ANALYSIS I, TUTORIAL 5 / Real-valued Function with a real variable Asymptotic Expansion

Exercice 1. Find the asymptotic expansion of order 4 near zero of

$$\frac{1}{1-x} - e^x, \quad \sin(x)\cos(2x), \quad \ln\left(\frac{\sin(x)}{x}\right), \quad \ln(1+\mathrm{sh}(x)).$$

Exercice 2. Find the asymptotic expansion of order n near x_0 of

- $\sin(2x) + \cos(x^2)$ with $n = 7$ and $x_0 = 0$.
- $e^{3x}\sin(2x)$ with $n = 4$ and $x_0 = 0$.
- $e^{-x}\ln(1+x)/\sin(x)$ with $n = 3$ and $x_0 = 0$.
- $\sqrt{x}$ with $n = 3$ and $x_0 = 2$.
- $\cos(x)$ with $n = 3$ and $x_0 = \pi/6$.
- $x^3 + 4x^2 + x - 1$ for any n and $x_0 = 0$.
- $x^3 + 4x^2 + x - 1$ with $n = 5$ and $x_0 = 1$.

Exercice 3. Using the asymptotic expansion, calculate the following limits

$$\lim_{x\to 0} \frac{\sin(3x)}{3x - \frac{3}{2}\sin(2x)}, \quad \lim_{x\to 0} \frac{\ln(\cos(2x))}{\ln(\cos(3x))}, \quad \lim_{x\to 0} \frac{\mathrm{sh}(x)}{\sin(x)}, \quad \lim_{x\to 0} \frac{1-\cos(x)+\ln(\cos(x))}{x^4},$$

$$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\mathrm{tg}(x)}, \quad \lim_{x\to 0} \frac{e^{x^2}-\cos(x)}{x^2}, \quad \lim_{x\to +\infty} \left(1+\frac{7}{x}\right)^x, \quad \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}.$$

Exercice 4. Show that

$$\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]: \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \leq \cos(x) \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}. \tag{1}$$

Exercice 5.

- Give the asymptotic expansion of order 4 near 0 of $\ln(\cos(x))$.
- Calculate the limit of $(\cos(x))^{1/x}$ when x tends to 0.
- Find the equation of the tangent T of the graph of f at $x_0 = 0$ so that

$$f(x) = \ln(\cos(x)) - \frac{2}{1+x}.$$

- Discuss the position of the tangent T by respect to the graph of f near x_0.

Exercice 6. Let f be the function defined over $\mathbb{R}$ with value in $\mathbb{R}$ by

$$f(x) = \sqrt{1+x+x^2}.$$

- Give the asymptotic expansion of f of order 2 near zero.
- Find the equation of the tangent to the graph of f at $(0, f(0))$.
- Discuss the position of the tangent T with respect to the graph of f near zero.

Exercice 7. Let f be the function defined as

$$f(x) = \begin{cases} x^3 \sin\left(\dfrac{1}{x}\right) & si \quad x \neq 0, \\ \\ 0 & si \quad x = 0. \end{cases}$$

- Show that f has asymptotic expansion of order two near zero.
- Discuss if f is two times differentiable at zero.

Exercice 8. Let $a \in \mathbb{R}$ and f_a be a real-valued function with a real variable defined by

$$f_a(x) = \operatorname{arctg}\left(\frac{x+a}{1-ax}\right).$$

- For $n \in \mathbb{N}$, provide the asymptotic expansion of order $2n-1$ of $f_a^{(1)}$ near zero.
- Find the asymptotic expansion of order $2n$ of f_a near zero, and provide the value of $f_a^{(k)}(0)$.

Exercice 9. Let f be the function defined as

$$f(x) = \frac{2-\sqrt{4+x^2}}{x-2+\sqrt{4+x^2}}.$$

- Provide the asymptotic expansion of order two near zero of f near zero.
- Calculate the limit of f as x goes to zero.
- Examine the position of the graph of f with respect to the tangent at zero.

Exercice 10. Let $f \in \mathcal{C}^2(\mathbb{R}_+^*, \mathbb{R})$ such that f and $f^{(2)}$ are bounded, we set

$$M_1 = \operatorname*{Sup}_{x\in]0,+\infty[} |f(x)|, \quad M_2 = \operatorname*{Sup}_{x\in]0,+\infty[} \left|f^{(2)}(x)\right|.$$

- Let $x, h \in \mathbb{R}_+^*$, show that
$$\left|f^{(1)}(x)\right| \leq \frac{h}{2}M_2 + \frac{2}{h}M_1.$$
- Show that the function Φ defined by
$$\forall h \in \mathbb{R}_+^* : \quad \Phi(h) = \frac{h}{2}M_2 + \frac{2}{h}M_1,$$
has a minimum on $\mathbb{R}_+^*$.
- Show that $f^{(1)}$ is bounded on $\mathbb{R}_+^*$.

Exercice 11. Let f be the function defined as

$$\forall x \in \mathbb{R}^* : \quad f(x) = \frac{1-\cos(x)+\ln(\cos(x))}{x^2}.$$

- Provide the asymptotic expansion of order two near zero of the function f.
- Calculate the limit of f at zero and provide the extension by continuity, if it exists, of f to $\mathbb{R}$.