Course Support - Calculus-M and MI-

Indeterminate forms

$\frac{0}{0}$	$\frac{\infty}{\infty}$	$\infty\times 0$	$+\infty-\infty$	0^0	1^∞	∞^0

Some usual limits

$\lim\limits_{x\to+\infty} x^n = +\infty,$
$\lim\limits_{x\to\pm\infty} \dfrac{1}{x^n} = 0$
$\lim\limits_{x\to-\infty} x^n = \begin{cases} +\infty \text{ if } n \text{ is even} \\ -\infty \text{ if } n \text{ is odd} \end{cases}$
$\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right) = 1$
$\lim\limits_{x\to+\infty} \left(\dfrac{\sin x}{x}\right) = 0$
$\lim\limits_{x\to 0} \left(\dfrac{\tan x}{x}\right) = 1$
$\lim\limits_{x\to 0} \left(\dfrac{\cos x - 1}{x}\right) = 0$
$\lim\limits_{x\to+\infty} \left(\dfrac{e^x}{x^n}\right) = +\infty, (n\in\mathbb{N})$
$\lim\limits_{x\to 0} \left(\dfrac{e^x - 1}{x}\right) = 1,$
$\lim\limits_{x\to-\infty} x^n e^x = 0^-, (n\in\mathbb{N})$
$\lim\limits_{x\to+\infty} x^n e^{-x} = 0^+, (n\in\mathbb{N})$
$\lim\limits_{x\to 0^+} (x\ln x) = 0^-$
$\lim\limits_{x\to+\infty} (x\ln x) = +\infty,$
$\lim\limits_{x\to+\infty} \left(\dfrac{\ln x}{x^n}\right) = 0^+, (n\in\mathbb{N})$
$\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$

Derivatives of usual functions:

Derivatives of trigonometric functions	
Particular case	Generalization
$\sin' x = \cos x$	$\sin'[f(x)] = f'(x)\cos[f(x)]$
$\cos' x = -\sin x$	$\cos'[f(x)] = -f'(x)\sin[f(x)]$
$\tan' x = \dfrac{1}{\cos^2 x}$	$\tan'[f(x)] = f'(x)\,\dfrac{1}{\cos^2[f(x)]}$
$\cot g' x = \dfrac{-1}{\sin^2 x}$	$\cot g'[f(x)] = -f'(x)\,\dfrac{1}{\sin^2[f(x)]}$

Derivative of the inverse of a function
$[f^{-1}(x)]' = \dfrac{1}{(f'\circ f^{-1})(x)}$

Derivatives of inverse trigonometric functions	
Particular case	Generalization
$\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$	$\arcsin'[f(x)] = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$\arccos'(x) = \dfrac{-1}{\sqrt{1-x^2}}$	$\arccos'[f(x)] = \dfrac{-f'(x)}{\sqrt{1-[f(x)]^2}}$
$\arctan'(x) = \dfrac{1}{1+x^2}$	$\arctan'[f(x)] = \dfrac{f'(x)}{1+[f(x)]^2}$
$\operatorname{arccot} g'(x) = \dfrac{-1}{1+x^2}$	$\operatorname{arccot} g'[f(x)] = \dfrac{-f'(x)}{1+[f(x)]^2}$

Derivatives of hyperbolic functions	
Particular case	Generalization
$sh'(x) = ch(x)$	$sh'[f(x)] = f'(x)\,ch[f(x)]$
$ch'(x) = sh(x)$	$ch'[f(x)] = f'(x)\,sh[f(x)]$
$th'(x) = 1 - th^2(x)$ $= \dfrac{1}{ch^2(x)}$	$th'[f(x)] = f'(x)\left[1 - th^2(f(x))\right]$ $= \dfrac{f'(x)}{ch^2(f(x))}$
$coth'(x) = 1 - coth^2(x)$	$coth'[f(x)] = f'(1 - coth^2 f(x))$
$= \dfrac{-1}{sh^2(x)}$	$= \dfrac{-f'(x)}{sh^2(f(x))}$

Derivatives of inverse hyperbolic functions	
Particular case	Generalization
$argch'(x) = \dfrac{1}{\sqrt{x^2-1}};$	$argch'[f(x)] = \dfrac{f'(x)}{\sqrt{[f(x)]^2-1}}$
$\arg sh'(x) = \dfrac{1}{\sqrt{x^2+1}};$	$\arg sh'[f(x)] = \dfrac{f'(x)}{\sqrt{[f(x)]^2+1}}$
$argth'(x) = \dfrac{1}{1-x^2};$	$argth'[f(x)] = \dfrac{f'(x)}{1-[f(x)]^2}$
$argcoth'(x) = \dfrac{1}{1-x^2};$	$argcoth'[f(x)] = \dfrac{f'(x)}{1-[f(x)]^2}$

Derivatives of power functions	
Particular case	Generalization
$(x^n)' = nx^{n-1}$ Ex: $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	$[f^n(x)]' = nf'(x)[f^{n-1}(x)]$ $\left(\sqrt{f(x)}\right)' = \frac{f'(x)}{2\sqrt{f(x)}}$

For $(a>0, \neq 1)$:

Derivatives of logarithmic functions	
Particular case	Generalization
$\ln' x = \dfrac{1}{x}$	$\ln'[f(x)] = \dfrac{f'(x)}{f(x)}$
$\log'_a x = \dfrac{1}{\ln a}.\left(\dfrac{1}{x}\right)$	$\log'_a[f(x)] = \frac{1}{\ln a}.\left(\dfrac{f'(x)}{f(x)}\right)$

For $(a>0, \neq 1)$

Derivatives of exponential functions	
Particular case	Generalization
$(e^x)' = e^x$	$\left(e^{[f(x)]}\right)' = [f'(x)].e^{[f(x)]}$
$(a^x)' = (\ln a).a^x$	$\left(a^{[f(x)]}\right)' = (\ln a).[f'(x)].a^{[f(x)]}$

Properties of natural logarithmic and exponential functions:

If $x\in]0,1[$ *alors* $\ln x < 0$
If $x\in]1,+\infty[$ *alors* $\ln x > 0$
$\forall x>0, Log_a(x) = \dfrac{\ln x}{\ln a} (a>0,\neq 1).$
$\forall x\in\mathbb{R}, a^x = e^{x\ln a}$ $(a>0,\neq 1).$

$\forall x, y > 0$:

$e^0 = 1, \quad e^1 = e = 2.718...$	$\ln 1 = 0, \quad \ln e = 1$
$\ln(x.y) = \ln x + \ln y$	
$\ln\dfrac{x}{y} = \ln x - \ln y$	Ex: $\ln\left(\dfrac{1}{x}\right) = -\ln x,$
$\ln(x)^y = y\ln x$	Ex: $\left(\ln\sqrt{x} = \frac{1}{2}\ln x\right)$

$\forall x, y \in \mathbb{R}$

	$\forall x \in D_u, e^{u(x)} > 0.$
$e^{(\ln x)} = x,\ (x > 0)$ and $\ln(e^y) = y$	
$e^x.e^y = e^{x+y}$	
$\dfrac{e^x}{e^y} = e^{x-y}$	Ex: $\left(\dfrac{1}{e^y} = e^{-y}\right)$,
$(e^x)^y = e^{xy}$	Ex:$\left(\sqrt{e^x} = e^{\frac{x}{2}}\right)$

Relationships between hyperbolic functions:

$\forall x \in \mathbb{R}, ch^2(x) - sh^2(x) = 1$	$\forall x \in \mathbb{R}, th(x) = \dfrac{e^{2x}-1}{e^{2x}+1}$
$\forall x \in \mathbb{R}, ch(x) > sh(x)$	$\forall x \in \mathbb{R}, ch(s) + sh(x) = e^x$
	$\forall x \in \mathbb{R}, ch(s) - sh(x) = e^{-x}$

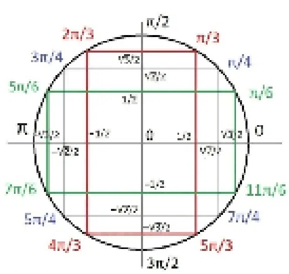
$ch(a+b) = ch\,(a)\,ch(b) + sh(a)sh(b)$
$sh(a+b) = sh\,(a)\,ch(b) + ch(a)sh(b)$
$th(a+b) = \dfrac{th\,(a) + th(b)}{1 + th(a)th(b)}$

$ch(2x) = ch^2(x) + sh^2(x) = 1 + 2sh^2(x) = 2ch^2(x) - 1$
$sh(2x) = 2sh(x)ch(x)$
$th(2x) = \dfrac{2th(x)}{1 + th^2(x)}$

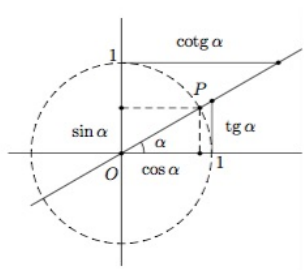
Relationships between hyperbolic functions and inverse hyperbolic functions:

$\forall x \in \mathbb{R}\backslash\,[-1, 1]:$

$sh(\arg\cot h(x) = \dfrac{\lvert x\rvert}{x\sqrt{x^2-1}}$	$ch(\arg\cot h(x) = \dfrac{\lvert x\rvert}{\sqrt{x^2-1}}$
$coth(\arg\cot h(x) = x$	$th(\arg\cot h(x) = \dfrac{1}{x}$



(a) The trigonometric circle.



(b) The axes of the main trigonometric functions

Relationships between trigonometric functions:

$\forall x \in \mathbb{R}, \sin^2 x + \cos^2 x = 1$	$\pi = 3,1415...$
$\tan x = \dfrac{\sin x}{\cos x}$	$\dfrac{1}{\cos^2 x} = 1 + \tan^2 x$
$\cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$	$\dfrac{1}{\sin^2 x} = 1 + \cot^2 x$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha.\tan\beta}$
$\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha.\tan\beta}$

$\cos\alpha + \cos\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$
$\cos\alpha - \cos\beta = -2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$
$\sin\alpha + \sin\beta = 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$
$\sin\alpha - \sin\beta = 2\sin\left(\dfrac{\alpha-\beta}{2}\right)\cos\left(\dfrac{\alpha+\beta}{2}\right)$

$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$
$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$
$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$
$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$

$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$
$\sin\alpha\sin\beta = \frac{-1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right]$
$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$\sin^2 x = \dfrac{1-\cos(2x)}{2}$	$\sin(2x) = 2\sin x\cos x$
$\cos^2 x = \dfrac{1+\cos(2x)}{2}$	$\cos(2x) = \cos^2 x - \sin^2 x$
	$= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$
$\tan^2 x = \dfrac{1-\cos(2x)}{1+\cos(2x)}$	$\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$

$\sin(3x) = 3\sin x - 4\sin^3 x$
$\cos(3x) = 4\cos^3 x - 3\cos x$
$\tan(3x) = \dfrac{3\tan x - \tan^3 x}{3\tan^2 x}$

$\sin(-x) = -\sin x$	$\sin(x+2k\pi) = \sin x$
$\cos(-x) = \cos x$	$\cos(x+2k\pi) = \cos x$
$\tan(-x) = -\tan x$	$\tan(x+k\pi) = \tan x$

$\sin(\frac{\pi}{2} - x) = \cos x$	$\sin(\frac{\pi}{2} + x) = \cos x$
$\cos(\frac{\pi}{2} - x) = \sin x$	$\cos(\frac{\pi}{2} + x) = -\sin x$
$\tan(\frac{\pi}{2} - x) = \cot x$	$\tan(\frac{\pi}{2} + x) = -\cot x$

$\sin(\pi - x) = \sin x$	$\sin(\pi + x) = -\sin x$
$\cos(\pi - x) = -\cos x$	$\cos(\pi + x) = -\cos x$
$\tan(\pi - x) = -\tan x$	$\tan(\pi + x) = \tan x$

$\cos x = 0 \Leftrightarrow x = \frac{\pi}{2} + k\pi$
$\sin x = 0 \Leftrightarrow x = k\pi \qquad (k \in \mathbb{Z})$
$\tan x = 0 \Leftrightarrow x = k\pi$

$\forall x \in \mathbb{R}, 2\sin(3x)\cos x = \sin(4x) + \sin(2x)$
$\forall x \in D_{\tan}, \sin x \leq x \leq \tan x$

Logarithmic functions of basis a :

Let be a a constant such that : $a > 0, a \neq 1$,

$$f: \quad]0,+\infty[\to \mathbb{R}$$
$$x \mapsto f(x) = \log_a(x)$$

Properties

1. $\log_a(x)$ is only defined if $x > 0$.
2. $\forall a \in I, \quad f(1) = 0$.
3. f is continuous on $]0,+\infty[$.
4. if $a > 1$ then f is strictly increasing and:

$\lim\limits_{x\to+\infty} f(x) = +\infty, \qquad \lim\limits_{\substack{x\to 0\\ >}} f(x) = -\infty.$

On the other hand, if $0 < a < 1$ then f is strictly decreasing and :

$\lim\limits_{x\to+\infty} f(x) = -\infty, \qquad \lim\limits_{\substack{x\to 0\\ >}} f(x) = +\infty.$

The Neperian logarithmic function

This is the case when : $a = e = 2.718...$,

$$f: \quad]0,+\infty[\to \mathbb{R}$$
$$x \mapsto f(x) = \ln x$$

$\forall x \in]0,+\infty[, \log_a(x) = \dfrac{\ln x}{\ln a}$

ln **is a bijection** from $]0,+\infty[$ towards $\mathbb{R}$.

It then admits a reciprocal function, which is the Neperian exponential function.

Exponential functions of basis a

$$f: \quad \mathbb{R} \to]0,+\infty[$$
$$x \mapsto f(x) = a^x$$

It is the only function defined over $\mathbb{R}$ that verifies $f(0) = 1$ and is equal to its derivative.

Properties

1. $\forall x \in \mathbb{R}, a^x > 0$.
2. $\forall a \in I, f(0) = 1, f(1) = a$.
3. f is continuous over $\mathbb{R}$.
4. If $a > 1$ then f is strictly increasing and:

$\lim\limits_{x\to+\infty} f(x) = +\infty, \qquad \lim\limits_{x\to-\infty} f(x) = 0.$

On the other hand, if $a < 1$ then f is strictly decreasing and:

$\lim\limits_{x\to+\infty} f(x) = 0, \qquad \lim\limits_{x\to-\infty} f(x) = +\infty.$

Natural exponential function

This is the case where: $a = e = 2.718...$

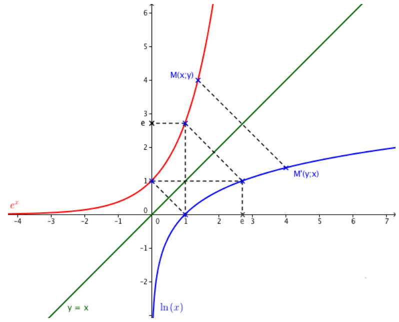
$$f: \quad \mathbb{R} \to]0,+\infty[$$
$$x \mapsto f(x) = e^x$$

$\forall x \in \mathbb{R}, a^x = e^{x \ln a}$.

e^x **is a bijection** from $\mathbb{R}$ to $]0,+\infty[$.

It admits a reciprocal function. This is the Natural logarithm.

Graphs of e^x and $\ln x$ on the same coordinate system:



Trigonometric Functions

The cosine function

a) Definition
$$\cos: \quad \mathbb{R} \to [-1,1]$$
$$x \mapsto f(x) = \cos x$$

b) Properties cos is:

1. $2\pi - \textit{periodic}$ i.e. $\forall x \in \mathbb{R}, \cos(x + 2k\pi) = \cos x.\ (k \in \mathbb{Z})$.
2. Continuous on $\mathbb{R}$ and even i.e. $\forall x \in \mathbb{R}, \cos(-x) = \cos x$.
3. Strictly decreasing on the domain $[0, \pi]$.

4.$\forall x \in \mathbb{R}, \cos'(x) = -\sin x$.

The sinus function

a) Definition
$$\sin: \quad \mathbb{R} \to [-1,1]$$
$$x \mapsto f(x) = \sin x$$

b) Properties sin is :

1. $2\pi - \textit{periodic}$ i.e. $\forall x \in \mathbb{R}, \sin(x + 2k\pi) = \sin x.\ (k \in \mathbb{Z})$.
2. Continuous on $\mathbb{R}$ and odd i.e. $\forall x \in \mathbb{R}, \sin(-x) = -\sin x$.
3. Strictly increasing on the domain $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

4.$\forall x \in \mathbb{R}, \sin'(x) = \cos x$.

The tangent function

a) Definition
$$\tan: \quad D_{\tan} \to \mathbb{R}$$
$$x \mapsto f(x) = \tan x = \frac{\sin x}{\cos x}$$

tan is defined only on the points where the cos does not vanish:

$\cos x = 0 \Leftrightarrow x = \left(\frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}$. So,

$D_{\tan} = \mathbb{R} \setminus \left\{x \in \mathbb{R} / x = \left(\frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}\right\}$.

b) Properties tan is :

1. $\pi - \textit{periodic}$ i.e. $\forall x \in \mathbb{R}, \tan(x + k\pi) = \tan x. \quad (k \in \mathbb{Z})$.
2. Continuous and odd on $D_{\tan}$ i.e.

$\forall x \in D_{\tan}, \tan(-x) = -\tan x$.

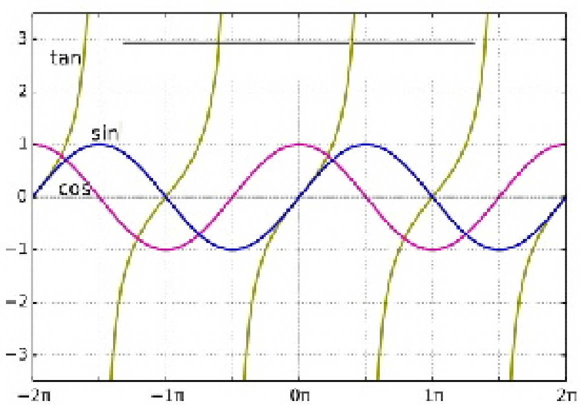
3. Strictly increasing on the domain $\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$.

4.$\forall x \in \mathbb{R}, \tan' x = \dfrac{1}{\cos^2 x} = 1 + \tan^2(x)$.

c) **Graphs**

Like all odd functions, the graphs of *sin* and *tan* are symmetrical with respect to the origin.

Like all even functions, the *cos* graph is symmetrical with respect to the ordinate axis.



Graphs of sin, cos and tan on the same coordinate system.

Inverse trigonometric functions

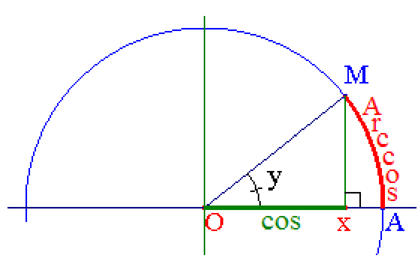
The arccosinus function

$$f: \begin{array}{l} [0,\pi] \to [-1,1] \\ x \mapsto f(x) = \cos x \end{array}$$

On $[0,\pi]$, cos is continuous and strictly decreasing but it is not even because its definition interval is not symmetrical with respect to 0. In addition, $f([0,\pi]) = [-1,1]$. It is therefore bijective and admits a reciprocal function, which we denote by arccos:

a) Definition $\arccos(x)$ is the unique arc between 0 and π whose cosine is x.

$$f^{-1}: \begin{array}{l} [-1,1] \to [0,\pi] \\ x \mapsto f^{-1}(x) = \arccos x \end{array}$$



The arc of arccos.

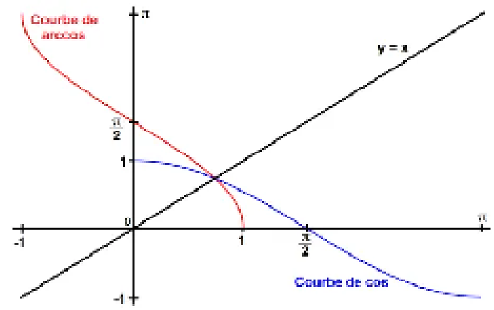
b) Properties *arccos* verifies:

1. $\forall x \in [-1,1], \cos(\arccos x) = x$.
2. $\forall x \in [0,\pi], \arccos(\cos x) = x$.
3. $\forall x \in [-1,1], y = \arccos x \Leftrightarrow \begin{cases} x = \cos y \\ y \in [0,\pi] \end{cases}$.
4. It is continuous, strictly decreasing, bijective but it is not even.
5. $\forall x \in]-1,1[, \arccos'(x) = \dfrac{-1}{\sqrt{1-x^2}}$

Examples 1) $\cos(\arccos \frac{1}{2}) = \frac{1}{2}$, 2) $\arccos(\cos \frac{\pi}{2}) = \frac{\pi}{2}$.
3) $\arccos(\cos \frac{5\pi}{4}) = \arccos(\cos \frac{3\pi}{4}) = \frac{3\pi}{4}$.

c) Graph

The graphs of a function and its reciprocal function are symmetrical with respect to the first bisector $(\Delta): y = x$.



Graphs of *cos* and of *arccos*.

The arcsinus function

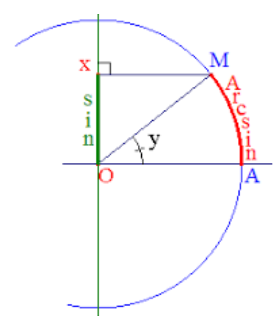
We consider the function

$$f: \begin{array}{l} \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1,1] \\ x \mapsto f(x) = \sin x \end{array}$$

On $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, it is continuous and strictly increasing. In addition, $f\left(\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right) = [-1,1]$. It is therefore a bijection and admits a reciprocal function, which we denote by arcsin:

a) Definition $arcsin(x)$ is the only arc between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.

$$f^{-1}: \begin{array}{l} [-1,1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ x \mapsto f^{-1}(x) = \arcsin x \end{array}$$



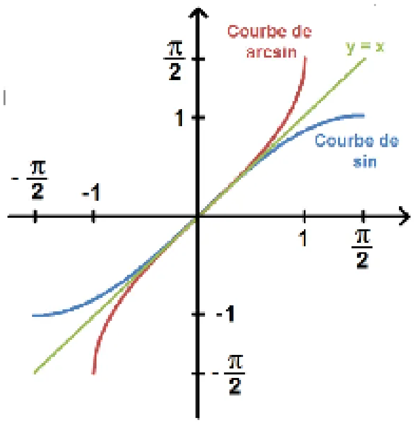
The arc of *arcsin*.

b) Properties *arcsin* verifies:

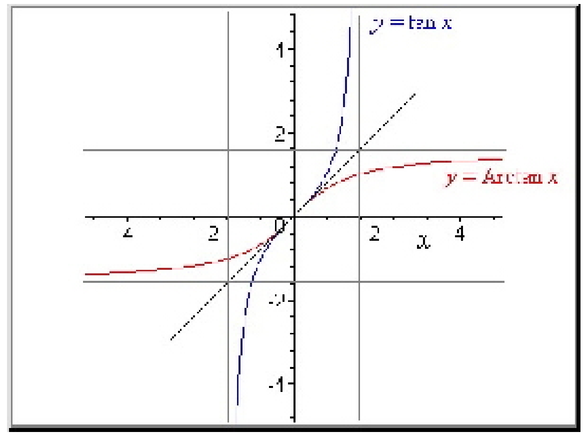
1. $\forall x \in [-1,1], \sin(\arcsin x) = x$.
2. $\forall x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \arcsin(\sin x) = x$.
3. $\forall x \in [-1,1], y = \arcsin x \Leftrightarrow \begin{cases} x = \sin y \\ y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \end{cases}$.
4. It is continuous, strictly increasing, bijective and odd on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.
5. $\forall x \in]-1,1[, \arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Examples 1) $\sin(\arcsin \frac{1}{3}) = \frac{1}{3}$, 2) $\arcsin(\sin 0) = 0$,
3) $\arcsin(\sin \frac{3\pi}{4}) = \arcsin(\sin \frac{\pi}{4}) = \frac{\pi}{4}$

c) Graph



Graphs of *sin* and of *arcsin*.



Graphs of *tan* and of *arctan*.

The arctangent function

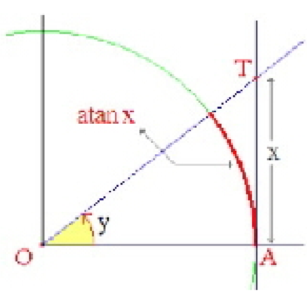
Consider the function:

$$f: \quad \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$$
$$x \mapsto f(x) = \tan x$$

On $\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$, it is continuous and strictly increasing. In addition $f\left(\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[\right) = \mathbb{R}$. It is therefore bijective and admits a reciprocal function, which we denote by arctan:

a) Definition $\arctan(x)$ is the unique arc between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.

$$f^{-1}: \quad \mathbb{R} \to \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$$
$$x \mapsto f^{-1}(x) = \arctan x$$



The arc of *arctan*.

b) Properties arctan verifies:

1. $\forall x \in \mathbb{R}, \tan(\arctan x) = x$.
2. $\forall x \in \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[, \arctan(\tan x) = x$.
3. $\forall x \in \mathbb{R}, y = \arctan x \Leftrightarrow \begin{cases} x = \tan y \\ y \in \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[\end{cases}$.
4. It is continuous, strictly increasing, bijective and odd on $\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$.
5. $\forall x \in \mathbb{R}, \arctan'(x) = \dfrac{1}{\sqrt{1+x^2}}$

Examples 1) $\tan(\arctan 5) = 5$, 2) $\arctan(\tan 0) = 0$
3) $\arctan(\tan \frac{5\pi}{4}) = \arctan(\tan \frac{\pi}{4}) = \frac{\pi}{4}$

c) Graph

Hyperbolic functions

Hyperbolic cosine function

$$f: \quad \mathbb{R} \to [1, +\infty[$$
$$x \mapsto f(x) = ch\ (x) = \frac{e^x + e^{-x}}{2}$$

Properties ch is :
1. even i.e. $\forall x \in \mathbb{R}, ch(-x) = ch\,(x)$.
2. Continuous on $\mathbb{R}$, $ch(0) = 1$, and $\forall x \in \mathbb{R}, ch(x) \geq 1$.
3. $\lim\limits_{x\to+\infty} ch(x) = \lim\limits_{x\to-\infty} ch(x) = +\infty$,
$\lim\limits_{x\to+\infty} \dfrac{ch(x)}{x} = +\infty$, $\lim\limits_{x\to-\infty} \dfrac{ch(x)}{x} = -\infty$.
4. ch is strictly increasing on $[0, +\infty[$.
5. ch constitute a bijection from $[0, +\infty[$ to $[1, +\infty[$.

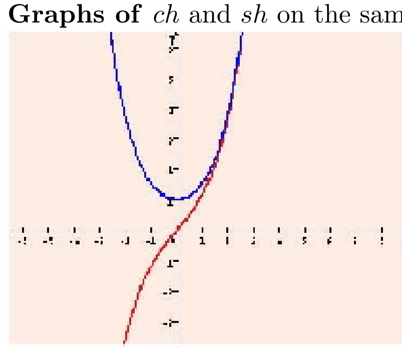
6. $\forall x \in \mathbb{R}, ch'(x) = sh\,(x)$

Hyperbolic sine function

$$f: \quad \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f(x) = sh\ (x) = \frac{e^x - e^{-x}}{2}$$

Properties sh is :
1. Odd i.e. $\forall x \in \mathbb{R}, sh(-x) = -sh\,(x)$.
2. Continuous over $\mathbb{R}, sh(0) = 0$ and
$\forall x \in]0, +\infty[\,, sh(x) > 0, \forall x \in]-\infty, 0[\,, sh(x) < 0$.
3. $\lim\limits_{x\to+\infty} sh(x) = +\infty$, $\lim\limits_{x\to-\infty} sh(x) = -\infty$,
$\lim\limits_{x\to+\infty} \dfrac{sh(x)}{x} = \lim\limits_{x\to-\infty} \dfrac{sh(x)}{x} = +\infty$.
4. Strictly increasing on $\mathbb{R}$.
5. sh constitute a bijection from $\mathbb{R}$ to $\mathbb{R}$.
6. $\forall x \in \mathbb{R}, sh'(x) = ch\,(x)$

Graphs of ch and sh on the same coordinate system :



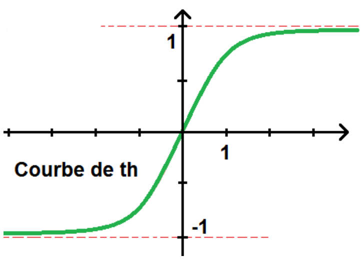
Hyperbolic tangent function

$$f: \quad \mathbb{R} \to]-1,1[$$
$$x \mapsto f(x) = th\ (x) = \frac{sh(x)}{ch(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Properties: th is **:**

1. Odd i.e. $\forall x \in \mathbb{R}, th(-x) = -th\,(x)$.
2. Continuous on $\mathbb{R}, th(0) = 0$.
3. $\lim\limits_{x \to +\infty} th(x) = 1, \ \lim\limits_{x \to -\infty} th(x) = -1$.
4. Strictly increasing on $\mathbb{R}$.
5. th constitute a bijection from $\mathbb{R}$ to $]-1,1[$.
6. $\forall x \in \mathbb{R}, th'(x) = \dfrac{1}{ch^2(x)} = 1 - th^2(x)$.

Graph of th.



Hyperbolic inverses functions

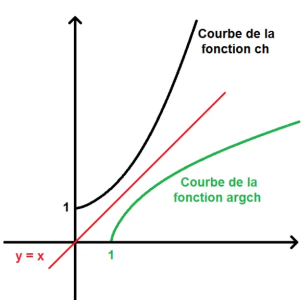
Hyperbolic cosine argument function

$$f: \quad [1,+\infty[\to [0,+\infty[$$
$$x \mapsto f(x) = \arg ch(x)$$

Properties $argch$ is :

1. Continuous over $\mathbb{R}$.
2. $\forall x \in [1,+\infty[, \arg ch(x) = \ln\left(x + \sqrt{x^2-1}]\right)$
3. Strictly increasing on $[0,+\infty[$.
4. $argch$ constitute a bijection from $[1,+\infty[$ to $[0,+\infty[$.
5. $\forall x \in]1,+\infty[, argch'(x) = \dfrac{1}{\sqrt{x^2-1}}$.

Graphs of ch and of $argch$



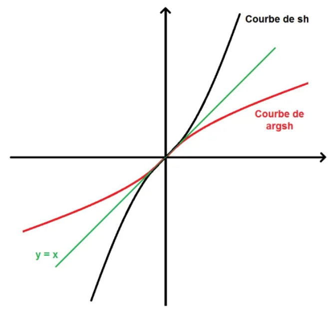
Hyperbolic sine argument function

$$f: \quad \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f(x) = \arg sh\ (x)$$

Properties $argsh$ is :

1. Odd i.e. $\forall x \in \mathbb{R}, \arg sh(-x) = -\arg sh\,(x)$.
2. Continuous on $\mathbb{R}$.
3. $\forall x \in \mathbb{R}, \arg sh(x) = \ln\left(x + \sqrt{x^2+1}\right)$.
4. Strictly increasing on $\mathbb{R}$.
5. $argsh$ constitute a bijection from $\mathbb{R}$ to $\mathbb{R}$.
6. $\forall x \in \mathbb{R}, \arg sh'(x) = \dfrac{1}{\sqrt{x^2+1}}$.

Graph of sh and $argsh$.



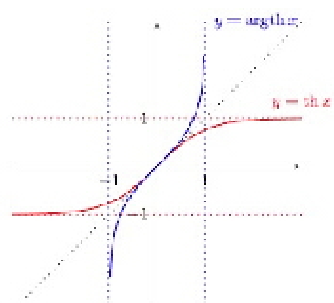
Hyperbolic tangent argument function

$$f^{-1}: \quad]-1,1[\to \mathbb{R}$$
$$x \mapsto f(x) = \arg th\ (x)$$

Properties $argth$ is:

1. Continuous and Odd on $]-1,1[, i.e. \forall x \in]-1,1[, \arg th(-x) = -\arg th\,(x)$.
2. $\forall x \in]-1,1[, \arg th(x) = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.
3. Strictly increasing on $]-1,1[$.
4. $\forall x \in]-1,1[, argth'(x) = \dfrac{1}{1-x^2}$

Graph of th and $argth$



Limited Expansion of some usual functions near 0

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + o(x^n)$$

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{(-1)^{n-1}x^n}{n} + o(x^n)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + ... + \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)}{n!}x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + o(x^n)$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + ... + (-1)^n x^{2n} + o(x^{2n})$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + ... + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)...(\frac{1}{2}-n+1)}{n!}x^n + o(x^n)$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 + \frac{x^3}{16} + ... + \frac{\frac{-1}{2}(\frac{-1}{2}-1)(\frac{-1}{2}-2)...(\frac{-1}{2}-n+1)}{n!}x^n + o(x^n)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + ... + \frac{(2n)!}{2^{2n}\,(n!)^2}x^{2n} + o(x^{2n})$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n})$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5)$$

$$ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^{2n}}{(2n)!} + o(x^{2n})$$

$$sh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} ... + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$$

$$th(x) = x - \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5)$$