2

Numerical sequences

Let $\mathbb{K}$ be a field, a numerical sequence $(u_n)_n$ is a map from $\mathbb{N}$ to $\mathbb{K}$ defined as

$$\begin{aligned} u : \mathbb{N} &\longrightarrow \mathbb{K} \\ n &\longmapsto u(n) = u_n. \end{aligned}$$

If $\mathbb{K} = \mathbb{Q}$ we say that the sequence is a rational one; if $\mathbb{K} = \mathbb{R}$ we say that the sequence is a real one, and if $\mathbb{K} = \mathbb{C}$ the sequence is called a complex one. For instance, the map defined from $\mathbb{N}$ to $\mathbb{R}$ as $u_n = \sin(n)$ for every $n \in \mathbb{N}$, is a real numerical sequence.

2.1. Bounded numerical sequences.

Let $(u_n)_n$ be a real numerical sequence, the set $\{u_n, \quad n \in \mathbb{N}\}$ is a subset of $\mathbb{R}$, hence we can use the notions and the results of Chapter 1, in particular the results about the upper bound and the lower one.

Definition 2.1. *Let $(u_n)_n$ be a real numerical sequence, we say that*

- *$(u_n)_n$ is upper-bounded if there exists $M \in \mathbb{R}$ such that $u_n \leq M$ for any $n \in \mathbb{N}$.*
- *$(u_n)_n$ is lower-bounded if there exists $m \in \mathbb{R}$ such that $u_n \geq m$ for any $n \in \mathbb{N}$.*
- *$(u_n)_n$ is bounded if there exists $U \in \mathbb{R}$ such that $|u_n| \leq U$ for any $n \in \mathbb{N}$.*

Example 2.1. The numerical sequence defined as $u_n = 2^n$ is lower-bounded by 1 but is not upper-bounded. The numerical sequence defined as $u_n = 1/n$ is lower-bounded by 0 and upper-bounded by 1. △

Definition 2.2. *Let $(u_n)_n$ be a real numerical sequence.*

- *We say that $(u_n)_n$ is increasing* (respectively strictly increasing) if for any $n \in \mathbb{N}$ we have $u_{n+1} \geq u_n$ (respectively $u_{n+1} > u_n$).*
- *We say that $(u_n)_n$ is decreasing (respectively strictly decreasing) if for any $n \in \mathbb{N}$ we have $u_{n+1} \leq u_n$ (respectively $u_{n+1} < u_n$).*
- *We say that $(u_n)_n$ is monotone (respectively strictly monotone) if it is increasing or decreasing (respectively strictly increasing or strictly decreasing).*

* The terminology "increasing" (respectively "strictly increasing") can be replaced by "non-decreasing" (respectively "increasing").

Example 2.2. Let the sequence $(u_n)_n$ defined over $\mathbb{N}$ by $u_n = n/(n^2+1)$. The calculus gives

$$u_{n+1} - u_n = \frac{-n^2 - n + 1}{[(n+1)^2 + 1][n^2 + 1]}.$$

The polynomial $-n^2 - n + 1$ is strictly negative for any $n \geq 1$, this leads to $u_{n+1} - u_n < 0$ for any $n \in \mathbb{N}$. Thus, the sequence $(u_n)_n$ is strictly decreasing. △

Example 2.3. Let the sequence $(u_n)_n$ defined as

$$\forall n \in \mathbb{N}^* : \quad u_n = \frac{1}{9^n} - \frac{1}{2n}.$$

The calculus provides

$$\forall n \in \mathbb{N}^* : \quad u_{n+1} - u_n = \frac{9^{n+1} - (16n^2 + 16n)}{9^{n+1} 2n(n+1)}. \tag{2.1}$$

We set $\alpha_n = 9^{n+1}$ and $\beta_n = 16n^2 + 16n$, by induction we show that $\alpha_n \geq \beta_n$ for every $n \in \mathbb{N}^*$. First, we have $\alpha_1 = 81 \geq 32 = \beta_1$. Second, we assume that $\alpha_n \geq \beta_n$ for some integer $n \in \mathbb{N}^*$. Since $\alpha_{n+1} - \beta_{n+1} = 9\alpha_n - \beta_{n+1}$, by using the induction hypothesis we get

$$\alpha_{n+1} - \beta_{n+1} \geq 9\beta_n - \beta_{n+1} = 16\big[8n^2 + 6n - 2\big]. \tag{2.2}$$

The polynomial $8n^2 + 6n - 2$ has two roots $-1/2$ and $1/8$; thus, $8n^2 + 6n - 2 \geq 0$ for any $n \in \mathbb{N}^*$. Ccombining this with (2.2) provides $\alpha_{n+1} \geq \beta_{n+1}$. In conclusion, for any $n \in \mathbb{N}^*$ we have $\alpha_n \geq \beta_n$, hence $9^{n+1} - (16n^2 + 16n) \geq 0$. By suing (2.1) we get $u_{n+1} \geq u_n$ for every $n \in \mathbb{N}^*$. Thus, $(u_n)_n$ is increasing. △

Proposition 2.1. *Let (u_n) and $(v_n)_n$ be real numerical sequences :*

- *If $(u_n)_n$ and $(v_n)_n$ are increasing (respectively decreasing); then, $(u_n + v_n)_n$ is increasing (respectively decreasing).*
- *If $(u_n)_n$ and $(v_n)_n$ are positive and increasing (respectively decreasing); then, $(u_n v_n)_n$ is increasing (respectively decreasing).*
- *If $(u_n)_n$ and $(v_n)_n$ are negative and increasing (respectively decreasing); then, $(u_n v_n)_n$ is decreasing (respectively increasing).*

2.2. Convergence of numerical sequences.

Studying the convergence of a numerical sequence means studying its behaviour when n goes to infinity.

Definition 2.3. *Le $(u_n)_n$ be a real numerical sequence, we say that $(u_n)_n$ converges; if there exists $l \in \mathbb{R}$ such that*

$$\forall \varepsilon > 0, \quad \exists\, N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N} : \quad n \geq N(\varepsilon) \Longrightarrow |u_n - l| \leq \varepsilon,$$

and we write $\lim\limits_{n\to+\infty} u_n = l$. Otherwise (means if l does not exist or is infinite), we say that the sequence diverges.

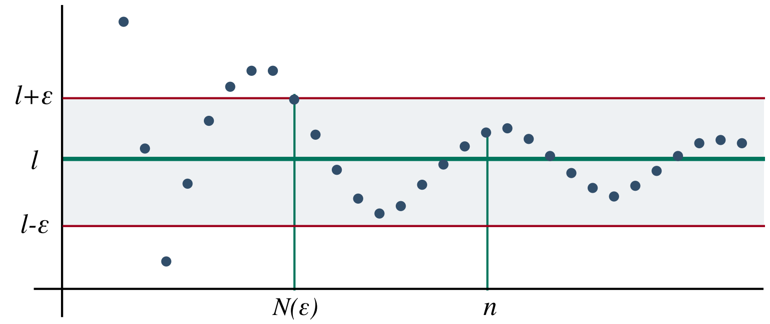

FIGURE 2.1 : Representation of the convergence of a sequence $(u_n)_n$ to l. ● : u_n.

Example 2.4. Let $(u_n)_n$ be defined by the expression

$$\forall n \in \mathbb{N} : \quad u_n = \frac{n+1}{n+2}.$$

We want to show that $\lim\limits_{n\to+\infty} u_n = 1$. The calculus provides

$$|u_n - 1| = \left|\frac{n+1}{n+2} - 1\right| = \frac{1}{n+2} \leq \frac{1}{n}.$$

We impose the condition $1/n \leq \varepsilon$, this yields $n \geq 1/\varepsilon$; we set $N(\varepsilon) = \text{floor}(1/\varepsilon) + 1$, thus

$$\forall \varepsilon > 0, \quad \exists\, N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N} : \quad n \geq N(\varepsilon) \Longrightarrow |u_n - 1| \leq \varepsilon.$$

Therefore, $(u_n)_n$ converges to 1 and we write $\underset{n\to+\infty}{\operatorname{Lim}} u_n = 1$. △

Theorem 2.1. *The limit of a numerical sequence, when it exists, is unique.*

Proof. Through proof by contradiction, let's assume that $(u_n)_n$ has two distinct limits l_1 and l_2 with $l_2 \neq l_1$. By using the definition of the limit, we get

$$\left.\begin{array}{l} \forall \varepsilon > 0, \quad \exists N_1(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \ n \geq N_1(\varepsilon) \Longrightarrow |u_n - l_1| \leq \varepsilon \\ \forall \varepsilon > 0, \quad \exists N_2(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \ n \geq N_2(\varepsilon) \Longrightarrow |u_n - l_2| \leq \varepsilon \end{array}\right\} \tag{2.3}$$

Let $N(\varepsilon) = \operatorname{Max}\{ N_1(\varepsilon), N_2(\varepsilon) \}$, the condition (2.3) gives

$$\forall \varepsilon > 0, \quad \exists N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow |u_n - l_1| \leq \varepsilon, \quad |u_n - l_2| \leq \varepsilon. \tag{2.4}$$

On the one hand, we have

$$|l_2 - l_1| = |(u_n - l_1) - (u_n - l_2)| \leq |(u_n - l_1)| + |(u_n - l_2)|,$$

using the condition (2.4), we get

$$\forall \varepsilon > 0, \quad \exists N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow |l_2 - l_1| \leq 2\varepsilon. \tag{2.5}$$

On the other hand, ε can take any non-negative value, since $l_2 \neq l_1$ we can choose $\varepsilon = |l_2 - l_1|/4$, this transforms the condition (2.5) into $|l_2 - l_1| \leq |l_2 - l_1|/2$; hence, $1 \leq 1/2$ and this a contradiction. This finishes the proof. □

Remark 2.1. *Theorem 2.1 states that in the case where a numerical sequence has two limits, it leads to the conclusion that the sequence diverges. For example, the sequence $(u_n)_n$ defined by $u_n = (-1)^n$ diverges since as n tends to infinity, we obtain two different values for $\underset{n\to+\infty}{\operatorname{Lim}} u_n$.*

Proposition 2.2. *Let $(u_n)_n$ be a real numerical sequence that converges to $l > 0$. Then, there exists $m \in \mathbb{R}_+^*$ and $N \in \mathbb{N}$ such that $u_n \geq m$ for every $n \geq N$.*

Proof. Since $(u_n)_n$ converges to l, then

$$\forall \varepsilon > 0, \quad \exists N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow |u_n - l| \leq \varepsilon,$$

The information $l > 0$ gives the possibility to choose $\varepsilon = l/2 \in \mathbb{R}_+^*$; hence, there exists $N(l/2) \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}: \quad n \geq N(l/2) \Longrightarrow |u_n - l| \leq l/2,$$

this affirms that $l/2 \leq u_n \leq 3l/2$ for any $n \geq N(l/2)$, hence $u_n > l/2 = m$ for any $n \geq N(l/2) = N$. □

Proposition 2.3. *Let $(u_n)_n$ be a convergent real numerical sequence, then $(u_n)_n$ is bounded.*

Proof. Assume that the sequence $(u_n)_n$ converges to l, then

$$\forall \varepsilon > 0, \quad \exists N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow |u_n - l| \leq \varepsilon.$$

Let $\varepsilon \in \mathbb{R}_+^*$ arbitrarily fixed, for $n \geq N(\varepsilon)$ we have $|u_n - l| \leq \varepsilon$; hence $l - \varepsilon \leq u_n \leq l + \varepsilon$. Thus

$$\forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow |u_n| \leq \operatorname{Max}\{|l - \varepsilon|, |l + \varepsilon|\} = M_1.$$

Therefore

$$|u_n| \leq \operatorname{Max}\{ |u_0|, |u_1|, \cdots, \left|u_{N(\varepsilon)}\right|, M_1\} = M \in \mathbb{R}_+,$$

hence $(u_n)_n$ is bounded. □

Proposition 2.4. *Let $\lambda \in \mathbb{R}$, $(u_n)_n$ and $(v_n)_n$ be two convergent numerical sequences. Then*

$$\lim_{n\to+\infty}(u_n+v_n)=\lim_{n\to+\infty}u_n+\lim_{n\to+\infty}v_n,\qquad \lim_{n\to+\infty}(\lambda u_n)=\lambda\lim_{n\to+\infty}u_n,\tag{2.6}$$

$$\lim_{n\to+\infty}(u_nv_n)=\Big(\lim_{n\to+\infty}u_n\Big)\Big(\lim_{n\to+\infty}v_n\Big),\qquad \lim_{n\to+\infty}|u_n|=\Big|\lim_{n\to+\infty}u_n\Big|.\tag{2.7}$$

Moreover, if the limit of v_n is non-zero, then

$$\lim_{n\to+\infty}\frac{u_n}{v_n}=\frac{\lim_{n\to+\infty}u_n}{\lim_{n\to+\infty}v_n}.\tag{2.8}$$

Proposition 2.5. *Let $(u_n)_n$ be a real numerical sequence. If $(u_n)_n$ is increasing (respectively decreasing) and upper-bounded (respectively lower-bounded), then $(u_n)_n$ converges.*

Proof. Assume that $(u_n)_n$ is increasing and upper-bounded. On the one hand, since $(u_n)_n$ is upper-bounded there exists $M\in\mathbb{R}$ such that $u_n\le M$ for every $n\in\mathbb{N}$; therefore, $\mathcal{A}=\{u_n;\quad n\in\mathbb{N}\}$ is an upper-bounded subset of $\mathbb{R}$. Hence, $\mathcal{A}$ has a supremum; so, for any $\varepsilon\in\mathbb{R}_+^*$ there exists $N(\varepsilon)$ such that $\mathrm{Sup}(\mathcal{A})-\varepsilon\le u_{N(\varepsilon)}$. Combining this with the fact that $(u_n)_n$ is increasing, yields

$$\forall n\ge N(\varepsilon):\quad \mathrm{Sup}(\mathcal{A})-\varepsilon\le u_n.\tag{2.9}$$

On the other hand, since $\mathrm{Sup}(\mathcal{A})$ is an upper bound then $u_n\le \mathrm{Sup}(\mathcal{A})\le \mathrm{Sup}(\mathcal{A})+\varepsilon$. Combining this with the condition (2.9) leads to $\mathrm{Sup}(\mathcal{A})-\varepsilon\le u_n\le \mathrm{Sup}(\mathcal{A})+\varepsilon$ for any $n\ge N(\varepsilon)$, hence

$$\forall\varepsilon>0,\quad \exists N(\varepsilon)\in\mathbb{N},\quad \forall n\in\mathbb{N}:\quad n\ge N(\varepsilon)\Longrightarrow |u_n-\mathrm{Sup}(\mathcal{A})|\le\varepsilon.$$

Hence, the sequence $(u_n)_n$ converges.
Assume now that $(u_n)_n$ is a decreasing lower-bounded sequence; thus, $(-u_n)_n$ is an increasing upper-bounded sequence. By applying the first assertion (already proved) of Proposition 2.5, we obtain $(-u_n)_n$ converges; thus, the sequence $(u_n)_n$ converges as well. □

Example 2.5. Let $(u_n)_n$ be the numerical sequence defined as

$$\forall n\in\mathbb{N}:\quad u_n=\sum_{k=1}^{n}2^{-k}[1+\sin(k)]$$

On the one hand, we have $u_{n+1}-u_n=2^{-(n+1)}[1+\sin(n+1)]\ge 0$ for every $n\in\mathbb{N}$; thus, the sequence $(u_n)_n$ is increasing. On the other hand

$$\forall n\in\mathbb{N}:\quad u_n=\sum_{k=1}^{n}2^{-k}[1+\sin(k)]\le 2\sum_{k=1}^{n}2^{-k}=\frac{1-2^{-n}}{1-2^{-1}}\le 2,$$

thus $(u_n)_n$ is an upper-bounded sequence. In conclusion, $(u_n)_n$ is a convergent sequence. △

Proposition 2.6. *Let $(u_n)_n$ and $(v_n)_n$ be two convergent numerical sequences such that for sufficiently large n, we have $u_n\le v_n$ (or $u_n<v_n$). Then*

$$\lim_{n\to+\infty}u_n\le\lim_{n\to+\infty}v_n.\tag{2.10}$$

Proposition 2.7. *Let $(u_n)_n$ and $(w_n)_n$ be two numerical sequences convergent to the same limit l_0, and $(v_n)_n$ be a numerical sequence satisfies $u_n\le v_n\le w_n$ for any integer $n\ge n_0$ for some $n_0\in\mathbb{N}$. Then, the sequences $(v_n)_n$ converges to l_0.*

Proof. First, we rewrite the hypothesis of the proposition as fellow

$$\exists\, n_0 \in \mathbb{N}, \quad \forall n \geq n_0 : \quad u_n \leq v_n \leq w_n, \quad \text{and} \quad \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} w_n = l_0. \tag{2.11}$$

By using Definition 2.3 page 2, the second part of (2.11) becomes as

$$\forall \varepsilon > 0, \quad \exists\, N_1(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \; n \geq N_1(\varepsilon) \Longrightarrow |u_n - l_0| \leq \varepsilon, \tag{2.12}$$
$$\forall \varepsilon > 0, \quad \exists\, N_2(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \; n \geq N_2(\varepsilon) \Longrightarrow |w_n - l_0| \leq \varepsilon. \tag{2.13}$$

We set $N_0(\varepsilon) = \text{Max}\{\, N_1(\varepsilon), N_2(\varepsilon)\,\}$, we get

$$\forall \varepsilon > 0, \quad \exists\, N_0(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow |u_n - l_0| \leq \varepsilon \quad \text{and} \quad |w_n - l_0| \leq \varepsilon. \tag{2.14}$$

From the relation (2.14), we can extract the following condition

$$\forall \varepsilon > 0, \quad \exists\, N_0(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow l_0 - \varepsilon \leq u_n \quad \text{and} \quad w_n \leq l_0 + \varepsilon,$$

and by using the first part of (2.11) we get $l_0 - \varepsilon \leq v_n \leq l_0 + \varepsilon$ for any $n \geq N(\varepsilon) = \text{Max}\{N_0(\varepsilon), n_0\}$, thus

$$\forall \varepsilon > 0, \quad \exists\, N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow |v_n - l_0| \leq \varepsilon,$$

hence $(v_n)_n$ converges to l_0. □

Example 2.6. Let $(u_n)_n$ be the sequence defined as

$$\forall n \in \mathbb{N}: \quad u_n = \frac{n\cos(\ln(n))}{n\ln(n) + 1}.$$

We can obtain the following inequalities :

$$0 \leq \left|\frac{n\cos(\ln(n))}{n\ln(n) + 1}\right| \leq \frac{1}{\ln(n)}.$$

Since $1/\ln(n)$ goes to zero as n tends to infinity : therefore, $\lim\limits_{n\to+\infty} |u_n| = 0$ and thus $\lim\limits_{n\to+\infty} u_n = 0$. △

Theorem 2.2 (d'Alembert criterion). *Let $(u_n)_n$ be a real numerical sequence such that*

$$\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = \alpha. \tag{2.15}$$

Then

- *If $\alpha < 1$, the sequence $(u_n)_n$ converges to zero.*
- *If $\alpha > 1$, the sequence $(u_n)_n$ diverges.*
- *If $\alpha = 1$, we have doubts about the nature of the sequence $(u_n)_n$.*

Proof. According to Definition 2.3 page 2, the condition (2.15) can be rewritten as fellow

$$\forall \varepsilon > 0, \quad \exists\, N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N}: \quad n \geq N(\varepsilon) \Longrightarrow \left|\left|\frac{u_{n+1}}{u_n}\right| - \alpha\right| \leq \varepsilon$$
$$\Longrightarrow (\alpha - \varepsilon)\,|u_n| \leq |u_{n+1}| \leq (\alpha + \varepsilon)\,|u_n|. \tag{2.16}$$

– If $\alpha = 0$, we choose $\varepsilon \in]0,1[$: the condition (2.16) provides $|u_{n+1}| \leq \varepsilon\,|u_n|$ for any $n \geq N(\varepsilon)$. By induction we can show that $|u_n| \leq \varepsilon^{n-N(\varepsilon)+1}\left|u_{N(\varepsilon)}\right|$ for any $n \geq N(\varepsilon)$; therefore, u_n has zero as limit at infinity.

– If $\alpha > 0$: by induction and by using (2.16), we can show that for any $\varepsilon \in]0, \alpha[$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for every $n \geq N(\varepsilon)$ we have

$$(\alpha - \varepsilon)^{n-N(\varepsilon)+1}\left|u_{N(\varepsilon)}\right| \leq |u_{n+1}| \leq (\alpha + \varepsilon)^{n-N(\varepsilon)+1}\left|u_{N(\varepsilon)}\right|. \tag{2.17}$$

- If $\alpha < 1$, we choose $\varepsilon \in]0, 1-\alpha[\cap]0, \alpha[$: hence, $0 < \alpha + \varepsilon < 1$ and $0 < \alpha - \varepsilon < 1$; thus

$$\operatorname{Lim}_{n\to+\infty} (\alpha - \varepsilon)^{n-N(\varepsilon)+1} = 0, \quad \operatorname{Lim}_{n\to+\infty} (\alpha + \varepsilon)^{n-N(\varepsilon)+1} = 0.$$

We combine this with the condition (2.17), we obtain u_n converges to 0.

- If $\alpha > 1$, we choose $\varepsilon \in]0, \alpha - 1[\cap]0, \alpha[$: hence, $\alpha + \varepsilon > 1$ and $\alpha - \varepsilon > 1$: thus

$$\operatorname{Lim}_{n\to+\infty} (\alpha - \varepsilon)^{n-N(\varepsilon)+1} = +\infty, \quad \operatorname{Lim}_{n\to+\infty} (\alpha + \varepsilon)^{n-N(\varepsilon)+1} = +\infty.$$

We combine this with the condition(2.17), we get u_n diverges.

- If $\alpha = 1$: Let $(u_n)_n$ and $(v_n)_n$ be the numerical sequences defined as

$$\forall n \in \mathbb{N}^* : \quad u_n = \frac{1}{n+1}, \quad v_n = n.$$

Therefore

$$\operatorname{Lim}_{n\to+\infty} \frac{u_{n+1}}{u_n} = \operatorname{Lim}_{n\to+\infty} \frac{n+1}{n+2} = 1, \quad \operatorname{Lim}_{n\to+\infty} \frac{v_{n+1}}{v_n} = \operatorname{Lim}_{n\to+\infty} \frac{n+1}{n} = 1.$$

For both sequences, we have $\alpha = 1$, but

$$\operatorname{Lim}_{n\to+\infty} u_n = 0, \quad \operatorname{Lim}_{n\to+\infty} v_n = +\infty,$$

then, $(u_n)_n$ converges and $(v_n)_n$ diverges. Therefore, when $\alpha = 1$, we cannot give any conclusion about the nature of the sequence. □

Example 2.7. Let $(u_n)_n$ be the numerical sequence defined by the expression $u_n = e^n/n!$ for every $n \in \mathbb{N}$. The calculus gives

$$\operatorname{Lim}_{n\to+\infty} \frac{u_{n+1}}{u_n} = \operatorname{Lim}_{n\to+\infty} \frac{n!e^{n+1}}{(n+1)i\,e^n} = \operatorname{Lim}_{n\to+\infty} \frac{e}{n+1} = 0 < 1.$$

Using the d'Alembert criterion provides $(u_n)_n$ converges to zero. △

Proposition 2.8 (Cauchy's n-th root criterion). *Let $(u_n)_n$ be a real numerical sequence such as*

$$\operatorname{Lim}_{n\to+\infty} \sqrt[n]{|u_n|} = \alpha. \tag{2.18}$$

- *If $\alpha < 1$, the sequence $(u_n)_n$ converges to zero.*
- *If $\alpha > 1$, the sequence $(u_n)_n$ diverges.*
- *If $\alpha = 1$, we have doubts about the nature of the sequence $(u_n)_n$.*

Proof. According to Definition 2.3 page 2, the condition (2.18) wan be rewritten as

$$\forall \varepsilon > 0, \quad \exists N(\varepsilon) \in \mathbb{N}, \quad \forall n \in \mathbb{N} : \quad n \geq N(\varepsilon) \Longrightarrow |\sqrt[n]{|u_n|} - \alpha| \leq \varepsilon. \tag{2.19}$$

- If $\alpha = 0$, we choose $\varepsilon \in [0, 1[$: the condition (2.19) yields $\sqrt[n]{|u_n|} \leq \varepsilon$, hence $|u_n| \leq \varepsilon^n$ for any $n \geq N(\varepsilon)$. Therefore, u_n has zero as limit at infinity.
- If $\alpha > 0$: the condition (2.19) provides that for any $\varepsilon \in]0, \alpha[$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for every $n \geq N(\varepsilon)$ we have

$$(\alpha - \varepsilon)^n \leq |u_n| \leq (\alpha + \varepsilon)^n \tag{2.20}$$

  - If $\alpha < 1$, we choose $\varepsilon \in]0, 1-\alpha[\cap]0, \alpha[$: this gives $0 < \alpha + \varepsilon < 1$ and $0 < \alpha - \varepsilon < 1$, hence

$$\operatorname{Lim}_{n\to+\infty} (\alpha - \varepsilon)^n = 0, \quad \operatorname{Lim}_{n\to+\infty} (\alpha + \varepsilon)^n = 0. \tag{2.21}$$

  The conditions (2.20)-(2.21) ensures that $|u_n|$ has zero as limit ; hence, $(u_n)_n$ converge to zero.

– If $\alpha > 1$, we choose $\varepsilon \in]0, \alpha - 1[\cap]0, \alpha[$: this gives $\alpha + \varepsilon > 1$ and $\alpha - \varepsilon > 1$, hence

$$\lim_{n\to+\infty} (\alpha - \varepsilon)^n = 0, \quad \lim_{n\to+\infty} (\alpha + \varepsilon)^n = 0. \tag{2.22}$$

The conditions (2.20)-(2.22) provies $\lim_{n\to+\infty} |u_n| = +\infty$; therefore, $(u_n)_n$ diverges.

– If $\alpha = 1$: we consider $(u_n)_n$ and $(v_n)_n$ the numerical sequences defined as

$$\forall n \in \mathbb{N} : \quad u_n = \frac{1}{n}, \quad v_n = n.$$

The sequence $(u_n)_n$ converges and the sequence $(v_n)_n$ diverges; furthermore, we have

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = n^{1/n} = e^{\ln(n)/n} = 1, \quad \lim_{n\to+\infty} \sqrt[n]{v_n} = n^{-1/n} = e^{-\ln(n)/n} = 1.$$

Therefore, when $\alpha = 1$ we cannot give any conclusion about the nature of the sequence. □

Definition 2.4 (Adjacent sequences). *Let $(u_n)_n$ and $(v_n)_n$ be two real numerical sequences. We say that $(u_n)_n$ and $(v_n)_n$ are adjacent; if $(u_n)_n$ is increasing, $(v_n)_n$ decreasing and* $\lim_{n\to+\infty} |u_n - v_n| = 0$.

Example 2.8. We consider the numerical sequences $(u_n)_n$ and $(v_n)_n$ defined as

$$\forall n \in \mathbb{N}^* : \quad u_n = 1 + \frac{1}{n}, \quad v_n = e^{-1/n} - \frac{1}{n^2}.$$

Since $(u_n)_n$ is decreasing, $(v_n)_n$ is increasing, and

$$\lim_{n\to+\infty} |u_n - v_n| = \lim_{n\to+\infty} \left| 1 + \frac{1}{n} - e^{-1/n} + \frac{1}{n^2} \right| = 0.$$

Then, $(u_n)_n$ and $(v_n)_n$ are adjacent sequences. △

Theorem 2.3. *If $(u_n)_n$ and $(v_n)_n$ are real adjacent sequences, then they converge to the same limit.*

Proof. Since $(u_n)_n$ and $(v_n)_n$ are adjacent, without losing the generality, we can suppose that $(u_n)_n$ is increasing and $(v_n)_n$ is decreasing. We define the sequence $(w_n)_n$ by the expression $w_n = u_n - v_n$, we have $u_{n+1} - u_n \geq 0$ (since u_n is increasing) and $v_{n+1} - v_n \leq 0$ (since v_n is decreasing); hence

$$w_{n+1} - w_n = (u_{n+1} - u_n) - (v_{n+1} - v_n) \geq 0,$$

thus, $(w_n)_n$ is increasing. Combining this with the information that $\lim_{n\to+\infty} w_n = 0$ (see Definition 2.4) we get $w_n \leq 0$ for any $n \in \mathbb{N}$, thus

$$\forall n \in \mathbb{N} : \quad u_n \leq v_n. \tag{2.23}$$

The condition (2.23), the information that $(u_n)_n$ is increasing and the fact that $(v_n)_n$ is decreasing; provide, $(u_n)_n$ is increasing upper-bounded by v_0 and $(v_n)_n$ is decreasing lower-bounded by u_0. Therefore, $(u_n)_n$ and $(v_n)_n$ are convergent and

$$0 = \lim_{n\to+\infty} w_n = \lim_{n\to+\infty} u_n - v_n = \lim_{n\to+\infty} u_n - \lim_{n\to+\infty} v_n.$$

Hence, u_n and v_n converge to the same limit. □

2.3. Subsequences of a numerical sequence.

A numerical sequence $(u_n)_n$ diverges if it has multiple limits (in which case we say that the limit does not exist) or if it has an infinite limit. This section explores the possibility of constructing a convergent sequence from a given one.

Definition 2.5. *Let $(u_n)_n$ and $(v_n)_n$ be a numerical sequences. We say that $(v_n)_n$ is a subsequence of $(u_n)_n$ if there exists a strictly increasing map g from $\mathbb{N}$ to $\mathbb{N}$ such that $v_n = u_{g(n)}$ for every $n \in \mathbb{N}$.*

Example 2.9. Let $(u_n)_n$ be the sequence defined as

$$\forall n \in \mathbb{N}^* : \quad u_n = \sin\left(\frac{(-1)^n}{n^2+1}\right).$$

Let the map g from $\mathbb{N}$ to $\mathbb{N}$ such as $g(n) = 2n$ for every $n \in \mathbb{N}$ (see that g is a strictly increasing). Then, the sequence $(v_n)_n$ defined as

$$v_n = u_{g(n)} = u_{2n} = \sin\left(\frac{1}{4n^2+1}\right),$$

is a subsequence of $(u_n)_n$. △

Definition 2.6. *We say that l is an adherent value of $(u_n)_n$ if there exists a subsequence of $(u_n)_n$ that converges to l.*

Lemma 2.1. *Let g be a strictly increasing map from $\mathbb{N}$ to $\mathbb{N}$; then, $g(n) \geq n$ for every $n \in \mathbb{N}$.*

Proof. By induction on the integer n. First, we have $g(0) \geq 0$ since g is from $\mathbb{N}$ to $\mathbb{N}$. Assume now that $g(n) \geq n$ for some integer n, by using the fact that g is strictly increasing we get $g(n+1) > g(n) \geq n$; therefore $g(n+1) > n$. Since $g(n+1), n \in \mathbb{N}$; then, $g(n+1) \geq n+1$. This finishes the proof. □

Theorem 2.4 (Bolzano − Weierstrass). *We can extract a convergent subsequence from any bounded real numerical sequence.*

2.4. Recursive sequences.

The Fibonacci's sequence defined by the relation $u_{n+2} = u_{n+1} + u_n$ is a recursive sequence since the construction of a term involves the previous terms of the sequence. This section will present the results related to this category of sequences, starting with the following definition.

Definition 2.7. *Let f be a function from $\mathscr{D} \subset \mathbb{R}$ to $\mathbb{R}$, and $(u_n)_n$ be a numerical sequence defined as*

$$\forall n \in \mathbb{N} : \quad u_{n+1} = f(u_n).$$

The sequence $(u_n)_n$ is called a recursive sequence of order one, a recursive sequence of order s is defined by giving a function g from $\mathscr{D} \subset \mathbb{R}^s$ to $\mathbb{R}$ and by the expression

$$\forall n \in \mathbb{N}, \quad n \geq s-1 : \quad u_{n+1} = g(u_n, u_{n-1}, \cdots, u_{n-s+1}). \tag{2.24}$$

Example 2.10.

− The Heron sequence : it is defined by two given constants $a, u_0 \in \mathbb{R}$ and by the relation

$$\forall n \in \mathbb{N} : \quad u_{n+1} = \frac{1}{2}\left(u_n + \frac{a}{u_n}\right).$$

− The arithmetical-geometrical sequence : it is defined by two given constants $a, b \in \mathbb{R}$ and the relation

$$\forall n \in \mathbb{N} : \quad u_{n+1} = au_n + b.$$

− Fibonacci sequence : it is defined by giving two constants $u_0, u_1 \in \mathbb{R}$ and the expression

$$\forall n \in \mathbb{N} : \quad u_{n+2} = au_{n+1} + bu_n.$$

− The geometrical sequence : it is defined by two given constants $a, u_0 \in \mathbb{R}$ and the relation

$$\forall n \in \mathbb{N} : \quad u_{n+1} = au_n.$$

△

Remark 2.2. *Let $(u_n)_n$ be a recursive numerical sequence of order s defined by the expression (2.24) and converge to l. Then, l can be determined by solving the equation $l = g(l, l, \cdots, l)$.*

Example 2.11. Let $(u_n)_n$ be the sequence defined by the relation

$$\forall n \in \mathbb{N}: \quad u_{n+1} = \sqrt{u_n + 1}, \quad u_0 = 1. \tag{2.25}$$

— *By induction on the integer n we show that $0 < u_n < 2$ for every $n \in \mathbb{N}$* : for $n = 0$ we have $u_0 = 1 \in]0, 2[$; assuming $u_n \in]0, 2[$ for some integer n, the relation (2.25) provides

$$\begin{aligned} u_n \in]1, 2[&\Longrightarrow \sqrt{u_n + 1} \in]\sqrt{2}, \sqrt{3}[\subset]1, 2[\\ &\Longrightarrow u_{n+1} \in]1, 2[. \end{aligned}$$

Therefore, $1 < u_n < 2$ for any $n \in \mathbb{N}$.

— *By induction on the integer n, we will show that $(u_n)_n$ is an increasing sequence* : the relation (2.25) gives $u_1 = \sqrt{2} > 1 = u_0$; assume $u_{n+1} > u_n$ for some $n \in \mathbb{N}$; thus

$$u_{n+1} = \sqrt{u_n + 1} < \sqrt{u_{n+1} + 1} = u_{n+2}.$$

Therefore, $u_{n+1} > u_n$ for every $n \in \mathbb{N}$; hence, $(u_n)_n$ is strictly increasing.

— *Determination of the limit of u_n* : according to the previous analysis, the sequence $(u_n)_n$ is a bounded increasing real numerical sequence, so it converges to some value $l \in \mathbb{R}$. By taking the limit at the level of the relation (2.25), we obtain

$$\operatorname*{Lim}_{n \to +\infty} u_{n+1} = \sqrt{\operatorname*{Lim}_{n \to +\infty} u_n + 1},$$

This provides $l = \sqrt{l + 1}$, hence $-l^2 + l + 1 = 0$. The polynomial $-l^2 + l + 1$ has a discriminant $\Delta = 5$, so it has two roots :

$$l_1 = \frac{1 - \sqrt{5}}{2} < 0, \quad l_2 = \frac{1 + \sqrt{5}}{2} \in]1, 2[.$$

Since $u_n \in]1, 2[$, then $(u_n)_n$ has l_2 as limit. △