4

Real-valued function with a real variable Differentiability

4.1. The slope of function's graph – Notion of differentiability.

The derivative represents the speed of change of the value of the function with respect to the variable of the same function. For instance, we consider f a function real-valued of a real variable defined over an open interval I and let $x_0, x_1 \in I$. Then, the growth rate Gr is given by

$$Gr(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

See that $Gr(x_1)$ represents the slope of the line (T_{x_0,x_1}) that join $\big(x_0, f(x_0)\big)$ and $\big(x_1, f(x_1)\big)$, this line has the following equation

$$(T_{x_0,x_1}) : \ y = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0).$$

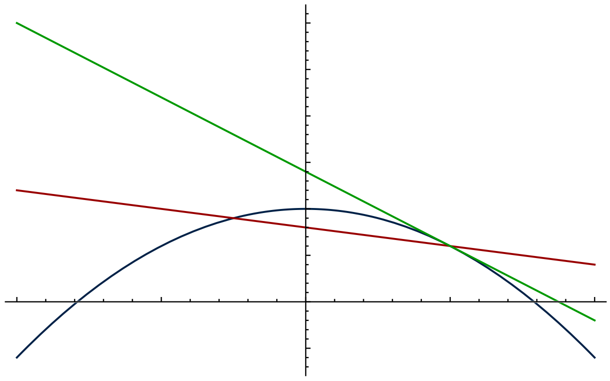

FIGURE 4.1 : $f(x) = 10 - 8x^2$
$(T_{1,-1/2})$: $y = -4x + 16$
(T_1) : $y = -16x + 28$

Definition 4.1 (The tangent). *Let f be a function defined on some neighbourhood v of $x_0 \in \mathbb{R}$. We call the tangente of the curve of f at x_0, noted (T_{x_0}), the limite of (T_{x_0,x_1}) when x_1 goes to x_0. In this case, $\big(x_0, f(x_0)\big)$ is the point of tangency. The graph's slope of f at x_0 will be identified as the tangent's slope of the graph at x_0.*

See that the existence of the limit of (T_{x_0,x_1}) when x_1 goes to x_0 is related to the existence of

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

The value of the previous limit, when it exists, we call it the derivative of the function f at x_0 ; by construction, this value represents the tangent's slope (and by convention, the slope of the graph) of f at x_0. In this first section, we define the derivative, left derivative, and right derivation and exhibit the relations between these concepts.

Definition 4.2. *Let $x_0 \in \mathbb{R}$ and f be a function defined over a neighbourhood of x_0. We say that f is differentiable at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \textit{exists and finite},$$

and we write

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) = f^{(1)}(x_0) = \frac{df}{dx}(x_0). \tag{4.1}$$

- *The value noted $f^{(1)}(x_0)$, or $f'(x_0)$, is called the derivative of the function f at x_0.*

If we consider the function $\varepsilon(\cdot)$ defined over a neighbourhood of zero by the expression

$$\varepsilon(h) = \frac{f(x_0+h) - f(x_0)}{h} - f^{(1)}(x_0).$$

On the one hand, we have $f(x_0+h) = f(x_0) + hf^{(1)}(x_0) + h\varepsilon(h)$, and on the other hand $\varepsilon(h)$ goes to zero as h does. Hence the following definition.

Definition 4.3. *Let $x_0 \in \mathbb{R}$ and f be a function defined over a neighbourhood of x_0. We say that f is differentiate at x_0 (or has a derivative at x_0) if there exists a real-valued function $\varepsilon(\cdot)$ defined on some neighbourhood of zero and there exists $L \in \mathbb{R}$ such that*

$$f(x_0+h) = f(x_0) + hL + h\varepsilon(h), \quad \operatorname*{Lim}_{h\to 0} \varepsilon(h) = 0.$$

The value L is called the derivative of f at x_0 and we write $L = f^{(1)}(x_0)$.

Example 4.1. Let the function f defined by

$$\forall x \in \mathbb{R}: \quad f(x) = \sin(x),$$

we want calculate the derivative of the function f at x_0, we recall the following identity

$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right),$$

we set $a = x_0 + h$ and $b = x_0$, we get

$$\operatorname*{Lim}_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \operatorname*{Lim}_{h\to 0} \frac{\sin(x_0+h) - \sin(x_0)}{h} = \operatorname*{Lim}_{h\to 0} \frac{\cos\left(\frac{2x_0+h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}},$$

we use the fact that $\sin(\alpha)/\alpha$ goes to zero as α does, we obtain

$$\operatorname*{Lim}_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \operatorname*{Lim}_{h\to 0} \cos\left(\frac{2x_0+h}{2}\right) \operatorname*{Lim}_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \cos(x_0).$$

Therefore, f is a differentiable and its derivative function is defined as $f^{(1)}(x) = \cos(x)$ for any $x \in \mathbb{R}$. $\triangle$

Proposition 4.1. *Let f be a real-valued function with a real variable. If f has a derivative at $x_0 \in \mathbb{R}$ then f is continuous at x_0.*

Proof. Let f a differentiable function at x_0 (see Definition 4.3), then there exists a real-valued function $\varepsilon(\cdot)$ defined on some neighbourhood of zero and there exists $L \in \mathbb{R}$ such that

$$f(x_0+h) = f(x_0) + hL + h\,\varepsilon(h), \quad \operatorname*{Lim}_{h\to 0} \varepsilon(h) = 0.$$

Therefore, $f(x_0+h)$ goes to $f(x_0)$ as x goes to x_0. Hence, f is continuous at x_0. $\square$

Remark 4.1. *The reciprocity of Proposition 4.1 is false. For instance, the function f defined as $f(x) = |x|$ is a continuous one over $\mathbb{R}$ but*

$$\operatorname*{Lim}_{h\to 0} \frac{f(h) - f(0)}{h} = \operatorname*{Lim}_{h\to 0} \frac{|h|}{h} = \pm 1,$$

hence, f does not have a derivative at zero.

Definition 4.4 (Left derivative − Right derivative *). *Let f be a real-valued function with a real variable defined over $I \subset \mathbb{R}$ and $x_0 \in I$.*

- *If f is defined over $]x_0 - a, x_0[$ for some $a \in \mathbb{R}_+^*$, we say that f has a left derivative at x_0 if there exists $L \in \mathbb{R}$ such as*
$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = L,$$
and we set $L = f^{(1)}(x_0^-) = f_-^{(1)}(x_0)$.
- *If f is defined over $]x_0, x_0 + a[$ for some $a \in \mathbb{R}_+^*$, we say that f has a right derivative at x_0 if there exists $L \in \mathbb{R}$ such as*
$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = L,$$
and we set $L = f^{(1)}(x_0^+) = f_+^{(1)}(x_0)$.

* The left derivative (respectively right derivative) terminology can be replaced by the left-hand derivative (respectively right-hand derivative)

Geometrically, the left derivative (respectively the right derivative) at x_0 gives the slope of the half tangent at the left side (respectively the right side) of the function's graph at $(x_0, f(x_0))$. More precisely, if the function f has a left derivative $f_-^{(1)}(x_0)$ at x_0, then the graph of f at x_0 has a half tangent from the left side noted $(T_{x_0^-})$ and defined by the expression

$$(T_{x_0^-}): \quad y = f_-^{(1)}(x_0)(x - x_0) + f(x_0), \quad x \leq x_0.$$

If the function f has a right derivative $f_+^{(1)}(x_0)$ at x_0, then the graph of f at x_0 has a half tangent from the right side noted $(T_{x_0^+})$ and defined by the expression

$$(T_{x_0^+}): \quad y = f_+^{(1)}(x_0)(x - x_0) + f(x_0), \quad x \geq x_0.$$

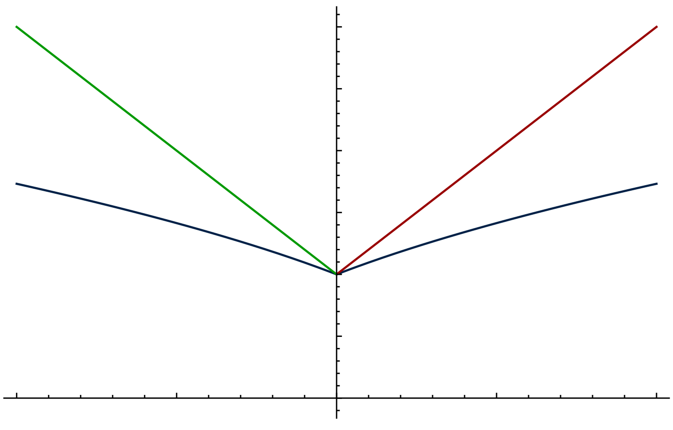

FIGURE 4.2 : $f(x) = \sqrt{|x| + 1}$, $(T_{0^-}): \ y = -x + 1$, $(T_{0^+}): \ y = x + 1$.

Proposition 4.2. *Let f be a real-valued function with a real variable defined over $I \subset \mathbb{R}$. The function f has a derivative at $x_0 \in I$ if and only if its right derivative at x_0 is equal to its left derivative at x_0.*

Example 4.2. For instance, the function f defined as $f(x) = |x|$ is a continuous one over $\mathbb{R}$ and

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = +1, \quad \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = -1.$$

Hence, at zero, f has left derivative $f_-^{(1)}(0) = f^{(1)}(0^-) = -1$ and right derivative $f_+^{(1)}(0) = f^{(1)}(0^+) = +1$. The left and right derivatives of $|x|$ at zero are not equal; hence, this function does not have a derivative at zero. △

4.2. Derivative Operation.

In this section, we will exhibit the rules and properties that govern the differentiation of functions involving addition, subtraction, multiplication, division, and composition. These rules provide powerful techniques to find the derivatives of more complex expressions by breaking them down into simpler components.

Theorem 4.1. *Let $\lambda \in \mathbb{R}$ and f, g be two differentiable functions at x_0. Then*

$$(\lambda f)^{(1)}(x_0) = \lambda f^{(1)}(x_0), \qquad (f+g)^{(1)}(x_0) = f^{(1)}(x_0) + g^{(1)}(x_0),$$

$$(fg)^{(1)}(x_0) = f^{(1)}(x_0)g(x_0) + f(x_0)g^{(1)}(x_0), \quad (f \circ g)^{(1)}(x_0) = g^{(1)}(x_0) f^{(1)}(g(x_0)),$$

$$\left(\frac{1}{g}\right)^{(1)}(x_0) = -\frac{g^{(1)}(x_0)}{g(x_0)^2}, \qquad \left(\frac{f}{g}\right)^{(1)}(x_0) = \frac{f(x_0)g^{(1)}(x_0) - f(x_0)g^{(1)}(x_0)}{g(x_0)^2}.$$

Proposition 4.3. *Let I be an interval of $\mathbb{R}$ and f be a continuous and strictly increasing (respectively strictly decreasing) function defined over I with value in $J = f(I)$. Then, there exists a unique continuous and strictly increasing (respectively strictly decreasing) function noted f^{-1} (called the inverse of f) defined over J with value in I satisfies*

$$\forall (x,y) \in I \times J : \quad (f^{-1} \circ f)(x) = f^{-1}\big(f(x)\big) = x, \quad (f \circ f^{-1})(y) = f\big(f^{-1}(y)\big) = y. \quad (4.2)$$

Moreover, if f is differentiable at x_0 and $f^{(1)}(x_0) \neq 0$ then g is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)^{(1)}(y_0) = \frac{1}{f^{(1)}\big(f^{-1}(y_0)\big)}.$$

Example 4.3. [The inverse of sin] Let the function f be defined as

$$\begin{aligned} f :]-\pi/2, +\pi/2[&\longrightarrow]-1, +1[\\ x &\longmapsto y = f(x) = \sin(x). \end{aligned}$$

The function f is continuous and différentiable over $]-\pi/2, +\pi/2[$, and $f^{(1)}(x) = \cos(x) > 0$ for any $x \in]-\pi/2, +\pi/2[$. Hence, f is a strictly increasing function, thus :

$$f(]-\pi/2, +\pi/2[) =]f(-\pi/2), f(+\pi/2)[=]-1, +1[.$$

Then, f is bijective from $]-\pi/2, +\pi/2[$ to $]-1,+1[$. So, f has an inverse noted $f^{-1}(y) = \arcsin(y)$ and

$$\forall y \in]-1, +1[: \quad \left(f^{-1}\right)^{(1)}(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{\cos\left(f^{-1}(y)\right)}. \qquad (4.3)$$

For $y \in]-1,+1[$ we have $f^{-1}(y) \in]-\pi/2, +\pi/2[$; therefore, $\cos\big(f^{-1}(y)\big) > 0$. Thus,

$$\forall y \in]-1, +1[: \quad \cos\left(f^{-1}(y)\right) = \sqrt{1 - \left(\sin\left(f^{-1}(y)\right)\right)^2} = \sqrt{1 - \left(f\left(f^{-1}(y)\right)\right)^2} = \sqrt{1-y^2},$$

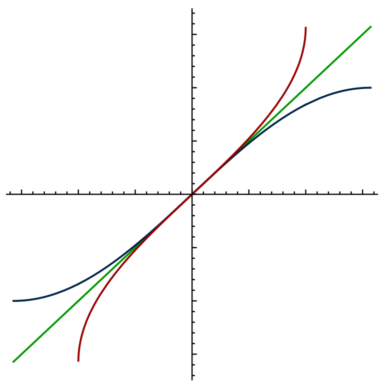

FIGURE 4.3

this transforms (4.3) into

$$\forall y \in]-1, +1[: \quad \arcsin^{(1)}(y) = \frac{1}{\sqrt{1-y^2}}.$$

Figure 4.3 represent the function sin over the interval $]-\pi/2, \pi/2[$ and the function arcsin over the interval $]-1, +1[$.

$y = x$
$f(x) = \sin(x)$
$f^{-1}(x) = \arcsin(x)$

△

Example 4.4. [The inverse of cos] Let the function f be defined as

$$\begin{aligned} f\,:\,]0,\pi[&\longrightarrow]-1,+1[\\ x &\longmapsto y = f(x) = \cos(x). \end{aligned}$$

The function f is continuous and différentiable over $]0,\pi[$ and $f^{(1)}(x) = -\sin(x) < 0$ for any $x \in]0,\pi[$. Hence, f is a strictly decreasing function, thus $f(]0,\pi[) =]f(0), f(\pi)[=]-1,+1[$. Then, f is bijective from $]0,\pi[$ to $]-1,+1[$. Hence, f has an inverse noted $f^{-1}(y) = \arccos(y)$ and

$$\forall y \in]-1,+1[: \quad \left(f^{-1}\right)^{(1)}(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = -\frac{1}{\sin\left(f^{-1}(y)\right)}. \tag{4.4}$$

For $y \in]-1,+1[$ we have $f^{-1}(y) \in]0,\pi[$; hence $\sin\big(f^{-1}(y)\big) > 0$. Thus,

$$\forall y \in]-1,+1[: \quad \sin\left(f^{-1}(y)\right) = \sqrt{1-\left(\cos\left(f^{-1}(y)\right)\right)^2} = \sqrt{1-\left(f\left(f^{-1}(y)\right)\right)^2} = \sqrt{1-y^2},$$

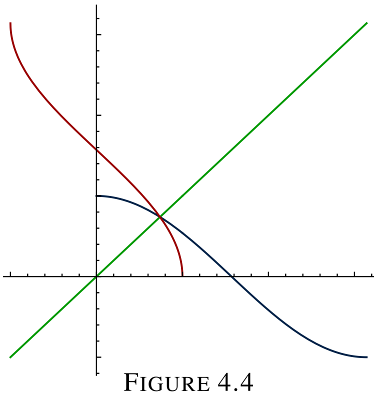

FIGURE 4.4

this transforms (4.4) into

$$\forall y \in]-1,+1[: \quad \arccos^{(1)}(y) = -\frac{1}{\sqrt{1-y^2}}.$$

Figure 4.4 represent the function cos over the interval $]0,\pi[$ and the function arccos over the interval $]-1,+1[$.

- $y = x$
- $f(x) = \cos(x)$
- $f^{-1}(x) = \arccos(x)$

△

Example 4.5. [The inverse of tan] Let the function f defined by

$$\begin{aligned} f\,:\,]-\pi/2,+\pi/2[&\longrightarrow]-\infty,+\infty[\\ x &\longmapsto y = f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}. \end{aligned}$$

The function f is continuous and différentiable over $]-\pi/2,+\pi/2[$ and $f^{(1)}(x) = 1/\big(\cos(x)\big)^2 > 0$ for any $x \in]-\pi/2,+\pi/2[$; so f is an increasing function. Thus

$$f(]-\pi/2,+\pi/2[) = \left] \lim_{\substack{x \to -\pi/2 \\ >}} f(x), \lim_{\substack{x \to +\pi/2 \\ <}} f(x) \right[=]-\infty,+\infty[.$$

Thus, f is bijective from $]-\pi/2,+\pi/2[$ to $\mathbb{R}$; hence, f has an inverse noted $f^{-1}(y) = \arctan(y)$ and

$$\forall y \in]-\infty,+\infty[: \quad \left(f^{-1}\right)^{(1)}(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = \left(\cos\left(f^{-1}(y)\right)\right)^2. \tag{4.5}$$

Using the fact that

$$\forall x \in \mathbb{R}: \quad (\cos(x))^2 = \frac{1}{1+(\tan(x))^2},$$

and we choose $x = f^{-1}(y)$, with $y \in \mathbb{R}$, we obtain

$$\forall y \in \mathbb{R}: \quad \left(\cos\left(f^{-1}(y)\right)\right)^2 = \frac{1}{1+\left(\tan\left(f^{-1}(y)\right)\right)^2} = \frac{1}{1+\left(f\left(f^{-1}(y)\right)\right)^2} = \frac{1}{1+y^2}.$$

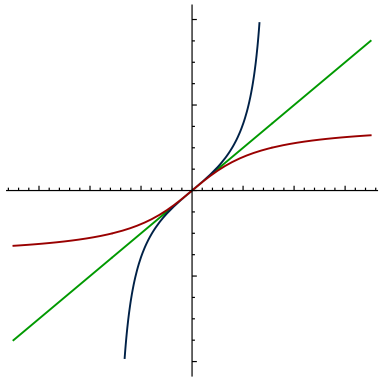

FIGURE 4.5

This with the condition (4.5) gives

$$\forall y \in]-\infty, +\infty[: \quad (\arctan)^{(1)}(y) = \frac{1}{1+y^2}.$$

Figure 4.5 represent the function tan over the interval $]-\pi/2, \pi/2[$ and the function arctan over the interval $]-\infty, +\infty[$.

- $y = x$
- $f(x) = \tan(x)$
- $f^{-1}(x) = \arctan(x)$

△

Example 4.6. [The inverse of cot] Let the function f defined by

$$\begin{aligned} f :\ ]0, \pi[&\longrightarrow]-\infty, +\infty[\\ x &\longmapsto y = f(x) = \cot(x) = \frac{\cos(x)}{\sin(x)}. \end{aligned}$$

The function f is continuous and différentiable over $]0, \pi[$ and $f^{(1)}(x) = -1/\big(\sin(x)\big)^2 < 0$ for any $x \in]0, \pi[$. Then, f is strictly decreasing function ; thus

$$f(]0, \pi[) = \Big] \lim_{\substack{x \to \pi \\ <}} f(x), \lim_{\substack{x \to 0 \\ >}} f(x) \Big[=]-\infty, +\infty[.$$

Thus, f is a bijective from $]0, \pi[$ to $]-\infty, +\infty[$. Hence, f has an inverse noted $f^{-1}(y) = \mathrm{arccot}(y)$ and

$$\forall y \in]-\infty, +\infty[: \quad \left(f^{-1}\right)^{(1)}(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = -\left(\sin\left(f^{-1}(y)\right)\right)^2. \tag{4.6}$$

Using the fact that

$$\forall x \in \mathbb{R} : \quad (\sin(x))^2 = \frac{1}{1 + (\cot(x))^2},$$

and we choose $x = f^{-1}(y)$, with $y \in \mathbb{R}$, we obtain

$$\forall y \in \mathbb{R} : \quad \left(\sin\left(f^{-1}(y)\right)\right)^2 = \frac{1}{1 + \left(\cot\left(f^{-1}(y)\right)\right)^2} = \frac{1}{1 + \left(f\left(f^{-1}(y)\right)\right)^2} = \frac{1}{1+y^2},$$

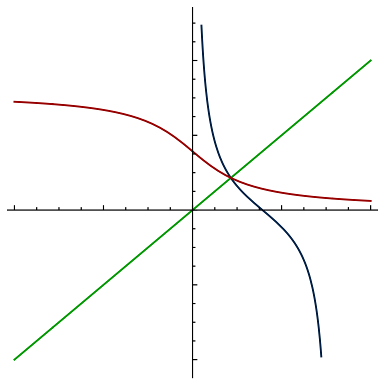

FIGURE 4.6

this with the condition (4.6) gives

$$\forall y \in]-\infty, +\infty[: \quad (\mathrm{arccot})^{(1)}(y) = -\frac{1}{1+y^2}.$$

Figure 4.6 represent the function cot over the interval $]-0, \pi[$ and the function arccot over the interval $]-\infty, +\infty[$.

- $y = x$
- $f(x) = \cot(x)$
- $f^{-1}(x) = \mathrm{arccot}(x)$

△

Example 4.7. [The inverse of sh] Let the function f be defined as

$$\begin{aligned} f :\ ]-\infty, +\infty[&\longrightarrow]-\infty, +\infty[\\ x &\longmapsto y = \mathrm{sh}(x) = \frac{e^x - e^{-x}}{2}. \end{aligned}$$

The function f is continuous monotonous and strictly increasing over $]-\infty,+\infty[$; thus,

$$f(]-\infty,+\infty[) = \left]\lim_{x\to-\infty} f(x), \lim_{x\to+\infty} f(x)\right[=]-\infty,+\infty[.$$

Therefore, f is a bijection between $\mathbb{R}$ and $\mathbb{R}$. Thus f has an inverse noted $f^{(-1)}(y) = \text{arcsh}(y)$, and

$$\forall y \in]-\infty,+\infty[: \quad \left(f^{-1}\right)^{(1)}(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{\text{ch}\left(f^{-1}(y)\right)}. \tag{4.7}$$

Since $\left(\text{ch}(x)\right)^2 - \left(\text{sh}(x)\right)^2 = 1$; then, $\text{ch}(x) = \sqrt{1+\left(\text{sh}(x)\right)^2}$, we set $x = f^{-1}(y)$ with $y \in \mathbb{R}$, we get

$$\forall y \in \mathbb{R}: \quad \text{ch}\left(f^{-1}(y)\right) = \sqrt{1-\left(\text{sh}\left(f^{-1}(y)\right)\right)^2} = \sqrt{1+\left(f\left(f^{-1}(y)\right)\right)^2} = \sqrt{1+y^2}.$$

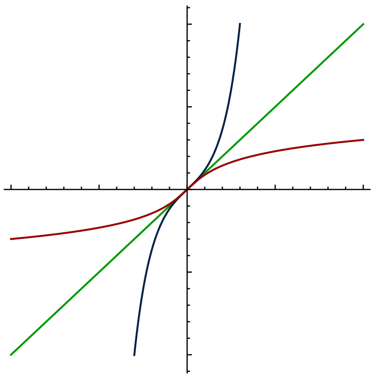

FIGURE 4.7

This transforms (4.7) to

$$\forall y \in]-\infty,+\infty[: \quad (\text{arcsh})^{(1)}(y) = \frac{1}{\sqrt{1+y^2}}.$$

Figure 4.7 represent the function sh over the interval $]-\infty,+\infty[$ and the function arcsh over the interval $]-\infty,+\infty[$.

- $y = x$
- $f(x) = \text{sh}(x)$
- $f^{-1}(x) = \text{arcsh}(x)$

△

Example 4.8. [The inverse of ch] Let f defined as

$$\begin{aligned} f: [0,+\infty[&\longrightarrow [1,+\infty[\\ x &\longmapsto y = f(x) = \text{ch}(x) = \frac{e^x+e^{-x}}{2}. \end{aligned}$$

See that the function f is continuous monotone and strictly increasing over $[0,+\infty[$. Thus,

$$f([0,+\infty[) = \left[f(0), \lim_{x\to+\infty} f(x)\right[= [1,+\infty[.$$

Therefore, f is a bijection from $\mathbb{R}_+$ to $[1,+\infty[$. Thus f has an inverse noted $f^{(-1)}(y) = \text{arcch}(y)$, and

$$\forall y \in [0,+\infty[: \quad \left(f^{-1}\right)^{(1)}(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{\text{sh}\left(f^{-1}(y)\right)}. \tag{4.8}$$

using the information $\left(\text{ch}(x)\right)^2 - \left(\text{sh}(x)\right)^2 = 1$ and $\text{sh}(x) \geq 0$ for $x \in \mathbb{R}_+$, we obtain $\text{sh}(x) = \sqrt{\left(\text{ch}(x)\right)^2 - 1}$.
We set $x = f^{-1}(y)$, with $y \in [1,+\infty[$, we get

$$\forall y \in [1,+\infty[: \quad \text{sh}\left(f^{-1}(y)\right) = \sqrt{\left(\text{ch}\left(f^{-1}(y)\right)\right)^2 - 1} = \sqrt{\left(f\left(f^{-1}(y)\right)\right)^2 - 1} = \sqrt{y^2-1}.$$

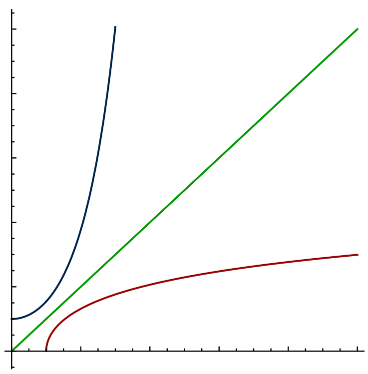

FIGURE 4.8

This transforms (4.8) to

$$\forall y \in [1,+\infty[: \quad (\text{arcch})^{(1)}(y) = \frac{1}{\sqrt{y^2-1}}.$$

Figure 4.8 represent the function ch over the interval $[0,+\infty[$ and the function arcch over the interval $[1,+\infty[$.

- $y = x$
- $f(x) = \text{ch}(x)$
- $f^{-1}(x) = \text{arcch}(x)$

△

Example 4.9. [The inverse of tanh] Let the function $f = \tanh$ (hyperbolic tangent) defined over $\mathbb{R}$ by

$$\begin{aligned} f :]-\infty, +\infty[&\longrightarrow]-1, +1[\\ x &\longmapsto y = f(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \end{aligned}$$

The function f is continuous, différentiable, and strictly increasing over $\mathbb{R}$. Thus, f is a function. Therefore,

$$f(]-\infty, +\infty[) = \left] \lim_{x\to-\infty} f(x), \lim_{x\to+\infty} f(x) \right[=]-1, +1[.$$

Thus, f is bijective from $\mathbb{R}$ to $]-1, +1[$. Hence, f has an inverse noted $f^{-1}(y) = \operatorname{arctanh}(y)$ and

$$\forall y \in]-1, +1[: \quad \left(f^{-1}\right)^{(1)}(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = \left(\operatorname{ch}\left(f^{-1}(y)\right)\right)^2. \tag{4.9}$$

Since for $x \in \mathbb{R}$ we have $(\operatorname{ch}(x))^2 = 1/\left(1 - (\tanh(x))^2\right)$, we set $x = f^{-1}(y)$ (with $y \in]-1, +1[$) we obtain

$$\forall y \in]-1, +1[: \quad \left(\operatorname{ch}\left(f^{-1}(y)\right)\right)^2 = \frac{1}{1 - \left(\tanh\left(f^{-1}(y)\right)\right)^2} = \frac{1}{1 - \left(f\left(f^{-1}(y)\right)\right)^2} = \frac{1}{1 - y^2}.$$

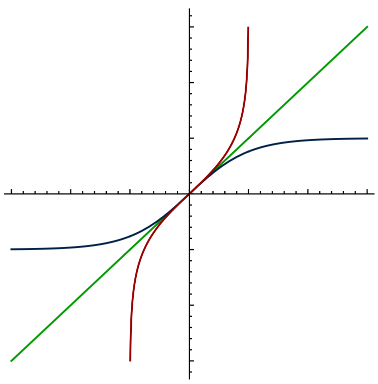

FIGURE 4.9

This transforms the condition (4.9) into

$$\forall y \in]-1, +1[: \quad (\operatorname{arctanh})^{(1)}(y) = \frac{1}{1 - y^2}.$$

Figure 4.9 represent the function tanh over the interval $]-\infty, +\infty[$ and the function arctanh over the interval $]1, +1[$.

$y = x$
$f(x) = \tanh(x)$
$f^{-1}(x) = \operatorname{arctanh}(x)$

△

Example 4.10. [The inverse of coth] The function coth (hyperbolic cotangent) is defined over $\mathbb{R}^*$ by the expression $\coth(x) = \operatorname{ch}(x)/\operatorname{sh}(x)$. Let the function f be defined as

$$\begin{aligned} f :]0, +\infty[&\longrightarrow]1, +\infty[\\ x &\longmapsto y = f(x) = \coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \end{aligned}$$

The function f is continuous, différentiable, and strictly decreasing over $]0, +\infty[$. Therefore

$$f(]0, +\infty[) = \left] \lim_{x\to+\infty} f(x), \lim_{\substack{x\to 0 \\ >}} f(x) \right[=]1, +\infty[.$$

Thus, f is bijective from $]0, +\infty[$ to $]1, +\infty[$. Hence, f has an inverse noted $f^{-1}(y) = \operatorname{arccoth}(y)$ and

$$\forall y \in]1, +\infty[: \quad \left(f^{-1}\right)^{(1)}(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = -\left(\operatorname{sh}\left(f^{-1}(y)\right)\right)^2. \tag{4.10}$$

Since $(\operatorname{sh}(x))^2 = 1/\left((\coth(x))^2 - 1\right)^2$, we set $x = f^{-1}(y)$, with $]1, +\infty[$, we obtain

$$\left(\operatorname{sh}\left(f^{-1}(y)\right)\right)^2 = -\frac{1}{\left(\coth\left(f^{-1}(y)\right)\right)^2 - 1} = -\frac{1}{\left(f\left(f^{-1}(y)\right)\right)^2 - 1} = \frac{1}{1 - y^2}.$$

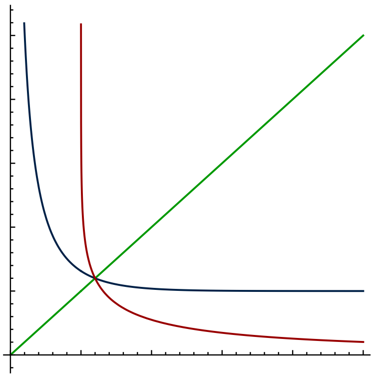

FIGURE 4.10

This transforms the condition (4.10) into

$$\forall y \in]1,+\infty[: \quad (\operatorname{arccoth})^{(1)}(y) = \frac{1}{1-y^2}.$$

Figure 4.10 represent the function coth over the interval $]0,+\infty[$ and the function arccoth over the interval $]1,+\infty[$.

△

Definition 4.5. *Let f be a real-valued function of a real variable defined over an interval I of $\mathbb{R}$. The derivative of order n, when it exists, of the function f is noted $f^{(n)}$ and is defined by*

$$f^{(n)} = \left(f^{(n-1)}\right)^{(1)}, \quad f^{(0)} = f.$$

Example 4.11. For instance, let f be defined over $\mathbb{R}$ as $f(x) = x^2 + x + 1$. Then,

$$f^{(1)}(x) = 2x + 1, \quad f^{(2)}(x) = 2, \quad f^{(3)}(x) = 0.$$

△

Definition 4.6. *Let f be a real-valued function with a real variable defined over an interval I of $\mathbb{R}$.*

- *We say that f is differentiable over I if it is differentiable at any point of I.*
- *Let $x_0 \in I$, the function $df(x_0)$ defined by*

$$\begin{aligned} df(x_0) : \mathbb{R} &\longrightarrow \mathbb{R} \\ h &\longmapsto \big(df(x_0)\big)(h) = f^{(1)}(x_0)h \end{aligned}$$

  is called the differentiable of f at x_0.
- *The set $\mathscr{C}^0(I,\mathbb{R})$ is the set of continuous function over I. The set $\mathscr{C}^n(I,\mathbb{R})$ is the set for the $n-$ times differentiable function where the $k-$th derivative is continuous for all $k = 1, \cdots, n$.*
- *We say that f is smooth over I, and we write $f \in \mathscr{C}^\infty(I,\mathbb{R})$, If $f \in \mathscr{C}^n(I,\mathbb{R})$ for every $n \in \mathbb{N}$, then*

$$\mathscr{C}^\infty(I;\mathbb{R}) = \bigcap_{n=0}^{\infty} \mathscr{C}^n(I;\mathbb{R}).$$

Example 4.12. The function $f(x) = x^3$ is a smooth one over $\mathbb{R}$, since $f^{(1)}(x) = 3x^2$, $f^{(2)}(x) = 6x$, $f^{(3)}(x) = 6$ and

$$\forall k \in \mathbb{N}, \quad k \geq 6: \quad f^{(k)}(x) = 0,$$

are all continuous over $\mathbb{R}$, thus we write $f \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$.
The function $g(x) = \sqrt{|x|}$ is a continuous function over $\mathbb{R}$ but does not have a derivative at zero; hence, this function is only $\mathscr{C}^0$ over $\mathbb{R}$ and is not a $\mathscr{C}^1$. △

Proposition 4.4 (Leibniz' formula). *Let $f, g \in \mathscr{C}^\infty(I,\mathbb{R})$. Then, for any $x \in I$, and $n \in \mathbb{N}^*$ we have*

$$\big(f(x)g(x)\big)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x) g^{(k)}(x). \tag{4.11}$$

such that $C_n^k = \dfrac{n!}{k!(n-k)!}$ for any $(n,k) \in \mathbb{N} \times \mathbb{N}$ with $k \leq n$.

Example 4.13. Let f be a function defined as $f(x) = e^{2x}\sin(x)$ for any $x \in \mathbb{R}$. Then

$$f^{(3)}(x) = \left(e^{2x}\sin(x)\right)^{(3)} = \sum_{k=0}^{3} C_3^k \left(e^{2x}\right)^{(3-k)} (\sin(x))^{(k)}$$

hence

$$\begin{aligned} f^{(3)}(x) = {} & C_3^0 \left(e^{2x}\right)^{(3)} (\sin(x))^{(0)} + C_3^1 \left(e^{2x}\right)^{(2)} (\sin(x))^{(1)} \\ & + C_3^2 \left(e^{2x}\right)^{(1)} (\sin(x))^{(2)} + C_3^3 \left(e^{2x}\right)^{(0)} (\sin(x))^{(3)} . \end{aligned}$$

The calculus gives $C_3^0 = 1$, $C_3^1 = 3$, $C_3^2 = 3$, $C_3^3 = 1$ and

$$\left(e^{2x}\right)^{(0)} = e^{2x}, \quad \left(e^{2x}\right)^{(1)} = 2e^{2x}, \quad \left(e^{2x}\right)^{(2)} = 4e^{2x}, \quad \left(e^{2x}\right)^{(3)} = 6e^{2x},$$

$$(\sin(x))^{(0)} = \sin(x),\ (\sin(x))^{(1)} = \cos(x),\ (\sin(x))^{(2)} = -\sin(x),\ (\sin(x))^{(3)} = -\cos(x).$$

Therefore

$$f^{(3)}(x) = \left(e^{2x}\sin(x)\right)^{(3)} = 8e^{2x}\sin(x) + 12e^{2x}\cos(x) - 6e^{2x}\sin(x) - e^{2x}\cos(x).$$

△

4.3. Derivative table.

The following table summarises the derivative of the usual function as well as the derivative of their inverses

f	$f^{(1)}$	f	$f^{(1)}$
x^n	nx^{n-1}	$a \in \mathbb{R}$	0
$\mathrm{Ln}(x)$	$\frac{1}{x}$	$\mathrm{Ln}(f(x))$	$\frac{f^{(1)}(x)}{f(x)}$
e^{ax}	ae^{ax}	$e^{f(x)}$	$f^{(1)}(x)e^{f(x)}$
$\cos(x)$	$-\sin(x)$	$\cos(f(x))$	$-f^{(1)}(x)\sin(f(x))$
$\sin(x)$	$\cos(x)$	$\sin(f(x))$	$f^{(1)}(x)\cos(f(x))$
$\tan(x)$	$\frac{1}{(\cos(x))^2}$	$\tan(f(x))$	$\frac{f^{(1)}(x)}{(\cos(f(x)))^2}$
$\mathrm{sh}(x)$	$\mathrm{ch}(x)$	$\mathrm{sh}(f(x))$	$f^{(1)}(x)\mathrm{ch}(f(x))$
$\mathrm{ch}(x)$	$\mathrm{sh}(x)$	$\mathrm{ch}(f(x))$	$f^{(1)}(x)\mathrm{sh}(f(x))$
$\tanh(x)$	$\frac{1}{(\mathrm{ch}(x))^2}$	$\tanh(f(x))$	$\frac{f^{(1)}(x)}{(\mathrm{ch}(f(x)))^2}$
$\coth(x)$	$-\frac{1}{(\mathrm{sh}(x))^2}$	$\coth(f(x))$	$-\frac{f^{(1)}(x)}{(\mathrm{sh}(f(x)))^2}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(f(x))$	$\frac{f^{(1)}(x)}{\sqrt{1-(f(x))^2}}$

$\arccos(x)$	$-\dfrac{1}{\sqrt{1-x^2}}$	$\arccos(f(x))$	$-\dfrac{f^{(1)}(x)}{\sqrt{1-(f(x))^2}}$
$\arctan(x)$	$\dfrac{1}{1+x^2}$	$\arctan(f(x))$	$\dfrac{f^{(1)}(x)}{1+(f(x))^2}$
$\operatorname{arccot}(x)$	$-\dfrac{1}{1+x^2}$	$\operatorname{arccot}(f(x))$	$-\dfrac{f^{(1)}(x)}{1+(f(x))^2}$
$\operatorname{arcsh}(x)$	$\dfrac{1}{\sqrt{1+x^2}}$	$\operatorname{arcsh}(f(x))$	$\dfrac{f^{(1)}(x)}{\sqrt{1+(f(x))^2}}$
$\operatorname{arcch}(x)$	$\dfrac{1}{\sqrt{x^2-1}}$	$\operatorname{arcch}(f(x))$	$\dfrac{f^{(1)}(x)}{\sqrt{(f(x))^2-1}}$
$\operatorname{arctanh}(x)$	$\dfrac{1}{1-x^2}$	$\operatorname{arctanh}(f(x))$	$\dfrac{f^{(1)}(x)}{1-(f(x))^2}$
$\operatorname{arccoth}(x)$	$\dfrac{1}{x^2-1}$	$\operatorname{arccoth}(f(x))$	$\dfrac{f^{(1)}(x)}{(f(x))^2-1}$

TABLE 4.2 : Derivative table

4.4. Fundamental theorems related to differentiability.

Definition 4.7. *Let f be a real-valued function defined over an interval I of $\mathbb{R}$ and let $x_0 \in I$. We say that :*

- *f has a local extrema at x_0, if f has a local maximum or a local minimum at x_0.*
- *f has a global extrema at x_0, if f has a global maximum or a global minimum at x_0.*

Example 4.14. For instance, the function f defined over $\mathbb{R}$ by

$$f(x) = \sqrt{|x|+1}$$

has a global extrema, a global minimum, at $x_0 = 0$ (see Figure 4.2 page 3). △

Theorem 4.2 (Fermat). *Let f be a real-valued function differentiable on a neighbourhood of $x_0 \in \mathbb{R}$. If f has an extrema at x_0, then $f^{(1)}(x_0) = 0$.*

Remark 4.2. *The function f can have an extrema at x_0 without be différentiable at x_0. For instance, the function f defined over $\mathbb{R}$ by $f(x) = \sqrt{|x|+1}$ has a minimum at $x_0 = 0$ but does not have a derivative at zero (see Figure 4.2 page 3).*

Theorem 4.3 (Rolle). *Let $a, b \in \mathbb{R}$ and $f \in \mathscr{C}^0([a,b],\mathbb{R}) \cap \mathscr{C}^1(]a,b[,\mathbb{R})$ such as $f(a) = f(b)$. Then, there exists $c \in]a,b[$ such that $f^{(1)}(c) = 0$.*

Example 4.15. Let's consider a point moving from the position $f(t_1)$ to the position $f(t_2)$ such that $f(t)$ represents the position at the instant t. We suppose the function f is continuous and differentiable over $[t_0, t_1]$. Therefore, we can apply the Rolle's Theorem and obtain

$$\exists\, t' \in]t_0, t_1[: \quad f(t_1) - f(t_0) = f^{(1)}(t').$$

Since $f^{(1)}(t') = v(t')$ is the instantaneous speed at the instant t'; thus, between the instant t_0 and t_1 there exists an instant t' in which the instantaneous velocity is equal to the average one. △

Theorem 4.4 (Lagrange mean value theorem*). *Let $a, b \in \mathbb{R}$ and $f \in \mathscr{C}^0([a,b],\mathbb{R}) \cap \mathscr{C}^1(]a,b[,\mathbb{R})$. Then*

$$\exists\, c \in]a,b[: \quad f(b) - f(a) = (b-a) f^{(1)}(c). \tag{4.12}$$

* Know alose as "Ifinite-increments theorem".

The Lagrange mean value theorem can be understood geometrically, let $(\mathscr{C})$ the curve defined as

$$(\mathscr{C}) = \left\{ \big(x, f(x)\big) : \quad x \in [a, b] \right\}.$$

The slope of the secant line $(L_{a,b})$ passing through $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ is

$$\frac{f(b) - f(a)}{b - a},$$

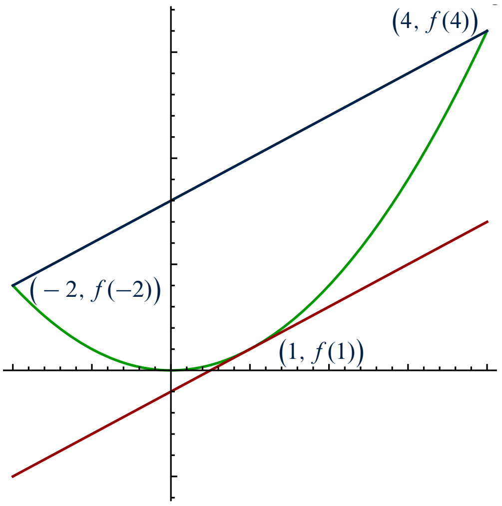

FIGURE 4.11

Theorem 4.4 show the existence of $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

Since $f'(c)$ is the slope of the tangent of the curve $(\mathscr{C})$ at $\big(c, f(c)\big)$ the Lagrange's mean value theorem states that the tangent at $\big(c.f(c)\big)$ is parallel to the secant line passing through $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$.
Figure 4.11 represent the function $f(x) = x^2$ over $[-2, 4]$, the tangent $(T_{(1,1)})$ at $(1, f(1))$ and the line joining $(-2, f(-2))$ and $(4, f(4))$

- $f(x) = x^2$
- $(L_{-2,4}) : \ 2x + 8$
- $(T_{(1,1)}) : \ 2x - 1$

Corollary 4.1. *Let $I \subset \mathbb{R}$ be an interval and $f \in \mathscr{C}^1(I, \mathbb{R})$.*

(a) If for any $x \in I$ we have $f^{(1)}(x) = 0$. Then, f is constant over I.
(b) If for any $x \in I$ we have $f^{(1)}(x) \geq 0$ (respectively $f^{(1)}(x) > 0$); then, f is increasing (respectively strictly increasing) over I.
(c) If for any $x \in I$ we have $f^{(1)}(x) \leq 0$ (respectively $f^{(1)}(x) < 0$); then, f is decreasing (respectively strictly decreasing) over I.

Example 4.16. Let $f(x) = (x + 1)/(x^2 + 1)$ for any $x \in \mathbb{R}$. The calculus gives

$$f^{(1)}(x) = \frac{-x^2 - 2x + 1}{x^2 + 1}.$$

The sign of $f^{(1)}(x)$ is the sign of the polynomial $P(x) = -x^2 - 2x + 1$ and $P(x)$ has a discriminant $\Delta = 8$; thus, this polynomial has two roots

$$x_1 = -1 - \sqrt{2}, \quad x_2 = -1 + \sqrt{2}.$$

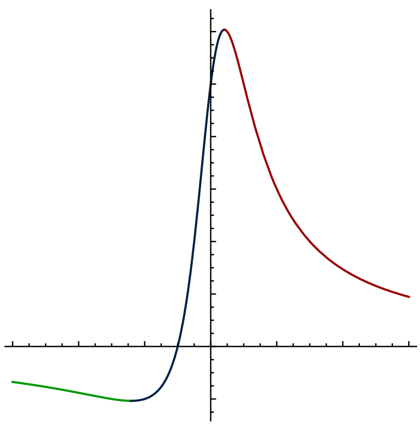

FIGURE 4.12

Therefore, $P(x) = -x^2 - x + 1 > 0$ if and only if $x \in]x_1, x_2[$ and $P(x) = -x^2 - x + 1 < 0$ if and only if $x \in]-\infty, x_1[\cup]x_2, +\infty[$. In conclusion : f is strictly decreasing over $]-\infty, x_1[$, strictly increasing over $]x_1, x_2[$ and strictly decreasing over $]x_2, +\infty[$.

Figure 4.12 represent the function f over the interval $[-6, +6]$.

- f over $[-6, x_1]$
- f over $[x_1, x_2]$
- f over $[x_2, +6]$

△

Proposition 4.5. *Let $f, g \in \mathscr{C}^1([a,b],\mathbb{R})$ such that $f(a) \leq g(a)$ and $f^{(1)}(x) \leq g^{(1)}(x)$ over $[a,b]$. Then*

$$\forall x \in [a,b] : \quad f(x) \leq g(x). \tag{4.13}$$

Example 4.17. We want show that $x \leq e^x$ for any $x \in \mathbb{R}_+$. Let's set $f(x) = x$ and $g(x) = e^x$, see that $f(0) = 0 \leq 1 = g(0)$ and $f^{(1)}(x) = 1 \leq e^x = g^{(1)}(x)$ for any $x \in \mathbb{R}_+$. Therefore, we apply Proposition 4.5 we get $f(x) \leq g(x)$ for any $x \in \mathbb{R}_+$. △

Lemma 4.1. *Let $f, g \in \mathscr{C}^1([a,b],\mathbb{R})$ such that $|f^{(1)}(x)| \leq g^{(1)}(x)$ for every $x \in [a,b]$. Then*

$$\forall x \in [a,b] : \quad |f(x) - f(a)| \leq g(x) - g(a). \tag{4.14}$$

Proposition 4.6. *Let $f, g \in \mathscr{C}^1([a,b],\mathbb{R})$ such that $|f^{(1)}(x)| \leq g^{(1)}(x)$ for every $x \in [a,b]$. Then*

$$\forall x, x_0 \in [a,b] : \quad |f(x) - f(x_0)| \leq |g(x) - g(x_0)|. \tag{4.15}$$

Proposition 4.7 (Cauchy's mean value Theorem). *Let $f, g \in \mathscr{C}^0([a,b],\mathbb{R}) \cap \mathscr{C}^1(]a,b[,\mathbb{R})$ such that $g^{(1)}(x) \neq 0$ for any $x \in]a,b[$. Then*

$$\exists c \in]a,b[: \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f^{(1)}(c)}{g^{(1)}(c)}. \tag{4.16}$$

Theorem 4.5 (L'Hôpital's rule*). *Let $x_0 \in \mathbb{R}$, v a neighbourhood of x_0 and $f, g \in \mathscr{C}^1(v,\mathbb{R})$ such that*

$$\operatorname*{Lim}_{x \to x_0} \frac{f^{(1)}(x)}{g^{(1)}(x)} = l \in \mathbb{R}.$$

Then

$$\operatorname*{Lim}_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = l.$$

* Also known as Bernoulli's rule.

Example 4.18. We want to calculate the limit of $\sin(x)/x$ as x goes to 0. First, we have

$$\operatorname*{Lim}_{x \to 0} \frac{(\sin(x))^{(1)}}{(x)^{(1)}} = \operatorname*{Lim}_{x \to 0} \cos(x) = 1,$$

thus,

$$\operatorname*{Lim}_{x \to 0} \frac{\sin(x)}{x} = \operatorname*{Lim}_{x \to 0} \frac{\sin(x) - \sin(0)}{x - 0} = 1.$$

△

Example 4.19. Let f, g be the functions defined as

$$g(x) = \sin(x), \quad f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & if \ x \neq 0 \\ 0 & if \ x \neq 0 \end{cases}.$$

We want to calculate the $f(x)/g(x)$ limit when x goes to 0. See that

$$\operatorname*{Lim}_{x \to 0} \frac{f(x)}{g(x)} = \operatorname*{Lim}_{x \to 0} \frac{f(x) - f(0)}{g(x) - g(0)}$$

and

$$f^{(1)}(0) = \operatorname*{Lim}_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \operatorname*{Lim}_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0, \quad g^{(1)}(0) = 1.$$

Therefore

$$\operatorname*{Lim}_{x \to 0} \frac{f(x)}{g(x)} = \operatorname*{Lim}_{x \to 0} \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f^{(1)}(0)}{g^{(1)}(0)} = 1.$$

△