Probabilistic Approach to Mean Field Games

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Outline

1. Mean Field Games
2. Algorithms for solving FBSDEs
   - Picard Iteration
   - Continuation in Time
   - Tree Algorithm
   - Grid Algorithm
3. Benchmark Examples
   - Toy Example
   - Trader Problem

Mean Field Games

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  1. For a fixed deterministic flow of probability measures $\mu = (\mu_t)_{0 \le t \le T}$ on the state space, solve the stochastic control problem:

$$\inf_{\alpha} J^{\mu}(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t^{\alpha}, \mu_t, \alpha_t)dt + g(X_t^{\alpha}, \mu_T)\right] \tag{1}$$

subject to

$$\begin{aligned} dX_t^{\alpha} &= b(t, X_t^{\alpha}, \mu_t, \alpha_t)dt + \sigma(t, X_t^{\alpha}, \mu_t, \alpha_t)dW_t \\ X_0 &= \xi \end{aligned} \tag{2}$$

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  2. Find the fixed point, μ, such that $\mathcal{L}(X_t^{\alpha}) = \mu_t$ for all $0 \le t \le T$.

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Weak Approach:

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$$\begin{aligned} dX_t &= b\left(t, X_t, \mathcal{L}(X_t), \hat{\alpha}\left(t, X_t, \mathcal{L}(X_t), \frac{Z_t}{\sigma}\right)\right) dt + \sigma dW_t \\ X_0 &= \xi \\ dY_t &= -f\left(t, X_t, \mathcal{L}(X_t), \hat{\alpha}\left(t, X_t, \mathcal{L}(X_t), \frac{Z_t}{\sigma}\right)\right) dt + Z_t dW_t \\ Y_T &= g(X_T, \mathcal{L}(X_T)) \end{aligned} \tag{3}$$

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X_0 &= \xi \\
dY_t &= -\left[\partial_x b((t, X_t, \mathcal{L}(X_t), \hat{\alpha}\left(t, X_t, \mathcal{L}(X_t), Y_t\right)))\, Y_t \right. \\
&\qquad \left. + \partial_x f\left(t, X_t, \mathcal{L}(X_t), \hat{\alpha}\left(t, X_t, \mathcal{L}(X_t), Y_t\right)\right)\right] dt + Z_t dW_t \\
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\end{aligned}
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- Time horizon $[0, T]$ with fixed time mesh $t_i = ih, i = 1, ..., N_t$.

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$$\mathbf{P}[\xi = x_m] = p_m, \quad \sum_{m=1}^{M} p_m = 1, \quad m = 1, ..., M$$

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- The decoupled forward-backward scheme is:

$$\begin{aligned}
Y^j_{t_i} &= \mathbb{E}_{t_i}[Y^j_{t_{i+1}} + h \cdot F(t_i, X^{j-1}_{t_i}, Y^{j-1}_{t_i}, Z^{j-1}_{t_i}, \mathcal{L}(X^{j-1}_{t_i}, Y^{j-1}_{t_i}, Z^{j-1}_{t_i}))] \\
Y^j_T &= g(X^{j-1}_T, \mathcal{L}(X^{j-1}_T)) \\
X^j_{t_{i+1}} &= X^j_{t_i} + h \cdot B(t_i, X^j_{t_i}, Y^j_{t_i}, Z^j_{t_i}, \mathcal{L}(X^j_{t_i}, Y^j_{t_i}, Z^j_{t_i})) + \sigma \Delta W_i \\
X^j_0 &= \xi
\end{aligned}$$

- Initialize (X^0, Y^0, Z^0), then do Picard iteration $j = 1, ..., J$.

Mapping of Picard Iteration

- Picard iteration mapping Φ is defined by:

$$\begin{aligned}\Phi : &(X^j, Y^j, Z^j, \mathcal{L}(X^j, Y^j, Z^j)) \\ \mapsto &(X^{j+1}, Y^{j+1}, Z^{j+1}, \mathcal{L}(X^{j+1}, Y^{j+1}, Z^{j+1}))\end{aligned} \tag{6}$$

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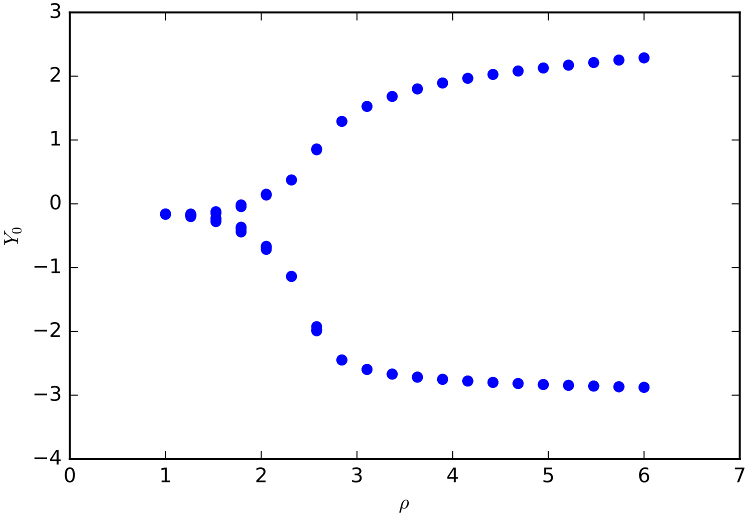
- Picard iteration convergence only guaranteed when ρ and T are not too large.

Bifurcation



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- Define global *solver* recursively:
  $solver[k](X_{T_k})$:
  1. Initialize (X,Y,Z)
  2. For $1 \leq j \leq J$
     1. $Y_{T_{k+1}} = solver[k+1](X_{T_{k+1}})$
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  where $solver[N] = g(X_T)$.
- Then $Y_0 = solver[0](\xi)$.

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Binomial Tree

- Binomial approximation of Brownian increments:

$$ {}^{\uparrow\downarrow}X^j_{t_{i+1}} = X^j_{t_i} + h \cdot B(t_i, X^j_{t_i}, Y^j_{t_i}, Z^j_{t_i}, \mathcal{L}(X^j_{t_i}, Y^j_{t_i}, Z^j_{t_i}) \pm \sigma\sqrt{h} \tag{8} $$

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- The backward scheme is easily calculated on the tree: the expectation conditional to the filtration of time t_i is given by the average of the value of "up" and "down" branches at time t_{i+1} from the node Y_{t_i}.

Outline

Grid for Discretized Distribution

- The uniform spatial grid of parameter $(\Delta x, x_{\min}, x_{\max})$ for the forward process is defined by:

$$\begin{aligned} \chi &= \{x_{min} = x_1 < \ldots < x_{N_x} = x_{max}\} \\ &= \{x_{min} + k\Delta x, k = 0, \ldots N_x - 1\} \end{aligned} \tag{9}$$

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- The distribution of X_{t_i} is defined by the distribution of its projection on the grid χ:

$$\begin{aligned}\mathcal{L}(X_{t_i}) \approx \mathcal{L}(\Pi(X_{t_i})) &= \{p_k | p_k = \mathbb{P}(\Pi(X_{t_i}) = x_k), k = 1, \ldots, N_x\} \\ &\approx \{p_k | p_k = \mathbb{P}(X_{t_i} \in B(x_k, \Delta x/2), k = 1, \ldots, N_x\}\end{aligned} \tag{10}$$

$B(x, \Delta x/2)$ the closed ball centered at x of radius $\Delta x/2$.

Scheme of Forward Process

- As before, the forward process is given by the Euler scheme with binomial approximation of Brownian increments:

$$ {}^{\uparrow\downarrow}\bar{X}_{t_{i+1}}^{t_i,x} = x + h \cdot b(x, \mu_i, Y_{t_i}, Z_{t_i}, \mathcal{L}(Y_{t_i}, Z_{t_i})) \pm \sigma\sqrt{h} \qquad (11) $$

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- For $x \in \chi$, the distribution of $X_{t_{i+1}}$ conditional to $\{X_{t_i} = x\}$ is approximated by $\mathcal{L}(\Pi(\bar{X}_{t_{i+1}}^{t_i,x}))$. Starting from the initial distribution $\mu^0 = \xi$, the sequence $(\mu^i)_{i=1,\dots,N_t}$ of discretized probability distributions is defined recursively as:

$$ \mu^{i+1} = \mathcal{L}(\Pi(\bar{X}_{t_{i+1}}^{t_i,x}) | \mathcal{L}(x) = \mu^i) \qquad (12) $$

Forward Distribution on the Grid

- For $l = 1, ..., N_x$:

$$\begin{aligned}
&\mathbb{P}_{t_i}[X_{t_{i+1}}^{t_i,x_k} \in B(x_l, \Delta x/2)] \\
\approx&\mathbb{P}_{t_i}[\Pi(\bar{X}_{t_{i+1}}^{t_i,x_k}) = x_l] = \mathbb{P}_{t_i}[\bar{X}_{t_{i+1}}^{t_i,x_k} \in B(x_l, \Delta x/2)] \\
\approx&1/2 \cdot [\mathbf{1}\{^{\uparrow}\bar{X}_{t_{i+1}}^{t_i,x_k} \in B(x_l, \Delta x/2)\} + \mathbf{1}\{^{\downarrow}\bar{X}_{t_{i+1}}^{t_i,x_k} \in B(x_l, \Delta x/2)\}]
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Forward Distribution on the Grid

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- To get the probability distribution $\mu_{i+1} = \{p_1^{i+1}, \dots, p_{N_x}^{i+1}\}$ at time t_{i+1}, we sum over k with respect to p_k^i:

$$p_l^{i+1} = \sum_{k=1}^{N_x} \frac{p_k^i}{2} [\mathbf{1}\{{}^{\uparrow}\bar{X}_{t_{i+1}}^{t_i,x_k} \in B(x_l, \Delta x/2)\} + \mathbf{1}\{{}^{\downarrow}\bar{X}_{t_{i+1}}^{t_i,x_k} \in B(x_l, \Delta x/2)\}] \tag{14}$$

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$$(Y_{t_i}, Z_{t_i}) \approx (u, v)(t_i, x, \mu^i) \text{ for } \Pi(X_{t_i}) = x \text{ and } \mathcal{L}(\Pi(X_{t_i})) = \mu^i \quad (15)$$

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- The backward scheme then becomes:

$$\begin{aligned} u(t_i, x, \mu_i) &\approx \frac{1}{2}[u(t_{i+1}, \Pi(^{\uparrow}\bar{X}^{t_i,x}_{t_{i+1}}), \mu_{i+1}) + u(t_{i+1}, \Pi(^{\downarrow}\bar{X}^{t_i,x}_{t_{i+1}}), \mu_{i+1})] \\ &\quad + h \cdot f(X_{t_i}, Y_{t_i}, Z_{t_i}, \mathcal{L}(X_{t_i}, Y_{t_i}, Z_{t_i})) \\ v(t_i, x, \mu_i) &\approx \frac{1}{2}h^{-1/2}[u(t_{i+1}, \Pi(^{\uparrow}\bar{X}^{t_i,x}_{t_{i+1}}), \mu_{i+1}) - u(t_{i+1}, \Pi(^{\downarrow}\bar{X}^{t_i,x}_{t_{i+1}}), \mu_{i+1})] \end{aligned} \quad (16)$$

Outline

Toy Example

$$
\begin{aligned}
dX_t &= -\rho Y_t dt + \sigma dW_t \\
X_0 &= x_0 = 2 \\
dY_t &= \arctan\left(\mathbb{E}(X_t)\right) dt + Z_t dW_t \\
Y_T &= \arctan\left(X_T\right)
\end{aligned}
\tag{17}
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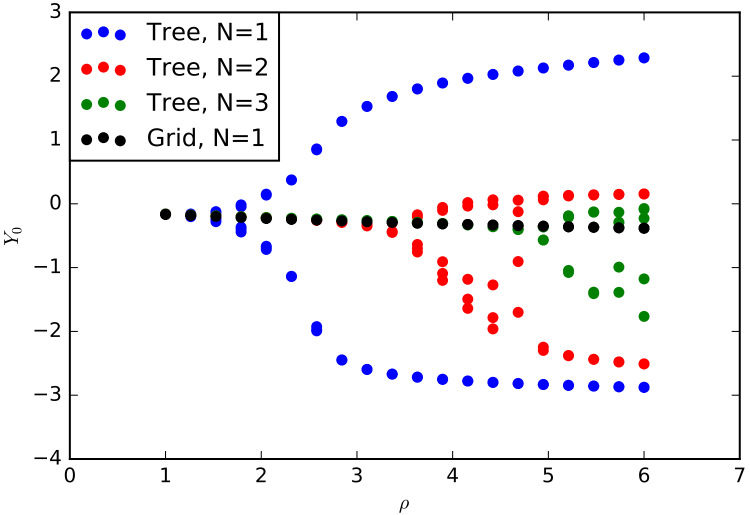
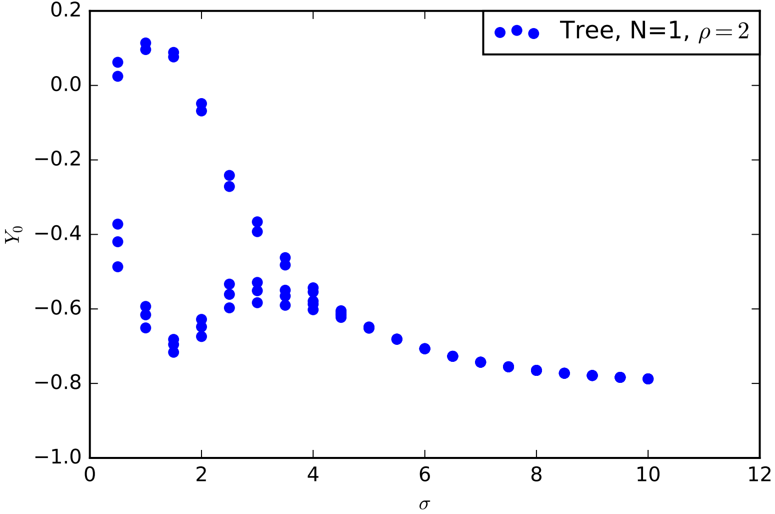
Toy Example: Continuation in Time



Toy Example: Effect of σ



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$$J(\alpha) = \mathbb{E}\left[\int_0^T \left(\frac{c_\alpha}{2}\alpha_t^2 + \frac{c_X}{2}X_t^2 - \bar{h}\mathbb{E}(\alpha_t)X_t\right) dt + \frac{c_g}{2}X_T^2\right] \tag{19}$$

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- This is an example of an *Extended Mean Field Game*.

Trader Example: Two Formulations

- Weak Formulation:

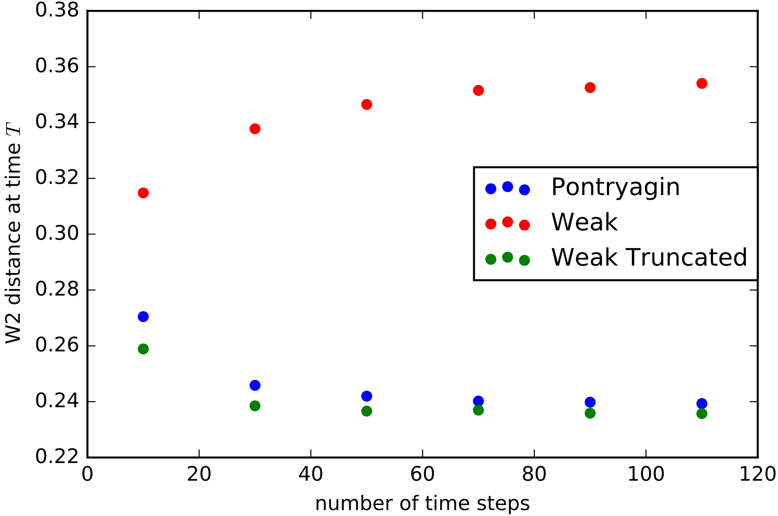
$$dX_t = -\frac{1}{c_\alpha}\frac{Z_t}{\sigma}dt + \sigma dW_t, X_0 = x_0$$

$$dY_t = -\left[\frac{c_X}{2}X_t^2 + \frac{\bar{h}}{c_\alpha}\frac{\mathbb{E}[Z_t]}{\sigma}X_t + \frac{1}{2c_\alpha}\left(\frac{Z_t}{\sigma}\right)^2\right]dt + Z_t dW_t, Y_T = c_g\frac{X_T^2}{2} \tag{20}$$

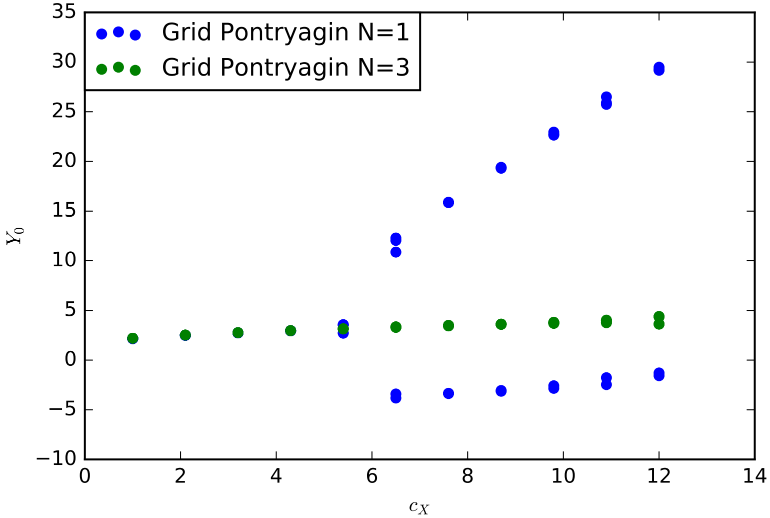
- Pontryagin Formulation:

$$\begin{aligned} dX_t &= -\frac{1}{c_\alpha}Y_t dt + \sigma dW_t, X_0 = x_0 \\ dY_t &= -(c_X X_t + \frac{\bar{h}}{c_\alpha}\mathbb{E}[Y_t])dt + Z_t dW_t, Y_T = c_g X_T \end{aligned} \tag{21}$$

Trader Example: W_2 Distance



Trader Example: Continuation in Time



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- Advantages of grid algorithm:
  1. For a given computational capacity, it is much easier to have a large number of time steps in grid algorithm than in tree algorithm where the complexity grows exponentially with number of time steps.

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- Future Work:
  1. Error analysis of two algorithms
  2. Estimation of grid parameters $(\Delta x, x_{\min}, x_{\max})$
  3. Investigate the effect of σ in the convergence of Picard iteration
  4. Application of tree algorithm to Mean Field Games with common noise

References

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Questions?