Sets

Solution Exercise 1. *A, B, C subsets of a set E.*

1. *To show this equivalence $A \cup B = B \Longleftrightarrow A \subset B$, we will show the two following implications :*
   (a) *Let's show first that : $A \cup B = B \Longrightarrow A \subset B$: Let $x \in E$ such that $x \in A$.*
   $$\begin{aligned} x \in A &\Longrightarrow x \in A \cup B \text{ because } A \cup B = \{x \in E, x \in A \text{ or } x \in B\} \\ &\Longrightarrow x \in B \text{ because the hypothasis is : } A \cup B = B. \end{aligned}$$
   hence $A \subset B$.
   (b) *let's show now the second implication : $A \subset B \Longrightarrow A \cup B = B$:*
      i. *Let $x \in E$ such that $x \in A \cup B$, we have then to show that $x \in B$ to deduce that $A \cup B \subset B$.*
      $$\begin{aligned} x \in A \cup B &\Longrightarrow x \in A \text{ or } x \in B \\ &\Longrightarrow x \in B \text{ because } A \subset B \text{ by hypothesis} \end{aligned}$$
      hence $A \cup B \subset B$.
      ii. *Let $x \in E$ such that $x \in B$. Let's show that $B \subset A \cup B$.*
      $$\begin{aligned} x \in B &\Longrightarrow x \in B \cup A \text{ this is true for any set } A. \\ &\Longrightarrow x \in A \cup B. \end{aligned}$$
      hence $B \subset A \cup B$.
   Conclusion $A \cup B = B$.
   The equivalence is proved.
2. *For this implication too $A \cap B = A \Longleftrightarrow A \subset B$, we will show two implications :*
   (a) *Let's first show that : $A \cap B = A \Longrightarrow A \subset B$: Let $x \in E$ such that $x \in A$.*
   $$\begin{aligned} x \in A &\Longrightarrow x \in A \cap B \text{ because it's assumed that } A = A \cap B \\ &\Longrightarrow x \in B \text{ by definition of the intersection} \end{aligned}$$
   Hence $A \subset B$.
   (b) *Let's now show that : $A \subset B \Longrightarrow A \cap B = A$. Let $x \in E$ such that $x \in A \cap B$.*
   $$\begin{aligned} x \in A \cap B &\Longrightarrow x \in A \text{ and } x \in B \\ &\Longrightarrow x \in A \end{aligned}$$
   then, $A \cap B \subset A$.
   $$\begin{aligned} x \in A &\Longrightarrow x \in B \text{ because } A \subset B \\ &\Longrightarrow x \in A \cap B \text{ because we had } x \in A \text{ at the beginning, and we obtained } x \in B \end{aligned}$$
   then $A \subset A \cap B$. Hence, $A \cap B = A$.
3. *Now we have to prove that : $A \cup B = A \cap C \Longleftrightarrow B \subset A \subset C$.*
   (a) *Let's start by the implication : $A \cup B = A \cap C \Longrightarrow B \subset A \subset C$. Let $x \in E$ such that $x \in B$.*
   $$\begin{aligned} x \in B &\Longrightarrow x \in A \cup B \text{ it's true with any subset } A \\ &\Longrightarrow x \in A \cap C \text{ because of the hypothesis} \\ &\Longrightarrow x \in A \text{ because of the definition of the intersection} \end{aligned}$$

then $B \subset A$. Let now $x \in E$ such that $x \in A$.

$$\begin{aligned} x \in A &\Longrightarrow x \in A \cup B \text{ it's true with any subset } B \\ &\Longrightarrow x \in A \cap C \text{ because of the hypothesis} \\ &\Longrightarrow x \in C \text{ because of the definition of the intersection} \end{aligned}$$

then $A \subset C$. Hence $B \subset A \subset C$.

(b) *Let's show now : $B \subset A \subset C \Longrightarrow A \cup B = A \cap C$.*
(b.1) Let's show that $A \cup B \subset A \cap C$. Let $x \in E$ such that $x \in A \cup B$.

$$\begin{aligned} x \in A \cup B &\Longrightarrow x \in A \text{ or } x \in B \\ &\Longrightarrow x \in A \text{ or } x \in A \text{ because } B \subset A \\ &\Longrightarrow x \in A \\ &\Longrightarrow x \in A \cap C \text{ because } A \subset C \Longleftrightarrow A = A \cap C (\text{above example}). \end{aligned}$$

then $A \cup B \subset A \cap C$.
(b.2) Let's show that $A \cap C \subset A \cup B$. Let $x \in E$ such that $x \in A \cap C$.

$$\begin{aligned} x \in A \cap C &\Longrightarrow x \in A \\ &\Longrightarrow x \in A \cup B, \text{ it's true with any subset } B \end{aligned}$$

then $A \cap C \subset A \cup B$. Hence $A \cup B = A \cap C$.

The equivalence is demonstrated.

4. *Now we deal with :* $\begin{cases} A \cap B = A \cap C \\ A \cup B = A \cup C \end{cases} \Longleftrightarrow B = C.$

(a) *Let's show the first implication "$\Longrightarrow$". Let $x \in E$ such that $x \in B$.*
(a.1) Let's show that $B \subset C$.

$$\begin{aligned} x \in B &\Longrightarrow x \in A \cup B \\ &\Longrightarrow x \in A \cup C, \text{ following the hypothesis} \\ &\text{but we started with } x \in B \text{, so, we obtain the next implication} \\ &\Longrightarrow x \in B \text{ and } x \in A \cup C \\ &\Longleftrightarrow x \in B \cap (A \cup C) \\ &\Longrightarrow x \in (B \cap A) \cup (B \cap C), \text{ distributivity of intersection over union} \\ &\Longrightarrow x \in (A \cap C) \cup (B \cap C), \text{ by hypothesis} \\ &\Longrightarrow x \in (A \cup B) \cap C, \text{ distributivity of intersection over union (inverted)} \\ &\Longrightarrow x \in C, \end{aligned}$$

then $B \subset C$.
(a.2) Let's show that $C \subset B$.

$$\begin{aligned} x \in C &\Longrightarrow x \in A \cup C \\ &\Longrightarrow x \in A \cup B, \text{ following the hypothesis} \\ &\text{but we started with } x \in C \text{, so, we obtain the next implication} \\ &\Longrightarrow x \in C \text{ and } x \in A \cup B \\ &\Longleftrightarrow x \in C \cap (A \cup B) \\ &\Longrightarrow x \in (C \cap A) \cup (C \cap B), \text{ distributivity of intersection over union} \\ &\Longrightarrow x \in (A \cap B) \cup (C \cap B), \text{ by hypothesis} \\ &\Longrightarrow x \in (A \cup C) \cap B, \text{ distributivity of intersection over union (inverted)} \\ &\Longrightarrow x \in B, \end{aligned}$$

then $C \subset B$. Hence $B = C$.

(b) *Now the other implication* $\Longleftarrow$.

$$B = C \Longrightarrow A \cup B = A \cup C$$
$$\text{and}$$
$$B = C \Longrightarrow A \cap B = A \cap C$$

5. *Let's now show the famous De Morgan's laws :* $\complement_E(A \cup B) = \complement_E A \cap \complement_E B$ *and* $\complement_E(A \cap B) = \complement_E A \cup \complement_E B$.
   Let $x \in E$ *such that* $x \in \complement_E(A \cup B)$.

$$\begin{aligned} x \in \complement_E(A \cup B) &\Longleftrightarrow x \notin A \cup B \\ &\Longleftrightarrow (x \notin A) \text{ and } (x \notin B) \\ &\Longleftrightarrow (x \in \complement_E A) \text{ and } (x \in \complement_E B) \\ &\Longleftrightarrow x \in \complement_E A \cap \complement_E B \end{aligned}$$

   Hence $\complement_E(A \cup B) = \complement_E A \cap \complement_E B$.
   Let $x \in E$ *such that* $x \in \complement_E(A \cap B)$.

$$\begin{aligned} x \in \complement_E(A \cap B) &\Longleftrightarrow x \notin A \cap B \\ &\Longleftrightarrow (x \notin A) \text{ or } (x \notin B) \\ &\Longleftrightarrow (x \in \complement_E A) \text{ or } (x \in \complement_E B) \\ &\Longleftrightarrow x \in \complement_E A \cup \complement_E B \end{aligned}$$

   Hence $\complement_E(A \cap B) = \complement_E A \cup \complement_E B$.

Solution Exercise 2.

$$E = \{-5, -1.1, \pi, 10\}$$

1. *We have* $cardE = 4$, *hence* $card\mathcal{P}(E) = 2^4 = 16$.
2. $\mathcal{P}(E) = \{\{-5\}, \{-1.1\}, \{\pi\}, \{10\}, \{-5, -1.1\}, \{-5, \pi\}, \{-5, 10\}, \{-1.1, \pi\}, \{-1.1, 10\}, \{\pi, 10\}, \{-5, -1.1, \pi\}, \{-5, -1.1, 10\}, \{-5, \pi, 10\}, \{-1.1, \pi, 10\}, \varnothing, E\}$
3. *The symboles and notations* $\in, \notin, \subset, \not\subset, \cap, \cup, \varnothing, E, =$, *are put as follows :*

$$-3 \notin E,\ -1.1 \notin \mathcal{P}(E),\ \{\pi\} \subset E,\ \{10\} \in \mathcal{P}(E),\ \varnothing \subset E,\ \varnothing \in \mathcal{P}(E),\ \{-5, \pi, 10\} \in \mathcal{P}(E)$$

$$\{-3, -1.1\} \cap \{-1.1, \pi\} = \{-1.1\},\ \{\{\pi\}\} \subset \mathcal{P}(E),\ \{-3, -1.1\} \not\subset E,\ \{\varnothing\} \subset \mathcal{P}(E)$$

Applications

Solution Exercise 3.

$$\begin{aligned} f :\,]-\sqrt{3}, +\infty[&\to \mathbb{R}_+ \\ x &\mapsto f(x) = x^2 - 3 \end{aligned}$$

1. $f(A) = \{f(x) \in \mathbb{R}_+, x \in A\}$ *and* $A = \{-\sqrt{2}, 0.5, 3, \pi, 8\}$
   Let's calculate the images of the elements of A by the application f :
   $f(-\sqrt{2}) = -1,\ f(0.5) = -2.75,\ f(3) = 6,\ f(\pi) = \pi - 3,\ f(8) = 61$ *but* $f(-\sqrt{2}) = -1 \notin \mathbb{R}_+$ *and* $f(0.5) = -2.75 \notin \mathbb{R}_+$ *and all the other images belong to* $\mathbb{R}_+$. *So :*

$$\begin{aligned} f(A) &= \{f(3), f(\pi), f(8)\} \\ &= \{6, \pi - 3, 61\} \end{aligned}$$

2. $f^{-1}(B) = f^{-1}([0,1]) = \{x \in]-\sqrt{3}, +\infty[, f(x) \in [0,1]\}$.
Let $x \in f^{-1}(B) = f^{-1}([0,1])$,

$$\begin{aligned} x \in f^{-1}([0,1]) &\Longleftrightarrow f(x) \in [0,1] \\ &\Longleftrightarrow 0 \leq f(x) \leq 1 \\ &\Longleftrightarrow 0 \leq x^2 - 3 \leq 1 \\ &\Longleftrightarrow 3 \leq x^2 \leq 4 \\ &\Longleftrightarrow -2 \leq x \leq -\sqrt{3} \text{ or } \sqrt{3} \leq x \leq 2 \\ &\Longleftrightarrow x \in [-2, -\sqrt{3}] \cup [\sqrt{3}, 2] \end{aligned}$$

but $[-2, -\sqrt{3}] \not\subset]-\sqrt{3}, +\infty[$ *Therefore*

$$f^{-1}(B) = [\sqrt{3}, 2]$$

Solution Exercise 4.

1. $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ *defined by :* $\forall x \in \mathbb{R}, f(x) = |x|$.
This application is not injective because for $x = -1 \in \mathbb{R}$ *and* $x' = 1 \in \mathbb{R}$ *we obtain* $f(-1) = f(1) = 1$. *This application is surjective. Indeed, let be* $y \in \mathbb{R}_+$ *then* $y = |x| \Longrightarrow y = x$ *or* $y = -x$, *it follows* $x = y$ *or* $x = -y$, *hencce* $\forall y \in \mathbb{R}_+, \exists x(= y) \in \mathbb{R},\ y = |x|$. *This application is not bijective because it is not injective.*
2. $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *defined by :* $\forall x \in \mathbb{R}_+, h(x) = x^2$.
This application is injective, let's prove it. Let be $x_1, x_2 \in \mathbb{R}_+$ *such that* $h(x_1) = h(x_2)$.

$$\begin{aligned} h(x_1) = h(x_2) &\Longrightarrow x_1^2 = x_2^2 \\ &\Longrightarrow \begin{cases} x_1 = x_2 \in \mathbb{R}_+ \\ \quad or \\ x_1 = -x_2 \notin \mathbb{R}_+ \end{cases} \\ &\Longrightarrow x_1 = x2. \end{aligned}$$

This application is not surjective. Let be $y = -1 \in \mathbb{R}$ *then* $x^2 = -1$ *has no solution, thus* $\exists y(= -1) \in \mathbb{R}, \forall x \in \mathbb{R}, x^2 \neq y$. *This application is not bijective because it is not surjective.*
3. $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by :* $\forall (x, y) \in \mathbb{R}^2, g(x, y) = (x, xy)$.
This application is not injective. Indeed, for $(x, y) = (0, 0)$ *and* $(x', y') = (0, 1)$, *we have* $g(0, 0) = (0, 0) = g(0, 1)$ *and* $(0, 0) \neq (0, 1)$. *This application is not surjective too, in fact for* $(s, t) = (0, 1) \in \mathbb{R}^2$,

$$\begin{aligned} g(x, y) = (0, 1) &\Longleftrightarrow (x, xy) = (0, 1) \\ &\Longrightarrow x = 0 \text{ and } xy = 1 \text{ which is impossible} \end{aligned}$$

hence, $\exists (s, t)(= (0, 1)) \in \mathbb{R}^2, \forall (x, y) \in \mathbb{R}^2, g(x, y) \neq (0, 1)$. *This application is not bijective, because it is neither injective or surjective.*
4. $k : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *defined by :* $\forall x \in \mathbb{R}_+, k(x) = \sqrt{x}$.
This application is injective. Let $x_1, x_1 \in \mathbb{R}_+$ *such that* $k(x_1) = k(x_2)$.

$$\begin{aligned} k(x_1) = k(x_2) &\Longrightarrow \sqrt{x_1} = \sqrt{x_2} \\ &\Longrightarrow x_1 = x_2. \end{aligned}$$

This application is surjective. Let be $y \in \mathbb{R}_+$ *then* $y = \sqrt{x} \Longrightarrow x = y^2 \in \mathbb{R}_+$ *then,* $\forall y \in \mathbb{R}_+, \exists x(= y^2) \in \mathbb{R}_+, y = \sqrt{x}$. *This application is bijective because it is injective and surjective.*

We have :

$$k : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \qquad \text{and} \qquad h : \mathbb{R}_+ \longrightarrow \mathbb{R}$$
$$x \mapsto k(x) = \sqrt{x} \qquad\qquad x \mapsto h(x) = x^2$$

so we can define the composition $h \circ k$ as folows :

$$\begin{aligned} h \circ k : \mathbb{R}_+ &\longrightarrow \mathbb{R} \\ x &\mapsto (h \circ k)(x) = h(k(x)) \\ &\qquad\qquad\quad = h(\sqrt{x}) \\ &\qquad\qquad\quad = (\sqrt{x})^2 \\ &\qquad\qquad\quad = x \end{aligned}$$

h is injective and k being bijective is then injective, we deuce then that $h \circ k$ is injective.

Solution Exercise 5.

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\mapsto f(x) = \frac{4x}{x^2+1} \end{aligned}$$

1. *Let $a \in \mathbb{R}^*$, then $f(a) = \dfrac{4a}{a^2+1}$ and $f\left(\dfrac{1}{a}\right) = \dfrac{4\frac{1}{a}}{\left(\frac{1}{a}\right)^2+1} = \dfrac{4}{\frac{1}{a}+a} = \dfrac{4a}{a^2+1}$, then $f(a) = f(\frac{1}{a})$.*

   If $a \neq 1, -1$ then $f(a) = f(\frac{1}{a})$ and $a \neq \dfrac{1}{a}$, thus f is not injective.
2.

$$\begin{aligned} h : [1, +\infty[&\longrightarrow \mathbb{R} \\ x &\mapsto h(x) = f(x) \end{aligned}$$

   (a) *Let $x, y \in [1, +\infty[$ such that $h(x) = h(y)$.*

$$\begin{aligned} h(x) = h(y) &\Longrightarrow \frac{4x}{x^2+1} = \frac{4y}{y^2+1} \\ &\Longrightarrow 4x(y^2+1) = 4y(x^2+1) \\ &\Longrightarrow xy^2 - yx^2 + x - y = 0 \\ &\Longrightarrow (y-x)xy + x - y = 0 \\ &\Longrightarrow (x-y)(-xy+1) = 0 \\ &\Longrightarrow \begin{cases} x - y = 0 \\ \qquad or \\ 1 - xy = 0 \end{cases} \\ &\Longrightarrow \begin{cases} x = y \\ \quad or \\ xy = 1 \end{cases} \\ &\Longrightarrow \begin{cases} x = y \\ \quad or \\ x = \dfrac{1}{y} \text{ because } y \neq 0 \end{cases} \end{aligned}$$

   for $y = 1$ then $x = 1$ i.e. $x = y$, but for $y \in]1, +\infty[$ then $x = \dfrac{1}{y} \in]0, 1[$ and hence $x \notin]1, +\infty[$ contradiction! so we obtain $x = y$, which means that h is injective.

(b) *To check if* $\forall x \in [1,+\infty[, h(x) \leq 2$, *let's calculate* $h(x)-2$ *and verify its sign.*

$$\begin{aligned}
h(x)-2 &= \frac{4x}{x^2+1}-2\\
&= \frac{4x-2x^2-2}{x^2+1}\\
&= -2\frac{x^2-2x+1}{x^2+1}\\
&= -2\frac{(x-1)^2}{x^2+1} \leq 0 \quad \forall x \in [1,+\infty[
\end{aligned}$$

hence $\forall x \in [1,+\infty[, h(x) \leq 2$.
It's easy to see that $\forall x \in [1,+\infty[, h(x) > 0$, *hence* $\forall x \in [1,+\infty[, 0 < h(x) \leq 2$, *then the application from* $[1,+\infty[$ *to* $]0,2]$, *which we still note* h *is injective.*

(c) *Let be now* $y \in]0,2]$, *then*

$$\begin{aligned}
h(x)=y \iff & \frac{4x}{x^2+1}=y\\
\implies & yx^2-4x+y=0\\
\implies & \Delta = 16-4y^2 = 4(2-y)(2+y) \geq 0 \text{ because } y \in]0,2]\\
\implies & x_1 = \frac{4+2\sqrt{4-y^2}}{2y} = \frac{2+\sqrt{4-y^2}}{y} \quad x_2 = \frac{4-2\sqrt{4-y^2}}{2y} = \frac{2-\sqrt{4-y^2}}{y}
\end{aligned}$$

Do $x_1, x_2 \in [1,+\infty[$ *? or at least one of them ? Let's check it :*

$$\begin{aligned}
x_1-1 &= \frac{2+\sqrt{4-y^2}}{y}-1\\
&= \frac{2+\sqrt{4-y^2}-y}{y}\\
&= \frac{\sqrt{2-y}(\sqrt{2-y}+\sqrt{2+y})}{y} \geq 0
\end{aligned}$$

then $x_1 \in [1,+\infty[$.

$$\begin{aligned}
x_2-1 &= \frac{2-\sqrt{4-y^2}}{y}-1\\
&= \frac{2-\sqrt{4-y^2}-y}{y}\\
&= \frac{\sqrt{2-y}(\sqrt{2-y}-\sqrt{2+y})}{y} \leq 0 \text{ because for } 0<y\leq 2 \text{ we have :}
\end{aligned}$$

$0 \leq 2-y < 2$ *and* $2 < 2+y$ *then* $2-y < 2+y$ *and* $\sqrt{2-y} < \sqrt{2+y}$.

Then $x_2 \notin [1,+\infty[$.

second method : *We can see that* $x_1, x_2 > 0$ *and* $x_1x_2 = 1$. *Reasoning by contradiction, let's suppose that* $x_2 \in [1,+\infty[$, *then* $x_1x_2 = 1 \implies x_2 = \frac{1}{x_1}$, *but* $x_1 \in [1,+\infty[$ *then* $x_2 = \frac{1}{x_1} \in]0,1]$ *contradiction.*

The case where $\Delta = 0$ *gives us* $y = 2$ *and here* $x = 1$.
Hence $\forall y \in]0,2], \exists x (= \frac{2+\sqrt{4-y^2}}{y}) \in [1,+\infty[, y = h(x)$, *that is* $h : [1,+\infty[\longrightarrow]0,2]$ *is surjective. We have showed before that* h *is injective then* $h : [1,+\infty[\longrightarrow]0,2]$ *is bijective. The reciprocal application is :*

$$\begin{aligned}
h^{-1} &:]0,2] \longrightarrow [1,+\infty[\\
& y \mapsto h^{-1}(y) = \frac{2+\sqrt{4-y^2}}{y}
\end{aligned}$$