IAS MATHEMATICS (OPT.)-2010

PAPER - I : SOLUTIONS

1(b) → IAS 2010 P-I

What is the null space of the differentiation transformation.

$$\frac{d}{dx} : P_n \longrightarrow P_n$$

where P_n is the space of all polynomials of degree $\le n$ over the real numbers? what is the null space of the second derivative as a transformation of P_n?

What is the null space of the k^{th} derivative?

Sol^n → $P_n = \{ a_0 + a_1 x + \cdots + a_k x^k \mid k \le n \ \& \ a_0, a_1, \ldots, a_k \in \mathbb{R} \}$.

$\frac{d}{dx} : P_n \longrightarrow P_n$ is differentiation transformation.

let $\frac{d}{dx} \equiv T$ and $p(x) = a_0 + a_1 x + \cdots + a_k x^k$

$N(T) = \{ P(x) \mid N(P(x)) = 0 \}$.

$N(P(x)) = 0$

$\Rightarrow \frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k) = 0$

$\Rightarrow a_1 + 2a_2 x + \cdots + k a_k x^{k-1} = 0$

$\Rightarrow a_1 = 0, \ a_2 = 0, \ldots \ a_k = 0$

$\Rightarrow P(x) = a_0$ (constant).

$\Rightarrow N(T) = \{ a_0 \mid a_0 \in \mathbb{R} \}$.

Now consider $T_1 = \frac{d^2}{dx^2} : P_n \longrightarrow P_n$ second derivative as transformation.

$N(T_1) = \{ P(x) = a_0 + a_1 x + \cdots + a_k x^k \ (k \le n) \mid T_1 P(x) = 0 \}$

$\because T_1(P(x)) = 0$

$\Rightarrow \frac{d^2}{dx^2}(a_0 + a_1 x + \cdots + a_k x^k) = 0$

$\Rightarrow \dfrac{d}{dx}(a_1 + 2a_2 x + 3a_3 x^2 + \cdots + k a_k x^{k-1}) = 0$

$\Rightarrow 2a_2 + 6a_3 x + \cdots + k(k-1) a_k x^{k-2} = 0$

$\Rightarrow a_2 = 0, \; a_3 = 0, \cdots, a_k = 0$

$\Rightarrow P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k = a_0 + a_1 x.$

$\Rightarrow N(T_1) = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}.$

$\Rightarrow N(T_1) =$ set of all linear equations

or set of all polynomials of degree ≤ 1

$\Rightarrow N(T_1) = P_1(x)$

Now let us consider,

$T_2 \equiv \dfrac{d^k}{dx^k} : P_n \longrightarrow P_n$ be the

transformation of P_n.

$N(T_2) = \{ P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_j x^j \, (j \le n) \mid T_2(P(x)) = 0 \}$

$\because T_2(x) = 0$

$\Rightarrow \dfrac{d^k}{dx^k}(a_0 + a_1 x_1 + \cdots + a_j x^j) = 0$ —— (1)

W.L.O.G. take $j \ge k$.

$\dfrac{d^k}{dx^k}(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + a_{k+1} x^{k+1} \cdots a_j x^j)$

$= \lfloor k \, a_k + (k+1)k(k-1) \cdots 2 \cdot a_{k+1} x$

$+ \cdots + j(j-1) \cdots (j-k+1) a_j x^{j-k}$

putting this value in eqn (1) we get

$\lfloor k \, a_k + (k+1)k \cdots 2 a_{k+1} x + \cdots j(j-1) \cdots (j-k+1) a_j x^{j-k} = 0$

$\Rightarrow a_k = a_{k+1} = \cdots = a_j = 0$

$\Rightarrow P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1}$

$\Rightarrow N(T_2) = \{ P(x) \mid P(x) = a_0 + a_1 x + \cdots a_j x^j$ where $j \le k-1 \}$

$\Rightarrow \boxed{N(T_2) = P_{k-1}}$

2(a) IAS 2010 P-I

Let $M = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. find the unique linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ so that M is the matrix of T with respect to the basis

$\beta = \{v_1 = (1,0,0), v_2 = (1,1,0), v = (1,1,1)\}$ of $\mathbb{R}^3$

and $\beta' = \{w_1 = (1,0), w_2 = (1,1)\}$ of $\mathbb{R}^2$.

Also find $T(x,y,z)$.

Sol^n

$T: \mathbb{R}^3 \to \mathbb{R}^2$ a linear transformation.

$\beta_1 = \{v_1 = (1,0,0), v_2 = (1,1,0), v_3(1,1,1)\}$

$\beta_2 = \{w_1 = (1,0), w_2 = (1,1)\}$ of $\mathbb{R}^2$

Given $[T: \beta_1, \beta_2] = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$\Rightarrow \quad T(V_1) = T(1,0,0) = 4w_1 + 0 \cdot w_2$ —— (i)

$T(V_2) = T(1,1,0) = 2w_1 + 1 \cdot w_2$ —— (ii)

$T(V_3) = T(1,1,1) = 1 \cdot w_1 + 3w_2$ —— (iii)

from (i) $T(V_1) = 4(1,0) + 0 \cdot w_2 = (4,0)$ —— (iv)

from (ii) $T(V_2) = 2(1,0) + (1,1) = (3,1)$ —— (V)

from (iii) $T(V_3) = 1(1,0) + 3(1,1) = (4,1)$ —— (vi)

$(x,y,z) \in \mathbb{R}^3$ and $\because \beta_1$ is basis of $\mathbb{R}^3$

$\Rightarrow \exists\ \alpha_1, \alpha_2, \alpha_3$ s.t.

$(x,y,z) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

$\Rightarrow (x,y,z) = \alpha_1(1,0,0) + \alpha_2(1,1,0) + \alpha_3(1,1,1)$

$\Rightarrow (x,y,z) = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3)$

$\Rightarrow \quad \alpha_3 = z$

$\alpha_2 + \alpha_3 = y \Rightarrow \alpha_2 = y - z$

$\alpha_1 + \alpha_2 + \alpha_3 = x \Rightarrow \alpha_1 = x - y$

$\Rightarrow (x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$

$\Rightarrow (x,y,z) = (x-y)V_1 + (y-z)V_2 + zV_3$

$\Rightarrow T(x,y,z) = T[(x-y)V_1 + (y-z)V_2 + zV_3]$

$= (x-y)T(V_1) + (y-z)T(V_2) + zT(V_3)$

($\because$ T is linear transformation)

putting the values of $T(V_1), T(V_2)$ & $T(V_3)$ from eqn (iv), (v) & (vi)

$\Rightarrow T(x,y,z) = (x-y)(4,0) + (y-z)(3,1) + z(4,1)$

$\Rightarrow T(x,y,z) = (4x-y+z, y)$

is required linear transformation.

3(a) IAS 2010 P.I → Let A and B be $n\times n$ matrices over reals. Show that $I-BA$ is invertible if $I-AB$ is invertible. Deduce that AB and BA have the same eigenvalues.

Soln → A and B be $n\times n$ matrices.

given $(I-AB)$ is invertible.

$\Rightarrow (I-AB)^{-1}$ exists.

Now consider,

$$(I-BA)\left(I+B(I-AB)^{-1}A\right)$$

$$= I-BA+\{B(I-AB)^{-1}A\}-\{BAB(I-AB)^{-1}A\}$$

$$= I-BA+B\{(I-AB)^{-1}-AB(I-AB)^{-1}\}A$$

$$= I-BA+B\{(I-AB)^{-1}(I-AB)\}A$$

$$= I-BA+BIA$$

$$= I-BA+BA$$

$$= I$$

$\Rightarrow (I-BA)(I+B(I-AB)^{-1}A)=I$

$\Rightarrow (I-BA)^{-1}=I+B(I-AB)^{-1}A$

$\Rightarrow (I-BA)$ is invertible.

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4(a)i IAS 2010 P-I

In the n-space, $\mathbb{R}^n$, determine whether or not the $\{e_1-e_2, e_2-e_3, e_{n-1}-e_n, e_n-e_1\}$ is linearly independent.

Sol^n

Let $A = \{e_1-e_2, e_2-e_3, \dots e_{n-1}-e_n, e_n-e_1\}$.

let $\exists\ \alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{R}$ s.t.

$$\alpha_1(e_1-e_2) + \alpha_2(e_1-e_2) + \dots + \alpha_{n-1}(e_{n-1}-e_n) + \alpha_n(e_n-e_1) = 0$$

$$\Rightarrow (\alpha_1-\alpha_n)e_1 + (\alpha_2-\alpha_1)e_2 + \dots + (\alpha_{n-1}-\alpha_{n-2})e_{n-1} + (\alpha_n-\alpha_{n-1})e_n = 0 \quad \text{———} (1)$$

$\because \{e_1, e_2, e_3 \dots, e_n\}$ standard basis of $\mathbb{R}^n$

$\Rightarrow \{e_1, e_2, e_3 \dots e_n\}$ linearly Independent.

from (i) —

$$\alpha_1-\alpha_n=0,\ \alpha_2-\alpha_1=0,\ \dots\ \alpha_{n-1}-\alpha_{n-2}=0 \text{ and } \alpha_n-\alpha_{n-1}=0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_{n-1} = \alpha_n = k$$

$$\Rightarrow \alpha_i \neq 0 \quad (\forall\ i = 1, 2, \dots n)$$

$\Rightarrow \{e_1-e_2, e_2-e_3, \dots, e_{n-1}-e_n, e_n-e_1\}$ is L.I.

4(a)ii → IAS 2010 P-I

Let T be a linear transformation from a vector space V over reals into V such that $T - T^2 = I$. Show that T is invertible.

Sol^n →

$\text{Ker } T = \{\alpha \in V \mid T(\alpha) = \hat{0}\}$

$0 \in \text{Ker } T$ ($\because$ by the property of linear transformation $T(0) = \hat{0}$)

$\Rightarrow \text{Ker } T \neq \phi$, Let $\alpha \in \text{ker } T$

now given $T - T^2 = I$

$\Rightarrow (T - T^2)(\alpha) = I(\alpha)$ ($\alpha \in \text{ker } T$)

$\Rightarrow T(\alpha) - T^2(\alpha) = I(\alpha)$

$\Rightarrow \hat{0} - T(T(\alpha)) = \alpha$ ($\because \alpha \in \text{Ker } T$)

$\Rightarrow -T(0) = \alpha$

$\Rightarrow \hat{0} = \alpha$

$\Rightarrow \text{ker } T = \{0\}$.

$\Rightarrow T$ is non-singular.

$\Rightarrow T$ is invertible.