Contents

Chapter 1

Probability calculation

1.1 Combinatorial Analysis

Combinatorial analysis is the mathematical theory of counting.

1.2 Basic Principle of Counting

Suppose that k experiments are to be performed. If experiment 1 has n_1 possible outcomes; for each of them, there are n_2 possible outcomes of experiment 2; ......; for each of them, there are n_k possible outcomes of experiment k, then there are totally

$$m = n_1 * n_2 * ... * n_k$$

possible outcomes of the k experiments.
Example: How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?
Solution: 26*26*26*10*10*10*10 = 175760000.
What if repetition among letters or numbers is prohibited?
Solution: 26*25*24*10*9*8*7 = 78624000.

1.2.1 Arrangement

a) Arrangement without repetition we call arrangement without repetition any set of k elements chosen from n one after the other without replacement (ordered arrangement) the number of these arrangements is

$$A_n^k = \frac{n!}{(n-k)!}$$

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Examples 1) a DNA sequence is made up of a chain of 4 nucleotides [A (Adenine), C (Cytosine), G (Guanine) and T (Thymine)]. There are different possible arrangements of two nucleotides or dinucleotides with k=2, n=4;

$$A_4^2 = \frac{4!}{(4-2)!} = 12$$

.

2) The number of words of 5 different letters (with or without meaning) formed with the 26 letters of the alphabet corresponds to k=5, n=26;

$$A_{26}^5 = \frac{26!}{(26-5)!} = 7893600$$

.

3)The number of committees (president, secretary, cashier) of 3 members that can be formed from 8 people corresponds to the number of possible arrangements with k=3, n=8 ;

$$A_8^3 = \frac{8!}{(8-3)!} = 336$$

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In the previous examples, the order of the elements is essential but a base or a letter of the alphabet can be found several times. We must therefore distinguish two types of arrangements with or without repetition.
b) Arrangement with repetition:it is any set of k elements chosen from n with replacement.

$$\tilde{A}_n^k = n^k$$

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Examples 1) Concerning the example of the DNA sequence, the expected number of dinucleotides if we assume that a base can be observed several times in the sequence (which corresponds to reality) is therefore? $4^2 = 16$.
2) How many 7-digit phone numbers are there? 10^7.
3) How many ways can we distribute 10 people over 3 counters? 3^{10}.
Remark in the case of arrangement with repetition, it is possible to have $k > n$.

1.2.2 Permutations

a) Permutations without repetition:it is an arrangement without repetition with $k = n$, the number of these permutations is then;

$$P_n = n!$$

.
The permutation of n objects constitutes a special case of arrangement without repetition of k elements among n with $k = n$.
Examples: 1) In a train with 10 different cars, there are 10! ways to put it together. Assuming the locomotive is still in the lead).
2) There are 7! different ways to divide a group of 7 people into a row of 7 chairs.
b)Permutation with repetition:it is an arrangement of n elements divided into k classes containing $n_1, n_2, ..., n_k$ elements respectively;

$$\tilde{P}_n = \frac{n!}{n_1!n_2!...n_k!}$$

.
$n_1 + n_2 + ... + n_K = n$, there is k classes.
Examples:
1) A class has 12 students, in how many ways can these 12 students take three different exams, knowing that 4 students take the same exam?
answer: $\tilde{P}_{12} = \frac{12!}{4!4!4!} = 34650$.
2) How many anagrams can be formed with the letters of the word "excellency"
answer:$\tilde{P}_{10} = \frac{10!}{4!2!2!1!1!} = 37800$.

1.2.3 Combinations

a) Combination with repetition:A combination of k elements chosen from n is an unordered arrangement of these k elements where each appears at most once, denoted ${C_n}^k$, the number of these combinations;

$$C_n^k = \frac{n!}{k!(n-k)!}$$

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Examples:
1) The formation of a delegation of 5 people from a group of 50 constitutes a combination with p=5 and n=50;

$$C_{50}^5 = \frac{50!}{5!45!} = 2118760$$

.

2) Let the numbers be 1, 2, 3, 4. By choosing 2 numbers, we can obtain 6 combinations, in Concerning the example of the DNA sequence, the number of dinucleotides expected if we make the hypothesis that the order is not important, then for p=2 and n=4;

$$C_4^2 = \frac{4!}{2!2!} = 6$$

.

b) Combination with repetition:A combination with repetition of k elements is a grouping in any order of k elements, these elements are not necessarily distinct in pairs and k is no longer necessarily less than or equal to n. We are looking for the number

$$\tilde{C}_n^k = \frac{(n+k-1)!}{k!(n-1)!}$$

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Examples

12 delegates from an association must elect a representative to the governing board. Two candidates present themselves; all delegates must vote for one or the other, we want to determine the number of possible votes, this is the number of combination with repetition n=2, p=12

$$\tilde{C}_2^{12} = \frac{13!}{12!} = 13$$

there are 13 possible votes.

The number of groups of 9 letters, with repetition, that can be formed with the 3 letters a, b, c;

$$\tilde{C}_3^9 = \frac{11!}{2!9!} = 55$$

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1.3 Probability theory

Introduction

We can classify the experiences into two large groups;

those whose outcome is predicted with certainty depending on established

physical laws and those whose outcome depends on chance, for which rigorous predictions cannot be made, these are random experiments.
Historically, the notion of probability emerged from simple examples generally taken from games of chance.

1.3.1 Concept of events, universe

A random experiment (trial) can present a finite or infinite number of outcomes (results).
Each of these outcomes is an elementary event E_i....
The set of these elementary events constitutes the universe Ω it is the set of all possible outcomes.
Examples:
1) Rolling an ordinary fair die n times $\Omega = \{E_1, ..., E_n\}$.
A non-elementary event is a set of several outcomes (part of Ω).
2) For the same random die-roll experiment with $n = 3$, we consider the events:
"obtaining even number", $A = \{2, 4, 6\}$.
"obtaining odd number"; $B = \{1, 3, 5\}$.
A et B are non-elementary event.

1.3.2 Algebra and event tribe

An event being an element of $\mathcal{P}(\Omega)$ obeys the laws of set theory.
$(\Omega, \mathcal{P}(\Omega))$ is called probability space.
If Ω is infinite, we can consider $\mathcal{A}$ a restricted family of $\mathcal{P}(\Omega)$. Thus $\mathcal{A}$, *Algebra* of parts of Ω which verifies:
$C1$- for every A in $\mathcal{A}$, $\bar{A} \in \mathcal{A}$.
$C2$- for every A, B in $\mathcal{A}$, $A \cup B \in \mathcal{A}$.
the condition $C2$ is equivalente to the condition $\acute{C}2$;
$\acute{C}2$- for every A, B in $\mathcal{A}$, $A \cap B \in \mathcal{A}$.
Examples:
1) The most basic algebra is $\{\Omega, \emptyset\}$.
Let $\Omega = \{a, b, c, d\}$, from the partition $\{a, b\}, \{c\}, \{d\}$ we construct the algebra $\mathcal{A}$ such that;
$\mathcal{A} = \{\emptyset, \{a, b\}, \{c\}, \{d\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}, \Omega\}$.
Properties of the algebra: P1- $\emptyset \in \mathcal{A}$ and $\Omega \in \mathcal{A}$.
P2- If $A_j \in \mathcal{A}$, for $1 \leq j \leq n$, we can demonstrate by induction that $\cup_{j=1}^{n} A_j \in \mathcal{A}$.

P3- If $A_j \in \mathcal{A}$, pour $1 \leq j \leq n$, we can demonstrate by passing to the complement that $\cap_{j=1}^{n} A_j \in \mathcal{A}$.
Some experiments can take place indefinitely, for example the random experiment "roll a die" then the event, A_n:"get 3 on the n^{th} thrown".
The event A: "get 3" will be then $A = \cup_{n=1}^{\infty} A_n$,hence the need to move from a finite union in P2 to a countable union; then we get:
If $A_n \in \mathcal{A}$ for every n,then $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$.
In thie case, we say that $\mathcal{A}$ is $\sigma - algebra$ or a *tribe*.
thus we can associate a probability to the space $(\Omega,\ \mathcal{A})$.

1.3.3 Event logic

*Certain event (contains all the results of the experiment);$E = \Omega$.
*Impossible event (never happens); $E = \emptyset$.
*The union of events, $A \cup B$; (A is achieved or B is achieved).
*The intersection of Event, $A \cap B$; (A is achieved and B is achieved).
*Two events that cannot be carried out at the same time are said to be incompatible.

$$\text{A and B are incompatibles (mutually exclusive)} \Leftrightarrow A \cap B = \emptyset.$$

*Complementarity, $\bar{A}$ complementary to A; A is not achieved.
A et $\bar{A}$ are incompatibles.

1.3.4 Probability

Once we have defined all the events in which we are interested, we will try to translate their chances of happening into numbers.

Definitions 1 *We call probability* $\mathbb{P}$ *on* $(\Omega,\ \mathcal{A})$
a map $\mathbb{P}$*:* $\mathcal{A} \rightarrow [0,\ 1]$*, such that:*
(i) $\mathbb{P}(\Omega) = 1$.
(ii) For any sequence A_n *of incompatible events;* A_n *and* A_m *in* $\mathcal{A}$*,*
$A_n \cap A_m = \emptyset$*, for* $m \neq n$*:*

$$\mathbb{P}\left(\cup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mathbb{P}\left(A_n\right).$$

Property called σ *additivity.*

Remark 1 *For any non-disjoint sequence of (A_n), we then have the Boolean inequality;*

$$\mathbb{P}\left(\cup_{n=0}^{\infty} A_n\right) \leq \sum_{n=0}^{\infty} \mathbb{P}\left(A_n\right).$$

$(\Omega, \mathcal{A}, \mathbb{P})$ is called *probability space.*

properties 1 *P1)* $\mathbb{P}(\emptyset) = 0$.
P2) $\mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A)$.
P3) $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$.
P4) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Examples:
1) Let's throw a coin in the air, this game constitutes a test, that is to say an experience whose result is uncertain. There are two possible eventualities; heads or tails.
If we consider the "getting heads" eventuality, among the two equally probable outcomes, there is only one that is favorable to "getting heads".
The probability of getting heads is equal to $\frac{1}{2}$.
More generally

$$\mathbb{P}(A) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

2) We roll a die, $\mathbb{P}(\text{get an odd number}) = 3/6$.
3) in an urn, there are 30 red balls, 20 black balls, 10 green balls indistinguishable to the touch and arranged randomly. We draw a ball.
$\mathbb{P}(\text{draw a green ball}) = 10/60$.
$\mathbb{P}(\text{draw a black ball}) = 20/60$.
$\mathbb{P}(\text{draw a red ball}) = 30/60$.
Random selection and equiprobable events
The purpose of selecting objects at random is to ensure that each has the same chance of being selected. This method of selection is called fair or unbiased, and the selection of any particular object is said to be equally likely or equiprobable.
When one object is randomly selected from n objects,

$$p_i = \frac{1}{n}, \text{ with } 1 \leq i \leq n,\ n = card(\Omega).$$

Example: 1) If we roll a well-balanced die, then the probability of a number between 1 and 6 occuring is:

$$p_i = \frac{1}{6}$$

avec $\Omega = \{1, 2, ..., 6\}$.
An urn contains 12 balls numbered from 1 to 12, we draw one at random, calculate the probability of drawing an even number or a multiple of three.
Solution:
*Express the events considered;
Note the “or” between these events.
State the total probability theorem to use.
Ask the question about mutually exclusive A : « draw an even number »= $\{2, 4, 6, 8, 10, 12\}$
B : « draw a factor of three » = $\{3, 6, 9, 12\}$
$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
$A \cap B = \{6, 12\}$.
$\mathbb{P}(A \cup B) = 7/12 + 4/12 - 2/12 = 2/3$.

1.3.5 Conditional probability

The word conditional is used to describe a probability that is dependent on some additional information given about an outcome or event. let A et B be two avents
we read $\mathbb{P}(A|B)$ as "the probability that A occurs, given that B occurs".
Introductory Example : The composition of an amphitheater of 200 students in a university is as follows:
130 students are girls.
100 students live with their families.
Among these 100 students who live with their families, 80 are girls.
We choose a student at random and we are interested in the three events:
A: "the student lives with his family".
B: "the student is a girl".
$A \cap B$ is a girl who lives with her family.
$\mathbb{P}(A) = 100/200$, $\mathbb{P}(B) = 130/200$,
$\mathbb{P}(A \cap B) = 80/200$.(number of girls who live with their parents/number of students)
If we know beforehand that the student is a girl then the reference set is made up of 130 student girls and the probability that she lives with her parents knowing that she is a girl becomes 80/130 (number of girls who live with their parents/number of student girls)= $\mathbb{P}(A|B)$.
Indeed: this probability is conditioned by the additional information, the event B “the student is a girl” is realized.
Notice that $80/130 = (80/200)/(130/200)$, a general result.

$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$.
The probability that the girl does not live with her parents among all student girls is 50/130, hence $\mathbb{P}(\bar{A}|B) = 1 - \mathbb{P}(A|B)$.

Definitions 2 *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, A an event such that $\mathbb{P}(A) \neq 0$, for any event B, we have*

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$$

properties:
1) $\mathbb{P}(\bar{B}|C) = 1 - \mathbb{P}(B|C)$.
2) $\mathbb{P}(B \setminus C|A) = \mathbb{P}(B|A) - \mathbb{P}(B \cap C|A)$.
3) $\mathbb{P}(B_1 \cup B_2|A) = \mathbb{P}(B_1|A) + \mathbb{P}(B_2|A) - \mathbb{P}(B_1 \cap B_2|A)$.
I a finite or infinite part of $\mathbb{N}$, $(B_i)_{i\in I}$ a sequence of incompatible events,

$$\mathbb{P}(\cup_{i\in I} B_i|A) = \sum_{i\in I} \mathbb{P}(B_i|A).$$

Example 2: Consider an urn containing 10 white, 20 red and 30 black balls. We draw two balls without replacement in the urn, what is the probability that the first ball is red and the second white? Solution:
Let be the events:
R:"draw a red ball", B:"draw a white ball".
$\mathbb{P}(R \cap B) = \mathbb{P}(R) * \mathbb{P}(B|R) = \frac{20}{60} * \frac{10}{59} = \frac{10}{117}$.
$\mathbb{P}(B|R) = \frac{10}{59}$,(59 balls remain in the urn, 10 of which are white).
Formula for compound probabilities:
Let $(A_i)_{1\leq i\leq n}$ be an avent sequence, $\mathcal{A}$ tel que

$$\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n) \neq 0.$$

Then we have $\mathbb{P}(\cap_{i=1}^n A_i) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)....\mathbb{P}(A_n|A_1 \cap A_2.... \cap A_{n-1})$.

theorem 1

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A) = \mathbb{P}(A|B) * \mathbb{P}(B) = \mathbb{P}(B|A) * \mathbb{P}(A)$$

If A and B are independent, then

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A) = \mathbb{P}(A) * \mathbb{P}(B)$$

Remark 2 *If A and B are independent, then $\mathbb{P}(A|B) = \mathbb{P}(A)$.*
The occurrence of one event does not modify the probability of occurrence of the other.

Warning: do not confuse incompatibility ($B \cap A = \emptyset$) and independence.

Definitions 3 *We say that* $\{A_1, ..., A_n\}$ *is a complete system if:*
*$\mathbb{P}(A_i) > 0$.
*$A_i \cap A_j = \emptyset$, *for* $i \neq j$*.(incompatible)*
*$\cup_{i=1}^{n} A_i = \Omega$

theorem 2 *(Bayes' theorem) Let* $\{A_1, ..., A_n\}$ *be a complete system. Let* B *be an event, the complete probability theorem;*

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\,\mathbb{P}(A_i)$$

Bayes' formula will be then be;

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\,\mathbb{P}(A_i)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i)\,\mathbb{P}(A_i)}$$

Example 1: Three urns are drawn at random, the composition of which is indicated in the table below.
Knowing that we have obtained a red ball, we ask ourselves what is the probability that it comes from urn U2.

	Red	Blue	Green
U1	3	4	1
U2	1	2	3
U3	4	3	2

$\mathbb{P}(R) = \sum_{i=1}^{3} \mathbb{P}(R \cap Ui) = \sum_{i=1}^{3} \mathbb{P}(R|Ui)\,\mathbb{P}(Ui)$
$= \frac{1}{3}(\frac{3}{8} + \frac{1}{6} + \frac{4}{9}) = \frac{71}{216}$
The posterior probability therefore being;
$\mathbb{P}(U2|R) = \frac{\mathbb{P}(R|U2)\mathbb{P}(U2)}{\mathbb{P}(R)} = \frac{\frac{1}{3} * \frac{1}{6}}{\frac{1}{3} * \frac{71}{72}}$
$= \frac{12}{71} < \mathbb{P}(U2)$.
Example 2: Three machines A, B, C produce respectively $40\%, 35\%$ and 25% of the total number of tablets manufactured by a pharmaceutical laboratory, each of these machines produces respectively $5\%, 6\%$et 3% of defective tablets.
a) We take a tablet at random, what is the probability that it is defective?
b) We take a tablet at random, we see that it is defective, what is the probability that it is produced by machine A?

Answer
$\mathbb{P}(A) = 0.4,\ \mathbb{P}(B) = 0.35,\ \mathbb{P}(C) = 0.25.$
D: "defective tablet".
$\mathbb{P}(D|A) = 0.05,\ \mathbb{P}(D|B) = 0.06,\ \mathbb{P}(D|C) = 0.03.$
The realization of D can be due to each of the three causes A, B, C, these three causes are incompatible;
$\mathbb{P}(D) = \mathbb{P}(A)\mathbb{P}(D|A) + \mathbb{P}(B)\mathbb{P}(D|B) + \mathbb{P}(C)\mathbb{P}(D|C)$
$= 0.4 * 0.05 + 0.35 * 0.06 + 0.25 * 0.03 = 0.0485.$
b) We know that the event "defective tablet" has occurred, we look for the probability of the cause "machine A";$\mathbb{P}(A|D) = \frac{\mathbb{P}(D|A)\mathbb{P}(A)}{\mathbb{P}(D)}$
$= \frac{0.4*0.05}{0.0485} = 0.41$