ANALYSIS I, TUTORIAL 1 / Real Numbers

Exercise 1. Calculate the following sums

$$S_1 = \sum_{k=1}^{n} k^2, \quad S_2 = \sum_{k=1}^{n} k^3.$$

Exercise 2. Prove that

- $|x+y| = |x| + |y|$ if and only if x, y have the same sign.
- $|x-y| = |x-z| + |z-y|$ fi and only if $x \le z \le y$ or $y \le z \le x$.

Exercise 3. Show the existence of a constant $C \in \mathbb{R}_+^*$ such as

$$\forall x \in \mathbb{R}_+ : \quad \left|\frac{2x+5}{x+2} - \sqrt{5}\right| \le C \mid x - \sqrt{5} \mid .$$

Exercise 4. Prove that $\sqrt[2]{2} + \sqrt[2]{3}$ and $\sqrt[3]{5} - \sqrt[2]{3}$ are irrationals.

Exercise 5. Show that, for any $x \in \mathbb{R} \setminus \mathbb{Q}$ and $(a, b, c, d) \in \mathbb{Q}^4$ such as $ad - cb \neq 0$ we have

$$y = \frac{ax+b}{cx+d} \notin \mathbb{Q}.$$

Exercise 6. Let $b, b' \in \mathbb{N}$ be prime numbers and $a, a' \in \mathbb{Q}$ such as $a + \sqrt{b} = a' + \sqrt{b'}$. Show that $a = a'$ and $b = b'$.

Exercise 7. Let $n, p \in \mathbb{N}$ such as $n > p$, and $r \in \mathbb{R} \setminus \{1\}$. Show that

$$\sum_{k=p}^{n} r^k = r^p \frac{1 - r^{n-p+1}}{1-r}$$

Exercise 8. For $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$ we set $\sqrt[n]{x} = x^{1/n}$. Show that

$$\forall x, y \in \mathbb{R}_+, \quad \forall n \in \mathbb{N}^* \setminus \{1\} : \quad \sqrt[n]{x+y} \le \sqrt[n]{x} + \sqrt[n]{y}\,.$$

Exercise 9. Show that

$$\forall n \in \mathbb{N}^* : \qquad \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}\,.$$

Deduce the integer part of

$$y_k = \frac{1}{2} \sum_{j=1}^{10^{2k}} \frac{1}{\sqrt{j}}.$$

Exercise 10. Prove that for $x \in]1, +\infty[$ we have

$$\left(\frac{\text{floor}(x)}{x}\right)^2 \le \frac{\text{floor}(x)}{x}, \quad \left(\frac{\text{floor}(x)}{x-1}\right)^2 > \frac{\text{floor}(x)}{x-1}.$$

Exercise 11. Find the upper-bound, lower-bound, maximum, and minimum of the following sets :

$$\mathcal{A} = \left\{ -1 + \frac{(-1)^n}{n+1}\,, \quad n \in \mathbb{N} \right\}, \qquad \mathcal{B} = \left\{ \frac{2n + (-1)^n}{n+1}\,, \quad n \in \mathbb{N} \right\}.$$

$$\mathcal{C} = \left\{ \frac{1}{3^n} - \frac{(-1)^n}{n}, \quad n \in \mathbb{N}^* \right\}, \qquad \mathcal{D} = \left\{ \frac{1 + (-1)^n}{n^2} - n, \quad n \in \mathbb{N}^* \right\}.$$

Exercise 12. Examine if the following assertions are true or false :

- Every subset of an upper-bounded (respectively lower-bounded) set is upper-bounded (respectively lower-bounded).
- The union of a finite family of upper-bounded (respectively lower-bounded) subsets is upper-bounded (respectively lower-bounded).

Exercise 13. Let A and B be two non-empty upper-bounded subsets of $\mathbb{R}$. We define the following sets :

$$A + B = \{x = a + b : \quad a \in A, b \in B\}, \qquad -A = \{x = -a : \quad a \in A\},$$

Show that

$$\mathrm{Sup}(A + B) = \mathrm{Sup}(A) + \mathrm{Sup}(B), \quad \mathrm{Inf}(-A) = -\mathrm{Sup}(A), \quad \mathrm{Sup}(A \cup B) = \max\{\mathrm{Sup}(A), \mathrm{Sup}(B)\}.$$

Exercise 14. Let $n \in \mathbb{N}^*$, $x_1, \ldots, x_n \in \mathbb{R}$ and $y_1, \ldots, y_n \in \mathbb{R}$, provide the following inequality

$$\Big(\sum_{i=1}^{n} x_i y_i\Big)^2 \leq \Big(\sum_{i=1}^{n} x_i^2\Big)\Big(\sum_{j=1}^{n} y_j^2\Big).$$

Exercise 15. Show that for any $n \in \mathbb{N}$ we have

$$\prod_{j=0}^{n} \Big(1 + \frac{1}{2j+1}\Big) > \sqrt{2n+3}\,, \quad \prod_{j=1}^{n} \Big(x^{2^j} + 1\Big) = \frac{x^{2^{n+1}} - 1}{x - 1}.$$

Exercise 16. Determine the set of upper bounds, the set of lower bounds, the maximum and minimum of

$$\mathcal{A} = \left\{ x \in \mathbb{R}^* : \quad -2 \leq x + \frac{1}{2x} \leq +2 \right\}.$$

Exercise 17. Let $\mathcal{A}$ be the subset of $\mathbb{R}$ defined as

$$\mathcal{A} = \left\{ x \in \mathbb{R} : \quad \frac{\mathrm{floor}(x)^4}{\mathrm{floor}(x)^2 + 2} \geq 1, \quad 0 < \mathrm{floor}(x) < 4 \right\}.$$

Find $\mathrm{Sup}(\mathcal{A})$, $\mathrm{Inf}(\mathcal{A})$ and determine the maximum element and the minimum element of the set $\mathcal{A}$.

Exercise 18. Let $x \in \mathbb{R}_+^*$ and $n \in \mathbb{N}^*$, Prove that

$$n\mathrm{floor}(x) \leq \mathrm{floor}(nx) < n\mathrm{floor}(x) + n.$$

Provide the integer part of $\mathrm{floor}(nx)/n$.