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Dedicated to the memory of S. Bach.

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The Author thanks V. Exalted.

Abstract. Replace this text with your own abstract.

Contents

University of Tlemcen **Academic year 2023-2024**
Faculty of Sciences **(L1 ING-INF)**
Department of Informatic **Algebra (First Year)**

Worksheet N°3/ "Equivalence relation- Order relation" Correction.

Exercise 01: (catch-up 22-23) The relation $\Re$ is defined in $\mathbb{R}^*$ as:

$$x\Re y \Longleftrightarrow x^2 - \frac{1}{x^2} = y^2 - \frac{1}{y^2}.$$

(1) Show that $\Re$ is an equivalence relation on $\mathbb{R}^*$.
a) $\Re$ is reflexive**?**

$\Re$ is reflexive $\Leftrightarrow \forall x \in \mathbb{R}^*, x\Re x$.

$$\forall x \in \mathbb{R}^*, x^2 - \frac{1}{x^2} = x^2 - \frac{1}{x^2} \Rightarrow x\Re x \Rightarrow \Re \text{ is reflexive.}$$

b) $\Re$ is symmetric**?**

$\Re$ is symmetric $\Leftrightarrow \forall x, y \in \mathbb{R}^*, x\Re y \Rightarrow y\Re x$.

$$\begin{aligned} \forall x, y &\in \mathbb{R}^*, x\Re y \Rightarrow x^2 - \frac{1}{x^2} = y^2 - \frac{1}{y^2}, \\ &\Rightarrow y^2 - \frac{1}{y^2} = x^2 - \frac{1}{x^2} \\ &\Rightarrow y\Re x, \\ &\Rightarrow \Re \text{ is symmetric.} \end{aligned}$$

c) $\Re$ is transitive**?**

$\Re$ est transitive $\Leftrightarrow \forall x, y, z \in \mathbb{R}^*, x\Re y$ and $y\Re z \Rightarrow x\Re z$.

$$\begin{aligned} \forall x, y, z &\in \mathbb{R}^*, x\Re y \text{ and } y\Re z, \\ &\Rightarrow x^2 - \frac{1}{x^2} = y^2 - \frac{1}{y^2} \text{ and } y^2 - \frac{1}{y^2} = z^2 - \frac{1}{z^2}, \\ &\Rightarrow x^2 - \frac{1}{x^2} = z^2 - \frac{1}{z^2} \Rightarrow x\Re z. \end{aligned}$$

<u>Conclusion:</u> $\Re$ is an equivalence relation in $\mathbb{R}^*$ because $\Re$ is reflexive, symmetrice and transitive.

(2) What is the equivalence class of $a \in \mathbb{R}^*$ for this equivalence relation.

$$cl(a) = \dot{a} = \{x \in \mathbb{R}^* / x\Re a\}.$$

$$\begin{aligned}
x\Re a &\Leftrightarrow x^2-\frac{1}{x^2}=a^2-\frac{1}{a^2}\\
&\Leftrightarrow x^2-a^2+\frac{1}{a^2}-\frac{1}{x^2}=0\\
&\Leftrightarrow x^2-a^2+\frac{x^2-a^2}{a^2x^2}=0\\
&\Leftrightarrow \left(x^2-a^2\right)\left(1+\frac{1}{a^2x^2}\right)=0.\\
&\Leftrightarrow \begin{cases} x^2-a^2=0\Rightarrow x=a \text{ or } x=-a,\\ \text{or}\\ \underbrace{1+\frac{1}{a^2x^2}}_{>0}=0 \text{ (is not suitable)}\end{cases}
\end{aligned}$$

So

$$cl(a)=\{a,-a\}\text{ .Remark: } cl(a)=cl(-a).$$

(3) Find the quotient set.

$$\{cl(a),a\in\mathbb{R}^*\}$$

$$\mathbb{R}^*/\Re=\left\{cl(a),a\in\mathbb{R}_+^*\right\}\overset{\text{or}}{=}\left\{cl(a),a\in\mathbb{R}_-^*\right\}.$$

Exercise 02: The relation R is defined in $\mathbb{Z}\times\mathbb{N}^*$ as:

$$(x,y)\,R\,(x',y')\Leftrightarrow xy'-x'y=0.$$

(1) Show that R is an equivalence relation on $\mathbb{Z}\times\mathbb{N}^*$.

a) R is reflexive**?**

R is reflexive $\Leftrightarrow\forall\,(x,y)\in\mathbb{Z}\times\mathbb{N}^*,(x,y)\,R\,(x,y)\,.$

$$\begin{aligned}&\forall\,(x,y)\in\mathbb{Z}\times\mathbb{N}^*\Rightarrow xy-xy=0\\&\Rightarrow(x,y)\,R\,(x,y)\Rightarrow R\text{ is reflexive.}\end{aligned}$$

b) R is symmetric**?**

R is symmetric $\Leftrightarrow\forall\,(x_1,y_1)\,,(x_2,y_2)\in\mathbb{Z}\times\mathbb{N}^*,$

$$(x_1,y_1)\,R\,(x_2,y_2)\Rightarrow(x_2,y_2)\,R\,(x_1,y_1)?$$

$\forall\,(x_1,y_1)\,,(x_2,y_2)\in\mathbb{Z}\times\mathbb{N}^*,$

$$\begin{aligned}&\text{If }\ (x_1,y_1)\,R\,(x_2,y_2)\Rightarrow x_1y_2-x_2y_1=0...(\times(-1))\\&\Rightarrow x_2y_1-x_1y_2=0,\\&\Rightarrow(x_2,y_2)\,R\,(x_1,y_1)\\&\Rightarrow R\text{ is symmetric.}\end{aligned}$$

c) R is transitive**?**

R is transitive $\Leftrightarrow\forall\,(x_1,y_1)\,,(x_2,y_2)\,,(x_3,y_3)\in\mathbb{Z}\times\mathbb{N}^*,$

$$\text{If}\begin{cases}(x_1,y_1)\,R\,(x_2,y_2)\\ \quad\text{and}\\ (x_2,y_2)\,R\,(x_3,y_3)\end{cases}\Rightarrow(x_1,y_1)\,R\,(x_3,y_3)?$$

$\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{Z} \times \mathbb{N}^*,$

$$\begin{aligned}
\text{If} \left\{ \begin{array}{c} (x_1, y_1) R (x_2, y_2) \\ \text{and} \\ (x_2, y_2) R (x_3, y_3) \end{array} \right. & \Rightarrow \left\{ \begin{array}{c} x_1 y_2 - x_2 y_1 = 0 \ldots (\times y_3) \\ \text{and} \\ +x_2 y_3 - x_3 y_2 = 0 \ldots (\times y_1) \end{array} \right. \\
\text{The summ} & \Rightarrow x_1 y_3 y_2 - x_3 y_1 y_2 = 0 \left(\underset{y_2 \in \mathbb{N}^*}{y_2 \neq 0} \right) \\
& \Rightarrow x_1 y_3 - x_3 y_1 = 0 \\
& \Rightarrow (x_1, y_1) R (x_3, y_3)
\end{aligned}$$

Conclusion: R is an equivalence relation in $\mathbb{Z} \times \mathbb{N}^*$.

(2) Identify $cl\,((1;2))$ and $cl\,((-1;2))$.

$$cl\,((1;2)) = \{(x;y) \in \mathbb{Z} \times \mathbb{N}^* / (x;y) R\,(1;2)\}.$$

$$(x;y) R\,(1;2) \Leftrightarrow 2x - y = 0 \Leftrightarrow x = \frac{y}{2} \Leftrightarrow y = 2x.$$

So

$$\begin{aligned}
cl\,((1;2)) &= \left\{ (x; \underset{\in \mathbb{N}^*}{2x}), x \in \mathbb{Z}_+^* \right\} \\
&\overset{\text{or}}{=} \left\{ (\underset{\in \mathbb{Z}}{\frac{y}{2}}; y), y \in 2\mathbb{N}^* \right\}, (2\mathbb{N}^* = \{2k, k \in \mathbb{N}^*\})
\end{aligned}$$

$$cl\,((-1;2)) = \{(x;y) \in \mathbb{Z} \times \mathbb{N}^* / (x;y) R\,(-1;2)\}.$$

$$(x;y) R\,(-1;2) \Leftrightarrow 2x + y = 0 \Leftrightarrow x = -\frac{y}{2} \Leftrightarrow y = -2x.$$

So

$$\begin{aligned}
cl\,((-1;2)) &= \left\{ (x; \underset{\in \mathbb{N}^*}{-2x}), x \in \mathbb{Z}_-^* \right\} \\
&\overset{\text{or}}{=} \left\{ (\underset{\in \mathbb{Z}}{-\frac{y}{2}}; y), y \in 2\mathbb{N}^* \right\}, (2\mathbb{N}^* = \{2k, k \in \mathbb{N}^*\})
\end{aligned}$$

Exercise 03 : Let E be a non-empty set and F a non-empty sub-set of E.

In $P(E)$ the power set of E, we defined $\Re$ by:

$$\forall (A, B) \in P(E) \times P(E), A \Re B \Leftrightarrow A \cap F = B \cap F.$$

(1) Prove that $\Re$ is an equivalence relation.

a) $\Re$ is reflexive?

$$\Re \text{ is reflexive} \Leftrightarrow \forall A \in P(E), A \Re A.$$

$$\begin{aligned}
\forall A &\in P(E), A \cap F = A \cap F \Rightarrow A \Re A \\
&\Rightarrow \Re \text{ is reflexive.}
\end{aligned}$$

b) $\Re$ is symmetric?

$$\Re \text{ is symmetric} \Leftrightarrow \forall A, B \in P(E), A \Re B \Rightarrow B \Re A.$$

$$\begin{aligned}\forall A,B &\in P(E), A\Re B \Rightarrow A\cap F = B\cap F\\ &\Rightarrow B\cap F = A\cap F\\ &\Rightarrow B\Re A\\ &\Rightarrow \Re \text{ is symmetric.}\end{aligned}$$

c) $\Re$ is transitive?

$\Re$ is transitive $\Leftrightarrow \forall A,B,C \in P(E), A\Re B$ and $B\Re C \Rightarrow A\Re C$.

$$\begin{aligned}\forall A,B,C &\in P(E), \left\{\begin{array}{c} A\Re B\\ \text{and}\\ B\Re C\end{array}\right. \Rightarrow \left\{\begin{array}{c} A\cap F = B\cap F\\ \text{and}\\ B\cap F = C\cap F\end{array}\right.\\ &\Rightarrow A\cap F = C\cap F\\ &\Rightarrow A\Re C\\ &\Rightarrow \Re \text{ is transitive.}\end{aligned}$$

Conclusion: $\Re$ is an equivalence relation because $\Re$ is reflexive, symmetric and transitive.

(2) What is the equivalence class of $\emptyset$.

$$cl(\emptyset) = \{A \in P(E)/A\Re\emptyset\}.$$

$$\forall (A,B) \in P(E)\times P(E), A\Re B \Leftrightarrow A\cap F = B\cap F.$$

$$\begin{aligned}A\Re\emptyset &\Leftrightarrow A\cap F = \emptyset\cap F\\ &\Leftrightarrow A\cap F = \emptyset.\end{aligned}$$

$$cl(\emptyset) = \left\{A/A \subset \overline{F}\right\} = \left\{A/A \in P(\overline{F})\right\}.$$

(3) Have they: $E \in Cl(\emptyset)$? Justify.

$$E \notin Cl(\emptyset) \text{ because } E \notin P(\overline{F}).$$

(4) Find $Cl(E)$. Deduce $Cl(F)$.

$$cl(E) = \{A \in E/A\Re E\}.$$

$$\begin{aligned}A\Re E &\Leftrightarrow A\cap F = E\cap F\\ &\Leftrightarrow A\cap F = F \text{ so } (F\subset A)\\ &\Leftrightarrow A = B\cup F \text{ with } B\subset \overline{F}\end{aligned}$$

So

$$cl(E) = \left\{A \in E/A = B\cup F \text{ with } B\subset\overline{F}\right\}.$$

If

$$B = \varnothing \Rightarrow A = F \Rightarrow F \in cl(E) \Rightarrow cl(F) = cl(E).$$

<u>Exercises of the order relation.</u>

Exercise 04: The relation Φ is defined in $\mathbb{N}^*$ as:

$$x\Phi y \Leftrightarrow \exists n \in \mathbb{N} \text{ such as: } x^n = y.$$

(1) Prove that Φ is an order relation.

a) Φ is reflexive?

$$\Phi \text{ is reflexive} \Leftrightarrow \forall x \in \mathbb{N}^*, x\Phi x.$$

$$\forall x \in \mathbb{N}^*, \exists n = 1 \in \mathbb{N}, x^n = x^1 = x \Rightarrow x\Phi x \Rightarrow \Phi \text{ is reflexive.}$$

b) Φ is antisymmetric?

$$\Phi \text{ is antisymmetric} \Leftrightarrow \forall x, y \in \mathbb{N}^*, [x\Phi y \text{ and } y\Phi x] \Rightarrow x = y.$$

$$\forall x, y \in \mathbb{N}^*, \begin{cases} x\Phi y \\ \text{and} \\ y\Phi x \end{cases} \Rightarrow \begin{cases} \exists n_1 \in \mathbb{N} \text{ such as: } x^{n_1} = y...(1) \\ \text{and} \\ \exists n_2 \in \mathbb{N} \text{ such as: } y^{n_2} = x...(2) \end{cases}$$

We replace (1) in (2):

$$\begin{aligned} (x^{n_1})^{n_2} &= x \Rightarrow x^{n_1 n_2} = x \Rightarrow n_1 n_2 = 1 \\ &\Rightarrow n_1 = n_2 = 1 \\ \overset{\text{in } (1)}{\Rightarrow} x &= y \Rightarrow \Phi \text{ is antisymmetric.} \end{aligned}$$

c) Φ is transitive?

$$\Phi \text{ is transitive} \Leftrightarrow \forall x, y, z \in \mathbb{N}^*, [x\Phi y \text{ and } y\Phi z] \Rightarrow x\Phi z.$$

$$\forall x, y, z \in \mathbb{N}^*, \begin{cases} x\Phi y \\ \text{and} \\ y\Phi z \end{cases} \Rightarrow \begin{cases} \exists n_1 \in \mathbb{N} \text{ such as: } x^{n_1} = y...(1) \\ \text{and} \\ \exists n_2 \in \mathbb{N} \text{ such as: } y^{n_2} = z...(2) \end{cases}$$

We replace (1) in (2):

$$\begin{aligned} (x^{n_1})^{n_2} &= z \Rightarrow x^{n_1 n_2} = z \\ &\Rightarrow \exists n_3 = n_1 n_2 \in \mathbb{N} \text{ such as: } x^{n_3} = z \\ &\Rightarrow x\Phi z \Rightarrow \Phi \text{ is transitive.} \end{aligned}$$

Conclusion: Φ is an order relation because Φ is reflexive, antisymmetric and transitive.

(2) Is it a total order? Justify.

Remainder of the definition of the total order $\leq$ in $\mathbb{R}$:

$$\forall x, y \in \mathbb{R}, x \leq y \text{ or } y \leq x.$$

Φ is a total order $\Leftrightarrow$

$$\forall x, y \in \mathbb{N}^*, x\Phi y \text{ or } y\Phi x.$$

Φ is a partial order.
because: If $x = 2$ and $y = 3$, we have n't:

$$2^{n_1} = 3 \text{ or } 3^{n_2} = 2 \Rightarrow \text{ neither } 2\Phi 3 \text{ nor } 3\Phi 2.$$

Scrambler:

$$[\exists n_1 \in \mathbb{N} \text{ such as: } x^{n_1} = y] \text{ or } [\exists n_2 \in \mathbb{N} \text{ such as: } y^{n_2} = x]$$

(3) Let $A = \{1, 4, 8\}$. Determine if they exist, $\max A$ and $\min A$ for the order Φ.

Remainder: For $\leq$ in E :

$\inf E$ but $\min E$ doesn't exist $\quad\quad$ $\sup E \in E = \max E$

$-\infty$ lower bounds $\quad]a \quad E$ set of $x \quad b] \quad$ upper bounds $+\infty$

$$M \text{ is an upper bound} \Leftrightarrow \forall u \in E, u \leq M.$$

$$\sup E \text{ is the smallest of the upper bounds.}$$

$$\begin{array}{lcl} \text{If } \sup E & \in & E \Rightarrow \max E = \sup E. \\ \text{If } \sup E & \notin & E \Rightarrow \max E \text{ doesn't exist.} \end{array}$$

$$m \text{ is a lower bound} \Leftrightarrow \forall u \in E, m \leq u.$$

$$\inf E \text{ is the largest of the lower bounds.}$$

$$\begin{array}{lcl} \text{If } \inf E & \in & E \Rightarrow \min E = \inf E. \\ \text{If } \inf E & \notin & E \Rightarrow \min E \text{ doesn't exist.} \end{array}$$

In the exercise:

$$A = \{1, 4, 8\}.$$

$$M \text{ is an upper bound} \Leftrightarrow \forall x \in A, x\Phi M.$$

$$\left\{ \begin{array}{c} 1\Phi M \Rightarrow \exists n_1 \in \mathbb{N} \text{ such as: } 1^{n_1} = M \Rightarrow M = 1. \\ \text{and} \\ 4\Phi M \Rightarrow \exists n_2 \in \mathbb{N} \text{ such as: } 4^{n_2} = M \Rightarrow M \in \{1, 4, 16, ...\} \\ \text{and} \\ 8\Phi M \Rightarrow \exists n_3 \in \mathbb{N} \text{ such as: } 8^{n_3} = M \Rightarrow M \in \{1, 8, 64, ...\} \end{array} \right.$$

So:

$$M = 1 \Rightarrow \sup A = 1 \in A \Rightarrow \max A = 1.$$

$$m \text{ is a lower bound} \Leftrightarrow \forall x \in A, m\Phi x.$$

$$\left\{ \begin{array}{l} m\Phi 1 \Rightarrow \exists n_1 \in \mathbb{N} \text{ such as: } m^{n_1} = 1 \\ \Rightarrow m \in \mathbb{N}, (n_1 = 0)\,. \\ \text{and} \\ m\Phi 4 \Rightarrow \exists n_2 \in \mathbb{N} \text{ such as: } m^{n_2} = 4 \\ \Rightarrow M \in \{2; 4\} \left(4^1 = 4, 2^2 = 4\right) \\ \text{and} \\ m\Phi 8 \Rightarrow \exists n_3 \in \mathbb{N} \text{ such as: } m^{n_3} = 8 \\ \Rightarrow M \in \{2; 8\} \left(2^3 = 8, 8^1 = 8\right) \end{array} \right.$$

So:

$$m = 2 \Rightarrow \inf A = 2 \notin A \Rightarrow \min A \text{ doesn't exist.}$$

Exercise 05: (Final exam 22-23) In $]1, +\infty[$, the relation $\Re$ is defined:

$$x\Re y \Leftrightarrow \frac{y}{y^2+1} \geq \frac{x}{x^2+1} \Leftrightarrow f(y) \geq f(x).$$

(1) Show that $\Re$ is an order relation in $I =]1;+\infty[$.

We put $f(x) = \frac{x}{x^2+1}$ defined in $\mathbb{R}$.

a) $\Re$ is reflexive?

$\Re$ is reflexive $\Leftrightarrow \forall x \in I, x\Re x$.

$$\forall x \in I, f(x) \geq f(x) \Rightarrow x\Re x \Rightarrow \Re \text{ is reflexive.}$$

b) $\Re$ is antisymmetric?

$$\forall x, y \in I, [x\Re y \text{ and } y\Re x] \Rightarrow x = y.$$

$$\begin{aligned} \forall x, y \quad &\in \quad I, \begin{cases} x\Re y \\ \text{and} \\ y\Re x \end{cases} \Rightarrow \begin{cases} f(y) \geq f(x) \\ \text{and} \\ f(x) \geq f(y) \end{cases} \\ &\Rightarrow \quad f(x) = f(y) \Rightarrow x = y \text{ if } f \text{ is injective in } I =]1;+\infty[.. \end{aligned}$$

$$f'(x) = \frac{\left(x^2+1\right) - (2x)\,x}{\left(x^2+1\right)^2} = \frac{1-x^2}{\left(x^2+1\right)^2} < 0,$$

before f is a continuous application and strictly decreasing that implies f is injective.

Conclusion: $\Re$ is antisymmetric.

c) $\Re$ is transitive?

$\Re$ is transitive $\Leftrightarrow \forall x, y, z \in I, [x\Re y \text{ and } y\Re z] \Rightarrow x\Re z$.

$$\begin{aligned} \forall x, y, z \quad &\in \quad I, \begin{cases} x\Re y \\ \text{and} \\ y\Re z \end{cases} \Rightarrow \begin{cases} f(y) \geq f(x) \\ \text{and} \\ f(z) \geq f(y) \end{cases} \\ &\Rightarrow \quad f(z) \geq f(x) \Rightarrow \Re \text{ is transitive.} \end{aligned}$$

Conclusion: $\Re$ is an order relation because $\Re$ is reflexive, antisymmetric and transitive.

(2) Is it a total order? Justify.

$\Re$ is a total order $\Leftrightarrow \forall x, y \in I, x\Re y$ or $y\Re x$.

$\Re$ is a total order because:
$\forall x, y \in I$ we have two cases:

$$\begin{aligned} \text{1st case} \quad &: \quad x \leq y \Rightarrow f(x) \geq f(y) \Rightarrow y\Re x \\ \text{2nd case} \quad &: \quad y \leq x \Rightarrow f(y) \geq f(x) \Rightarrow x\Re y. \end{aligned}$$

before f is strictly decreasing.

(3) Let $A = \{2;7;8\}$.

Determine if they exist $\sup A$ and $\inf A$.

Calculate

$$f(2) = \frac{2}{4+1} = \frac{2}{5} = 0.4; f(7) = \frac{7}{50} = 0.14; f(8) = \frac{8}{65} = 0.12;$$

$$\begin{aligned} [f(2) \quad &\geq \quad f(7) \Rightarrow 7\Re 2] \text{ and } [f(7) \geq f(8) \Rightarrow 8\Re 7] \\ &\Rightarrow \quad 8\Re 7\Re 2 \end{aligned}$$

$$\Rightarrow \inf A = \min A = 8 \text{ and } \sup A = \max A = 2.$$

Remark: If we have the question: Find the upper bounds and the lower bounds.

$$M \text{ is an upper bound for } A \Leftrightarrow \forall x \in A, x\Re M.$$
$$m \text{ is a lower bound for } A \Leftrightarrow \forall x \in A, m\Re x.$$

But $\Re$ is a total order:

$$\begin{aligned} 2\Re M &\Leftrightarrow f(M) \geq f(2) \overset{f \text{ is strictly decreasing}}{\Rightarrow} M \leq 2. \\ m\Re 8 &\Leftrightarrow f(8) \geq f(m) \overset{f \text{ is strictly decreasing}}{\Rightarrow} 8 \leq m. \end{aligned}$$

Conclusion:

1) The set of the upper bounds is $]1;2]$.
2) The set of the upper bounds is $[8;+\infty[$.

<u>Exercise 06:</u> In $\mathbb{R}^2$, the relation $\leq$ is defined:

$$(x,y) \leq (x',y') \Leftrightarrow x \leq x' \text{ and } y \leq y'.$$

(1) Show that it is an order relation. Is it a total order?

a) $\leq$ is reflexive**?**

$\leq$ is reflexive $\Leftrightarrow \forall (x,y) \in \mathbb{R}^2, (x,y) \leq (x,y)$.

$$\begin{aligned} \forall (x,y) &\in \mathbb{R}^2 \Rightarrow x \leq x \text{ and } y \leq y \Rightarrow (x,y) \leq (x,y) \\ &\Rightarrow \leq \text{ is reflexive.} \end{aligned}$$

b) $\leq$ is it antisymmetric**?**

$\leq$ is antisymmetric $\Leftrightarrow \forall (x,y), (x',y') \in \mathbb{R}^2$,

$$[(x,y) \leq (x',y') \text{ and } (x',y') \leq (x,y)] \Rightarrow (x,y) = (x',y').$$

Let $(x,y), (x',y') \in \mathbb{R}^2$,

$$\begin{aligned} (x,y) &\leq (x',y') \text{ and } (x',y') \leq (x,y), \\ &\Rightarrow [x \leq x' \text{ and } y \leq y'] \text{ and } [x' \leq x \text{ and } y' \leq y], \\ &\Rightarrow x = x' \text{ and } y = y' \Rightarrow (x,y) = (x',y'), \\ &\Rightarrow \leq \text{ is antisymmetric.} \end{aligned}$$

c) $\leq$ is it transitive**?**

$\leq$ is transitive $\Leftrightarrow \forall (x,y), (x',y'), (x'',y'') \in \mathbb{R}^2$,

$$(x,y) \leq (x',y') \text{ and } (x',y') \leq (x'',y'') \Rightarrow (x,y) \leq (x'',y'').$$

Let $(x,y), (x',y'), (x'',y'') \in \mathbb{R}^2$,

$$\begin{aligned} (x,y) &\leq (x',y') \text{ and } (x',y') \leq (x'',y''), \\ &\Rightarrow [x \leq x' \text{ and } y \leq y'] \text{ and } [x' \leq x'' \text{ and } y' \leq y''], \\ &\Rightarrow x \leq x'' \text{ and } y \leq y'', \\ &\Rightarrow (x,y) \leq (x'',y'') \Rightarrow \leq \text{ is transitive.} \end{aligned}$$

Conclusion: $\leq$ is an order relation because $\leq$ is reflexive, antisymmetric and transitive.

2)$\leq$ is a total order if and only if:

$$\forall (x,y), (x',y') \in \mathbb{R}^2, (x,y) \leq (x',y') \text{ or } (x',y') \leq (x,y).$$

$\leq$ is a partial order because from for example: $(6,1)$ and $(1,5)$

$$\text{neither } (6,1) \leq (1,5) \text{ nor } (1,5) \leq (6,1).$$

(3) Specify two lower bounds, two upper bounds, supremum, infimum, maximum and minimum of the part:

$$A = \{(1,2)\,;(3,1)\}\,.$$

i) (M_1, M_2) is an upper bound of $A \Rightarrow \forall\,(x,y) \in A, (x,y) \leq (\ M_1, M_2)\,,$

$$\Rightarrow \begin{cases} (1,2) \leq (M_1; M_2) \Rightarrow 1 \leq M_1 \text{ and } 2 \leq M_2, \\ \qquad\qquad\text{and} \\ (3,1) \leq (M_1; M_2) \Rightarrow 3 \leq M_1 \text{ and } 1 \leq M_2. \end{cases}$$

$\Rightarrow 3 \leq M_1$ and $2 \leq M_2$.

So the set of the upper bounds is:

$$S_1 = \left\{(M_1; M_2) \in \mathbb{R}^2, \text{ with: } 3 \leq M_1 \text{ and } 2 \leq M_2\right\}.$$

$\Rightarrow \sup A = (3;2) \notin A \Rightarrow \max A$ doesn't exist.

<u>Remark:</u>

$$\forall\,(M_1; M_2) \in S_2, (3;2) \leq (M_1; M_2) \Rightarrow \sup A = (3;2)\,.$$

ii) (m_1, m_2) is a lower bound of $A \Rightarrow \forall\,(x,y) \in A, (m_1; m_2) \leq (x;y)\,,$

$$\Rightarrow \begin{cases} (m_1; m_2) \leq (1;2) \Rightarrow m_1 \leq 1 \text{ and } m_2 \leq 2, \\ \qquad\qquad\text{and} \\ (m_1; m_2) \leq (3;1) \Rightarrow m_1 \leq 3 \text{ and } m_2 \leq 1. \end{cases}$$

$\Rightarrow m_1 \leq 1$ and $m_2 \leq 1$.

So the set of the lower bounds is:

$$S_2 = \left\{(m_1; m_2) \in \mathbb{R}^2, \text{ with: } m_1 \leq 1 \text{ and } m_2 \leq 1\right\}.$$

$\Rightarrow \inf A = (1;1) \notin A \Rightarrow \min A$ doesn't exist.

<u>Remark:</u>

$$\forall\,(m_1; m_2) \in S_2, (m_1; m_2) \leq (1;1) \Rightarrow \inf A = (1;1)\,.$$

Exercise 07: In $\mathbb{R}^2$ let $\Re$ be a relation defined by:

$$(x,y)\,\Re\,(x',y') \Leftrightarrow (x < x') \text{ or } (x = x' \text{and } y \leq y')\,.$$

(1) Show that it is an order relation. Is it a total order?

a) $\Re$ is reflexive**?**

$\Re$ is reflexive $\Leftrightarrow \forall\,(x,y) \in \mathbb{R}^2, (x,y)\,\Re\,(x,y)\,.$

$$\forall\,(x,y) \quad \in \quad \mathbb{R}^2 \Rightarrow \underbrace{\underset{\text{False}}{(x < x)} \text{ or } \underset{\text{True}}{(x = x \text{ and } y \leq y)}}_{\text{True}} \Rightarrow (x,y)\,\Re\,(x,y)$$

$$\Rightarrow \quad \Re \text{ is reflexive.}$$

b) $\Re$ is it antisymmetric**?**

$\Re$ is antisymmetric $\Leftrightarrow \forall\,(x,y)\,,(x',y') \in \mathbb{R}^2,$

$$[(x,y)\,\Re\,(x',y') \text{ and } (x',y')\,\Re\,(x,y)] \Rightarrow (x,y) = (x',y')\,.$$

Let $(x;y), (x';y') \in \mathbb{R}^2$,

$$\begin{cases} (x,y)\,\Re\,(x',y') \Rightarrow (x<x') \text{ or } (x=x'\text{and } y\le y') \\ \text{and} \\ (x',y')\,\Re\,(x,y) \Rightarrow (x'<x) \text{ or } (x'=x \text{ and } y'\le y) \end{cases}$$

$$\Rightarrow \begin{cases} (x<x') \text{ and } (x'<x) \quad \text{(is not suitable)} \\ (x<x') \text{ and } (x'=x \text{ and } y'\le y) \quad \text{(is not suitable)} \\ (x=x'\text{and } y\le y') \text{ and } (x'<x) \quad \text{(is not suitable)} \\ (x=x'\text{and } y\le y') \text{ and } (x'=x \text{ and } y'\le y) \Rightarrow x=x'\text{and } y=y' \end{cases}$$

$$\Rightarrow \quad (x,y)=(x',y')$$
$$\Rightarrow \quad \Re \text{ is antisymmetric.}$$

c) $\Re$ is it transitive**?**

$\Re$ is transitive $\Leftrightarrow \forall (x,y), (x',y'), (x'',y'') \in \mathbb{R}^2$,

$$(x,y)\,\Re\,(x',y') \text{ and } (x',y')\,\Re\,(x'',y'') \Rightarrow (x,y)\,\Re\,(x'',y'').$$

Let $(x,y), (x',y'), (x'',y'') \in \mathbb{R}^2$,

$$\begin{cases} (x,y)\,\Re\,(x',y') \Rightarrow (x<x') \text{ or } (x=x'\text{and } y\le y') \\ \text{and} \\ (x',y')\,\Re\,(x'',y'') \Rightarrow (x'<x'') \text{ or } (x'=x'' \text{ and } y'\le y'') \end{cases}$$

$$\Rightarrow \begin{cases} (x<x') \text{ and } (x'<x'') \Rightarrow x<x'' \\ (x<x') \text{ and } (x'=x'' \text{ and } y'\le y'') \Rightarrow x<x'' \\ (x=x'\text{and } y\le y') \text{ and } (x'<x'') \Rightarrow x<x'' \\ (x=x'\text{and } y\le y') \text{ and } (x'=x'' \text{ and } y'\le y'') \Rightarrow x=x''\text{and } y\le y'' \end{cases}$$

$\Rightarrow \quad x<x''$ or $[x=x''\text{and } y\le y'']$

$\Rightarrow \quad (x,y)\,\Re\,(x'',y'') \Rightarrow$ $\Re$ est transitive.

Conclusion: $\Re$ is an order relation because $\Re$ is reflexive, antisymmetric and transitive.

(1) $\Re$ is a tatal order $\Leftrightarrow \forall (x,y), (x',y') \in \mathbb{R}^2$ we have:

$$[x<x'\text{or } (x=x' \text{ and } y\le y')] \text{ or } [x'<x \text{ or } (x'=x \text{ and } y'\le y)]$$

So,

$$(x,y)\,\Re\,(x',y') \text{ or } (x',y')\,\Re\,(x,y).$$

(2) Let $A=\{(-1;1), (2;-1)\}$, Find $\sup A, \inf A, \max A$, and $\min A$.

1st method: Before $\Re$ is a total order we have:

$$(-1;1)\,\Re\,(2;-1),$$
$$\Rightarrow \sup A \in A \Rightarrow \max A = (2;-1)$$
$$\text{and } \inf A \in A \Rightarrow \min A = (-1;1).$$

2nd method:

a) (M_1,M_2) is an upper bound of $A \Leftrightarrow \forall (x,y) \in A, (x,y)\,\Re\,(M_1,M_2)$**,**

$$\Rightarrow \begin{cases} (-1,1)\,\Re\,(M_1,M_2) \Leftrightarrow (-1<M_1) \text{ or } (-1=M_1 \text{ and } 1\le M_2), \\ \text{and } (2,-1)\,\Re\,(M_1,M_2) \Leftrightarrow (2<M_1) \text{ or } (2=M_1 \text{ and } -1\le M_2), \end{cases}$$

$$\Rightarrow \begin{cases} (-1<M_1) \text{ and } (2<M_1) \Rightarrow (2<M_1), \\ (-1<M_1) \text{ and } (2=M_1 \text{ and } -1\le M_2) \Rightarrow (2=M_1 \text{ and } -1\le M_2) \\ (-1=M_1 \text{ and } 1\le M_2) \text{ and } (2<M_1) \quad \text{(is not suitable)} \\ (-1=M_1 \text{ and } 1\le M_2) \text{ and } (2=M_1 \text{ and } -1\le M_2) \quad \text{(is not suitable)} \end{cases}$$

Conclusion: the set of the upper bounds is:

$$S_1 = \{(M_1, M_2) \,/\, (2 < M_1) \text{ or } (2 = M_1 \text{ and } -1 \leq M_2)\}.$$

$\sup A = (2, -1)$ because:

$$\forall (M_1, M_2) \in S_1, (2, -1) \Re (M_1, M_2).$$

$$(2, -1) \in A \Rightarrow \max A = (2, -1).$$

b) (m_1, m_2) is a lower bound of $A \Leftrightarrow \forall (x, y) \in A, (m_1, m_2) \Re (x, y)$,

$$\Rightarrow \begin{cases} (m_1, m_2) \Re (-1, 1) \Leftrightarrow (m_1 < -1) \text{ or } (m_1 = -1 \text{ and } m_2 \leq 1), \\ \text{and } (m_1, m_2) \leq (2, -1) \Leftrightarrow (m_1 < 2) \text{ or } (m_1 = 2 \text{ and } m_2 \leq -1), \end{cases}$$

$$\Rightarrow \begin{cases} (m_1 < -1) \text{ and } (m_1 < 2) \Rightarrow (m_1 < -1), \\ (m_1 < -1) \text{ and } (m_1 = 2 \text{ and } m_2 \leq -1), \text{ (is not suitable)} \\ (m_1 = -1 \text{ and } m_2 \leq 1) \text{ and } (m_1 < 2) \Rightarrow (m_1 = -1 \text{ et } m_2 \leq 1) \\ (m_1 = -1 \text{ and } m_2 \leq 1) \text{ and } (m_1 = -2 \text{ and } m_2 \leq -1) \text{ (is not suitable)} \end{cases}$$

Conclusion: the set of the lower set is:

$$S_2 = \{(m_1, m_2) \,/\, (m_1 < -1) \text{ or } (m_1 = -1 \text{ and } m_2 \leq 1)\}.$$

$\inf A = (-1, 1)$ because:

$$\forall (m_1, m_2) \in S_2, (m_1, m_2) \leq (-1, 1).$$

$$(-1, 1) \in A \Rightarrow \min A = (-1, 1).$$