Analysis I, Tutorial 2 / Numerical Sequences

Exercise 1. Let (u_n) and (v_n) be real numerical sequences such that $(u_n)_n$ is increasing and positive, and $(v_n)_n$ is decreasing and negative. Show that $(u_n v_n)_n$ is decreasing.

Exercise 2. Find the limit of the following numerical sequences

$$u_n = 1 + \frac{1}{a^n}, \quad v_n = 5^n - 3^n, \quad w_n = \frac{2^n - 2}{3^n - 1}, \quad z_n = \frac{\cos(n)}{n+1},$$

$$x_n = \frac{n + (-1)^n}{n^2 + 1}, \quad y_n = \left(\sin\left(\frac{1}{n}\right)\right)^n, \quad l_n = \frac{1}{n} + \left(\frac{1}{3}\right)^n, \quad k_n = \frac{2^n}{n!},$$

$$g_n = \sqrt{n+1} - \sqrt{n}, \quad p_n = \frac{\text{floor}(\sqrt{n})}{n}, \quad q_n = \frac{n!}{n^n}.$$

Exercise 3. Let $(u_n)_{n\in\mathbb{N}}$ be the numerical sequence defined as

$$\forall n \in \mathbb{N}: \quad u_n = n \sum_{k=1}^{2n+1} \frac{1}{n^2 + k},$$

Provide the following inequalities :

$$\forall n \in \mathbb{N}: \quad n\frac{2n+1}{(n+1)^2} \leq u_n \leq n\frac{2n+1}{n^2+1}.$$

Show that $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence and find its limit.

Exercise 4. Let $(u_n)_{n\in\mathbb{N}^*}$ be the numerical sequence defined by the expression

$$\forall n \in \mathbb{N}^*: \quad v_n = \frac{1}{n} \sum_{k=0}^{n-1} \cos\left(\frac{1}{\sqrt{n+k}}\right),$$

Establish the following inequalities :

$$\forall n \in \mathbb{N}^*: \quad \cos\left(\frac{1}{\sqrt{n}}\right) \leq u_n \leq \cos\left(\frac{1}{\sqrt{2n-1}}\right).$$

Show that $(v_n)_{n\in\mathbb{N}}$ is a convergent sequence and find its limit.

Exercise 5. By using the definition of the limit of numerical sequence, show that :

$$\lim_{n\to+\infty} \left(\frac{1}{2}\right)^n = 0, \quad \lim_{n\to+\infty} \frac{n+1}{n+2} = 1, \quad \lim_{n\to+\infty} \frac{n+1}{n^2+1} = 0.$$

Exercise 6. Let (u_n) and (v_n) be real numerical sequences such that $(u_n)_n$ converges to l_1 and $(v_n)_n$ converges to l_2. Show that $(u_n v_n)_n$ converges to $l_1 l_2$.

Exercise 7. Let (u_n) be a real numerical sequence converges to $l \in \mathbb{N}$ and g be a strictly increasing map from $\mathbb{N}$ to $\mathbb{N}$. Show that for any $n \in \mathbb{N}$ we have $g(n) \geq n$ and $v_n = u_{g(n)}$ converges to l.

Exercise 8. Let (u_n) be a real numerical sequence converges to l such that $|l| \in]0, 1[$, show that

$$\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty} \sqrt[n]{|u_n|}.$$

Exercise 9. Let $(u_n)_n$, $(v_n)_n$ and $(\mathcal{S}_n)_n$ be the numerical sequences defined as

$$\forall n \in \mathbb{N}: \quad u_{n+1} = \frac{1}{6}u_n + 2, \quad u_0 = 1, \quad v_n = u_n - 3, \quad \mathcal{S}_n = \sum_{k=0}^{n} v_k$$

- Show that $(v_n)_n$ is a geometrical sequence, and provide the terms u_n and v_n as function of n.
- Calculate the limit of u_n (respectively v_n) when n goes to infinity.
- Provide $\mathcal{S}_n$ as function of n and find its limit when n goes to infinity.

Exercise 10. Provide the value of the general term of the sequence $(u_n)_{n\in\mathbb{N}^*}$ in the following cases

- The sequence $(u_n)_{n\in\mathbb{N}^*}$ is defined by

$$\forall n \in \mathbb{N}^*: \quad u_{n+1} = \frac{1}{n}u_n, \quad u_1 \in \mathbb{R}^*.$$

- The sequence $(u_n)_{n\in\mathbb{N}^*}$ is defined as

$$\forall n \in \mathbb{N}^*: \quad u_{n+2} = \frac{n+1}{n^2}u_n, \quad u_1 \in \mathbb{R}^*.$$

Exercise 11. Let $(u_n)_n$ be a numerical sequence defined as

$$u_0 = 1, \quad \forall n \in \mathbb{N}: \quad u_{n+1} = 2u_n + 1 - n.$$

Show that $u_n \geq n$ for every $n \in \mathbb{N}$, and deduce that $(u_n)_n$ diverges.

Exercise 12. Let $(u_n)_n$ and $(v_n)_n$ be two numerical sequences defined as

$$\forall n \in \mathbb{N}: \quad u_{n+1} = \frac{1}{2}\sqrt{u_n^2 + 12}, \quad v_n = u_n^2 - \alpha.$$

- Provide $\alpha \in \mathbb{R}$ so that $(v_n)_n$ be a geometrical sequence.
- Based on the value of α determined at the level of the previous question, calculate the limit of $(u_n)_n$.

Exercise 13. Let $(u_n)_n$ be a numerical sequence defined as

$$u_0 = \frac{3}{2}, \quad \forall n \in \mathbb{N}: \quad u_{n+1} = u_n^2 - 2u_n + 2$$

- Show that for any $n \in \mathbb{N}$ we have $u_n \in [1, 2]$.
- Show that $(u_n)_n$ is a decreasing sequence.
- Show that $(u_n)_n$ is a convergent sequence and calculate its limit.

Exercise 14. Let $(u_n)_n$ be a sequence and f be a function such as

$$\forall x \in I = \left[\frac{2}{\sqrt{3}}, +\infty\right[, \quad f(x) = \frac{1}{2}x + \frac{2}{3x}, \qquad \forall n \in \mathbb{N}: \quad u_{n+1} = f(u_n), \quad u_0 = 2.$$

- Show that $f(x) \in I$ for every $x \in I$.
- Show that $u_n \in I$ for every $n \in \mathbb{N}$.
- Show that $(u_n)_n$ is a decreasing sequence.
- Show that $(u_n)_n$ is a convergent sequence and calculate its limit.

Exercise 15. Let $(u_n)_n$ be a numerical sequence defined as

$$\forall n \in \mathbb{N}: \quad u_{n+1} = \frac{u_n}{1 + u_n}, \quad u_0 = \frac{1}{2}.$$

- Show that $0 < u_n < 1$ for every $n \in \mathbb{N}$.
- Show that $(u_n)_n$ is increasing sequence.
- Show that $(u_n)_n$ converges and calculate its limit.