AVG Lab 1: Image Rectification (supplement document)

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1 Metric Rectification

Consider a square on a plane π_1 in the world coordinate as shown in Fig.1. This square is imaged by a camera, resulting in the image π_2. This imaging process can be interpreted as applying a projectivity (i.e. a 3x3 nonsingular matrix) H_P to every point x^w on π_1, so that it is mapped to x on π_2 according to Eq.(1). Note that in this equation, x^w and x are homogeneous coordinate.

$$\mathrm{x} = H_P \cdot \mathrm{x}^w \tag{1}$$

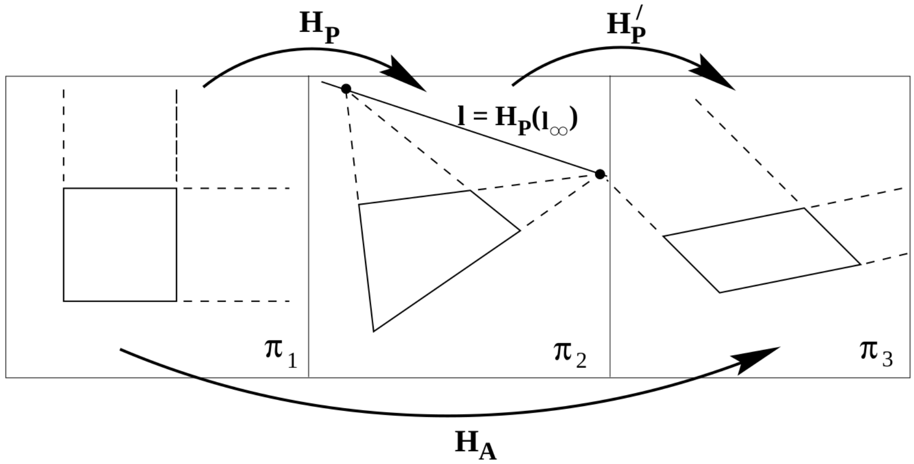

Figure 1: Affine rectification scheme

Due to projective distortion, the image of the square on π_2 is now a polygon, thus its edges are no neither parallel nor orthogonal. As a result the image l of the line at infinity l_∞ is not at infinity anymore, but visible on image plane π_2.

In the first exercise, you find the projectivity $H_P^{'}$ (constructed based on $\mathrm{l} = [\mathrm{l}_0, \mathrm{l}_1, \mathrm{l}_2]$ as in Eq.(2)) such that affinely rectify the original image π_2. In other word, mapping every point x on π_2 to $\mathrm{x}^{'}$ on π_3 using $H_P^{'}$ as Eq.(3), the parallelism is recover.

$$H_P^{'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathrm{l}_0 & \mathrm{l}_1 & \mathrm{l}_2 \end{bmatrix} \tag{2}$$

$$\mathrm{x}^{'} = H_P^{'} \cdot \mathrm{x} \tag{3}$$

1.1 Identify the dual conic on affine image

To do the metric rectification, you need to locate the dual conic $C_{\infty}^{*'}$ on the affinely rectified image π_3. To be precise, $C_{\infty}^{*'}$ is the image of the dual conic C_{∞}^{*} on π_1 under the imaging process that maps π_1 to π_3.

Before progressing further, let's clarify the imaging process that maps π_1 to π_3. Substitute Eq.(1) into Eq.(3),

$$\mathrm{x}^{'} = H_P^{'} \cdot H_P \cdot \mathrm{x}^w = H_A \cdot \mathrm{x}^w \tag{4}$$

According to Eq.(4), such an imaging process that maps a point x^w on π_1 to the point $\mathrm{x}^{'}$ on π_3 is characterized by the projectivity H_A. Because π_3 is an affine image (i.e. parallelism is restored on π_3), H_A has to be an affinity, thus having the form in Eq.(5). Note: an affinity is a special class of projectivity which preserves the parallelism.

$$H_A = \begin{bmatrix} K_{2\times 2} & t_{2\times 1} \\ 0_{1\times 2} & 1 \end{bmatrix} \tag{5}$$

Returning to the dual conic C_{∞}^{*} on π_1 and its image $C_{\infty}^{*'}$ on π_3 under the mapping characterized by H_A, these two are related by

$$C_{\infty}^{*'} = H_A \cdot C_{\infty}^{*} \cdot H_A^T \tag{6}$$

Since C_{∞}^{*} is the dual conic on π_1 in world coordinate (or Euclidean coordinate), its has the form

$$C_{\infty}^{*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

Substitute Eq.(5) and Eq.(7) into Eq.(6),

$$C_{\infty}^{*'} = \begin{bmatrix} K \cdot K^T & 0_{2\times 1} \\ 0_{1x2} & 0 \end{bmatrix} = \begin{bmatrix} S & 0_{2\times 1} \\ 0_{1\times 2} & 0 \end{bmatrix} \tag{8}$$

Here $K \cdot K^T$ is S. Because of this definition, S is symmetric matrix

$$S = \begin{bmatrix} s_0 & s_1 \\ s_1 & s_2 \end{bmatrix} \tag{9}$$

Let m and n be a pair of orthogonal line on π_1. The cosine of the angle between them is calculated by

$$\mathrm{m}^T \cdot C_{\infty}^{*} \cdot \mathrm{n} = 0 \tag{10}$$

Notice that in Eq.(10), m and n are in homogeneous coordinate.

Now let's checkout the cosine of the angle between $\mathrm{m}^{'}$ and $\mathrm{n}^{'}$ which are the image on π_3 of m and n respectively under the mapping H_A.

$$\cos\left(\angle\left(\mathrm{m}^{'}, \mathrm{n}^{'}\right)\right) = \mathrm{m}^{'T} \cdot C_{\infty}^{'*} \cdot \mathrm{n}^{'} \tag{11}$$

In Eq.(11), replace

$$\begin{aligned} \mathrm{m}^{'} &= H_A^{-T} \cdot \mathrm{m} \\ \mathrm{n}^{'} &= H_A^{-T} \cdot \mathrm{n} \\ C_{\infty}^{*'} &= H_A \cdot C_{\infty}^{*} \cdot H_A^T \end{aligned} \tag{12}$$

to get,

$$\mathrm{m}^{'T} \cdot C_\infty^{'*} \cdot \mathrm{n}^{'} = \mathrm{m}^T \cdot C_\infty^* \cdot \mathrm{n} = 0 \tag{13}$$

Eq.(13) means the angle of the image of a pair of lines are equal to angle of pair. More importantly, this equation provides a constraint on elements of matrix S (defined in Eq.(9)). Expand Eq.(13)

$$\begin{bmatrix} \mathrm{m}_0^{'} & \mathrm{m}_1^{'} & \mathrm{m}_2^{'} \end{bmatrix} \cdot \begin{bmatrix} s_0 & s_1 & 0 \\ s_1 & s_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{n}_0^{'} \\ \mathrm{n}_1^{'} \\ \mathrm{n}_2^{'} \end{bmatrix} = 0 \tag{14}$$

The equation above can be arranged into

$$\begin{bmatrix} \mathrm{m}_0^{'}\mathrm{n}_0^{'} & \mathrm{m}_0^{'}\mathrm{n}_1^{'} + \mathrm{m}_1^{'}\mathrm{n}_0^{'} & \mathrm{m}_1^{'}\mathrm{n}_1^{'} \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} = 0 \tag{15}$$

With Eq.(15), one pair of image of orthogonal lines $\mathrm{m}^{'}$ and $\mathrm{n}^{'}$ provide a constraint on $[s_0, s_1, s_2]$. If we have two pairs, we can stack the constraints vertically from matrix C so that S can be found be solving the following homogenous equation

$$C_{2\times 3} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} = 0 \tag{16}$$

If C is full rank (two rows of C are linearly independent), $[s_0, s_1, s_2]$ is the null vector of C and can be found using the SVD trick (which will be explained later).

1.2 The projectivity for metric rectification

In the last section, we have found the matrix S which in turn defines the dual conic on π_3 - $C_\infty^{'*}$. To metrically rectify the image π_3, we need to find a projectivity H such that H maps $C_\infty^{'*}$ backs to C_∞^*.

$$H \cdot C_\infty^{'*} \cdot H^T = C_\infty^* \tag{17}$$

Expanding equation above, the matrix H which we are looking for needs to satisfy

$$H \cdot \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \cdot H^T = H \cdot \begin{bmatrix} K \cdot K^T & 0 \\ 0 & 0 \end{bmatrix} \cdot H^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{18}$$

It can be verified that H defined by Eq.(19) satisfied Eq.(18)

$$H = \begin{bmatrix} K^{-1} & 0_{2\times 1} \\ 0_{1\times 2} & 0 \end{bmatrix} \tag{19}$$

All left for us to do is to find K, this is done by decompose S into the form

$$S = K \cdot K^T \tag{20}$$

Equation above can be solve for K by first performing the eigen decomposition of S as in Eq.(21)

$$S = Q \cdot \Sigma \cdot Q^T = Q \cdot \sqrt{\Sigma} \cdot \left(Q \cdot \sqrt{\Sigma}\right)^T \tag{21}$$

In Eq.(21), Q is the matrix such that each column of Q is an eigvector os S. Σ is the diagonal matrix whose diagonal is made of eigenvalues of S. Comparing Eq.(20) and Eq.(21), K can be found

$$K = Q \cdot \sqrt{\Sigma} \tag{22}$$

Once we have K, the projectivity H that does the metric rectification is

$$H = \begin{bmatrix} \sqrt{\Sigma}^{-1} \cdot Q^T & 0_{2\times1} \\ 0_{1\times2} & 0 \end{bmatrix} \tag{23}$$

2 SVD trick for finding null vector of a matrix

This section will explain how we can use SVD to solve homogenous equation like Eq.(16). Assume we want to find vector s such that

$$C \cdot \mathrm{s} = 0 \tag{24}$$

Eq.(24) is homogeneous in the sense that if s is a nontrivial solution ($\mathrm{s} \neq 0$), $\alpha \cdot \mathrm{s}$ is also a solution with α is a nonzero scalar.

Since Eq.(24) is homogeneous, additional constraint on s can be made to result in unique solution. A popular constraint is forcing s to have unique length. Taking this unique length constraint into account, Eq.(24) becomes

$$\begin{aligned} C \cdot s &= 0 \\ \text{such that} \|\mathrm{s}\| &= 1 \end{aligned} \tag{25}$$

The equation above can be rewritten as an optimization problem

$$\mathrm{s}^* = \arg\min_{\mathrm{s}} \|C \cdot \mathrm{s}\| \tag{26}$$

such that $\|\mathrm{s}^*\| = 1$

Perform the SVD for C

$$C_{m\times n} = U_{m\times m} \cdot \Sigma_{m\times n} \cdot V^T_{n\times n} \tag{27}$$

here U and V are orthogonal and unitary matrix. Assume $n > m$

$$\Sigma = \begin{bmatrix} \sigma_0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{m-1} & 0 & \cdots & 0 \end{bmatrix} \tag{28}$$

In matrix Σ, $\sigma_i \geq \sigma_j$ if $i < j$.

Since U is an orthogonal and unitary matrix,

$$\|C \cdot \mathrm{s}\| = \left\|U \cdot \Sigma \cdot V^T \cdot \mathrm{s}\right\| = \left\|\Sigma \cdot V^T \cdot \mathrm{s}\right\| \tag{29}$$

Let $V = [\mathrm{v}_0, \mathrm{v}_1, \cdots, \mathrm{v}_{n-1}]$, here v_i is a column vector of size $n \times 1$. The product between V^T and s is

$$V^T \cdot \mathrm{s} = \begin{bmatrix} \mathrm{v}_0^T \cdot \mathrm{s} \\ \mathrm{v}_1^T \cdot \mathrm{s} \\ \vdots \\ \mathrm{v}_{n-1}^T \cdot \mathrm{s} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} \tag{30}$$

In equation above, $y_i = \mathrm{v}_i^T \cdot \mathrm{s}$ is a scalar. Substitute Eq.(30) into Eq.(29),

$$\|C \cdot \mathrm{s}\| = \left\| \Sigma \cdot \begin{bmatrix} y_0 & y_1 & \cdots & y_{n-1} \end{bmatrix}^T \right\| = \sqrt{\Sigma_{i=0}^{m-1} (\sigma_i y_i)^2} \tag{31}$$

The norm of $C \cdot \mathrm{s}$ calculated by Eq.(31) can be minimized to 0 if $y_i = 0$ for all value of i in the range of 0 to (m - 1). This can be done if

$$\mathrm{s} = \mathrm{v}_{n-1} \tag{32}$$

given the fact that all vectors in V are mutually orthogonal.