Table des Matières

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Hello students, I am MESSIRDI in charge of the algebra module, before starting I have just given you some instructions. Knowing that before in your high school career you never studied algebra and the big problem that arises in a general way in mathematics is the understanding of notions and not preservation. Before most teachers tell the students if you have this question you answer like that and even the student if he asks the question how? the teacher says take the technique without going further. that's the problem. So for the algebra subject we will give you the tools to better understand especially the reasoning aspect. Let's start by giving you an idea of the program for the first semester.

Preface

This book was the result of teaching experience in the field of algebra for years in the common trunks of various technical and mathematical disciplines (SETI – MI – ST- SM- 1st year preparatory school), through which I identified the weaknesses and difficulties encountered by the student, and tried to simplify and overcome these difficulties by giving a simplified explanation of the lesson followed by targeted exercises with a sequence of ideas and a global concept.

I saw that students have gaps to learn mathematics especially in the reasoning aspect that is to say understanding things but how do I start writing the answer? Hence the great problem for the student is the writing of ideas.

For this I tried to guide the student by a style quite simple to understand whose object is to give a complete and formal explanation. Each chapter is structured around a mini course that simplifies the information followed by examples of applications and a variety of exercises that serve to illustrate the theory and to simplify the concepts and make them free of complexity and difficulty.

I have divided the book into seven chapters, each of which consists of a complete summary of the lesson followed by various exercises that include most of the questions used to illustrate the theory. In the first chapter, I gave a comprehensive overview on the modes of reasoning, the notions of logic and the theory of sets, in the second chapter I gave a summary on relationships, in the third I touched on applications and everything related to them, for the fourth I recalled and added notions about complex numbers, the fifth is a formal chapter that gives a general idea about algebraic structures that are useful especially in the chapter of vector spaces, in the penultimate chapter I have touched on most of the questions on arithmetic in $\mathbb{Z}$, the last chapter and in the last chapter I have presented polynomials and rational fractions which are very useful especially in the courses of integration methods.

In the end, I hope I have managed to present the ideas in a simplified form, as I can only thank all those who contributed directly or indirectly to this work, including my colleagues at Tlemcen University.

Chapitre 1

Elementary logic-Some types of reasoning

1.1 Elementary logic

You know from experience that a mathematics course consists of a series of statements, called definitions or propositions. Definitions are laid down a priori and propositions must be demonstrated with the help of definitions or other already established propositions. In any problem we find hypotheses and questions and to answer these questions we use the given hypotheses or well known theorems. It is this approach, which consists in moving logically through the various stages of mathematical reasoning. However, we thought it useful to identify some rules of universal logic.

1.1.1 Concept of proposition

Définition 1 *A proposition is an assertion (a statement) that can be made unambiguously (undoubtedly) if it is true or false. for example* $2 > 1$ *is a true proposition;*$2-5$ *is a natural integer, is a false proposition; but* $A \subset B$ *is not a proposition if we do not have data on the two sets* A *and* B*.*

The propositions are noted in the letters: $P, Q, R, ...$or by indexed letters: $P_1, P_2, P_3, ...$ Subsequently we associate to a true proposition the letter " V " or the number " 1 ", and a false proposition by " F " or " 0 ".

1.1.2 Negation of a proposition

If P is a proposition, we note the negation of P by (not P) or $\overline{P}$, which is true if P is false and the contrary.

1.1.3 Logical connectors

To two propositions P and Q, we can associate a third, which is defined by a logical connector or connectors between these two propositions.

Conjunction

Définition 2 *We call conjunction of two propositions P and Q, the proposition noted $P \wedge Q$ (that reads P and Q), knowing that it is true, if P and Q are true, and false in other cases. Two propositions are said to be incompatible, if their conjunction is false.*

Exemple 3 *If two pieces of information are given to an individual, then the total information is only true if both pieces of information are true.*

disjunction

Définition 4 *A disjunction of two propositions P and Q is a proposition rated $P \vee Q$ (that reads P or Q), knowing that it is true if either is true.*

Exemple 5 *If a teacher has given us two questions, but he will take the best score of both, then the student has the full score if he answers just one of the two questions.*

Remarque 6 *Saying $P \vee Q$ does not mean that the two propositions P and Q are mutually exclusive, in other words, the "or" here allows us to give any of the following three cases:*
i) We have P but not Q.
ii) We have Q but not P.
iii) We have P and we have Q.
Here, for example, we find that a parallelogram is a quadrilateral whose sides are parallel and equal two to two or if their diagonals are median (its diagonals cut in the middle).

Implication

Définition 7 *The implication of two propositions P and Q, is the proposition noted $P \Rightarrow Q$ (which reads P implies Q), which is false in the only case where P is true and Q is false. In the implication $P \Rightarrow Q$, P plays the role of hypotheses and Q plays the role of conclusion.*

Exemple 8 *In an exam the teacher gives an exercise consisting of hypotheses and questions. In nature he makes true assumptions and he wants true answers so that the grade will be complete, then the student will have the full grade in three cases:*
1- In the normal state, that is, the hypotheses are true and the answers also.
2- The teacher has made a mistake in the assumptions and the student fails to find the solution (statements of an exercise are false).
3- The teacher made a mistake in the assumptions and the student found the solution because he is brilliant or he used something other than the assumptions.
It is in the latter case where the implication is false.
4- In this case the student did not receive the grade because: The assumptions are true and the answers are false.

Remarque 9 *Implication is used if the following expression is used in a question:*

Prove that if you have P then you have Q.

That is, mathematically we write:

$$\underbrace{P}_{Hypotheses} \Rightarrow \underbrace{Q}_{Problems}.$$

Equivalence

Définition 10 *Two propositions are said to be equivalent, which is $P \iff Q$ (reads P is equivalent to Q) if the two propositions have the same truth value, that is, the equivalence is true only if both are simultaneously true, or both are simultaneously false. We must always see* *equivalence as two senses of implication, that is, $P \Longleftrightarrow Q$ is exactly:*

$$P \Rightarrow Q \text{ and } Q \Rightarrow P.$$

Exemple 11

$$\underbrace{(1=2)}_{False} \Leftrightarrow \underbrace{(3=4)}_{False} \text{ is a true proposition because both are false.}$$

Remarque 12 *Equivalency is used if the expression is found in a question:*
*Prove that you have P **if and only if** you have Q. This means:*

$$P \Leftrightarrow Q.$$

1.1.4 Overview of connectors and their truth

These formulae are given in the following table (The truth table):

P	Q	$\bar{P}$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Longleftrightarrow Q$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

To show that a proposal is true or false we can use the truth table. If the total proposition is combinations between two propositions or their negations then in the combinations of truths between these two propositions we have **4 lines**, but if we have three then we have **8 lines**. In the general case, the number of rows is 2^n, where n is the number of propositions used (see the letters used in the total proposal). On the other hand for columns it is necessary to lay down all the sub-propositions which constructed the total proposal. To tell if the proposal is true or false just see the last column of the table (total proposal), if all the results are true then the proposal is true and if one of the results is false then the proposal is false.

Remarque 13 *In the filling of the table that uses three propositions, we start with the 1^{st}*

column we divide it by two above we have 1 *and below we have* 0*, then the* 2^{nd} *column we divide it by* 4 *in the filling it is two by two (of* 1 *and* 0*) and in the* 3^{rd} *is one by one (*1 *then a* 0*) to have all the combinations of truth between three propositions.*

Exemple 14 *Is the following proposition true or false?*

$$[H_1] : (P \Rightarrow Q) \Leftrightarrow \left(\overline{Q} \Rightarrow \overline{P}\right).$$

Indeed:

P	Q	$\overline{Q}$	$\overline{P}$	$P \Rightarrow Q$	$\overline{Q} \Rightarrow \overline{P}$	$[H_1]$
1	1	0	0	1	1	1
1	0	1	0	0	0	1
0	1	0	1	1	1	1
0	0	1	1	1	1	1

So the $[H_1]$ *proposition is true.*

Exemple 15 *Is the following proposition true or false?*

$$[H_2] : \left[\left(P \vee \overline{Q}\right) \Rightarrow R\right] \Leftrightarrow \overline{P \wedge R}.$$

P	Q	R	$\overline{Q}$	$P \vee \overline{Q}$	$P \wedge R$	$\overline{P \wedge R}$	$\left[\left(P \vee \overline{Q}\right) \Rightarrow R\right]$	$[H_2]$
1	1	1	0	1	1	0	1	0
1	1	0	0	1	0	1	0	0
1	0	1	1	1	1	0	1	0
1	0	0	1	1	0	1	0	0
0	1	1	0	0	0	1	1	1
0	1	0	0	0	0	1	1	1
0	0	1	1	1	0	1	1	1
0	0	0	1	1	0	1	0	0

which implies that proposition $[H_2]$ *is false.*

1.1.5 Tautology

Définition 16 *A tautology is a proposition that is true in all cases.*

Exemple 17 *Verify if the proposition:*

$$(P \Rightarrow Q) \vee (Q \Rightarrow P),$$

is a tautology?

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \vee (Q \Rightarrow P)$
1	1	1	1	1
1	0	0	1	1
0	1	1	0	1
0	0	1	1	1

So it's a tautology.

Exemple 18 *The proposition $P \vee \overline{P}$ is a tautology because:*

P	$\overline{P}$	$P \vee \overline{P}$
1	0	1
0	1	1

This proposition is called the law of excluded third parties (either the first or the second and not a third).

1.1.6 Antinomy

Définition 19 *An antinomy is a proposition that is false in all cases.*

Exemple 20 *The proposition $P \wedge \overline{P}$ is an antinomy because:*

P	$\overline{P}$	$P \wedge \overline{P}$
1	0	0
0	1	0

This proposition is called the law of ***contradiction.***

1.1.7 The negation of connectors

The negations of the connectors are given by the following formulas:

(1) $\overline{P \wedge Q}$ is $\overline{P} \vee \overline{Q}$ and we write: $\overline{P \wedge Q} \Longleftrightarrow \overline{P} \vee \overline{Q}$.

(2) $\overline{P \vee Q}$ is $\overline{P} \wedge \overline{Q}$ and we write: $\overline{P \vee Q} \Longleftrightarrow \overline{P} \wedge \overline{Q}$.

(3) $\overline{P \Rightarrow Q}$ is $P \wedge \overline{Q}$ and we write: $\overline{P \Rightarrow Q} \Longleftrightarrow P \wedge \overline{Q}$.

(4) $\overline{P \Leftrightarrow Q}$ is $\left[P \wedge \overline{Q} \text{ or } Q \wedge \overline{P}\right]$ and we write:

$$\overline{P \Leftrightarrow Q} \Leftrightarrow \left[P \wedge \overline{Q} \text{ or } Q \wedge \overline{P}\right].$$

(5) $\overline{\overline{P}} \Leftrightarrow P$.

Caution: Do not write $=$ instead of $\Leftrightarrow$.

(6) From (3) and (5):

$$(P \Rightarrow Q) \Leftrightarrow \left(\overline{\overline{P \Rightarrow Q}}\right) \Leftrightarrow \overline{P \wedge \overline{Q}} \Leftrightarrow \overline{P} \vee Q.$$

Remarque 1.1 *The first two forms (1) and (2) are known as the laws of* MORGAN.

Explanation of these results

(1) For the negation of the conjunction: the negation of two true information is one of the two false.

(2) For the negation of the disjunction: The negation of one is true is the both are false.

(3) For the negation of the implication, we give the following example:

If you say to me, "If you give me a key, then I open this door," then the denial of that sentence for me to contradict you is, "I give you a key and you can't open the door," that key is no longer the right key to the door. Another example if one of the students has hypotheses and will answer the questions then the negation is that he has the hypotheses and fails to answer the questions.

(4) For the equivalence it is enough to write it in the form of two meanings of implications and one negates the conjunction of the two implications.

1.1.8 Properties of connectors

Let P, Q and R three propositions.

(1) $(P \vee P) \Rightarrow P$.

(2) $P \Rightarrow (P \vee Q)$.

(3) $(P \wedge Q) \Leftrightarrow (Q \wedge P)$ and $(P \vee Q) \Leftrightarrow (Q \vee P)$ **(The commutativity)**.

(4) $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ and $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ **(Associativity).**

(5) $(P \wedge Q) \vee R \Leftrightarrow (P \vee R) \wedge (Q \vee R)$ and $(P \vee Q) \wedge R \Leftrightarrow (P \wedge R) \vee (Q \wedge R)$ **(Distributivity from one to the other)**.

(6) $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$ **(Transitivity)**.

(7) $(P \Rightarrow Q) \Rightarrow [(R \vee P) \Rightarrow (R \vee Q)]$.

1.1.9 Logical quantifiers

Either a set E and a specified property P. The following two questions can be asked:

a)Are there elements of E that is verifing this property?

b)If yes, is the property true for all elements or for a single element?

To formulate the answers to these two questions we introduce symbols called **quantifiers**, these are:

Existential quantifier

It is written $\exists$ and means: that there is at least one element of E having the property P, for example:

$$\exists x \in \mathbb{R}, x^2 + x - 2 = 0.$$

Means that there is at least a real number such as:

$$x^2 + x - 2 = 0, \text{ for example } x = -2.$$

In addition if the property $P(x)$ is checked for a single element $x \in E$ we write:

$$\exists! x \in E, P(x), \text{(Here we have existence and uniqueness)}$$

For example:

$$x^2 - 2x + 1 = 0 \Leftrightarrow (x-1)^2 = 0,$$

so the only solution to this equation is $x = 1$.

Universal quantifier

Which is written $\forall$ (reads: For all) and means that any element of E checks P, for example:

$$\forall x \in \mathbb{R},\ x^2 + 2x + 1 \geq 0,$$

means that each actual number verifies the written inequality because:

$$x^2 + 2x + 1 = (x+1)^2 \geq 0.$$

1.1.10 The negation of quantifiers

(1) The negation of ($\forall x \in E, P(x)$ is true) is:

$$\exists x \in E,\ P(x) \text{ is false.}$$

(2) The negation of ($\exists x \in E, P(x)$ is true) is:

$$\forall x \in E, P(x) \text{ is false.}$$

(3) The negation of ($\exists! x \in E,\ P(x)$ is true) is:

$$\underbrace{(\forall x \in E, P(x) \text{ is false})}_{\text{Negation of the existence}} \text{ or } \underbrace{(\exists x_1 \neq x_2 \in E, P(x_1) \text{ and } P(x_2) \text{ are true})}_{\text{Negation of uniqueness}}.$$

Properties 1.1 *Let E a set and $P(x)$ a proposition whose values of truths are based on elements x of E.*

(1) $(\forall x \in E, P(x) \wedge Q(x)) \Leftrightarrow ((\forall x \in E, P(x)) \wedge (\forall x \in E, Q(x)))$.

(2) $(\forall x \in E, P(x) \vee Q(x)) \Leftarrow ((\forall x \in E, P(x)) \vee (\forall x \in E, Q(x)))$.

(3) $(\exists x \in E, P(x) \wedge Q(x)) \Rightarrow ((\exists x \in E, P(x)) \wedge (\exists x \in E, Q(x)))$.

(4) $(\exists x \in E, P(x) \vee Q(x)) \Leftrightarrow ((\exists x \in E, P(x)) \vee (\exists x \in E, Q(x)))$.

This means that you can distribute $\forall$ on « $\wedge$ » and $\exists$ on « $\vee$ », but you can't distribute $\forall$ on « $\vee$ » and $\exists$ on « $\wedge$ ».

Now if the proposition $P(x, y)$ depends on two variables x and y.

(5) $((\forall x \in E), (\forall y \in E), P(x, y)) \Leftrightarrow ((\forall y \in E), (\forall x \in E), P(x, y))$.

(6) $((\exists x \in E), (\exists y \in E), P(x, y)) \Leftrightarrow ((\exists y \in E), (\exists x \in E), P(x, y))$.

This means that we can swap quantifiers of the same nature.

(7) $((\exists x \in E)/(\forall y \in E, P(x, y))) \Rightarrow (\forall y \in E, \exists x \in E/P(x, y))$.

(8) $(\forall y \in E, \exists x \in E/P(x, y)) \not\Rightarrow ((\exists x \in E)/(\forall y \in E, P(x, y)))$.

In the last formula, the x is not the same for all y. This means that we cannot swap quantifiers of different natures in all cases.

Exemple 21 *Either the results table for* Ali *and* Sara*, knowing that the proposition:*

$$P(x, y): \text{ The student has an average of greater than or equal to } 10.$$

	Algebra1	*Analysis1*	*PHYSIQUE1*	*INFO1*
Ali	08	13	12	15
Sara	14	10	16	07

We let the sets $E = \{$Ali, Sara$\}$ and $F = \{$Algebra1, Analysis1, PHYSIQUE1, INFO1$\}$.

(1) $\forall x \in E, \forall y \in F : P(x, y)$ *is false because:*

For $x =$ Ali and $y =$ Algebra1 the rating is strictly lower than 10.

(2) $\forall x \in E, \exists y \in F : P(x,y)$ *is true because:*

Each student has at least one grade above or equal to 10*.*

(3) $\exists x \in E, \forall y \in F : P(x,y)$ *is false because:*

Both students do not have all grades above or equal to 10*.*

(4) $\exists x \in E, \exists y \in F : P(x,y)$ *is true because:*

Both students have at least a grade greater than or equal to 10*.*

1.2 Some types of reasoning

It is important to find a way or a method to answer a certain problem, for this we are inspired by some techniques called reasoning.

1.2.1 Direct reasoning

Direct reasoning is used to show a type of implication:

$$P \Rightarrow Q,$$

where P is the hypotheses (assumptions) and Q is the problems or conclusions. For this we apply techniques from given hypotheses and known theorems to find our problems.

Exemple 1.1 *Let's show that if* $n \in \mathbb{N}$ *is even then* n^2 *is even.*

$$\begin{aligned} n \textit{ is even} &\Rightarrow \exists k_1 \in \mathbb{N} \textit{ such as: } n = 2k_1, \\ &\Rightarrow n^2 = (2k_1)^2 = 4k_1^2 = 2\left(2k_1^2\right) = 2k_2 \textit{ with } k_2 = \left(2k_1^2\right) \in \mathbb{N} \\ &\Rightarrow n^2 \textit{ is even.} \end{aligned}$$

1.2.2 Reasoning by contradiction

Generally, the search for an answer to a problem is based on the given hypotheses or known theorems, but sometimes one can find a reasoning other than the direct path. One draws on the reasoning by the absurd, which supposes that the negation of the problem is true, and then one comes to a contradiction with the given hypotheses, either one of the known theorems, or one of the axioms, that is, what was proposed is false, which implies that the problem is true. In other words:

$$(P \Rightarrow Q) \Longleftrightarrow \left(\overline{Q} \Rightarrow \text{contradiction}\right).$$

Exemple 1.2 *Let's show that:*

$$\forall n \in \mathbb{N}, n^2 \textit{ is even} \Rightarrow n \textit{ is even.}$$

By contradiction assumes that n is not even, therefore:

$$\begin{aligned} n \textit{ is odd} &\Rightarrow \exists k \in \mathbb{N} \textit{ such as: } n = 2k+1, \\ &\Rightarrow n^2 = 2\left(2k^2 + 2k\right) + 1 \\ &\Rightarrow n^2 = 2k' + 1 \textit{ with } k' = \left(2k^2 + 2k\right) \\ &\Rightarrow n^2 \textit{ is odd, contradicts the hypothesis,} \end{aligned}$$

where n is even.

Exemple 1.3 *Prove that $\frac{\ln 2}{\ln 3}$ is an irrational number $\left(\frac{\ln 2}{\ln 3} \notin \mathbb{Q}\right)$.*
<u>Reminder:</u> *x is a rational number ($x \in \mathbb{Q}$) if it checks:*

$$x = \frac{p}{q}; p, q \in \mathbb{Z}, q \neq 0 \textit{ such as: } (p \wedge q) = 1,$$

$(p \wedge q) = 1$ means that p and q are relatively prime (first among them) (the only common divisor between them is 1).
To show that $\frac{\ln 2}{\ln 3} \notin \mathbb{Q}$, by contradiction, assumes that:

$$\begin{aligned} \frac{\ln 2}{\ln 3} &\in \mathbb{Q} \Rightarrow \exists p, q \in \mathbb{N}^* \textit{ such as } (p \wedge q) = 1 \textit{ and } \frac{\ln 2}{\ln 3} = \frac{p}{q}, \\ &\Rightarrow \ln 2^q = \ln 3^p \Rightarrow 2^q = 3^p \textit{ (contradiction),} \end{aligned}$$

because 2^q is even and 3^p is odd, hence $\frac{\ln 2}{\ln 3}$ is an irrational number.

1.2.3 Contrapositive

One calls contraposed of an implication $P \Rightarrow Q$, the implication (not $Q \Rightarrow$ not P). As remark the contraposed is a particular case of reasoning by contradiction for that the reasoning by contradiction is usually used (because not P is exactly a contradiction with one of the hypotheses).

Finally the contraposed is:

$$(P \Rightarrow Q) \Leftrightarrow \left(\overline{Q} \Rightarrow \overline{P}\right).$$

Exemple 1.4 *Let's show that:*

$$\left[n = p^2, p \in \mathbb{N}\right] \Rightarrow \left[2n \neq q^2, \forall q \in \mathbb{N}\right].$$

Indeed:

$$\begin{aligned} \exists q \quad &\in \quad \mathbb{N}, 2n = q^2 \Rightarrow n = \frac{q^2}{2} \\ &\Rightarrow \quad n \neq p^2, \ \textit{(this is the contraposed)} \end{aligned}$$

because if:

$$\begin{aligned} n \quad &= \quad p^2 \Rightarrow \frac{q^2}{2} = p^2 \Rightarrow 2 = \frac{q^2}{p^2} \\ &\Rightarrow \quad \sqrt{2} = \frac{p}{q} \in \mathbb{Q}, \ \textit{which is false.} \end{aligned}$$

1.2.4 Counter-Example

It is used to prove that the proposition or a property is not always true by giving
an example where it is false.

Exemple 22 *Prove that for all* $x \in \mathbb{R}, f(x) = x^2 - 3x - 4 \geq 0$ *is false, because if we give for exemple* $x = 0$*, we have* $f(0) < 0$.

1.2.5 Proof by induction

We use the reasoning by induction in the case of a relation or a formula dependent on an index $n \in \mathbb{N}$ **(on a natural integer and no other)**. So to show that a property is true for any natural integer n greater than or equal to an integer n_0, we check that it is heriditary (i.e.: if it is true for any integer, then it is true for the next integer). It suffices then that it is true for the first integer n_0 to deduce that it is true for any integer n greater than or equal to n_0.

Recap: To show that a relationship (R_n), is true for all integer $n \geq n_0$ by induction, the following two steps are followed:

1st step: It is shown to be true for the first index, i.e.: (R_{n_0}) is true.

2nd Step: We assume that (R_n) is true for a fixed integer $n \in \mathbb{N}$ (is called the inductive hypothesis) and show that (R_{n+1}) is true.

Exemple 23 *Knowing that a natural integer m is divisible by 7 is equivalent to:*

$$\exists k \in \mathbb{N} \text{ such as: } m = 7k.$$

We prove by induction that:

$$\forall n \in \mathbb{N}, 3^{2n} - 2^n \text{ is divisible by } 7.$$

Note this property by (R_n).

By induction, we prove that (R_n) is true for all $n \in \mathbb{N}$.

1st step: *For $n = 0$:*

$$3^0 - 2^0 = 0 = 0 \times 7,$$

which implies that $3^0 - 2^0$ is divisible by 7, therefore (R_0) is true.

2nd Step: *We assume that (R_n) is true, that is $3^{2n} - 2^n$ is divisible by 7 $\left(\exists k_1 \in \mathbb{N}, 3^{2n} - 2^n = 7k_1\right)$ and show that (R_{n+1}) is also, that is:*

$$3^{2(n+1)} - 2^{n+1} \text{ is divisible by } 7 \text{ ?}$$

Indeed:

$$\begin{aligned}
3^{2(n+1)} - 2^{n+1} &= 3^2 \times 3^{2n} - 2 \times 2^n = 9 \times 3^{2n} - 2 \times 2^n = 2\left(3^{2n} - 2^n\right) + 7 \times 3^{2n}, \\
&= 2 \times 7k_1 + 7 \times 3^{2n} \textit{ (based on the inductive hypothesis)}, \\
&= 7\left(2k_1 + 3^{2n}\right) = 7k_2 \textit{ with } k_2 = 2k_1 + 3^{2n},
\end{aligned}$$

Hence:

$$3^{2(n+1)} - 2^{n+1} \textit{ is divisible by } 7.$$

Conclusion:

$$\forall n \in \mathbb{N}, 3^{2n} - 2^n \quad \textit{is divisible by } 7.$$

Exemple 24 *We prove by induction that:*

$$\forall n \in \mathbb{N}, 4^n + 6n - 1 \textit{ is a multiple of } 9. \qquad (P_n)$$

1st step: *For* $n = 0$:

$$\begin{aligned}
&4^0 - (6 \times 0) - 1 = 0 = 0 \times 9, \\
&\Rightarrow 4^0 - (6 \times 0) - 1 \textit{ is a multiple of } 9,
\end{aligned}$$

that means that (P_0) *is true.*

2nd Step: *We assume that* (P_n) *is true for a fixed integer* n $(4^n + 6n - 1 = 9k_1, k_1 \in \mathbb{N})$ *and show that* (P_{n+1}) *is also, that is:*

$$4^{n+1} + 6(n+1) - 1 \textit{ is a multiple of } 9.$$

Indeed:

$$4^{n+1} + 6(n+1) - 1 = 4 \times 4^n + 6n - 1 + 6$$

$$= 1 \times 4^n + 6n - 1 + 3 \times 4^n + 6$$

$$= 9k + 3 \times 4^n + 6 \textit{ (based on the inductive hypothesis)}$$

$$= 9k + 3(9k_1 - 6n + 1) + 6 \; (4^n + 6n - 1 = 9k_1)$$

$$= 9(k_1 + 3k_1 - 2n + 1) = 9k_2 \textit{ with } k_2 = 4k_1 - 2n + 1,$$

$$\Rightarrow 4^{n+1} + 6(n+1) - 1 \textit{ is a multiple of } 9,$$

so (R_{n+1}) *is true.*

Conclusion:

$$\forall n \in \mathbb{N}, 4^n + 6n - 1 \textit{ is a multiple of } 9.$$

Chapitre 2

Sets and Applications

2.1 set theory

A set is made up of material objects, or phenomena, or signs, or abstract identities, brought together by virtue of a common property.

A set is an entity of a different nature from the elements that make it up. A point set is not a point, even if it contains only one point.

Some particularly important sets are designated by specific letters, Note:

$\mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$ (set of natural integers).
$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ (set of relative integers).
$\mathbb{Q}$: Rational numbers.
$\overline{\mathbb{Q}}$: Irrational numbers.
$\mathbb{R}$: The set of real numbers.
$\mathbb{C}$: The set of complex numbers.

Alternatively, a set or a part can be designated by specifying their particular properties P verified by an element x of this part, for example:

$$E = \{x/x \in \mathbb{Q} \text{ and } 1 \leq x \leq 5\}.$$

2.1.1 Vocabulary and ratings (notations)

If a is an element of Set E, write $a \in E$, state a "a is an element of E" or "a belongs to E". The negation of the previous statement is noted $a \notin E$ (a is not an element of E or say a does not belong to E). It is noted that a set that does not contain any elements is said to be the empty set, noted: $\emptyset$.

2.1.2 Inclusion - subset

Let E and F be two sets. If all the elements of the set E belong to the set F we say that E is included in F, or else E is a subset of F and we write $E \subset F$. To show in the general case that $E \subset F$, it is enough to take any element x of E and we show that this element belongs to F.

$$(E \subset F) \Leftrightarrow (x \in E \Rightarrow x \in F)\,. \tag{2.1}$$

Exemple 25 $\mathbb{N} \subset \mathbb{R}$.

Exemple 26 $\emptyset \subset E$, *with* E *is any set.*

Preuve: We have:

$$\left[\underbrace{a \in \emptyset}_{\text{False proposition}} \Rightarrow a \in E\right],$$

this implication is true because the first proposition is false, which asserts that the implication is true, then by (2.1) we have $\emptyset \subset E$. ■

2.1.3 Equality of two sets

Let E and F be two sets. To proof that $E = F$, just show that $E \subset F$ and $F \subset E$.

2.1.4 Power set (Sets of parts)

Définition 27 ***The power set*** *of a set* E*, noted* $\wp(E)$ *is the set whose elements are the subsets of* E*, such that* $\emptyset \in \wp(E)$ *and* $E \in \wp(E)$*, in addition the number of elements of the power set* $\wp(E)$ *is* 2^n *where* n *is the number of elements of* E*.*

Remarque 28 *Despite that the elements are sets we write $A \in \wp(E)$ if $A \subset E$.*

Exemple 29 *If $E = \{1, 2, 3\}$, then:*

$$\wp(E) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, E\}.$$

Exemple 30 *If $F = \{5, 6, 7, 8\}$, so:*

$$\wp(F) = \left\{ \begin{array}{c} \emptyset, \{5\}, \{6\}, \{7\}, \{8\}, \{5, 6\}, \{5, 7\}, \{5, 8\} \\ \{6, 7\}, \{6, 8\}, \{7, 8\}, \{5, 6, 7\}, \{5, 6, 8\}, \{6, 7, 8\}, \{5, 7, 8\}, F \end{array} \right\}.$$

Exemple 31 *If $G = \{2\}$, then the power set is:*

$$\wp(G) = \{\emptyset, \{2\}\}.$$

On the other hand for $\wp(\wp(G))$ we have:

$$\wp(\wp(G)) = \{\emptyset, \{\emptyset\}, \{\{2\}\}, \{\emptyset, \{2\}\}\}.$$

Remarque 32 *We can note the power set by $P(E)$ instead of $\wp(E)$.*

2.1.5 Intersection

Définition 33 *The intersection of two parts, A and B, is the subset formed by the elements belonging to each of the parts considered. This intersection is designated by the notation $A \cap B$ (We pronounce "A and B" or "A intersected with B"), for that:*

$$(x \in A \cap B) \Leftrightarrow (x \in A \text{ and } x \in B).$$

The intersection can be reduced to the empty part, in this case we say that the sets are disjunct.

2.1.6 Union

Définition 34 *The whole of all the elements belonging to at least one of the parts A and B, is said the union of these parts, noted: $A\cup\ B$ (We pronounce "A union B"). We have:*

$$(x \in A \cup B) \Leftrightarrow (x \in A \text{ or } x \in B).$$

Properties 2.1 *Let A, B and C be tree subsets of a set E, then we have the following properties:*

(1) $A \cap B \subset A$.

(2) $A \subset A \cup B$.

(3) $A \cap A = A \cup A = A$.

(4) $A \cap \varnothing = \varnothing$.

(5) $A \cup \varnothing = A$.

(6) $A \cap B = B \cap A$ *and* $A \cup B = B \cup A$.

(7) $(A \cap B) \cap C = A \cap (B \cap C)$ *and* $(A \cup B) \cup C = A \cup (B \cup C)$.

(8) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(9) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(10) $A \cap B = B$ *if and only if* $B \subset A$.

(11) $A \cup B = B$ *if and only if* $A \subset B$.

2.1.7 Partitions

Définition 35 ***A partition*** *of a set E is made by classifying the elements of E into separate **non-empty** subsets two to two E_1, E_2, ..., $E_n, n \in \mathbb{N}^*$, such that any element of E is classified. It is rated by $P(E)$, and in particular:*

$$E = \underset{1\leq i\leq n}{\cup} E_i \text{ and } E_i \cap E_j = \emptyset, \forall i \neq j.$$

Remarque 36 *A partition of a set is not unique.*

Exemple 37 *Let* $E = \{1, 2, 3\}$, *we have the different partitions:*

$P(E) = \{\{1\}, \{2\}, \{3\}\}$	$P(E) = \{\{1, 2\}, \{3\}\}$	$P(E) = \{\{1, 3\}, \{2\}\}$
$P(E) = \{\{2, 3\}, \{1\}\}$	$P(E) = \{\{1, 2, 3\}\}.$	

2.1.8 Complement

Définition 38 *Let* $E \subset F$, *the* ***complement*** *of the set* E *in the set* F, *is the set in the union with* E *equals* F, *and the intersection with* E *equals the empty set. We note it by:* C_F^E *or* $\overline{E}$, *so we have:*

$$E \cup C_F^E = F \text{ and } E \cap C_F^E = \emptyset.$$

Exemple 39 *For:*

$$E = \{1, 2, 3\} \text{ and } F = \{1, 2, 3, 4, 5, 6\},$$

we have: $C_F^E = \{4, 5, 6\}.$

2.1.9 Cartesian product

Définition 40 *Cartesian product of two sets* E *and* F *is the set of ordered pairs of type* (x, y) *with* $x \in E$ *and* $y \in F$, *noted* $E \times F$, *that is:*

$$E \times F = \{(x, y) / x \in E \text{ and } y \in F\}.$$

Exemple 41 $E = \{1, 2, 3\}$ *and* $F = \{4, 5\}$, *then:*

$$E \times F = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}.$$

Remarque 42 *For all two differents sets* E *and* F*:*

$$E \times F \neq F \times E.$$

2.1.10 Sum

Définition 43 *The sum of two sets E and F is the rated set $E+F$ defined by:*

$$E+F=\{\alpha=x+y, x\in E \text{ and } y\in F\}.$$

Exemple 44 $E=\{1,2,3\}$ *and* $F=\{1,4,5\}$ *then:*

$$E+F=\{2,5,3,6,7,4,8\}.$$

2.1.11 Difference

Définition 45 *The difference between two sets E and F is the set noted $E\setminus F$ or $E-F$ defined by:*

$$E-F=\{x\in E \text{ with } x\notin F\}.$$

Exemple 46 $E=\{1,2,3\}$ *and* $F=\{1,4,5\}$ *then:*

$$E-F=\{2,3\}.$$

2.1.12 Symmetric difference

Définition 47 *The symmetric difference between two sets E and F is the set:*

$$\begin{aligned} E\triangle F &= (E-F)\cup(F-E), \\ &= (E\cup F)-(E\cap F), \\ &= \{(x\in E \text{ and } x\notin F) \text{ or } (x\in F \text{ and } x\notin E)\}. \end{aligned}$$

Exemple 48 $E=\{1,2,3\}$ *and* $F=\{1,4,5\}$ *then:*

$$E\triangle F=(E-F)\cup(F-E)=\{2,3\}\cup\{4,5\}=\{2,3,4,5\}.$$

2.1.13 Example

Lets E and F two subsets of G, we show that:

$$(1)\overline{E}\cup\overline{F}=\overline{E\cap F}.$$

$" \Rightarrow "$ It shows that:

$$\overline{E}\cup\overline{F}\subset\overline{E\cap F}?$$

$$\begin{aligned}
x &\in \overline{E}\cup\overline{F}\Rightarrow x\in\overline{E} \text{ or } x\in\overline{F},\\
&\Rightarrow (x\in G \text{ and } x\notin E) \text{ or } (x\in G \text{ and } x\notin F),\\
&\Rightarrow x\in G \text{ and } (x\notin E \text{ or } x\notin F),\\
&\Rightarrow x\in G \text{ and } x\notin E\cap F,\\
&\Rightarrow x\in\overline{E\cap F}.
\end{aligned}$$

$" \Leftarrow "$ It shows that:

$$\overline{E\cap F}\subset\overline{E}\cup\overline{F}?$$

$$\begin{aligned}
x &\in \overline{E\cap F}\Rightarrow x\in G \text{ and } x\notin E\cap F,\\
&\Rightarrow x\in G \text{ and } (x\notin E \text{ or } x\notin F),\\
&\Rightarrow (x\in G \text{ and } x\notin E) \text{ or } (x\in G \text{ and } x\notin F),\\
&\Rightarrow x\in\overline{E} \text{ or } x\in\overline{F},\\
&\Rightarrow x\in\overline{E}\cup\overline{F}.
\end{aligned}$$

Similarly we can prove that:

$$(2)\overline{E}\cap\overline{F}=\overline{E\cup F},$$

but we use the equivalence directly:

$$\begin{aligned} x &\in \overline{E} \cap \overline{F} \Leftrightarrow x \in \overline{E} \text{ and } x \in \overline{F}, \\ &\Leftrightarrow (x \in G \text{ and } x \notin E) \text{ and } (x \in G \text{ and } x \notin F), \\ &\Leftrightarrow x \in G \text{ and } (x \notin E \text{ and } x \notin F), \\ &\Leftrightarrow x \in G \text{ and } x \notin E \cup F, \\ &\Leftrightarrow x \in \overline{E \cup F}. \end{aligned}$$

2.2 Applications

2.2.1 Concept of application

Définition 49 *Given two sets E and F, we define an application f of E in F by giving ourselves a rule allowing to match to any element of E a determining element of F. Otherwise, we say that f is an application of E in F if for each $x \in E$ there is a unique $y \in F$ associated to it. We often note the applications by: $f, g, h, \ldots$. Moreover if $x \in E$, $f(x)$ denotes the image of x ander f, and writes:*

$$\begin{aligned} f &: E \rightarrow F \\ x &\longmapsto f(x) = y. \end{aligned}$$

It is said that x is the antecedent (pre-image) of y ander f, E is the starting set and F is the arrival set or codomain of f. Formally, using predicate logic:

$$f : E \rightarrow F,$$

is an application if:

$$\forall x \in E, \exists! y \in F \text{ such as } f(x) = y.$$

In addition, we have what we call the application graph noted G_f given by:

$$G_f = \{(x, y) \in E \times F / y = f(x)\}.$$

Exemple 50

$$\begin{aligned} f \quad &: \quad \mathbb{R} \to \mathbb{R} \\ x \quad &\longmapsto \quad f(x) = 6x + 3, \textit{ is an application.} \end{aligned}$$

Remarque 51 *An application f is a function of E in F whose domain definition D_f is equal to E.*

2.3 Restriction of an application-Extention of an application

Définition 52 ***(Restriction-Extention)*** *Let $f : E \to F$ be an application and A and B two subsets of E and F respecively. In this case it is said that $g : A \to B$ is the restriction of f to A and f is the extention of g to E.*

2.4 Equality of two applications

Définition 53 *To show that two applications $f$and g are equal, we show that they have the same starting set E and the same arrival set F , in addition:*

$$\forall x \in E, f(x) = g(x).$$

2.5 Composition of applications

Définition 54 *Let $f : E_1 \to F_1$ and $g : E_2 \to F_2$ be two applications. So the composite application f followed by g noted $g \circ f$ exists if we have the condition $f(E_1) \subset E_2$ with:*

$$\begin{aligned} g \circ f &: E_1 \to F_2 \\ &x \mapsto (g \circ f)(x) = g(f(x)). \end{aligned}$$

Exemple 55 *Let:*

$$\begin{aligned} f \quad &: \quad \mathbb{N} \to \mathbb{N} \\ x \quad &\mapsto \quad f(x) = 2x, \end{aligned}$$

and

$$\begin{aligned} g \quad &: \quad \mathbb{N} \to \mathbb{N} \\ x \quad &\mapsto \quad g(x) = \begin{cases} \frac{x}{2} \text{ if } x \text{ is even,} \\ \frac{x+1}{2} \text{ if } x \text{ is odd.} \end{cases} \end{aligned}$$

$f \circ g$ *and* $g \circ f$ *existe and we have:*

$$\begin{aligned} f \circ g &: \mathbb{N} \to \mathbb{N} \\ & x \mapsto f(g(x)) = \begin{cases} f\left(\frac{x}{2}\right) \text{ if } x \text{ is even,} \\ f\left(\frac{x+1}{2}\right) \text{ if } x \text{ is odd.} \end{cases} \\ & \qquad = \begin{cases} x \text{ if } x \text{ is even,} \\ x+1 \text{ if } x \text{ is odd.} \end{cases} \end{aligned}$$

On the other hand:

$$\begin{aligned} g \circ f &: \mathbb{N} \to \mathbb{N} \\ & x \mapsto g(f(x)) = g(2x) = x \text{ because } 2x \text{ is even.} \end{aligned}$$

Remarque 56 *In the general case:* $g \circ f \neq f \circ g$ *(see example). Knowing that in cases one sense of the composite exists and the other does not exist.*

Exemple 57

$$\begin{aligned} f \quad &: \quad \mathbb{N} \to \mathbb{N} \\ x \quad &\mapsto \quad f(x) = \sqrt{x}, \end{aligned}$$

and

$$\begin{aligned} g &: \mathbb{Z} \to \mathbb{Z} \\ x &\mapsto 2x. \end{aligned}$$

(1) For $f \circ g$ exists because $\mathbb{N} \subset \mathbb{Z}$ and we have:

$$\begin{aligned} f \circ g &: \mathbb{N} \to \mathbb{Z} \\ x &\mapsto (f \circ g)(x) = f(g(x)) = f(2x) = \sqrt{2x}. (x \in \mathbb{N}) \end{aligned}$$

(2) $g \circ f$ does n't exist because for $x \in \mathbb{Z}^-, g(f(x))$ does n't exist.

2.6 Image of a subset

Définition 58 *Let $f : E \to F$ be an application and A a subset of E. Then the image of A by f is defined by:*

$$f(A) = \{f(x), x \in A\},$$

hence:

$$y \in f(A) \Leftrightarrow \exists x \in A, y = f(x).$$

Exemple 59

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}^+ \\ x &\mapsto f(x) = |x|, \end{aligned}$$

and

$$A = \{-1, 1, -2, 2, -3, 3\}.$$

Therefore:

$$f(A) = \{1, 2, 3\}.$$

2.7 Injective applications

Définition 60 *Let $f : E \to F$ be an application.*

$$f \text{ is } \boldsymbol{injective} \Leftrightarrow \forall x_1, x_2 \in E, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

Or,

$$\forall x_1, x_2 \in E, f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ (The contraposed).}$$

This means that each $y \in F$ admits at most one antecedent $x \in E$.

Exemple 61

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto f(x) = 2x, \end{aligned}$$

and

$$\begin{aligned} g &: \mathbb{R} \to \mathbb{R}^+ \\ x &\mapsto g(x) = |x|. \end{aligned}$$

(1) f is injective because:

$$\forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2 \Rightarrow 2x_1 \neq 2x_2 \Rightarrow f\ (x_1) \neq f\ (x_2).$$

(2) g is n't injective because for example:

$$2 \neq -2 \text{ but } f(2) = f(-2) = 2.$$

2.8 Surjective applications

Définition 62 *Let $f : E \to F$ be an application. Then:*

$$f \text{ is } \boldsymbol{surjective} \Leftrightarrow \forall y \in F, \exists x \in E \text{ such as: } f(x) = y.$$

That is to say each element of the set of arrival admits at least one antecedent.

Remarque 63 *To show that an application is surjective it is enough to find the x according to the y, and see if x exists in the set E for all $y \in F$.*

Exemple 64

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto f(x) = |x|. \end{aligned}$$

f is n't surjective because for:

$$y \in \mathbb{R}^-, \forall x \in \mathbb{R} : f(x) = |x| \neq y.$$

Exemple 65

$$\begin{aligned} g &: \mathbb{N} \to \mathbb{N} \\ x &\mapsto g(x) = \begin{cases} \frac{x}{2} \text{ if } x \text{ is even,} \\ \frac{x+1}{2} \text{ if } x \text{ is odd.} \end{cases} \end{aligned}$$

g is surjective because:

$$\forall y \in \mathbb{N}, \exists x = 2y \in \mathbb{N} \text{ avec } f(x) = f(2y) = \frac{2y}{2} = y.$$

Remarque 66 *There are other techniques to show injectivity and surjectivity, which uses the notion of derivative and the table of variations.*

Proposition 2.1 *Let$f : E \to F$ be an application. If f is a strictly monotonous function (strictly increasing or strictly decreasing), then f is injective. On the other hand if $f(E) = F$, then f is surjective.*

Exemple 2.1 *Let f be an application defined by:*

$$\begin{array}{rcl} f & : & [-1,+\infty[\to \mathbb{R} \\ x & \mapsto & f(x) = \dfrac{1}{\sqrt{x^2+2x+2}}, \end{array}$$

which is well defined on $[-1,+\infty[$, *because:*

$$x^2+2x+2 = (x+1)^2+1 > 0.$$

f *is continuous on*$[-1,+\infty[$ *(even on* $\mathbb{R}$*) because it is the quotient of two continuous functions. Moreover, it is derivable with:*

$$\begin{aligned} f'(x) &= \frac{-\frac{2x+2}{2\sqrt{x^2+2x+2}}}{(x^2+2x+2)} = -\frac{2x+2}{2(x^2+2x+2)\sqrt{x^2+2x+2}} \\ &= -\frac{x+1}{(x^2+2x+2)\sqrt{x^2+2x+2}} \leq 0, \end{aligned}$$

hence the table of variations:

x	-1		$+\infty$
$f'(x)$		$-$	
$f(x)$	1	$\searrow$	0

So we notice that f is strictly decreasing so it is injective. On the other hand:

$$f([-1,+\infty[) =]0,1],$$

but in the example, the arrival set is $\mathbb{R}$ *so f is not surjective.*

2.9 Bijectivite application

Définition 67 *Let f an application of a set E in a set F. f is bijective if and only if f is both injective and surjective, or say:*

$$\forall y \in F, \exists! x \in E \text{ such as: } f(x) = y.$$

Exemple 68

$$\begin{aligned} f &: \mathbb{R}^+ \to \mathbb{R}^+ \\ x &\mapsto f(x) = x^2. \end{aligned}$$

f is bijective, because:

$$\forall y \in \mathbb{R}^+, \exists! x = \sqrt{y} \in \mathbb{R}^+ \text{ because: } y = x^2 = f(x).$$

2.10 Inverse application

Définition 69 *Let f be an application of a set E in a set F. The necessary and sufficient condition for the reverse to exist is that the application f is bijective. In this case the inverse application noted f^{-1} is defined by F in E, which has for each element y, we associate a unique element x.*

Exemple 70

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto f(x) = 3x + 5. \end{aligned}$$

It's very simple to check that f is bijective, so we have:

$$y = 3x + 5 \Rightarrow x = \frac{y-5}{3},$$

then:

$$\begin{aligned} f^{-1} &: \mathbb{R} \to \mathbb{R} \\ y &\mapsto \frac{y-5}{3}. \end{aligned}$$

or write:

$$\begin{aligned} f^{-1} &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto \frac{x-5}{3}, \end{aligned}$$

(changing variable roles is not important).

2.11 Inverse Image of a Subset

Définition 71 *Let f be an application of a set E in a set F and B a part of F. Then the inverse image of B by f is defined by:*

$$f^{-1}(B) = \{x \in E \ / \ f \ (x) \in B\}.$$

Exemple 72

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}^{+} \\ x &\mapsto f(x) = |x| \ \text{ and } B = \{1, 2, 3\}. \end{aligned}$$

Therefore:

$$f^{-1}(B) = \{-1, -2, -3, 1, 2, 3\}.$$

Remarque 73 *The inverse image of an element exists unless the application is bijective, but the inverse image of a set exists in all cases.*

Exemple 74

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}^{+} \\ x &\mapsto f(x) = x^2 \textit{ and } A = \{4\}. \end{aligned}$$

We have:

$$f(2) = f(-2) = 4 \textit{ with } 2 \neq -2,$$

so f is not injective so is not bijective then: $f^{-1}(4)$ doesn't exist, but $f^{-1}(\{4\}) = \{-2, 2\}$.

2.12 application properties

Let $f : E \to F, \forall A, B \in P(E)$ and $C, D \in P(F)$, we have the following properties:

(1) $A \subset B \Rightarrow f(A) \subset f(B)$.

Preuve:

$$y \in f(A) \Rightarrow \exists x \in A \text{ such as, } f(x) = y,$$

then:

$$\begin{aligned} \exists x &\in B \text{ such as, } f(x) = y \text{ because: } A \subset B \\ &\Rightarrow y \in f(B), \end{aligned}$$

therefore:

$$f(A) \subset f(B).$$

■

(2) $f(A \cup B) = f(A) \cup f(B)$.

Preuve: (a) Prove that: $f(A\cup B)\subset f(A)\cup f(B)$.

Let $y\in f(A\cup B)$, then:

$$\begin{aligned}
&\exists x\in A\cup B \text{ such as: } f(x)=y,\\
&\Rightarrow \exists x\in A \text{ or } \exists x\in B \text{ such as: } f(x)=y,\\
&\Rightarrow (\exists x\in A \text{ such as: } f(x)=y) \text{ or } (\exists x\in B \text{ such as: } f(x)=y),\\
&\Rightarrow y\in f(A) \text{ or } y\in f(B)\\
&\Rightarrow y\in f(A)\cup f(B).
\end{aligned}$$

(b) Prove that: $f(A)\cup f(B)\subset f(A\cup B)$.

$$\begin{aligned}
&\text{Let } y\in f(A)\cup f(B)\Rightarrow y\in f(A) \text{ or } y\in f(B),\\
&\Rightarrow (\exists x\in A \text{ such as: } f(x)=y) \text{ or } (\exists x\in B \text{ such as: } f(x)=y),\\
&\Rightarrow (\exists x\in A \text{ or } \exists x\in B) \text{ such as: } f(x)=y,\\
&\Rightarrow \exists x\in A\cup B \text{ such as: } f(x)=y,\\
&\Rightarrow y\in f(A\cup B).
\end{aligned}$$

■

(3) a) $f(A\cap B)\subset f(A)\cap f(B)$ Equality only takes place if f is injective.

Preuve: Let $y\in f(A\cap B)$, then:

$$\begin{aligned}
&\exists x\in A\cap B \text{ such as: } f(x)=y\\
&\Rightarrow (\exists x\in A \text{ and } \exists x\in B) \text{ such as: } f(x)=y,\\
&\Rightarrow (\exists x\in A \text{ such as: } f(x)=y) \text{ and } (\exists x\in B \text{ such as: } f(x)=y),\\
&\Rightarrow y\in f(A) \text{ and } y\in f(B),\\
&\Rightarrow y\in f(A)\cap f(B).
\end{aligned}$$

■

b) Prove that if f is injective than:

$$f(A\cap B)=f(A)\cap f(B).$$

Preuve: Just show that: $f(A) \cap f(B) \subset f(A \cap B)$?

Let $y \in f(A) \cap f(B)$, then:

$$y \in f(A) \quad \text{and } y \in f(B),$$

so,

$$(\exists x_1 \in A \text{ such as: } f(x_1) = y) \text{ and } (\exists x_2 \in B \text{ such as: } f(x_2) = y),$$

but f is injective then:

$$\begin{aligned}
&x_1 = x_2 = x, \\
&\Rightarrow \exists x \in A \text{ and } \exists x \in B \text{ such as: } f(x) = y, \\
&\Rightarrow \exists x \in A \cap B \text{ such as: } f(x) = y \\
&\Rightarrow y \in f(A \cap B).
\end{aligned}$$

Conclusion: If f is injective then: $f(A \cap B) = f(A) \cap f(B)$. ■

Remarque 75 *The equality only takes place if f is injective.*

Exemple 76 $A = \{0, \pi\}, B = \{0, 3\pi\}$ *and* $f(x) = \cos x$ *(f is n't injective). We have:*

$$f(A) = \{1, -1\} \text{ and } f(B) = \{1, -1\},$$

then:

$$f(A \cap B) = \{1\} \text{ and } f(A) \cap f(B) = \{1, -1\},$$

which implies that:

$$f(A) \cap f(B) \not\subset f(A \cap B).$$

(4) $C \subset D \Rightarrow f^{-1}(C) \subset f^{-1}(D)$.

Preuve:

$$\text{If } x \in f^{-1}(C) \Rightarrow \exists y \in C \text{ such as, } f(x) = y,$$

then:

$$\begin{aligned} \exists x &\in D \text{ such as, } f(x) = y \text{ because: } C \subset D \\ &\Rightarrow x \in f^{-1}(D), \end{aligned}$$

so,

$$f^{-1}(C) \subset f^{-1}(D).$$

■

(5) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

Preuve: Let $x \in f^{-1}(C \cup D)$, therefore:

$$\begin{aligned} &\exists y \in C \cup D \text{ such as, } f(x) = y \\ &\Leftrightarrow \exists y \in C \text{ or } y \in D \text{ such as, } f(x) = y, \\ &\Leftrightarrow (\exists y \in C \text{ such as, } f(x) = y) \text{ or } (\exists y \in D \text{ such as, } f(x) = y), \\ &\Leftrightarrow x \in f^{-1}(C) \text{ or } x \in f^{-1}(D) \\ &\Leftrightarrow x \in f^{-1}(C) \cup f^{-1}(D). \end{aligned}$$

Conclusion:

$$\forall C, D \subset F, f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

■

(6) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Preuve: Let $x \in f^{-1}(C \cap D)$, therefore:

$$\begin{aligned} &\exists y \in C \cap D \text{ such as, } f(x) = y \\ &\Leftrightarrow \exists y \in C \text{ and } y \in D \text{ such as, } f(x) = y, \\ &\Leftrightarrow (\exists y \in C \text{ such as, } f(x) = y) \text{ and } (\exists y \in D \text{ such as, } f(x) = y), \\ &\Leftrightarrow x \in f^{-1}(C) \text{ and } x \in f^{-1}(D), \\ &\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D). \end{aligned}$$

Conclusion:

$$\forall C, D \subset F, f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

■

2.13 Involution

Définition 77 *An involution is a bijection of a set E on itself, which is equal to its inverse, that is:*

$$\forall x \in E, f(x) = f^{-1}(x).$$

Hence:

$$f[f(x)] = x \text{ or else: } f \circ f = I,$$

where I is the identity application given by:

$$\forall x \in E, I(x) = x.$$

Exemple 78

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto f(x) = x, \end{aligned}$$

is an involution.

Chapitre 3

Equivalence and Order Relations

3.1 Notion of the binary relationship

We call relationship of E to F any process combining elements of E elements of F, generally noted by $\Re, S, \Gamma, \Phi, \ldots$.

Let $\Re$ a relationship from E to F. If $u \in E$ is related to $v \in F$, note it by: $u\Re v$.

All couples $(u, v) \in E \times F$ verifying a relationship $\Re$ is called the **graph** of $\Re$.

If $E = F$, a relationship of E to E is called **binary** relation on E. For example equality is a binary relationship on any set E.

Remarque 79 *The elements of E noted u, v and w are generally:*

(1) Numbers in $(\mathbb{N}, \mathbb{Z}, \mathbb{R}, \ldots)$ *so they can be replaced by: x, y and z.*

(2) Couples i.e.: (x, y) *of which the indices may be used, namely:* $(x_1, y_1), (x_2, y_2)$ *and* (x_3, y_3).

(3) Sets can therefore be replaced by: X, Y and Z.

3.1.1 Properties of binary relations in a set

Let $\Re$ be a binary relation in a set E and u, v, w are elements of E.

Reflexivity

Définition 80 $\Re$*is reflexive if and only if:*

$$\forall u \in E, u\Re u.$$

Exemple 81 *Let $\Re$ be a relationship defined on $\mathbb{Z}$ by:*

$$x\Re y \Leftrightarrow 3 \text{ divide } (x-y).$$

Reminder: *a divide $b \Leftrightarrow \exists k \in \mathbb{Z} : b = ka$.*

Then we have:

$$\forall\ x \in \mathbb{Z}, x - x = 0 = 0 \times 3,$$

hence 3 divides $(x-x)$, so $x\Re x$, and consequently $\Re$ is reflexive.

Symmetry

Définition 82 *$\Re$ is symmetric if and only if:*

$$\forall u, v \in E,\ u\Re v \Rightarrow v\Re u.$$

Exemple 83 *Let $\Re$ be the relationship defined on $\mathbb{R}$ by:*

$$x\Re y \Leftrightarrow (x-y) \text{ is a multiple of } 2.$$

So we have:

$$\forall x, y \in \mathbb{R}, x\Re y \Leftrightarrow (x-y) \text{ is a multiple of } 2.$$

wich implies that:

$$(y-x) \text{ is a multiple of } 2 \Rightarrow y\Re x,$$

and as a result $\Re$ is symmetric.

Antisymmetry

Définition 84 *$\Re$ is antisymmetric if and only if:*

$$\forall u, v \in E, (u\Re v \text{ and } v\Re u) \Rightarrow u = v.$$

Exemple 85 *Let $\Re$ be the relationship defined on $\mathbb{N}^*$ by:*

$$a\Re b \Leftrightarrow a \text{ divide } b.$$

Let $a, b \in \mathbb{N}^*$, we have:

$$a\Re b \Leftrightarrow a \text{ divide } b \Rightarrow \exists k_1 \in \mathbb{N}^*, b = k_1 a,$$

on the other hand we have:

$$b\Re a \Leftrightarrow b \text{ divide } a \Rightarrow \exists k_2 \in \mathbb{N}^*, a = k_2 b,$$

Thus,

$$a = k_2 k_1 a \Rightarrow k_2 k_1 = 1 \Rightarrow k_2 = k_1 = 1 \Rightarrow a = b,$$

wich implies that $\Re$ is antisymmetric.

Transitivity

Définition 86 *$\Re$ is transitive if and only if:*

$$\forall u, v, w \in E, (u\Re v \text{ and } v\Re w) \Rightarrow u\Re w.$$

Exemple 87 *Let $\Re$ be the relationship defined on $\mathbb{N} \times \mathbb{N}$ by:*
$\forall (x, x'), (y, y') \in \mathbb{N} \times \mathbb{N}$:

$$(x, x') \Re (y, y') \Leftrightarrow x + x' = y + y'.$$

Then we have: for all $(x, x'), (y, y')$ and $(z, z') \in \mathbb{N} \times \mathbb{N}$,

$$(x, x') \Re (y, y') \Leftrightarrow x + x' = y + y',$$

and

$$(y, y') \Re (z, z') \Leftrightarrow y + y' = z + z',$$

which implies that:

$$x + x' = z + z',$$

hence $(x, x') \Re (z, z')$, *and therefore* $\Re$ *is transitive.*

3.2 Equivalence Relations

3.2.1 Definition of an equivalence relation

Définition 88 *A relation defined in a set* E *is called an equivalence relation if and only if it is: Reflexive, symmetric and transitive. Moreover, if* $u\Re v$, *with* $\Re$ *is an equivalence relationship, then* u *is said to be equivalent to* v *modulo* $\Re$.

Exemple 89 *The relation* $\Re$ *defined on* $\mathbb{Z}$ *by:*

$$x\Re y \Leftrightarrow 3 \text{ divide } (x-y) \text{ is an equivalence relation on } \mathbb{Z}.$$

3.2.2 The equivalence class

Définition 90 *The equivalence class of a given element* u *for an equivalence relation* $\Re$ *defined on* E *is the set of elements* v *equivalent to that element. It is noted as:* $\dot{u}$ *or* $cl(u)$, *with:*

$$\dot{u} = \{v \in E / u\Re v\} \text{ (we can write } v\Re u \text{ because } \Re \text{ is symmetric).}$$

Exemple 91 *Let the equivalence relationship defined on* $\mathbb{Z}$ *by:*

$$x\Re y \Leftrightarrow 3 \text{ divide } (x-y).$$

Then for example:

$$cl(2) = \{x \in \mathbb{Z} / x\Re 2\},$$

$$\begin{aligned}
x\Re 2 &\Leftrightarrow 3 \text{ divide } (x-2) \\
&\Leftrightarrow \exists k \in \mathbb{Z} : x - 2 = 3k, \\
&\Leftrightarrow x = 3k + 2, \\
&\Leftrightarrow cl(2) = \{3k + 2, k \in \mathbb{Z}\}, \\
&\Leftrightarrow cl(2) = \{..., -7, -4, -1, 2, 5, 8, ...\}.
\end{aligned}$$

Properties 3.1 *If $a \in \dot{x}$, then $\dot{a} = \dot{x}$.*

Preuve: (1) If $v \in \dot{a} \Rightarrow v\Re a$ but $a \in \dot{x}$ then $a\Re x$, by transitivity $v\Re x$, which implies that: $v \in \dot{x}$,i.e.: $\dot{a} \subset \dot{x}$.

(2) If $v \in \dot{x} \Rightarrow v\Re x$ but $a \in \dot{x}$ then $x\Re a$, par transitivity $v\Re a$, which implies that: $v \in \dot{a}$,i.e.: $\dot{x} \subset \dot{a}$. ■

3.2.3 Quotient set

Définition 92 *Let E be a set and let $\Re$ an equivalence relation. The quotient set of E by $\Re$ is the set of equivalence classes with respect to $\Re$ and is denoted $E/\Re$. So we have:*

$$E/\Re = \{cl(u), u \in E\}.$$

Proposition 3.1 *The quotient set form a partition of E.*

Preuve: (1) Since $\Re$ is reflexive, we have:

$$\forall u \in E, u\Re u,$$

then $u \in \dot{u}$, which implies that:

$$\forall u \in E, \dot{u} \neq \emptyset.$$

(2) we have:

$$\bigcup_{u \in E} \dot{u} = E.$$

Because the $\dot{u}$ are subsets of E, so $\cup\dot{u} \subset E$ and each element $u \in E$ verifies $u\Re u$ (reflexivity) so $u \in \dot{u} \subset \cup\dot{u}$, which implies that: $E \subset \cup\dot{u}$.

(3) Finally if $\dot{u} \neq \dot{v}$ then $\dot{u} \cap \dot{v} = \emptyset$ because if there is an element $a \in \dot{u} \cap \dot{v}$ we will have $a\Re u$ and $v\Re a$ hence $v\Re u$ because the relation is transitive. Thus $\dot{u} = \dot{v}$ (contradiction). ■

3.3 Order relation

3.3.1 Definition of an order relation

Définition 93 *A binary relation defined in a set E is called an* ***order relation*** *if it is: Reflexive, antisymmetric and transitive.*

Exemple 94 *Let $\Re$ be a relation defined in $\mathbb{N}^*$ by:*

$$p\Re q \Leftrightarrow \left(\exists n \in \mathbb{N}^* \text{ such as } p^n = q\right).$$

(1) The reflexivity, we have:

$$\forall p \in \mathbb{N}^*, p^1 = p \Rightarrow p\Re p \Rightarrow \Re \text{ est reflexive.}$$

(2) Antisymmetry:

$$\forall p, q \in \mathbb{N}^*, p\Re q \text{ and } q\Re p,$$

which implies that:

$$\begin{aligned}
\exists n_1, n_2 &\in \mathbb{N}^*, p^{n_1} = q \text{ and } q^{n_2} = p, \\
&\Rightarrow q^{n_1 n_2} = q \Rightarrow n_1 n_2 = 1 \Rightarrow n_1 = n_2 = 1, \\
&\Rightarrow p = q \Rightarrow \Re \text{ est antisymmetric.}
\end{aligned}$$

(3) Transitivity:

$$\forall p, q, r \in \mathbb{N}^*, p\Re q \text{ and } q\Re r,$$

which implies that:

$$\begin{aligned}
\exists n_1, n_2 &\in \mathbb{N}^*, p^{n_1} = q \text{ and } q^{n_2} = r, \\
&\Rightarrow p^{n_1 n_2} = r, \\
&\Rightarrow \left(\exists m = n_1 n_2 \in \mathbb{N}^* \text{ such as } p^m = r\right), \\
&\Rightarrow p\Re r \Rightarrow \Re \text{ est transitive.}
\end{aligned}$$

Conclusion:

$\Re$ is an order relation because it is reflexive, antisymmetric and transitive.

3.3.2 Total Orders and Partial Orders

Définition 95 *A total order is an order relation in which every pair of elements is comparable. That means,*

$$\forall u, v \in E, u\Re v \text{ or } v\Re u.$$

On the other hand if:

$$\exists u, v \in E \text{ as we have neither } u\Re v \text{ nor } v\Re u.$$

Then $\Re$ is a partial order.

Exemple 96 *Let $\Re$ be a relation defined in $\mathbb{N}^*$ by:*

$$p\Re q \Leftrightarrow (p \text{ divide } q)\,.$$

$\Re$ a partial order because:

$$\text{For } p = 2 \text{ and } q = 3 \text{ we have neither } 2\Re 3 \text{ nor } 3\Re 2.$$

Exemple 97 *Let S be a relation defined in $\mathbb{R}$ by:*

$$pSq \Leftrightarrow p \leq q.$$

S a total order because:

$$\forall p, q \in \mathbb{R}, p \leq q \text{ or } q \leq p, \text{ so } pSq \text{ or } qSp.$$

3.3.3 Upper bound - Lower bound

Définition 98 *Let E be a set with an order relation $\Re$, then:*

(1) M is an upper bound of E, if: $\forall u \in E, u\Re M$.

(2) m is a lower bound of E, if: $\forall u \in E, m\Re u$.

3.3.4 Supremum (least upper bound)- Infimum(greatest lower bound)

Définition 99 *Let E be a set with an order relation $\Re$, then* ***supremum*** *of a set E is the smallest of the upper bound of E (we use $\Re$), noted $\sup E$. On the other hand the* ***infimum*** *is the largest of the lower bound of E (we use $\Re$), noted $\inf E$. Otherwise one has:*
(1) $\forall M$ an upper bound of E, $(\sup E)\,\Re M$.
(2) $\forall m$ a lower bound of E, $m\Re\,(\inf E)$.

3.3.5 Maximum - minimum

Définition 100 *Let E be a set with an order relation $\Re$, then if the supremum of a set E belongs to E, then the maximum exists and it is equal to the supremum of E, if not the maximum does not exist.*
On the other hand if the infimum of a set E belongs to E, then the minimum exists and it is equal to the infimum of E, if not the minimum does not exist.
The maximum is noted by: $\max E$ and the minimum by: $\min E$.

Exemple 101 *In $I = [2, 5[$ provided with an order relation $\Re$ defined by:*

$$x\Re y \Leftrightarrow x \leq y.$$

(1) $\Re$ is a total order and we have:

$$]-\infty, 2] \text{ is the set of the upper bound of } I.$$
$$[5, +\infty[\text{ is the set of the lower bound of } I.$$

(2) We have: $\sup I = 5$ and $\inf I = 2$.
(3) $\sup I = 5 \notin I \Rightarrow \max I$ doesn't exist.
(4) $\inf\ I = 2 \in I \Rightarrow \min I = 2$.