# ANALYSIS I, TUTORIAL 3 / Real-valued Function with a real variable Differentiability

**Exercise 1.** Provide the derivative of the following functions and exhibit their domain.

$$f(x) = \frac{1}{\sqrt[3]{x^2}} - \frac{1}{\sqrt{x^3}}, \quad g(x) = \sin\left(\cos(3x)\right), \quad h(x) = 2^{\ln(x)} + 3^x.$$

**Exercise 2.** Provide the $n-$th derivative of the following functions

$$f(x) = e^{2x}, \quad g(x) = \frac{1}{1+x}, \quad h(x) = \sin(x).$$

**Exercise 3.** Let $f$ be a real-valued function with a real variable defined by

$$\forall x \in \mathbb{R}: \quad f(x) = \frac{x}{1+|x|}.$$

- Provide the domain, the parity and the derivative of $f$.
- Provide the limit of $f$ at its domain's boundary.
- Find the largest sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}$ such as the maps $g$ defined as

$$\begin{aligned} g : \mathcal{A} &\longrightarrow \mathcal{B} \\ x &\longmapsto g(x) = f(x), \end{aligned}$$

  be a bijective one ; in this case, exhibit the inverse of $g$.

**Exercise 4.**

- Show that for any $x \in [-1, +1]$ we have $\arcsin(x) + \arccos(x) = \pi/2$.
- Show that for any $x \in [0, +1]$ we have $\arcsin(x) + \arcsin\left(\sqrt{1-x^2}\right) = \pi/2$.

**Exercise 5.** [The inverse of the hyperbolic sinus] Let $f$ be the function defined over $\mathbb{R}$ with value in $\mathbb{R}$ and satisfies

$$f(x) = \mathrm{sh}(x) = \frac{e^x - e^{-x}}{2}.$$

Show that $f$ has an inverse, denoted it arcsh, and provide the derivative of $f^{-1}$.

**Exercise 6.** [The inverse of the hyperbolic consinus] Let $f$ be the function defined over $\mathbb{R}$ with value in $\mathbb{R}_+^*$ and satisfies

$$f(x) = \mathrm{sh}(x) = \frac{e^x + e^{-x}}{2}.$$

Show that $f$ has an inverse, denoted it arcch, and provide the derivative of $f^{-1}$.

**Exercise 7.** [The inverse of the hyperbolic tangent] Let $f$ be the function defined over $\mathbb{R}$ with value in $]-1, 1[$ and satisfies

$$f(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Show that $f$ has an inverse, denoted it arctanh, and provide the derivative of $f^{-1}$.

**Exercise 8.** Study the differentiability of the following functions

$$f(x) = x|x|, \quad g(x) = \frac{1}{1+|x|}, \quad h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & if \quad x \neq 0, \\ 0 & if \quad x = 0. \end{cases}$$

**Exercise 9.** Let $a, b, c \in \mathbb{R}$, we define the function $f$ as

$$f(x) = \begin{cases} \sqrt{x} & if \quad x \in [0; 1], \\ ax^2 + bx + c & if \quad x > 1. \end{cases}$$

Find $a, b, c$ so that the function $f$ be differentiable over $\mathbb{R}_+^*$.

**Exercise 10.**

- Show that the equation $\cos(x) - x = 0$ has a solution in $[0, 1]$.
- Show that the equation $x + e^x = 0$ has a unique solution in $\mathbb{R}$.
- Provide an example of function $g$ from $]0, 1[$ to $]0, 1[$ without fixed point.

**Exercise 11.** Let $n \in \mathbb{N}^*$, we defined the function $\mathscr{L}_n$ over $I = [0, 1]$ by

$$\forall x \in I : \quad \mathscr{L}_n(x) = x^n \sin(\pi x).$$

Show the existence of $x_n \in ]0, 1[$ such that $\mathscr{L}_n'(x_n) = 0$, write $\mathscr{L}_n(x)$ in terms of $\mathscr{L}_n'(x)$, and calculate the limit of $\mathscr{L}_n(x_n)$ when $n$ tends to $+\infty$.

**Exercise 12.** Let the function $f(x) = \mathrm{Ln}(x)$ defined over $\mathbb{R}_+^*$ with value in $\mathbb{R}$.

- Show that

$$\forall x \in \mathbb{R}_+^* : \quad \frac{1}{1+x} < f(x+1) - f(x) < \frac{1}{x}.$$

- Calculate the value of the following limits

$$\lim_{x \to +\infty} \sqrt{x}\left(\mathrm{Ln}(x+1) - \mathrm{Ln}(x)\right), \quad \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x.$$

**Exercise 13.** Let the function $f$ defined over $\mathbb{R}$ with value in $\mathbb{R}$ by

$$f(x) = \begin{cases} \frac{3 - x^2}{2} & si \quad x < 1, \\ \frac{1}{x} & si \quad x \geq 1. \end{cases}$$

Show that $f$ is continuous and differentiable on $\mathbb{R}$ and there exists $x_0 \in ]0, 2[$ such that

$$2f^{(1)}(x_0) = f(2) - f(0).$$

**Exercise 14.** Show that for ever $x \in \mathbb{R}_+$ we have $x \leq e^x$ and prove that

$$\forall \alpha \in \mathbb{R}_+, \quad \exists C_\alpha \in \mathbb{R}_+^*, \quad \exists K_\alpha \in \mathbb{R}_+^*, \quad \forall x \in \mathbb{R}_+ : \quad C_\alpha x^\alpha \leq e^x, \quad K_\alpha x^\alpha \geq \mathrm{Ln}(x).$$

Show that for any $s \in \mathbb{R}_+^*$ we have

$$\lim_{x \to +\infty} \frac{e^x}{x^s} = +\infty, \quad \lim_{x \to +\infty} \frac{x^s}{\mathrm{Ln}(x)} = +\infty.$$

**Exercise 15.** Let $n$ be an integer greater than or equal to two; we define the fucntion $f_n$ from $\mathbb{R}$ to $\mathbb{R}$ by

$$\forall x \in \mathbb{R} : \quad f_n(x) = x^n + x^{n-1} + x^2 + x - 1.$$

- Show that $f_n$ has a unique root noted $u_n$ in $\mathbb{R}_+^*$.
- Show that $u_n \in ]0, 2/3[$ for every $n \in \mathbb{N}^*$.
- Show that the sequence $(u_n)_n$ is strictly increasing.
- Show that the sequence $(u_n)_n$ converges and calculate its limit.