence a_n in A that converges to sup A. Also there is a sequence in A that converges to $\inf A$.

Proof. a) Let $r = \sup A$. If r' is also a least upper bound of A, then in particular, r' is an upper bound, so $r \leq r'$. Similarly, $r' \leq r$ since r is also an upper bound of A, and r' is least by assumption. It follows that $r' = r$. The proof that the greatest lower bound is unique is essentially the same.

(b) We prove the assertion about $\sup A$ and leave the one about $\inf A$ as an exercise. First, if $s = \sup A$ is in A (e.g., if A is a finite set), then the constant sequence $a_n = s$ converges to s (trivially), and the proof is finished. Next, suppose that s is not in A (hence, A is infinite). If $\sup A = \infty$, then for every positive integer n there is an element a_n of S such that $a_n \geq n$. It follows that a_n diverges to ∞ and thus to $\sup A$. Finally, let $\sup A = s < \infty$. Then there is an element of $a_1 \in A$ such that $s - a_1 < 1$; if not, then $s \geq a + 1$ for all $a \in A$, and thus s is not the least upper bound. Therefore, $s - 1 < a_1 < s$ Similarly, there is $a_2 \in A$ such that $a_2 > s - 1/2$ and so on; for every n, there is $a_n \in A$ such that $s - \frac{1}{n} < a_n < s$ Since $\lim_{n\to\infty}(s - 1/n) = s$, the squeeze theorem implies that $\lim_{n\to\infty} a_n = s$. □

3.3 The Stolz-Cesaro Theorem

Theorem 3.17. *If $(b_n)_n$ is a sequence of positive real numbers, such that $\sum_{n=1}^{\infty} b_n = \infty$, then for any sequence $(a_n)_n \subset \mathbb{R}$ one has the inequalities:*

$$\limsup_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \limsup_{n\to\infty} \tfrac{a_n}{b_n} \tag{3.2}$$

$$\liminf_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \geq \liminf_{n\to\infty} \tfrac{a_n}{b_n}. \tag{3.3}$$

In particular, if the sequence $(a_n/b_n)_n$ has a limit, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Proof. It is quite clear that we only need to prove (3.2), since the other inequality follows by replacing a_n with $-a_n$.

The inequality (3.2) is trivial, if the right-hand side is $+\infty$. Assume then that the quantity $L = \limsup_{n\to\infty} (a_n/b_n)$ is either finite or $-\infty$, and let us fix for the moment some number $\ell > L$. By the definition of limsup, there exists some index $k \in \mathbb{N}$, such that

$$\frac{a_n}{b_n} \leq \ell, \quad \forall n > k. \tag{3.4}$$

Using (3.4) we get the inequalities

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \cdots + a_k + \ell \left(b_{k+1} + b_{k+2} + \ldots b_n \right), \quad \forall n > k.$$

If we denote for simplicity the sums $a_1 + \cdots + a_n$ by A_n and $b_1 + \cdots + b_n$ by B_n, the above inequality reads:

$$A_n \leq A_k + \ell \left(B_n - B_k \right), \quad \forall n > k,$$

so dividing by B_n we get

$$\frac{A_n}{B_n} \leq \ell + \frac{A_k - \ell B_k}{B_n}. \tag{3.5}$$

Since $B_n \to \infty$, by fixing k and taking limsup in (3.5), we get $\limsup_{n\to\infty} (A_n/B_n) \leq \ell$. In other words, we obtained the inequality

$$\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \ell, \quad \forall \ell \geq L,$$

which in turn forces

$$\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq L$$

□

Remark. An equivalent formulation of the above Theorem is as follows: If $(y_n)_n$ is a strictly increasing sequence with $\lim_{n\to\infty} y_n = \infty$, then for any sequence $(x_n)_n$, the following inequalities hold:

$$\limsup_{n\to\infty} \frac{x_n}{y_n} \leq \limsup_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$
$$\liminf_{n\to\infty} \frac{x_n}{y_n} \geq \liminf_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

In particular, if the sequence $\left(\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right)_n$ has a limit, then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}.$$

Indeed (assuming all the y_n 's are positive, which happens anyway for n large enough), if we consider the sequences $(a_n)_n$ and $(b_n)_n$, defined by $a_1 = x_1, b_1 = y_1$, and $a_n = x_n - x_{n-1}$, $b_n = y_n - y_{n-1}, n \geq 2$, then everything is clear, since $x_n = a_1 + \cdots + a_n$ and $y_n = b_1 + \cdots + b_n$.

The Stolz-Cesaro Theorem has numerous applications in Calculus. Below are three of the most significant ones.

Theorem 3.18. *Cesaro's Theorem*
For any sequence $(a_n)_n \subset \mathbb{R}$ *one has the inequalities:*

$$\limsup_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \limsup_{n\to\infty} a_n$$
$$\liminf_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \liminf_{n\to\infty} a_n.$$

In particular, if the sequence $(a_n)_n$ *has a limit, then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \lim_{n\to\infty} a_n.$$

Proof. Particular case of Stolz-Cesaro Theorem with $b_n = 1$. □

Remark 3.7. *An equivalent formulation of the above Theorem (proven using the alternative version of Stolz-Cesaro Theorem) is as follows: For any sequence* $(x_n)_n$, *the following inequalities hold:*

$$\limsup_{n\to\infty} \frac{x_n}{n} \leq \limsup_{n\to\infty} (x_n - x_{n-1})$$
$$\liminf_{n\to\infty} \frac{x_n}{n} \geq \liminf_{n\to\infty} (x_n - x_{n-1})$$

In particular, if the sequence $(x_n - x_{n-1})_n$ *has a limit, then*

$$\lim_{n\to\infty} \frac{x_n}{n} = \lim_{n\to\infty} (x_n - x_{n-1}).$$

3.4 Sequences defined by recursion formulas

Definition 3.9. *Let* $f : D \subset \mathbb{R} \to \mathbb{R}$. *A recursive sequence is a sequence in which terms are defined using one or more previous terms which are given by* $u_0 \in D$ *and the relation* $\forall n \in \mathbb{N} : u_{n+1} = f(u_n)$

We suppose that $f(D) \subset D$, and so the sequence is well defined.

Example 3.17. *Let* $u_{n+1} = \sqrt{u_n + 2}$, $u_0 = 1$, *we have* $f(x) = \sqrt{x+2}, D = [-2, +\infty[$ *et* $f(D) = [0, +\infty[\subset D$. *Thus* u_n *is well defined.*

Lemma 3.19. *If* f *is continuous on* D *and the sequence* $\{u_n\}$ *converges to* $l \in D$, *then* $l = f(l)$.

Theorem 3.20. *Let* $\{u_n\}$ *be a real sequence recursively defined by* $u_{n+1} = f(u_n)$.

(a) If f is nondecreasing then the sequence (u_n) is monotone. More precisely (u_n) is monotone

1) If $u_0 \leqslant u_1$, then the sequence is nondecreasing.

2) If $u_0 \geqslant u_1$, then the sequence is nonincreasing.

(b) If f is nonincreasing then the sequence $u_{n+1} \leqslant u_n$, is positive and negative alternatively. We set $g = fof$, so g is nondecreasing, we can easily verify that the sequences (u_{2n}) and (u_{2n+1}) defined by $u_{2n+2} = f(f(u_{2n})), \quad u_2 = f(u_1)$ and $u_{2n+1} = f(f(u_{2n-1})), \quad u_1 \quad$ given, are oppositely monotone such as that $g(u_1) - u_1 = f(f(u_1)) - u_1$ and $g(f(u_1)) - f(u_1) = f(f(f(u_1))) - f(u_1)$ have different signs.

CHAPTER

4

REAL FUNCTIONS OF REAL VARIABLES

4.1 Introduction

Let $D \in \mathbb{R}$ i.e D is a subset of the real numbers. Often we need to associate with $x \in D$ a new real number which we denote at the moment by $f(x)$. For example the absolute value.

Definition 4.1. *Let $D \in \mathbb{R}$. A function $f : D \longmapsto R$ is a rule which assigns to every $x \in D$ exactly one real value $f(x)$. For this we write $x \longmapsto f(x)$ and say that x is mapped onto $f(x)$, or $f(x)$ is the value of f at x.*

We call D the domain of the function f, sometimes we write $D(f)$ or D_f instead of D. So $D_f = \{x \in D : f(x) \text{ exists}\}$.

The set of all functions is denoted by $\mathcal{F}(D, \mathbb{R})$.

Definition 4.2. *The set $\{y = f(x), x \in D\}$ is called the range of f and is denoted by $H(f)$. Variable x is called argument or independent variable and variable y is called dependent.*

Definition 4.3. *Let $f, g \in \mathcal{F}(D, \mathbb{R})$ and $\lambda \in \mathbb{R}$. We define the following important operations:*

1. $(f+g)(x) = f(x) + g(x)$ $(\forall x \in D)$
2. $(f.g)(x) = f(x).g(x)$ $(\forall x \in D)$
3. $(\lambda g)(x) = \lambda g(x)$ $(\forall \lambda \in \mathbb{R}, \forall x \in D)$
4. *If* $\forall x \in D : f(x) \neq 0$, $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$.

Definition 4.4. *Let f and g be real functions with domains $D(f)$ and $D(g)$. Let $H(f) \subset D(g)$. Then under the composition of function f and g we understand function h defined by $\forall x \in D(h) : h(x) = g(f(x))$, with $D(h) = D(f)$.*

Notation: $h = gof$.

Definition 4.5. *Two functions f and g are equal $(f = g)$, if*
(i) $D(f) = D(g)$
(ii) $\forall x \in D(f) : f(x) = g(x)$

Remark 4.1. *In general $gof \neq fog$.*

Definition 4.6. *Graph of function f is a set of ordered pairs of real numbers $(x, f(x))$, where $x \in D(f)$. We write*

$$graphf = \{(x, f(x))/x \in D(f)\}$$

Even and odd functions

Definition 4.7. *Let $D \subset \mathbb{R}$ such that $(\forall x \in D) \Longrightarrow (-x \in D)$*
We say that function $f : D \longrightarrow \mathbb{R}$ is even if and only if $(\forall x \in D), (f(-x) = f(x))$.
We say that function $f : D \longrightarrow \mathbb{R}$ is odd if and only if $(\forall x \in D), (f(-x) = -f(x))$.

Periodic functions

Definition 4.8. *A function $f : D \longrightarrow \mathbb{R}$ is called periodic if $\exists \quad T > 0$ such that*
– $\forall x \in D \Longrightarrow x \pm T \in D, f(x+T) = f(x)$
– $\forall x \in D, f(x+T) = f(x)$. *Number T is called a period of f. The smallest positive period is called primitive.*

Remark 4.2. *Let $T > 0$ be a period of f, then*

$$\forall x \in \mathbb{R}, \forall n \in \mathbb{Z} : f(x + nP) = f(x).$$

Theorem 4.1. *(i) If f is periodic with period P and function g such that $H(f) \subset D(g)$ then a composition $h(x) = g(f(x))$ is periodic with the same period p. (ii) If f is periodic with period p and $a \in \mathbb{R}, a \neq 0$; then function $g(x) = f(ax)$ is periodic with period $\frac{P}{a}$.*

Proof. Just do it. □

Bounded Functions

Definition 4.9. *Let $f : D \to \mathbb{R}$ and let $f(D)$ the set of all the values of f.*

We say that function f is bounded above on its domain D if $f(D)$ is bounded above i.e. $\exists M \in \mathbb{R}, \forall x \in D : f(x) \leqslant M$.

We say that function f is bounded below on its domain D if $f(D)$ is bounded below i.e. $\exists m \in \mathbb{R}, \forall x \in D : f(x) \geqslant m$.

We say that function f is bounded on its domain D if $f(D)$ is bounded i.e. $\exists A \in \mathbb{R}, \forall x \in D : |f(x)| \leqslant A$.

Remark 4.3. *If f is bounded on D, then $f(D)$ admits a supremum M and an infimum m. We denote $M = \sup\limits_{x \in D} f(x)$, and $m = \inf\limits_{x \in D} f(x)$. We have*

$$\sup_{x \in D} f(x) = M < +\infty \Longleftrightarrow \begin{cases} \forall x \in D : f(x) \leqslant M \\ \forall \varepsilon > 0, \exists x_0 \in D : f(x_0) > M - \varepsilon \end{cases}$$

$$\inf_{x \in D} f(x) = m > -\infty \Longleftrightarrow \begin{cases} \forall x \in D : f(x) \geqslant m \\ \forall \varepsilon > 0, \exists x_0 \in D : f(x_0) < m + \varepsilon \end{cases}$$

Monotone functions

Definition 4.10. *Consider $f : D \subset \mathbb{R} \to \mathbb{R}$, and set $M \subset D$.*

1) f is nondecreasing on $M \Leftrightarrow \forall x_1, x_2 \in M,\ x_1 \leqslant x_2 \Longrightarrow f(x_1) \leqslant f(x_2)$

2) $f$is nonincreasing on $M \Leftrightarrow \forall x_1, x_2 \in M,\ x_1 \leqslant x_2 \Longrightarrow f(x_1) \geqslant f(x_2)$

3) f is increasing on $M \Leftrightarrow \forall x_1, x_2 \in M,\ x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$

4) f is decreasing on $M \Leftrightarrow \forall x_1, x_2 \in M,\ x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$

Definition 4.11. *If f satisfies any of condition $(1-4)$ we call it monotone. If f has property (3) or (4), we call it strictly monotone.*

Corollary 4.1. *We can check that in case where f is increasing or decreasing then we have $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$, so f is injective.*
A sum of two increasing (decreasing) functions is an increasing (decreasing) function.

Inverse functions

Definition 4.12. *Let $f : D(f) \longleftrightarrow \mathbb{R}$ be an injective function with range $H(f)$. Inverse function of f (denoted f^{-1}) is defined by the relation $y = f(x) \Longleftrightarrow x = f^{-1}(y)$. Obviously the domain $D(f^{-1}) = H(f)$ and range $H(f^{-1}) = D(f)$.*

Remark 4.4. *(i) Graph of f^{-1} is symmetric to the graph of f with respect to a line $y = x$.*
(ii) $\forall x \in D(f) : f^{-1}(f(x)) = x$.
(iii)$\forall y \in D(f^{-1}) = H(f) : f(f^{-1}(y)) = y$.
(iv) $(f^{-1})^{-1} = f$.

Lemma 4.2. *Let A, B be two subset of $\mathbb{R}$. Let $f : A \longrightarrow B$ be a bejective and strictly monotonic function. The f^{-1} is strictly monotonic function, the same monotonic as f.*

Proof. WLOG, we can suppose that f is strictly increasing. let y_1, y_2 be two elements of B such that $y_1 < y_2$, then we prove that $f^{-1}(y_1) < f^{-1}(y_2)$. Since f is bejective then there exist x_1, x_2 such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Let us proceed by contra-positive, we suppose that $x_1 \geq x_2$ then $y_1 = f(x_1) \geq f(x_2) = y_2$ which is a contradiction with $y_1 < y_2$. □

Lemma 4.3. *Let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a strictly monotonic function such that $f(I)$ is an interval, then f is necessary continuous on I.*

Proof. WLOG, we can suppose that I is not a trivial interval $I = \emptyset$, $I = \{a\}$ and we suppose that f is strictly increasing. Let $a \in I$, if $a > inf(I)$, we prove that $\lim\limits_{x \to a^-} f(x) = f(a)$ and if $a < sup(I)$, we prove that $\lim\limits_{x \to a^+} f(x) = f(a)$. Indeed let $a \in I$, if $a > inf(I)$, by monotonic limit theorem, there exists a real l such that $\lim\limits_{x \to a^-} f(x) = l$, and $l \leq f(a)$. Actually, $l = sup\{f(x), x \in I, x < a\}$, we aim to prove that $l = f(a)$. By contrapositive, suppose that $l < f(a)$, then there exists $l < m < f(a)$, there exists $\alpha \in I$ such that $\alpha < a$. By the hypothesis that f is increasing, we get $f(\alpha) \leq l < m < f(a)$. $f(I)$ is an interval, so there exists $c \in I$ such that $f(c) = m$.

- If $c \geq a$, and f is supposed to be increasing, hence $f(c) \geq f(a) > m$,
- If $c < a$, monotonic limit gives $f(c) < l < m$, which is a contradiction, thus $f(a) = l$.

Similarly, with the right limit. □

Lemma 4.4. *Let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a continuous and injective function. Then f is strictly monotonic.*

Proof. By contrapositive principal, we suppose that f isn't strictly monotone, thus

- $\exists x, y \in I : x < y$ and $f(x) \geq f(y)$
- $\exists x', y' \in I : x' < y'$ and $f(x') \leq f(y')$

The segments $[x, x'], [y, y']$ are defined by $[x, x'] = \{tx + (1-t)x', t \in [0,]\}$ and $[y, y'] = \{ty + (1-t)y', t \in [0,1]\}$. Then let us define the following functions:

$$\alpha : [0,1] \longrightarrow \mathbb{R} : t \to tx + (1-t)x',$$

$$\beta : [0,1] \longrightarrow \mathbb{R} : t \to ty + (1-t)y'.$$

$\alpha(t)$ and $\beta(t)$ belong to the interval I. Now we consider the function defined by

$$\phi : [0,1] \longrightarrow \mathbb{R} : t \to f(\alpha(t)) - f(\beta(t)).$$

(1) α, β, f are continuous so ϕ,
(2) $\phi(0) = f(x) - f(y) \geq 0$ and $\phi(1) = f(x') - f(y') \leq 0$. The mean theorem value implies that there exists $t_0 \in]0,1[$ such that $\phi(t_0) = 0$, which means that $f(\alpha(t_0) = f(\beta(t_0)$. Contradiction (f is injective). □

The inverse function theorem for strictly monotonic function

Theorem 4.5. *Let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. Set $J = f(I)$. Then two of the following properties implie the third one.*

1- J is an interval and $f : I \longrightarrow J$ is a bejection function.

2- f is strictly monotonic on I.

3- f is continuous on I.

more; if 1, 2 and 3 are satisfied, then the inverse function $f^{-1} : J \longrightarrow I$ is continuous, strictly monotonic, the same as f.

Proof.
- If 1 and 2 are satisfied then f is continuous (Lemma (4.26)).
- If 1 and 3 are satisfied then f is strictly monotonic (Lemma (4.27)).

- If 2 and 3 are satisfied then J is an interval (MTV theorem). f is strictly monotonic, thus f is injective and by the way bejective.

If 1, 2 and 3 are satisfied the by (Lemma (4.25)), $f^{-1} : J \longrightarrow I$ is strictly monotonic, the same as f. $f^{-1} : J \longrightarrow I$ realize a bejection from J on I, so f^{-1} satisfies 1 and 2, hence f^{-1} is continuous. $\square$

Inverse image

Let $f : A \longrightarrow B$ be a function, and let $U \subset B$ be a subset. The inverse image (or, preimage) of U is the set $f^{-1}(U) \subset A$ consisting of all elements $a \in A$ such that $f(a) \in U$.

The inverse image commutes with all set operations: For any collection $\{U_i\}_{i\in I}$ of subsets of B, we have the following identities for

(1) Unions:

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}\left(U_i\right)$$

(2) Intersections:

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}\left(U_i\right)$$

and for any subsets U and V of B, we have identities for

(3) Complements:

$$\left(f^{-1}(U)\right)^{\complement} = f^{-1}\left(U^{\complement}\right)$$

(4) Set differences:

$$f^{-1}(U\backslash V) = f^{-1}(U)\backslash f^{-1}(V)$$

(5) Symmetric differences:

$$f^{-1}(U\triangle V) = f^{-1}(U)\triangle f^{-1}(V)$$

In addition, for $X \subset A$ and $Y \subset B$, the inverse image satisfies the miscellaneous identities

(6) $\left(f|_X\right)^{-1}(Y) = X \cap f^{-1}(Y)$

(7) $f\left(f^{-1}(Y)\right) = Y \cap f(A)$

(8) $X \subset f^{-1}(f(X))$, with equality if f is injective. Let $f : A \longrightarrow B$ be a function, and

let $U \subset A$ be a subset. The direct image (or, simply, image) of U is the set $f(U) \subset B$ consisting of all elements of B which equal $f(u)$ for some $u \in U$.

Direct images satisfy the following properties:

(1) Unions: For any collection $\{U_i\}_{i\in I}$ of subsets of A,

$$f\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f\left(U_i\right).$$

(2) Intersections: For any collection $\{U_i\}_{i\in I}$ of subsets of A,

$$f\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f\left(U_i\right).$$

(3) Set difference: For any $U, V \subset A$,

$$f(V\backslash U) \supset f(V)\backslash f(U).$$

In particular, the complement of U satisfies $f\left(U^{\complement}\right) \supset f(A)\backslash f(U)$.

(4) Subsets: If $U \subset V \subset A$, then $f(U) \subset f(V) \subset B$.

(5) Inverse image of a direct image: For any $U \subset A$,

$$f^{-1}(f(U)) \supset U$$

with equality if f is injective.

(6) Direct image of an inverse image: For any $V \subset B$,

$$f\left(f^{-1}(V)\right) \subset V$$

with equality if f is surjective.

Local maximum, local minimum Local maximum and minimum are the points of the functions, which give the maximum and minimum range. The local maxima and local minima can be computed by finding the derivative of the function. The first derivative test and the second derivative test are the two important methods of finding the local maximum and local minimum.

Definition 4.13. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and $x_0 \in E$.*

– x_0 is said to be local maximum of f if there exists $\alpha > 0$ such that f is nondecreasing on $]x_0 - \alpha, x_0[$ and nonincreasing on $]x_0, x_0 + \alpha[$.

– x_0 is said to be local minimum of f if there exists $\alpha > 0$ such that f is nonincreasing on

$]x_0-\alpha, x_0[$ *and nondecreasing on* $]x_0, x_0+\alpha[$.

Order relation on $\mathcal{F}(D,\mathbb{R})$.

Definition 4.14. *Let $f, g : D \subset \mathbb{R} \to \mathbb{R}$.*
1) f is said to be positive $f \geqslant 0$, if : $\forall x \in D : f(x) \geqslant 0$ (resp. negative, $f \leqslant 0$ if $\forall x \in D : f(x) \leqslant 0$.
2) f is said to be greater then $g, (f \geqslant g)$, if: $\forall x \in D : f(x) \geqslant g(x)$.
3) f is said to be smaller then $g, (f \leqslant g)$, if: $\forall x \in D : f(x) \leqslant g(x)$.

Remark 4.5. *We can easily check that this order relation isn't total.*

4.2 Limit of a Function

The basic idea underlying the concept of the limit of a function f at a point x_0 is to study the behaviour of f at points close to, but not equal to, x_0. We illustrate this with the following simple examples. Suppose that the velocity $v(m/s)$ of a falling object is given as a function $v = v(t)$ of time t. If the object hits the ground in $t = 2$, then $v(2) = 0$. Thus to find the velocity at the time of impact, we investigate the behaviour of $v(t)$ as t approaches 2, but is not equal to 2. Neglecting air resistance, the function $v(t)$ is given as follows :

$$v(t) = \begin{cases} 32t, 0 \leqslant t < 2, \\ 0, t \geqslant 2. \end{cases}$$

Our intuition should convince us that $v(t)$ approaches $64m/s$ as t approaches 2 , and that this is the velocity upon impact. As another example, consider the function $f(x) = xsin\left(\frac{1}{x}\right)$, $x \neq 0$. Here the function f is not defined at $x = 0$. Thus to investigate the behaviour of f at 0 we need to consider the values $f(x)$ for x close to, but not equal to 0 . Since

$$|f(x)| = \left|xsin\left(\frac{1}{x}\right)\right| \leqslant |x|$$

for all $x \neq 0$, our intuition again should tell us that $f(x)$ approaches 0 as x approaches 0. We now make this idea of (x) approaching a value L as x approaches a point x_0 precise. In order that the definition be meaningful, we must require that the point p be a limit point of the domain of the function f .

Definition 4.15. *Let $x_0 \in \mathbb{R}$ an accumulation point of a subset $D \neq \phi \subset \mathbb{R}$ and $f : D \mapsto \mathbb{R}$ a function defined on a neighbourhood of x_0 except may be at x_0. The function f has a limit at x_0 if there exists $l \in \mathbb{R}$ such that*

$$\forall \varepsilon > 0, \exists \delta (x_0, \varepsilon) > 0, \forall x : 0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

Customary we write: $\lim_{x \to x_0} f(x) = l$ or $f(x) \to l$, $x \to x_0$
Shortly

$$\lim_{x \to x_0} f(x) = l \Leftrightarrow \forall \varepsilon > 0, \exists \delta (x_0, \varepsilon) > 0, \forall x : 0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

Uniqueness of the limit

Theorem 4.6. *If f admits a limit then it is unique.*

Proof. Just do it. □

One-Sided Limits
It is possible for a function to fail to have a limit at a point and yet appear to have limits on one side. If we ignore what is happening on the right for a function, perhaps it will have a " left-hand limit." This is easy to achieve

Definition 4.16. *We say that a function f has right-hand limit (resp. left-hand limit) at x_0 if $\forall \varepsilon > 0, \exists \delta > 0, \forall x : x_0 < x < x_0 + \delta$ (resp. $x_0 - \delta < x < x_0$) $\Rightarrow |f(x) - l| < \varepsilon$*
we note

$$\lim_{x \to x_0 + 0} f(x) = f(x_0 + 0), \lim_{x \to x_0 - 0} f(x) = f(x_0 - 0).$$

We have

$$\lim_{x \to x_0} f(x) = l \Longleftrightarrow \lim_{x \to x_0 + 0} f(x) = \lim_{x \to x_0 - 0} f(x) = l$$

Limit at infinity

Definition 4.17. *Let $x_0 = +\infty$ be an accumulation point of a given subset D. Then*

$$\lim_{x \to +\infty} f(x) = l \Leftrightarrow \forall \varepsilon > 0, \exists A > 0, \forall x > A \Rightarrow |f(x) - l| < \varepsilon$$

Let $x_0 = -\infty$ *be an accumulation point of a given subset* D*. Then*

$$\lim_{x\to-\infty} f(x) = l \Leftrightarrow \forall\varepsilon > 0, \exists A > 0, \forall x < -A \Rightarrow |f(x) - l| < \varepsilon$$

Infinite limits

Definition 4.18. *Let* $x_0 \in \mathbb{R}$ *be an accumulation point of a given subset* D *and* $f : D \to \mathbb{R}$*. Then*

$$\lim_{x\to x_0} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists\delta > 0, \forall x : 0 < |x - x_0| < \delta \Rightarrow f(x) > A$$

$$\lim_{x\to x_0} f(x) = -\infty \Leftrightarrow \forall A > 0, \exists\delta > 0, \forall x : 0 < |x - x_0| < \delta \Rightarrow f(x) < -A$$

Main limit's theorems

Our first theorem allows us to reduce the question of the existence of the limit of a function to one concerning the existence of limits of sequences. As we will see, this result will be very useful in subsequent proofs, and also in showing that a given function does not have a limit at a point x_0.

Theorem 4.7. *Let* $f : D \subset \mathbb{R} \to \mathbb{R}$ *be a given function and* x_0 *be an accumulation point of a given subset* D*. Then*
1) $\lim\limits_{x\to x_0} f(x) = l$
if and only if
2) For all sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in D/\{x_0\}$ *and* $\lim\limits_{n\to+\infty} x_n = x_0$ *we obtain* $\lim\limits_{n\to+\infty} f(x_n) = l$, *(*$l$ *finite or not.)*

Proof. Since x_0 is a limit point of D, there exists a sequence $x_n \in D$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} x_n = x_0$.

Suppose $\lim\limits_{x\to x_0} f(x) = l$. Let x_n be any sequence in D with $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} x_n = x_0$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to x_0} f(x) = l$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{for all} \quad x \in D, 0 < |x - x_0| < \delta. \tag{4.1}$$

Since $\lim\limits_{n\to+\infty} x_n = x_0$, for the above δ, there exists a positive integer n_0 such that $|x_n - x_0| < \delta$, for all $n \geqslant n_0$. Thus if $n \geqslant n_0$, by (4.1), $|f(x_n) - l| < \varepsilon$. Therefore $\lim\limits_{n\to+\infty} f(x_n) = l$.
Conversely, suppose that $\mathop{lim}\limits_{n\to+\infty} f(x_n) = l$ for every sequence $x_n \in D$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} x_n = x_0$. Suppose $\lim\limits_{x\to x_0} f(x) \neq l$. Then there exists an $\varepsilon > 0$ such that for

every $\delta > 0$, there exists an $x \in D$ with $0 < |x - x_0| < \delta$ and $|f(x) - l| \geqslant \varepsilon$. For each $n \in \mathbb{N}$, take $\delta = \frac{1}{n}$. Then for each n, there exists $x_n \in D$ such that

$$|x_n - x_0| < \frac{1}{n} \quad \textit{and} \quad |f(x_n) - l| \geqslant \varepsilon.$$

Thus $x_n \longrightarrow x_0$, but $f(x_n)$ does not converge to l. This contradiction proves the result.

□

Some limit laws We now state some limit laws for functions.

Theorem 4.8. *Let $f, g : D \subset \mathbb{R} \to \mathbb{R}$ be real functions, and x_0 an accumulation (cluster) point of D. Suppose that $\lim\limits_{x \to x_0} f(x) = l_1$, $\lim\limits_{x \to x_0} g(x) = l_2$, $l_1, l_2 \in \mathbb{R}$.*

Then

1) $\lim\limits_{x \to x_0} (f(x) \pm g(x)) = l_1 + l_2$

2) $\lim\limits_{x \to x_0} (\lambda f(x)) = \lambda l_1 \ (\lambda \in \mathbb{R})$

3) $\lim\limits_{x \to x_0} (f(x) g(x)) = l_1 l_2$

4) $\lim\limits_{x \to x_0} |f(x)| = |l|$

5) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{l_1}{l_2}$, *if* $l_2 \neq 0$.

Proof. The proofs are left as an exercises. (To prove the results, use the sequential criterion for limits and the limits laws for sequences). □

Theorem 4.9. *Let $f, g : D \subset \mathbb{R} \to \mathbb{R}$ be functions such that $\forall x \in D : f(x) \leqslant g(x)$. Suppose that $\lim\limits_{x \to x_0} f(x) = l_1$, and $\lim\limits_{x \to x_0} g(x) = l_2$. then $l_1 \leqslant l_2$.*

Proof. Set $F(x) = g(x) - f(x)$ and $L = l_2 - l_1$. It is sufficient to prove that $L \geqslant 0$. We prove the contrapositive. Suppose then that $L < 0$. Let $\varepsilon > 0$ be such that $L + \varepsilon < 0$. Then since $\lim\limits_{x \to x_0} (g - f)(x) = L$, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $(g - f)(x) = L + \varepsilon < 0$. Hence, $(g - f)(x) < 0$ for some $x \in D$. We can give another proof using the sequential criterion for limits. □

Corollary 4.2. *$f : D \subset \mathbb{R} \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Suppose that $M_1 \leqslant f(x) \leqslant M_2$ for all $x \in D$ and suppose that $\lim\limits_{x \to x_0} f(x) = L$. Then $M_1 \leqslant L \leqslant M_2$.*

Corollary 4.3. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function and x_0 be an accumulation point of D. If $\lim\limits_{x \to x_0} f(x) = l \in \mathbb{R}$, then $\exists \delta > 0$ such that for $0 < |x - x_0| < \delta$ the functionf is bounded.*

Theorem 4.10. *Let $f, g : D \subset \mathbb{R} \to \mathbb{R}$ be a functions and x_0 be an accumulation point of D. If g is bounded on D and $\lim\limits_{x \to x_0} f(x) = 0$, then $\lim\limits_{x \to x_0} f(x) g(x) = 0$.*

Proof. Use squeeze technique. □

Theorem 4.11. *Suppose that the limit $\lim\limits_{x \to x_0} f(x) = l$, exists. Then $\lim\limits_{x \to x_0} |f(x)| = |l|$.*

Since maxima and minima can be expressed in terms of absolute values, there is a corollary that is sometimes useful.

Corollary 4.4. *Suppose that the limits $\lim\limits_{x \to x_0} f(x) = L$, and $\lim\limits_{x \to x_0} g(x) = M$, exist and that x_0 is a point of accumulation of $D_f \cap D_g$. Then $\lim\limits_{x \to x_0} max\{f(x), g(x)\} = max\{L, M\}$, and $\lim\limits_{x \to x_0} min\{f(x), g(x)\} = min\{L, M\}$.*

Proof. The first follows from the identity $max\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$ and the theorem on limits of sums and the theorem on limits of absolute values. In the same way the second assertion follows from $min\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$. □

4.3 Continuous functions

Let f be a function defined on an interval $[a, b]$. We shall now consider the behaviour of f at points of $[a, b]$.

Continuity at a Point

Definition 4.19. *A function f is said to be continuous at a point $x_0, a < x_0 < b$, if*

$$\lim_{x \to x_0} f(x) = f(x_0)$$

In other words, the function is continuous at x_0, if for each $\varepsilon > 0, \exists \delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon, \text{ when } |x - x_0| < \delta$$

Definition 4.20. *A function f is said to be continuous from the left at x_0, if*

$$\lim_{x \to x_0 - 0} f(x) = f(x_0)$$

Also f is continuous from the right at x_0, if

$$\lim_{x \to x_0 + 0} f(x) = f(x_0)$$

Clearly a function is continuous at x_0 if and only if it is continuous from the left as well as from the right.

Definition 4.21. *A function f defined on a closed interval $[a, b]$ is said to be continuous at the end point a if it is continuous from the right at a,*

i.e.,

$$\lim_{x \to a+0} f(x) = f(a)$$

Also the function is continuous at the end point b of $[a, b]$ if

$$\lim_{x \to b-0} f(x) = f(b)$$

Thus a function f is continuous at a point x_0 if

(i) $\lim_{x \to x_0} f(x)$ exists, and

(ii) limit equals the value of the function at $x = x_0$.

Continuity in an Interval

A function f is said to be continuous in an interval $[a, b]$ if it is continuous at every point of the interval.

Discontinuous Functions

A function is said to be discontinuous at a point x_0 of its domain if it is not continuous there. The point x_0 is then called a point of discontinuity of the function.

4.4 Types of discontinuities

(i) A function f is said to have a removable discontinuity at $x = x_0$ if $\lim_{x \to x_0} f(x)$ exists but is not equal to the value $f(x_0)$ (which may or may not exist) of the function. Such a discontinuity can be removed by assigning a suitable value to the function at $x = x_0$.

(ii) f is said to have a discontinuity of the first kind at $x = x_0$ if $\lim_{x \to x_0 - 0} f(x)$ and $\lim_{x \to x_0 + 0} f(x)$ both exist but are not equal.

(iii) f is said to have a discontinuity of the first kind from the left at $x = x_0$ if $\lim_{x \to x_0 - 0} f(x)$ exists but is not equal to $f(x_0)$

Discontinuity of the first kind from the right is similarly defined.

(iv) f is said to have a discontinuity of the second kind at $x = x_0$ if neither $\lim_{x \to x_0 - 0} f(x)$ nor $\lim_{x \to x_0 + 0} f(x)$ exists.

(v) f is said to have a discontinuity of the second kind from the left at $x = x_0$ if $\lim_{x \to x_0 - 0} f(x)$ does not exist.

Discontinuity of the second kind from the right may be defined similarly.

Theorems on Continuity

Theorem 4.12. *A function f defined on an interval I is continuous at a point $x_0 \in I$ if for every sequence $\{c_n\}$ in I converging to x_0, we have*

$$\lim_{n \to \infty} f(c_n) = f(x_0)$$

Proof. First let us suppose that the function f is continuous at a point $x_0 \in I$, and $\{c_n\}$ is a sequence in I such that $\lim_{n \to \infty} c_n = x_0$.

Since f is continuous at x_0, therefore, for any given $\varepsilon > 0, \exists$ a $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon, \text{ when } 0 < |x - x_0| < \delta \tag{4.2}$$

Again, since $\lim_{n \to \infty} c_n = x_0$, therefore, $\exists$ a positive integer m, such that

$$|c_n - x_0| < \delta, \forall n \geq m \tag{4.3}$$

From (4.2), putting $x = c_n$, we have

$$\begin{aligned} & |f(c_n) - f(x_0)| < \varepsilon, \text{ when } |c_n - x_0| < \delta \\ \Rightarrow \quad & |f(c_n) - f(x_0)| < \varepsilon, \quad \forall n \geq m \text{ [using 2]} \\ \Rightarrow \quad & \text{the sequence } \{f(c_n)\} \text{ converges to } f(x_0) \end{aligned}$$

or

$$\lim_{n \to \infty} f(c_n) = f(x_0)$$

Let us now suppose that f is not continuous at x_0, we shall now show that though there exists a sequence (c_n) in I converging to x_0 yet the sequence $(f(c_n))$ does not converge to $f(x_0)$.

Since f is not continuous at x_0, therefore, there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in I$, such that

$$|f(x) - f(x_0)| \geq \varepsilon, \text{ when } |x - x_0| < \delta$$
$$\lim c_n = x_0.$$

Also by taking $\delta = 1/n$, we find that for each positive integer n, there is a $c_n \in I$, such that

$$|f(c_n) - f(x_0)| \geq \varepsilon, \text{ when } |c_n - x_0| < \frac{1}{n}$$

Thus, the sequence $(f(c_n))$ does not converge to $f(x_0)$, while the sequence (c_n) converges to x_0. $\square$

Remark 4.6. *If* $\lim c_n = x_0 \Rightarrow \lim f(c_n) \neq f(x_0)$, *then* f *is not continuous at* x_0.

Theorem 4.13. *If* f, g *be two functions continuous at a point* x_0, *then the functions* $f + g, f - g, fg$ *are also continuous at* x_0 *and if* $g(x_0) \neq 0$, *then* f/g *is also continuous at* x_0.

The proof is left as an exercise (use the sequential criterion for limits and the laws for sequences).

Example 4.1. *Examine the following function for continuity at the origin:*

$$f(x) = \begin{cases} \frac{xe^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Now

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{xe^{1/x}}{1 + e^{1/x}} = 0$$

and

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x}{e^{-1/x} + 1} = 0$$

Thus,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = \lim_{x \to 0} f(x) = 0$$

Also

$$\lim_{x \to 0} f(x) = f(0)$$

Thus, the function is continuous at the origin.

Example 4.2. *Show that the function defined as:*

$$f(x) = \begin{cases} \frac{\sin 2x}{x}, & \textit{when } x \neq 0 \\ 1, & \textit{when } x = 0 \end{cases}$$

has removable discontinuity at the origin.

Solution. Now

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 = 2$$

so that

$$\lim_{x \to 0} f(x) \neq f(0)$$

Hence, the limit exists, but is not equal to the value of the function at the origin. Thus the function has a removable discontinuity at the origin.

Note. The discontinuity can be removed by re-defining the function at the origin such as $f(0) = 2$.

Example 4.3. *Show that the function defined by*

$$f(x) = \begin{cases} x \sin 1/x, & \textit{when } x \neq 0 \\ 0, & \textit{when } x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution. Now

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0$$

so that

$$\lim_{x \to 0} f(x) = f(0)$$

Hence, f *is continuous at* $x = 0$.

Example 4.4. *A function* f *is defined on* $[0, 1]$ *by*

$$f(x)=\begin{cases}-x^2 & \textit{if } x\le 0\\ 5x-4 & \textit{if } 0<x\le 1\\ 4x^2-3x & \textit{if } 1<x<2\\ 3x+4 & \textit{if } x\ge 2\end{cases}$$

Examine f for continuity at $x=0,1,2$. Also discuss the kind of discontinuity, if any.
Solution. Now

$$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\left(-x^2\right)=0$$
$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}(5x-4)=-4$$

so that

$$\lim_{x\to 0-} f(x)=f(0)\neq\lim_{x\to 0+} f(x)$$

Thus the function has a discontinuity of the first kind from the right at the origin.
Again

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(5x-4)=1$$
$$\lim_{x\to 1+} f(x)=\lim_{x\to 1+}\left(4x^2-3x\right)=1$$

so that

$$\lim_{x\to 1-} f(x)=\lim_{x\to 1+} f(x)=1=f(1)$$
$$\lim_{x\to 1} f(x)=f(1)$$

Thus the function is continuous at $x=1$.
Again

$$\lim_{x\to 2-} f(x)=\lim_{x\to 2-}\left(4x^2-3x\right)=10$$
$$\lim_{x\to 2+} f(x)=\lim_{x\to 2+}(3x+4)=10$$

Also $f(2)=10$
$\Rightarrow$

$$\lim_{x\to 2} f(x)=f(2)$$

Thus, the function is continuous at $x = 2$.

Example 4.5. *Is the function f, where $f(x) = \frac{x-|x|}{x}$ continuous?*

Solution. For $x < 0, f(x) = \frac{x+x}{x} = 2$, continuous.

For $x > 0, f(x) = \frac{x-x}{x} = 0$, continuous. The function is not defined at $x = 0$.

Thus, $f(x)$ is continuous for all x except zero.

Example 4.6. *Discuss the kind of discontinuity, if any of the function defined as follows:*

$$f(x) = \begin{cases} \frac{x-|x|}{x}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0 \end{cases}$$

Solution. The function is continuous at all points except possibly the origin.

Let us test at $x = 0$.

Now

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x+x}{x} = 2$$
$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x-x}{x} = 0$$

and

$$f(0) = 2$$

Thus the function has discontinuity of the first kind from the right at $x = 0$.

Example 4.7. *If $[x]$ denotes the largest integer $\leq x$, then discuss the continuity at $x = 3$ for the function*

$$f(x) = x - [x], \quad \forall x \geq 0$$

Solution. Now

$$\lim_{x \to 3-} f(x) = \lim_{x \to 3-} \{x - [x]\} = 3 - 2 = 1$$
$$\lim_{x \to 3+} f(x) = \lim_{x \to 3+} \{x - [x]\} = 3 - 3 = 0$$

and

$$f(3) = 0$$

Thus the function has a discontinuity of the first kind from the left at $x = 3$.

Note. The function is continuous at all points except the integer value $1, 2, 3, \ldots$

Example 4.8. *Prove that the Dirichlet's function f defined on* $\mathbf{R}$ *by*

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational} \end{cases}$$

is discontinuous at every point.

Solution. First, let a be any rational number so that $f(a) = -1$.

Since in any interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n, such that $|a_n - a| < \frac{1}{n}$.

Thus, the sequence $\{a_n\}$ converges to a. But $f(a_n) = 1$ for all n, and $f(a) = -1$, so that

$$\lim_{n \to \infty} f(a_n) \neq f(a)$$

Thus, the function is discontinuous at any rational number a.

Hence the function is discontinuous at all rational points.

Next, let b be any irrational number. For each positive integer n we can choose a rational number b_n, such that $|b_n - b| < \frac{1}{n}$. Thus, the sequence (b_n) converges to b.

But $f(b_n) = -1$ for all n and $f(b) = 1$.

Therefore $\lim_{n \to \infty} f(b_n) \neq f(b)$.

Hence, the function is discontinuous at all irrational points.

Example 4.9. *Show that the function $f(x)$ defined on $\mathbb{R}$ by*

$$f(x) = \begin{cases} \ \ x, \text{ when } x \text{ is irrational} \\ -x, \text{ when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$.

Solution. First, let $a \neq 0$ be any rational number, so that $f(a) = -a$. Since in every interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n such that

Thus the sequence (a_n) converges to a.

$$|a_n - a| < \frac{1}{n}$$

But

$$\lim_{n\to\infty} f\left(a_n\right) = \lim_{n\to\infty} a_n = a$$

Thus

$$\lim_{n\to\infty} f\left(a_n\right) \neq f(a), a \neq 0$$

so that, the function is discontinuous at any rational number, other than zero.

In a similar way the function may be shown to be discontinuous at every irrational point.

It may be seen from above, that the function is continuous at $x = 0$ (i.e., $a = 0$). However, it can be shown to be continuous at $x = 0$ as follows:

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$ (or any $\delta < \varepsilon$), then

$$|x| < \delta \Rightarrow |f(x) - f(0)| = |-x| = |x| < \varepsilon, \text{ when } x \text{ is rational and}$$
$$|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \varepsilon, \text{ when } x \text{ is irrational.}$$

Thus

$$|x| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$$

or

$$\lim_{x\to 0} f(x) = f(0)$$

Hence, the function is continuous at $x = 0$.

4.4.1 Continuous functions on closed intervals

We shall now study some properties of functions which are continuous on closed intervals. In fact, we shall show that a function which is continuous on a closed interval, is bounded, attains its bounds and assumes every value between the bounds.

Theorem 4.14. *If a function is continuous in a closed interval, then it is bounded therein.*

Proof. Let f be a function defined and continuous in a closed interval I.

We shall show that if the function f is not bounded, then it fails to be continuous at some point of the closed interval I

Let, if possible, f be not bounded above, so that for each positive integer $n \exists$ a point $x_n \in I$ such that $f(x_n) > n$

Now $\{x_n\}$, being a sequence in the closed interval I, is bounded and has at least one limit point, say ξ.

A closed interval is a closed set and so $\xi \in I$.

Further, since ξ is a limit point of the sequence (x_n), therefore, there exists (Bolzano-Weirestrass theorem) a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to \xi$ as $k \to \infty$.

Also since $f(x_{n_k}) > n_k$, for all k, therefore the sequence $(f(x_{n_k}))$ diverges to ∞.

Thus, there exists a point ξ of I such that a sequence (x_{n_k}) in I converges to ξ, but

$$\lim_{k \to \infty} f(x_{n_k}) \neq f(\xi)$$

Thus, f is not continuous at ξ, which is a contradiction and hence the function is bounded above. By considering a function $-f$, it can be shown in a similar way that the function f is also bounded below. Hence, the function is bounded. □

Theorem 4.15. *If a function f is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$.*

Proof. If f is a constant function, then evidently, it attains its bounds at every point of the interval.

Let f be a function which is not a constant.

Since f is continuous on the closed interval $[a, b]$, therefore, it is bounded. Let m and M be the infimum and supremum of f. It is to be shown that $\exists$ points, α, β of $[a, b]$ such that

$$f(\alpha) = m, \quad f(\beta) = M$$

Let us consider the case of the supremum.

Suppose f does not attain the supremum M so that the function does not take the value M for any point $x \in [a, b]$, i.e.,

$$f(x) \neq M, \text{ for any } x \in [a, b]$$

Now consider the function

$$g(x) = \frac{1}{M - f(x)}, \quad \forall x \in [a, b]$$

which is positive for all values of x in $[a, b]$.

Evidently the function g is continuous and so bounded in $[a,b]$.

Let $k(>0)$ be its supremum

$$\begin{aligned} & \frac{1}{M-f(x)} \leq k, \forall x \in [a,b] \\ \Rightarrow \quad & f(x) \leq M - \frac{1}{k}, \forall x \in [a,b] \end{aligned}$$

which contradicts the hypothesis that M is the supremum of f in $[a,b]$. Hence, our supposition that f does not attain the value M leads to a contradiction and therefore f attains its supremum for at least one value in $[a,b]$.

It may similarly be shown that the function also attains its infimum m.

Hence, the function attains its bounds at least once in $[a,b]$.

□

Note It may be observed from the two preceding theorems, that the function f, continuous on the closed interval $[a,b]$, has the least and the greatest values m and M, i.e., the range set of f is bounded with m and M as its smallest and greatest elements. Thus the range set of f is a subset of $[m,M]$. We shall, in fact, show later that the range set of f is $[m,M]$ itself and that f takes up every value between m and M.

4.4.2 Examples

1. The function $f(x) = \frac{1}{1+|x|}$, for real x, is continuous and bounded and attains its supremum for $x=0$ but does not attain the infimum.

2. The function $f(x) = -\frac{1}{1+|x|}$, for all $x \in \mathbb{R}$, is continuous and bounded, attains its infimum but not the supremum.

3. The function $f(x) = x$, for all $x \in]0,1[$ is continuous and bounded but attains neither the infimum nor the supremum.

Theorem 4.16. *If a function f is continuous at an interior point c of an interval $[a,b]$ and $f(c) \neq 0$, then $\exists$ a $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$, for every $x \in]c-\delta, c+\delta[$.*

Proof. Since the function f is continuous at an interior point c of $[a,b]$, therefore for any $\varepsilon > 0, \exists$ a $\delta > 0$, such that

$$|f(x) - f(c)| < \varepsilon, \quad \forall x \in]c-\delta, c+\delta[$$

or

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \quad \forall x \in]c - \delta, c + \delta[$$

When $f(c) > 0$, taking ε to be less than $f(c)$, we find that

$$f(x) > 0, \forall x \in]c - \delta, c + \delta[$$

When $f(c) < 0$, taking ε to be less than $-f(c)$, we find that

$$f(x) < 0, \forall x \in]c - \delta, c + \delta[$$

Hence the theorem.

□

Corollary 4.5. *If f is continuous at the end point b of $[a, b]$ and $f(b) \neq 0$, then there exists an interval $]b - \delta, b]$, such that $f(x)$ has the sign of $f(b)$ for all x in $]b - \delta, b]$*

A similar result holds for continuity at a.

Note When c is an interior point of the interval, the theorem may be restated as:

Theorem 4.17. *If a function f is continuous at an interior point c of an interval $[a, b]$ and $f(c) \neq 0$, then there exists a neighbourhood N of c wherein $f(x)$ has the same sign as $f(c)$, for all $x \in N$.*

Theorem 4.18. *(intermediate value theorem) If a function f is continuous on a closed interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one point $\alpha \in]a, b[$ such that $f(\alpha) = 0$.*

Proof. Let us suppose that $f(a) > 0$ and $f(b) < 0$.

Let S consist of those points of $[a, b]$ for which $f(x)$ is positive, i.e.,

$$S = \{x : a \leq x \leq b \wedge f(x) > 0\}$$

Now

$$f(a) > 0 \Rightarrow a \in S \Rightarrow S \neq \phi$$

Also S is bounded by a and b.

By the order completeness property, S has the supremum, say α, where $a \leq \alpha \leq b$. We shall now show that

(i) $\alpha \neq a, \alpha \neq b$, and

(ii) $f(\alpha) = 0$

(i) First we show that $\alpha \neq a$

Since $f(a) > 0$, therefore $\exists$ a $\delta > 0$ such that

$$f(x) > 0, \quad \forall x \in [a, a+\delta[$$

$\Rightarrow \quad [a, a+\delta[\subseteq S$

$\Rightarrow$ the supremum α of S is greater than or equal to $a + \delta$

$\Rightarrow \alpha \neq a$

Now we shall show that $\alpha \neq b$.

Since $f(b) < 0$, therefore $\exists$ a $\delta > 0$ such that

$$f(x) < 0, \quad \forall x \in]b-\delta, b]$$

$\Rightarrow b - \delta$ is an upper bound of S

$\Rightarrow \alpha \leq b - \delta \Rightarrow \alpha \neq b$

(ii) We shall now show that $f(\alpha) \not> 0$ and $f(\alpha) \not< 0$.

If $f(\alpha) > 0$, then $\exists$ a $\delta > 0$ such that

$$f(x) > 0, \quad \forall x \in]\alpha-\delta, \alpha+\delta[$$
$$\Rightarrow \quad]\alpha-\delta, \alpha+\delta[\subseteq S$$

Let us choose a positive $\delta_2 < \delta$ such that $\alpha + \delta_2 \in]\,\alpha - \delta, \alpha + \delta\,[\Rightarrow$ a member $\alpha + \delta_2$ of S is greater than the supremum α of S, which is a contradiction.

Therefore

$$f(\alpha) \not> 0$$

Let now $f(\alpha) < 0$, so that $\exists$ a $\delta_1 > 0$ such that

$$f(x) < 0, \quad \forall x \in]\alpha - \delta_1, \alpha + \delta_1[\tag{4.4}$$

Again, since α is the supremum of S, therefore, there exists a member β of S, where $\alpha - \delta_1 < \beta \leq \alpha$ such that

$$f(\beta) > 0$$

But from (4.4), $f(\beta) < 0$, which is a contradiction.

Therefore $\quad f(\alpha) \nless 0$

Thus it follows that $f(\alpha) = 0$.

$\square$

Theorem 4.19. *If a function f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then it assumes every value between $f(a)$ and $f(b)$*

Proof. Let A be any number between $f(a)$ and $f(b)$. We shall show that there exists a number $c \in]a, b[$ such that $f(c) = A$. Consider a function ϕ defined on $[a, b]$ such that

$$\phi(x) = f(x) - A$$

Clearly $\phi(x)$ is continuous on $[a, b]$.

Also

$$\phi(a) = f(a) - A, \text{ and } \phi(b) = f(b) - A$$

so that $\phi(a)$ and $\phi(b)$ are of opposite signs.

Thus the function ϕ is continuous on $[a, b]$ and $\phi(a)$ and $\phi(b)$ are of opposite signs; therefore, by the previous theorem, $\exists c \in]a, b[$ such that

$$\Rightarrow$$

$$\begin{aligned} \phi(c) &= 0 \\ f(c) - A &= 0 \Rightarrow f(c) = A, \end{aligned}$$

$\square$

Corollary 4.6. *A function f, which is continuous on a closed interval $[a, b]$, assumes every value between its bounds.*

Proof. Since the function f is continuous on the closed interval $[a, b]$, therefore, it is bounded and attains its bounds on $[a, b]$, i.e., $\exists$ two numbers α, β in $[a, b]$ such that

$$f(\alpha) = M \text{ and } f(\beta) = m,$$

where M and m are, respectively, the supremum and the infimum of f.

Since f is continuous on $[a, b]$, therefore, it is continuous on $[\beta, \alpha]$ or $[\alpha, \beta]$ as the case may be, and consequently assumes every value between $f(\alpha)$ and $f(\beta)$.

Thus the function assumes every value between its bounds.

We may sum up in other words:

The range of a continuous function whose domain is a closed interval is as well a closed interval. Or, in still better words:

The image of a closed interval under a continuous function (mapping) is a closed interval.

$\square$

4.4.3 Uniform continuity

Let f be a function defined on an interval I. Then by definition, the function is continuous at any point $x_0 \in I$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon, \text{ when } |x - x_0| < \delta.$$

For continuity at any other point $d \in I$, for the same ε, a $\delta_1 > 0$ would exist (not necessarily equal to δ). There is in fact a δ corresponding to each point of I. The number δ in general depends on the selection of ε and the point x_0. However, if a δ could be found which depends only on ε and not on the selection of the point x_0, such a δ would work for the whole interval I on which f is continuous. In such a case, f is said to be uniformly continuous on I. Thus, the notion of uniform continuity is global in character in as much as we talk of uniform continuity only on an interval.

The notion of continuity is, however, local in character in as much as we can talk of continuity at a point.

It may seem to a beginner that the infimum of the set consisting of δ 's corresponding to different points of I would work for the whole of I. But the infimum may be zero. In general, therefore, a δ which may work for the entire interval may not exist, so that every continuous function may not be uniformly continuous.

Definition 4.22. *A function f defined on an interval I is said to be uniformly continuous on I if to each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x_2) - f(x_1)| < \varepsilon, \textit{ for arbitrary points } x_1, x_2 \textit{ of } I$$
$$\textit{for which } |x_1 - x_2| < \delta$$

Theorem 4.20. *A function which is uniformly continuous on an interval is continuous on that interval.*

Let a function f be uniformly continuous on an interval I, so that for a given $\varepsilon > 0$, there corresponds a $\delta > 0$ such that

$|f(x_1) - f(x_2)| < \varepsilon$, where x_1, x_2 are any two points of I for which

$$|x_1 - x_2| < \delta$$

Let $x \in I$, then on taking $x_1 = x$, we find that for $\varepsilon > 0, \exists \delta > 0$ such that

$$|f(x) - f(x_2)| < \varepsilon, \text{ when } |x - x_2| < \delta.$$

Hence the function is continuous at every point $x_2 \in I$, i.e., the function f is continuous on I.

Theorem 4.21. *A function which is continuous on a closed interval is also uniformly continuous on that interval.*

Proof. Let a function f be continuous on a closed interval I. Let, if possible, f be not uniformly continuous on I. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there are numbers $x, y \in I$ for which

$$|f(x) - f(y)| \nless \varepsilon, \text{ when } |x - y| < \delta$$

In particular for each positive integer n, we can find real numbers x_n, y_n in I such that

$$|f(x_n) - f(y_n)| \nless \varepsilon, \text{ when } |x_n - y_n| < 1/n$$

Now (x_n) and (y_n) being sequences in the closed interval I, they are bounded and so each has at least one limit point, say ξ and η respectively.

As a closed interval is a closed set,

therefore

$$\xi \in I, \eta \in I$$

Further since ξ is a limit point of (x_n), there exists a convergent subsequence (x_{n_k}) of (x_n), such that $x_{n_k} \to \xi$.

Similarly, there exists a convergent subsequence (y_{n_k}) of (y_n) such that $y_{n_k} \to \eta$.

Again from above, we find that

$$|f(x_{n_k}) - f(y_{n_k})| \not< \varepsilon, \text{ when } |x_{n_k} - y_{n_k}| < 1/n_k \leq 1/k$$

The second inequality shows that

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k}$$
$$\xi = \eta$$

From the first inequality we find that in case the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ converge, the limits to which they converge are different.

We thus have two sequences (x_{n_k}) and (y_{n_k}) both of which converge to ξ but $(f(x_{n_k}))$ and $(f(y_{n_k}))$ do not converge to the same limit.

So f is not continuous at ξ, for, otherwise, the two sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ would converge to the same point $f(\xi)$.

Thus we arrive at a contradiction and so the hypothesis that f is not uniformly continuous on I is false.

Hence f is uniformly continuous on I. □

Theorem 4.22. *If $f, g : (a,b) \longrightarrow \mathbb{R}$ are both uniformly continuous, then $f+g$ and $f-g$ (as functions from (a,b) to $\mathbb{R}$) are also uniformly continuous. $fg : (a,b) \longrightarrow \mathbb{R}$ is uniformly continuous.*

Theorem 4.23. *Let f and g be uniformly continuous on $\mathbb{R}$. Then their composition $f \circ g$ is also uniformly continuous on $\mathbb{R}$.*

Definition 4.23. *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined to be Lipschitz if there exists a constant $K > 0$ such that for all $a, b \in \mathbb{R}$, we have $|f(a) - f(b)| \leq K|a-b|$*

Theorem 4.24. *A Lipschitz function is uniformly continuous. A periodic and continuous function is uniformly continuous.*

Example 4.10. *Show that the function $f(x) = 1/x$ is not uniformly continuous on $]0,1]$.*

Solution. Clearly the function is continuous on $]0,1]$.

It will be uniformly continuous on the given interval if for a given $\varepsilon > 0, \exists$ a $\delta > 0$, independent of the choice of points x and c in $]0,1]$, such that

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon, \text{ when } |x-c| < \delta$$

or

$$\left|\frac{c-x}{cx}\right| < \varepsilon, \text{ when } c-\delta < x < c+\delta \tag{4.5}$$

If we take $c = \delta$*, then the interval* $]c-\delta, c+\delta[$ *becomes* $]0, 2\delta[$*. Also condition (4.5) must hold for any* x *in this interval.*

But

$$\frac{\delta - x}{\delta x} \rightarrow \infty \text{ as } x \rightarrow 0,$$

i.e., if we choose x *sufficiently close to zero, then condition (4.5)) is violated.*

Hence $1/x$ *is not uniformly continuous on* $]0, 1]$.

Example 4.11. *Show that the function* $f(x) = x^2$ *is uniformly continuous on* $[-1, 1]$.

Solution. Let x_1, x_2 *be any two points of* $[-1, 1]$*, then*

$$|f(x_1) - f(x_2)| = \left|x_1^2 - x_2^2\right| = |x_1 - x_2| \cdot |x_1 + x_2| < \varepsilon$$
$$\text{when } |x_1 - x_2| < \frac{1}{2}\varepsilon = \delta$$

(where δ *is independent of the choice of* x_1, x_2 *).*

Thus for any $\varepsilon > 0, \exists$ *a* $\delta = \frac{1}{2}\varepsilon$ *such that for any choice of* x_1, x_2 *in* $[-1, 1]$*, we have*

$$|f(x_1) - f(x_2)| < \varepsilon, \text{ when } |x_1 - x_2| < \frac{1}{2}\varepsilon = \delta$$

Thus the function f *is uniformly continuous on* $[-1, 1]$.

Inverse functions

Lemma 4.25. *Let* A, B *be two subset of* $\mathbb{R}$*. Let* $f : A \longrightarrow B$ *be a bejective and strictly monotonic function. The* f^{-1} *is strictly monotonic function, the same monotonic as* f.

Proof. WLOG, we can suppose that f is strictly increasing. let y_1, y_2 be two elements of B such that $y_1 < y_2$, then we prove that $f^{-1}(y_1) < f^{-1}(y_2)$. Since f is bejective then there exist x_1, x_2 such that $f(x_1) = y_1$and $f(x_2) = y_2$. Let us proceed by contra-positive, we suppose that $x_1 \geq x_2$ then $y_1 = f(x_1) \geq f(x_2)y_2$ which is a contradiction with $y_1 < y_2$. □

Lemma 4.26. *let* I *be an interval of* $\mathbb{R}$ *and* $f : I \Longrightarrow \mathbb{R}$ *be a monotonic function such that* $f(I)$ *be an interval, then* f *is necessary continuous on* I.

Proof. WLOG, we can suppose that I is not a trivial interval $I = \emptyset, \quad I = a$ and we suppose that f is strictly increasing. Let $a \in I$, if $a > inf(I)$, we prove that $\lim_{x \to a^-} f(x) = f(a)$ and if $a < sup(I)$, we prove that $\lim_{x \to a^+} f(x) = f(a)$. Indeed let $a \in I$, if $a > inf(I)$, by monotonic limit theorem, there exists a real l such that $\lim_{x \to a^-} f(x) = l$, such that $l < f(a)$. Actually, $l = sup\{f(x), x \in I, x < a\}$, we aim to prove that $l = f(a)$. By contrapositive, suppose that $l < f(a)$, then there exists $l < m < f(a)$, there exists $\alpha \in I$ such that $\alpha < a$. By the hypothesis that f is increasing, we get $f(\alpha) \leq l < m < f(a)$. $f(I)$ is an interval, so there exists $c \in I$ such that $f(c) = m$.

- If $c \geq a$, and f is supposed to be increasing, hence $f(c) \geq f(a) > m$,
- If $c < a$, monotonic limit gives $f(c) < f(a) < m$, which is a contradiction, thus $f(a) = l$.

Similarly, with the right limit. □

Lemma 4.27. *let I be an interval of $\mathbb{R}$ and $f : I \Longrightarrow \mathbb{R}$ be a continuous and injective function. Then f is strictly monotonic.*

Proof. By contrapositive principal, we suppose that f isn't strictly monotone, thus

- $\exists x, y \in I : x < y$ and $f(x) \geq f(y)$
- $\exists x', y' \in I : x' > y'$ and $f(x') \leq f(y')$

The segments $[x, x'], [y, y']$ are defined by $[x, x'] = \{tx + (1-t)x', t \in [0,]\}$ and $[y, y'] = \{ty + (1-t)y', t \in [0,1]\}$. Then let us define the following functions:

$$\alpha : [0,1] \longrightarrow \mathbb{R} : t \to tx + (1-t)x', \quad \beta : [0,1] \longrightarrow \mathbb{R} : t \to ty + (1-t)y'.$$

$\alpha(t) and \beta(t)$ belong to the interval I. Now we consider the function defined by

$$\phi : [0,1] \longrightarrow \mathbb{R} : t \to f(\alpha(t)) - f(\beta(t)).$$

(1) α, β, f are continuous so ϕ,
(2) $\phi(0) = f(x) - f(y) \geq 0$ and $\phi(1) = f(x') - f(y') \leq 0$. The mean theorem value implies that there exists $t_0 \in]0,1[$ such that $\phi(t_0) = 0$, which means that $f(\alpha(t_0) = f(\beta(t_0)$. Contradiction (f is injective). □

The inverse function theorem for strictly monotonic function

Theorem 4.28. *let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. Set $J = f(I)$. Then two of the following properties implie the third one.*

1- J is an interval and $f : I \longrightarrow J$ is a bejection function.

2- f is strictly monotonic on I.

3- f is continuous on I.

more; if 1, 2 and 3 are satisfied, then the inverse function $f^{-1} : J \longrightarrow I$ is continuous, strictly monotonic, the same as f.

Proof.

- If 1 and 2 are satisfied then f is continuous (Lemma (4.26)).
- If 1 and 3 are satisfied then f is strictly monotonic (Lemma (4.27)).
- If 2 and 3 are satisfied then J is an interval (MTV theorem). f is strictly monotonic, thus f is injective and by the way bejective.

If 1, 2 and 3 are satisfied the by (Lemma (4.25)), $f^{-1} : J \longrightarrow I$ is strictly monotonic, the same as f. $f^{-1} : J \longrightarrow I$ realize a bejection from J on I, so f^{-1} satisfies 1 and 2, hence f^{-1} is continuous. □

The fact that the domain of f must be an interval is a necessary condition, see for example the following.

Example 4.12. *Let $g : [0, 1] \cup]2, 3] \longrightarrow [0, 2]$ defined by*

$$g(x) = \begin{cases} x, \text{ if } 0 \leq x \leq 1 \\ x - 1, \text{ if } 2 < x \leq 3 \end{cases}$$

The inverse is

$$g^{-1}(x) = \begin{cases} y, \text{ if } 0 \leq y \leq 1 \\ y + 1, \text{ if } 1 < y \leq 2 \end{cases}$$

and it is not continuous because of a jump at $y = 1$.

Theorem 4.29. *Let f be a continuous and increasing function on the interval $[a, b]$. Let $\alpha = f(a)$, and $\beta = f(b)$. Then*

(1) The image of $[a, b]$ by f is equal to the interval $[\alpha, \beta]$ ($f([a, b]) = [\alpha, \beta]$)

(2) There exists an inverse function $x = g(y)$ of f continuous and increasing on $[\alpha, \beta]$.

Remark 4.7. *The inverse of a decreasing and continuous function f on $[a, b]$ is a decreasing and continuous function on $[\alpha, \beta]$, where $\alpha = f(a)$, $\beta = f(b)$. One can consider the function $-f$.*

Big O and Little o Notation-Bachmann–Landau notation

It is often useful to talk about the rate at which some function changes as its argument grows (or shrinks), without worrying to much about the detailed form. This is what the O($\cdot$) and o($\cdot$) notation lets us do. Let x_0 be an accumulation of a subset D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$.

Definition 4.24. *We say that f is negligible compared to g or is ultimately smaller than, when $x \to x_0$ and we note $f = o(g)$(little o) if :*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x : |x - x_0| < \delta \Rightarrow |f(x)| < \varepsilon |g(x)|$$

Corollary 4.7. *It results from definition (4.24) that if g does not vanish on x_0 neighbourhood's then:*

$$f = o(g)(x \to x_0) \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

If $g = 1$, then

$$f = o(1)(x \to x_0) \Leftrightarrow \lim_{x \to x_0} f(x) = 0$$

Definition 4.25. *Let f and g be two functions defined on the interval $[a, +\infty[$. we set by definition $f = \circ(g)(x \to +\infty) \Leftrightarrow \forall \varepsilon > 0, \exists A > 0, \forall x : x > A \Rightarrow |f(x)| \leqslant \varepsilon |g(x)|$*

Definition 4.26. *Let x_0 be an accumulation of a subset D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$. f is said to be equivalent to g, when $x \to x_0$ and we note by $f \sim g$ if $f - g = \circ(1)$, $x \to x_0$.*

Remark 4.8. *If $\forall x \in D/\{x_0\}$: $g(x) \neq 0$. then $f \sim g$, $x \to x_0 \Longleftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$.*

Example 4.13. $\lim_{x \to 0} \frac{a^x - 1}{x \ln a} = \lim_{y \to 0} \left(\frac{e^y - 1}{y} \right) = 1$, $y = x \ln a \Longleftrightarrow a^x - 1 = x \ln a + \circ(x)$, $x \to 0 \Longleftrightarrow a^x - 1 \sim x \ln a$, $x \to 0$.

Theorem 4.30. *Let $x \in D/\{x_0\} : g(x) \neq 0$, $g_1(x) \neq 0$ and $g \sim g_1$, $x \to x_0$. Then for all function $f : D \to \mathbb{R}$, one has*

$$\lim_{x \to x_0} (f(x) g(x)) = \lim_{x \to x_0} (f(x) g_1(x))$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)}{g_1(x)}.$$

Definition 4.27. *Let x_0 be an accumulation of a subset D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$. We say that, g is "of the same order" as f, and they "grow at the same rate", or "shrink at the same rate", we say also f is dominated by g when $x \to x_0$ and we write $f = O(g)$ if*

$\exists M > 0, \exists \delta > 0, \forall x : |x - x_0| < \delta \Rightarrow |f(x)| \leqslant M |g(x)|.$

If $x_0 = +\infty$, then f is dominated by g when $x \to +\infty$ if $\exists M > 0, \exists A > 0, \forall x : x > A \Rightarrow |f(x)| \leqslant M |g(x)|.$

Remark 4.9. *IF $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} \in \mathbb{R}$ (exists) then $f = O(g)$ $(x \to x_0)$.*

Remark 4.10. *Reminder that, the notation $f = O(1)$ on D means that f is bounded on D.*

CHAPTER

5

DIFFERENTIABLE FUNCTIONS

We begin with the definition of the derivative of a function.

Definition 5.1. *Let $I \subset \mathbb{R}$ be an interval and let $x_0 \in I$. We say that $f : I \to \mathbb{R}$ is differentiable at x_0 or has a derivative at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. We say that f is differentiable on I if f is differentiable at every point in I.

By definition, f has a derivative at x_0 if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \varepsilon.$$

Derivative function

If f is differentiable at x_0, we will denote $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ by $f'(x_0)$, that is,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The rule that sends x_0 to the number $f'(x_0)$ defines a function on a possibly smaller subset $J \subset I$. The function $f' : J \to \mathbb{R}$ is called the derivative of f.

Example 5.1. *Let $f(x) = 1/x$ for $x \in (0, \infty)$. Prove that $f'(x) = -\frac{1}{x^2}$*

Example 5.2. *Let $f(x) = \sin(x)$ for $x \in \mathbb{R}$. Prove that $f'(x) = \cos(x)$.*

Solution

Recall that

$$\sin(x) - \sin(x_0) = 2 \sin\left(\frac{x - x_0}{2}\right) \cos\left(\frac{x + x_0}{2}\right)$$

and that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$. Therefore,

$$\begin{aligned}
\lim_{x \to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{2 \sin\left(\frac{x - x_0}{2}\right) \cos\left(\frac{x + x_0}{2}\right)}{x - x_0} \\
&= \lim_{x \to x_0} \left(\frac{\sin\left(\frac{x - x_0}{2}\right)}{\frac{x - x_0}{2}}\right) \cos\left(\frac{x + x_0}{2}\right) \\
&= 1 \cdot \cos(x_0) = \cos(x_0).
\end{aligned}$$

Hence $f'(x_0) = \cos(x_0)$ for all x_0 and thus $f'(x) = \cos(x)$.

Example 5.3. *Prove by definition that $f(x) = \frac{x}{1+x^2}$ is differentiable on $\mathbb{R}$.*

Solution

$$\begin{aligned}
\frac{f(x) - f(x_0)}{x - x_0} &= \frac{\frac{x}{1+x^2} - \frac{x_0}{1+x_0^2}}{x - x_0} \\
&= \frac{x\left(1 + x_0^2\right) - x_0\left(1 + x^2\right)}{\left(1 + x^2\right)\left(1 + x_0^2\right)\left(x - x_0\right)} \\
&= \frac{1 - x_0 x}{\left(1 + x_0^2\right)\left(1 + x^2\right)}.
\end{aligned}$$

Now

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \frac{1 - x_0^2}{\left(1 + x_0^2\right)^2}.$$

Hence, $f'(x_0)$ exists for all $x_0 \in \mathbb{R}$ and the derivative function of f is

$$f'(x) = \frac{1 - x^2}{\left(1 + x^2\right)^2}.$$

Example 5.4. *Prove that $f'(x) = \alpha$ if $f(x) = \alpha x + b$.*

Solution

We have that $f(x) - f(x_0) = \alpha x - \alpha x_0 = \alpha(x - x_0)$. Therefore, $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \alpha$. This proves that $f'(x) = \alpha$ for all $x \in \mathbb{R}$.

Example 5.5. *Compute the derivative function of $f(x) = |x|$ for $x \in \mathbb{R}$.*

Solution If $x > 0$ then $f(x) = x$ and thus $f'(x) = 1$ for $x > 0$. If $x < 0$ then $f(x) = -x$ and therefore $f'(x) = -1$ for $x < 0$. Now consider $x_0 = 0$. We have that

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{|x|}{x}.$$

We claim that the limit $\lim_{x\to 0} \frac{|x|}{x}$ does not exist and thus $f'(0)$ does not exist. To see this, consider $x_n = 1/n$. Then $(x_n) \to 0$ and $f(x_n) = 1$ for all n. On the other hand, consider $y_n = -1/n$. Then $(y_n) \to 0$ and $f(y_n) = -1$. Hence, $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, and thus the claim holds by the Sequential criterion for limits. The derivative function f' of f is therefore defined on $A = \mathbb{R}\backslash\{0\}$ and is given by

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x > 0. \end{cases}$$

Hence, even though f is continuous at every point in its domain $\mathbb{R}$, it is not differentiable at every point in its domain. In other words, continuity is not a sufficient condition for differentiability.

Right-hand derivative, left-hand derivative

Let $f : I \longmapsto \mathbb{R}$ be function and suppose that there exists $\delta > 0 :]x_0 - \delta, x_0] \subset I$.

Definition 5.2. *If $\lim\limits_{x\to x_0^-} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists (finite), we say that f is left hand differentiable at x_0, or has a left hand derivative at x_0. Noted by $f'(x_0 - 0)\left(f'_-(x_0)\right)$*

Let $f : I \longmapsto \mathbb{R}$ be function and suppose that there exists $\delta > 0 : [x_0, x_0 + \delta, [\subset I$.

Definition 5.3. *IF $\lim\limits_{x\to x_0^+} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists (finite), we say that f is right hand differentiable at x_0, or has a right hand derivative at x_0. Noted by $f'(x_0 + 0)\left(f'_+(x_0)\right)$.*

Remark 5.1. *A function f is differentiable at x_0 if and only if both the right-hand derivative and left-hand derivative at x_0 exist and both of these derivatives are equal.*

Example 5.6. $f : x \longmapsto |x|$ *defined on* $\mathbb{R}$ *is not differentiable at* 0.

Theorem 5.1. *Suppose that* $f : I \to \mathbb{R}$ *is differentiable at* x_0. *Then* f *is continuous at* x_0.

Proof. To prove that f is continuous at x_0 we must show that $\lim_{x\to x_0} f(x) = f(x_0)$. By assumption $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ exists, and clearly $\lim_{x\to x_0}(x- x_0) = 0$. Hence we can apply the Limits laws and compute

$$\begin{aligned}
\lim_{x\to x_0} f(x) &= \lim_{x\to x_0} (f(x) - f(x_0) + f(x_0)) \\
&= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{(x - x_0)} (x - x_0) + f(x_0) \right) \\
&= f'(x_0) \cdot 0 + f(x_0) \\
&= f(x_0)
\end{aligned}$$

and the proof is complete.

□

Theorem 5.2. *Let* $f : I \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *be differentiable at* $x_0 \in I$. *The following hold:*

(i) If $\alpha \in \mathbb{R}$ *then* (αf) *is differentiable and* $(\alpha f)'(x_0) = \alpha f'(x_0)$.

(ii) $(f + g)$ *is differentiable at* x_0 *and* $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.

(iii) fg *is differentiable at* x_0 *and* $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

(iv) If $g(x_0) \neq 0$ *then* (f/g) *is differentiable at* x_0 *and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Proof. Parts (i) and (ii) are straightforward. We will prove only (iii) and (iv). For (iii), we have that

$$\begin{aligned}
\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0}.
\end{aligned}$$

Now $\lim_{x\to x_0} g(x) = g(x_0)$ because g is differentiable at x_0. Therefore,

$$\lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} g(x) + \lim_{x\to x_0} f(x_0)\frac{g(x)-g(x_0)}{x-x_0}$$
$$= f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

To prove part (iv), since $g(x_0) \neq 0$, then there exist a δ-neighborhood $J = (x_0-\delta, x_0+\delta)$ such that $g(x) \neq 0$ for all $x \in J$. If $x \in J$ then

$$\begin{aligned}
\frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x)}}{x-x_0} &= \frac{f(x)g(x_0) - g(x)f(x_0)}{g(x)g(x_0)(x-x_0)} \\
&= \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - g(x)f(x_0)}{g(x)g(x_0)(x-x_0)} \\
&= \frac{\frac{f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0} - \frac{f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}}{g(x)g(x_0)}
\end{aligned}$$

Since $g(x_0) \neq 0$, it follows that

$$\lim_{x\to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x)}}{x-x_0} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

and the proof is complete.

□

We now move to the Chain Rule.

Theorem 5.3. *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be functions such that $f(I) \subset J$ and let $x_0 \in I$. If $f'(x_0)$ exists and $g'(f(x_0))$ exists then $(g \circ f)'(x_0)$ exists and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.*

Proof. Suppose that there exists a neighborhood of x_0 where $f(x) \neq f(x_0)$. Otherwise, the composite function $(g \circ f)(x)$ is constant in a neighborhood of x_0, and then clearly differentiable at x_0. Consider the function $h : J \to \mathbb{R}$ defined by

$$h(y) = \begin{cases} \frac{g(y)-g(f(x_0))}{y-f(x_0)}, & y \neq f(x_0) \\ g'(f(x_0)), & y = f(x_0). \end{cases}$$

Now

$$\begin{aligned}\lim_{y\to f(x_0)} h(y) &= \lim_{y\to f(x_0)} \frac{g(y)-g(f(x_0))}{y-x_0}\\ &= g'(f(x_0))'\\ &= h(f(x_0)).\end{aligned}$$

Hence, h is differentiable at $f(x_0)$ and therefore h is at $f(x_0)$. Now,

$$\frac{g(f(x))-g(f(x_0))}{x-x_0} = h(f(x))\frac{f(x)-f(x_0)}{x-x_0}$$

and therefore

$$\begin{aligned}\lim_{x\to x_0}\frac{g(f(x))-g(f(x_0))}{x-x_0} &= \lim_{x\to x_0} h(f(x))\frac{f(x)-f(x_0)}{x-x_0}\\ &= h(f(x_0))f'(x_0)\\ &= g'(f(x_0))f'(x_0).\end{aligned}$$

Therefore, $(g\circ f)'(x_0) = g'(f(x_0))f'(x_0)$ as claimed.

$\square$

Example 5.7. *Compute $f'(x)$ if*

$$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x = 0.\end{cases}$$

Where is $f'(x)$ continuous?

Solution

When $x \neq 0, f(x)$ is the composition and product of differentiable functions at x, and therefore f is differentiable at $x\neq 0$. For instance, on $A=\mathbb{R}\backslash\{0\}$, the functions $1/x, \sin(x)$ and x^2 are differentiable at every $x\in A$. Hence, if $x\neq 0$ we have that

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Consider now $x_0 = 0$. If $f'(0)$ exists it is equal to

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{x}\\ &= \lim_{x\to 0} x\sin\left(\frac{1}{x}\right).\end{aligned}$$

Using the Squeeze Theorem, we deduce that $f'(0) = 0$. Therefore,

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

From the above formula obtained for $f'(x)$, we observe that when $x \neq 0$ f' is continuous since it is the product/difference/composition of continuous functions. To determine continuity of f' at $x = 0$ consider $\lim_{x\to 0} f'(x)$. Consider the sequence $x_n = \frac{1}{n\pi}$, which clearly converges to $x_0 = 0$. Now, $f'(x_n) = \frac{2}{n\pi}\sin(n\pi) - \cos(n\pi)$. Now, $\sin(n\pi) = 0$ for all n and therefore $f'(x_n) = -\cos(n\pi) = (-1)^{n+1}$. The sequence $f'(x_n)$ does not converge and therefore $\lim_{x\to 0} f'(x)$ does not exist. Thus, f' is not continuous at $x = 0$.

Example 5.8. *Compute $f'(x)$ if*

$$f(x) = \begin{cases} x^3\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Where is $f'(x)$ continuous?

Solution

When $x \neq 0, f(x)$ is the composition and product of differentiable functions, and therefore f is differentiable at $x \neq 0$. For instance, on $A = \mathbb{R}\backslash\{0\}$, the functions $1/x, \sin(x)$ and x^3 are differentiable on A. Hence, if $x \neq 0$ we have that

$$f'(x) = 3x^2\sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right).$$

Consider now $x_0 = 0$. If $f'(0)$ exists it is equal to

$$\begin{aligned} \lim_{x\to 0}\frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x\to 0}\frac{x^3\sin\left(\frac{1}{x}\right)}{x} \\ &= \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \end{aligned}$$

and using the Squeeze Theorem we deduce that $f'(0) = 0$. Therefore,

$$f'(x) = \begin{cases} 3x^2\sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

When $x \neq 0$, f' is continuous since it is the product/difference/composition of continuous functions. To determine continuity of f' at $x_0 = 0$ we consider the limit $\lim_{x\to 0} f'(x)$. Now $\lim_{x\to 0} 3x^2 \sin\left(\frac{1}{x}\right) = 0$ using the Squeeze Theorem, and similarly $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$ using the Squeeze Theorem. Therefore, $\lim_{x\to 0} f'(x)$ exists and is equal to 0 , which is equal to $f'(0)$. Hence, f' is continuous at $x = 0$, and thus continuous everywhere.

Example 5.9. *Consider the function*

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \in \mathbb{Q}\backslash\{0\} \\ x^2 \cos\left(\frac{1}{x}\right), & x \notin \mathbb{Q} \\ 0, & x = 0. \end{cases}$$

Show that $f'(0) = 0$.

5.0.1 Some theorems

Definition 5.4. *Let* $f : I \to \mathbb{R}$ *be a function and let* $x_0 \in I$.

(i) We say that f *has a relative maximum at* x_0 *if there exists* $\delta > 0$ *such that* $f(x) \leq f(x_0)$ *for all* $x \in (x_0 - \delta, x_0 + \delta)$.

(ii) We say that f *has a relative minimum at* x_0 *if there exists* δ *such that* $f(x_0) \leq f(x)$ *for all* $x \in (x_0 - \delta, x_0 + \delta)$.

A point $x_0 \in I$ *is called a critical point of* $f : I \to \mathbb{R}$ *if* $f'(x_0) = 0$ *. The next theorem says that a relative maximum/minimum of a differentiable function can only occur at a critical point.*

Theorem 5.4. *(* ***Pierre Fermat: 1601-1665)*** *Let* $f : I \to \mathbb{R}$ *be a function and let* x_0 *be an interior point of* I. *Suppose that* f *has a relative maximum (or minimum) at* x_0. *If* f *is differentiable at* x_0 *then* x_0 *is a critical point of* f, *that is,* $f'(x_0) = 0$.

Proof. Suppose that f has a relative maximum at x_0; the relative minimum case is similar. Then for $x \neq x_0$, it holds that $f(x) - f(x_0) \leq 0$ for $x \in (x_0 - \delta, x_0 + \delta)$ and some $\delta > 0$. Consider the function $h : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases}$$

Then the function h is continuous at $x_0 = 0$ because $\lim_{x\to x_0} h(x) = h(x_0)$. Now for $x \in A =$

$(x_0, x_0+\delta)$ it holds that $h(x) \leq 0$ and therefore $f'(x_0) = \lim_{x\to x_0} h(x) \leq 0$. Similarly, for $x \in B = (x_0-\delta, x_0)$ it holds that $h(x) \geq 0$ and therefore $0 \leq f'(x_0)$. Thus $f'(x_0) = 0$. $\square$

Corollary 5.1. *If $f : I \to \mathbb{R}$ has a relative maximum (or minimum) at x_0 then either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.*

Example 5.10. *The function $f(x) = |x|$ has a relative minimum at $x = 0$, however, f is not differentiable at $x = 0$.*

Theorem 5.5. ***(Michele Rolle: 1652-1719)*** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $]a,b[$. If $f(a) = f(b)$ then there exists $x_0 \in (a,b)$ such that $f'(x_0) = 0$.*

Proof. Since f is continuous on $[a,b]$ it achieves its maximum and minimum at some point x^* and x_*, respectively, that is $f(x_*) \leq f(x) \leq f(x^*)$ for all $x \in [a,b]$. If f is constant then $f'(x) = 0$ for all $x \in (a,b)$. If f is not constant then $f(x_*) < f(x^*)$. Since $f(a) = f(b)$ it follows that at least one of x_* and x^* is not contained in $\{a,b\}$, and hence there exists $x_0 \in \{x_*, x^*\}$ such that $f'(x_0) = 0$. $\square$

Remark 5.2. *Rolle remains true even when the interval is open, provided that $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to b^-} f(x)$.*

We now state and prove the main result of this section.

Theorem 5.6. ***(Mean Value theorem: Lagrange 1736-1813)*** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $x_0 \in (a,b)$ such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.*

Proof. If $f(a) = f(b)$ then the result follows from Rolle's Theorem ($f'(x_0) = 0$ for some $x_0 \in (a,b)$). Let $h : [a,b] \to \mathbb{R}$ be the line from $(a, f(a))$ to $(b, f(b)$, that is,

$$h(x) = f(a) + \frac{f(b)-f(a)}{(b-a)}(x-a)$$

and define the function

$$g(x) = f(x) - h(x)$$

for $x \in [a,b]$. Then $g(a) = f(a) - f(a) = 0$ and $g(b) = f(b) - f(b) = 0$, and thus $g(a) = g(b)$. Clearly, g is continuous on $[a,b]$ and differentiable on (a,b), and it is straightforward to verify that $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. By Rolle's Theorem, there exists $x_0 \in (a,b)$ such that $g'(x_0) = 0$, and therefore $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Theorem 5.7. ***Extended mean value theorem: Cauchy theorem 1789-1857)***

Let f, g be two continuous functions on $[a, b]$ and differentiables on $]a, b[$. If g' does not vanish on $]a, b[$, then there exists a point $x_0 \in]a, b[$ such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(x_0)}{g'(x_0)}$.

Proof. Consider the function

$$h(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a)).$$

This is continuous on $[a, b]$ and differentiable on (a, b), with

$$h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)).$$

Note that $h(a) = 0 = h(b)$. By Rolle's Theorem, there a spot x_0 where $h'(x_0) = 0$. □

Remark 5.3. *Lagrange theorem is a special case of Cauchy theorem: $g(x) = x, x \in [\, a, b]$.*

Example 5.11. *Prove that $\forall x > 0 : e^x > 1 + x + \dfrac{x^2}{2}$.*

Proof. Cauchy theorem applied with $f(x) = e^u, g(u) = 1 + u + \dfrac{u^2}{2}, u \in [\, 0, x]$. Then

$$\exists x_0 \in]0, x[: \frac{e^x - e^0}{1 + x + \frac{x^2}{2} - 1} = \frac{e^{x_0}}{1 + x_0} \text{ but } \frac{e^{x_0}}{1 + x_0} > 1, \forall x_0 > 0.$$

So

$$\frac{e^x - 1}{x + \frac{x^2}{2}} > 1 \Rightarrow e^x > 1 + x + \frac{x^2}{2}.$$

□

L'Hospital's Rule And Indeterminate Forms

L'Hospital's Rule (Guillaume L'Hospital (1661-1704)) tells us that if we have an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, all we need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Theorem 5.8. *Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x = a$, with $f(a) = g(a) = 0$. Suppose that f and g are differentiable, and that f' and g' are continuous. Finally, suppose that $g'(a) \neq 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.$$

Also,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

and

$$\lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f'(x)}{g'(x)}$$

Proof. Since that $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then, for any $x, f(x) = f(x) - f(a)$ and $g(x) = g(x) - g(a)$. But then,

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \\
&= \lim_{x\to a} \frac{[f(x) - f(a)]/(x-a)}{[g(x) - g(a)]/(x-a)} \\
&= \frac{\lim_{x\to a}([f(x) - f(a)]/(x-a))}{\lim_{x\to a}([g(x) - g(a)]/(x-a))} \\
&= \frac{f'(a)}{g'(a)},
\end{aligned}$$

since, by definition, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$. Since f' and g' are assumed to be continuous, this is also

$$\frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

□

This version is easy to prove, and is good enough to compute limits like

$$\lim_{x\to 0} \frac{\sin(2x)}{x + x^2}.$$

However, it isn't good enough to compute limits like

$$\lim_{x\to 0} \frac{1 - \cos(2x)}{x^2}$$

since in that case $g'(0) = 0$. To solve problems like the last one, we need the following version.

Theorem 5.9. *Suppose that f and g are continuous on a closed interval $[a, b]$, and are differentiable on the open interval (a, b). Suppose that $g'(x)$ is never zero on (a, b), and that $\lim_{x\to a^+} f'(x)/g'(x)$ exists, and that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Then*

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. By assumption, f and g are differentiable to the right of a, and the limits of f and g as $x \to a^+$are zero. Define $f(a)$ to be zero, and likewise define $g(a) = 0$. Since these values agree with the limits, f and g are continuous on some half-open interval $[a, b)$ and differentiable on (a, b).

For any $x \in (a, b)$, we have that f and g are differentiable on (a, x) and continuous on $[a, x]$. By the extended MVT, there is a point c between a and x such that $f'(c)g(x) = f'(x)g(c)$. In other words, $f'(c)/g'(c) = f(x)/g(x)$. Also, as x approaches a, c also approaches a, since c is somewhere between x and a. But then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(c)}{g'(c)} = \lim_{c\to a^+} \frac{f'(c)}{g'(c)}$$

That last expression is the same as $\lim_{x\to a^+} f'(x)/g'(x)$. □

Note that this theorem doesn't require anything about $g'(a)$, just about how g' behaves to the right of a. An analogous theorem applies to the limit as $x \to a^-$(and requires f and g and f' and g' to be defined on an interval that ends at a, rather than one that starts at a). You can combine the two to get a theorem about an overall limit as $x \to a$.

The conclusion of L'Hôpital's Rule relates one limit (of f/g) to another limit (of f'/g'), and not to the value of $f'(a)/g'(a)$. This is what allows the theorem to be used recursively to solve problems.

The inverse is not always true, see the following

Example 5.12. *We have* $\lim\limits_{x\to 0} \dfrac{x^2 \sin \frac{1}{x}}{\sin x} = \lim\limits_{x\to 0} \dfrac{x^2 \sin \frac{1}{x}}{x} = \lim\limits_{x\to 0} x \sin \dfrac{1}{x} = 0,$

despite that $\lim\limits_{x\to 0} \dfrac{\left(x^2 \sin \frac{1}{x}\right)'}{(\sin x)'} = \lim\limits_{x\to 0} \dfrac{2x \sin \frac{1}{x} - \cos \frac{1}{x}}{\cos x}$ *does not exist.*

Remark 5.4. *If* $\dfrac{f'(x)}{g'(x)}$ *is an indeterminate form such* $\frac{0}{0}$ *and we suppose that* $f'(x)$ *and* $g'(x)$ *satisfy L'Hopital theorem hypothesis, then* $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\dfrac{f''(x)}{g''(x)}$.

Theorem 5.10. *Extended L'Hôpital's Rule Suppose that* f *and* g *are two functions well defined and differentiable on a neighbourhood of a point* a*, may be except at* a *and assume that* $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$*, and that* $g(x) \neq 0$ *et* $g'(x) \neq 0$ *on the candidate neighbourhood. So, if* $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ *exists, then* $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ *exists and* $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

Remark 5.5. *If* $a = \infty$ *the transformation* $x = \dfrac{1}{t}$ *guide us to the case* $a = 0$,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{t\to 0}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t\to 0}\frac{\left(f\left(\frac{1}{t}\right)\right)'}{\left(g\left(\frac{1}{t}\right)\right)'} = \lim_{t\to 0}\frac{-\frac{1}{t^2}f'\left(\frac{1}{t}\right)}{-\frac{1}{t^2}g'\left(\frac{1}{t}\right)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

Theorem 5.11. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b)*. If* $f'(x) = 0$ *for all* $x \in (a, b)$ *then* f *is constant on* $[a, b]$.

Proof. Let $y \in (a, b]$. Now f restricted to $[a, y]$ satisfies all the assumptions needed in the Mean Value Theorem. Therefore, there exists $x_0 \in (a, y)$ such that $f'(x_0) = \frac{f(y)-f(a)}{y-a}$. But $f'(x_0) = 0$ and thus $f(y) = f(a)$. This holds for all $y \in (a, b]$ and thus f is constant on $[a, b]$ □

Corollary 5.2. *If* $f, g : [a, b] \to \mathbb{R}$ *are continuous and differentiable on* (a, b) *and* $f'(x) = g'(x)$ *for all* $x \in (a, b)$ *then* $f(x) = g(x) + C$ *for some constant* C.

The sign of the derivative f' determines where f is increasing or decreasing.

Theorem 5.12. *Suppose that* $f : I \to \mathbb{R}$ *is differentiable.*

(i) Then f *is increasing if and only if* $f'(x) \geq 0$ *for all* $x \in I$.

(ii) Then f *is decreasing if and only if* $f'(x) \leq 0$ *for all* $x \in I$.

Proof. Suppose that f is increasing. Then for all $x, x_0 \in I$ with $x \neq x_0$ it holds that $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$ and therefore $f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \geq 0$. Hence, this proves that $f'(x) \geq 0$ for all $x \in I$.

Now suppose that $f'(x) \geq 0$ for all $x \in I$. Suppose that $x < y$. Then by the Mean Value Theorem, there exists $x_0 \in (x, y)$ such that $f'(x_0) = \frac{f(y)-f(x)}{y-x}$. Therefore, since $f'(x_0) \geq 0$ it follows that $f(y) - f(x) \geq 0$.

Part (ii) is proved similarly. $\square$

Derivative of inverse functions

Theorem 5.13. *Let $f :]a, b[\to \mathbb{R}$, $-\infty \leqslant a < b \leqslant +\infty$ be a function that is both invertible and differentiable. Let $f\left(]a, b[\right) =]c, d[\,(-\infty \leqslant c < d \leqslant +\infty)$ et $f^{-1} :]c, d[\to]a, b[$, the inverse function of f. If f is differentiable at x_0 such that $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0)$ and satisfies $\left(f^{-1}\right)'\left(f(x_0)\right) = \dfrac{1}{f'(x_0)}$.*

Proof. Use composition derivative formula. $\square$

5.1 Higher order derivatives

Let $f :]a, b[\longmapsto \mathbb{R}$ be a differentiable function on $]a, b[$. Let $g = f' :]a, b[\longmapsto \mathbb{R}$ be the derivative function by setting $g(x) = f'(x), \forall x \in]a, b[$.

Definition 5.5. *At $x_0 \in]a, b[$ the derivative of the derivative of a function f if it exists, is called the second derivative of that function and is denoted by one of the symbols $f''(x_0), \dfrac{d^2}{dx^2} f(x_0)$.*

If the derivative of $n \in \mathbb{N}$ exists, then, we denote it by $f^n(x_0)$ or $\dfrac{d^n}{dx^n} f(x_0)$ and if for all $x \in]a, b[$, $f^n(x)$ exists, then the derivative of order $n+1$ is defined by $f^{(n+1)}(x) = \left(f^{(n)}(x)\right)'$, if it exists.

Example 5.13. $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$,

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right), (\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right)$$

By induction we have

For $n = 1$, *one has* $(\sin x)' = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$, *is true.*

Suppose that it is true for n.

So $(\sin x)^{(n+1)} = \left(\sin^{(n)} x\right)' \left(\sin\left(x + n\dfrac{\pi}{2}\right)\right)' = \cos\left(x + n\dfrac{\pi}{2}\right) = \sin\left(x + n\dfrac{\pi}{2} + \dfrac{\pi}{2}\right) = \sin\left(x + (n+1)\dfrac{\pi}{2}\right)$

which is true too.

Technical fact

If $y = f(x)$ is a function of time that describes the position of a moving object, then:

1. The first derivative represents the velocity of the object.

2. The second derivative represents the acceleration of the object.

3. The third derivative represents the jerk of the object.

Definition 5.6. *Let* $A \subset \mathbb{R}$*. Then* $f \in C^{(n)}(A)$*,* $(n \in \mathbb{N}) \Longleftrightarrow \forall x \in A,\ f^{(n)}(x)\,(n \in \mathbb{N})$ *exists and* $f^{(n)} \in C(A)\,(n \in \mathbb{N})$. f *is said to be* n *times continuously differentiable.*

Definition 5.7. $C^{(\infty)}(A) = \bigcap\limits_{n=1}^{\infty} C^{(n)}(A)$

$f \in C^{(\infty)}(A) \Longleftrightarrow \forall n \in \mathbb{N} : f \in C^{(n)}(A)$, f *is said to be infinitely differentiable.*

Example 5.14. *Functions* $\sin x, \cos x, e^x, x \in \mathbb{R}$ *belong to* $C^{(\infty)}(\mathbb{R})$.

Theorem 5.14. *(Leibniz formula) Let* $\{f, g\} \subset C^{(n)}(]a, b[)$ *for* $n \in \mathbb{N}$*. Then* $fg \in C^{(n)}(]a, b[)$*, and we have*

$$(fg)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)}.$$

Proof. By induction. □

Example 5.15. *Calculate* $\left(x^2 \sin 2x\right)^{(10)}$. *One has* $f(x) = x^2, g(x) = \sin 2x$ *and* $f'(x) = 2x,\ f''(x) = 2,\ f'''(x) = ... = f^{10}(x) = 0$

$$\begin{aligned} g'(x) &= 2\sin\left(2x + \frac{\pi}{2}\right) = 2\cos 2x,\ g''(x) = 2^2\sin\left(2x + 2\frac{\pi}{2}\right) = 2^2(-\sin 2x), ..., \\ g^{(10)}(x) &= 2^{10}\sin\left(2x + 10\frac{\pi}{2}\right) = 2^{10}\sin(2x + 5\pi) = -2^{10}\sin 2x \end{aligned}$$

So

$$\begin{aligned} \left(x^2 \sin 2x\right)^{(10)} &= f^{(0)}g^{(10)} + C_{10}^1 f' g^{(9)} + C_{10}^2 f'' g^{(8)} + C_{10}^3 f'' g^{(7)} \\ &= -2^{10}x^2\sin 2x + 10(2x)\,2^9\sin\left(2x + 9\frac{\pi}{2}\right) + \frac{10.9}{2}\sin\left(2x + 8\frac{\pi}{2}\right) \\ &= -2^{10}x^2\sin 2x + 2^{10}10x\cos 2x + 8^8.10.9\sin 2x \\ &= -2^{10}\left(x^2\sin 2x - 10x\cos 2x - \frac{45}{2}\sin 2x\right). \end{aligned}$$

5.2 Taylor Polynomials (Brook Taylor-1685-1731)

Definition 5.8. *Let $x_0 \in [a, b]$ and suppose that $f : [a, b] \to \mathbb{R}$ is such that the derivatives $f'(x_0), f^{(2)}(x_0), f^{(3)}(x_0), \ldots, f^{(n)}(x_0)$ exist for some positive integer n. Then the polynomial*

$$\begin{aligned} P_n(x) = & f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \\ & \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \end{aligned}$$

is called the nth order Taylor polynomial of f based at x_0. Using summation convention, $P_n(x)$ can be written as

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k .$$

By construction, the derivatives of f and P_n up to order n are identical at x_0 :

$$\begin{aligned} P_n(x_0) &= f(x_0) \\ P_n^{(1)}(x_0) &= f^{(1)}(x_0) \\ \vdots &= \quad \vdots \\ P^{(n)}(x_0) &= f^{(n)}(x_0) . \end{aligned}$$

It is reasonable then to suspect that $P_n(x)$ is a good approximation to $f(x)$ for points x near x_0. If $x \in [a, b]$ then the difference between $f(x)$ and $P_n(x)$ is

$$R_n(x) = f(x) - P_n(x)$$

and we call $R_n(x)$ the nth order remainder based at x_0. Hence, for each $x^* \in [a, b]$, the remainder $R_n(x^*)$ is the error in approximating $f(x^*)$ with $P_n(x^*)$. You may be asking yourself why we would need to approximate $f(x)$ if the function f is known and given.

Example 5.16. *If say $f(x) = \sin(x)$ then why would we need to approximate say $f(1) = \sin(1)$ since any basic calculator could easily compute $\sin(1)$? Well, what your calculator is actually computing is an approximation to $\sin(1)$ using a (rational) number such as $P_n(1)$ and using a large value of n for accuracy (although modern numerical algorithms for computing trigonometric functions have superseded Taylor approximations but Taylor approximations are a good start). Taylor's theorem provides an expression for the remainder term $R_n(x)$ using the derivative $f^{(n+1)}$.*

Theorem 5.15. *Let $f:[a,b]\to\mathbb{R}$ be a function such that for some $n\in\mathbb{N}$ the functions $f, f^{(1)}, f^{(2)},\ldots,f^{(n)}$ are continuous on $[a,b]$ and $f^{(n+1)}$ exists on (a,b). Fix $x_0\in[a,b]$. Then for any $x\in[a,b]$ there exists c between x_0 and x such that*

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\left(x-x_0\right)^{n+1}.$$

This is Lagrange form of the remainder.

Proof. If $x=x_0$ then $P_n\left(x_0\right)=f\left(x_0\right)$ and then x_0 can be chosen arbitrarily. Thus, suppose that $x\neq x_0$, let $m=\frac{f(x)-P_n(x)}{\left(x-x_0\right)^{n+1}}$, and define the function $g:[a,b]\to\mathbb{R}$ by

$$g(t) = f(t) - P_n(t) - m\left(t-x_0\right)^{n+1}.$$

Since $f^{(n+1)}$ exists on (a,b) then $g^{(n+1)}$ exists on (a,b). Moreover, since $P^{(k)}\left(x_0\right) = f^{(k)}\left(x_0\right)$ for $k=0,1,\ldots,n$ then $g^{(k)}\left(x_0\right)=0$ for $k=0,1,\ldots,n$. Now $g(x)=0$ and therefore since $g\left(x_0\right)=0$ by Rolle's theorem there exists c_1 in between x and x_0 such that $g'\left(c_1\right)=0$. Now we can apply Rolle's theorem to g' since $g'\left(c_1\right)=0$ and $g'\left(x_0\right)=0$, and therefore there exists c_2 in between c_1 and x_0 such that $g''\left(c_2\right)=0$. By applying this same argument repeatedly, there exists c in between x_0 and c_{n-1} such that $g^{(n+1)}(c)=0$. Now,

$$g^{(n+1)}(t) = f^{(n+1)}(t) - m(n+1)!$$

and since $g^{(n+1)}(c)=0$ then

$$0 = f^{(n+1)}(c) - m(n+1)!$$

from which we conclude that

$$f(x) - P(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\left(x-x_0\right)^{n+1}$$

and the proof is complete. □

Example 5.17. *Consider the function $f:[0,2]\to\mathbb{R}$ given by $f(x)=\ln(1+x)$. Use P_4 based at $x_0=0$ to estimate $\ln(2)$ and give a bound on the error with your estimation.*

Solution

Note that $f(1) = \ln(2)$ and so the estimate of $\ln(2)$ using P_4 is $\ln(2) \approx P_4(1)$. To determine P_4 we need $f(0), f^{(1)}(0), \ldots, f^{(4)}(0)$. We compute

$$\begin{aligned} f^{(1)}(x) &= \frac{1}{1+x} & f^{(1)}(0) &= 1 \\ f^{(2)}(x) &= \frac{-1}{(1+x)^2} & f^{(2)}(0) &= -1 \\ f^{(3)}(x) &= \frac{2}{(1+x)^3} & f^{(3)}(0) &= 2 \\ f^{(4)}(x) &= \frac{-6}{(1+x)^4} & f^{(4)}(0) &= -6. \end{aligned}$$

Therefore,

$$P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4.$$

Now $P_4(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$ and therefore

$$\ln(2) \approx P_4(1) = \frac{7}{12}.$$

The error is $R_4(1) = f(1) - P_4(1)$ which is unknown but we can approximate it using Taylor's theorem. To that end, by Taylor's theorem, for any $x \in [0, 2]$ there exists c in between $x_0 = 0$ and x such that

$$\begin{aligned} R_4(x) &= \frac{f^{(5)}(c)}{5!}x^5 \\ &= \frac{1}{5!}\frac{24}{(1+c)^5}x^4 \\ &= \frac{1}{5(1+c)^5}. \end{aligned}$$

Therefore, for $x = 1$, there exists $0 < c < 1$ such that

$$R_4(1) = \frac{1}{5(1+c)^5}.$$

Therefore, a bound for the error is

$$|R_4(1)| = \left|\frac{1}{5(1+c)^5}\right| \leq \frac{1}{5}$$

since $1 + c > 1$.

Corollary 5.3. *Let $n \in \mathbb{N}$ and $\{a_0, a_1, ..., a_n\} \subset \mathbb{R}$. Then for all $x_0 \in \mathbb{R}$, the polynomial* $p(x) = a_0 + a_1 x^2 + ... + a_n x^n$, $x \in \mathbb{R}$. *(1)*
Can be rewritten by the form
$p(x) = b_0 + b_1 (x - x_0) + b_2 (x - x_0)^2 + ... + b_n (x - x_0)^n, x \in \mathbb{R}$. *(2)*
Where $\{b_0, b_1, ..., b_n\} \subset \mathbb{R}$.

Proof. Set $x = x_0$ in the equation (2), we obtain $b_0 = p(x_0)$. Derivation of (2), we get

$$p'(x) = b_1 + 2b_2 (x - x_0) + ... + nb_n (x - x_0)^{n-1}$$

Then by setting $x = x_0, b_1 = p'(x_0)$.
The second derivative, one has $p''(x) = 2!b_2 + ... + n(n-1) b_n (x - x_0)^{n-2}$ and for $x = x_0$ we obtain $p''(x_0) = 2!b_2$, thus $b_2 = \dfrac{p''(x_0)}{2!}$, In order to determine the others coefficients of (2), we repeat the same technique, we get the general formula

$$b_k = \frac{p^{(k)}(x_0)}{k!} \quad (k = 0, 1, 2, ..., n) \qquad (3)$$

Finally, we obtain Taylor polynomial by introducing coefficients from the equation (3) in the development(2).

$$\begin{aligned} p(x) &= p(x_0) + p'(x_0)(x - x_0) + \frac{p''(x_0)}{2!}(x - x_0)^2 + ... + \frac{p^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{k=0}^{n} \frac{p^{(k)}(x_0)}{k!}(x - x_0)^k \qquad (4) \end{aligned}$$

□

Example 5.18. *Expand the polynomial* $p(x) = x^4 - 5x^3 + 5x^2 + x + 2$ *according to the powers of* $x - 2$.
We have $x_0 = 2$, *then*

$$p'(x) = 4x^3 - 15x^2 + 10x + 1, \; p''(x) = 12x^2 - 30x + 10, \; p'''(x) = 24x - 30, \; p^{(4)}(x) = 24$$

and

$$p(2) = 0, p'(2) = -7, p''(2) = -2, p'''(2) = 18, p^{(4)}(2) = 24$$

so

$$\begin{aligned}
p(x) &= p(2) + \frac{p'(2)}{1!}(x-2) + \frac{p''(2)}{2!}(x-2)^2 + \frac{p'''(2)}{3!}(x-2)^3 + \frac{p^{(4)}(2)}{4!}(x-2)^4 \\
&= -7(x-2) + (-1)(x-2)^2 + 3(x-2)^3 + (x-2)^4 \\
&= -7(x-2) - (x-2)^2 + 3(x-2)^3 + (x-2)^4
\end{aligned}$$

Taylor polynomial with Young remainder or Peano remainder

Let $f :]a,b[\longmapsto \mathbb{R}, \quad x_0 \in]a,b[$, suppose that, for $n \in \mathbb{N}$

1) $\forall x \in]a,b[, \ f^{(n-1)}(x)$ exists

2) $f^{(n)}(x_0)$ exists

Then Taylor polynomial with Young (Peano) form of the remainder. $f(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o((x-x_0)^n), \ x \longrightarrow x_0$.

Remark 5.6. *In the particular case when $x_0 = 0$, we obtain Maclaurin polynomial with Young (Peano) form of the remainder.*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + ... + \frac{f^n(0)}{n!}x^n + o(x^n), \ x \longrightarrow 0.$$

Example 5.19. $f(x) = \sin x$. *We have*

$$\sin^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right), \quad then \quad f^{(n)}(0) = \sin n\frac{\pi}{2} = \begin{cases} 0, & n = 2k \\ (-1)^k, & n = 2k+1 \end{cases} \quad (k \in \mathbb{N})$$

Thus Maclaurin polynomial with Young (Peano) form of the remainder, takes the following form

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right), x \longrightarrow 0.$$

Remark 5.7. *Lagrange form of the remainder can take the form*

$$R_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1}.$$

Example 5.20. *Let* $n \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + r_n(x)$, *where*

$$R_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1} = \frac{e^{\theta x}}{(n+1)!}x^{n+1}, \theta \in \left]0, 1\right[.$$

Example 5.21. *Let* $n \in \mathbb{N}, \forall x \in \mathbb{R}$, $\sin x = x + \frac{x^3}{3!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x)$, *where*

$$R_{2n+1}(x) = (-1)^{n+1} \frac{\sin \theta x}{(2n+2)!}x^{2n+2}, \theta \in \left]0, 1\right[.$$

Example 5.22. *Let* $n \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $\cos x = 1 + \frac{x^2}{2!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x)$, *where*

$$R_{2n}(x) = (-1)^{n+1} \frac{\sin \theta x}{(2n+1)!}x^{2n+1}, \theta \in \left]0, 1\right[.$$

Example 5.23. *Let* $n \in \mathbb{N}^*, \forall x > -1$, $\ln(1+x) = x - \frac{x^2}{2} + ... + (-1)^{n-1} \frac{x^n}{n} + r_n(x)$, *where*

$$R_n(x) = (-1)^n \frac{1}{(n+1)(1+\theta x)^{n+1}}x^{n+1}, \theta \in \left]0, 1\right[.$$

Example 5.24. *Let* $n \in \mathbb{N}$, *for all* $x > -1$,
$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + ... + \frac{\alpha(\alpha-1)...\alpha(\alpha-n+1)}{n!}x^n + R_n(x)$,
where

$$R_n(x) = \frac{\alpha(\alpha-1)...\alpha(\alpha-n)}{(n+1)!}(1+\theta x)^{\alpha-n-1}x^{n-1}, \theta \in \left]0, 1\right[.$$

Local and absolute extrema

An extremum (or extreme value) of a function is a point at which a maximum or minimum value of the function is obtained in some interval. A local extremum (or relative extremum) of a function is the point at which a maximum or minimum value of the function in some open interval containing the point is obtained.

An absolute extremum (or global extremum) of a function in a given interval is the point at which a maximum or minimum value of the function is obtained. Frequently, the interval given is the function's domain, and the absolute extremum is the point corresponding to the maximum or minimum value of the entire function.

Theorem 5.16. *Suppose that* x_0 *is a stationary (critical) point of a given function* f $(i, e : f'(x) = 0)$

and suppose that the second derivative of f is continuous in some neighbourhood of x_0.

If $f''(x_0) < 0$, then f admits a local maximum local at x_0;

If $f''(x_0) > 0$, then f admits a local minimum local at x_0.

Theorem 5.17. *Suppose that $f'(x_0) = f''(x_0) = ... = f^{(n)}(x_0) = 0$ and suppose that $f^{(n+1)}(x) \neq 0$ is continuous in some neighbourhood of x_0. If $n+1$ is even and $f^{(n+1)}(x_0) < 0$, then f admits a local maximum at x_0; If $n+1$ is even and $f^{(n+1)}(x_0) > 0$, then f admits a local minimum at x_0; If $n+1$ is odd, then f does not have any local extrema at x_0.*

Example 5.25. *Let be the function $f(x) = e^x + e^{-x} + 2\cos x$, $f'(x) = e^x - e^{-x} - 2\sin x$, remark that $x = 0$ is a stationary point. Then*

$$f''(x) = e^x + e^{-x} - 2\cos x, f''(0) = 0$$

$$f'''(x) = e^x - e^{-x} + 2\sin x, f'''(0) = 0$$

$$f^{(4)}(x) = e^x + e^{-x} + 2\cos x, f^{(4)}(0) = 4 > 0, (n+1) = 4$$

so $x = 0$ is a local minimum point.

The following theorem is stated under weak conditions then the above one.

Theorem 5.18. *Suppose that the function f on an interval $]x_0 - \delta, x_0 + \delta[$ $(\delta > 0)$ and differentiable on $]x_0 - \delta, x_0[$ and on $]x_0, x_0 + \delta[$. Assume that*

$f''(x) \geqslant 0, (\textit{resp.}, \leqslant 0)$ *on* $]x_0 - \delta, x_0[$ *(3)*

$f''(x) \leqslant 0, (\textit{resp.}, \geqslant 0)$ *on* $]x_0, x_0 + \delta[$ *(4)*

Then the function f admits a local maximum (resp. local minimum) at x_0. Remark that the existence of $f'(x_0)$ isn't mandatory.

Example 5.26. *Find the extrema points of the function $f(x) = x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}$*

First we calculate derivative: $f'(x) = \dfrac{\frac{1}{3} - x}{\sqrt[3]{x^2(1-x)}} = 0$

we can easily check that $x_1 = \dfrac{1}{3}$ is a stationary point. The derivative at points $x_2 = 0$ et $x_3 = 1$ does not exist. Let $0 < \delta < \dfrac{1}{3}$, then:

$f'\left(\frac{1}{3} - \delta\right) > 0,\ f'\left(\frac{1}{3} + \delta\right) < 0$

$f'(-\delta) > 0,\ f'(\delta) > 0$

$f'(1-\delta) < 0,\ f'(1+\delta) > 0$

So for $x_1 = \dfrac{1}{3}$ *the function admits a maximum, for* $x_2 = 0$*, there is no extrema. For* $x_3 = 1$ *the function admits a minimum.*

Bounds of a function

Suppose we have to find the maximum (resp, minimum) of a continuous function on an interval $[a, b]$ three cases only are possible:

1) $x_0 = a$ 2) $x_0 = b$ 3) $x_0 \in]a, b[$

If $x_0 \in]a, b[$, then the function f admits a local extrema at x_0, that is a critical point (either stationary or a point such that the derivative does not exist).

If $\{x_1, ..., x_n\}$, is a finite set, then

$$\max_{x\in[a,b]} f(x) = \max\{f(a), f(b), f(x_1), ..., f(x_n)\}$$
$$\min_{x\in[a,b]} f(x) = \min\{f(a), f(b), f(x_1), ..., f(x_n)\}$$

Example 5.27. *Find local extrema of the function* $f(x) = x^3 - 3x + 3$ *on* $\left[-3, \dfrac{3}{2}\right]$

We have $f'(x) = 3x^2 - 3 = 0 \Leftrightarrow x_1 = -1, x_2 = 1$

Since $f(-1) = 5,\ f(1) = 1,\ f(-3) = -15,\ f\left(\dfrac{3}{2}\right) = \dfrac{15}{8}$

then $\max\limits_{x\in\left[-3,\frac{3}{2}\right]} f(x) = 5$ *et* $\min\limits_{x\in\left[-3,\frac{3}{2}\right]} f(x) = -15.$

Convexity of a curve, point of inflection

There are different types of functions. They are classified according to the categories. One such category is the nature of the graph. Depending upon the nature of the graph, the functions can be divided into two types namely, convex Function and concave Function Both the concavity and convexity can occur in a function once or more than once. The point where the function is neither concave nor convex is known as inflection point or the point of inflection.

Definition 5.9. *If a curve opens in an upward direction or it bends up to make a shape like a cup, it is said to be concave up or convex down. If a curve bends down or resembles a cap, it is known as concave down or convex up. In other words, the tangent lies underneath the curve if the slope of the tangent increases by the increase in an independent variable.*

Remark 5.8. *If* f *is a differentiable function, then when* $f'' \geq 0$*, we have a portion of the graph where the gradient is increasing, so the graph is convex at this section. When* $f'' \leq 0$*, we have a portion of the graph where the gradient is decreasing, so the graph is concave at this section. .*

Definition 5.10. *The point of inflection or inflection point is a point in which the concavity of the function changes. It means that the function changes from concave down to concave up or vice versa. In other words, the point in which the rate of change of slope from increasing to decreasing manner or vice versa is known as an inflection point. Those points are certainly not local maxima or minima, but they are stationary points too.*

Example 5.28. *Let the curve defined by* $y = 1 + \sqrt[3]{x}$.

Check the concavity (convexity) at points $A(-1, 0)$, $B(1, 2)$. *On a* $y''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$, $y''(-1) = \frac{2}{9} > 0$, $y''(1) < 0$, *so at point* A, *the curve is concave up or convex down, and at the point* B, *it is concave down or convex up.*

Corollary 5.4. *If* x_0 *is a point of inflection of a curve* $y = f(x)$ *and if the second derivative* f'' *exists at* x_0, *then one has necessary* $f''(x_0) = 0$.

Theorem 5.19. *If* f *is such that the derivative* f''' *is continuous at* x_0 *and* $f''(x_0) = 0$ *with* $f'''(x_0) \neq 0$, *then the curve* $y = f(x)$ *admits a point of inflection at* x_0.

Example 5.29. *Consider* $y = x^3 - 3x^2 - 9x + 9$. *We have* $y' = 3x^2 - 6x - 9$, $y'' = 6x - 6$. *Then* $y''(1) = 0$ *and* $y'''(1) = 6 \neq 0$. *So* $x = 1$ *is an inflection point.*

Remark 5.9. *A curve can admit an inflection point at a given point* x_0 *despite that* $f''(x_0)$ *does not exist.*

Theorem 5.20. *Let* f *be a function such that:* $f''(x_0) = ... = f^{(n)}(x_0) = 0$, $f^{(n+1)}$ *is continuous at* x_0 *and* $f^{(n+1)}(x_0) \neq 0$, *if* n *is odd, then the curve* $y = f(x)$ *is convex up or convex down according to* $f^{(n+1)}(x_0) < 0$ *or* $f^{(n+1)}(x_0) > 0$. *If* n *is even, then* x_0 *is a point of inflection.*

Example 5.30. *Let* $y = x^5$. *we have* $y'(x) = 5x^4$,

$y'' = 20x^3,$ $\quad y''(0) = 0$

$y'''(x) = 60x^2,$ $\quad y'''(0) = 0$

$y^{(4)}(x) = 60.2x,$ $\quad y^{(4)}(0) = 0$

$y^{(5)}(x) = 120,$ $\quad y^{(5)}(0) = 120 \neq 0$

$n = 4$ *even, then* $x_0 = 0$ *is an inflection point.*

5.3 Convex Functions

Definition 5.11. *A function $f : I = [a, b] \longrightarrow \mathbb{R}$ is said to be convex, if* $\forall(x_1, x_2) \in I, \forall\lambda \in [0, 1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda)f(x_2)$$

This means that for all x_1 and x_2 in I, the line segment between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is always above or on the curve f.

Definition 5.12. *f is said to be concave if $-f$ is convex, which means* $\forall(x_1, x_2) \in I, \forall\lambda \in [0, 1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \geqslant \lambda f(x_1) + (1-\lambda)f(x_2)$$

Definition 5.13. *A function $f : I \longrightarrow \mathbb{R}$ is said to be strictly convex if* $\forall(x_1, x_2) \in I, x_1 \neq x_2, \forall\lambda \in [0, 1]$

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2)$$

Definition 5.14. *We say that $f : I \longrightarrow \mathbb{R}$ is strictly concave if*
$\forall(x_1, x_2) \in I, x_1 \neq x_2, \forall\lambda \in [0, 1]$

$$f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda)f(x_2)$$

Examples:

1. The function $f : x \longrightarrow x^2$ is convex on $\mathbb{R}$

Indeed, for all x, y of I and for all $\lambda \in [0,1]$, we have

$$\begin{aligned} [\lambda x+(1-\lambda)\,y]^2 &\leqslant \lambda x^2+(1-\lambda)\,y^2 \\ &\Longleftrightarrow \\ \lambda^2x^2+2\lambda\,(1-\lambda)\,xy+(1-\lambda)^2\,y^2 &\leqslant \lambda x^2+(1-\lambda)\,y^2 \\ &\Longleftrightarrow \\ \lambda^2\left[-x^2+2xy-y^2\right]-\lambda\left[-x^2+2xy-y^2\right] &\leqslant 0 \\ &\Longleftrightarrow \\ \lambda\,(1-\lambda)\left[-x^2+2xy-y^2\right] &\leqslant 0 \\ &\Longleftrightarrow \\ -\lambda\,(1-\lambda)\ \ (x-y)^2 &\leqslant 0 \end{aligned}$$

which is true.

2. Affine functions are both convex and concave. But not strictly.

3. Exponential function is strictly convex on $\mathbb{R}$.

4. Logarithm function is strictly concave on $\mathbb{R}^+$.

5.4 Convex function properties

Definition 5.15. *Let $f : I \longrightarrow \mathbb{R}$ and $\alpha \in I$, we define a new application $\varphi_\alpha : I\setminus\{\alpha\} \longrightarrow \mathbb{R}$ by :*

$$\varphi_\alpha = \frac{f(x)-f(\alpha)}{x-\alpha}$$

φ_α which is the growth rate.

Lemma 5.21 (Three slopes inequalities)**.** *Suppose that function $f : I \longrightarrow \mathbb{R}$ is convex then for all $(x,y,z) \in I^3$, $x < z < y$, we have*

$$\frac{f(z)-f(x)}{z-x} \leqslant \frac{f(y)-f(x)}{y-x} \leqslant \frac{f(y)-f(z)}{y-z}.$$

Conversely, if for all x, y, z de I such that $x < z < y$, at least one of the two inequalities is assumed, then f is convex.

Proposition 5.1. *A function $f : I \longrightarrow \mathbb{R}$ is convex if and only if, for all $\alpha \in I$ the following function φ_α is nondecreasing.*

Proof. First, Suppose that f is convex, and show that φ_α is nondecreasing.

Let $(x, y, z) \in I^3$ such that $x < y$ and $z \in [x, y]$, there exists $\lambda \in [0, 1]$ such that $z = \lambda x + (1 - \lambda)\, y$. Indeed, we have $\lambda = \frac{y-z}{y-x}$, so

$$f(z) \leqslant \lambda f(x) + (1-\lambda)\, f(y) \iff f(z) - f(y) \leqslant \lambda\,(f(x) - f(y))$$

$$\iff f(z) - f(y) \leqslant \frac{y-z}{y-x}\,(f(x) - f(y))$$

$$\iff \frac{f(y) - f(z)}{y-z} \geqslant \frac{f(y) - f(x)}{y-x}$$

$$\iff$$

$$\varphi_x(y) \leqslant \varphi_z(y) \text{ because } y - z < 0,$$

in other hand $1 - \lambda = \frac{z-x}{y-x}$ then

$$f(z) \leqslant \lambda f(x) + (1-\lambda)\, f(y)$$

$$\iff$$

$$f(z) - f(x) \leqslant (1-\lambda)\,(f(y) - f(x))$$

$$\iff$$

$$f(z) - f(x) \leqslant \frac{z-x}{y-x}\,(f(y) - f(x))$$

$$\iff$$

$$\frac{f(z) - f(x)}{z-x} \leqslant \frac{f(y) - f(x)}{y-x}$$

$$\iff$$

$$\varphi_x(z) \leqslant \varphi_x(y) \text{ because } z - x > 0,$$

So, φ_α is nondecreasing.

By the way, the theorem of three slopes inequalities is proved. □

Conversely. Suppose that the application φ_α is nondecreasing for all $\alpha \in I$, we will show that f is convex.

Let $x, y, z \in I^3$ which $x < z < y$ and $\lambda \in [0, 1]$.

Set $z = \lambda x + (1 - \lambda)\, y$, such that $\lambda = \frac{y-z}{y-x}$ and $1 - \lambda = \frac{z-x}{y-x}$

Since φ_α is nondecreasing, so

$$\begin{aligned}
&\varphi_x(y) \leqslant \varphi_z(y) \\
&\qquad \iff \frac{f(y)-f(x)}{y-x} \leqslant \frac{f(y)-f(z)}{y-z} \\
&\qquad \iff f(y)-f(z) \geqslant \frac{y-z}{y-x}\left(f(y)-f(x)\right) \\
&\qquad \iff f(z)-f(y) \leqslant \frac{y-z}{y-x}\left(f(x)-f(y)\right), \text{ because } \quad y-z<0 \\
&\qquad \iff f(z)-f(y) \leqslant \lambda\left(f(x)-f(y)\right) \\
&\qquad \iff f(z) \leqslant \lambda f(x) + (1-\lambda) f(y) \\
&\qquad \iff f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y).
\end{aligned}$$

On other hand

$$\begin{aligned}
&\varphi_x(z) \leqslant \varphi_x(y) \\
&\qquad \iff \frac{f(z)-f(x)}{z-x} \leqslant \frac{f(y)-f(x)}{y-x} \\
&\qquad \iff f(z)-f(x) \leqslant \frac{z-x}{y-x}\left(f(y)-f(x)\right) \text{ because } z-x>0 \\
&\qquad \iff f(z)-f(x) \leqslant (1-\lambda)\left(f(y)-f(x)\right) \\
&\qquad \iff f(z) \leqslant \lambda f(x) + (1-\lambda) f(y) \\
&\qquad \iff f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y),
\end{aligned}$$

thus f is convex. □

Proposition 5.2. *Let $f : I \longrightarrow \mathbb{R}$ be a continuous function and differentiable on I, then f is convex if and only if f' is nondecreasing.*

Proof. We start by the necessary condition. Suppose that f is convex. Let $x, y \in I$ such that $x < y$. For all $z \in [x, y]$

$$\frac{f(z)-f(x)}{z-x} \leqslant \frac{f(z)-f(y)}{z-y}$$

let z approach x, we get

$$f'(x) \leqslant \frac{f(x)-f(y)}{x-y}$$

similarly, when z goes to y

$$\frac{f(x)-f(y)}{x-y} \leqslant f'(y)$$

so $f'(x) \leqslant f'(y)$.

Focus now on the sufficient condition. Let us prove the convexity of f).

For $\lambda \in [0,1]$ and $a = \lambda x + (1-\lambda)\, y$ By the mean theorem value applied on $[x, a]$ and on $[a, y]$, there exist $c_1 \in]x, a[$ and $c_2 \in]a, y[$ such that

$$\frac{f(x)-f(a)}{x-a} = f'(c_1) \text{ and } \frac{f(y)-f(a)}{y-a} = f'(c_2)$$

since the function f' is nondecreasing, therefore

$$\frac{f(x)-f(a)}{x-a} \leqslant \frac{f(y)-f(a)}{y-a}$$

and this implies that $f(a) \leqslant \lambda f(x) + (1-\lambda) f(y)$, as seen in proposition (5.2) □

Corollary 5.5.

$$f \text{ convex on } I \iff f' \text{ nondecreasing on } I \iff f'' \geqslant 0 \text{ on } I$$

Example 5.31. *Let $f : x \longmapsto x^h$ for $h \in]0, +\infty[$. f belongs to C^∞ and we have*

$$f'(x) = hx^{h-1} \text{ nondecreasing } \forall x > 0$$

and

$$f''(x) = h\,(h-1)\, x^{h-2} \geqslant 0 \;\forall x > 0$$

then

$$f \text{ convex } \iff h \leqslant 0 \text{ or } h \geqslant 1$$

$$f \text{ concave } \iff h \in [0,1]$$

Asymptote of a curve An asymptote is a straight line that constantly approaches a given curve but does not meet at any infinite distance. In other words, Asymptote is a line that a curve approaches as it moves towards infinity. The curves visit these asymptotes but never overtake them.

Definition 5.16. *The line $x = a$ is said to be a vertical asymptote of a continuous curve $y = f(x)$ if at least one the two limits $\lim\limits_{x \longrightarrow a^+} f(x)$, $\lim\limits_{x \longrightarrow a^-} f(x)$ is infinite.*

If a curve $y = f(x)$ is defined for $x > A$ (resp.$x < A$), then $y = ax+b$ is said to be oblique asymptote of the curve $y = f(x)$ for $x \longrightarrow +\infty$(resp.$x \longrightarrow -\infty$) if $f(x) = ax + b + \alpha(x)$ where $\lim\limits_{x \longrightarrow +\infty} \alpha(x) = 0$(resp.$x \longrightarrow +\infty$) (in other words $|f(x) - ax - b|$ is infinitely small with respect to pour $x \longrightarrow +\infty$(resp.$x \longrightarrow -\infty$)).

A horizontal asymptote is a horizontal line, $y = a$, that has the property that either: $\lim\limits_{x \longrightarrow +\infty} f(x) = a$ or $\lim\limits_{x \longrightarrow -\infty} f(x) = a$. This means, that as x approaches positive or negative infinity, the function tends to a constant value a.

Example 5.32. *Consider $y = \dfrac{8}{x-2}$. The line $x = 2$ is a vertical asymptote because $\lim\limits_{x \longrightarrow 2^+} \dfrac{8}{x-2} = +\infty$, $\lim\limits_{x \longrightarrow 2^-} \dfrac{8}{x-2} = -\infty$.*

Example 5.33. *Let $y = \dfrac{x^2+x}{x-1} = 0$. Since $f(x) = x + 2 + \dfrac{2}{x-1}$ et $\lim\limits_{x \longrightarrow \infty} \dfrac{2}{x-1} = 0$, then the line $y = x + 2$ is an oblique asymptote of the curve $y = \dfrac{x^2+x}{x-1}$ for $x \longrightarrow +\infty$ and for ($x \longrightarrow -\infty$).*

Theorem 5.22. *Let $y = f(x)$ be a given curve. $y = f(x)$ admits an oblique asymptote of the form $y = ax + b$ for $x \longrightarrow +\infty$ ($x \longrightarrow -\infty$) if and only if the following limits*

$$\lim_{\substack{x \longrightarrow +\infty \\ (x \longrightarrow -\infty)}} \frac{f(x)}{x} = a, \lim_{\substack{x \longrightarrow +\infty \\ (x \longrightarrow -\infty)}} [f(x) - ax] = b$$

exist.

Remark 5.10. *The existence of the limits*

$$\lim_{\substack{x \longrightarrow +\infty \\ (x \longrightarrow -\infty)}} \frac{f(x)}{x} = a, \lim_{\substack{x \longrightarrow +\infty \\ (x \longrightarrow -\infty)}} [f(x) - ax] = b$$

is necessary. Indeed for the curve

$y = \sqrt{x}\,(x \geqslant 0)$ we have $\lim\limits_{x \longrightarrow +\infty} \dfrac{\sqrt{x}}{x} = 0 = a$ and $\lim\limits_{x \longrightarrow +\infty} [\sqrt{x} - 0x] = +\infty$, which means $b = +\infty$, so this curve does not have asymptote.

Example 5.34. *Let the curve defined by* $y = xe^{\frac{1}{x^2}}$

1) *Vertical asymptote*

$$\lim_{x\longrightarrow 0^+} xe^{\frac{1}{x^2}} = \lim_{y\longrightarrow +\infty} \frac{e^{y^2}}{y} = +\infty, \lim_{x\longrightarrow 0^-} xe^{\frac{1}{x^2}} = \lim_{y\longrightarrow -\infty} \frac{e^{y^2}}{y} = -\infty.$$

then $x = 0$ *is a vertical asymptote.*

2) *oblique asymptote*

$$a = \lim_{x\longrightarrow \pm\infty} x\frac{e^{\frac{1}{x^2}}}{x} = \lim_{x\longrightarrow \pm\infty} xe^{\frac{1}{x^2}} = 1,$$

$$b = \lim_{x\longrightarrow \pm\infty} \left[xe^{\frac{1}{x^2}} - x\right] = \lim_{x\longrightarrow \pm\infty} \left[x\left(1 + \frac{1}{x^2}\right) - x\right] = 0.$$

thus $b = 0$*. So* $y = x$ *is an oblique asymptote of the considered curve.*

CHAPTER

6

ELEMENTARY FUNCTIONS

Introduction

In this chapter, we propose to introduce the so-called Elementary functions.

$$e^x, \log x, a^x, \sin x, \cos x.$$

The reader is already familiar with these functions but this acquaintance is based on a treatment which was essentially based on intuitive and less rigorous geometrical considerations. Even the question of existence was ignored.

We shall base the study of these functions on the set of real numbers as a complete ordered field, the notion of limit and the convergence of series. Starting from the definitions of these functions, their basic properties will be studied. It is very important to notice here that there is many ways to introduce exponential and logarithm functions. We focus only on two approaches.

6.1 First approach

6.1.1 Logarithm

Theorem 6.1. *There exists a unique function,* $\ln :]0,+\infty[\to \mathbb{R}$ *such that :*

$$\ln'(x) = \frac{1}{x} \quad \textit{(for all } x > 0\Big) \qquad \textit{and} \qquad \ln(1) = 0.$$

This function verifies (for all $a, b > 0$ *) :*

1. $\ln(a \times b) = \ln a + \ln b$,
2. $\ln\left(\frac{1}{a}\right) = -\ln a$,
3. $\ln(a^n) = n \ln a$, (*for all* $n \in \mathbb{N}$)
4. *ln is a continuous function, increasing and define a (one to one) bijection from* $]0,+\infty[$ on $\mathbb{R}$,
5. $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$,
6. *the function* $\ln$ *is concave and* $\ln x \leqslant x - 1$ *(for all* $x > 0$ *).*

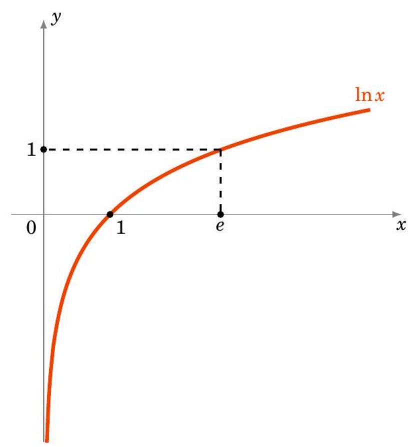
Proof. Integral theory ensures the existence and uniqueness : $\ln(x) = \int_1^x \frac{1}{t} dt$.

1. Set $f(x) = \ln(xy) - \ln(x)$ where $y > 0$ is fixed. Then $f'(x) = y \ln'(xy) - \ln'(x) = \frac{y}{xy} - \frac{1}{x} = 0$. Thus, the derivative of $x \mapsto f(x)$ is equal to zero, therefore, the function is constant et equal to $f(1) = \ln(y) - \ln(1) = \ln(y)$. So $\ln(xy) - \ln(x) = \ln(y)$.
2. From a side: $\ln\left(a \times \frac{1}{a}\right) = \ln a + \ln \frac{1}{a}$, but from the second side: $\ln\left(a \times \frac{1}{a}\right) = \ln(1) = 0$. So $\ln a + \ln \frac{1}{a} = 0$.
3. By induction.
4. In is differentiable, so continuous and $\ln'(x) = \frac{1}{x} > 0$ therefore ln is increasing. Since $\ln(2) > \ln(1) = 0$ then $\ln(2^n) = n \ln(2) \to +\infty$ (when $n \to +\infty$). Thus $\lim_{x \to +\infty} \ln x = +\infty$. From $\ln x = -\ln \frac{1}{x}$ we deduce $\lim_{x \to 0} \ln x = -\infty$. Using the theorem on increasing and continuous function we get that, $\ln :]0,+\infty[\to \mathbb{R}$ is bijective (one-to-one function).

5. $\lim_{x\to 0}\frac{\ln(1+x)}{x}$ is the derivative of ln at the point $x_0 = 1$, so it exists and equals $\ln'(1) = 1$.

6. $\ln'(x) = \frac{1}{x}$ is decreasing, so the function ln is concave. Let $f(x) = x - 1 - \ln x$; $f'(x) = 1 - \frac{1}{x}$. f attains its minimum at $x_0 = 1$. Then $f(x) \geqslant f(1) = 0$. So $\ln x \leqslant x - 1$.

$\square$



Remark 6.1. *ln is called natural logarithm function, which is characterized by* $\ln(e) = 1$.

Definition 6.1. *Given a positive real number a such that $a \neq 1$, the logarithm of a positive real number x with respect to base a is the exponent by which a must be raised to yield x. In other words, the logarithm of x to base a is the unique real number y such that $b^y = x$..*

The logarithm is denoted $\log_a$ (pronounced as "the logarithm of x to base a", "the base-a logarithm of x", or most commonly "the log, base a, of x").

An equivalent and more succinct definition is that the function $\log_a$ is the inverse function to the function $x \mapsto b^x$. More precisely we define

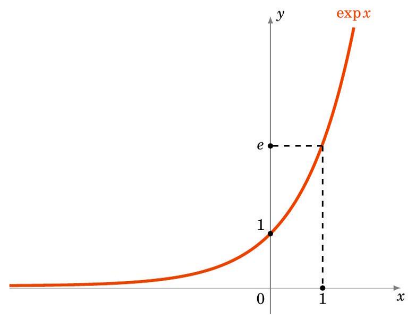
$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

such that $\log_a(a) = 1$.

Remark 6.2. *If $a = 10$ we obtain the decimal logarithm $\log_{10}$ that verifies $\log_{10}(10) = 1$ (and so $\log_{10}(10^n) = n$). For some purposes we use : $x = 10^y \Longleftrightarrow y = \log_{10}(x)$ in computer sciences $\log_2(2^n) = n$ is widely used.*

6.1.2 Exponential

Definition 6.2. *The inverse function of* $\ln :]0, +\infty[\to \mathbb{R}$ *is called exponential function, noted* $\exp : \mathbb{R} \to]0, +\infty[$*.*



For $x \in \mathbb{R}$.

Proposition 6.1. *Exponential function verifies the following properties:*

- $\exp(\ln x) = x$ *for all* $x > 0$ *and* $\ln(\exp x) = x$ *for all* $x \in \mathbb{R}$

$-\exp(a + b) = \exp(a) \times \exp(b)$

$-\exp(nx) = (\exp x)^n$

$-\exp : \mathbb{R} \to]0, +\infty[$ *is continuous, increasing where* $\lim_{x\to-\infty} \exp x = 0$ *and* $\lim_{x\to+\infty} \exp = +\infty$.

- *Exponential function is differentiable and* $\exp' x = \exp x$*, for all* $x \in \mathbb{R}$*. It is convex and* $\exp x \geqslant 1 + x$

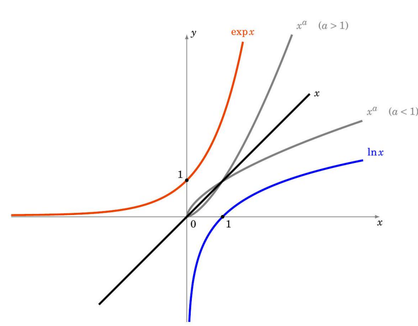
Proof. Exponential function is the natural logarithm inverse.

□

Remark 6.3. *The exponential function is the unique function which verifies* $\exp'(x) = \exp(x)$ *(for all* $x \in \mathbb{R}$ *) and* $\exp(1) = e$*, where* e *satisfies* $\ln e = 1$*.*

Proposition 6.2.

$$\lim_{x\to+\infty} \frac{\ln x}{x} = 0 \qquad et \qquad \lim_{x\to+\infty} \frac{\exp x}{x} = +\infty$$



Proof. 1. One has from previous $\ln x \leqslant x-1$ (for all $x>0$). Therefore $\ln x \leqslant x$ thus $\frac{\ln\sqrt{x}}{\sqrt{x}} \leqslant 1$. Then

$$0 \leqslant \frac{\ln x}{x} = \frac{\ln\left(\sqrt{x}^2\right)}{x} = 2\frac{\ln\sqrt{x}}{x} = 2\frac{\ln\sqrt{x}}{\sqrt{x}}\frac{1}{\sqrt{x}} \leqslant \frac{2}{\sqrt{x}}$$

which implies that $\lim_{x\to+\infty}\frac{\ln x}{x}=0$.

2. We have $\exp x \geqslant 1+x$ (for all $x\in\mathbb{R}$). So $\exp x \to +\infty$ (when $x\to+\infty$).

$$\frac{x}{\exp x} = \frac{\ln(\exp x)}{\exp x} = \frac{\ln u}{u}.$$

We conclude using (1). ☐

6.2 Second approach

Exponential functions

The power series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots \tag{6.1}$$

is everywhere convergent for real x. We proceed now to examine in detail the function represented by this series.

Definition 6.3. *The function represented by the power series (6.1) is called the Exponential function, denoted, provisionally, by expx. Thus*

$$\begin{aligned} &expx = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots \\ &exp(0)=1 \end{aligned} \tag{6.2}$$

and

$$exp(1) = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}+\ldots \tag{6.3}$$

The series on the right hand side of (6.3) converges to a number which lies between 2 and 3. This number is denoted by e, the Exponential base and is the same number as represented by

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Thus $exp(1) = e$.

The Additional Formula

The function $expx$, defined by (6.2) is continuous and differentiable any number of times, for every x.

By differentiation, we get

$$exp'(x) = expx$$
$$exp''(x) = expx$$
$$\vdots$$
$$exp^{(n)}(x) = expx$$

Further we state (justification may be seen expanding by Taylor's Theorem)* that

$$exp\,(x_1 + x_2) = exp\,(x_1) \cdot exp\,(x_2)$$

$*expx = exp\,(x_1) + \frac{exp(x_1)}{1!}\,(x - x_1) + \ldots$, for all values of x and x_1.

Replacing x by $x_1 + x_2$, we get

$exp\,(x_1 + x_2) = exp\,(x_1)\left\{1 + \frac{x_2}{1!} + \frac{x_2^2}{2!} + \ldots\right\} = exp\,(x_1) \cdot exp\,(x_2)$ This formula is called the Addition formula for the exponential function. It gives further

$$\begin{aligned} exp\,(x_1 + x_2 + x_3) &= exp\,(x_1 + x_2) \cdot exp\,(x_3) \\ &= exp\,(x_1) \cdot exp\,(x_2) \cdot exp\,(x_3) \end{aligned}$$

and repetition of the process gives, for any positive integer q,

$$exp\,(x_1 + x_2 + \ldots + x_q) = exp\,(x_1) \cdot exp\,(x_2) \ldots exp\,(x_q) \tag{6.4}$$

If $x_1 = x_2 = x_3 = \ldots = x_q = x$, we get

$$exp(qx) = exp(x)^q \tag{6.5}$$

Hence for $x = 1$,

$$exp(q) = \{exp(1)\}^q = e^q, \text{ for any positive integer } q$$

But since $exp(0) = 1$, therefore the above relation holds for $q = 0$ also.

Hence $exp(q) = e^q$ holds for all integers ≥ 0.

Again replacing each x by p/q in (6.5), we get

$$exp\left(q\frac{p}{q}\right) = \left\{exp\left(\frac{p}{q}\right)\right\}^q \text{ for positive integers, } p, q$$

or

$$exp(p/q) = \{exp(p)\}^{1/q} = e^{p/q} \quad [\because exp(p) = e^p]$$

Hence $exp(m) = e^m$, for all rational numbers $m \geq 0$.

For any positive irrational number ξ there always exists a sequence (x_n) of positive rational terms, converging to ξ.

Now for each n

$$exp\,(x_n) = e^x n.$$

When $n \to +\infty$, the left hand side tends to $exp(\xi)$, and the right hand side to e^ξ, so that we get

$$exp(\xi) = e^\xi$$

$$expx = e^x, \text{ for real } x \geq 0 \tag{6.6}$$

Again by Addition formula,

$$expx \cdot exp(-x) = exp(x - x) = exp(0) = 1 \tag{6.7}$$

Thus we conclude that $expx \neq 0$, for any real x, and that for $x \geq 0$,

$$exp(-x) = \frac{1}{expx} = \frac{1}{e^x} = e^{-x},$$

Consequently, $expx = e^x$ holds for all real x.

Monotonicity

By definition

$$expx > 0, \forall x > 0$$

so that from (6.7) it follows that

$$exp(-x) > 0, \quad \forall x > 0$$

Hence $expx > 0$, for all real x.

Again by definition, for real x,

$$expx \to +\infty, \text{ as } x \to +\infty$$

Hence (6.7) shows that

$$expx \to 0 \text{ as } x \to -\infty$$

Also by definition,

$$0 < x_1 < x_2 \Rightarrow exp\left(x_1\right) < exp\left(x_2\right)$$

Also it follows from (6.7) that

$$exp\left(-x_2\right) < exp\left(-x_1\right), \text{ when } -x_2 < -x_1 < 0$$

Hence the function exp is strictly increasing from 0 to $+\infty$ on the whole real line.

Note. By definition $e^x > \frac{x^{n+1}}{(n+1)!}$, for $x > 0$, so that $x^n e^{-x} < \frac{(n+1)!}{x}$.

$\therefore \lim_{x \to +\infty} x^n e^{-x} = 0$, for all n

This fact we express by saying that e^x tends to $+\infty$ "faster" than any power of x, as $x \to +\infty$.

Logarithmic functions (base e)

Since the exponential function exp is strictly increasing on the set $\mathbb{R}$ of real numbers (i.e., $exp : \mathbb{R} \to \mathbb{R}^+$is one-one, onto), it has inverse function ln (or $\log_e$) which is also strictly increasing and whose domain of definition is $\mathbb{R}^+(= exp(\mathbb{R}))$, the set of positive reals. Thus ln is defined by

$$exp\{ln(y)\} = y, (y > 0)$$

or

$$ln\{expx\} = x, (x \text{ real }) \tag{6.8}$$

or equivalently, for any real x,

$$\left.\begin{array}{r} expx = y \Rightarrow ln(y) = x \\ e^x = y \Rightarrow \log_e y = x \end{array}\right\}$$

Thus the logarithmic function ln (or $\log_e$) is defined for positive values only of the variable.

By definition,

$$\left.\begin{array}{c} exp(-x) = \frac{1}{y} \Rightarrow ln\left(\frac{1}{y}\right) = -x = -ln(y) \\ exp(0) = 1 \Rightarrow ln(1) = 0 = \log_e 1 \\ exp(1) = e \Rightarrow ln(e) = 1 = \log_e e \end{array}\right\}$$

Again

$\because$

$$expx \to +\infty \text{ as } x \to +\infty$$

and

$expx \to 0$ as $x \to -\infty$

$ln(x) \to +\infty$ as $x \to +\infty$

$ln(x) \to -\infty$ as $x \to 0$

$\therefore$ Writing $u = exp\left(x_1\right), v = exp\left(x_2\right)$ or $ln(u) = x_1, ln(v) = x_2$ in (4), we get

$exp\left(x_1 + x_2\right) = uv$

$\Rightarrow ln(uv) = x_1 + x_2 = ln(u) + ln(v)$

which is a familiar property of the logarithmic function and which makes logarithms a useful tool for computation.

Since the function exp is differentiable, therefore, its inverse function ln is also differentiable.

Hence differentiating (6.8), we get

$$ln'\{expx\} \cdot expx = 1$$

Writing $expx = y$, we get

$$ln'(y) = \frac{1}{y}$$

which implies that

$$ln(y) = \int_1^y \frac{dx}{x} \tag{6.9}$$

Quite often (6.9) is taken as the definition of the logarithmic function and thus the starting point of the theory of the logarithmic and the exponential functions.

Note. In theoretical investigations, it is always more convenient to use the so-called natural logarithms, that is to say, those with the base e. Hence in our further discussion, $\log x$ shall always stand for $ln(x)$ or $\log_e x$.

Generalised Power Functions

The meaning of a^x is well understood when a is any positive real number and x is any rational number. We shall now give a meaning to a^x when x is any real number whatsoever. We define thus:

Definition 6.4. $a^x = exp(x \log a)$*, for all* x *and* $a > 0$*.*

Evidently the range of a^x is the set $\mathbb{R}^+$of positive reals, i.e.,

$$a^x > 0, \forall x$$

Therefore $a^x \cdot a^y = exp(x \log a) \cdot exp(y \log a)$

$$= exp\{(x + y) \log a\} = a^{x+y}$$

Thus $\quad a^x \cdot a^y = a^{x+y}$

Let us now verify that this definition of a^x is consistent with that already known to us for x, an integer or a rational number.

(i) Let $x = n$, a positive integer. Therefore

$$\begin{aligned} a^n &= exp(n \log a) = exp[\log a + \log a + \ldots n \text{ times }] \\ &= exp(\log a) \cdot exp(\log a) \ldots n \text{ times} \\ &= a \cdot a \ldots n \text{ times} \end{aligned}$$

(ii) Now let $x = -n, n$ being a positive integer.

Therefore

$$\begin{aligned} a^{-n} &= exp(-n \log a) \\ &= exp[(-\log a) + (-\log a) + \ldots n \text{ times }] \\ &= exp\left[\log \frac{1}{a} + \log \frac{1}{a} + \ldots n \text{ times } \right] \\ &= exp\left(\log \frac{1}{a}\right) \cdot exp\left(\log \frac{1}{a}\right) \ldots n \text{ times} \\ &= \frac{1}{a} \cdot \frac{1}{a} \ldots n \text{ times} \end{aligned}$$

Thus, $exp(x \log a)$ has the same meaning as a^x when x is an integer.

(iii) Let now $x = p/q$, where p, q are integers, and q is positive.

Now

$$exp\left(\frac{p}{q} \log a\right) = a^{p/q}$$

$$\left[exp\left(\frac{p}{q} \log a\right)\right]^q = a^p = exp(p \log a)$$

so that $exp\left(\frac{p}{q} \log a\right)$ is q th root of $exp(p \log a)$.

Thus, $a^{p/q}$ is a q th root of a^p.

Hence, the definition holds good when x is a rational number.

Thus, the above definition of a^x agrees with what is already known to us about a^x.

Logarithmic Functions (any base)

Definition 6.5. $a^x = y \Leftrightarrow \log_a y = x.$

Since y is always positive, therefore the logarithmic function, $\log_a$, is defined for positive values only of the variable.

Evidently

$$a^{-x} = \frac{1}{y}$$
$$\log_a \frac{1}{y} = -x = -\log_a y$$

Also, from definition,

It may be easily shown that

$$\log_a 1 = 0, \log_a a = 1$$
$$\log_a x + \log_a y = \log_a(xy)$$
$$\log_a x - \log_a y = \log_a(x/y)$$
$$\log_a x^y = y \log_a x$$
$$\log_b x \cdot \log_a b = \log_a x$$
$$\log_b a \cdot \log_a b = 1$$

6.2.1 Trigonometric functions

We are now in a position to introduce rigorously the circular functions, employing purely the arithmetical methods. For this purpose, we consider the power series, everywhere convergent (absolutely and uniformly) and the functions represented by them.

Definition.

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^2}{2!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \quad \forall x$$
$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \quad \forall x$$

Each of these series represents a function everywhere continuous and differentiable any number of times in succession. The properties of these functions will be established, taking as starting point their expansions in series form, and it will be seen finally that these coincide with the functions $\cos x$ and $\sin x$ with which we are familiar from elementary studies, i.e., $C(x) = \cos x$ and $S(x) = \sin x$.

Properties of the Functions $(C(x), S(x))$

(i) The functions $C(x)$ and $S(x)$ are continuous and derivable for all x; in fact it may easily be seen that

$$C'(x) = -S(x) \text{ and } S'(x) = C(x)$$

(ii) From definitions,

$$\begin{aligned} &S(0) = 0, C(0) = 1 \\ &S(-x) = -x - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \ldots + (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} + \ldots \\ &= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots\right] = -S(x) \forall x \end{aligned}$$

Similarly, $C(-x) = C(x) \forall x$.

(iii) The Addition Theorems. These functions, like the exponential function, satisfy simple addition theorems, by means of which they can then be further examined.

First Method. By Taylor's expansion for any two variables, x_1 and x_2 (since the two series converge everywhere absolutely).

$$\begin{aligned} C(x_1 + x_2) &= C(x_1) + \frac{C'(x_1)}{1!} x_2 + \frac{C''(x_1)}{2!} x_2^2 + \ldots \\ &= C(x_1) - \frac{S(x_1)}{1!} x_2 - \frac{C(x_1)}{2!} x_2^2 + \frac{S(x_1)}{3!} x_2^3 + \ldots \end{aligned}$$

As this series is absolutely convergent, we may rearrange it in any way we please. Therefore

$$\begin{aligned} C(x_1 + x_2) &= C(x_1) \left\{1 - \frac{x_2^2}{2!} + \frac{x_2^4}{4!} - \ldots\right\} - S(x_1) \left\{x_2 - \frac{x_2^3}{3!} + \frac{x_2^5}{5!} - \ldots\right\} \\ &= C(x_1) \cdot C(x_2) - S(x_1) \cdot S(x_2) \end{aligned}$$

Similarly,

$$S(x_1 + x_2) = S(x_1) \cdot C(x_2) + C(x_1) \cdot S(x_2)$$

Second Method. For any fixed value of x_2, consider the functions

$$\begin{aligned} f(x_1) &= S(x_1 + x_2) - S(x_1) \cdot C(x_2) - C(x_1) \cdot S(x_2) \\ g(x_1) &= C(x_1 + x_2) - C(x_1) \cdot C(x_2) + S(x_1) \cdot S(x_2) \end{aligned}$$

Differentiating with respect to x_1, we get

$$f'(x_1) = C(x_1+x_2) - C(x_1)\cdot C(x_2) + S(x_1)\cdot S(x_2) = g(x_1)$$
$$g'(x_1) = -S(x_1+x_2) + S(x_1)\cdot C(x_2) + C(x_1)\cdot S(x_2) = -f(x_1)$$

therefore
$$\frac{d}{dx_1}\left[f^2(x_1) + g^2(x_1)\right] = 2f(x_1)f'(x_1) + 2g(x_1)g'(x_1)$$
$$= 2f(x_1)g(x_1) - 2g(x_1)f(x_1) = 0, \forall x_1$$

$\Rightarrow f^2(x_1) + g^2(x_1)$ is a constant, $\forall x_1$

Hence for all x_1

$$\begin{aligned} f^2(x_1) + g^2(x_1) &= f^2(0) + g^2(0) = 0 \\ \Rightarrow \quad f(x_1) &= 0, \quad g(x_1) = 0 \\ \text{therefore} \qquad C(x_1+x_2) &= C(x_1)\cdot C(x_2) - S(x_1)\cdot S(x_2) \end{aligned}$$

$$\Rightarrow \quad f(x_1) = 0, g(x_1) = 0$$

and

$$S(x_1+x_2) = S(x_1)\cdot C(x_2) + C(x_1)\cdot S(x_2)$$

The form of these theorems coincides with that of the addition theorems for the functions cosine and sine, with which we are clearly acquainted from an elementary standpoint. With the help of these theorems, we shall now show that the functions C and S satisfy all the other so called purely trigonometrical formulae-in fact C and S are same as the functions cosine and sine. We note, in particular:

(a) Changing x_2 to $-x_2$,

$$C(x_1-x_2) = C(x_1)\cdot C(x_2) + S(x_1)\cdot S(x_2)$$
$$S(x_1-x_2) = S(x_1)\cdot C(x_2) - C(x_1)\cdot S(x_2)$$

(b) Writing $x_2 = -x_1$, we deduce that

$$C^2(x_1) + S^2(x_1) = 1 \text{ or } C^2(x) + S^2(x) = 1, \forall x$$
$$\Rightarrow |S(x)| \le 1, |C(x)| \le 1, \forall x$$

(c) Replacing x_1 and x_2 by x,

$$C(2x) = C^2(x) - S^2(x)$$
$$S(2x) = 2S(x) \cdot C(x)$$

Theorem 6.2. *There exists a positive number π, such that*

$$C(\pi/2) = 0 \text{ and } C(x) > 0, \text{ for } 0 \leq x < \pi/2$$

Proof. Consider the interval $[0, 2]$.

We know $C(0) = 1 > 0$; we shall now show that $C(2) < 0$.

Now

$$\begin{aligned} C(2) &= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \dots \\ &= 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) - \frac{2^6}{6!}\left(1 - \frac{2^2}{7.8}\right) - \dots \end{aligned}$$

Since the brackets are all positive, we have

$$C(2) < 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) = -\frac{1}{3}$$

so that $C(2)$ is negative.

Thus, the continuous function $C(x)$ is positive at 0 and negative at 2 .

$C(x)$ vanishes at least once between 0 and 2 (by the Intermediate-value theorem). Further, since $S(x)$ is positive in $[0, 2]$, where

$$S(x) = x\left(1 - \frac{x^2}{2.3}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6.7}\right) + \dots$$

therefore, the derivative $(-S(x))$ of $C(x)$ is always negative for all values of x between 0 and 2 . Consequently $C(x)$ is a (strictly) monotonic decreasing function in $[0, 2]$, and can therefore, vanish at only one point in $[0, 2]$. Thus, there exists one and only one root of the equation $C(x) = 0$ lying between 0 and 2 . Denoting this root by $\pi/2$, we see that $\pi/2$ is the least positive root of the equation $C(x) = 0$. □

Clearly $C(x) > 0$, when $0 \leq x < \pi/2$.

Using the above results, we deduce that

(a) $S(x) > 0$, when $0 < x \leq \pi/2$.

Since the derivative of $S(x)$ is non-negative in $[0,\ \pi/2]$, therefore, $S(x)$ is a strictly monotonic increasing function. Also since $S(0) = 0$, therefore, $S(x)$ is positive for $0 < x \leq$

$\pi/2$.

(b) As $C^2(\pi/2) + S^2(\pi/2) = 1$ and $C(\pi/2) = 0$,

$\Rightarrow \quad S^2(\pi/2) = 1 \Rightarrow S(\pi/2) = \pm 1$

But, by Lagrange's Mean Value Theorem,

$S(\pi/2) - S(0) = (\pi/2)C(\alpha) > 0$, where $0 < \alpha < \pi/2$

$\Rightarrow \quad S(\pi/2) = 1$

(c) $C(\pi) = 2C^2(\pi/2) - 1 = -1$

$$S(\pi) = 2S(\pi/2)C(\pi/2) = 0$$

(d) $C(2\pi) = 1, S(2\pi) = 0$.

(e) $C(\pi/2) = 2C^2(\pi/4) - 1$.

rejecting the negative sign, as $C(\pi/4)$ is positive.

Similarly, $S(\pi/4) = 1/\sqrt{2}$

(f) It finally follows from the addition theorems that for all x,

$$\begin{aligned}
S\left(\frac{1}{2}\pi - x\right) &= C(x), \quad & C\left(\frac{1}{2}\pi - x\right) &= S(x) \\
S\left(\frac{1}{2}\pi + x\right) &= C(x), \quad & C\left(\frac{1}{2}\pi + x\right) &= -S(x) \\
S(\pi + x) &= -S(x), \quad & C(\pi + x) &= -C(x) \\
S(\pi - x) &= S(x), \quad & C(\pi - x) &= -C(x) \\
S(2\pi + x) &= S(x), \quad & C(2\pi + x) &= C(x)
\end{aligned}$$

Thus, we see that the functions $C(x)$ and $S(x)$ exactly coincide with the functions $\cos x$ and $\sin x$ respectively, and so we shall henceforth use $\cos x$ and $\sin x$ in place of $C(x)$ and $S(x)$ respectively.

The Functions $\tan x, \cot x$

The function $\tan x$ and $\cot x$ are defined as usual by the ratios

$$\tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}$$

and as functions they, therefore, represent nothing new. The expansions in power series for these functions are also not so simple. A few of the coefficients of the expansions could be easily obtained by division, but that gives us no insight into any relationships.

Clearly $\tan x$ is defined, continuous and derivable for all values of x except those for which the denominator, $\cos x$, vanishes, which is the case for $x=\frac{1}{2}(2n+1)\pi, n$ being any integer, positive, negative or zero.

We have

$$\tan(\pi+x)=\tan x,$$

so that, $\tan x$ is a periodic function with period π.

Also we may easily show that when $x \neq \frac{1}{2}(2n+1)\pi$,

$$\frac{d}{dx}\tan x=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{1}{\cos^2 x}$$

Theorem 6.3. *Show that*

$$\lim_{x\to\frac{1}{2}\pi-0}\tan x=\infty, \lim_{x\to\frac{1}{2}\pi+0}\tan x=-\infty.$$

Proof. Let k be any positive number.

As $\lim_{x\to\pi/2}\sin x=1, \exists\delta_1>0$, such that $\left(\text{ taking } \varepsilon=\frac{1}{2}\right)$,

$$\frac{1}{2}<\sin x, \quad \forall x\in\left[\frac{1}{2}\pi-\delta_1, \frac{1}{2}\pi+\delta_1\right]$$

Again, since, $\lim_{x\to\pi/2}\cos x=0$, therefore, $\exists\delta_2>0$, such that

$$-\frac{1}{2k}<\cos x<\frac{1}{2k}, \quad \forall x\in\left[\frac{1}{2}\pi-\delta_2, \frac{1}{2}\pi+\delta_2\right]$$

As $\cos x$ is positive for $x\in[0,\pi/2[$, and negative for $x\in]\pi/2,\pi]$, we have

$$0<\cos x<\frac{1}{2k}, \quad \forall x\in\left[\frac{1}{2}\pi-\delta_2, \frac{1}{2}\pi\right[$$
$$-\frac{1}{2k}<\cos x<0, \quad \forall x\in\left]\frac{1}{2}\pi, \frac{1}{2}\pi+\delta_2\right]$$

Let $\delta=\min(\delta_1,\delta_2)$ therefore from (i) and (ii),

and from (i) and (iii),

$$\tan x=\frac{\sin x}{\cos x}>k, \quad \forall x\in\left[\frac{1}{2}\pi-\delta, \frac{1}{2}\pi\right[$$

$$\tan x=\frac{\sin x}{\cos x}<-k, \quad \forall x\in\left]\frac{1}{2}\pi, \frac{1}{2}\pi+\delta\right]$$

□

Inverse Trigonometric Functions $\cos^{-1} y, \sin^{-1} y, \tan^{-1} y$

We will denote the inverse trigonometric functions by

$\sin^{-1}, \cos^{-1}, \tan^{-1}, \text{cotan}^{-1}$,

or:

$\sin^{inv}, \cos^{inv}, \tan^{inv}, \text{cotan}^{inv}$,

or even:

arcsin , arccos , arctan , arccoth.

$\cos^{-1} y$ function

Since, as may be easily seen, $\cos x$ strictly decreases from $+1$ to -1 as x increases from 0 to π, the function cos is invertible and its inverse, denoted as $\cos^{-1}$, is a function with domain $[-1, 1]$ and range $[0, \pi]$. We write

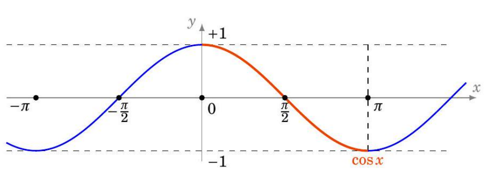
$$y = \cos x \Leftrightarrow x = \cos^{-1} y.$$

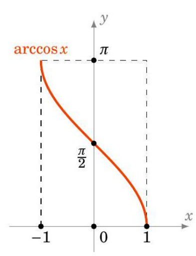
Definition 6.6. *Given y (where $-1 \leq y \leq 1$), $\cos^{-1} y$ is that x which lies between 0 and $\pi (0 \leq x \leq \pi)$ and $\cos x = y$.*

$\cos^{-1} y$ is derivable in the open interval $]-1, 1[$, with $-1/\sqrt{1-y^2}$ as its derivative. In fact, we have $\therefore \quad \frac{dx}{dy} \cdot \frac{dy}{dx} = 1$, and $x = \cos^{-1} y, y = \cos x$

$$\frac{d}{dy}\left(\cos^{-1} y\right) = \frac{1}{\frac{d}{dx}\cos x} = -\frac{1}{\sin x} = \frac{-1}{\sqrt{1-y^2}}, y \neq \pm 1.$$

$$\text{arccos} : [-1, 1] \to [0, \pi]$$





$\sin^{-1} y$ function

Since $\sin x$ is a strictly increasing function in $[-\pi/2, \pi/2]$, with range $[-1, 1]$, therefore, the function sin is invertible and its inverse function is denoted by $\sin^{-1}$, with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.

Also

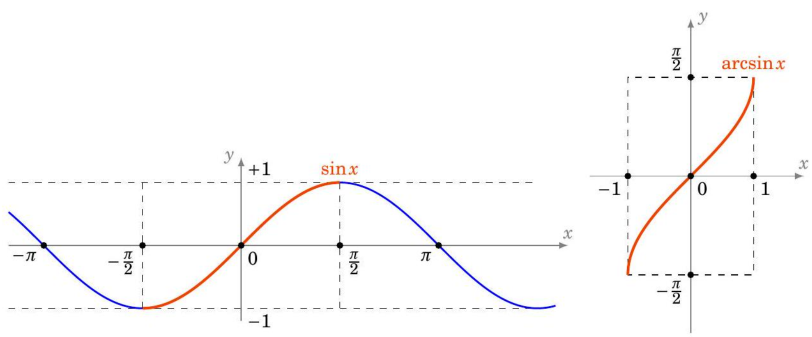
$$y = \sin x \Leftrightarrow x = \sin^{-1} y$$

Definition 6.7. *Given y where $-1 \leq y \leq 1, \sin^{-1} y$ is that x which lies between $-\pi/2$ and $\pi/2, (-\pi/2 \leq x \leq \pi/2)$, and $\sin x = y$.*

It may be shown as before that $\sin^{-1} y$ is derivable in the open interval $]\ 1, 1[$ and

$$\frac{d}{dy} \sin^{-1} y = \frac{1}{\sqrt{1-y^2}}, y \neq \pm 1$$

$$\text{arcsin} : [-1, 1] \to \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$$



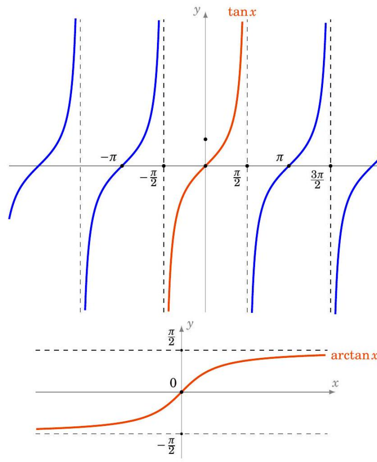
$\tan^{-1} y$ function
Since $\tan x$ is strictly monotonic with domain $]-\pi/2, \pi/2[$ and range $]-\infty, \infty[$, the function is invertible, we have

$$y = \tan x \Leftrightarrow x = \tan^{-1} y$$

so that $\tan^{-1} y$ is a function with domain $]-\infty, \infty[$ and range $]-\pi/2, \pi/2[$.

Definition 6.8. *For any number $y, \tan^{-1} y$ is that x which lies between $-\pi/2$ and $\pi/2 (-\pi/2 < x < \pi/2)$ and $\tan x = y$.*

$$\text{arctan} : \mathbb{R} \to]-\frac{\pi}{2}, +\frac{\pi}{2}[$$



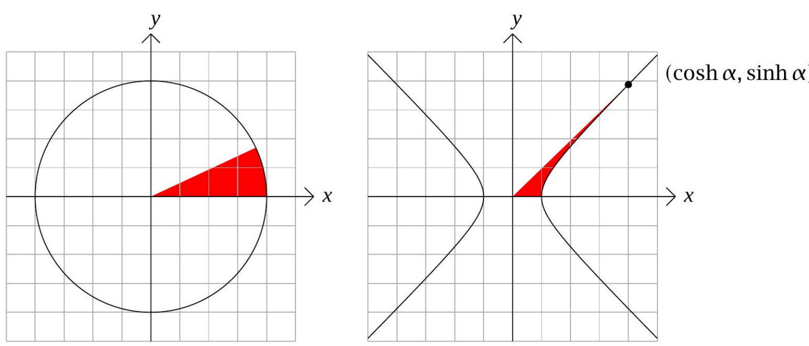
It may be seen that

$$\frac{d}{dy}\tan^{-1} y = \frac{1}{1+y^2}, \forall y.$$

6.2.2 Hyperbolic Functions

The trigonometric functions $\cos\alpha$ and $\cos\alpha$ are defined using the unit circle $x^2 + y^2 = 1$ by measuring the distance α in the counter-clockwise direction along the circumference of the circle. The area of the sector so determined is $\frac{\alpha}{2}$, so we can equivalently say that $\cos\alpha$ and $\cos\alpha$ are derived from the unit circle $x^2 + y^2 = 1$ by measuring off a sector (shaded red) of area $\frac{\alpha}{2}$. The other four trigonometric functions can then be defined in terms of cos and sin.

Similarly, we may define hyperbolic functions $\cosh\alpha$ and $\sinh\alpha$ from the "unit hyperbola" $x^2 - y^2 = 1$ by measuring off a sector (shaded red) of area $\frac{\alpha}{2}$ to obtain a point P whose x - and y-coordinates are defined to be $\cosh\alpha$ and $\sinh\alpha$.



Since at this point we do not yet know how to compute the areas of most curved regions, we must take it on faith that the six hyperbolic functions may be expressed simply in terms of the exponential function:

$$\sinh\alpha = \tfrac{e^\alpha - e^{-\alpha}}{2}$$
$$\cosh\alpha = \tfrac{e^\alpha + e^{-\alpha}}{2}$$
$$\tanh\alpha = \tfrac{\sinh\alpha}{\cosh\alpha} = \tfrac{e^\alpha - e^{-\alpha}}{e^\alpha + e^{-\alpha}}$$
$$\operatorname{cotanh}\alpha = \tfrac{\cosh\alpha}{\sinh\alpha} = \tfrac{e^\alpha + e^{-\alpha}}{e^\alpha - e^{-\alpha}}$$
$$\operatorname{sech}\alpha = \tfrac{1}{\cosh\alpha} = \tfrac{2}{e^\alpha + e^{-\alpha}}$$
$$\operatorname{cosech}\alpha = \tfrac{1}{\sinh\alpha} = \tfrac{2}{e^\alpha - e^{-\alpha}}$$

Note that the domains of sinh, cosh, tanh, and sech are $(-\infty, \infty)$ and the domains of cotanh and cosech are $(-\infty, 0) \cup (0, \infty)$. We can check that the point $\left(\tfrac{e^\alpha+e^{-\alpha}}{2}, \tfrac{e^\alpha-e^{-\alpha}}{2}\right)$ lies on the unit hyperbola:

$$\left(\frac{e^\alpha + e^{-\alpha}}{2}\right)^2 - \left(\frac{e^\alpha - e^{-\alpha}}{2}\right)^2 = \frac{e^{2\alpha} + 2 + e^{-2\alpha}}{4} - \frac{e^{2\alpha} - 2 + e^{-2\alpha}}{4} = \frac{4}{4} = 1$$

"Pythagorean" Identities and some laws

This gives us the first important hyperbolic function identity:

$$\cosh^2\alpha - \sinh^2\alpha \equiv 1$$

This may be used to derive two other identities relating the two other pairs of hyperbolic functions:

$$1 - \tanh^2\alpha = \operatorname{sech}^2\alpha \quad \text{and} \quad \operatorname{cotanh}^2\alpha - 1 = \operatorname{cosech}^2\alpha$$

It is clear that sinh, tanh, cotanh x and cosech are odd functions, while cosh, cotanh , and sech are even, so we have the corresponding identities:

$$\sinh(-x) = -\sinh x, \tanh(-x) = -\tanh x$$
$$\operatorname{cotanh}(-x) = -\operatorname{cotanh} x, \operatorname{cosech}(-x) = -\operatorname{cosech} x$$
$$\cosh(-x) = \cosh x, \operatorname{sech}(-x) = \operatorname{sech} x.$$

We can use the above formulas for the hyperbolic functions in terms of e^x to derive analogs of the identities for the trigonometric functions:

$$\sinh\alpha\cosh\beta = \tfrac{e^\alpha - e^{-\alpha}}{2}\tfrac{e^\beta + e^{-\beta}}{2} = \tfrac{\left(e^\alpha - e^{-\alpha}\right)\left(e^\beta + e^{-\beta}\right)}{4} = \tfrac{e^{\alpha+\beta} + e^{\alpha-\beta} - e^{-\alpha+\beta} - e^{-\alpha-\beta}}{4}$$
$$\sinh\beta\cosh\alpha = \tfrac{e^\beta - e^{-\beta}}{2}\tfrac{e^\alpha + e^{-\alpha}}{2} = \tfrac{\left(e^\beta - e^{-\beta}\right)\left(e^\alpha + e^{-\alpha}\right)}{4} = \tfrac{e^{\beta+\alpha} + e^{\beta-\alpha} - e^{-\beta+\alpha} - e^{-\beta-\alpha}}{4}$$

Adding these two products gives:

$\sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}-e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4} + \frac{e^{\beta+\alpha}+e^{\beta-\alpha}+e^{-\beta+\alpha}-e^{-\beta-\alpha}}{4} = \frac{2e^{\alpha+\beta}-2e^{-\alpha-\beta}}{4} = \frac{e^{\alpha+\beta}-e^{-\alpha-\beta}}{2} = \frac{e^{(\alpha+\beta)}-e^{-(\alpha+\beta)}}{2} = \sinh(\alpha+\beta)$

and subtracting these two products gives:

$\sinh\alpha\cosh\beta - \sinh\beta\cosh\alpha = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}-e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4} - \frac{e^{\beta+\alpha}+e^{\beta-\alpha}+e^{-\beta+\alpha}-e^{-\beta-\alpha}}{4} = \frac{2e^{\alpha-\beta}-2e^{-(\alpha-\beta)}}{4} = \frac{e^{\alpha-\beta}-e^{-(\alpha-\beta)}}{2} = \sinh(\alpha-\beta)$

Similarly,

$\cosh\alpha\cosh\beta = \frac{e^{\alpha}+e^{-\alpha}}{2}\frac{e^{\beta}+e^{-\beta}}{2} = \frac{\left(e^{\alpha}+e^{-\alpha}\right)\left(e^{\beta}+e^{-\beta}\right)}{4} = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}+e^{\beta-\alpha}+e^{-\alpha-\beta}}{4}$

$\sinh\alpha\sinh\beta = \frac{e^{\alpha}-e^{-\alpha}}{2}\frac{e^{\beta}-e^{-\beta}}{2} = \frac{\left(e^{\alpha}-e^{-\alpha}\right)\left(e^{\beta}-e^{-\beta}\right)}{4} = \frac{e^{\alpha+\beta}-e^{\alpha-\beta}-e^{\beta-\alpha}+e^{-\alpha-\beta}}{4}$

Adding these two products gives

$\cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}+e^{\beta-\alpha}+e^{-\alpha-\beta}}{4} + \frac{e^{\alpha+\beta}-e^{\alpha-\beta}-e^{\beta-\alpha}+e^{-\alpha-\beta}}{4}$
$= \frac{2e^{\alpha+\beta}+2e^{-\alpha-\beta}}{4} = \frac{e^{\alpha+\beta}+e^{-(\alpha+\beta)}}{2} = \cosh(\alpha+\beta)$

and subtracting them gives:

$\cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}+e^{\beta-\alpha}+e^{-\alpha-\beta}}{4} - \frac{e^{\alpha+\beta}-e^{\alpha-\beta}-e^{\beta-\alpha}+e^{-\alpha-\beta}}{4} = \frac{2e^{\alpha-\beta}+2e^{-\alpha+\beta}}{4} = \frac{e^{\alpha-\beta}+e^{-(\alpha-\beta)}}{2} = \cosh(\alpha-\beta)$

Summarizing, we have four identities:

$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha$ $\sinh(\alpha-\beta) = \sinh\alpha\cosh\beta - \sinh\beta\cosh\alpha$

$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$

$\cosh(\alpha-\beta) = \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$

which are almost exactly parallel to those for the trigonometric functions and may be used to derive sum and difference formulas for the other four hyperbolic functions.

Letting $\beta = \alpha$, we get:

$\sinh 2\alpha = 2\sinh\alpha\cosh\alpha$

$\cosh 2\alpha = \cosh^2\alpha + \sinh^2\alpha = 1 + 2\sinh^2\alpha = 2\cosh^2\alpha - 1$, so

$\cosh^2\alpha = \frac{\cosh 2\alpha+1}{2}$ and $\sinh^2\alpha = \frac{\cosh 2\alpha-1}{2}$, and thus:

$\cosh\alpha = \sqrt{\frac{\cosh 2\alpha+1}{2}}$ and $\sinh\alpha = \sqrt{\frac{\cosh 2\alpha-1}{2}}$ $\cosh\frac{\alpha}{2} = \sqrt{\frac{\cosh\alpha+1}{2}}$ and $\sinh\frac{\alpha}{2} = \sqrt{\frac{\cosh\alpha-1}{2}}$

Derivatives

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x + (-e^{-x})}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) =$$
$$\frac{\cosh x(\sinh x)' - \sinh x(\cosh x)'}{\cosh^2 x} = \frac{\cosh x\cosh x - \sinh x\sinh x}{\cosh^2 x} =$$

$$\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{cotanh} x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) =$$
$$\frac{\sinh x(\cosh x)' - \cosh x(\sinh x)'}{\sinh^2 x} = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} =$$
$$\frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}(\cosh x)^{-1} =$$
$$(-1)(\cosh x)^{-2}(\cosh x)' = (-1)(\cosh x)^{-2}\sinh x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = \frac{d}{dx}(\sinh x)^{-1} =$$
$$(-1)(\sinh x)^{-2}(\sinh x)' = (-1)(\sinh x)^{-2}\cosh x = -\operatorname{cosech} x \operatorname{cotanh} x$$

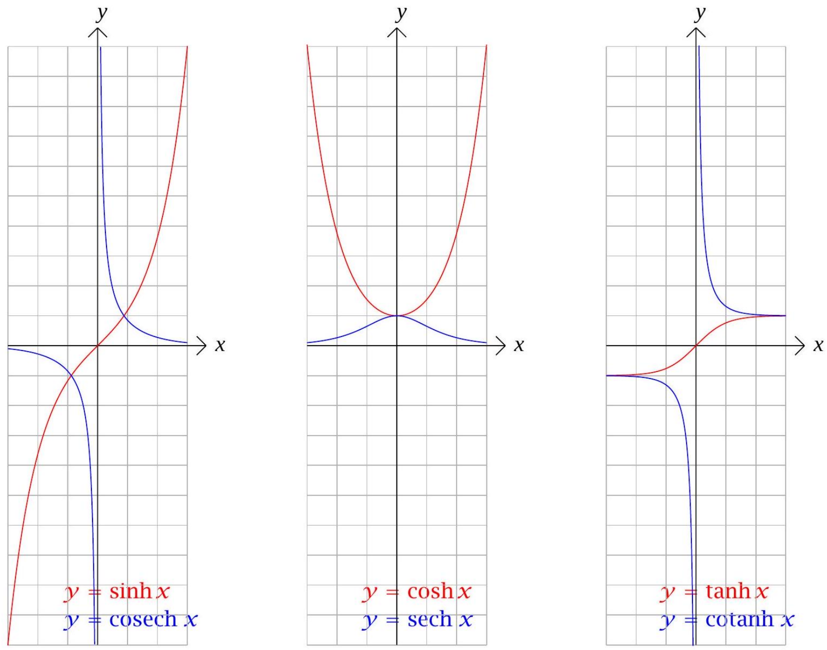
Then we can summarize them as:

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \frac{d}{dx}(\operatorname{cotanh} x) = -\operatorname{cosech}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \quad \frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \operatorname{cotanh} x$$

The domains and ranges are summarized in the next table:

function	domain	Range
sinh	$(-\infty, \infty)$	$(-\infty, \infty)$
cosh	$(-\infty, \infty)$	$[1, \infty)$
tanh	$(-\infty, \infty)$	$(-1, 1)$
cotanh	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
sech	$(-\infty, \infty)$	$(0, 1])$
cosech	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Graphs of the Hyperbolic Functions



Inverse Hyperbolic Functions

sinh, tanh, cotanh and cosech are one-to-one, but cosh and sech are not. For the purpose of defining the inverse of cosh and sech we will restrict their domains to $[0, \infty)$

We will denote the inverse hyperbolic functions by

$\sinh^{-1}, \cosh^{-1}, \tanh^{-1}, \operatorname{cotanh}^{-1}, \operatorname{sech}^{-1}$, and $\operatorname{cosech}^{-1}$

or:

$\sinh^{inv}, \cosh^{inv}, \tanh^{inv}, \operatorname{cotanh}^{inv}, \operatorname{sech}^{inv}$, and $\operatorname{cosech}^{inv}$

or even:

arcsinh , arccosch , arctanh , arccothh , arcsech , and arccosech .

The usual Cancellation Laws hold in the appropriate domains:

$$\begin{aligned}
\sinh\left(\sinh^{-1} x\right) &= x & \sinh^{-1}(\sinh x) &= x \\
\cosh\left(\cosh^{-1} x\right) &= x & \cosh^{-1}(\cosh x) &= x \\
\tanh\left(\tanh^{-1} x\right) &= x & \tanh^{-1}(\tanh x) &= x \\
\operatorname{cotanh}(\operatorname{cotanh}^{-1} x) &= x & \operatorname{cotanh}^{-1}(\operatorname{cotanh} x) &= x \\
\operatorname{sech}\left(\operatorname{sech}^{-1} x\right) &= x & \operatorname{sech}^{-1}(\operatorname{sech} x) &= x \\
\operatorname{cosech}(\operatorname{cosech}^{-1} x) &= x & \operatorname{cosech}^{-1}(\operatorname{cosech} x) &= x
\end{aligned}$$

The derivatives of the inverse hyperbolic functions may be found the same way the derivatives of the inverse trigonometric functions were found: by differentiating the left-hand Cancellation Laws above: for example let us differentiating $\sinh\left(\sinh^{-1} x\right) = x$ we get

$$\begin{aligned}
&\cosh\left(\sinh^{-1} x\right)\left(\sinh^{-1} x\right)' = 1, \text{ so} \\
&\left(\sinh^{-1} x\right)' = \frac{1}{\cosh\left(\sinh^{-1} x\right)}.
\end{aligned}$$

Using the identity $\cosh^2 x - \sinh^2 x = 1$ we get

$$\begin{aligned}
&\cosh^2 x = 1 + \sinh^2 x, \text{ so} \\
&\cosh x = \sqrt{1 + \sinh^2 x} \text{ and therefore} \\
&\cosh\left(\sinh^{-1} x\right) = \sqrt{1 + \sinh^2\left(\sinh^{-1} x\right)} = \sqrt{1 + \left(\sinh\left(\sinh^{-1} x\right)\right)^2} \\
&= \sqrt{1 + x^2}
\end{aligned}$$

Thus we have $\left(\sinh^{-1} x\right)' = \frac{1}{\sqrt{1+x^2}}$

One may similarly derive the derivatives of the other hyperbolic functions:

$$\begin{aligned}
&\left(\cosh^{-1} x\right)' = \frac{1}{\sqrt{x - 1^2}} \\
&\left(\tanh^{-1} x\right)' = \left(\text{cotanh}^{-1} x\right)' = \frac{1}{1 - x^2} \\
&\left(\text{sech}^{-1} x\right)' = -\frac{1}{x\sqrt{1 - x^2}} \\
&\left(\text{cosech}^{-1} x\right)' = \frac{-1}{|x|\sqrt{1 + x^2}}
\end{aligned}$$

Example 6.1. *Solve the equation* $\sinh y = x$ *for* y *in terms of* x*. We have* $\sinh y = \frac{e^y - e^{-y}}{2} = x$*, so*

$e^y - e^{-y} = 2x$ *or* $e^y - 2x - e^{-y} = 0$*. Multiplying both sides of this equation by* e^y *we get:*

$$\begin{aligned}
&(e^y)^2 - 2xe^y - 1 = 0, \text{ a quadratic equation in } e^y \text{ which has solution} \\
&e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2} = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}
\end{aligned}$$

Since $x - \sqrt{x^2 + 1} < 0$ *and we must have* $e^y > 0$*, we get*

$e^y = x + \sqrt{x^2 + 1}$*.*

Taking logarithms of both sides of this equation, we get

$y = \ln\left(x + \sqrt{x^2 + 1}\right)$*, so we have*

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

Similarly,

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad \textit{and} \qquad \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

We then have
$\operatorname{cosech}^{-1} x = \sinh^{-1}\frac{1}{x} = \ln\left(\frac{1}{x} + \sqrt{\left(\frac{1}{x}\right)^2 + 1}\right) = \ln\left(\frac{1}{x} + \sqrt{\frac{1+x^2}{x^2}}\right) =$
$\ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$
Similarly
$\operatorname{cotanh}^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ *and* $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

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