University of Tlemcen **Academic year 2023-2024**
Faculty of Sciences **(L1 ING-INF)** **Semester 2**
Department of Informatic **Algebra (First Year)**

Worksheet N°3/ "Vector Spaces"

Exercice 01 : (Corrected in the course)Determine whether a subset of a vector space is a subspace?

$$\begin{aligned} (1) A &= \left\{(x,y) \in \mathbb{R}^2 / y = x^3\right\} \ (2)\ B = \left\{(x,y) \in \mathbb{R}^2 /\ x^2 + y^2 \leq 2\right\}. \\ (3) C &= \{(x,y,z)\,; 2x + 3y - 5z = 0\} \\ (4)\ D &= \{f \in E_1; f(0) = f'(0) = \ 0\}.(E_1 = \mathcal{F}(\mathbb{R},\mathbb{R}) \text{ the set of derivable functions}) \\ (5)\ G &= \{P \in \mathbb{R}_n[X] /\ P(1) = P'(1)\}.\mathbb{R}_n[X]: \text{ set polynomials of degree} \leq n. \end{aligned}$$

Exercice 02 :

(1) Are the following vectors linearly independent in E?

$$(1)\, v_1(1,2,-1), v_2(3,-2,2), v_3(-1,4,-1); E = \mathbb{R}^3.$$

$$(2)\, v_1(7,8,-1), v_2(9,-2,12), v_3(-1,4,-1), v_4(-5,5,-1); E = \mathbb{R}^3.$$

$$(3) P_1(X) = 2 - X, P_2(X) = 3 + X + X^2, P_3(X) = X - X^2 + 2X^3; E = \mathbb{R}_3[X].$$

$$\begin{aligned} (4) P_2(X) &= 4, P_2(X) = 2 - X, P_3(X) = 3 + X + X^2, \\ P_4(X) &= X^2 + 2X^3, P_5 = 5X^3; E = \mathbb{R}_3[X]. \end{aligned}$$

(2) Write the vector $v = (7,-2,4)$ as a linear combination of:

$$v_1 = (1,-1,1), v_2 = (1,2,-3) \text{ et } v_3 = (2,-5,4).$$

(3) In $E = \mathbb{R}_2[X]$ the vector space of polynomials of degree less than or equal to 2 and with real coefficients, do the following vectors spans E?

$a)$ $P = X^2 - 5X + 1$ and $Q = X + 7$.
$b)$ $P = X^2 - 6,\ Q = 7X + 1$ and $R = 3X^2 + 8$.
$c)$ $P = 5, Q = 2X - 1$ and $R = X(X - 1)$.

Exercice 03 : Let $\mathbb{R}[X]$ the $\mathbb{R}$-vector space of polynomials with real coefficients and P_3 le the sub-set of $\mathbb{R}[X]$ such as $H_3 = \{P \in \mathbb{R}[X] / \deg P \leq 3\}$.

(1) Show that H_3 is a subspace of $\mathbb{R}[X]$.

(2) Let $Q_0 = 3, Q_1 = 1 - 2X, Q_2 = 3X - X^2$ and $Q_3 = 4X^2 - 5X^3$ be a polynomials.

a) verify that $B_2 = \{Q_0, Q_1, Q_2, Q_3\}$ is a basis of P_3.

b) Determine the coordinates of the polynomial $P = 1 + 6X - 4X^2 - 2X^3$ in B_2.

Exercice 04 : (Corrected in the course) Let:

$$\begin{aligned} E_1 &= \left\{(a,b,c) \in \mathbb{R}^3; a = 5c\right\}, \; E_2 = \left\{(a,b,c) \in \mathbb{R}^3; a + 3b - 7c = 0\right\} \\ \text{and } E_3 &= \{(0,2c,c)\,; c \in \mathbb{R}\}. \end{aligned}$$

(1) Show that E_i with $i = 1,2,3$ are a subspace of $\mathbb{R}^3$.

(2) Show that $\mathbb{R}^3 = E_1 + E_2, \mathbb{R}^3 = E_2 + E_3$ and $\mathbb{R}^3 = E_1 + E_3$.

(3) In which case is the sum is a direct sums?

Exercice 05 : (Corrected in the course) In $\mathbb{R}^3$ let the two subset:

$$E_1 = \left\{(3a - 5b, 2b - 4a, 3a) \in \mathbb{R}^3 /\ a, b \in \mathbb{R}\right\} \text{ and } E_2 = \left\{(4c, c, 6c) \in \mathbb{R}^3 /\ c \in \mathbb{R}\right\}.$$

(1) Show that E_1 and E_2 are a subspace of $\mathbb{R}^3$.

(2) determine a basis B_1 of E_1 and a basis B_2 of E_2.

(3) deduce $\dim E_1$ and $\dim E_2$.

(4) Show that: $\mathbb{R}^3 = E_1 + E_2$.

(5) deduce if the sum is a direct sum or not.

e 06 : (Additional) In $\mathbb{R}_8[X]$ the vector space of polynomials of degree less than or equal to 8 and with real coefficients,. We put: $E_0 = \{P \in \mathbb{R}_8[X] \ / \ P(0) = 0\}$,
$E_p = \{P \in \mathbb{R}_8[X] \ / \ \forall X \in \mathbb{R},\ P(X) = P(-X)\}$and $E_i = \{P \in \mathbb{R}_8[X] \ / \ P(X) = -P(-X)\}$.

(2) Show that: $\mathbb{R}_8[X] = E_0 + E_p$ and $\mathbb{R}_8[X] = E_i + E_p$.

(3) In which case the sum is a direct sum?