3

Real-valued function of a real variable Limit and Continuity

A map f from a set $\mathbb{E}$ (the domain of the map) to a set $\mathbb{F}$ (the codomain of the map) is a relation that associates to each x from $\mathbb{E}$ a unique y in $\mathbb{F}$ such as $y = f(x)$. In the case where $\mathbb{E} \subseteq \mathbb{R}$ and $\mathbb{F} = \mathbb{R}$, we say that we have a real-valued function of a real variable. The variable x is called the preimage of y, the variable $y = f(x)$ is called the image, f is called the name of the function, and we write

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto y = f(x). \end{aligned}$$

Definition 3.1 (Domain of a function – Graph of a function). *The subset of $\mathbb{R}$ formed by the elements $x \in \mathbb{R}$ for which $f(x)$ exists is called the domain of the function f, and it is denoted $\mathscr{D}_f$. We write*

$$\mathscr{D}_f = \{ x \in \mathbb{R} / \quad \exists\, y \in \mathbb{R} : \quad y = f(x) \}.$$

The graph of the function f, denoted as G_f, is defined as the set given by

$$G_f = \{ (x, f(x)) : \quad x \in \mathscr{D}_f \}.$$

The relation f defined from $\mathbb{R}$ to $\mathbb{R}$ by

$$f(x) = \frac{\sin(x)}{x},$$

is a function with a domain $\mathscr{D}_f = \mathbb{R}^*$, for the graphe of f see Figure 3.1. On the other hand, the relation g defined from $\mathbb{R}$ to $\mathbb{R}$ by

$$(g(x))^2 - x^2 = 0, \qquad (3.1)$$

it is not a function since for a given x, there are two values $\pm g(x)$ that satisfy (3.1).

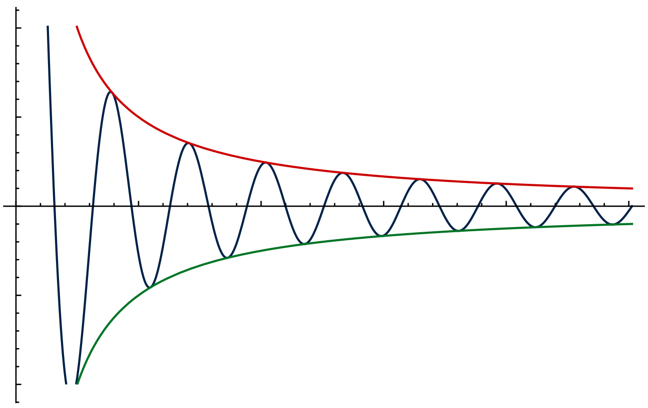

FIGURE 3.1 : $\frac{\sin(x)}{x}$, $\frac{1}{x}$, $-\frac{1}{x}$.

Remark 3.1. *We remark that the function f defined as*

$$\begin{aligned} f : \mathscr{D}_f &\longrightarrow \mathbb{R} \\ x &\longmapsto y = f(x). \end{aligned}$$

is a map. In other words, every function is a mapping over its domain.

3.1. Direction of variation, the monotonicity of a function..

In this section, we will provide definitions related to the direction of variation, the parity and the periodicity of a function. The concept of function's upper-bound and lower-bound will be exposed and follow by some theorems concerning the existence of the supremum and infimum.

Definition 3.2. *Let f be a function with $\mathscr{D}_f$ as a domain, and I an interval such that $I \subset \mathscr{D}_f$.*

- *We say that f is constant on I if there exists $y_0 \in \mathbb{R}$ such that $f(x) = y_0$ for any $x \in I$.*
- *We say that f is increasing*(respectively strictly increasing) on I, if for every $x_1, x_2 \in I$ such that $x_1 < x_2$ we have $f(x_1) \leq f x_2)$ (respectively $f(x_1) < f(x_2)$).*
- *We say that f is decreasing (respectively strictly decreasing) on I, if for every $x_1, x_2 \in I$ such that $x_1 < x_2$ we have $f(x_1) \geq f(x_2)$ (respectively $f(x_1) > f(x_2)$).*
- *We say that f is monotonous (respectively strictly monotonous) on I if it is either increasing or decreasing (respectively strictly increasing or strictly decreasing) on I.*

* In some books, the terminology "increasing" (respectively "strictly increasing") is changed by "non-decreasing" (respectively "increasing").

For instance, the function f defined over $\mathbb{R}_+$ by $f(x) = 2$, is constant one. The function g defined by over $\mathbb{R}_+$ by $g(x) = x^2$ is strictly increasing, since for $x_1, x_2 \in \mathbb{R}_+$ such that $x_2 > x_1$ we have

$$g(x_2) = x_2^2 > x_2 x_1 > x_1^2 = g(x_1).$$

The function h defined on $\mathbb{R}_+^*$ by

$$h(x) = \frac{1}{x}$$

is strictly decreasing.

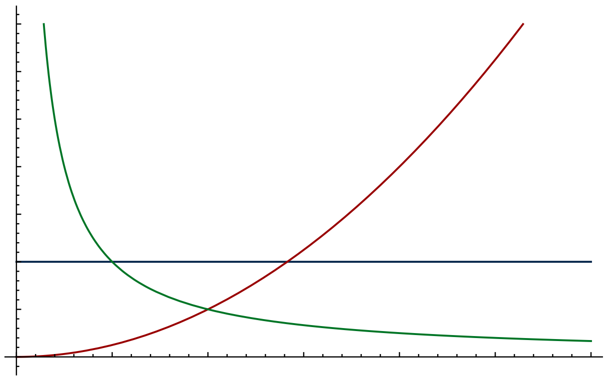

FIGURE 3.2 : $f(x) = 2$, $\frac{1}{x}$, x^2.

Definition 3.3 (Parity and periodicity of function). *Let f be a function with a domain $\mathscr{D}_f$, We say that :*

- *f is an odd one on $I \subset \mathscr{D}_f$ if for every $x \in I$ we have $-x \in I$ and $f(-x) = f(x)$.*
- *f is even one on $I \subset \mathscr{D}_f$ if for every $x \in I$ we have $-x \in I$ and $f(-x) = -f(x)$*
- *f is a periodic one $\mathscr{D}_f = \mathbb{R}$ if there exists $\rho \in \mathbb{R}_+^*$ such as for every $x \in \mathbb{R}$ we have $f(x+\rho) = f(x)$. The smallest strictly positive value of ρ satisfies the previous condition called the period of f.*

Let the functions f and g defined over $[-\pi; \pi]$ as

$$f(x) = \cos(x), \quad g(x) = \sin(x).$$

The function f is an odd one; hence, its graph is symmetric with respect to the axis (OY). The function g is an even one; thus, its graph is symmetric with respect to the origin O. The functions $\sin(\cdot)$ and $\cos(\cdot)$ defined over $\mathbb{R}$ are periodic functions with 2π as period (See Figure 3.3).

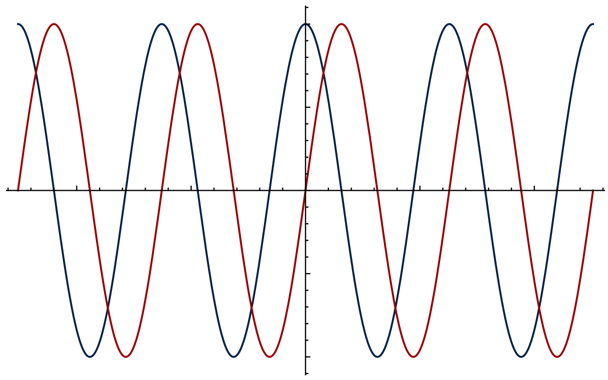

FIGURE 3.3 : $\sin(x)$, $\cos(x)$.

Definition 3.4. *Let f be a fonction with a domain $\mathscr{D}_f$.*

- *We say that f is upper-bounded on $I \subset \mathscr{D}_f$, if there exists $M \in \mathbb{R}$ such as $f(x) \leq M$ for any $x \in I$.*
- *We say that f is lower-bounded on $I \subset \mathscr{D}_f$, if there exists $m \in \mathbb{R}$ such as $f(x) \geq M$ for any $x \in I$.*
- *We say that f is bounded on $I \subset \mathscr{D}_f$, if there exists $M \in \mathbb{R}_+^*$ such as $|f(x)| \geq M$ for any $x \in I$.*

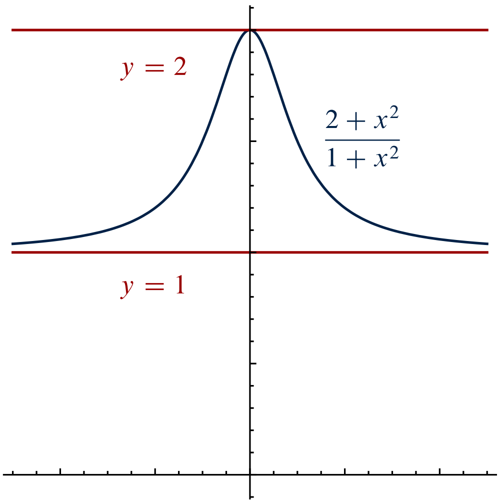

FIGURE 3.4

Example 3.1. Let the function f defined over $\mathbb{R}$ by

$$f(x) = \frac{x^2+2}{x^2+1}.$$

For any $x \in \mathbb{R}$ we have $x^2 + 1 \geq 1$, hence

$$0 \leq \frac{1}{x^2+1} \leq 1.$$

Therefore,

$$\forall x \in \mathbb{R}: \quad 1 \leq f(x) \leq 2.$$

Thus, the function f is upper and lower bounded then is bounded. The graph of a bounded function can be sandwiched between two vertical lines. In this example, these lines have equations $y = 1$ and $y = 2$, (see Figure 3.4). △

Definition 3.5. *Let f be a function with $\mathscr{D}_f$ as a domain and let $I \subset \mathscr{D}_f$ be an interval. We say that f is a Lipschitz function over I if*

$$\exists k \in \mathbb{R}_+^*, \quad \forall x_1, x_2 \in I: \quad |f(x_1) - f(x_2)| \leq k\,|x_1 - x_2|,$$

The constant k is called the Lipschitz constant of the function f; in this case, the function f is called a $k-$Lipschitz one. Moreover, if $k \in]0,1[$, we say that f is a contracted function.

Example 3.2. Let the function f defined over $\mathbb{R}$ by the expression

$$f(x) = \frac{x}{x^2+1}.$$

For any $x_1, x_2 \in \mathbb{R}$ we have

$$|f(x_1) - f(x_2)| = \left|\frac{x_1x_2+1}{(x_1^2+1)(x_2^2+1)}\right| |x_1 - x_2| \leq \frac{|x_1|\,|x_2|+1}{(x_1^2+1)(x_2^2+1)}\,|x_1 - x_2|. \tag{3.2}$$

We remark that $(|x_1| + |x_2|)^2 + |x_1|\,|x_2| + x_1^2x_2^2 + 1 \geq 1$, this leads to $x_1^2 + x_2^2 - |x_1|\,|x_2| + x_1^2x_2^2 + 1 \geq 1$. Hence $(x_1^2+1)(x_2^2+1) \geq 1 + |x_1|\,|x_2|$, thus

$$\frac{1+|x_1|\,|x_2|}{(x_1^2+1)(x_2^2+1)} \leq 1,$$

This with the condition (3.2) gives $|f(x_1) - f(x_2)| \leq |x_1 - x_2|$ for every $x_1, x_2 \in \mathbb{R}$; therefore, f is a Lipschitz function with $k = 1$ as a Lipschitz constant (we say that f is a $1-$Lipschitz function). △

Definition 3.6. *Let f be a real-valued function defined over an interval I and $x_0 \in I$. We say that f has*

- *Local maximum at x_0 if there exists a neighbourhood v of x_0 such that $f(x) \leq f(x_0)$ for any $x \in v \cap I$.*
- *Local minimum at x_0 if there exists a neighbourhood v of x_0 such that $f(x) \geq f(x_0)$ for any $x \in v \cap I$.*
- *Global minimum (or just minimum) at x_0 if $f(x) \geq f(x_0)$ for any $x \in \mathscr{D}_f$.*
- *Global maximum (or just maximum) at x_0 if $f(x) \leq f(x_0)$ for any $x \in \mathscr{D}_f$.*

Example 3.3. The function $f(x) = x^2 + 1$ has 1 as a minimum at 0, since $f(x) \geq 1$ for any $x \in \mathbb{R}$. △

3.2. The limit of function.

In this section, we will discuss the concept of the limit of a function. This concept is fundamental to give meaning to continuity and differentiation.

Definition 3.7 (Finite limit at finite point, see Figure 3.5). *Let f be a function defined on $v(x_0) \setminus x_0$, where $v(x_0)$ is a neighbourhood of x_0. We say that f has a limit l at x_0, and we write $\lim\limits_{x \to x_0} f(x) = l$, if*

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v(x_0) \setminus \{x_0\} : \quad |x - x_0| \leq \eta(\varepsilon) \Longrightarrow |f(x) - l| \leq \varepsilon,$$

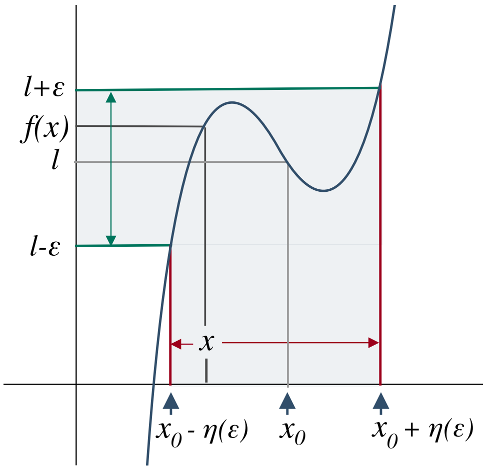

FIGURE 3.5 : The output ($y-$coordinate) approaches l as the input ($x-$coordinate) approaches x_0.

Example 3.4. Let's consider the function f defined on $\mathbb{R}$ by $f(x) = x^2$. We want to show that f has 1 as a limit at 1. Let $\varepsilon \in \mathbb{R}_+^*$, and we work within a neighbourhood of 1 where the function f is defined. For example, let's take $v(1) =]0, 2[$. In this interval, we have

$$|f(x) - 1| = \left|x^2 - 1\right| = |x + 1|\,|x - 1| \leq 2\,|x - 1|\,.$$

We set $2\,|x - 1| \leq \varepsilon$, which implies $|x - 1| \leq \varepsilon/2$, and let $\eta(\varepsilon) = \varepsilon/2$. Thus,

$$\forall \varepsilon > 0, \quad \exists \eta(\varepsilon) > 0, \quad \forall x \in v(1) : \quad |x - 1| \leq \eta(\varepsilon) \Longrightarrow |f(x) - 1| \leq \varepsilon.$$

Therefore, we conclude that $\lim\limits_{x \to 1} f(x) = 1$. △

Definition 3.8 (Finite left-limit – Finite right-limit*). *Let f be a function defined over $v =]x_0 - a, x_0[$ for some $a \in \mathbb{R}_+^*$. We say that*

- *f has l as left-limit at x_0, and we write $\lim\limits_{\substack{x \to x_0 \\ <}} f(x) = l$, if*

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v : \quad 0 < x_0 - x \leq \eta(\varepsilon) \Longrightarrow |f(x) - l| \leq \varepsilon.$$

- *f has l as right-limit at x_0, and we write $\lim\limits_{\substack{x \to x_0 \\ >}} f(x) = l$, if*

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v : \quad 0 < x - x_0 \leq \eta(\varepsilon) \Longrightarrow |f(x) - l| \leq \varepsilon.$$

* The terminology left-limit (respectively right-limit) can be replaced by left-hand limit (respectively right-hand limit).

Example 3.5. Let's consider the function f defined as

$$\forall x \in \mathbb{R}^+ : \quad f(x) = \sqrt{x}\,.$$

The function f is only defined from the right of 0 ; so, we work on an interval of the form $[0, a[$ with $a \in \mathbb{R}_+^*$. Let's set $|f(x) - 0| \leq \varepsilon$, where $\varepsilon \in \mathbb{R}_+^*$ is arbitrary, this gives $\sqrt{x} \leq \varepsilon$; thus, $0 \leq x - 0 \leq \varepsilon^2 = \eta(\varepsilon)$. Hence

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in]0, a[: \quad 0 \leq x - 0 \leq \eta(\varepsilon) \Longrightarrow |f(x) - 0| \leq \varepsilon.$$

This affirms that f has 0 as right-limit at 0. △

Proposition 3.1. *Let $x_0 \in \mathbb{R}$, $v(x_0)$ be a neighbourhood of x_0 and f be a function defined over $v(x_0) \setminus \{x_0\}$. Then, f has a limit at x_0 if and only if the left-limit and the right-limit of f at x_0 are equals.*

Proof.

- *Proof of the "if" part* : we suppose that f has a limit at x_0 ; thus, there exists $l \in I$ such that

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v(x_0) \setminus \{x_0\} : \quad |x - x_0| \leq \eta(\varepsilon) \Longrightarrow |f(x) - l| \leq \varepsilon, \tag{3.3}$$

hence the two followings assertions

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v(x_0) \setminus \{x_0\} : \quad 0 < x_0 - x \leq \eta(\varepsilon) \Longrightarrow |f(x) - l| \leq \varepsilon, \tag{3.4}$$
$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v(x_0) \setminus \{x_0\} : \quad 0 < x - x_0 \leq \eta(\varepsilon) \Longrightarrow |f(x) - l| \leq \varepsilon. \tag{3.5}$$

On the one hand, the condition (3.4) affirms that f has l as left-limit at x_0. On the other hand, the condition (3.5) ensures that f has l as right-limit at x_0 (see Definition 3.8).

- *Proof of the "only if" part* : assume that f has a left-limit at x_0 equal to the right-limit at x_0. Hence the tow assertions (3.4)–(3.5) for some $l \in \mathbb{R}$, this yields (3.3) ; thus, f has a limit at x_0 (see Definition 3.7).

□

Definition 3.9 (Infinite left-limit). *Let f be a function defined over $v =]x_0 - a, x_0[$ for some $a \in \mathbb{R}_+^*$.*

- *We say that f has $+\infty$ as left-limit at x_0, and we write $\underset{\substack{x \to x_0 \\ <}}{\text{Lim}} f(x) = +\infty$, if*

$$\forall A > 0, \quad \exists\, \eta(A) > 0, \quad \forall x \in v : \quad 0 < x_0 - x \leq \eta(A) \Longrightarrow f(x) \geq A.$$

- *We say that f has $-\infty$ as left-limit at x_0, and we write $\underset{\substack{x \to x_0 \\ <}}{\text{Lim}} f(x) = -\infty$, if*

$$\forall A > 0, \quad \exists\, \eta(A) > 0, \quad \forall x \in v : \quad 0 < x_0 - x \leq \eta(A) \Longrightarrow f(x) \leq -A.$$

Definition 3.10 (Infinite right-limit). *Let f be a function defined over $v =]x_0, x_0 + a[$ for some $a \in \mathbb{R}_+^*$.*

- *We say that f has $+\infty$ as right-limit at x_0, and we write $\underset{\substack{x \to x_0 \\ >}}{\text{Lim}} f(x) = +\infty$, if*

$$\forall A > 0, \quad \exists\, \eta(A) > 0, \quad \forall x \in v : \quad 0 < x - x_0 \leq \eta(A) \Longrightarrow f(x) \geq A.$$

- *We say that f has $-\infty$ as right-limit at x_0, and we write $\underset{\substack{x \to x_0 \\ >}}{\text{Lim}} f(x) = -\infty$, if*

$$\forall A > 0, \quad \exists\, \eta(A) > 0, \quad \forall x \in v : \quad 0 < x - x_0 \leq \eta(A) \Longrightarrow f(x) \leq -A.$$

Example 3.6. Let f be a real-valued function with a real variable defined as

$$\forall x \in \mathbb{R}^* : \quad f(x) = \frac{1}{x}.$$

Let $v_1 =]-1, 0[$ and $A \in \mathbb{R}_+^*$. We set $f(x) \leq -A$, this gives $0 < -x < 1/A = \eta(A)$, hence

$$\forall A > 0, \quad \exists\, \eta(A) > 0, \quad \forall x \in v_1 \setminus \{0\} : \quad 0 < 0 - x < \eta(A) \Longrightarrow f(x) \leq -A,$$

thus f has $-\infty$ as left-limit at $x_0 = 0$ (See Figure **??**).
Let $v_2 =]0, 1[$ and $A \in \mathbb{R}_+^*$. We set $f(x) \geq A$, this gives $0 < x \leq 1/A = \eta(A)$, hence

$$\forall A > 0, \quad \exists\, \eta(A) > 0, \forall x \in v_2 \setminus \{0\} : \quad 0 < x - 0 < \eta(A) \Longrightarrow f(x) \geq A,$$

so f has $+\infty$ as right-limit at $x_0 = 0$ (See Figure **??**). △

Remark 3.2. *The left-limit of f at x_0 can be noted as $\lim\limits_{x\to x_0^-} f(x)$ or as $f(x_0^-)$ as well as $f_-(x_0)$. The right-limit of f at x_0 can be noted as $\lim\limits_{x\to x_0^+} f(x)$ or as $f(x_0^+)$ as well as $f_+(x_0)$.*

Definition 3.11 (Infinite limit at finite point). *Let $x_0 \in \mathbb{R}$, $v(x_0)$ be a neighbourhood of x_0 and f be a function defined over $v(x_0) \setminus \{x_0\}$.*

- *We say that f has $+\infty$ as limit at x_0, and we write $\lim\limits_{x\to x_0} f(x) = +\infty$, if*

$$\forall A > 0, \quad \exists\, \eta(A) > 0, \quad \forall x \in v(x_0) \setminus \{x_0\} : \quad |x - x_0| \le \eta(A) \Longrightarrow f(x) \ge A,$$

- *We say that f has $-\infty$ as limit at x_0, and we write $\lim\limits_{x\to x_0} f(x) = -\infty$, if*

$$\forall A > 0, \quad \exists\, \eta(A) > 0, \quad \forall x \in v(x_0) \setminus \{x_0\} : \quad |x - x_0| \le \eta(A) \Longrightarrow f(x) \le -A.$$

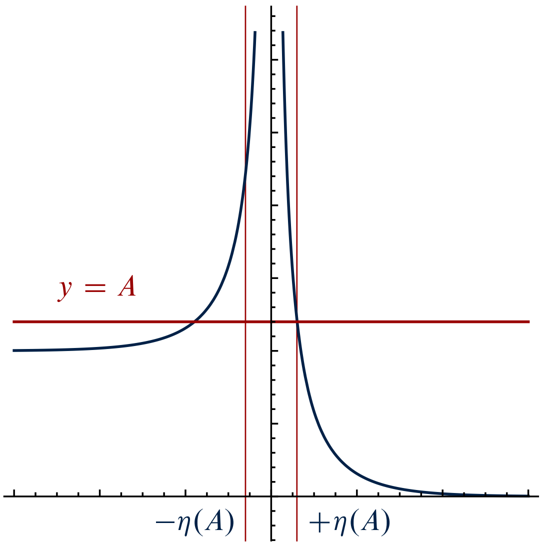

FIGURE 3.6 : $f(x) = 1/|1 - e^x|$

Example 3.7. [See Figure 3.6] Let f be defined on $\mathbb{R}^*$ by $f(x) = 1/|1 - e^x|$. For $A \in \mathbb{R}_+^*$ we set $f(x) \ge A > 0$, hence $|1 - e^x| \le 1/A$. So

$$\ln(1 - 1/A) \le x \le \ln(1 + 1/A).$$

We work on $v(0) =]-1/2, +1/2[$ and we put

$$\eta(A) = \mathrm{Min}\{\,|\ln(1 - 1/A)|\,, |\ln(1 + 1/A)|\,\},$$

we get

$$\begin{aligned} &\forall A > 0, \quad \exists\, \eta(A) > 0, \quad \forall x \in v(0) \setminus \{0\} : \\ &\qquad |x - 0| \le \eta(A) \Longrightarrow f(x) \ge A, \end{aligned}$$

therefore

$$\lim_{x\to 0} f(x) = +\infty.$$

△

Definition 3.12 (Finite limit at infinity).

- *Let f be a function defined over a neighbourhood of $+\infty$ of the form $V = [a, +\infty[$, we say that f has l as limite when x goes to $+\infty$, and we write $\lim\limits_{x\to+\infty} f(x) = l$, if*

$$\forall \varepsilon > 0, \quad \exists\, B(\varepsilon) > 0, \quad \forall x \in V : \quad x \ge B(\varepsilon) \Longrightarrow |f(x) - l| \le \varepsilon.$$

- *Let f be a function defined over a neighbourhood of $-\infty$ of the form $V =]-\infty, a]$, we say that f has l as limit when x goes to $-\infty$, and we write $\lim\limits_{x\to-\infty} f(x) = l$, if*

$$\forall \varepsilon > 0, \quad \exists\, B(\varepsilon), \quad \forall x \in V : \quad x \le -B(\varepsilon) \Longrightarrow |f(x) - l| \le \varepsilon.$$

Example 3.8. Let the function f defined over $\mathbb{R}$ by

$$f(x) = \frac{x}{1 + x^2}.$$

We have $|f(x)| = |x|/(1 + x^2) \le 1/|x|$, we set $1/|x| \le \varepsilon$, this leads to $|x| \ge 1/\varepsilon = B(\varepsilon)$. Thus

$$\begin{aligned} &\forall \varepsilon > 0, \quad \exists\, B(\varepsilon) > 0, \quad \forall x \in [1, +\infty[: \quad x \ge B(\varepsilon) \Longrightarrow |f(x) - 0| \le \varepsilon, \\ &\forall \varepsilon > 0, \quad \exists\, B(\varepsilon) > 0, \quad \forall x \in]-\infty, 1] : \quad x \le -B(\varepsilon) \Longrightarrow |f(x) - 0| \le \varepsilon, \end{aligned}$$

this affirms that f has zero as limite at $-\infty$ and $+\infty$. △

Definition 3.13 (Infinite limit at infinity).

- *Let f be a function defined over a neighbourhood of $-\infty$ of the form $V =]-\infty, a]$.*
  - *We say that f has $+\infty$ as limite when x goes to $-\infty$, and we write $\lim\limits_{x\to-\infty} f(x) = +\infty$, if*
  $$\forall A > 0, \quad \exists B(A) > 0, \quad \forall x \in V : \quad x \leq -B \Longrightarrow f(x) \geq A,$$
  - *We say that f has $-\infty$ as limit when x goes to $-\infty$, and we write $\lim\limits_{x\to-\infty} f(x) = -\infty$, if*
  $$\forall A > 0, \quad \exists B(A) > 0, \quad \forall x \in V : \quad x \leq -B \Longrightarrow f(x) \leq -A,$$
- *Let f be a function defined over a neighbourhood of $+\infty$ of the form $V = [a, +\infty[$.*
  - *We say that f has $+\infty$ as limite when x goes to $+\infty$, and we write $\lim\limits_{x\to+\infty} f(x) = +\infty$, if*
  $$\forall A > 0, \quad \exists B(A) > 0, \quad \forall x \in V : \quad x \geq B \Longrightarrow f(x) \geq A,$$
  - *We say that f has $-\infty$ as limit when x goes to $+\infty$, and we write $\lim\limits_{x\to+\infty} f(x) = -\infty$, if*
  $$\forall A > 0, \exists B(A) > 0, \quad \forall x \in V : \quad x \geq B \Longrightarrow f(x) \leq -A,$$

Example 3.9. We consider the function f defined by $f(x) = e^x$ for any $x \in \mathbb{R}$. Let $A \in \mathbb{R}_+^*$ and we set $f(x) \geq A$, this gives $x \geq \ln(A) = B(A)$; so
$$\forall A > 0, \quad \exists B(A) > 0, \quad \forall x \in [1, +\infty[: \quad x \geq B(A) \Longrightarrow f(x) \geq A,$$
and this affirms that that f has $+\infty$ and $+\infty$. △

Proposition 3.2. *Let $x_0 \in \mathbb{R}$, $v(x_0)$ be a neighbourhood of x_0 and f, g be two real-valued functions of a real variable defined over $v(x_0) \setminus \{x_0\}$ such that $f(x) \leq g(x)$ (respectively $f(x) < g(x)$) for any $x \in v(x_0) \setminus \{x_0\}$. Then*
$$\lim_{x\to x_0} f(x) \leq \lim_{x\to x_0} g(x). \tag{3.6}$$

Proposition 3.3. *Let f, g be real-valued functions with a real variable and $a, b, x_0, \lambda \in \mathbb{R}$, such that $]a, b[\subset \mathscr{D}_f \cap \mathscr{D}_g$, $x_0 \in]a, b[$ and f, g have finite limits at x_0. Then*
$$\lim_{x\to x_0} \lambda f(x) = \lambda \lim_{x\to x_0} f(x), \tag{3.7}$$
$$\lim_{x\to x_0} f(x) + g(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x), \tag{3.8}$$
$$\lim_{x\to x_0} f(x)g(x) = \big[\lim_{x\to x_0} f(x)\big]\big[\lim_{x\to x_0} g(x)\big]. \tag{3.9}$$

Remark 3.3. *The results (3.7)–(3.8)–(3.9) stay true in the case where $x_0 = +\infty$ (respectively $x_0 = -\infty$) and the proof stay similar*. In the same way, we can prove the following assertions :*

- *If f has an infinite limit and g has a non-zero finite limit, then*
$$\lim_{x\to x_0} f(x) + g(x) = \pm\infty, \quad \lim_{x\to x_0} f(x)g(x) = \pm\infty.$$
- *If f and g have infinite limits with same sign, then*
$$\lim_{x\to x_0} f(x) + g(x) = \pm\infty, \quad \lim_{x\to x_0} f(x)g(x) = +\infty.$$
- *If the functions f and g have an infinite limits with different sign, then*
$$\lim_{x\to x_0} f(x)g(x) = -\infty.$$

* Change the condition $|x - x_0| \leq \eta(\varepsilon)$ by $x \geq B(\varepsilon)$ (respectively $x \leq -B(\varepsilon)$).

Let the functions $f(x) = \sin(x)$ that has 0 as limit when x goes to $x_0 = 0$ and $g(x) = 1/|x|$ that has $+\infty$ as limit when x goes to $x_0 = 0$. The relations given at the level of Proposition 3.3 and Remark 3.3 don't give information about the following limit

$$\operatorname*{Lim}_{x\to 0} \frac{\sin(x)}{|x|}.$$

For more clarification, let $a \in \mathbb{R}_+^*$ and f be the function defined as $f(x) = (1+x)^{a/x}$ for every $x \in \mathbb{R}_+^*$. See that $f(x) = e^{a\mathrm{Ln}(1+x)/x}$, since $\mathrm{Ln}(1+x)/x$ goes to 1 as x tends to 0, then

$$\operatorname*{Lim}_{x\to 0} f(x) = \operatorname*{Lim}_{x\to 0}(1+x)^{a/x} = \operatorname*{Lim}_{x\to 0} e^{a\mathrm{Ln}(1+x)/x} = e^a.$$

On the one hand, if we calculate the limit of $1+x$ and a/x separately, the calculus of the limit of $(1+x)^{ax}$ at 0 gives "1^∞". On the other hand, the previous calculus yields "e^a is the limit of $(1+x)^{a/x}$ at 0". Therefore, we cannot provide a fixed value of the *form* 1^∞ (since the previous limit depends on the choice of a). Here we say that we have an *indeterminate form*. We summarise the different situations in Table 3.1.

La forme	Limite à calculé	Limite de f	Limite de g
$-\infty+\infty$	$\mathrm{Lim}\,[f(x)+g(x)]$	$\mathrm{Lim}\ f(x) = +\infty$	$\mathrm{Lim}\ g(x) = -\infty$
$0/0$	$\mathrm{Lim}\,[f(x)/g(x)]$	$\mathrm{Lim}\ f(x) = 0$	$\mathrm{Lim} g(x) = 0$
∞/∞	$\mathrm{Lim}\,[f(x)/g(x)]$	$\mathrm{Lim}\ f(x) = \infty$	$\mathrm{Lim}\ g(x) = \infty$
$0\times\infty$	$\mathrm{Lim}\,[f(x)g(x)]$	$\mathrm{Lim}\ f(x) = 0$	$\mathrm{Lim}\ g(x) = \infty$
1^∞	$\mathrm{Lim}\ f(x)^{g(x)}$	$\mathrm{Lim}\ f(x) = 1$	$\mathrm{Lim}\ g(x) = \infty$
∞^0	$\mathrm{Lim}\ f(x)^{g(x)}$	$\mathrm{Lim}\ f(x) = \infty$	$\mathrm{Lim}\ g(x) = 0$
0^0	$\mathrm{Lim}\ f(x)^{g(x)}$	$\mathrm{Lim}\ f(x) = 0$	$\mathrm{Lim}\ g(x) = 0$

TABLE 3.1 : The different cases of indeterminate forms.

Example 3.10.

- The polynomial case : Let $f(x)$ defined as

$$f(x) = \sum_{k=0}^{n} a_k x^k, \quad a_n \neq 0.$$

The function f can be rewritten as

$$\forall x \in \mathbb{R}^* : \quad f(x) = x^n L(x), \quad L(x) = a_n + \sum_{k=0}^{n-1} a_k x^{k-n}.$$

see that x^{k-n} goes to zero as x tends to $\pm\infty$ (since k is integer between 0 and $n-1$) ; therefore, $L(x)$ has a_n as limit at $\pm\infty$. Hence

$$\operatorname*{Lim}_{x\to\pm\infty} f(x) = \operatorname*{Lim}_{x\to\pm\infty} a_n x^n.$$

- The polynomial fraction case : Let $f(x)$ defined as

$$f(x) = \frac{P(x)}{Q(x)}, \quad P(x) = \sum_{k=0}^{n} a_k x^k, \quad Q(x) = \sum_{k=0}^{m} b_k x^k, \quad a_n \neq 0, \quad b_m \neq 0.$$

see that

$$\operatorname*{Lim}_{x\to\pm\infty} \sum_{k=0}^{n} a_k x^k = \operatorname*{Lim}_{x\to\pm\infty} a_n x^n, \quad \operatorname*{Lim}_{x\to\pm\infty} \sum_{k=0}^{m} b_k x^k = \operatorname*{Lim}_{x\to\pm\infty} b_m x^m,$$

thus

$$\operatorname*{Lim}_{x\to\pm\infty} f(x) = \operatorname*{Lim}_{x\to\pm\infty} \frac{a_n}{b_m} x^{n-m}.$$

– The root function case : Let the function f defined as $f(x) = \sqrt{x} - \sqrt{2x}$ for every $x \in \mathbb{R}_+^*$. We have

$$f(x) = \sqrt{x} - \sqrt{2x} = \frac{(\sqrt{x} - \sqrt{2x})(\sqrt{x} + \sqrt{2x})}{\sqrt{x} + \sqrt{2x}} = \frac{-x}{\sqrt{x} + \sqrt{2x}} = -\frac{\sqrt{x}}{1 + \sqrt{2}},$$

therefore

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} -\frac{\sqrt{x}}{1 + \sqrt{2}} = -\infty.$$

– The integer part case : Let $f(x) = \text{floor}(x)/x$. We have $x - 1 < \text{floor}(x) \leq x$; thus

$$\frac{x-1}{x} < f(x) \leq 1.$$

By tacking the limit, we obtain $f(x)$ goes to 1 as x tends to $+\infty$.

– The exponential case : Let $f(x) = x^{1/x} = e^{\text{Ln}(x)/x}$. Since $\text{Ln}(x)/x$ goes to zero as x tends to $+\infty$, then

$$\lim_{x \to +\infty} x^{1/x} = 1.$$

△

3.3. The continuity of function.

In this section, we will present the notion of continuous function. Geometrically, the concept of continuity is related to the function's graph not breaking. With this initial definition :

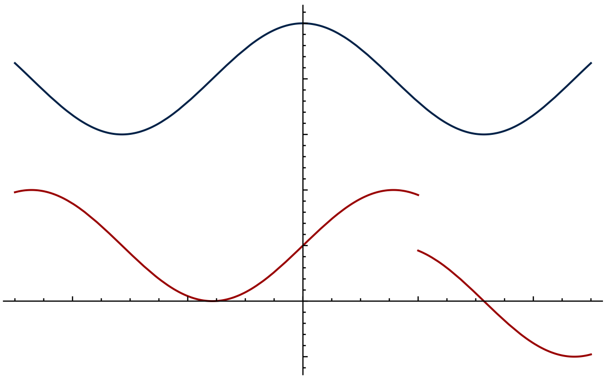

FIGURE 3.7 : — $f(x)$, — $g(x)$.

– The function f defined over $\mathbb{R}$ by

$$f(x) = \cos(x) + 4,$$

is continuous (Figure 3.7).

– The function g defined over $\mathbb{R}$ by

$$g(x) = \begin{cases} \sin(x) + 1 & \text{if} \quad x \leq 2, \\ \sin(x) & \text{if} \quad x > 2, \end{cases}$$

is discontinuous (Figure 3.7).

Definition 3.14 (The continuity). *Let f be a function defined over a neighbourhood $v(x_0)$ of x_0. We say that f is continuous at x_0, and we write $\lim_{x \to x_0} f(x) = f(x_0)$, if*

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v(x_0) : \quad |x - x_0| \leq \eta(\varepsilon) \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.$$

Example 3.11. Let the function f defined over $\mathbb{R}_+$ by $f(x) = \sqrt{x}$. Let $x_0 \in \mathbb{R}_+^*$ and $v(x_0) =]x_0 - a, x_0 + a[$, with $0 < a < x_0$, be a neighbourhood of x_0. Then

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{|x - x_0|}{\sqrt{x_0 - a} + \sqrt{x_0}}.$$

We set $|x - x_0|/(\sqrt{x_0 - a} + \sqrt{x_0}) \leq \varepsilon$, this gives $|x - x_0| \leq (\sqrt{x_0 - a} + \sqrt{x_0})\varepsilon = \eta(\varepsilon)$. In conclusion

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v(x_0) : \quad |x - x_0| \leq \eta(\varepsilon) \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon,$$

therefore f is continuous at $x_0 \in \mathbb{R}_+^*$. △

Definition 3.15 (Left continuity – Right continuity*). *Let f be a real-valued function of a real variable.*

- *We say that f is left-continuous at x_0, If f is defined over $v =]x_0 - a, x_0]$ for some $a \in \mathbb{R}_+^*$ and*

$\underset{\substack{x \to x_0 \\ <}}{\text{Lim}} f(x) = f(x_0)$. *In another word*

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v: \quad 0 < x_0 - x \leq \eta(\varepsilon) \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.$$

- *We say that f is right-continuous at x_0, if f is defined over $v = [x_0, x_0 + a[$ for some $a \in \mathbb{R}_+^*$ and* $\underset{\substack{x \to x_0 \\ >}}{\text{Lim}} f(x) = f(x_0)$. *In another word*

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v: \quad 0 < x - x_0 \leq \eta(\varepsilon) \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.$$

* The terminology "left-continuous" (respectively "right-continuous") is replaced by "continuous from the left" (respectively "continuous from the right").

Example 3.12. Let the function f defined over $\mathbb{R}_+$ by $f(x) = \sqrt{x}$, let $x_0 = 0$ and we work on the interval $v(x_0) = [0; a[$ with $a > 0$. We set $|\sqrt{x} - \sqrt{0}| = \sqrt{x} \leq \varepsilon$, hence $0 \leq x < \sqrt{\varepsilon}$. We choose $\eta(\varepsilon) = \sqrt{\varepsilon}$, thus

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in [0; a[: \quad 0 \leq x - 0 \leq \eta(\varepsilon) \Longrightarrow \left|\sqrt{x} - \sqrt{0}\right| \leq \varepsilon,$$

therefore the function $f(x) = \sqrt{x}$ is right-continuous at $x_0 = 0$. △

Proposition 3.4. *Let $x_0 \in \mathbb{R}$ and f be a function defined over a neighbourhood of x_0. Then, f is continuous at x_0 if and only if it is continuous from the left and the right at x_0.*

Proof.

- *Proof of the "if" part* : let f be a function continuous at x_0, thus there exists an interval $v =]x_0 - a, x_0 + a[$ of x_0, with $a \in \mathbb{R}_+^*$, such that

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v: \quad |x - x_0| \leq \eta(\varepsilon) \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon, \tag{3.10}$$

hence the two followings assertions

$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v: \quad 0 < x_0 - x \leq \eta(\varepsilon) \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon, \tag{3.11}$$
$$\forall \varepsilon > 0, \quad \exists\, \eta(\varepsilon) > 0, \quad \forall x \in v: \quad 0 < x - x_0 \leq \eta(\varepsilon) \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon. \tag{3.12}$$

On the one hand, The condition (3.11) affirms that the function f is left-continuous at x_0. On the other hand, the condition (3.12) ensures that f is right-continuous at x_0 (see Definition 3.15).

- *Proof of the "only if" part* : we suppose that the function f is continuous from the right and from the left at x_0. Hence the assertions (3.11)–(3.12), and this yield (3.10). Therefore, the function f is continuous at x_0, (see Definition 3.14).

□

Definition 3.16 (first kind and second kind discontinuity). *Let f be a real-valued function with a domain $\mathscr{D}_f \subset \mathbb{R}$ and let $x_0 \in \overline{\mathscr{D}_f}$ (the closure of $\mathscr{D}_f$, see Definition **??** page **??**). Then*

- *We say that x_0 is a point of discontinuity of the first kind for f if*

$$\left|\underset{x \to x_0^+}{\text{Lim}} f(x)\right| < +\infty, \quad \left|\underset{x \to x_0^-}{\text{Lim}} f(x)\right| < +\infty.$$

If $\underset{x \to x_0^+}{\text{Lim}} f(x) = \underset{x \to x_0^-}{\text{Lim}} f(x)$ we say that we have a removable discontinuity; otherwise, we say that we have a jump discontinuities.

- *We say that x_0 is a point of discontinuity of the second kind (or an infinite discontinuity) for f if*

$$\left|\underset{x \to x_0^+}{\text{Lim}} f(x)\right| = +\infty \quad \text{or} \quad \left|\underset{x \to x_0^-}{\text{Lim}} f(x)\right| = +\infty.$$

Example 3.13. [Figure 3.8]

- The function f defined over $\mathbb{R}^*$ by the expression $f(x) = 1/|x|$ has an infinite discontinuity (a second kind discontinuity) at $x_0 = 0$.
- The function g defined over $\mathbb{R}$ by

$$g(x) = \begin{cases} e^x + 3 & \text{if } x \leq 2, \\ \sqrt{x-2} + 2 & if\ x > 2, \end{cases}$$

has a jump discontinuity (a first-kind discontinuity) at $x_0 = 2$.

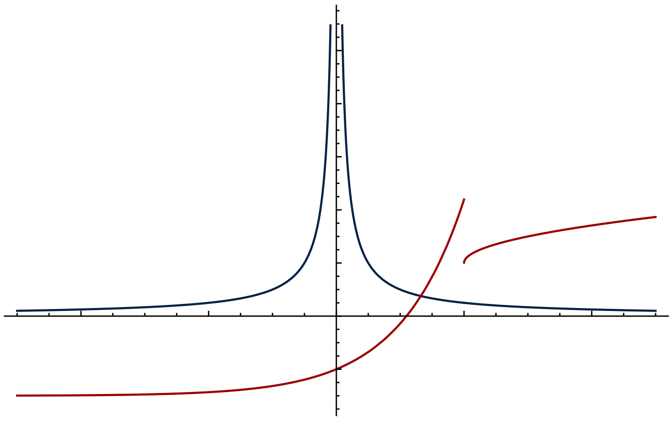

FIGURE 3.8 : $f(x)$, $g(x)$.

△

Definition 3.17 (Continuous extension of a function). *Let f be a continuous function over $I \setminus \{x_0\}$, and $x_0 \in I$ be a limite point of I such that $\lim\limits_{x \to x_0} f(x) = l \in \mathbb{R}$. Then, the function g defined over I by*

$$g(x) = \begin{cases} f(x) & if\ x \in I \setminus \{x_0\} \\ l & if\ x = x_0, \end{cases}$$

is called the continuous extension of the function f at x_0.

Example 3.14. The function f defined on $I = \mathbb{R}^*$ by the expression

$$f(x) = \frac{\sin(x)}{x}$$

has a continuous extension defined over $\mathbb{R}$ by

$$g(x) = \begin{cases} \sin(x)/x & if\ x \neq 0 \\ 1 & if\ x = 0. \end{cases}$$

△

Definition 3.18 (The image of a function*). *Let f be a function with a domain $\mathscr{D}_f$. We call the image of the function f the set* Im f *defined as*

$$\operatorname{Im} f = \{y \in \mathbb{R},\quad \exists x \in \mathscr{D}_f : \quad y = f(x)\} = \{y = f(x) : \quad x \in \mathscr{D}_f\}.$$

* In older books, the word "Range" means "Codomain" [?]. In modern books, the word "Range" means "Image" [?].

Example 3.15. Let f be the function defined by $f(x) = 1 + \sqrt{x}$. The function f has as domain the set $\mathscr{D}_f = \mathbb{R}_+$; moreover, this function is strictly increasing. Therefore

$$\operatorname{Im} f = [f(0), \lim_{x \to +\infty} f(x)[= [1, +\infty[.$$

△

Definition 3.19 (the image of a function over an interval). *Let f be a function defined over the interval $[a, b] \subset \mathbb{R}$, we call the image of the function f over the interval $[a, b]$ the set noted $f([a, b])$ and defined by*

$$f([a,b]) = \{y \in \mathbb{R}/ \quad \exists x \in [a,b] : \quad y = f(x)\},$$

and we set

$$\sup_{x \in [a,b]} f(x) = \operatorname{Sup} f([a,b]), \qquad \inf_{x \in [a,b]} f(x) = \operatorname{Inf} f([a,b]).$$

Example 3.16. Let the function f defined over $\mathbb{R}$ by $f(x) = e^x$, et let $I = [0,1]$. The function f is a increasing one over I, thus

$$f(I) = f([0,1]) = [f(0), f(1))] = [1, e].$$

△

Theorem 3.1 (Weierstrass). *Let $I = [a,b] \subset \mathbb{R}$ be a closed bounded interval and f be a real-valued function continuous over I. Then, f is bounded over I, and it reaches its lower and upper bounds. More precisely*

$$\exists l_1, l_2 \in I: \quad \operatorname*{Inf}_{x\in I} f(x) = f(l_1), \quad \operatorname*{Sup}_{x\in I} f(x) = f(l_2). \tag{3.13}$$

Remark 3.4. *We can summarise the Weierstrass Theorem (Theorem 3.1) as follows : The image of a function over an interval is an interval.*

Theorem 3.2 (The Intermediate Value Theorem). *Let f be a real-valued unction continuous over $[a,b] \subset \mathbb{R}$ such as $f(a) \neq f(b)$, we denote by $\alpha = \mathrm{Min}\{f(a), f(b)\}$ and $\beta = \mathrm{Max}\{f(a), f(b)\}$. Then*

$$\forall y \in [\alpha, \beta], \quad \exists x \in [a,b]: \quad y = f(x).$$

Proposition 3.5. *Let f be a continuous real-valued function with a real variable defined over an interval $I \subset \mathbb{R}$. Then f is an injective map over I if and only if it is strictly monotonous over I.*

Theorem 3.3 (Brouwer). *Let f be a function continuous over $[a,b]$ such that $f([a,b]) \subset [a,b]$. Then ; there exists $\overline{x} \in [a,b]$ such that $f(\overline{x}) = \overline{x}$ (we say that $\overline{x}$ is a fixed point of f). Moreover, if f is a contracted function, then $\overline{x}$ is unique.*

Proof. The function f is continued one over $[a,b]$, then the function f reaches its minimum and its maximum (see Theorem 3.1 page 12), hence

$$\exists l_1, l_2 \in [a,b]: \quad f(l_1) = \operatorname*{Min}_{x\in[a,b]} f(x), \quad f(l_2) = \operatorname*{Max}_{x\in[a,b]} f(x).$$

We combine this with the fact that the function f satisfies $f([a,b]) \subset [a,b]$, we obtain

$$\exists l_1, l_2 \in [a,b]: \quad f(l_1) = \operatorname*{Min}_{x\in[a,b]} f(x) \geq a, \quad f(l_2) = \operatorname*{Max}_{x\in[a,b]} f(x) \leq b. \tag{3.14}$$

We defined the function g over the interval $[a,b]$ by the expresion $g(x) = f(x) - x$, the condition (3.14) becomes as

$$\exists l_1, l_2 \in [a,b]: \quad g(l_1) = \operatorname*{Min}_{x\in[a,b]} g(x) \geq 0, \quad g(l_2) = \operatorname*{Max}_{x\in[a,b]} g(x) \leq 0,$$

therefore 0 is between $g(l_1)$ and $g(l_2)$, we use the intermediate value theorem (Theorem 3.2 page 12) we get the existence of $\overline{x}$ between l_1 and l_2 (and thus $\overline{x} \in [a,b]$ since $l_1, l_2 \in [a,b]$) such as $g(\overline{x}) = 0$, this with the expression of g leads to $f(\overline{x}) = \overline{x}$.
Lets suppose now that f is contracted (See Definition 3.5 page 3), then there exists $k \in [0,1[$ such as

$$\forall x, y \in [a,b]: \quad |f(x) - f(y)| \leq k|x - y|. \tag{3.15}$$

We suppose that f has two fixed points $\overline{x}_1, \overline{x}_2 \in [a,b]$ such that $\overline{x}_1 \neq \overline{x}_2$, thus $f(\overline{x}_1) = \overline{x}_1$ and $f(\overline{x}_2) = \overline{x}_2$. Combining this with (3.15) we obtain $|\overline{x}_1 - \overline{x}_2| \leq k|\overline{x}_1 - \overline{x}_2|$ and therefore $1 \leq k$ and this contradict the information that f is contracted. □