5

Real-valued function with real variable
Asymptotic expansion

5.1. Taylor's formulas.

Theorem 5.1 (Taylor's formula with Lagrange's remainder*). *Let* $]a, b[\subset \mathbb{R}$ *and* $f \in \mathscr{C}^{n+1}(]a, b[, \mathbb{R})$. *Then, for every* $x, x_0 \in]a, b[$ *there exists* c *between* x *and* x_0 *such that*

$$f(x) = f(x_0) + \left(\sum_{k=1}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)\right) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c). \tag{5.1}$$

* When $x_0 = 0$, Taylor's formula is also know as MacLaurin's formula.

Example 5.1. Let the function $f(x) = e^x$, see that f is $\mathscr{C}^\infty$ over $\mathbb{R}$ and $f^{(k)}(x) = e^x$ for any $k \in \mathbb{N}^*$ and every $x \in \mathbb{R}$. Thus, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have the existence of c between 0 and x such that

$$f(x) = f(0) + \left(\sum_{k=1}^{n} \frac{(x-0)^k}{k!} f^{(k)}(0)\right) + \frac{(x-0)^{n+1}}{(n+1)!} f^{(n+1)}(c),$$

therefore

$$e^x = \underbrace{1 + \left(\sum_{k=1}^{n} \frac{x^k}{k!}\right)}_{S} + \underbrace{\frac{x^{n+1}}{(n+1)!} e^c}_{R}.$$

See that the sum S approximates e^x near $x_0 = 0$, and R represents this approximation's error. If we work over $[-1, +1]$, the fact that c is between 0 and x gives $c \in]-1, +1[$; hence

$$|R| = \left|\frac{x^{n+1}}{(n+1)!} e^c\right| \leq \frac{1}{(n+1)!} \leq \frac{1}{2^n}. \tag{5.2}$$

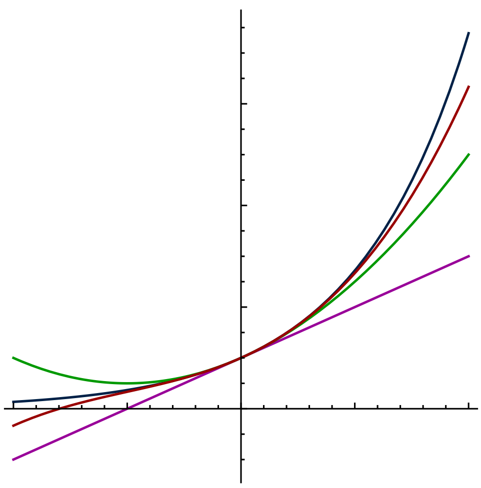

FIGURE 5.1

Thus, to approximate e^x over $[-1, +1]$ with a precession of $10^{-\alpha}$, it suffices to impose the condition $1/2^n \leq 10^{-\alpha}$ (this leads immediately to $|R| \leq 10^{-\alpha}$); hence, $n \geq \alpha \mathrm{Ln}(10)/\mathrm{Ln}(2)$. The value

$$N = \mathrm{floor}\left(\alpha \frac{\mathrm{Ln}(10)}{\mathrm{Ln}(2)}\right) + 1,$$

represents the minimum iteration to do to reach an approximative value of e^x with an error less than $10^{-\alpha}$ for any $x \in [-1, +1]$.

Figure 5.1 represent over $[-1, +1]$ the following functions

- e^x
- $1 + x + (x^2/2!) + (x^3/3!)$
- $1 + x + (x^2/2!)$
- $1 + x$

△

Remark 5.1. *Since c is between x and x_0 ; then, there exists $\theta \in]0,1[$ such that $c = (1-\theta)x + \theta x_0$. Hence, the expression (5.1) becomes as*

$$f(x) = f(x_0) + \left(\sum_{k=1}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) \right) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}\big((1-\theta)x + \theta x_0\big).$$

Definition 5.1 (Landau's notation *). *Let f and g be functions defined on some neighbourhood $v(a)$ of $a \in \mathbb{R} \cup \{-\infty; +\infty\}$. If g does not vanish on $v(a)$, we say that f is negligible compared to g if*

$$\operatorname*{Lim}_{n \to a} \frac{f(x)}{g(x)} = 0,$$

and we write $f(x) \underset{x\to a}{=} \mathrm{o}(g(x))$ (read "$f(x)$ is little−o of $g(x)$" near a). If $g(x) = (x-a)^n$, we write $f(x) = \mathrm{o}\big((x-a)^n\big)$.

* Know also as "Bachmann−Landau notation", as well as "asymptotic notation".

Example 5.2. Let f, g be the functions defined near zero as

$$f(x) = \sin(x) - x + \frac{x^3}{6}, \quad g(x) = x^3.$$

By applying L'Hôpital's rule, we get

$$\operatorname*{Lim}_{x\to 0} \frac{f(x)}{g(x)} = \operatorname*{Lim}_{x\to 0} \frac{\sin(x) - x}{x^3} + \frac{1}{6} = \operatorname*{Lim}_{x\to 0} \frac{\cos(x) - 1}{3x^2} + \frac{1}{6} = \operatorname*{Lim}_{x\to 0} -\frac{\sin(x)}{6x} + \frac{1}{6} = 0.$$

Thus, $f = \mathrm{o}(g)$; hence

$$\sin(x) = x - \frac{x^3}{6} + \mathrm{o}(x^3).$$

△

Lemma 5.1.

(a) Let $\lambda, x, x_0 \in \mathbb{R}^*$ and $n, m \in \mathbb{N}$ such that $n \leq m$, we set $x_1 = x - x_0$. Then

$$\lambda \mathrm{o}(x_1^n) = \mathrm{o}(x_1^n), \quad \mathrm{o}(x_1^n) + \mathrm{o}(x_1^m) = \mathrm{o}(x_1^n), \quad \mathrm{o}(x_1^n)\mathrm{o}(x_1^m) = \mathrm{o}(x_1^{n+m}) = \mathrm{o}(x_1^n) = \mathrm{o}(x_1^m).$$

(b) For any $x_0 \in \mathbb{R}$ and X near x_0 we have

$$\int_{x_0}^{X} \mathrm{o}\big((x-x_0)^n\big) dx = \mathrm{o}\big((X-x_0)^{n+1}\big). \tag{5.3}$$

(c) For any bounded function g near x_0 we have

$$g(x)\, \mathrm{o}\big((X-x_0)^{n+1}\big) = \mathrm{o}\big((X-x_0)^{n+1}\big). \tag{5.4}$$

Theorem 5.2 (Taylor−Young's formula). *Let $x_0 \in \mathbb{R}$, $v(x_0)$ be a neighbourhood and f be a continuous function n−times differentiable over $v(x_0)$. Then*

$$\forall\, x \in v(x_0) : \quad f(x) = f(x_0) + \left(\sum_{k=1}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) \right) + \mathrm{o}\big((x-x_0)^n\big).$$

If the function f is differentiable n−times at x_0 only (and not near x_0), the induction proof of Theorem 5.2 does not work. More precisely, if we consider the previous hypotheses, there is a problem in applying the induction for the case $n = 2$. Indeed, in this case, contrary to the argument mentioned above, we do not know if $f^{(1)}$ is continuous (while for $n > 2$, there is no problem since f is differentiable $(n-1)$−times, thus $f^{(1)}$ is $(n-2)$−times differentiable, hence continuous for any $n \geq 2$). Theorem 5.3 gives same result as the Theorem 5.2 with fewer constraints (Theorem 5.2 becomes a particular case of Theorem 5.3).

Theorem 5.3 (Taylor–Young's formula with less conditions). *Let $x_0 \in \mathbb{R}$, $v(x_0)$ be a neighbourhood of x_0 and f be a continuous function $(n-1)$–times differentiable on $v(x_0)$ and n–times differentiable at x_0. Then*

$$\forall\, x \in v(x_0): \quad f(x) = f(x_0) + \left(\sum_{k=1}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) \right) + \mathrm{o}\big((x-x_0)^n\big).$$

The following table summarises Taylors-Young's formulas near $x_0 = 0$ of the usual function.

$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \mathrm{o}(x^n) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \mathrm{o}(x^3)$
$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + \mathrm{o}(x^n) = 1 + x + x^2 + x^3 + \mathrm{o}(x^3)$
$\mathrm{Ln}(1-x) = -\sum_{k=0}^{n} \frac{x^k}{k} + \mathrm{o}(x^n) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \mathrm{o}(x^3)$
$(1+x)^\alpha = \sum_{k=0}^{n} \frac{\alpha_k}{k!} x^k + \mathrm{o}(x^n), \quad \alpha_0 = 1, \quad \forall k \in \mathbb{N}_n^*: \quad \alpha_k = \prod_{l=0}^{k-1} (\alpha - l)$
$\sqrt{1+x} = 1 + \sum_{k=1}^{n} \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}(k-1)!k!} x^k + \mathrm{o}(x^n) = 1 + \frac{x}{2} - \frac{3}{8}x^2 + \frac{1}{16}x^3 + \mathrm{o}(x^3)$
$\sin(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \mathrm{o}(x^{2n+1}) = x - \frac{x^3}{3!} + \mathrm{o}(x^3)$
$\cos(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + \mathrm{o}(x^{2n}) = 1 - \frac{x^2}{2!} + \mathrm{o}(x^3)$
$\mathrm{ch}(x) = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + \mathrm{o}(x^{2n}) = 1 + \frac{x^2}{2!} + \mathrm{o}(x^3)$
$\mathrm{sh}(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + \mathrm{o}(x^{2n+1}) = x + \frac{x^3}{3!} + \mathrm{o}(x^3)$
$\arcsin(x) = \sum_{k=0}^{n} \frac{(2k)!}{4^k (k!)^2} \frac{x^{2k+1}}{2k+1} + \mathrm{o}(x^{2n+2}) = x + \frac{x^3}{6} + \mathrm{o}(x^3)$
$\arccos(x) = \frac{\pi}{2} - \sum_{k=0}^{n} \frac{(2k)!}{4^k (k!)^2} \frac{x^{2k+1}}{2k+1} + \mathrm{o}(x^{2n+2}) = \frac{\pi}{2} - x - \frac{x^3}{6} + \mathrm{o}(x^3)$
$\arctan(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + \mathrm{o}(x^{2n+1}) = x - \frac{x^3}{3} + \mathrm{o}(x^3)$
$\mathrm{arcsh}(x) = \sum_{k=0}^{n} (-1)^k \frac{(2k)!}{4^k (k!)^2} \frac{x^{2k+1}}{2k+1} + \mathrm{o}(x^{2n+2}) = x - \frac{x^3}{6} + \mathrm{o}(x^3)$
$\mathrm{arcth}(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{2k+1} + \mathrm{o}(x^{2n+1}) = x + \frac{x^3}{3} + \mathrm{o}(x^3)$

TABLE 5.2 : Taylor–Young's asymptotic table

5.2. Asymptotic expansion.

Let f be the function defined over $\mathbb{R}$ by the following expression

$$\forall x \in \mathbb{R}^* : \quad f(x) = x + x^3 \sin\left(\frac{1}{x}\right), \quad f(0) = 0.$$

On the one hand, f is continuous over $\mathbb{R}^*$ and $f(x)$ goes to $0 = f(0)$ as x does; thus f is a $\mathscr{C}^0$ function over $\mathbb{R}$. On the other hand,

$$\forall x \in \mathbb{R}^* : \quad f^{(1)}(x) = 1 + 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right), \quad \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 1,$$

therefore, f is differentiable over $\mathbb{R}$ and

$$\forall x \in \mathbb{R}^* : \quad f^{(1)}(x) = 1 + 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right), \quad f^{(1)}(0) = 1.$$

See that $f^{(1)}$ is continuous over $\mathbb{R}^*_-$, over $\mathbb{R}^*_+$ and $f^{(1)}(x)$ goes to $1 = f^{(1)}(0)$ as x does; thus $f^{(1)}$ is a a $\mathscr{C}^0$ function over $\mathbb{R}$; thus, f is $\mathscr{C}^1$ over $\mathbb{R}$. On the other hand

$$\lim_{x\to 0} \frac{f^{(1)}(x) - f^{(1)}(0)}{x - 0} = \lim_{x\to 0} 3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) = -\lim_{x\to 0} \cos\left(\frac{1}{x}\right).$$

The previous limit does not exists; indeed, let $(u_n)_n$ and $(v_n)_n$ be the sequences defined as

$$\forall n \in \mathbb{N}^* : \quad u_n = \frac{2}{(4n+1)\pi}, \quad v_n = \frac{1}{2\pi n}.$$

See that both previous sequences go to zero as n goes to infinity and

$$\lim_{n\to+\infty} \cos\left(\frac{1}{u_n}\right) = 0 \neq 1 = \lim_{n\to+\infty} \cos\left(\frac{1}{v_n}\right).$$

This ensures that $\cos(1/x)$ does not have at zero. Thus, f does not have a second derivative at zero. Hence, we can not apply Taylor's formula to approximate f with a polynomial of degree two near zero. Now, see that f can be rewritten as

$$f(x) = x + x^2 \varepsilon(x), \quad \varepsilon(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & if \ x \neq 0 \\ 0 & if \ x = 0 \end{cases}$$

moreover, $\varepsilon(x)$ goes to zero as x does; thus $f(x) = x + \mathrm{o}(x^2)$. In conclusion, there exist functions that do not satisfy the condition of Taylor's theorem but can be approximated by polynomials.

Definition 5.2 (asymptotic expansion). *Let f be a real-valued function with a real variable and $x_0 \in \mathbb{R}$, we say that f has an asymptotic expansion of order $n \in \mathbb{N}$ near x_0, if there exists a neighbourhood $v(x_0)$ of x_0 and constants c_k with $k \in \mathbb{N}_n$ such as*

$$\forall x \in v(x_0) : \quad f(x) = \sum_{k=0}^{n} c_k (x - x_0)^k + \mathrm{o}\big((x - x_0)^n\big). \tag{5.5}$$

The previous definition affirms that if f has an asymptotic expansion near x_0, then f has a finite limit at x_0 equal to c_0; therefore, if the limit of f at x_0 does not exist or is infinite, then f does not have an asymptotic expansion near x_0. On the other hand, Lemma 5.1 page 2 (*assertion (a)*) ensures that $\mathrm{o}\big((x - x_0)^n\big) = \mathrm{o}\big((x - x_0)^p\big)$ for every $p \in \mathbb{N}$ such as $p \leq n$; Thus, if f has an asymptotic expansion near x_0 given by the expression (5.5); then, f has an asymptotic expansion of order p near x_0 for every integer $p \leq n$ and

$$f(x) = \sum_{k=0}^{p} c_k (x - x_0)^k + \mathrm{o}\big((x - x_0)^p\big).$$

It is easy to notice that if f satisfying Taylor's theorem conditions; then, f has an asymptotic expansion.

Remark 5.2. *The terminology "asymptotic expansion of order n of f near x_0" refers to the formulas*

$$f(x) = \sum_{k=1}^{n} c_k \psi_k(x) + \mathrm{o}\big((x-x_0)^n\big),$$

such as $\psi_1, \cdots, \psi_n$ are given functions and $c_1, \cdots, c_n$ are constants. We consider the case where $\psi_k(x) = (x-x_0)^k$; hence, in this chapter, the "asymptotic expansion of order n of f near x_0" terminology refers to the expression (5.5).

Theorem 5.4 (uniqueness of asymptotic expansion). *Let f be a real-valued function with a real variable and $x_0 \in \mathbb{R}$. Then, asymptotic expansion of order n near x_0 of f, when it exists, is unique.*

It is easy to check that if f has an asymptotic expansion of order one near x_0 of the form

$$f(x) = c_0 + c_1(x-x_0) + \mathrm{o}(x-x_0),$$

then c_0 is the value of f (respectively : of the continuous extension of f) at x_0 and a_1 is the derivative of f (respectively : the derivative of the continuous extension of f) at x_0. On the other hand, if f has an asymptotic expansion of order two near x_0 of the form

$$f(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \mathrm{o}\big((x-x_0)^2\big),$$

we can't affirm that f has a second derivative at x_0 (see the example at the beginning of the current section).

Proposition 5.1. *Let $x_0, \lambda \in \mathbb{R}$ and f, g be functions have asymptotic expansion of order n near x_0 of the form*

$$f(x) = \sum_{k=0}^{n} c_k(x-x_0)^k + \mathrm{o}\big((x-x_0)^n\big), \quad g(x) = \sum_{k=0}^{n} d_k(x-x_0)^k + \mathrm{o}\big((x-x_0)^n\big). \tag{5.6}$$

Then

(a) λf has asymptotic expansion of order n near x_0 and

$$(\lambda f)(x) = \sum_{k=0}^{n} \lambda c_k(x-x_0)^k + \mathrm{o}\big((x-x_0)^n\big).$$

(b) $f+g$ has asymptotic expansion of order n near x_0 and

$$(f+g)(x) = \sum_{k=0}^{n} (c_k+d_k)(x-x_0)^k + \mathrm{o}\big((x-x_0)^n\big).$$

(c) fg has asymptotic expansion of order n near x_0 and

$$(fg)(x) = \left(\sum_{k=0}^{n} c_k(x-x_0)^k\right)\left(\sum_{k=0}^{n} d_k(x-x_0)^k\right) + \mathrm{o}\big((x-x_0)^n\big) \tag{5.7}$$

Example 5.3. Let f and g be defined near 0 by

$$f(x) = \sin(x), \quad g(x) = \cos(x).$$

The asymptotic expansion of order 3 of f and g near zero is

$$f(x) = \sin(x) = x - \frac{x^3}{3!} + \mathrm{o}(x^3), \quad g(x) = \cos(x) = 1 - \frac{x^2}{2!} + \mathrm{o}(x^3).$$

– *Asymptotic expansion of order 3 near 0 of $f+g$:*

$$f(x) + g(x) = \left(x - \frac{x^3}{3!}\right) + \left(1 - \frac{x^2}{2!}\right) + \mathrm{o}(x^3) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \mathrm{o}(x^3).$$

– *Asymptotic expansion of order 3 near 0 of fg :*

$$f(x)g(x) = \left(x - \frac{x^3}{3!}\right)\left(1 - \frac{x^2}{2!}\right) + \mathrm{o}(x^3) = x - \frac{x^3}{2!} - \frac{x^3}{3!} + \frac{x^5}{3!2!} + \mathrm{o}(x^3) = x - \frac{4}{3}x^3 + \mathrm{o}(x^3).$$

△

Proposition 5.2. *Let $x_0 \in \mathbb{R}$ and f be a function satisfies $f(x_0) \neq 0$ and has asymptotic expansion of order n near x_0 of the form*

$$f(x) = \sum_{k=0}^{n} c_k (x - x_0)^k + \mathrm{o}\big((x - x_0)^n\big). \tag{5.8}$$

Then

$$\frac{1}{f(x)} = c_0 \sum_{l=0}^{n} (-1)^l \left(\sum_{k=1}^{n} \frac{c_k}{c_0} (x - x_0)^k \right)^l + \mathrm{o}\big((x - x_0)^n\big). \tag{5.9}$$

Example 5.4. We want to find the asymptotic expansion of order 3 near 0 of the function f defined near zero by the expression :

$$f(x) = \frac{1}{2 - \sin(x)}.$$

Since $X = \sin(x)/2$ goes to zero as x does, and when X is near 0 we have

$$\frac{1}{1 - X} = 1 + X + X^2 + X^3 + \mathrm{o}(X^3),$$

then

$$f(x) = \frac{1}{2} \frac{1}{1 - \frac{\sin(x)}{2}} = \frac{1}{2}\left[1 + \frac{\sin(x)}{2} + \left(\frac{\sin(x)}{2}\right)^2 + \left(\frac{\sin(x)}{2}\right)^3\right] + \mathrm{o}(x^3). \tag{5.10}$$

Using the asymptotic expansion of order 3 near 0 of $\sin(x)$ (see Table 5.2) transforms (5.10) to

$$\begin{aligned}
f(x) &= \frac{1}{2}\left[1 + \frac{1}{2}\left(x - \frac{x^3}{3!}\right) + \frac{1}{2^2}\left(x - \frac{x^3}{3!}\right)^2 + \frac{1}{2^3}\left(x - \frac{x^3}{3!}\right)^3\right] + \mathrm{o}(x^3) \\
&= \frac{1}{2}\left[1 + \frac{1}{2}\left(x - \frac{x^3}{3!}\right) + \frac{1}{2^2}\left(x^2 + \frac{x^6}{3!^2} - 2\frac{x^5}{3!}\right) + \frac{1}{2^3}\left(x^3 - 3\frac{x^5}{3!} + 3\frac{x^7}{3!^2} - \frac{x^9}{3!^3}\right)\right] + \mathrm{o}(x^3) \\
&= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{48} + \mathrm{o}(x^3).
\end{aligned}$$

△

Proposition 5.3. *Let $x_0 \in \mathbb{R}$, g be a function has asymptotic expansion of order n near x_0 and f be a function has asymptotic expansion of order m near $y_0 = g(x_0)$. We set*

$$f(y) = \sum_{l=0}^{n} c_l (y - y_0)^l + \mathrm{o}\big((y - y_0)^n\big), \quad g(x) = \sum_{k=0}^{n} d_k (x - x_0)^k + \mathrm{o}\big((x - x_0)^n\big). \tag{5.11}$$

Then $f \circ g$ has asymptotic expansion of order n near x_0 and

$$(f \circ g)(x) = \sum_{k=0}^{n} c_k \left(\sum_{l=1}^{n} d_l (x - x_0)^l \right)^k + \mathrm{o}\big((x - x_0)^n\big). \tag{5.12}$$

Example 5.5. We want to find the asymptotic expansion of order 3 near 0 of $f \circ g$ such that

$$f(x) = \sin(x), \quad g(x) = \cos(x).$$

The asymptotic expansion of order 3 of f and g near zero is

$$f(x) = \sin(x) = x - \frac{x^3}{3!} + \mathrm{o}(x^3), \quad g(x) = \cos(x) = 1 - \frac{x^2}{2!} + \mathrm{o}(x^3). \tag{5.13}$$

Thus

$$(f \circ g)(x) = \sin\left(1 - \frac{x^2}{2!} + \mathrm{o}(x^3)\right) = \sin\left(1 - \frac{x^2}{2!}\right) + \mathrm{o}(x^3).$$

See that $1 - x^2/2!$ does not go to zero as x does; therefore, we can not apply the expansion of f given by (5.13). For solve this problem recall that $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$, thus

$$\big(f \circ g\big)(x) = \sin\left(1 - \frac{x^2}{2!}\right) + \mathrm{o}(x^3) = \sin(1)\cos\left(\frac{x^2}{2!}\right) + \cos(1)\sin\left(\frac{x^2}{2!}\right) + \mathrm{o}(x^3).$$

Using the expression (5.13) we get

$$\begin{aligned}\big(f \circ g\big)(x) &= \sin(1)\left[1 - \frac{1}{2!}\left(\frac{x^2}{2!}\right)^2\right] + \cos(1)\left[\frac{x^2}{2!} - \frac{1}{3!}\left(\frac{x^2}{2!}\right)^3\right] + \mathrm{o}(x^3)\\ &= \sin(1) - \frac{\sin(1)}{8}x^4 + \frac{\cos(1)}{2!}x^2 - \frac{\cos(1)}{24}x^6 + \mathrm{o}(x^3)\\ &= \sin(1) + \frac{\cos(1)}{2!}x^2 + \mathrm{o}(x^3).\end{aligned}$$

△

Proposition 5.4. *Let $x_0 \in \mathbb{R}$ and f be a function that has asymptotic expansion of order n near x_0 of the form*

$$f(x) = \sum_{k=0}^{n} c_k (x - x_0)^k + \mathrm{o}\big((x - x_0)^n\big). \tag{5.14}$$

Then

$$\int_{x_0}^{X} f(x)dx = \sum_{k=0}^{n} \frac{c_k}{k+1}(x - x_0)^{k+1} + \mathrm{o}\big((x - x_0)^{n+1}\big). \tag{5.15}$$

Example 5.6. See that near 0 and for any $n \in \mathbb{N}$ we have

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + \mathrm{o}(x^n).$$

Thus

$$\mathrm{Ln}(1 - X) = -\int_0^X \frac{1}{1-x}dx = -\sum_{k=0}^{n}\int_0^X x^k dx + \mathrm{o}(X^{n+1}) = -\sum_{k=0}^{n}\frac{X^{n+1}}{n+1} + \mathrm{o}(X^{n+1}).$$

△

Proposition 5.5. *Let $x_0 \in \mathbb{R}$ and f be a $\mathscr{C}^n$ function on a neighbourhood of x_0 and has asymptotic expansion of order n near x_0 of the form*

$$f(x) = \sum_{k=0}^{n} c_k (x - x_0)^k + \mathrm{o}\big((x - x_0)^n\big). \tag{5.16}$$

Then, $f^{(1)}$ has an asymptotic expansion of order $n-1$ near x_0 of the form

$$f^{(1)}(x) = \sum_{l=1}^{n} k c_k (x - x_0)^{k-1} + \mathrm{o}\big((x - x_0)^{n-1}\big). \tag{5.17}$$

Definition 5.3. *Let $a \in \mathbb{R}^*+$ and f be a function defined over $]-\infty, -a]$, or $[a, +\infty[$ or $]-\infty, -a] \cup [a, +\infty[$. We say that f has an asymptotic expansion of order n near infinity if the function $g(x) = f(1/x)$ has an asymptotic expansion of order n near zero.*

Example 5.7. Let f be the function defined as

$$\forall x \in \mathbb{R}^* \setminus \{-2\}: \quad f(x) = \frac{x^2 - 2}{x^2 + 2x}.$$

We set $X = 1/x$ and we defined the function g as

$$\forall X \in \mathbb{R} \setminus \left\{-\frac{1}{2}\right\} : \quad g(X) = f\left(\frac{1}{X}\right) = \frac{1-2X^2}{1+2X} = 1 - X - X\frac{1}{1+2X}.$$

See that as X goes to zero $2X$ does; thus, near zero we have

$$\frac{1}{1+2X} = \sum_{k=0}^{n}(-2)^k X^k + \mathrm{o}(X^n),$$

hence

$$g(X) = 1 - X - \sum_{k=0}^{n}(-2)^k X^{k+1} + \mathrm{o}(X^{n+1}) = 1 - 2X + \sum_{k=1}^{n}(-2)^k X^{k+1} + \mathrm{o}(X^{n+1}).$$

Now we change X by $1/x$ and we use the information that $g(X) = f(1/X)$ we get

$$f(x) = 1 - \frac{2}{x} + \sum_{k=1}^{n}\frac{(-2)^k}{x^{k+1}} + \mathrm{o}\left(\frac{1}{x^{n+1}}\right).$$

△

5.3. Application of asymptotic expansion.

Definition 5.4. *Let f, g be functions defined near x_0. We say that f is equivalent to g near x_0, and we set $f \underset{x_0}{\sim} g$ (or $f \sim g$ if there is no confusion), if $f(x)/g(x)$ goes to one as x_0 tends to x_0.*

Lemma 5.2. *Let f be a function with an asymptotic expansion of order n near x_0 of the form*

$$f(x) = \sum_{k=0}^{n} a_k(x-x_0)^k + \mathrm{o}\big((x-x_0)^n\big), \quad \exists k \in \mathbb{N}_n : \quad a_k \neq 0.$$

Then, near x_0 we have

$$f(x) \sim \sum_{k=0}^{n} a_k(x-x_0)^k.$$

Proposition 5.6. *Let f, g be functions such that*

$$f(x) = \sum_{k=0}^{n} a_k(x-x_0)^k + \mathrm{o}\big((x-x_0)^n\big), \quad g(x) = \sum_{k=0}^{n} b_k(x-x_0)^k + \mathrm{o}\big((x-x_0)^n\big),$$

such that for some $k_1, k_2 \in \mathbb{N}_n$ we have $a_{k_1} \neq 0$, and $b_{k_2} \neq 0$. Then

$$\operatorname*{Lim}_{x\to x_0}\frac{f(x)}{g(x)} = \operatorname*{Lim}_{x\to x_0}\left(\sum_{k=0}^{n} a_k(x-x_0)^k\right) \Big/ \left(\sum_{k=0}^{n} b_k(x-x_0)^k\right).$$

Remark 5.3. *See that if f and g is as in Proposition 5.6, then we have an indeterminate form when calculating the limit of $f(x)/g(x)$ as x goes to x_0. Thus, the asymptotic expansion combined with the Proposition 5.6 helps remove the indeterminate form $0/0$.*

Example 5.8.

– See that $\big(1-\cos(x)\big)/\sin(x)$ gives an indeterminate form at 0; on the other hand, Table 5.2 page 3 gives

$$\sin(x) = x + \mathrm{o}(x^2), \quad 1-\cos(x) = \frac{x^2}{2} + \mathrm{o}(x^2).$$

Therefore

$$\operatorname*{Lim}_{x\to 0}\frac{1-\cos(x)}{\sin(x)}=\operatorname*{Lim}_{x\to 0}\frac{\frac{x^2}{2}}{x}=0.$$

– See that $\big((1+x)^{1/x}-e\big)/x$ gives an indeterminate form at 0 ; on the other hand, using Table 5.2 we get

$$(1+x)^{1/x}=e^{\mathrm{Ln}(1+x)/x}=e^{1-(x/2)+\mathrm{o}(x)}=ee^{-(x/2)}+\mathrm{o}(x)=e-e\frac{x}{2}+\mathrm{o}(x).$$

Therefore

$$\operatorname*{Lim}_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=\operatorname*{Lim}_{x\to 0}\frac{\left(e-e\frac{x}{2}\right)-e}{x}=\frac{1}{2}.$$

△

Let f be a function that has an asymptotic expansion of order one near x_0 of the form

$$f(x)=c_0+c_1(x-x_0)+\mathrm{o}(x-x_0). \tag{5.18}$$

The function f has a continuous extension at x_0 defined as

$$\tilde{f}(x)=\begin{cases}c_0+c_1(x-x_0)+\mathrm{o}(x-x_0) & if\ x\neq x_0,\\ c_0 & if\ x=x_0.\end{cases} \tag{5.19}$$

If f is defined at x_0, then f equals $\tilde{f}$.

Proposition 5.7. *Let f be a function that has an asymptotic expansion of order one near x_0 as in (5.18) and continuous extension defined as in (5.19). Then, $y=c_0+c_1(x-x_0)$ is the equation of the graph's tangent of $\tilde{f}$ at (x_0,c_0). Moreover, if f has an asymptotic expansion of order k_0 near x_0, for some $k_0\in\mathbb{N}^*\setminus\{1\}$, of the form*

$$f(x)=c_0+c_1(x-x_0)+c_{k_0}(x-x_0)^{k_0}+\mathrm{o}\big((x-x_0)^{k_0}\big),\quad c_{k_0}\neq 0. \tag{5.20}$$

Then

(a) If k_0 is even and $a_{k_0}>0$ (respectively $a_{k_0}<0$), the graph of $\tilde{f}$ is locally above (respectively bellow) the tangent at (x_0,c_0).

(b) If k_0 is odd, then the $\tilde{f}$ curve intersects the tangent at (x_0,c_0) (we say that we have an inflexion point tangent).

Example 5.9. Let f be the function defined over $\mathbb{R}$ as

$$\forall x\in\mathbb{R}^*:\quad f(x)=\frac{\sin(x)}{x},\quad f(0)=1.$$

See that f has an asymptotic expansion of order two near zero of the form :

$$f(x)=1-\frac{x}{2!}+\frac{x2}{3!}+\mathrm{o}(x^2).$$

Thus, the equation of the tangent to the curve of f at $(0,f(0))=(0,0)$ is $y=1-(x/2)$; on the other hand, since the coefficient of $x^2/3!$ is positive near zero ; then, locally near zero the curve of f is above this tangent. △