SLICE KNOTS AND KNOT CONCORDANCE
SELECTED HOMEWORK SOLUTIONS

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SUMMER 2023, MPIM/UNI BONN

The exam will consist of homework problems (a subset of those at the end of this document) and definitions. Below, you will find some solutions to these problems. As always, we encourage you to find solutions that are satisfying to you! Please use the Discord to discuss the problems and find additional solutions from your classmates.

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Last updated: July 31, 2023

Lecture 1. Motivation and Overview

Exercise 1.1. 🥰 Let K be a knot in S^3 and νK denote a tubular neighbourhood of a (smooth) knot $K \subseteq S^3$, i.e. νK is diffeomorphic to $S^1 \times D^2$.

(1) Show that $S^3 \smallsetminus \nu K$ is a homology circle, i.e. $H_*(S^3 \smallsetminus \nu K; \mathbb{Z}) \cong H_*(S^1; \mathbb{Z})$.
(2) Show that $\pi_1(S^3 \smallsetminus \nu K)$ is normally generated by an arbitrary meridian of K, i.e. it is generated by the set of conjugates of the meridian.

Do these generalise to higher-dimensional knots $S^n \hookrightarrow S^{n+2}$, or knots with arbitrary codimension?

Solution to (1). I will give two arguments for (1). The first is an argument I learned from my algebraic topology course, and although it does a great job of answering the question, it's hard (for me) to see the generator of $H_1(S^3 \setminus \nu K)$. Here goes... Split K into a pair of arcs with common boundary a pair of points p and q. Consider the (contractible) open subspaces $S^3 \setminus K_\pm$; these subspaces cover $S^3 \setminus \{p, q\} \simeq S^2$ and have intersection $S^3 \setminus \nu K$. The associated reduced Mayer-Vietoris sequence (which you should write on the nearest scrap of paper) gives the isomorphism

$$\widetilde{H}_{n+1}(S^2) \cong \widetilde{H}_n(S^3 \setminus \nu K)$$

In other words, $S^3 \setminus \nu K$ is a homology 1-sphere. △

Solution to (1). An approach where the topology and algebra jive better is as follows. In particular, since we know that the only homology is in $H_1(S^3 \setminus \nu K)$, the most interesting information comes from the final paragraph of the proof. In any case, consider (nice open spaces containing) the subspaces νK and $S^3 \setminus \nu K$. These spaces cover S^3 and have intersection $\partial \nu K$, homeomorphic to a torus. The relevant portions of the associated reduced Mayer-Vietoris sequences are

$$\cdots \to H_3(\partial \nu K) \longrightarrow H_3(S^3 \setminus \nu K) \oplus H_3(\nu K) \longrightarrow H_3(S^3) \xrightarrow{\partial} H_2(\partial \nu K) \to \cdots \tag{1.1}$$

$$\cdots \to H_3(S^3) \xrightarrow{\partial} H_2(\partial \nu K) \longrightarrow H_2(S^3 \setminus \nu K) \oplus H_2(\nu K) \longrightarrow H_2(S^3) \to \cdots \tag{1.2}$$

$$\cdots \to H_2(S^3) \xrightarrow{\partial} H_1(\partial \nu K) \xrightarrow{i_* \oplus j_*} H_1(S^3 \setminus \nu K) \oplus H_1(\nu K) \longrightarrow H_1(S^2) \to \cdots \tag{1.3}$$

We claim that (1.1) shows $H_3(S^3 \setminus \nu K) = 0$. To see this, note that $H_3(\partial \nu K) = 0$ and ∂ is an isomorphism (Alexander duality blah blah blah). By exactness, we must have $H_3(S^3 \setminus \nu K) = 0$. A similar argument using (1.2) shows $H_2(S^3 \setminus \nu K) = 0$; the map ∂ is an isomorphism and $H_2(S^3) = 0$.

Finally, we claim that (1.3) shows $H_1(S^3 \setminus \nu K) \cong \mathbb{Z}$ and is generated by the meridian of K. We have $H_2(S^3) = 0 = H_1(S^2)$ so $i_* \oplus j_*$ is an isomorphism. Now, $\partial \nu K$ is homeomorphic to a 2-torus, and in particular, consider the homeomorphism taking the meridian and longitude $m, \ell \subset T^2$ to the meridian μ and Seifert longitude λ of $\partial \nu K$. This homeomorphism produces an isomorphism $H_1(\partial \nu K) \cong \langle [\mu], [\lambda] \rangle$ and with respect to this basis, we may calculate $i_* \oplus j_*$, which we claim is $i_* \oplus j_* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have $i_*([\lambda]) = 0$ because the 1-cycle λ is a boundary (of the 2-chain represented by the Seifert surface for K). Similarly, we have $j_*([\mu]) = 0$ because μ bounds a disk in νK. Thus, for $i_* \oplus j_*$ to be an isomorphism, we must have $i_*([\mu]) = 1$ and $j_*([\lambda]) = 1$. It follows that $H_1(S^3 \setminus \nu K) \cong \langle \mu \rangle$, as desired. △

Solution to (2). For this part of the problem, we again present two solutions. The first approach begins similarly to the second proof of (1) by considering the pair of (nice, open subspaces containing) the subspaces νK and $S^3 \setminus \nu K$. The Seifert-Van Kampen theorem expresses $\pi_1(S^3) = 1$ as the free product with amalgamation of $\pi_1(S^3 \setminus \nu K) = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_n \rangle$ and $\pi_1(\nu K) = \langle b \rangle$ with respect to $\pi_1(\partial \nu K) = \langle m, \ell \rangle$. That is, for inclusion maps $i : \partial \nu K \to S^3 \setminus \nu K$ and $j : \partial \nu K \to \nu K$ we have the presentation

$$\pi_1(S^3) = \langle a_1, \ldots, a_n, b \mid r_1, \ldots, r_n, i_*(m) = j_*(m) = 1, b = j_*(\ell) = i_*(\ell) \rangle$$

The relation $b = j_*(\ell) = i_*(\ell)$ allows us to reduce the presentation because it tells us that $b = i_*(\ell) \subset \nu K$ can be expressed entirely as a product of the a_i's. Note that $i_*(m) = \mu$ is a meridian of K, and the relation $\mu = i_*(m) = 1$ is (by definition) telling us to mod out by $\langle\langle \mu \rangle\rangle$, the normal subgroup genereated by μ. All together, the above presentation becomes

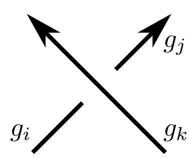
$$\pi_1(S^3) = \langle a_1, \dots, a_n \mid r_1, \dots, r_n, \mu = 1 \rangle = \pi_1(S^3 \setminus \nu K)/\langle\langle \mu \rangle\rangle$$

Since the presentation is also a presentation for the trivial group $\pi_1(S^3) = 1$, we must have that $\langle\langle \mu \rangle\rangle$ is the entire group $pi_1(S^3 \setminus \nu K)$, as desired. △

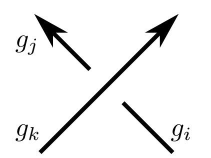
Solution to (2). The second approach (again) is slightly more geometric. The result follows quickly from a Wirtinger presentation of K, which we "recall" here (please note that we did not expect you to know this, but if you already do, then it might be a nice, geometric approach to solving the problem). For more details, this is covered very nicely in Chapter 11 of Lickorish's text *An introduction to knot theory.* A Wirtinger presentation of an oriented knot is a presentation

$$\pi_1(S^3 \setminus \nu k) = \langle g_1, \dots, g_m \mid r_1, \dots, r_n \rangle$$

obtained through the following process. Choose an oriented knot diagram D of the oriented knot K, drawn in the usual manner as a collection of oriented arcs, separated by the undercrossings of the projection. For each arc, create a loop g_i that travels from a chosen basepoint (away from the diagram) to a point near the arc, positively loops around the arc once (with linking number 1), and returns along the original path to the basepoint. For each crossing, record a relation r_i



$$1 = g_k g_i g_k^{-1} g_j^{-1}$$



$$1 = g_k^{-1} g_i g_k g_j^{-1}$$

using the above rule. Lickorish justifies these relations by stating:

> Each relator, when equated to the identity, asserts that the two generators corresponding to the under-passing arc are conjugate by means of the over-passing generator or its inverse (that choice being determined by the sign of the crossing).

The triviality of these relators can also be seen, for example, in Figure 1, which we have adapted from Chapter 4 of Stillwell's text *Classical topology and combinatorial group theory.* One should, of course, check that this is a sufficient number of generators and relators to produce this group etc etc etc. The point, however, is that this presentation gives a direct argument of the claim. Each generator is a conjugate of the next generator: if we enumerate the arcs in the diagram $a_0, \dots, a_{n-1}$ in order of the orientation (i.e., in our figures and relations, $j = i + 1 \mod n$), then each g_i is a conjugate of g_{i+1}. Consequently, they are all conjugates of eachother. Any meridian can be represented by one of these generators, so the claim follows immediately. △

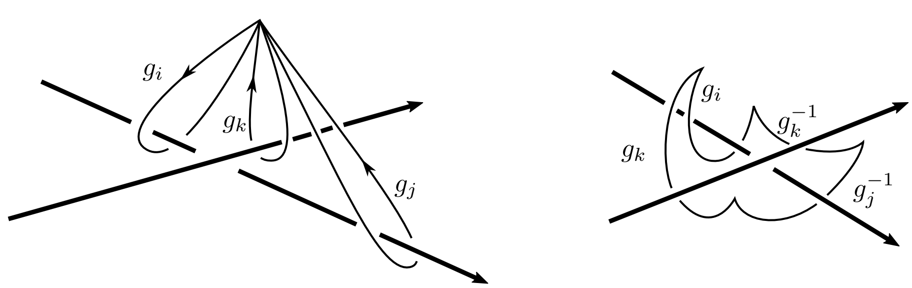


FIGURE 1. Adapted from a figure in Stillwell's text: the generators and relations from a Wirtinger presentation.

Lecture 2. Definitions and Examples

Exercise 2.1. 🥰 Prove that the knots in Figure 2 are smoothly slice. Are they also ribbon?

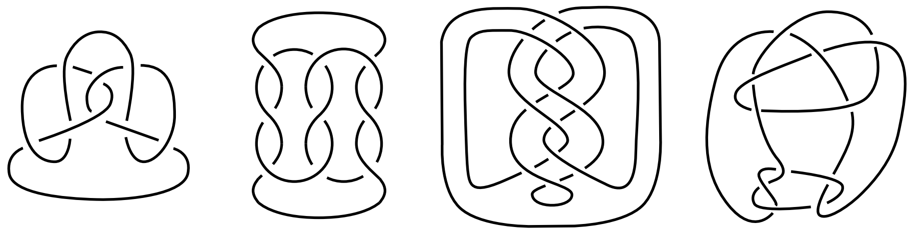
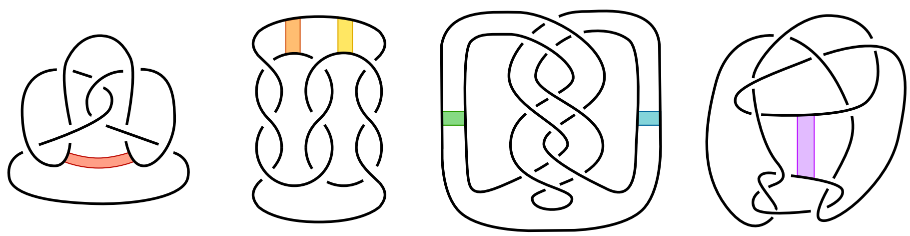
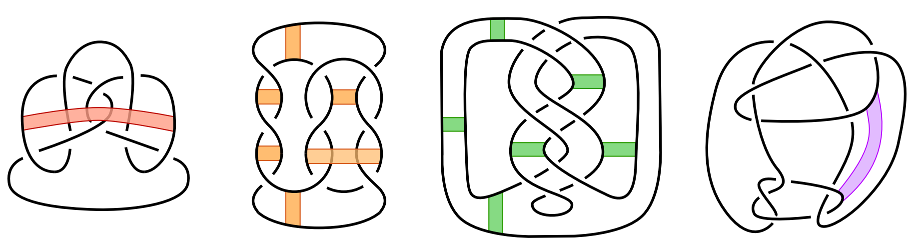


Figure 2. Some slice knots, 8_{20} (left), 9_{46} (middle two; but are the same?), and the Kinoshita–Terasaka knot $11n42$ (right).

Solution. Any of the bands listed below describe a saddle from the given knot to a two-component unlink. Please note, I do not mean to apply multiple saddles at the same time! There are often multiple slice disks for the same knot, and in certain cases, they are not equivalent (i.e., the disks are not isotopic rel boundary). The middle two examples have a pair of non-isotopic slice disks, so I have given one saddle for each disk.



Also note that your solution might use a saddle in a different location! It is extremely likely that this saddle describes one of the surfaces I've listed here. To see this, you can *slide* the band along the knot so that it matches the band given here. I've drawn a few of these below.



LECTURE 3. ALGEBRAIC CONCORDANCE

Exercise 3.2. 🥰 Consider the knots in Figure 3, namely, the Stevedore knot 6_1 and the family of twist knots T_n. Recall that in class we examined the trefoil T_{-1} and the figure-eight knot T_1. In fact, Stevedore's knot is the twist knot T_2. Calculate the Seifert form for these knots, and use it to calculate the associated Alexander polynomial and signature. Which twist knots can you now obstruct from being topologically slice? Please feel free to use Wolfram Alpha or other computational tools.

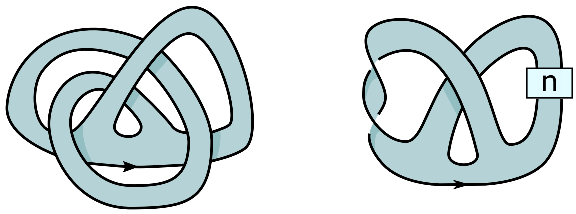


FIGURE 3. The knot 6_1 (left) and the twist knots T_n (right) and their Seifert surfaces F and F_n, respectively.

Solution. There are many potential choices you will make in doing this problem yourself, so don't worry if your Seifert matrix looks different than what is written here. We do, however, recommend using the Seifert surfaces F and F_n hinted at with the original diagrams in the statement of the exercise, namely, those illustrated above.

To begin, we calculate our collection of invariants. Bases for $H_1(F)$ and $H_1(F_n)$ consist of the cores of the two bands from either Seifert surface. After choosing orientations for these basis and calculating some linking numbers, the associated Seifert matrices for F and F_n are, respectively:

$$V = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} \qquad\qquad V_n = \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}$$

To calculate the signature, we diagonalize these matrices using our favorite computer program (or if you have a few minutes and some extra brain energy, by hand):

$$V' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \qquad\qquad V_n' = \begin{pmatrix} -1 & 0 \\ 0 & n \end{pmatrix}$$

The signatures are $\sigma(6_1) = \sigma(T_{n\geq 0}) = 0$ and $\sigma(T_{n\leq 0}) = -2$. To calculate the Alexander polynomials, consider the following matrices:

$$V - tV^T = \begin{pmatrix} 1-t & -1+2t \\ -2+t & 0 \end{pmatrix} \qquad\qquad V_n - tV_n^T = \begin{pmatrix} -1+t & 1 \\ -t & n-tn \end{pmatrix}$$

Taking the determinant of these matrices, we obtain the associated Alexander polynomials:

$$\Delta_t(6_1) = 2t^2 - 5t + 2 \qquad\qquad \Delta_t(T_n) = -nt^2 + (2n+1)t - n$$

Note that for $n = 2$, we have $\Delta_t(6_1) \doteq \Delta_t(T_2)$. Finally, the determinants of our knots are

$$\det(6_1) = 9 \qquad\qquad \det(T_n) = 4n+1$$

We are now ready to analyze the sliceness of these knots. The signature obstructs $T_{n<0}$ from being topologically slice (must be 0). The determinant obstructs any T_n with non-square $4n+1$ from being topologically slice (must be square). Similarly, for the Alexander polynomial to factor, we must have square discrimnant, however, the discriminant turns out to also be $4n+1$, and this line of reasoning gets hazy in the algebra. Thus, the topological sliceness of T_n remains a mystery when $4n+1$ is a square. We note that $4n+1$ is square if and only if n is a product of consecutive natural numbers, i.e., it is a pronic number. See also Exercise 4.2 below. □

Exercise 3.5. 🥰 Given a knot K, we form the positive (untwisted) Whitehead double of K, denoted $\mathrm{Wh}^+(K)$, by taking a push-off K^+ of K whose linking number with K is 0 (why does such a knot exist?), reversing the orientation on K^+, and connecting K and K^+ by a clasp (see Figure 4). Prove that $\Delta_t(\mathrm{Wh}^+(K)) = 1$ for every K.

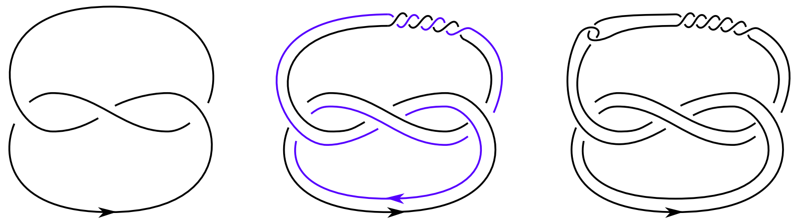
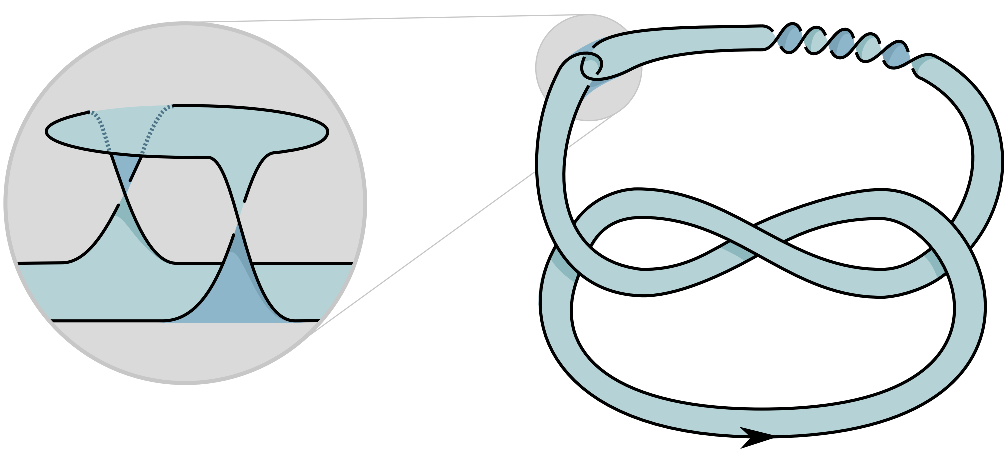


FIGURE 4. The process of Whitehead doubling: the positive trefoil $K = 3_1$ (left); the link formed by K and the reversed push-off rK^+ (middle); the Whitehead double $\mathrm{Wh}(3_1)$ of 3_1 (right).

Solution. To construct a Seifert surface for the Whitehead double of an arbitrary knot K, we build the surface in stages matching the construction of $\mathrm{Wh}(K)$. First, note that K and its push-off K^+ cobound an annulus obtained by running a band between the two knots. Next, locally attach a band to this annulus by gluing one end to K, adding two right-hand twists, and gluing the other end to K^+, as illustrated in the figure below (left). The resulting surface S is a Seifert surface for $\mathrm{Wh}(K)$, as it is oriented, etc, and has boundary $\partial S = \mathrm{Wh}(K)$.



Essentially, the hard part is done. This surface has genus 1 (why?), so a basis for $H_1(S)$ can be represented by two loops: one loop a which runs through the core of the annulus from the above construction, and another loop b which runs across the attached band. Note that, regardless of the orientation we choose for a and b, we have $\mathrm{lk}(a, a^+) = 0$ because $\mathrm{lk}(K, K^+) = 0$. Similarly, $\mathrm{lk}(a, b^+) = 0$ because $a \cup b^+$ is a split link (i.e., the two components of the link can be separated by a 3-ball). Now, if we choose the orientations carefully, we get $\mathrm{lk}(b, a^+) = 1$ because $a^+ \cup b$ is a positive Hopf link. With this orientation, $\mathrm{lk}(b, b^+) = -1$ because $b \cup b^+$ is a negative Hopf link. Thus, the Seifert matrix for S is

$$V = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

Thus, we have $\det(V - tV^T) = t$, and recalling that the Alexander polynomial is considered up to multiplication by $\pm t^n$, we conclude that $\Delta_t(\mathrm{Wh}(K)) \doteq 1$, as desired.

Lecture 4. Algebraic Sliceness, Intersection Forms, and Linking Forms

Exercise 4.1. 🥰 Draw a Venn diagram of knots that are smoothly slice, topologically slice, and algebraically slice. We will come back to this diagram throughout the semester.

Solution. You've got this one ;-)

Exercise 4.2. 🥰 Determine which twist knots T_n are algebraically slice (c.f., Exercise 3.2).

Solution. Our goal is to show that the twist knot T_n is algebraically slice if and only if n is a pronic number, i.e., n is a product of consecutive natural numbers, i.e., n belongs to the set

$$P = \{m(m+1) \mid m \in \mathbb{N}\} = \{0, 2, 6, 12, 20, 30, 42, 56, 72, ...\}$$

In Exercise 3.2, the signature and Alexander polynomial were used to obstruct the non-triangular twist knots $T_{n \notin P}$ from being topologically slice; in this exercise, we will show that these twist knots are not algrebraically slice either. On the other hand, for $n \in P$, our result differs from Exercise 3.2, where the signature and Alexanter polynomial were inconclusive. The natural follow-up questions is, are any of the (algebraically slice) twist knots $T_{n \in P}$ topologically slice? We answered this question in the negative using Casson-Gordon invariants, producing our first examples of algebraically slice knots that are not topologically slice.

Returning to the above question, to determine which T_n are algebraically slice, we will determine the hyperbolicity of their Seifert matrices. As in our proof from class that *topologically slice implies algebraically slice*, a Seifert matrix V associated to a Seifert surface F for a knot is hyperbolic if there is a subspace of $H_1(F) \cong \mathbb{Z}^{2g}$ with basis $x_1, \ldots, x_g$ such that $x_i V x_j^T = 0$.

With the above in mind, the Seifert matrix V_n from Exercise 3.2 associated to the Seifert surface F_n for T_n is hyperbolic if and only if there exists non-trivial $(x \;\; y) \in \mathbb{Z}^2 \cong H_1(F_n)$ such that

$$(x \;\; y) V_n (x \;\; y)^T = 0$$

or equivalently, $-x^2 + xy + ny^2 = 0$. We applying the change of variables $z = \frac{x}{y}$, which is valid because $y \neq 0$ (note that if $y = 0$, then $x = 0$ by the above equation). This yields the equation $-z^2 + z + n = 0$, which has solutions

$$z = \frac{-1 \pm \sqrt{1 + 4n}}{2}$$

These solutions must be rational (since x and y are integral), so $1 + 4n$ must be a square number. Noting that $4n + 1$ is square if and only if n is pronic, we have shown that T_n is algebraically slice if and only if $n \in P$, as claimed. ∎

Lecture 5. Alexander module, Blanchfield forms, twisted intersection forms, and Casson-Gordon signatures

Exercise 5.1. 🥰 Compute the Alexander module for the twist knots from Figure 3.

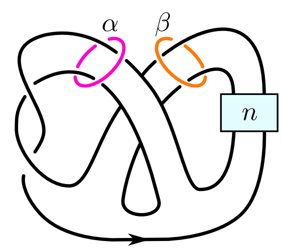


Figure 5. The twist knots T_n, with a genus one Seifert surface F_n, showing the curves α and β.

Solution. Recall from Exercise 3.2 that $V_n = \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}$ is a Seifert matrix for the twist knot T_n. Recall also that the Alexander module has presentation matrix $V - tV^T$. In other words, we have generators α and β (shown in Figure 5) and relators $(-1+t)\alpha + \beta$ and $-t\alpha + n(1-t)\beta$. Using the first relator to substitute β into the second relator, we see that the Alexander module is cyclic (with generator α), with a single relator,

$$\mathcal{A}(T_n) \cong \mathbb{Z}[t^{\pm 1}]/(nt^2 + (1-2n)t + n).$$

Exercise 5.2. 🥰

(1) Give two knots which have isomorphic Alexander modules but whose Blanchfield forms are non-isomorphic and not even equal in the algebraic concordance group.
(2) Give two knots which have non-isomorphic Blanchfield forms, but which are equivalent in the algebraic concordance group.
(3) Think of some candidate pairs of knots which have equivalent Seifert forms/Alexander modules/Blanchfield forms, but appear to be distinct.

Solution. Let K_1 be the right-handed trefoil and $K_2 = -K_1$ be the left-handed trefoil. Then their Seifert martices are negatives of each other, so the Alexander modules are isomorphic (you could also consider the isomorphism induced by the map from S_3 to S^3 with the reversed orientation). The signatures of K_1 and K_2 are different (what are they?), so the knots are not equal in the algebraic concordance group.

Let K_3 be the unknot and K_4 be the stevedore knot. We know that K_3 has trivial Alexander module, while the stevedore knot has non-trivial (cyclic, by the previous exercise) Alexander module. So the Blanchfield forms are not even on the same module, so they must be non-isomorphic. Since K_4 is slice (and therefore algebraically slice), we know that K_3 and K_4 are equal in the algebraic concordance group.

For part (3), consider the twisted Whitehead doubles (what are these?) vs the twist knots.

Lecture 6. Branched covers and Casson-Gordon signatures as sliceness obstructions

Exercise 6.1. 🥰 Prove that the double branched covers of S^3 branched along twist knots are lens spaces.

Solution. This is similar to what we did in class for the trefoil. Think of S^3 as the union of two copies of B^3 glued along the boundary. Call these B and B' for convenience. Then the twist knot is formed by the union of a pair of arcs $(A_1, A_2) \subseteq B$ and a pair of arcs $(A_1', A_2') \subseteq B'$, as shown in Figure 6. Note that the triple (B, A_1, A_2) is diffeomorphic to $(D^2 \times [0,1], x \times [0,1], y \times [0,1])$, for two points $x, y \in D^2$. So we can take the double branched cover Y of $(D^2 \times [0,1], x \times [0,1], y \times [0,1])$, branched over the two arcs. As we saw in class, the branched cover of D^2, branched along two points is an annulus, so the branched cover Y is diffeomorphic to $S^1 \times [0,1] \times [0,1]$, i.e. a solid torus. Similarly we have Y', the branched double cover of B', branched along A_1' and A_2'. By the same argument as above, Y' is also a solid torus. Then by construction the branched double cover of S^3 branched along a twist knot is a union of two solid tori along their boundary, and so by definition a lens a space. □

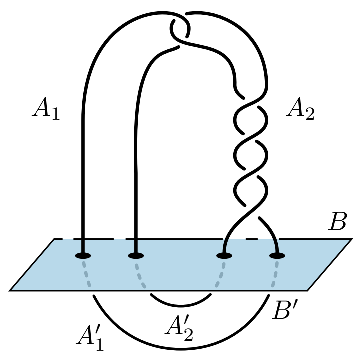


FIGURE 6. The twist knot T_n, shown as a union of arcs $A_1, A_2 \subseteq B$ and $A_1', A_2' \subseteq B'$, where B and B' are copies of B^3 so that $B \cup B' = S^3$.

Lecture 7. Alexander polynomial one knots are topologically slice

Exercise 7.1. 🥰 Let $K \subseteq S^3$ be a knot and $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$. Compute the homology groups $H_*(S^3_{p/q}(K); \mathbb{Z})$.

Solution. First the easy ones. Of course $H_i(S^3_{p/q}(K); \mathbb{Z}) = \mathbb{Z}$ for $i = 0, 3$, and $H_i(S^3_{p/q}(K); \mathbb{Z}) = 0$ for $i \geq 4$, since $S^3_{p/q}(K)$ is a closed, connected 3-manifold.

From a previous exercise, you know that the knot exterior $S^3 \setminus \nu K$ is a homology circle. In particular, $H_1(S^3 \setminus \nu K; \mathbb{Z}) \cong \mathbb{Z}$, generated by the meridian. Let μ and λ denote the meridian and longitude for K respectively.

We think of constructing the manifold $S^3_{p/q}(K)$ by first attaching the 2-cell corresponding to the meridional disc of the surgery solid torus (the solid torus we glue to $S^3 \setminus \nu K$) to form the surgery), and then the remaining 3-cell.

To compute $H_1(S^3_{p/q(K)}; \mathbb{Z})$, we need only worry about the 2-cell being added. From an easy Mayer-Vietoris argument (and the definition of Dehn surgery), we see that the 2-cell adds a relator $p\mu + q\lambda$ to $H_1(S^3 \setminus \nu K)$, so we have that $H_1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p$, with generator μ. (Note that the longitude λ is trivial in $H_1(S^3 \setminus \nu K)$, since, e.g. it bounds the Seifert surface in that complement.) In case $p = \pm 1$, we get $H_1(S^3_{p/q}(K); \mathbb{Z}) \cong 0$. In case $p = 0$, we get $H_1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}$. Otherwise we get a finite cyclic group.

We can also compute $H_2(S^3_{p/q}(K); \mathbb{Z})$ directly like above. Alternatively, use Poincaré duality to see that $H_2(S^3_{p/q}(K); \mathbb{Z}) \cong H^1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathrm{Hom}(H_1(S^3_{p/q}(K); \mathbb{Z}), \mathbb{Z})$, and compute accordingly. Specifically, for $p = 0$, we get $H_2(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}$. For all other values of p, we get $H_2(S^3_{p/q}(K); \mathbb{Z}) \cong 0$.

Exercise 7.2. 🥰 Confirm that if a knot $K \subseteq S^3$ is topologically slice then the 0-surgery M_K is the boundary of a compact, connected 4-manifold W such that

(1) the inclusion induced map $\mathbb{Z} \cong H_1(M_K) \to H_1(W)$ is an isomorphism;
(2) the fundamental group $\pi_1(W)$ is normally generated by the meridian $\mu_K \subseteq M_K = \partial W$, i.e. generated by μ_K and its conjugates;
(3) and $H_2(W) = 0$.

Solution. Let W denote the slice disc complement. In other words, let $\Delta \subseteq B^4$ be a slice disc bounded by K, and let W denote the complement $B^4 \setminus \nu\Delta$ of a tubular neighbourhood $\nu\Delta$ of Δ. We discussed in class why $M_k \cong \partial W$.

Then (2) follows from the Seifert-van Kampen theorem since since we obtain B^4 (which is simply connected) from W by gluing in a thickened 2-cell to W along a meridian of K. To see (3), consider the Mayer-Vietoris sequence for B^4 as the union $W \cup \nu\Delta = B^4$. Notice that $W \cap \nu\Delta \cong D^2 \times S^1$. Then we have

$$H_2(D^2 \times S^1; \mathbb{Z}) \to H_2(W; \mathbb{Z}) \oplus H_2(\nu\Delta; \mathbb{Z}) \to H_2(B^4; \mathbb{Z}),$$

so we see that $H_2(W; \mathbb{Z}) = 0$.

From the same sequence, we also get that $H^2(W; \mathbb{Z}) \cong H^3(W; \mathbb{Z}) = 0$. Therefore, $H_1(W, \partial W; \mathbb{Z}) \cong H_2(W, \partial W; \mathbb{Z}) = 0$. Apply these to the long exact sequence for the pair (W, M_K), recalling that $\partial W = M_K$:

$$H_2(W, M_K; \mathbb{Z}) \to H_1(M_K; \mathbb{Z}) \to H_1(W; \mathbb{Z}) \to H_1(W, M_K; \mathbb{Z})$$

This yields (1). □

Lecture 8. Braids and the Slice-Bennequin Inequality

Exercise 8.3. 🥰 Consider the braid $\beta = \sigma_1^3(\sigma_2^{-1}\sigma_1\sigma_2)(\sigma_2\sigma_1\sigma_2^{-1})^2$ and its closure $\widehat{\beta} = 10_{127}$. This is an example of a braid that is quasipositive (why?) but not strongly quasipositive. Prove that 10_{127} is not smoothly slice by applying the slice Bennequin inequality.

Solution. This is a quasipositive knot because the braid words $w_1 = \sigma_1^3$ and $w_2 = \sigma_2^{-1}\sigma_1\sigma_2$ and $w_3 = \sigma_2\sigma_1\sigma_2^{-1}$ are all conjugates. It is not strongly quasipositive because w_3 is not of the correct form for a strongly quasipositive knot, namely, words of the correct form are obtained by right-multiplication by braid generators and left-multiplication by their inverses.

To apply the slice-Bennequin inequality, first record the number of braid strands $n = 3$ and the writhe of the braid $wr(\beta) = 6$. The latter can be obtained as the sum of the exponents in the braid word: $3 - 1 + 1 + 1 + 2 + 2 - 2 = 6$. The resulting inequality is

$$\chi_4(\widehat{\beta}) \leq 3 - 6$$

which guarantees that this braid closure is not smoothly slice. □

Exercise 8.4. 🥰 Use the theorems from class to prove the *Milnor conjecture*: the smooth 4-genus of the (p, q)-torus knot is $\frac{1}{2}(p-1)(q-1)$.

Solution. To begin, note that the (p, q)-torus link $T_{p,q}$ has the following braid representative

$$\beta = (\sigma_1\sigma_2 \dots \sigma_{q-1})^p$$

There's probably a super formal argument that proves this, but you can kinda see it after you draw the above braid diagram (the braid lives in a solid tube, its closure lives in a solid torus, and it can be radially projected to the boundary of the solid torus - the result wraps the correct number of times around the meridian and longitude).

Anyway, there are q strands in β and its writhe is $p(q-1)$, which can be seen as the sum of the exponents of the braid generators in β. Moreover, β is a positive braid and, therefore, is also strongly quasipositive. The theorems in class told us that for these knots, we have equality in the slice Bennequin inequality, so

$$\chi_4(\widehat{\beta}) = q - p(q-1)$$

We can convert this to a statement about the smooth 4-genus of a knot via the formula $\chi_4(K) = 1 - 2g_4(K)$. Thus,

$$g_4(K) = \frac{1}{2}(1 - q + p(q-1)) = \frac{1}{2}(p-1)(q-1)$$

as desired. □

Lecture 9. Khovanov Homology

Exercise 9.1. 🥰 Find five distinct cycles in the Khovanov chain complex of the negative Hopf link diagram D in Figure 7. You may reuse the cycles we found in class. Why does this not contradict the fact that $\mathrm{Kh}(D) \cong \mathbb{Z}^4$?

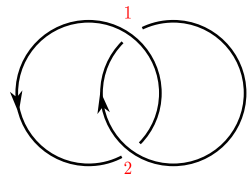


Figure 7. A diagram for the (negative) Hopf link with a chosen orientation and enumeration of its crossings.

Solution. Note that $n = 2$, $n_+ = 0$, and $n_- = 2$. Thus, with respect to the homological grading, the chain complex looks something like

$$0 \xrightarrow{d^{-3}} \mathcal{C}^{-2}(3_1) \xrightarrow{d^{-2}} \mathcal{C}^{-1}(3_1) \xrightarrow{d^{-1}} \mathcal{C}^{0}(3_1) \xrightarrow{d^{0}} 0$$

To begin, we can write out all possible labeled smoothings (on a board or something - not here) to find that the chain group is generated by 12 elements. One can compute the differential of each labeled smoothing to find five distinct cycles:

- The labeled smoothing $\alpha = (00, --)$ is a cycle because d^{-2} decreases the number of components in the 00-smoothing by merging two minus-labeled components, making $d^{-2}(\alpha) = 0$. Just for fun, we note that α represents a nontrivial Khovanov homology class because $d^{-3} \equiv 0$, meaning it cannot be a boundary.
- The labeled smoothing $\beta_\ell := (11, \ell)$ for any label ℓ is a cycle because $d^0 \equiv 0$. Within their respective quantum gradings, we have:
  - (0) β_{++} is not a boundary because there are no labeled smoothings in bigrading $(0, -2)$, so we must have $d^{0,-2} \equiv 0$
  - (-2) β_{+-} and β_{-+} each represent nontrivial Khovanov homology classes (this can be checked with some linear algebra). However, they are linearly dependent, because $d^{-1,-2}((01, +)) = \beta_{+-} + \beta_{-+}$.
  - (-4) β_{--} is a boundary because $d^{-1,-4}((01, -)) = \beta_{--}$

Thus, by analyzing the chains with only one labeled smoothing, we find that these five cycles generate a rank three subgroup of $\mathrm{Kh}(3_1)$. The specific bigradings are:

$$\mathrm{Kh}^{-2,-6}(3_1) = \langle \alpha \rangle$$
$$\mathrm{Kh}^{0,0}(3_1) = \langle \beta_{++} \rangle$$
$$\mathrm{Kh}^{0,-2}(3_1) = \langle \beta_{+-} \rangle = \langle \beta_{-+} \rangle$$

Another summand can be obtained by considering $\mathbb{Z}$-linear combinations of labeled smoothings. In particular, the chain $\gamma = (00, +-) - (00, -+)$ is a cycle. Again, it cannot be a boundary because the differential entering this homological grading is trivial. One can quickly check the bigrading:

$$\mathrm{Kh}^{-2,-4}(3_1) = \langle \gamma \rangle$$

Finally, we note that, throughout our work, we have found precisely four generators in homology, despite the (however many) number of cycles. □

Exercise 9.2. For the next two problems, consider the diagram D for the right-hand trefoil in Figure 8 together with the labeled smoothing $\alpha \in \mathcal{C}(D)$. Calculate the homological and quantum gradings for α. Show that α represents a nontrivial Khovanov homology class.

(*Hint*: show that α is a cycle ($d(\alpha) = 0$) but not a boundary ($\alpha \neq d(\beta)$ for any β). The latter can be done by writing out the differential in the relevant bigrading as a matrix A and showing there is no solution to $Ax = b$, where b is the vector representing α.)

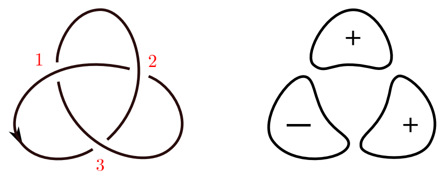


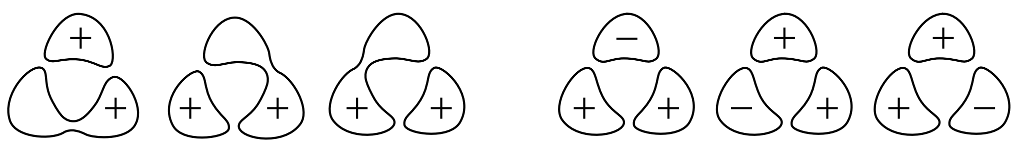
FIGURE 8. A diagram D for 3_1 with enumerated crossings (left) and a chain in $\mathcal{C}(D)$ representing a torsion Khovanov homology class in $\mathrm{Kh}(D)$ (right).

Solution. To begin, note that this oriented diagram has $n = 3$, $n_+ = 3$, and $n_- = 0$. Moreover, $\alpha = (111, + - +)$ if we enumerate the components in the smoothing accordingly. Thus, the homological and quantum gradings are

$$h(\alpha) = 3 - n_- = 3$$

$$q(\alpha) = 2 - 1 + h + 3 = 7$$

Because α is in an extreme homological grading, it is a cycle. To see that α is not a boundary, we will show it is not in the image of $d^{2,7} : \mathcal{C}^{2,7} \to \mathcal{C}^{3,7}$. There are three labeled smoothings x_1, x_2, x_3 with homological grading 2 and quantum grading 7, illustrated below on the left. In particular, these are the only three because there are no other smoothings with homological grading 2, and their quantum grading must satisfy both $7 = q = v_+ - v_- + h + 3$ and $v_+ + v_- = 2$. Similarly, we can check that $\mathcal{C}^{3,7}$ is generated by the three labeled smoothings $\beta_1, \beta_2, \beta_3$ illustrated below on the right.



x_1 x_2 x_3 β_1 β_2 β_3

With respect to the enumeration of these generators, we see that β is a boundary if there is an integral solution $x = c_1 x_1 + c_2 x_2 + c_3 x_3$ to the following matrix equation $Ax = b$.

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Working over $\mathbb{Q}$, we will argue that there is a unique non-integral solution, and therefore, cannot be an integral solution. Looking at the augmented matrix $A \mid b$ and row reducing gives

$$\left(\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -1 \end{array}\right)$$

The solution must have $-2c_3 = -1$, and backtracking gives $c_1 = c_2 = c_3 = \frac{1}{2}$. Therefore, there is a unique, non-integral solution, so no integral solution exists. We conclude that β is not a boundary, and as a result, represents a non-trivial Khovanov homology class. $\square$

Exercise 9.3. 🥰 Prove that $\alpha \in \mathcal{C}(D)$ represents a *torsion* Khovanov homology class in the Khovanov homology of the given diagram. (*Hint*: find a $\beta \in \mathcal{C}(D)$ such that $d(\beta) = 2\alpha$.)

Solution. In the previous problem, we saw that $x = (\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2})$ was a solution to $Ax = b$. By linearity, $y = 2x = (1\ 1\ 1)$ satisfies

$$Ay = A(2x) = 2(Ax) = 2b$$

Since y represents the chain $\beta = x_1 + x_2 + x_3$, we must have $d(\beta) = 2\alpha$. This can be verified directly as well, but we leave that at an exercise to the reader. $\square$

Lecture 10. Lee homology and the s-invariant

Exercise 10.1. 🥰 Prove that the s-invariant is a concordance invariant. You may use without proof that $s(K_1 \# K_2) = s(K_1) + s(K_2)$ and $s(\overline{K}) = -s(K)$. (*Hint*: recall the proof that the signature σ of a knot is a concordance invariant.)

Solution. Suppose that $K \sim J$ are concordant links. We will prove that $s(K) = s(J)$. By assumption, we also have $K\# - J$ is a slice knot (see, e.g., Exercise 2.3). Since $|s(K\# - J)| \leq 2g_4(K\# - J)$ we have $s(K\# - J) = 0$. Moreover, via the assumed properties, we also have

$$0 = s(K\# - J) = s(K) + s(-J) = s(K) - s(J)$$

Thus, $s(K) = s(J)$, concluding the proof that s is a concordance invariant.

To be completely careful, we should also check that $s(rK) = s(K)$, which is hidden within the above calculation. Namely, we note that

$$s(-J) = s(\overline{rJ}) = -s(rJ) = -s(J)$$

To see this, recall that rJ simply changes the orientation of the knot. This change has no effect on the homological or quantum gradings (n_+ and n_- are the same as before), so the isomorphism $\mathrm{Lee}(J) \to \mathrm{Lee}(rJ)$ given by $\mathfrak{s}_o \mapsto \mathfrak{s}_o$ and $\mathfrak{s}_{\overline{o}} \mapsto \mathfrak{s}_{\overline{o}}$ is $(0,0)$-bigraded. It follows that $s(J) = s(rJ)$, as desired. ☐

Exercise 10.2. 🥰 Let K be a positive knot. Recall from class that $|s(K)| = 2g_4(K) = 2g_3(K)$. In the proof of this fact, we explicitly calculated the s-invariant of K as the quantity

$$s(K) = n - k + 1$$

where n is the number of crossings in K and k is the number of components in the orientation-induced smoothing of K. Use these facts to reprove the Milnor conjecture (c.f., Exercise 8.4).

Solution. Let $T_{p,q}$ denote the (p,q)-torus knot; recall the braid representative β from Exercise 8.4. The orientation induced smoothing produces the all 0-smoothing. A simple picture shows that the all 0-smoothing connects the ith endpoint on the left of the braid with the ith endpoint on the right of the braid, so the number of components in this smoothing will match the number of strands in the braid, which is $k = q$. Again by the picture, we can see that the braid has p sections consisting of $q-1$ positive crossings, so $n = p(q-1)$. The given formula becomes

$$s(K) = p(q-1) - q + 1 = p(q-1) - (q-1) = (p-1)(q-1)$$

and in combination with the given equalities, we see that

$$g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$$

☐

Lecture 11. Alexander polynomial one knots (part II) and obstructions from Donaldson's theorem

Exercise 11.1. 🥰 Compute the Arf invariant for the twist knots.

Solution. Reusing the Seifert surface F_n for the twist knot T_n from Exercise 3.2, we note that the basis we chose was symplectic. Thus, the Arf invariant is the mod-2 product of the diagonal entries in the Seifert matrix

$$V_n = \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}$$

The resulting Arf invariant is $\mathrm{Arf}(T_n) = n \mod 2$.

Exercise 11.3. 🥰 Complete the Venn diagram of Exercise 4.1 with examples in all the regions of the diagram. Write down which invariants/constructions could be used to show that your given knots lie in the relevant spot in the diagram.

Solution. You've got this one ;-)

Lecture 12. Special topics

Exercise 12.1. 🥰 Let K and J be knots such that the 0-traces $X_0(K)$ and $X(J)$ are diffeomorphic. Suppose that $s(K) = 2$. Show that J is not smoothly slice.

Remark: This was the strategy used by Lisa Piccirillo to show that the Conway knot is not slice. See https://arxiv.org/abs/1808.02923 and https://www.quantamagazine.org/graduate-student-solves-decades-old-conway-knot-problem-20200519/ for more details.

Solution. From the trace embedding lemma, we know that K is slice if and only if J is slice. This is because when either knot is slice, the associated 0-trace embeds into S^4, and precomposing with the diffeomorphism $X_0(K) \cong X_0(J)$ gives an embedding of the 0-trace of the other knot into S^4, implying (by the trace embedding lemma) that the other knot is slice. Since $s(K) = 2$, we know that K cannot be slice, implying J is not slice, as desired. $\square$

Exam Questions

The final exam will include problems from the following list, which is a proper subset of the **green** problems from above, as well as definitions. We will provide some solutions to these problems before the date of the exams, however, we highly recommend that you also produce proofs that you are happy with and encourage you to work with others via the TA sessions and discord.

(1) 1.1
(2) 2.1
(3) 3.2, 3.5
(4) 4.1, 4.2
(5) 5.1, 5.2
(6) 6.1
(7) 7.1, 7.2
(8) 8.3, 8.4
(9) 9.1, 9.2
(10) 10.1
(11) 11.1, 11.3
(12) 12.1