Abou Bekr Belkaid University
Faculty of Sciences – Tidjani Haddam
Informatics Department

ANALYSIS I, TUTORIAL 3 / Real-valued Function with a real variable Limit and Continuity

Exercise 1. Determine if the following expressions are functions or not :

- f_1 defined over $[0, \pi/2]$ as $f_1(x)^2 + \cos(x) = 0$.
- f_2 defined over $\mathbb{R}$ as $f_2(x)^3 + e^{2x+1} = 0$.

Exercise 2. Determine the domain of the following functions :

$$f_1(x) = \frac{x+1}{x^4 - x^2 + 1}, \quad f_2(x) = \cos\left(\frac{1}{1-e^{x^2}}\right), \quad f_3(x) = \frac{\sin(x)}{\cos(2x)}, \quad f_4(x) = \sqrt{1 - 2\sin(2x)},$$
$$f_5(x) = \frac{1}{e^{2x} + e^x - 1}, \quad f_6(x) = \sqrt{\cos(x)^2 + \cos(x) + 1}, \quad f_7(x) = \ln(1 - \sin(x)).$$

Exercise 3. Using the definition of the limit, show that :

$$\lim_{x\to 0} \frac{1}{1+x} = 1, \quad \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0, \quad \lim_{x\to 2} \frac{1}{(x-2)^2} = +\infty, \quad \lim_{x\to 0^+} \frac{[x]}{x} = 0.$$

Exercise 4. Determine the set of $x \in \mathbb{R}$ such as $[x] \in \{0, -3, 1/2\}$ and represent the graph of the function f defined over $[-1, 1]$ with value in $\mathbb{R}$ by $f(x) = [2x]$.

Exercise 5. Calculate the following limits :

$$\lim_{x\to 3} \frac{1}{2x-6}, \quad \lim_{x\to 0^+} \left(1 + \frac{1}{\sqrt{x}}\right)(x-3), \quad \lim_{x\to 3} \frac{1-4x}{x-3}, \quad \lim_{x\to 2} \frac{x^3-8}{4-2x},$$
$$\lim_{x\to +\infty} \frac{\sqrt{x} + 2 - 3x}{x+1}, \quad \lim_{x\to -\infty} \frac{2x+5}{\sqrt{1-x}}, \quad \lim_{x\to +\infty} \left[\frac{1}{x}\right] \ln(x), \quad \lim_{x\to +\infty} \frac{[x]}{[2x]},$$
$$\lim_{x\to +\infty} e^{2x} - e^x, \quad \lim_{x\to +\infty} \sqrt{x-1} - \sqrt{x+1}, \quad \lim_{x\to 0} x^2 \sin\left(e^x + \ln(x)\right), \quad \lim_{x\to 1} \frac{x^3-1}{x^2-1},$$
$$\lim_{x\to 0} \sqrt{\frac{1}{x^2} - 1} - \frac{1}{x}, \quad \lim_{x\to +\infty} \frac{xe^x + 2e^x - 5}{e^x - 3}, \quad \lim_{x\to +\infty} \frac{x^2 + x\sin(x)}{x^2 + x\cos(x)}, \quad \lim_{x\to 0} x^2 + \frac{\sqrt{x^2}}{x},$$
$$\lim_{x\to +\infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}, \quad \lim_{x\to -\infty} \sqrt{x^2 + 5x} - \sqrt{x^2 - 1}, \quad \lim_{x\to 0} \frac{\sin(x)}{\sin(3x)}, \quad \lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin(x)}.$$

Exercise 6. Let f be a real-valued function with a real variable defined and increasing over $\mathbb{R}$ satisfies $(f \circ f)(x) = x$ for every $x \in \mathbb{R}$. Show that $f(x) = x$ for any $x \in \mathbb{R}$.

Exercise 7. *[From : Claude Bernard Lyon 1 University, fall 2023-2024] Let f, g and h be the function defined by*

$$f : \begin{array}{ccc} \mathbb{R} \setminus \{1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{x}{x-1} \end{array} \qquad g : \begin{array}{ccc} \mathbb{R}^* & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x} \end{array} \qquad h : \begin{array}{ccc} \mathbb{R} \setminus \{1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x-1} \end{array}$$

we set respectively by G_g, G_g and G_h the graphe of f, g and h.

- *Provide $a, b \in \mathbb{R}$ so that*

$$\forall x \in \mathbb{R} \setminus \{1\} : \quad \frac{x}{x-1} = a + \frac{b}{x-1}.$$

- *Let T_1 the translation by the vector $\overrightarrow{u} = (0,1)$ in the plan (xoy), show that the image of G_g by T_1 is G_h.*
- *Let T_2 the translation by the vector $\overrightarrow{v} = (0,1)$ in the plan (xoy), show that the image of G_h is G_f.*
- *Deduce the transformation that transform G_g into G_f.*

Exercise 8.

- Let f be an even real-valued function with a real variable defined at 0, show that $f(0) = 0$.
- Show that if f is odd and even at the same time, then f is zero everywhere.

Exercise 9. Show that every real-valued function with a real variable can be decomposed as the sum of an odd function and an even one.

Exercise 10. Study the continuity of the following functions :

$$f(x) = \begin{cases} x\left[\dfrac{1}{x}\right] & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0. \end{cases} \quad , \quad g(x) = \begin{cases} \dfrac{x^2-1}{\sqrt{x}-1} & \text{if} \quad x \geq 1, \\ 4\dfrac{\sin(x-1)}{x-1} & \text{if} \quad x < 1. \end{cases}$$

Exercise 11. Let f be the function defined by

$$f(x) = [x] + \sqrt{x - [x]}\,.$$

- Show that for any $x \in \mathbb{R}$ we have $x - [x] \in [0, 1[$.
- Determine $\mathscr{D}_f$ the domain of the function f.
- Study the continuity of the function f at $x_0 \in \mathbb{R}$.
- Show that for any $x \in \mathscr{D}_f$ we have $0 \leq f(x) - x \leq 1$, and provide the limit of f at $+\infty$ and $-\infty$.

Exercise 12. Study the possible continuous extension of the following functions

$$f(x) = \frac{\sin(x)}{x}, \quad g(x) = \frac{x(x-1)}{\sqrt{1-x}-\sqrt{x}}, \quad h(x) = \frac{\sin(x)}{\sqrt{x}-x}$$

$$k(x) = \begin{cases} \dfrac{\ln(x)}{x-1} & \text{if} \quad x > 1, \\ \dfrac{\sin(x-1)}{x-1} & \text{if} \quad x < 1. \end{cases} \quad , \quad l(x) = \begin{cases} \dfrac{\cos(x)-1}{x} & \text{if} \quad x > 0, \\ \dfrac{e^x-1}{x} & \text{if} \quad x < 0. \end{cases}$$

Exercise 13. Let $a > 0$, f and g be functions defined over $v =]x_0 - a, x_0 + a[$ such that f bounded over v and $g(x)$ goes to zero as x tends to x_0. Show that $f(x)g(x)$ has zero as limit at x_0.

Exercise 14. Let f be a real-value function with a real variable continuous over $\mathbb{R}$ satisfies :

$$\operatorname*{Lim}_{x\to-\infty} f(x) = l_1 \in \mathbb{R}, \quad \operatorname*{Lim}_{x\to+\infty} f(x) = l_2 \in \mathbb{R}.$$

Show that f is bounded over $\mathbb{R}$.

Exercise 15. Show that any real-valued polynomial with a real variable and odd degree has a real root.