Course Algabra 1 : Chapter 3, **Binary relations on a set**

1^{st} year Licence LMD **Informatique**

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Chapter 3

Binary relations on a set

3.1 Definitions

Definition 1. *We call a **relation** from E to F any process associating elements of E with elements of F. We denote it for example: $\mathcal{R}, \mathcal{S}, \mathcal{T}, \ldots$.*

Let $\mathcal{R}$ be a relation from E to F. If $x \in E$ is in relation with $y \in F$, we will denote this by:

$$x\mathcal{R}y$$

If a is not in relation with b, we denote it : $a \not{\mathcal{R}} b$.

*The set of pairs $(x, y) \in E \times F$ verifying a relation $\mathcal{R}$ is called the **graph** of $\mathcal{R}$. It is generally denoted $\mathcal{G}$. We therefore have:*

$$(x, y) \in \mathcal{G} \Longleftrightarrow x\mathcal{R}y$$

*If $E = F$, a relation from E to F is called a **binary relation** on E.*

Example 1.

1. *Equality is a binary relation on any set E.*
2. *On the set of natural integers $\mathbb{N}$, division is a binary relation formulated by x divides y. $(2, 8)$ belongs to the graph of this relation, but $(3, 4)$ does not.*
3. *Inclusion is a binary relation on $\mathcal{P}(E)$.*

3.2 Properties

Let $\mathcal{R}$ be a binary relation on a set E and x, y, z elements of E.

1. **Reflexivity :** $\mathcal{R}$ is reflexive if :
$$\forall x \in E,\ x\mathcal{R}x$$
2. **Symmetry :** $\mathcal{R}$ is symmetrical if :
$$\forall x \in E,\ \forall y \in E,\ x\mathcal{R}y \Longrightarrow y\mathcal{R}x$$
3. **Antisymmetry:** $\mathcal{R}$ is Antisymmetric si :
$$\forall x \in E,\ \forall y \in E,\ (x\mathcal{R}y \text{ and } y\mathcal{R}x) \Longrightarrow x = y$$
4. **Transitivity :** $\mathcal{R}$ is transitive if :
$$\forall x \in E,\ \forall y \in E,\ \forall z \in E,\ (x\mathcal{R}y \text{ and } y\mathcal{R}z) \Longrightarrow x\mathcal{R}z$$

Example 2. *Let the relation $\mathcal{R}$, called* equality, *on $\mathbb{N}$, be defined by:*

$$\forall m, n \in \mathbb{N},\ m\mathcal{R}n \Longleftrightarrow m = n$$

1. *The* equality *relation is reflexive on $\mathbb{N}$, because we have $\forall m \in \mathbb{N},\ m = m$, that is to say $m\mathcal{R}m$.*
2. *The* equality *relation is symmetric on $\mathbb{N}$, because we have $\forall m, n \in \mathbb{N},\ m = n \Longrightarrow n = m$, that is to say $m\mathcal{R}n \Longrightarrow n\mathcal{R}m$.*
3. *The* equality *relation is transitive on $\mathbb{N}$, because we have $\forall l, m, n \in \mathbb{N},\ (l = m \text{ and } m = n) \Longrightarrow l = n$, that is to say $(l\mathcal{R}m \text{ and } m\mathcal{R}n) \Longrightarrow l\mathcal{R}n$.*

Example 3. *Let E be a set and let the relation $\mathcal{T}$, called* inclusion, *on $\mathcal{P}(E)$, be defined by :*

$$\forall A, B \in \mathcal{P}(E),\ A\mathcal{T}B \Longleftrightarrow A \subset B$$

1. *The* inclusion *relation is reflexive on $\mathcal{P}(E)$, because we have*

   $$\forall A \in \mathcal{P}(E),\ A \subset A$$

   that is to say $A\mathcal{T}A$.
2. *The* inclusion *relation is transitive on $\mathcal{P}(E)$, because we have*

   $$\forall A, B, C \in \mathcal{P}(E),\ (A \subset B \text{ and } B \subset C) \Longrightarrow A \subset C$$

   that is to say $(A\mathcal{T}B \text{ et } B\mathcal{T}C) \Longrightarrow A\mathcal{T}C$.
3. *The* inclusion *relation is not symmetrical on $\mathcal{P}(E)$, because it suffices to choose $A, B \in \mathcal{P}(E)$ such that $A \subset B$ strictly and we obtain $B \not\subset A$. Instead, the inclusion is antisymmetric, because if $A \subset B$ and $B \subset A$ then $A = B$.*

3.3 Equivalence Relation

3.3.1 Definition

Definition 2. *Let $\mathcal{R}$ be a binary relation on a set E. $\mathcal{R}$ is called an* ***equivalence relation*** *if it is :*

1. *Reflexive*
2. *Symmetrical*
3. *Transitive*

Example 4. *The relation called* equality *on $\mathbb{N}$ is an equivalence relation, but the relation called* inclusion *on $\mathcal{P}(E)$ is not an equivalence relation because it is not symmetrical.*

Example 5. *We define the following relation $\mathcal{P}$ on $\mathbb{Z}$:*

$$\forall s, t \in \mathbb{Z},\ s\mathcal{P}t \Longleftrightarrow s - t \text{ is divisible by } 2.$$

1. *The relation $\mathcal{P}$ is reflexive on $\mathbb{Z}$, because we have $\forall s \in \mathbb{Z},\ s - s = 0$, and 0 is divisible by 2. That is to say $s\mathcal{P}s$.*

2. *The relation $\mathcal{P}$ is symmetrical on $\mathbb{Z}$, because, for $s, t \in \mathbb{Z}$:*

$$\begin{aligned} s\mathcal{P}t &\Longrightarrow s - t \text{ divisible by } 2 \\ &\Longrightarrow \exists k \in \mathbb{Z}, s - t = 2k \\ &\Longrightarrow t - s = -2k \\ &\Longrightarrow t - s = 2k^{'}, k^{'} \in \mathbb{Z} \\ &\Longrightarrow t - s \text{ is divisible by } 2 \\ &\Longrightarrow t\mathcal{P}s \end{aligned}$$

3. *The relation $\mathcal{P}$ is transitive on $\mathbb{Z}$. Indeed, let $s, t, u \in \mathbb{Z}$.*

$$\begin{aligned} s\mathcal{P}t \text{ and } t\mathcal{P}u &\Longrightarrow s - t \text{ is divisible by } 2 \text{ and } t - u \text{ is divisible by } 2 \\ &\Longrightarrow \exists k, k^{'} \in \mathbb{Z},\ s - t = 2k \text{ and } t - u = 2k^{'} \\ &\Longrightarrow s - u = 2k^{''},\ \text{with } k^{''} = k + k^{'} \in \mathbb{Z} \\ &\Longrightarrow s - u \text{ is divisible by } 2 \\ &\Longrightarrow s\mathcal{P}u. \end{aligned}$$

3.3.2 Equivalence class

Definition 3. *Let $\mathcal{R}$ be an equivalence relation on a set E. We call the **equivalence class** of an element $x \in E$ the following set, denoted $\bar{x}$ or $\dot{x}$ or C_x or $Cl(x)$:*

$$\bar{x} = \{y \in E,\ y\mathcal{R}x\}$$

Example 6. *Let E be a set defined by :*

$$E = \{\text{The students of the 1st year Licence Informatique 2024/2025 at the university of Tlemcen}\}$$

We note the students of E by letters : $a, b, c, d, \ldots$.
We define on E the relation $\mathcal{R}$ as follow :

$$a\mathcal{R}b \Longleftrightarrow a \text{ belongs to the same } b \text{ group}$$

1. *Let's show that $\mathcal{R}$ si an equivalence relation on E.*
   (a) *Let $a \in E$, then a belongs to the same a group, that is to say $a\mathcal{R}a$ i.e. $\mathcal{R}$ is reflexive.*
   (b) *Let $a, b \in E$ such that $a\mathcal{R}b$*

$$\begin{aligned} a\mathcal{R}b &\Longrightarrow a \text{ belongs to the same } b \text{ group} \\ &\Longrightarrow b \text{ belongs to the same } a \text{ group} \\ &\Longrightarrow b\mathcal{R}a. \\ &\Longrightarrow \mathcal{R} \text{ is symmetrical.} \end{aligned}$$

   (c) *Let $a, b, c \in E$ such that $a\mathcal{R}b$ and $b\mathcal{R}c$*

$$\begin{aligned} a\mathcal{R}b \text{ et } b\mathcal{R}c &\Longrightarrow a \text{ belongs to the same } b \text{ group } \text{ and } b \text{ belongs to the same } c \text{ group} \\ &\Longrightarrow a \text{ belongs to the same } c \text{ group} \\ &\Longrightarrow a\mathcal{R}c \\ &\Longrightarrow \mathcal{R} \text{ is transitive.} \end{aligned}$$

   then $\mathcal{R}$ is an equivalence relation on E.

2. *Let's look for the equivalence classes of all the elements of E according to the relation $\mathcal{R}$.*

   - *Let $a \in E$*

     $$\overline{a} = \{x \in E,\ x\mathcal{R}a\} = \{x \in E, x \text{ belongs to the same } a \text{ group }\}$$

     Suppose that a belongs to the group $G1$, then $\overline{a} = \{$All the students of the group $G1\}$, we then put this group aside.

   - *Let $b \in E$*

     $$\overline{b} = \{x \in E,\ x\mathcal{R}b\} = \{x \in E, x \text{ belongs to the same } b \text{ group }\}$$

     Suppose that b belongs to the group $G2$, then $\overline{b} = \{$All the students of the group $G2\}$, we remark that b cannot belong to the group $G1$ because we put it aside.

   - *We thus continue to go through all the students of the set E to include each of them, without forgetting anyone, in a single equivalence class, which is nothing other than the group to which he belongs. We then obtain 16 equivalence classes.*

Properties 1.
Let $\mathcal{R}$ be an equivalence relation on a set E.

1. $\forall x \in E\ x \in \overline{x}$.

2. $\forall x, y \in E,\ x\mathcal{R}y \Longleftrightarrow \bar{x} = \bar{y}$.

3. *The set of all equivalence classes modulo $\mathcal{R}$ on a set E, is called the* ***quotient set*** *of E by $\mathcal{R}$, and denoted $E/_{\mathcal{R}}$. It forms a partition of E.*

$$E/_{\mathcal{R}} = \{\bar{x},\ x \in E\}.$$

Proof.

1. *$\mathcal{R}$ being an equivalence relation, it is then reflexive, that is to say : for every $x \in E$, $x\mathcal{R}x$, then $x \in \overline{x}$.*

2. *Let $a \in \bar{x}$, then $a\mathcal{R}x$, but by hypothesis $x\mathcal{R}y$ therefore by transitivity $a\mathcal{R}y$, that is to say $a \in \bar{y}$, from which $\bar{x} \subset \bar{y}$. By an identical but symmetrical process we obtain that $\bar{y} \subset \bar{x}$. Therefore $\bar{x} = \bar{y}$. For the reciprocal implication, for $a \in \bar{x} = \bar{y}$, we have $a\mathcal{R}x$ and by symmetry we obtain $x\mathcal{R}a$, we also have $a\mathcal{R}y$, therefore by transitivity $x\mathcal{R}y$.*

3. *For all $x \in E$, we have $\bar{x} \neq \varnothing$, since $x \in \bar{x}$, given the reflexivity of $\mathcal{R}$.*
   It is clear that $\cup\bar{x} = E$, for $x \in E$.
   Finally, if $\bar{x} \neq \bar{y} \Longrightarrow \bar{x} \cap \bar{y} = \varnothing$. Indeed, let us reason by contradiction. Suppose $\bar{x} \neq \bar{y}$ and $a \in \bar{x} \cap \bar{y}$ then $a\mathcal{R}x$ and $a\mathcal{R}y$, from which $x\mathcal{R}y$ i.e. $\bar{x} = \bar{y}$ which is contrary to our hypothesis.

Example 7.
Let the equivalence relation denoted $\mathcal{R}$ be defined on $\mathbb{Z}$ by:

$$\forall x, y \in \mathbb{Z}, y\mathcal{R}x \Longleftrightarrow 3 \text{ divides } y - x$$

Determine the equivalence classes modulo $\mathcal{R}$ and the quotient set $\mathbb{Z}/_{\mathcal{R}}$.
Let $x \in \mathbb{Z}$. The equivalence class of x modulo $\mathcal{R}$ is therefore :

$$\bar{x} = \{y \in \mathbb{Z}, y\mathcal{R}x\} = \{y \in \mathbb{Z}, 3 \text{ divides } y - x\}$$
$$3 \text{ divides } y - x \Longleftrightarrow y - x = 3k,\ k \in \mathbb{Z}$$
$$\Longleftrightarrow y = 3k + x,\ k \in \mathbb{Z}$$
$$\Longrightarrow \bar{x} = \{y \in \mathbb{Z}, y = 3k + x,\ k \in \mathbb{Z}\}$$

Now the writing $y = 3k + x$ is nothing other than the **Euclidean division in** $\mathbb{Z}$ of an integer y by 3, whose remainder is x which must verify the condition $0 \leq x < 3$, and since $x \in \mathbb{Z}$, the only values that x can take are therefore : $0, 1, 2$. We deduce that the equivalence classes are therefore: $\bar{0}, \bar{1}, \bar{2}$.

The quotient set $\mathbb{Z}/\mathcal{R}$ is therefore: $\mathbb{Z}/\mathcal{R} = \{\bar{0}, \bar{1}, \bar{2}\}$, this set forms a partition of $\mathbb{Z}$.

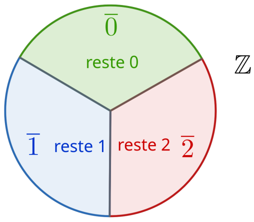

Figure 3.1: Partition of $\mathbb{Z}$ formed by the quotient set $\mathbb{Z}/3\mathbb{Z}$

Remark 1 (**Congruence**)**.** *This equivalence relation is famous and has a name which is* ***congruence modulo*** 3 *and we note it :*

$$y\mathcal{R}x \Longleftrightarrow y \equiv x[3]$$

and we say that y *is* ***congruent*** *to* x ***modulo*** 3. *Its quotient set is noted :*

$$\mathbb{Z}/3\mathbb{Z}$$

This notion of congruence generalizes to any integer $n \in \mathbb{N}^*$ *by saying that* y *is congruent to* x *modulo* n *and we obtain the notations :*

$$y \equiv x[n] \qquad , \qquad \mathbb{Z}/n\mathbb{Z}$$

3.4 Ordre relation

3.4.1 Definitions

Definition 4. *Let* $\mathcal{T}$ *be a binary relation on a set* E. $\mathcal{T}$ *is said to be an* ***order relation*** *if it is :*

1. *Reflexive*
2. *Antisymmetric*
3. *Transitive*

We say that $(E, \mathcal{T})$ *is an ordered set.*

Example 8.

1. *The relation called* ***inequality****, denoted* $\leq$ *on the set* $\mathbb{R}$ *is an order relation. Indeed, we have :*
   (a) $\forall x \in \mathbb{R},\ x \leq x$, *hence* $\leq$ *is reflexive.*
   (b) $\forall x, y \in \mathbb{R},\ x \leq y$ *et* $y \leq x \Longrightarrow x = y$, *hence* $\leq$ *is antisymmetric.*
   (c) $\forall x, y, z \in \mathbb{R},\ x \leq y$ *et* $y \leq z \Longrightarrow x \leq z$, *that is to say that* $\leq$ *is transitive.*
2. *The so-called* ***inclusion*** *relation , denoted* $\subset$, *on the set* $\mathcal{P}(E)$, *is an order relation, because :*
   (a) $\forall A \in \mathcal{P}(E),\ A \subset A$, *hence* $\subset$ *is reflexive.*
   (b) $\forall A, B \in \mathcal{P}(E),\ A \subset B$ *and* $B \subset A \Longrightarrow A = B$, *hence* $\subset$ *is antisymmetric.*
   (c) $\forall A, B, C \in \mathcal{P}(E),,\ A \subset B$ *and* $B \subset C \Longrightarrow A \subset C$, *that is* $\subset$ *is transitive.*

3.4.2 Total order, partial order

Definition 5. *Let $(E, \mathcal{T})$ be an ordered set.*

1. *Two elements x, y of E are said to be* ***comparable*** *for $\mathcal{T}$ if, and only if :*

$$x\mathcal{T}y \qquad ou \qquad y\mathcal{T}x$$

2. *We say that $\mathcal{T}$ is a* ***total order*** *relation if, and only if, the elements of E are all comparable two by two, that is to say :*

$$\forall x, y \in E, \qquad x\mathcal{T}y \qquad or \qquad y\mathcal{T}x$$

otherwise, if

$$\exists x, y \in E \qquad x\not{\mathcal{T}}y \qquad and \qquad y\not{\mathcal{T}}x$$

we then say that $\mathcal{T}$ is a ***partial order*** *relation.*

Example 9.

1. *The order relation $\leq$ on $\mathbb{R}$ is a total order relation on the same set, since for all x, y elements of $\mathbb{R}$, we have $x \leq y$ or $y \leq x$.*

2. *Let the set $E\{-1, 0, 3\}$ be. The order relation $\subset$ on $\mathcal{P}(E)$ is a partial order relation on $\mathcal{P}(E)$, because for the elements $\{-1\} \in \mathcal{P}(E)$ and $\{3\} \in \mathcal{P}$ we have*

$$\{-1\} \not\subset \{3\} \qquad and \qquad \{3\} \not\subset \{-1\}$$

in other words, there are two elements of $\mathcal{P}(E)$ which are not comparable for $\subset$.

3.4.3 Remarkable sets

Definition 6. *Let $(E, \leq)$ be an ordered set and let $A \subset E$.*

1. *Let $x \in E$. We say that x is an* ***upper bound*** *(resp.* ***lower bound****) of A in E if and only if :*

$$\forall a \in A,\ a \leq x \qquad (\text{resp. } \forall a \in A,\ x \leq a).$$

2. *We say that A is* ***bounded above*** *(resp.* ***bounded below****) in E if and only if A admits at least one upper bound (resp. lower bound) in E.*

3. *Let $\alpha \in A$. We say that α is the* ***greatest*** *(resp.* ***smallest****) element of A if and only if :*

$$\alpha \in A \text{ and } \forall a \in A,\ a \leq \alpha \qquad (\text{resp. } \alpha \in A \text{ and } \forall a \in A,\ \alpha \leq a).$$

Definition 7. *Let $(E, \leq)$ be an ordered set and let $A \subset E$.*

1. *If the set $\mathrm{Maj}_E(A)$ of upper bound of A in E admits a smallest element M, then M is called the* ***smallest upper bound*** *of A in E and is denoted $\sup_E(A)$.*

2. *If the set $\mathrm{Min}_E(A)$ of lower bounds of A in E has a greatest element m, then m is called the* ***greatest lower bound*** *of A in E and is denoted $\inf_E(A)$.*