University of Tlemcen **Academic year 2023-2024**
Faculty of Sciences **(L1 ING-INF)**
Department of Informatic **Algebra (First Year)**

Worksheet N°1/ "Logic - Modes of Reasoning"

"*Correction*"

Exercise 01: Solve in $\mathbb{R}$ the following equations and inequations:

(1) $x^2-3x+1=0$	(2) $x^2-2\sqrt{3}x+3=0$	(3) $2x^2-4x+6=0$
(4) $x^2-3x+1>0$	(5) $x^2-3x+1\leq 0$	(6) $x^2-3x+6>0$
(7) $x^2-2\sqrt{3}x+3>0$	(8) $\sqrt{x^2-3x}>2$	(9) $\sqrt{x^2+x-2}>1$

$$(1)\, x^2-3x+1=0,$$

we calculate the discriminant:

$$\triangle=(-3)^2-4\,(1)=5>0,$$

so we have two solutions:

$$x_1=\frac{3-\sqrt{5}}{2} \text{ and } x_2=\frac{3+\sqrt{5}}{2}.$$

Finally,

$$S_1=\left\{\frac{3-\sqrt{5}}{2},\frac{3+\sqrt{5}}{2}\right\}.\text{(Set of solutions)}$$

(2)

$$\begin{aligned} x^2-2\sqrt{3}x+3 &= 0\Leftrightarrow \left(x-\sqrt{3}\right)^2=0\text{(remarkable identities)} \\ &\Leftrightarrow x-\sqrt{3}=0\Leftrightarrow x=\sqrt{3}. \end{aligned}$$

Finally

$$S_2=\left\{\sqrt{3}\right\}.$$

(3)

$$(3)\, 2x^2-4x+6=0\Leftrightarrow x^2-2x+3=0,$$

we calculate the discriminant:

$$\triangle=-8<0,$$

so the equation does not admit solutions.

$$S=\emptyset.\text{(empty)}$$

(4)

$$(4)\, x^2-3x+1>0,$$

we calculate the discriminant:

$$\triangle = 5 > 0,$$

so we have two solutions:

$$x_1 = \frac{3-\sqrt{5}}{2} \text{ and } x_2 = \frac{3+\sqrt{5}}{2}.$$

The sign of the polynomial

$$\xrightarrow[\text{sign of } a \qquad \text{opposite of the sign of } a \qquad \text{sign of } a]{-\infty \qquad + \quad x_1 \qquad - \qquad x_2 \qquad + \qquad +\infty}$$

$$S_4 = \left]-\infty, x_1\right[\cup \left]x_2, +\infty\right[.$$

(5)

$$(5)\, x^2 - 3x + 1 \leq 0,$$

The same polynomial

$$S_5 = [x_1, x_2].$$

(6)

$$(6)\, x^2 - 3x + 6 > 0,$$

we calculate the discriminant:

$$\triangle = -15 < 0,$$

knowing that if the discriminant is negative then the sign of the polynomial is the sign of a (coefficient of x^2), then

$$x^2 - 3x + 1 > 0, \forall x \in \mathbb{R} \text{ (For all } x \text{ belong in } \mathbb{R}),$$

so

$$S_6 = \mathbb{R}.$$

(7)

$$(a-b)^2 = a^2 - 2ab + b^2.$$

$$(7)\, x^2 - 2\sqrt{3}x + 3 > 0 \Leftrightarrow \left(x - \sqrt{3}\right)^2 > 0,$$

which is true except for $x = \sqrt{3}$ (we have equality), then

$$S_7 = \left]-\infty, \sqrt{3}\right[\cup \left]\sqrt{3}, +\infty\right[= \mathbb{R} - \left\{\sqrt{3}\right\}.$$

(8)

$$R : \sqrt{x} > -2.$$

$$D_R = [0, +\infty[, \forall x \in D_R, R \text{ is true} \Rightarrow S_R = D_R.$$

$$H : \sqrt{x} > 2.$$

$$D_H = [0, +\infty[\,, \left[\sqrt{x} > 2 \Leftrightarrow \left(\sqrt{x}\right)^2 > (2)^2 \Leftrightarrow x > 4\right] \Rightarrow S_H =]4, +\infty[\,.$$

$$(8)\ \sqrt{x^2 - 3x} > 2,$$

First look for the domain of definition of the inequality (the square root).

$$x^2 - 3x \geq 0 \Leftrightarrow x\,(x-3) \geq 0$$

$$\xrightarrow[\text{sign of } a \quad \text{opposite of sign of } a \quad \text{sign of } a]{-\infty \qquad + \ \ 0 \quad - \qquad 3 \qquad + \qquad +\infty}$$

$$D_8 =]-\infty, 0] \cup [3, +\infty[\,.$$

Therefore in D_8 we have:

$$S_8 \subset D_8.$$

$$\begin{aligned} \sqrt{x^2 - 3x} &> 2 \Leftrightarrow \left(\sqrt{x^2 - 3x}\right)^2 > 2^2 \\ &\Leftrightarrow x^2 - 3x > 4 \Leftrightarrow x^2 - 3x - 4 > 0, \end{aligned}$$

we calculate the discriminant:

$$\triangle = 25 > 0,$$

so we have two solutions:

$$x_1 = \frac{3 - \sqrt{25}}{2} = -1 \text{ and } x_2 = \frac{3 + \sqrt{25}}{2} = 4.$$

The sign is of the polynomial

$$\xrightarrow[\text{sign of } a \quad \text{opposite of sign of } a \quad \text{sign of } a]{-\infty \qquad + \quad -1 \quad - \qquad 4 \qquad + \qquad +\infty}$$

$$S_8' =]-\infty, -1[\cup]4, +\infty[\,.$$

with,

$$D_8 =]-\infty, 0] \cup [3, +\infty[\,.$$

Then the set of solutions is

$$S_8 = S_8' \cap D_8 = S_8' \text{ because } \left(S_8' \subset D_8\right).$$

(9)

$$(9)\ \sqrt{x^2 + x - 2} > 1.$$

First look for the domain of definition of the inequality (the square root).

$$x^2 + x - 2 \geq 0,$$

$$\triangle = 9 > 0$$

$$\Leftrightarrow x_1 = \frac{-1-3}{2} = -2 \text{ and } x_2 = \frac{-1+3}{2} = 1$$

$$\underset{\text{sign of } a \qquad \text{opposite of sign of } a \quad \text{sign of } a}{\overrightarrow{-\infty \qquad + \quad -2 \quad - \qquad 1 \qquad + \qquad +\infty}}$$

$$D_9 = \left]-\infty, -2\right] \cup \left[1, +\infty\right[.$$

Therefore in D_9 we have:

$$\sqrt{x^2+x-2} > 1 \Leftrightarrow x^2+x-2 > 1 \Leftrightarrow x^2+x-3 > 0,$$

we calculate the discriminant:

$$\triangle = 13 > 0,$$

so we have two solutions:

$$x_1 = \underbrace{\frac{-1-\sqrt{13}}{2}}_{-2,3} \text{ and } x_2 = \underbrace{\frac{-1+\sqrt{13}}{2}}_{1,3}.$$

The sign is of the polynomial

$$\underset{\text{sign of } a \qquad \text{opposite of sign of } a \quad \text{sign of } a}{\overrightarrow{-\infty \qquad + \quad x_1 \quad - \qquad x_2 \qquad + \qquad +\infty}}$$

$$S_9' = \left]-\infty, x_1\right[\cup \left]x_2, +\infty\right[.$$

Then the set of solutions is

$$S_9 = S_9' \cap D_9 = S_9' \text{ because } \left(S_9' \subset D_9\right).$$

Exercise 02: Let P, Q, R be three assertions.

(1) Draw up the truth table of the following assertion:

$$(A) : \underbrace{\left[(P \wedge Q) \Leftrightarrow \overline{R}\right]}_{H_1} \Rightarrow \underbrace{\left[\bar{Q} \vee (Q \Rightarrow P)\right]}_{H_2}.$$

P	Q	R	$\bar{Q}$	$\overline{R}$	$P \wedge Q$	$Q \Rightarrow P$	H_1	H_2	(A)
1	1	1	0	0	1	1	0	1	1
1	1	0	0	1	1	1	1	1	1
1	0	1	1	0	0	1	1	1	1
1	0	0	1	1	0	1	0	1	1
0	1	1	0	0	0	0	1	0	0
0	1	0	0	1	0	0	0	0	1
0	0	1	1	0	0	1	1	1	1
0	0	0	1	1	0	1	0	1	1

(2) Without using the truth table let us show that this proposition is true or false.

Remark:

$$(1)\ \underbrace{\underbrace{H}_{F} \wedge \underbrace{K}}_{F} \text{ or } \underbrace{\underbrace{H} \wedge \underbrace{K}_{F}}_{F}$$

$$(2)\ \underbrace{\underbrace{H}_{T} \vee \underbrace{K}}_{T} \text{ or } \underbrace{\underbrace{H} \vee \underbrace{K}_{T}}_{T}$$

$$(3)\ \underbrace{\underbrace{H}_{F} \Rightarrow \underbrace{K}}_{T} \text{ or } \underbrace{\underbrace{H} \Rightarrow \underbrace{K}_{T}}_{T}$$

$(4)\ \underbrace{\underbrace{H} \Leftrightarrow \underbrace{K}}$ Not exist without using the truth table.

$$\left(Q \wedge \overline{R}\right) \Rightarrow \underbrace{\left[(P \Rightarrow R) \vee \left(P \wedge \bar{R}\right)\right]}_{H_1}.$$

$$K \vee \overline{K} \text{ is True.}$$

Such that we have:

$$\left(\overline{P \Rightarrow R}\right) \Leftrightarrow \left(P \wedge \bar{R}\right),$$

so,

$$H_1 \Leftrightarrow (P \Rightarrow R) \vee \left(\overline{P \Rightarrow R}\right),$$

is true in all cases, which implies that the implication is always true without seeing the first member.

$$\underbrace{... \Rightarrow \underbrace{...}_{\text{True}}}_{\text{True}}$$

Example: For the connector or ($\vee$)

$$\left(\bar{P} \Rightarrow \bar{Q}\right) \vee \left(P \Rightarrow \bar{Q}\right),$$

We have:

$$\underbrace{\left(\bar{P} \Rightarrow \bar{Q}\right) \vee \underbrace{\left(P \Rightarrow \bar{Q}\right)}_{\text{True if } P \text{ is false}}}_{\text{True}} \quad \text{or} \quad \underbrace{\underbrace{\left(\bar{P} \Rightarrow \bar{Q}\right)}_{\text{True if } P \text{ is True}} \vee \left(P \Rightarrow \bar{Q}\right)}_{\text{True}}$$

Example for the implication:

$$\underbrace{\underbrace{\underbrace{\left[(P \Rightarrow R) \wedge \left(P \wedge \bar{R}\right)\right]}_{H_1}}_{\text{False}} \Rightarrow \left(Q \wedge \overline{R}\right)}_{\text{False}}.$$

(3) Say if the following assertions are true or false and write their negation.

$$(a)\, \forall x \in \mathbb{R}, x < 5 \Rightarrow x^2 < 25.$$

is false because for $x = -6$ for example we have $-6 < 5$ but $(-6)^2 > 25$.

$$\left(\overline{P \Rightarrow R}\right) \Leftrightarrow \left(P \wedge \bar{R}\right).$$

The negation is:

$$\exists x \in \mathbb{R}, (x < 5) \wedge \left(x^2 \geq 25\right).$$

$$(b)\, \forall x, y \in \mathbb{R}, x > y \Rightarrow x^2 > y^2.$$

is false because for $x = 1$ and $y = -6$, we have $1 > -6$ but $(1)^2 < (-6)^2$.
The negation is:

$$\left(\bar{b}\right)\, \exists x, y \in \mathbb{R}, (x > y) \wedge \left(x^2 \leq y^2\right).$$

$$(c)\, \forall x, y \in \mathbb{R}^+, x < y \Rightarrow \sqrt{x} < \sqrt{y}.$$

If the difference $\sqrt{x} - \sqrt{y}$ is calculated we find:

$$\sqrt{x} - \sqrt{y} = \frac{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}{\underbrace{\left(\sqrt{x} + \sqrt{y}\right)}_{>0 \text{ because } x<y \text{ and } x,y\in\mathbb{R}^+}} = \frac{(x - y) < 0}{\left(\sqrt{x} + \sqrt{y}\right) > 0} < 0,$$

which gives $\sqrt{x} < \sqrt{y}$.The negation is:

$$(\bar{c})\, \exists x, y \in \mathbb{R}^+, (x < y) \wedge \left(\sqrt{x} \geq \sqrt{y}\right).$$

$$(d) : (\forall x \in \mathbb{R})\, (\exists y \in \mathbb{Z})\, ; (3x + y \leq 0).$$

$$E(\alpha) - 1 < E(\alpha) \leq \alpha < E(\alpha) + 1.$$

$$3x + y \leq 0 \Leftrightarrow y \leq -3x,$$

so (d) is true, for it is enough to take $y = E(-3x)$.(here make a small reminder on the integer part.)The negation is:

$$(\bar{d}) : (\exists x \in \mathbb{R})\, (\forall y \in \mathbb{Z})\, ; (3x + y > 0).$$

$$(e)\, \forall x, y \in \mathbb{R}^*, \exists n \in \mathbb{N}, nx > y.$$

$$nx > y$$

We have two cases

First case: If $x > 0$, then

$$nx > y \Leftrightarrow n > \frac{y}{x},$$

If

$$\frac{y}{x} > 0 \Rightarrow n = E\left(\frac{y}{x}\right) + 1.$$

but if

$$\frac{y}{x} < 0 \Rightarrow n = 0.$$

In this case we take $n = \max(0, E\left(\frac{y}{x}\right) + 1)$.

second case: If $x < 0$, then

$$nx > y \Leftrightarrow n < \frac{y}{x},$$

the problem here is when $\frac{y}{x} < 0$, then the n does n't exist.

Conclusion: (e) is false. The negation is:

$$(\overline{e})\, \exists x, y \in \mathbb{R}^*, \forall n \in \mathbb{N}, nx \leq y.$$

$$(f) : (\exists x \in \mathbb{R})\, (\forall y \in \mathbb{N})\, ; -5x + 2y > 1.$$

The negation of (f) is:

$$\overline{(f)} : (\forall x \in \mathbb{R})\, (\exists y \in \mathbb{N})\, ; -5x + 2y \leq 1,$$

for this we have

$$-5x + 2y \leq 1 \Leftrightarrow y \leq \frac{1 + 5x}{2},$$

the problem here is when $\frac{1+5x}{2} < 0$, then the y does n't exist.(make the same example if $y \in \mathbb{R}$)

Conclusion:

$$\overline{(f)} \text{ is false so } (f) \text{ is true.}$$

$$(g)\, \forall x \in \mathbb{R}, \exists y \in \mathbb{R}^*, x^2 + 2xy + 3 > 0.$$

For

$$x^2 + 2xy + 3 = x^2 + (2y)\, x + 3 = 0.$$

$$\triangle = (2y)^2 - 4\,(1)\,(3) = 4y^2 - 12 = 4(y^2 - 3),$$

The sign of $y^2 - 3$ is

$$\underrightarrow{-\infty \qquad + \qquad -\sqrt{3} \qquad - \qquad \sqrt{3} \qquad + \qquad +\infty}$$

So if we take $y \in \left[-\sqrt{3}, \sqrt{3}\right]$(For example $y = 1$), this gives $\triangle \leq 0$ and consequently

$$x^2 + 2xy + 3 > 0, \forall x \in \mathbb{R},$$

then (g) is true.

$$(h)\, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}^*, x^2 + 2xy + 3 > 0.$$

We have:

$$\left(\overline{h}\right) \exists x \in \mathbb{R}, \exists y \in \mathbb{R}^*, x^2 + 2xy + 3 \leq 0.$$

$\left(\overline{h}\right)$ is true for: $x = 1$ and $y = -5$, so (h) is false.

Exercise 03 : Prove that:

(1) $\forall n \in \mathbb{N}^*, \sqrt{n^2+1}$ is not an integer ($\sqrt{n^2+1} \notin \mathbb{N}$).

Let us prove by the absurdity (contradiction) that:

$$\forall n \in \mathbb{N}^*, \sqrt{n^2+1} \text{ is a not natural integer.}$$

We suppose by contradiction that there is $p \in \mathbb{N}^*$ such as:

$$\begin{aligned}
\sqrt{n^2+1} &= p \in \mathbb{N}^* \Leftrightarrow n^2 + 1 = p^2 \\
&\Leftrightarrow n^2 - p^2 = -1 \\
&\Leftrightarrow \underbrace{(n-p)}_{\in\mathbb{Z}}\underbrace{(n+p)}_{\in\mathbb{N}} = -1 \\
&\Leftrightarrow \begin{cases} n - p = 1 \text{ and } \underbrace{n + p = -1}_{\text{impossible because } n,p\in\mathbb{N}^*} \\ \text{or} \\ n - p = -1 \text{ and } \underbrace{n + p = 1}_{\text{impossible because } n,p\in\mathbb{N}^*} \end{cases}
\end{aligned}$$

hence the contradiction.

(2) (Additional) $\forall n \in \mathbb{N}^*, \sqrt{n+1} + \sqrt{n}$ is not an integer.

1st Method: Let us prove by (contradiction) that:

$$\begin{aligned}
\exists \alpha &\in \mathbb{N}^*, \sqrt{n+1} + \sqrt{n} = \alpha \text{ ...(1)} \\
&\Rightarrow \frac{\left(\sqrt{n+1} + \sqrt{n}\right)\left(\sqrt{n+1} - \sqrt{n}\right)}{\left(\sqrt{n+1} - \sqrt{n}\right)} = \alpha \\
&\Rightarrow \frac{\left(\left(\sqrt{n+1}\right)^2 - \left(\sqrt{n}\right)^2\right)}{\left(\sqrt{n+1} - \sqrt{n}\right)} = \alpha \\
&\Rightarrow \frac{1}{\sqrt{n+1} - \sqrt{n}} = \alpha \in \mathbb{N}^* \\
&\Rightarrow \sqrt{n+1} - \sqrt{n} = \frac{1}{\alpha} \text{ ...(2)} \\
(1) - (2) &\Rightarrow 2\sqrt{n} = \left(\alpha - \frac{1}{\alpha}\right) \Rightarrow n = \left(\frac{\alpha^2 - 1}{2\alpha}\right)^2 \\
&\Rightarrow 4n\alpha^2 = \alpha^4 - 2\alpha^2 + 1 \\
&\Rightarrow \alpha^4 - (2 + 4n)\,\alpha^2 + 1 = 0\,(\alpha, n \in \mathbb{N}),
\end{aligned}$$

if we pose $X = \alpha^2 \in \mathbb{N}^*$, we have:

$$X^2 - (2+4n)X + 1 = 0$$

$$\triangle = (2+4n)^2 - 4 = 16n^2 + 16n > 0, \mathbf{n} \in \mathbb{N}^*$$

so:

$$\begin{aligned} X_1 &= \frac{(2+4n) - \sqrt{\triangle}}{2} = \frac{(2+4n) - 4\sqrt{n^2+n}}{2} \\ &= (1+2n) - 2\sqrt{n^2+n} \notin \mathbb{N}. \end{aligned}$$

and

$$\begin{aligned} X_2 &= \frac{(2+4n) + \sqrt{\triangle}}{2} = \frac{(2+4n) + 4\sqrt{n^2+n}}{2} \\ &= (1+2n) + 2\sqrt{n^2+n} \notin \mathbb{N}. \end{aligned}$$

because if:

$$\begin{aligned} X_1, X_2 &\in \mathbb{N} \Rightarrow (X - X_1)(X - X_2) = X^2 - (X_1 + X_2)X + X_1 X_2 \\ &= X^2 - (2+4n)X + 1 \\ &\Rightarrow X_1 X_2 = 1 \Rightarrow X_1 = X_2 = 1 \\ &\Rightarrow (1+2n) - 2\sqrt{n^2+1} = 1 \text{ and } (1+2n) + 2\sqrt{n^2+1} = 1 \\ &\Rightarrow 2(n - \sqrt{n^2+1}) = 0 \text{ and } 2(n + \sqrt{n^2+1}) = 0 \\ &\Rightarrow (n - \sqrt{n^2+1}) = 0 \text{ and } (n + \sqrt{n^2+1}) = 0 \text{ (impossible in the two cases)} \end{aligned}$$

which is the contradiction.

2nd method: Let us prove by the absurdity (contradiction) that:

$$\begin{aligned} \sqrt{n+1} + \sqrt{n} &= \alpha \in \mathbb{N}^* \Rightarrow \sqrt{n+1} = \alpha - \sqrt{n} \\ &\Rightarrow n + 1 = \left(\alpha - \sqrt{n}\right)^2 \Rightarrow n + 1 = \alpha^2 - 2\alpha\sqrt{n} + n \\ &\Rightarrow 1 = \alpha^2 - 2\alpha\sqrt{n} \Rightarrow \sqrt{n} = \frac{\alpha^2 - 1}{2\alpha} \\ &\Rightarrow n = \left(\frac{\alpha^2 - 1}{2\alpha}\right)^2 = \frac{\alpha^4 - 2\alpha^2 + 1}{4\alpha^2} \\ &\Rightarrow 4n\alpha^2 = \alpha^4 - 2\alpha^2 + 1 \\ &\Rightarrow \alpha^4 - (2+4n)\alpha^2 + 1 = 0\,(\alpha, n \in \mathbb{N}), \end{aligned}$$

if we pose $X = \alpha^2 \in \mathbb{N}^*$, we have:

$$X^2 - (2+4n)X + 1 = 0$$

$$\triangle = (2+4n)^2 - 4 = 16n^2 + 16n > 0, \mathbf{n} \in \mathbb{N}^*$$

so:

$$\begin{aligned} X_1 &= \frac{(2+4n)-\sqrt{\triangle}}{2} = \frac{(2+4n)-4\sqrt{n^2+n}}{2} \\ &= (1+2n)-2\sqrt{n^2+n} \notin \mathbb{N}. \end{aligned}$$

and

$$\begin{aligned} X_2 &= \frac{(2+4n)+\sqrt{\triangle}}{2} = \frac{(2+4n)+4\sqrt{n^2+n}}{2} \\ &= (1+2n)+2\sqrt{n^2+n} \notin \mathbb{N}. \end{aligned}$$

because if:

$$\begin{aligned} X_1, X_2 &\in \mathbb{N} \Rightarrow (X-X_1)(X-X_2) = X^2-(X_1+X_2)X+X_1X_2 \\ &= X^2-(2+4n)X+1 \\ &\Rightarrow X_1X_2 = 1 \Rightarrow X_1 = X_2 = 1 \\ &\Rightarrow (1+2n)-2\sqrt{n^2+1} = 1 \text{ and } (1+2n)+2\sqrt{n^2+1} = 1 \\ &\Rightarrow 2(n-\sqrt{n^2+1}) = 0 \text{ and } 2(n+\sqrt{n^2+1}) = 0 \\ &\Rightarrow (n-\sqrt{n^2+1}) = 0 \text{ and } (n+\sqrt{n^2+1}) = 0 \text{ (impossible in the two cases)} \end{aligned}$$

which is the contradiction.

Remark: If a and b are two solutions of : $\alpha x^2+\beta x+\mu$, so $\alpha ab = \mu$.
because:

$$\alpha x^2+\beta x+\mu = \alpha(x-a)(x-b) = \alpha x^2-\alpha(a+b)x+\alpha ab.$$

(3) $\forall n \in \mathbb{N}^*, 7^n+6n-1$ is a multiple of 12. (R_n)

By induction we have:

1st step: to $n=1$:

$$\begin{aligned} &7^1+(6\times 1)-1 = 12 = 12\times 1, \\ &\Rightarrow 7^1+(6\times 1)-1 \text{ is a multiple of } 12, \\ &\Rightarrow R_1 \text{ is true.} \end{aligned}$$

2nd Step: We suppose that (R_n) is true for a fixed $n \in \mathbb{N}$ (the induction hypothesis) so we have:

$$7^n+6n-1 = 12k, k \in \mathbb{N}.$$

$$4^{n+1}$$

, and prove that (R_{n+1}) is also,i.e.:

$$7^{n+1}+6(n+1)-1 \text{ is a multiple of } 12?$$

Indeed:

$$\begin{aligned}&7^{n+1}+6\,(n+1)-1=7\times 7^{n}+6n+6-1\\&=7(12k-6n+1)+6n+5 \text{ (the induction hypothesis)}\\&=7\times 12k-36n+12=12\underbrace{(7k-4n+1)}_{k'},\\&\Rightarrow 7^{n+1}+6\,(n+1)-1 \text{ is a multiple of } 12,\end{aligned}$$

$\Rightarrow (R_{n+1})$ is true.
Conclusion:

$$\forall n\in\mathbb{N}, 7^{n}+6n-1 \text{ is a multiple of } 12.$$

(3*) $\forall n\in\mathbb{N}, x^{n}-y^{n}$ is a multiple of $(x-y)\,.\,(R_n)$
By induction reasoning:

1st step: for $n=0$,

$$x^{n}-y^{n}=x^{0}-y^{0}=1-1=0=0\times(x-y)\,,$$

then $x^{0}-y^{0n}$ is a multiple of $(x-y)$, so (R_0) is true.

2nd Step: We suppose that (R_n) is true for a fixed $n\in\mathbb{N}$ (the induction hypothesis) so we have:

$$x^{n}-y^{n}=k\,(x-y)\,,k\in\mathbb{Z}$$

and prove that (R_{n+1}) is also,i.e.:

$$x^{n+1}-y^{n+1}=k'\,(x-y)\,.$$

We have:

$$\begin{aligned}x^{n+1}-y^{n+1}&=(x-y+y)\,.x^{n}-y.y^{n}\\&=(x-y)\,.x^{n}+y\,(x^{n}-y^{n})\\&=(x-y)\,.x^{n}+y.k.\,(x-y)\\&=(x-y)\underbrace{[x^{n}+y.k]}_{k'},\end{aligned}$$

so $x^{n+1}-y^{n+1}$ is multiple of $(x-y)$, then (R_{n+1}) is true.

Conclusion: $\forall n\in\mathbb{N}, x^{n}-y^{n}$ is a multiple of $(x-y)\,.$

(4) $\forall n\in\mathbb{N}^{*}, 2^{n-1}\leq n!, (R_n)$ with $n!=1\times 2\times ...\,(n-2)\times(n-1)\times n$ and $0!=1$.

For example :

$$\begin{aligned}2!&=1\times 2=2\\5!&=1\times 2\times 3\times 4\times 5=120.\\1!&=1.\end{aligned}$$

1st step: for $n = 1$,

$$\begin{aligned} 2^{1-1} &= 2^0 = 1 \text{ and } 1! = 1 \\ &\Rightarrow 2^{1-1} \leq 1! \Rightarrow (R_1) \text{ is true.} \end{aligned}$$

2nd Step: We suppose that (R_n) is true for a fixed $n \in \mathbb{N}$ (the induction hypothesis), so we have:

$$2^{n-1} \leq n!,$$

and prove that (R_{n+1}) is also, that is:

$$2^n \leq (n+1)!.$$

Remark:

$$\begin{aligned} 5! &= 1 \times 2 \times 3 \times 4 \times 5 = 5.(4!). \\ 10! &= 10.(9!). \\ (n+1)! &= (n+1).(n!). \end{aligned}$$

Indeed:

$$\begin{aligned} 2^n &= \underbrace{2^1 \times 2^{n-1} \leq 2 \times n!}_{\text{Induction hypotheses}} \leq (n+1).n! = (n+1)!, \text{ because } n \geq 1, (n+1 \geq 2) \\ &\Rightarrow (R_{n+1}) \text{ is true.} \end{aligned}$$

Conclusion:

$$\forall n \in \mathbb{N}^*, 2^{n-1} \leq n!.$$

(5) Remark:

$$\begin{aligned} \sum_{k=1}^{n} f(k) &= f(1) + f(2) + f(3) + ... + f(n). \\ (1) \sum_{k=1}^{3} f(k) &= f(1) + f(2) + f(3). \\ (2) \sum_{k=1}^{1} f(k) &= f(1). \\ (3) \sum_{k=1}^{n+1} f(k) &= \underbrace{f(1) + f(2) + f(3) + ... + f(n)}_{\sum_{k=1}^{n} f(k)} + f(n+1) \\ &= \sum_{k=1}^{n} f(k) + f(n+1). \end{aligned}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.(R_n)$$

step 1: for $n = 1$,

$$\sum_{k=1}^{1} k^2 = 1^2 = 1 \text{ and } \frac{1 \times (1+1)(2 \times 1 + 1)}{6} = 1,$$

$\Rightarrow R_1$ is true.

step 2: We suppose that (R_n) is true for a fixed $n \in \mathbb{N}$ (the induction hypothesis),so

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

and prove that (R_{n+1}) is also, that is:

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}?$$

We have:

$$\begin{array}{l} \sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 \\ = \frac{n(n+1)(2n+1)}{6} + (n+1)^2, \\ \text{(from the induction hypothesis)} \\ = (n+1)\left[\frac{n(2n+1)}{6} + (n+1)\right], \\ = (n+1)\left(\frac{2n^2+7n+6}{6}\right) \\ = \frac{(n+1)(n+2)(2n+3)}{6}, \\ \Rightarrow (R_{n+1}) \text{ is true.} \end{array}$$

Conclusion:

$$\forall n \in \mathbb{N}^*, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise 04 :

(1) Let a and p be two natural integers, prove that:

$$(p \text{ prime integer and } p \text{ divide } a^2) \Rightarrow p \text{ divide } a.$$

$$\begin{array}{rcl} a \times b & = & (p_1 \times p_2 \times ...) \times (q_1 \times q_2 \times ...) \\ & = & h \times .... \end{array}$$

$$a \times a = h \times ...$$

If p prime integer and p divide a^2 then:

$$\begin{array}{rcl} \exists k & \in & \mathbb{N} \text{ such as: } a^2 = k \times p \\ & \Rightarrow & a \times a = k \times p, \text{ with } p \text{ is a prime integer} \\ & \Rightarrow & p \text{ divide } a. \end{array}$$

(or we have)

$$\begin{cases} a \text{ divide } p \text{ and } k \text{ divide } a \text{ contradiction with } p \text{ est prime integer,} \\ \text{or } p \text{ divide } a \text{ et } a \text{ divide } k, \end{cases}$$

so p divide a.

(2) (a) If p is a prime integer then $\sqrt{p}$ is an irrational number.

(In frensh we say)si p est premier alors $\sqrt{p}$ est un nombre irrationnel.

Remark: q is a rational number if:

$$q = \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{Z}^*, (a \wedge b) = 1 \text{ (} a \text{ and } b \text{ are relatively prime).}$$

By contradiction we suppose that: $\sqrt{p}$ is a rational number, so:

$$\begin{aligned} \exists a, b \quad &\in \quad \mathbb{N}^*, (a \wedge b) = 1 \text{ et } \sqrt{p} = \frac{a}{b} \Rightarrow p = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}, \\ &\Rightarrow \quad p \times b^2 = a^2 \Rightarrow p \text{ divide } a^2 = a \times a, \\ &\Rightarrow \quad p \text{ divide } a \text{ ....(1) because } p \text{ prime integer,} \\ &\Rightarrow \quad a = k \times p, k \in \mathbb{N}, \\ &\Rightarrow \quad p \times b^2 = p \times p \times k^2 \\ &\Rightarrow \quad b^2 = p \times k^2 \\ &\Rightarrow \quad p \text{ divide } b^2 \text{, but } p \text{ prime integer,} \\ &\Rightarrow \quad p \text{ divide } b, ...(2) \\ &\Rightarrow \quad p \neq 1 \text{ (is a prime integer) who is a divisor commun from } a \text{ and } b, \end{aligned}$$

contradiction with $(a \wedge b) = 1 \Rightarrow \sqrt{p}$is an irrational number.

(b) deduce (en déduire in frensh) that, $\sqrt{2} + \sqrt{3}$ is irrational number.

By contradiction we suppose that: $\sqrt{2} + \sqrt{3} = \beta$ est rationnal,

Remark:

$$\alpha \in \mathbb{Q}^* \Rightarrow \alpha = \frac{a}{b}, a, b \in \mathbb{Z}^* \Rightarrow \frac{1}{\alpha} = \frac{b}{a} \in \mathbb{Q}^*.$$

$$\begin{aligned} &\Rightarrow \quad \frac{1}{\sqrt{2} + \sqrt{3}} \in \mathbb{Q}^* \Rightarrow \left[\frac{1}{\sqrt{2} + \sqrt{3}} \times \frac{(\sqrt{2} - \sqrt{3})}{(\sqrt{2} - \sqrt{3})}\right] \in \mathbb{Q} \\ &\Rightarrow \quad \sqrt{3} - \sqrt{2} \in \mathbb{Q}^*. \end{aligned}$$

The sum of the two number $(\sqrt{2} + \sqrt{3})$ and $\left(\sqrt{3} - \sqrt{2}\right)$ give $2\sqrt{3} \in \mathbb{Q}$,
$\Rightarrow \sqrt{3} \in \mathbb{Q}$ (contradiction).
That implies: $\sqrt{2} + \sqrt{3}$ is an irrationnal number.

(3) Prove that:

$$\sqrt{2}+\sqrt{3}+\sqrt{6}\notin\mathbb{Q}.$$

By contradiction we suppose that:

$$\begin{aligned}
\sqrt{2}+\sqrt{3}+\sqrt{6} &= \alpha\in\mathbb{Q}\\
&\Rightarrow \sqrt{2}+\sqrt{3}=\alpha-\sqrt{6}\\
&\Rightarrow \left(\sqrt{2}+\sqrt{3}\right)^2=\left(\alpha-\sqrt{6}\right)^2\\
&\Rightarrow 5+2\sqrt{6}=\alpha^2+6-2\sqrt{6}\\
&\Rightarrow \sqrt{6}=\frac{\alpha^2+1}{4}\in\mathbb{Q},
\end{aligned}$$

$\Rightarrow \sqrt{2}+\sqrt{3}\in\mathbb{Q}$ contradiction with (5).

Sincere wishes you success (MESSIRDI BACHIR)