Propositional Logic Lecture1

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Introduction

- **Mathematical logic**
  - A discipline which is interested in studying methods and models of valid reasoning
- It is split into 04 branches:
  - Theory of models
  - Theory of proof
  - Set theory
  - Computability (recursion thesory)

The birth of Mathematical Logic (ML)

- **Paradoxes (19 & 20 centuries)**
  - certain formalized mathematical domains contained unsound axioms or even paradoxes.
  - geometry (initially done by Euclid), set theory (initially done by Cantor),…
- **Russell's paradox:** let R be the set of all sets that do not belong to themselves;
- a naive question :does R belong to itself or not?
  - If the answer is yes, then, as by definition the members of this set do not belong to themselves, R does not belong to itself →contradiction.
  - If the answer is no, then it has the property required to belong to itself→ contradiction again.
  - $R=\{X| X\notin X\}$, from this definition we will have: $R\in R \Leftrightarrow R\notin R$
- **Postulate of parallelism (geometry)**

Berry's paradox

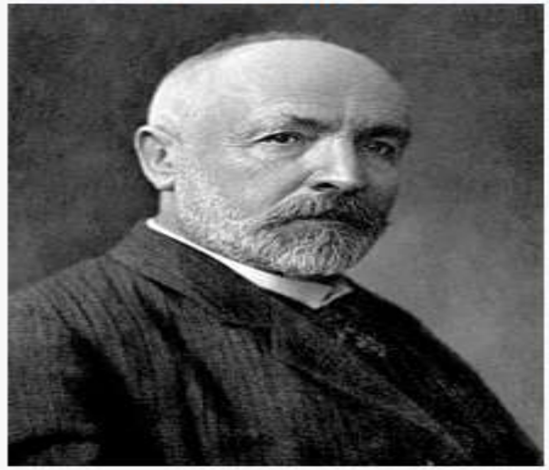
- **English version**
- "the smallest positive integer not definable in fewer than twelve words"
- Another (more precise) variation: use sixty letters in place of twelve words.
- **French version**
- « Le plus petit entier naturel non descriptible par une expression de quinze mots ou moins. »
- Ce nombre appartient-il à l'ensemble des entiers naturels descriptibles par une expression de quinze mots ou moins ?

ML Pioneers

David Hilbert

1862-1943

Georg Cantor

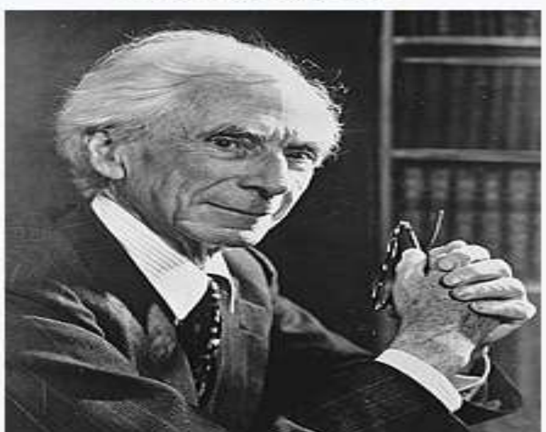


1845-1918

Gottlob Frege

1848-1925

The Earl Russell



1872-1970

Kurt Gödel

1906-1978

Alan Turing

1912-1954

The birth of ML

- Are the mathematical foundations consistent (free from errors) ?
- Hilbert Program (1900)
- A list of 23 problems to be solved
  - Consistency of arithmetic (properties of natural numbers)
  - Continuum hypothesis
  - Diophantine equations
  - ….
- Godel's incompleteness theorem (1931)
  - Any sufficiently expressive axiomatic theory (containing at least arithmetic) is incomplete
  - We can't even prove the consistency of arithmetic

Definitions

- Proposition (assertion): declarative statement that can be either true or false
- Example:
- The earth is round
- The amphitheater is closed
- Everyone is present
- 3+4=8

…

Definitions

- The definition excludes imperative, exclamatory, and interrogative statements
- Examples:
  - Close the window!
  - Is the window closed?
  - What a beautiful window!
  - He is present
  - N is prime
  - $x+y > 10$
  - This expression is false (self-reference)

Definitions

- Predicate: n-ary relation defined on the Cartesian product of a set of domains
- Ex: friend(X,Y), with X,Y $\in$ L2CLASS
  - friend$\equiv${(etu1, etu3), (etu2, etu9), (etu3, etu1), (etu9, etu2)…}.
- Note: after replacing free variables with concrete individuals, the predicate becomes a proposition

Logic of propositions

- The proposition calculus goes through the following stages:
  - How to write the formulas?

    Syntactic aspects
  - How to determine the truth value of a formula?
    Semantic aspects
  - How to (automatically) demonstrate new results?
    Deductive aspects

Syntax

- the vocabulary (alphabet) of propositional logic is composed of:
  - all connectors: $\{\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg, (,)\}$
  - the set of symbols of propositional variables **(atomic propositions)** denoted as P= { p, q, r, s, …}.
- To construct a well formed formula (wff) we apply the grammar rules:
- Any formula $F \in Prop$ can be defined as follows:
- $F \equiv p$, with p an atomic proposition
- $F \equiv \neg H$, with $H \in Prop$
- $F \equiv H \wedge I$, or $F \equiv H \vee I$, or $F \equiv H \Rightarrow I$, or $F \equiv H \Leftrightarrow I$.

with $H, I \in Prop$

Syntax

Examples

- p
- $(q \Rightarrow p) \Leftrightarrow (s \Rightarrow p)$
- $\neg(p \vee (r \wedge \neg s) \vee t)$
- $\neg\neg p$

Counter examples:

- $pq \vee$
- $\Rightarrow p$

Syntax

- Priority:

+ $\neg$

$\wedge, \vee$

\- $\Rightarrow, \Leftrightarrow$

Semantics (model theory)

- The semantics of a propositional formula is its truth value, i.e., 1 or 0
- We calculate the meaning using the notion of interpretation
- An interpretation I is a function δ:P→{0, 1}, and can be extended to all propositional formulas:
- δ: Prop $\longrightarrow$ {0,1}
- it assigns a truth value to each atomic proposition; in addition, it represents the semantics of the connectives

Semantics

Example:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg (p \wedge q)$	$(\neg p \vee \neg q)$	F
1	1	1	0	1	0	0	1
0	1	1	0	0	1	1	1
1	0	0	1	0	1	1	1
0	0	0	1	0	1	1	1

Each row of the truth table represents a possible interpretation
The interpretation ensuring a 1 is called a model.
The interpretation ensuring a 0 is called an antimodel

Types of propositional formulas

- **Satisfiable (consistent) formula**
  - F is satisfiable iff $\exists\ \delta$ such that: $\delta(F)=1$
  - In other words, There is at least one row giving 1 in the truth table of F
- **Examples**
  - $(s \Rightarrow p)$
  - $(\neg p \vee (r \wedge \neg s))$

Types of propositional formulas

- **Unsatisfiable formula (antilogy, inconsistent, contradictory)**
  - F is unsatisfiable iff $\forall\ \delta: \delta(F)=0$
  - In other words, all rows in the truth table give 0s
- **Examples**
  - $(p \land \neg p)$
  - $(\neg p \Leftrightarrow p)$

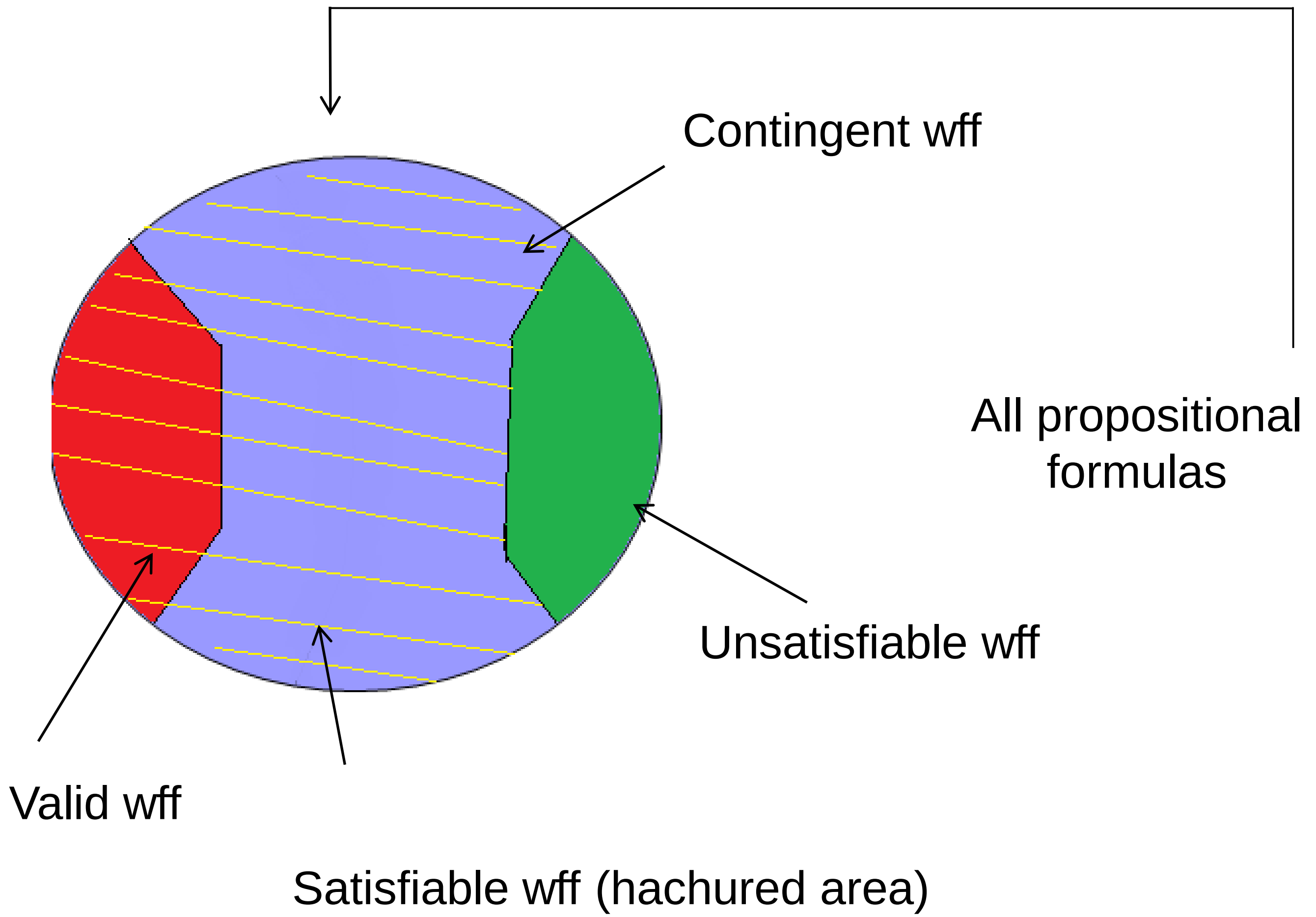
Types of propositional formulas

- **Valid formula (tautology)**
  - F is valid iff $\forall \delta: \delta(F)=1$
  - In other words, all rows in the truth table give 1s
- **Examples**
  - $(p \vee \neg p)$
  - $(\neg p \Rightarrow \neg s) \Rightarrow (s \Rightarrow p)$

Types of propositional formulas

- **Contingent formula**
  - F is contingent iff $\exists$ δ such that: $\delta(F)=1$, and δ' such that: $\delta'(F)=0$
  - In other words, there is at least one row which gives 1 and another which gives 0, in the truth table of F
- **examples**
  - $(s \Rightarrow p)$
  - $(p \vee (r \wedge \neg s))$

Formula Classes



Logical (semantic) consequence

- Let A={F1,….Fn} be a set of well-formed formulas ‘wff', similarly G is an ‘wff'.
- G is a logical consequence of A iff:
  - $\forall\delta$: if $\delta(F1)=1$ and $\delta(F2)=1$…and $\delta(Fn)=1$

    Then $\delta(G)=1$
    - In other words every model of A is also a model of G
- We note $A \models G$
  - Example: $\{s \Rightarrow p, \neg p)\} \models \neg s$
  - Note: A logical consequence of $\emptyset$ is a tautology

Logical equivalence

- Let A and G be two well-formed ‘wff' formulas, A is logically equivalent to B iff:
  - A $\models$G and
  - G $\models$ A

Normal forms

- **Literal**
  - It is an atomic proposition or its negation
  - Examples: $\neg p$, q…
- **Clause**
  - It is a disjunction of literals
  - Examples: $(P1\vee \neg P2\vee \neg P3 \vee \neg P4)$, $(\neg P1\vee \neg P2)$,…
- **Conjunctive Normal Form**
  - It is a conjunction of disjunction of literals
  - Ex: $(\neg P1\vee \neg P2\vee \neg P3)\wedge(P3\vee \neg P2)\wedge(P1\vee \neg P3)$
- **Disjunctive Normal Form**
  - It is a disjunction of conjunction of literals
  - Ex: $(\neg P1\wedge P3)\vee(\neg P3\wedge \neg P2)\vee(P1\wedge P2\wedge P3)$

END