SLICE KNOTS AND KNOT CONCORDANCE
HOMEWORK PROBLEMS

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The exercises come in three color/label combinations: **green** 🥰, **orange** 🤨, and **red** 🥺. The **green** exercises should be attempted if you are seeing this material for the first time. Prerequisites are courses in introductory geometric and algebraic topology. **Orange** exercises are for those of you who are already comfortable with some of the terminology and concepts; they may require nontrivial input from outside this course, which we will mention in the body of the exercise. **Red** exercises are challenge problems, which we hope are fun to play with and think about, regardless of whether you eventually reach a solution. Please feel free to work on your preferred subset of problems; in particular, they will not be graded.

As mentioned in class, the exam will consist of homework problems (a subset of those at the end of this document) and definitions.

Please use the Discord to discuss the problems!

Contents

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Lecture 1. Motivation and Overview

Let νK denote a tubular neighbourhood of a (smooth) knot $K \subseteq S^3$, i.e. νK is diffeomorphic to $S^1 \times D^2$.

Exercise 1.1. 🥰 Let K be a knot in S^3.

(1) Show that $S^3 \smallsetminus \nu K$ is a homology circle, i.e. $H_*(S^3 \smallsetminus \nu K; \mathbb{Z}) \cong H_*(S^1; \mathbb{Z})$.
(2) Show that $\pi_1(S^3 \smallsetminus \nu K)$ is normally generated by an arbitrary meridian of K, i.e. it is generated by the set of conjugates of the meridian.

Do the above generalise to higher-dimensional knots $S^n \hookrightarrow S^{n+2}$? How about knots with arbitrary codimension?

Exercise 1.2. 🥰 Show that the 3-torus $S^1 \times S^1 \times S^1$ cannot be obtained by Dehn surgery on S^3 along a knot. Which other 3-manifolds can you think of that cannot be obtained as Dehn surgery on a knot?

Here is the paper I mentioned in class (with a spoiler for this exercise in the introduction): https://arxiv.org/abs/1408.1508

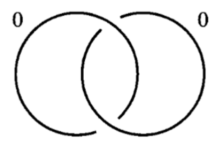

Figure 1. (Simultaneous) 0-framed surgery on the components of the Hopf link.

Exercise 1.3. 🤨 Let H denote the Hopf link. Show that the result of doing (simultaneous) 0-framed Dehn surgery on both components of the Hopf link is also S^3. This is usually denoted by the framed link shown in Figure 1.

Hint: Let νH denote the union of tubular neighbourhoods for each component of H. What is $S^3 \smallsetminus \nu H$? Recall also that the 0-framing curve for a knot is the longitude.

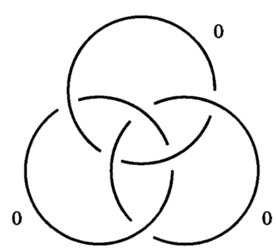

Figure 2. (Simultaneous) 0-framed surgery on the components of the Borromean rings.

Exercise 1.4. 🥺 Let B denote the Borromean rings. Show that the result of doing (simultaneous) 0-framed Dehn surgery on all three components of B is the 3-torus $S^1 \times S^1 \times S^1$. This is usually denoted by the framed link shown in Figure 2.

To convince yourself you believe this, maybe you should start by finding some of the S^1-factors and $S^1 \times S^1$-factors in the 3-torus. Or you might compute the homology of the result of Dehn surgery to confirm it's the expected group(s).

Lecture 2. Definitions and Examples

Exercise 2.1. 🥰 Prove that the knots in Figure 3 are smoothly slice. Are they also ribbon?

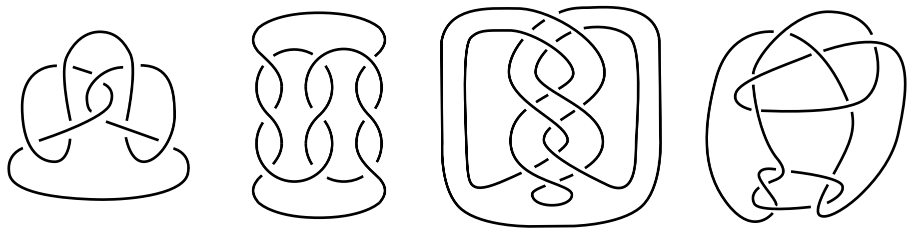

Figure 3. Some slice knots, 8_{20} (left), 9_{46} (middle two; but are the same?), and the Kinoshita–Terasaka knot $11n42$ (right).

Exercise 2.2. 🥰 For an oriented knot $K : S^1 \hookrightarrow S^3$, we define the following knots:

- the *reverse* of K, denoted rK, is the knot obtained by reversing tho orientation of S^1;
- the *mirror* of K, denoted $\overline{K}$, is the knot obtained by reversing the orientation of S^3;
- the *inverse* of K is the knot $-K := r\overline{K}$.

Prove K is slice if and only if rK is slice if and only if $\overline{K}$ is slice. Here we use *slice* to mean either smoothly slice or topologically slice throughout; your proof will likely apply to either category.

Exercise 2.3. 🥰 Prove that K is smoothly concordant to J if and only if $K\# -J$ is smoothly slice. Conclude that $[K]+[-K]=0$ in $\mathscr{C}^{\mathrm{sm}}$. The statement also holds topologically, however, you do not need to give a proof (although please indicate in your proof where smoothness was used).

Exercise 2.4. 🤨 Let K be a *nontrivial* knot in S^3. View the 4-ball as the cone of S^3, i.e., the quotient space $B^4 = S^3\times[0,1]/S^3\times\{0\}$. Prove that the *cone of the knot* $C(K) := K\times[0,1]/K\times\{0\}$ is an embedded disc in B^4 but not a slice disc (i.e., is not locally flat).

Your proof might use the fact from classical knot theory that a knot $K \subseteq S^3$ is trivial if and only if the knot group $\pi_1(S^3 \smallsetminus \nu K) \cong \mathbb{Z}$.

Exercise 2.5. 🤨 Prove that every knot in S^3 bounds a smoothly embedded disc in S^4. Does this imply that every knot is smoothly slice? In your proof, identify what fails if we require the disc to be a subspace of $S^4_- = \{(x_1,x_2,x_3,x_4 \mid x_4 \leq 0\}$ (note that $S^4_- \cong B^4$). Separately, although perhaps additionally, prove that any S^1 in S^4 bounds a smoothly embedded disc in S^4.

Exercise 2.6. 🥺 A knot $K \subseteq S^3$ is *homotopy ribbon* if there exists a topologically locally-flat disc $D \subseteq B^4$ bounded by K such that the inclusion induced map $\pi_1(S^3 \setminus \nu K) \to \pi_1(B^4 \setminus \nu D)$ is surjective. Prove that ribbon implies homotopy ribbon.

Exercise 2.7. 🥺 Explore why (or prove that) not every slice disc is isotopic to a ribbon disc. Why does this not contradict the Slice-Ribbon conjecture?

Lecture 3. Algebraic Concordance

Exercise 3.1. 🥰 In class we learned two definitions for linking number. Here is another definition based on link diagrams: given a diagram for a 2-component link $L = L_1 \sqcup L_2$, assign to each crossing between the two components (but not their self-crossings) a sign $\pm$ based on the convention:

If $n_\pm$ is the number of $\pm$ marked crossings, the linking number $lk(L_1, L_2) = \frac{1}{2}(n_+ - n_-)$. Use this definition to recalculate the linking number of the Hopf link and the Whitehead link. (See also Problem 3.9.)

Exercise 3.2. 🥰 Consider the knots in Figure 4, namely, the Stevedore knot 6_1 and the family of twist knots T_n. Recall that in class we examined the trefoil T_{-1} and the figure-eight knot T_1. In fact, Stevedore's knot is the twist knot T_2. Calculate the Seifert form for these knots, and use it to calculate the associated Alexander polynomial and signature. Which twist knots can you now obstruct from being topologically slice? Please feel free to use Wolfram Alpha or other computational tools.

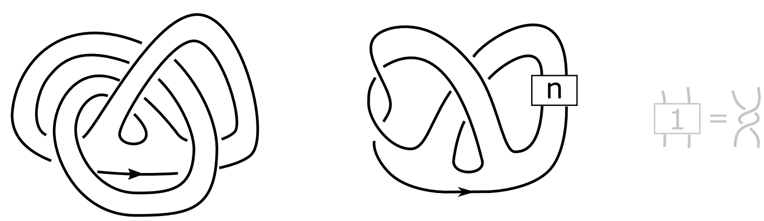

Figure 4. The knot 6_1 (left) and the twist knots T_n (right) obtained by adding n full twists in the box. For $n > 0$ we use a right-handed twist (as illustrated for $n = 1$); for $n < 0$ we use a left-handed twist; for $n = 0$ we use two parallel strands.

Exercise 3.3. 🥰 Let K, J be knots with Seifert matrices V, W, respectively. Prove the following:

(1) $K\#J$ has Seifert matrix $V \oplus W$.
(2) rK has Seifert matrix V^T.
(3) $\overline{K}$ has Seifert matrix $-V^T$.

Conclude the following properties hold:

(a) the signature is additive: $\sigma(K\#J) = \sigma(K) + \sigma(J)$.
(b) the Alexander polynomial is multiplicative: $\Delta_t(K\#J) \doteq \Delta_t(K)\Delta_t(J)$.
(c) $\sigma(rK) = \sigma(K)$ and $\Delta_t(rK) = \Delta_t(K)$.
(d) $\sigma(\overline{K}) = -\sigma(K)$ and $\Delta_t(\overline{K}) = \Delta_t(K)$.
(e) $\sigma(-K) = -\sigma(K)$ and $\Delta_t(-K) = \Delta_t(K)$.

Exercise 3.4. 🥰 Recall that a topologically slice knot K admits a hyperbolic Seifert matrix. Using this fact, prove the following hold for topologically slice knots:

(a) the Fox-Milnor condition: $\Delta_t(K) \doteq f(t)f(t^{-1})$ for some $f \in \mathbb{Z}[t^\pm]$.
(b) the signature of K vanishes: $\sigma(K) = 0$.

You may use the following without proof: for $2n \times 2n$ matrices A, B, and C, we have

$$\sigma\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma(A) + \sigma(B) \qquad \det\begin{pmatrix} 0 & A \\ B & C \end{pmatrix} = \det(A)\det(B)$$

Note that the (slice) knots $K\# -K$ satisfy the Fox-Milnor condition; to see this, use Problem 3.3 and the fact that the Alexander polynomial is symmetric: $\Delta_t(K) = \Delta_{t^{-1}}(K)$.

Exercise 3.5. 🥰 Given a knot K, we form the positive (untwisted) Whitehead double of K, denoted $\mathrm{Wh}^+(K)$, by taking a push-off K^+ of K whose linking number with K is 0 (why does such a knot exist?), reversing the orientation on K^+, and connecting K and K^+ by a clasp (see Figure 5). Prove that $\Delta_t(\mathrm{Wh}^+(K)) = 1$ for every K. *Hint*: every Whitehead double bounds a genus-1 Seifert surface.

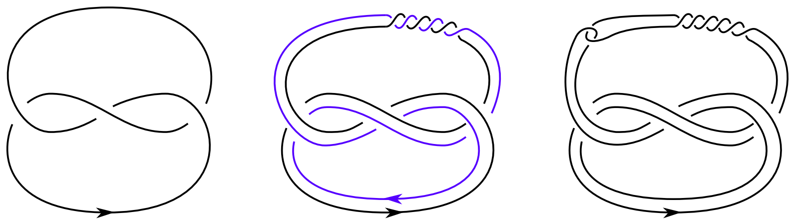

FIGURE 5. The process of Whitehead doubling: the positive trefoil $K = 3_1$ (left); the link formed by K and the reversed push-off rK^+ (middle); the Whitehead double $\mathrm{Wh}(3_1)$ of 3_1 (right).

Exercise 3.6. 🤨 Show the algebraic concordance group $(\mathscr{G}, \oplus)$ is an abelian group by proving:

(a) cobordism is an equivalence relation
(b) $\oplus$ is well-defined
(c) determine what is the identity element
(d) the inverse of $[A]$ is $-[A] := [-A]$
(e) $\oplus$ is commutative

It may be helpful to prove one or both of the following:

(f) if M and N are hyperbolic, then $M \oplus N$ is hyperbolic
(g) if M and $M \oplus N$ are hyperbolic, then N is hyperbolic

Exercise 3.7. 🤨 Prove that any Seifert matrix for a knot satisfies $\det(V - V^T) = \pm 1$.

Hint: it may be helpful to prove that, in the definition of the Seifert pairing, we have $\mathrm{lk}(x, y^+) = \mathrm{lk}(x^-, y^+) = \mathrm{lk}(x^-, y)$. For spoilers, see Rolfsen 8B4-8B8.

Exercise 3.8. 🤨 The knots in Figure 6 are infamous for their sliceness-ness. To see this, try finding slice disks for these knots (but don't try too hard 😉). Next, show that we cannot use our algebraic techniques to obstruct sliceness (i.e., they have Alexander polynomials that satisfying the Fox-Milnor condition and have vanishing signatures).

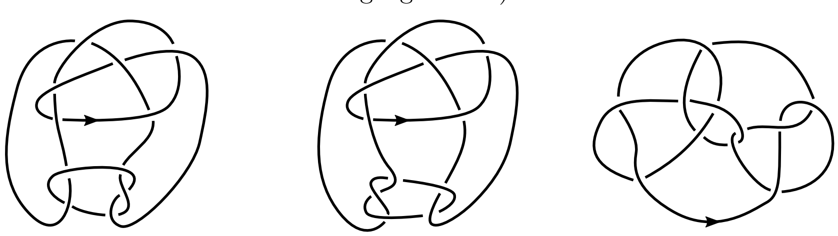

FIGURE 6. The knots 11n34 aka the Conway knot (left), 11n42 aka the Kinoshita-Terasaka knot (middle), and 11n45 (right). These knots have vanishing signature and their Alexander polynomials satisfy the Fox-Milnor condition. How will we know if they are slice!?

Exercise 3.9. 🥺 Prove that the three definitions of linking number we have discussed (via a Seifert surface, surfaces in B^4, and link diagrams) are equivalent.

Lecture 4. Algebraic Sliceness, Intersection Forms, and Linking Forms

Exercise 4.1. 🥰 Draw a Venn diagram of knots that are smoothly slice, topologically slice, and algebraically slice. We will come back to this diagram throughout the semester.

Exercise 4.2. 🥰 Determine which twist knots T_n are algebraically slice (compare this with your solution to Problem 3.2).

Exercise 4.3. 🥰 Recall from class that for each symmetric, prime, integral polynomial p, we obtain a homomorphism $\phi_p : \mathscr{C} \to \mathbb{Z}_2$ defined by $[K] \mapsto r_i \bmod 2$, where r_i denotes the degree of the polynomial p in the prime factorisation of the Alexander polynomial $\Delta_t(K)$ over $\mathbb{Z}$.

Prove that the knots K_a in Figure 7 generate a $\mathbb{Z}_2^\infty$ summand of $\mathscr{G}$ by proving they are linearly independent and have order 2 (note that $K_1 = 4_1$ was discussed in class).

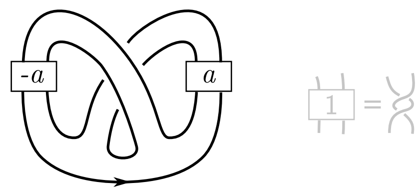

Figure 7. A family of knots K_a generating a $\mathbb{Z}_2^\infty$ summand in $\mathscr{G}$.

Exercise 4.4. 🥰 Let $p, q > 0$ be relatively prime integers. Recall from class that the lens space $L(p,q)$ is obtained as the union $S^1 \times D^2 \cup_\phi S^1 \times D^2$, where ϕ is the diffeomorphism $S^1 \times \partial D^2 \to S^1 \times \partial D^2$, mapping $pt \times \partial D^2$ to $p\lambda + q\mu$, where $\lambda = S^1 \times pt$ for $pt \in \partial D^2$ is the longitude of the solid torus and $\mu = pt \times \partial D^2$ is the meridian.

(1) Show that the lens space $L(p,q)$ is obtained by $\frac{p}{q}$-framed Dehn surgery on S^3 along the unknot.
(2) Compute the homology groups $H_*(L(p,q);\mathbb{Z})$. Find a nice representative of a generator of $H_1(L(p,q);\mathbb{Z}) \cong \mathbb{Z}/p$.
(3) Show that the linking form on $L(p,q)$ is given by

$$\begin{aligned} lk_{L(p,q)} \colon \mathbb{Z}/p \times \mathbb{Z}/p &\to \mathbb{Q}/\mathbb{Z} \\ (a,b) &\mapsto \frac{q}{p}ab \end{aligned}$$

Exercise 4.5. 🤨 Prove that the twist knots $T_{n<0}$ generate a $\mathbb{Z}^\infty$ summand in $\mathscr{G}$ (i.e., are linearly independent) by computing their Tristram-Levine signatures σ_z.

Exercise 4.6. 🤨 Let M be a closed, connected, oriented 3-manifold. Given a smooth embedding $M \hookrightarrow S^4$, prove that the torsion subgroup of $H_1(M;\mathbb{Z})$ splits as a direct sum of two subgroups G_1, G_2 such that the linking form lk_M vanishes on each G_i. (Hint: $S^4 \smallsetminus M$ has two components (why?); use the Mayer-Vietoris sequence and that the intersection form on S^4 is trivial.)

For which p, q is there a smooth embedding $L(p,q) \hookrightarrow S^4$?

Exercise 4.7. 🥺 (Half lives, half dies) For Y^3 a compact, oriented 3-manifold with ∂Y a genus-g surface, prove the inclusion induced map $H_1(\partial Y;\mathbb{Q}) \to H_1(Y;\mathbb{Q})$ has kernel and image of rank $g = \frac{1}{2}H_1(\partial Y;\mathbb{Q})$.

Lecture 5. Alexander module, Blanchfield form, twisted intersection forms, Casson-Gordon signatures

Exercise 5.1. 🥰 Let $X = S^1$ and R be a ring with involution. Think of $\pi_1(X) \cong \mathbb{Z}$ as the multiplicative group generated by t. Suppose we have a homomorphism $\varphi\colon \mathbb{Z}[\pi_1(X)] = \mathbb{Z}[t^{\pm 1}] \to R$.

(1) Write down the cellular chain complex of $\mathbb{Z}[t^{\pm 1}]$-modules for the universal cover $\widetilde{X}$.
(2) Compute $H_*^{\varphi}(X; R)$ when $R = \mathbb{Z}$ and $\varphi(t) = 1$.
(3) Compute $H_*^{\varphi}(X; R)$ when $R = \mathbb{Z}[t^{\pm 1}]$ and $\varphi(t) = t$.

Exercise 5.2. 🥰 Compute the Alexander module for the twist knots from Figure 4.

Exercise 5.3. 🥰 Let $\Lambda := \mathbb{Z}[t^{\pm 1}]$. Let $K \subseteq S^3$ be an arbitrary knot. Use the formula for the Blanchfield form to establish the following properties, for all $x, y \in \mathcal{A}(K)$ and $p(t) \in \Lambda$. For $p(t) \in \Lambda$, let $\overline{p(t)}$ denote $p(t^{-1})$. For $\frac{p(t)}{q(t)} \in \mathbb{Q}(t)$, let $\overline{\left(\frac{p(t)}{q(t)}\right)}$ denote $\frac{\overline{p(t)}}{\overline{q(t)}} = \frac{p(t^{-1}}{q(t^{-1})}$.

(1) $\mathcal{B}\ell(p(t)x, y) = p(t)\mathcal{B}\ell(x, y)$, i.e. $\mathcal{B}\ell$ is Λ-linear in the first variable.
(2) $\mathcal{B}\ell(x, p(t)y) = \mathcal{B}\ell(x, y)\overline{p(t)}$, i.e. $\mathcal{B}\ell$ is Λ-conjugate linear in the second variable.

A form with the above two properties is said to be *sesquilinear*, which means "1.5"-linear.

You may use that, since t acts by an orientation preserving homeomorphism on the infinite cyclic cover $\widehat{S^3 \setminus \mathring{\nu} K}$, we know $tx \cdot ty = x \cdot y$, where $\cdot$ denotes the algebraic intersection.

Exercise 5.4. 🥰

(1) Give two knots which have isomorphic Alexander modules but whose Blanchfield forms are non-isomorphic and not even equal in the algebraic concordance group. *Hint:* Consider changing orientations on knots. Obstruct algebraic concordance by looking at easy to compute algebraic invariants (rather than difficult to compute Blanchfield forms).
(2) Give two knots which have non-isomorphic Blanchfield forms, but which are equivalent in the algebraic concordance group.
(3) Think of some candidate pairs of knots which have equivalent Seifert forms/Alexander modules/Blanchfield forms, but appear to be distinct.

We didn't define this in class, but two knots K and J have isomorphic Blanchfield forms if there is an isomorphism $f\colon \mathcal{A}(K) \to \mathcal{A}(J)$ such that $\mathcal{B}\ell(x, y) = \mathcal{B}\ell(f(x), f(y))$ for all $x, y \in \mathcal{A}(K)$.

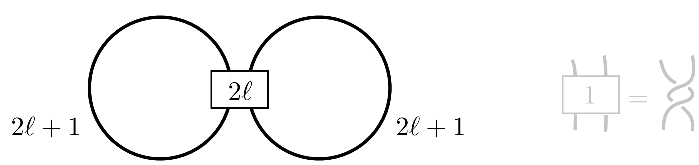

Figure 8. A Dehn surgery diagram for a 3-manifold M_ℓ, for $\ell \geq 1$.

Exercise 5.5. 🥰 Compute the homology groups for the 3-manifolds given by the Dehn surgery diagram in Figure 8. Specifically, give a presentation matrix for $H_1(M_\ell; \mathbb{Z})$ and give a set of generators.

🥺Curiously, each M_ℓ is a lens space. Can you guess which one?

Exercise 5.6. 🤨 The definition of the Blanchfield form given in class is difficult to compute with. There is however an equivalent definition which is much easier to work with in practice, which we describe here. Let $K \subseteq S^3$ be a knot, with a Seifert surface F with genus g and corresponding Seifert matrix V. Let $\Lambda := \mathbb{Z}[t^{\pm 1}]$. Then the Blanchfield form is isometric to

$$\begin{aligned}\Lambda^{2g}/(V - tV^T) \times \Lambda^{2g}/(V - tV^T) &\to \mathbb{Q}(t)/\Lambda \\ (v, w) &\mapsto \overline{v}^T(1-t)(tV - V^T)^{-1}w\end{aligned}$$

(For many details about why this is isometric to the Blanchfield form as defined in class, see https://arxiv.org/abs/1512.04603)

Note that the Alexander module of the trefoil knot (a twist knot) is cyclic. Use the formula above to compute $\mathcal{B}\ell(1,1)$ where 1 generates the Alexander module.

Exercise 5.7. 🤨 Let X be a connected CW complex. Given an epimorphism $\varphi\colon \pi_1(X) \to G$, denote by $\widehat{X}$ the associated G-cover of X corresponding to the kernel of φ. Then φ endows $\mathbb{Z}[G]$ with a right $\mathbb{Z}[\pi_1(X)]$-module structure.

(1) Show that the covering map $\widetilde{X} \to \widehat{X}$ induces a chain isomorphism
$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}) \cong C_*(\widehat{X}),$$
where $C_*(\widehat{X})$ is interpreted as a $\mathbb{Z}[G]$-module.

(2) Conclude that $H_*^{\varphi}(X; \mathbb{Z}[G])$ and $H_*(\widehat{X})$ are canonically isomorphic.

Lecture 6. Branched covers and Casson-Gordon signatures as sliceness obstructions

Exercise 6.1. 🥰 Prove that the double branched covers of S^3 branched along twist knots are lens spaces. See also Exercise 6.6.

Exercise 6.2. 🥰 Let $K \subseteq S^3$ be a knot. Define:

- $X(K) = S^3 \smallsetminus \mathring{\nu} K$ the knot exterior
- $M_K = S^3_0(K)$ the result of 0-framed Dehn surgery
- $X_n(K)$ the n-fold cyclic cover of $X(K)$
- $M_n(K)$ the n-fold cyclic cover of M_K
- $\Sigma_n(K)$ the n-fold cyclic branched cover of S^3 branched along K.

Now do the following:

(1) Prove that $X_n(K)$, $M_n(K)$, and $\Sigma_n(K)$ exist and are unique upto homeomorphism.
(2) Prove that $M_n(K)$ and $\Sigma_n(K)$ are both obtained by gluing on a solid torus to $X_n(K)$. What are the (different) gluing maps?
(3) Show that $H_1(M_n(K);\mathbb{Z}) \cong H_1(X_n(K);\mathbb{Z})$ and $H_1(X_n(K);\mathbb{Z}) \cong H_1(\Sigma_n(K),\mathbb{Z}) \oplus \mathbb{Z}$.

Exercise 6.3. 🤨 Let Y be a closed, oriented, connected 3-manifold and $\chi\colon \pi_1(Y) \to \mathbb{Z}/m$ be a homomorphism for $m \geq 1$. Prove that the Casson-Gordon signature is well-defined, using the following facts:

- Signatures (twisted and untwisted) are invariant under bordism. In other words if (W,ψ) and (W',ψ') are equal in $\Omega_4(\mathbb{Z}_m)$ then $\sigma^{\psi}(W) = \sigma^{\psi'}(W')$.
- The bordism group $\Omega_4(\mathbb{Z}/m) \cong \Omega_4(*)$, and the latter is generated by $\mathbb{CP}^2$.

As part of your proof, convince yourself that that the twisted and untwisted signatures agree for a simply connected 4-manifold.

Hint: The standard trick in these proofs is to glue together two possible (rational) null bordisms for (Y,χ) and then use facts about closed 4-manifolds. Skip this problem if you haven't encountered bordism groups before (or first go learn about bordism groups, they are cool!)

Exercise 6.4. 🤨 Construct a 3-fold branched cover $f\colon \Sigma \to D^2$ with the following properties:

(1) The branching set $B \subseteq D^2$ consists of two points $\{x,y\}$.
(2) $f^{-1}(x) = \{x_1,x_2\}$ and $f^{-1}(y) = \{y_1,y_2\}$, where x_1 and y_1 have branching index 1 while x_2 and y_2 have branching index 2.

Identify the surface Σ.

Exercise 6.5. 🥺 Let $K \subseteq S^3$ be a slice knot with slice disc Δ. Let p be a prime and $r \geq 1$. Let $\Sigma_n(\Delta)$ denote the n-fold cyclic branched along Δ, for any $n \geq 1$.

(1) Prove that $H_*(\Sigma_{p^r}(K);\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$, i.e. $\Sigma_{p^r}(K)$ is a *rational homology sphere.*
(2) Prove that $\Sigma_n(\Delta)$ exists for any n.
(3) Prove that $H_*(\Sigma_{p^r}(\Delta);\mathbb{Q}) \cong H_*(B^4;\mathbb{Q})$, i.e. $\Sigma_{p^r}(\Delta)$ is a *rational homology ball.*

Hint: The standard way to do this problem is by using something called the *exact sequence of an infinite cyclic cover*, or a *Wang sequence*. Feel free to look up what that is.

Exercise 6.6. 🥺 Prove that the double branched cover of S^3 branched along the twist knot T_ℓ is given by the 3-manifold M_ℓ in Figure 8, via the following steps:

(1) Prove that $+1$-framed Dehn surgery on the unknot yields S^3.
(2) Prove that the knot in Figure 9(a) is the twist knot T_ℓ.
(3) Observe that T_ℓ is also given by the diagram Figure 9(b).
(4) Give a Dehn surgery diagram of the double branched cover of T_ℓ, using the diagram in Figure 9(b). *Hint:* If all has gone well, you have produced the diagram in Figure 8.

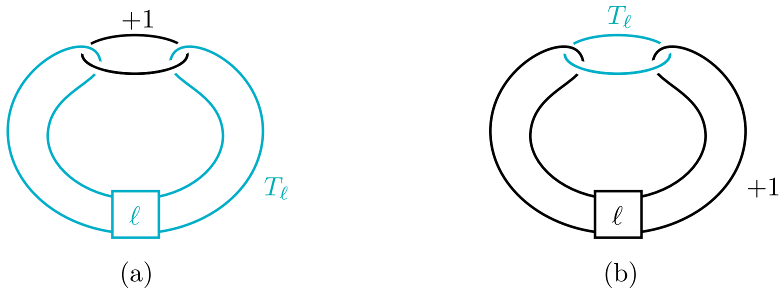

FIGURE 9. Two equivalent diagrams for the twist knots T_ℓ in S^3.

Construct the double branched by first constructing the unbranched double cover of the knot complement by cutting along a Seifert surface and gluing. Figure out where the surgery curve lifts in this cover. At this point you will have the link (in S^3 (why?)) on which you should do Dehn surgery. Then think carefully about what the framings ought to be.

(5) Try to show that M_ℓ is the lens space $L(4\ell + 1, 2)$, or that it is a lens space $L(4\ell + 1, q)$ for some q. (By the way, I have no idea how to do this sub-problem, at least the first part, without using tricks (friendly once you know them, but hard to figure out on your own) to manipulate Dehn surgery diagrams.)

Lecture 7. Alexander polynomial one knots are topologically slice

Exercise 7.1. 🥰 Let $K \subseteq S^3$ be a knot and $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$. Compute the homology groups $H_*(S^3_{p/q}(K); \mathbb{Z})$.

Exercise 7.2. 🥰 Confirm that if a knot $K \subseteq S^3$ is topologically slice then the 0-surgery M_K is the boundary of a compact, connected 4-manifold W such that

1. the inclusion induced map $\mathbb{Z} \cong H_1(M_K) \to H_1(W)$ is an isomorphism;
2. the fundamental group $\pi_1(W)$ is normally generated by the meridian $\mu_K \subseteq M_K = \partial W$, i.e. generated by μ_K and its conjugates;
3. and $H_2(W) = 0$.

Exercise 7.3. 🥰 Consider the two following statements.

1. Every homotopy 4-sphere is homeomorphic to S^4.
2. Every homotopy 4-ball with boundary S^3 is homeomorphic to B^4.

Prove that the two statements above are equivalent. Does the analogue hold in other dimensions? In the smooth category?

Exercise 7.4. 🤨 Let $K \subseteq S^3$ be a knot with Seifert matrix V. Show that the homology $H_1(\Sigma_2(K); \mathbb{Z})$ is presented by the matrix $V + V^T$. Conclude that $H_1(\Sigma_2(K); \mathbb{Z})$ is a finite group with odd order equal to the determinant of K.

Hint: Construct the double branched cover by first constructing the cyclic double cover of the knot complement, by cutting along a Seifert surface and gluing (similar to what we did for the infinite cyclic cover, but using, of course, only two fundamental domains). Feel free to use, without proof, that $V - V^T$ has determinant ± 1 (see Exercise 3.7) and that $\Delta_K(1) = \pm 1$.

Lecture 8. Braids and the Slice-Bennequin Inequality

Exercise 8.1. 🥰 Find two braids that have the same closure. Conjecture at a few relations between braids with equivalent closures.

Exercise 8.2. 🥰 The (p, q)-torus link is formed by taking a curve on the standard torus in S^3 that wraps p-times along the meridian and q-times along the longitude. For example, the right-hand trefoil is the $(3, 2)$-torus knot. Here are some facts about torus links:

- a torus link is a knot if and only if p and q are relatively prime,
- a torus knot is trivial if and only if p or q is ± 1.

Find a braid representative for the (p, q)-torus knot. Show that the Whitehead double of a non-trivial positive torus knot ($p > 1$, $q > 1$) is not smoothly slice.

Exercise 8.3. 🥰 Consider the braid $\beta = \sigma_1^3(\sigma_2^{-1}\sigma_1\sigma_2)(\sigma_2\sigma_1\sigma_2^{-1})^2$ and its closure $\widehat{\beta} = 10_{127}$. This is an example of a braid that is quasipositive (why?) but not strongly quasipositive. Prove that 10_{127} is not smoothly slice by applying the slice Bennequin inequality.

Exercise 8.4. 🥰 Use the theorems from class to prove the *Milnor conjecture*: the smooth 4-genus of the (p, q)-torus knot is $\frac{1}{2}(p-1)(q-1)$. Torus knots were defined in Exercise 8.2.

Lecture 9. Khovanov Homology

Exercise 9.1. 🥰 Find five distinct cycles in the Khovanov chain complex of the Hopf link diagram D in Figure 10. You may reuse the cycles we found in class. Why does this not contradict the fact that $\mathrm{Kh}(D) \cong \mathbb{Z}^4$?

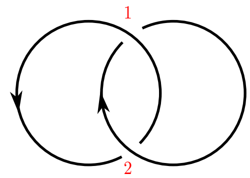

Figure 10. A diagram for the Hopf link with a chosen orientation and enumeration of its crossings.

Exercise 9.2. 🥰 For the next two problems, consider the diagram D for the right-hand trefoil in Figure 11 together with the labeled smoothing $\alpha \in \mathcal{C}(D)$. Calculate the homological and quantum gradings for α. Show that α represents a nontrivial Khovanov homology class.

(*Hint*: show that α is a cycle ($d(\alpha) = 0$) but not a boundary ($\alpha \neq d(\beta)$ for any β). The latter can be done by writing out the differential in the relevant bigrading as a matrix A and showing there is no solution to $Ax = b$, where b is the vector representing α.)

Exercise 9.3. 🥰 Prove that $\alpha \in \mathcal{C}(D)$ represents a *torsion* Khovanov homology class in the Khovanov homology of the given diagram. (*Hint*: find a $\beta \in \mathcal{C}(D)$ such that $d(\beta) = 2\alpha$.)

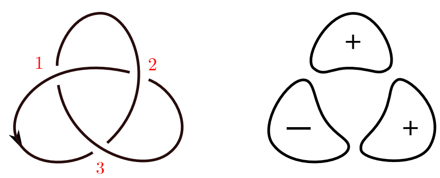

Figure 11. A diagram D for 3_1 with enumerated crossings (left) and a chain in $\mathcal{C}(D)$ representing a torsion Khovanov homology class in $\mathrm{Kh}(D)$ (right).

Exercise 9.4. 🥰 Prove that the (co)differential in the Khovanov chain complex is $(1, 0)$-bigraded, that is, it increases the homological grading by 1 and preserves the quantum grading. (Do not prove that $d \circ d = 0$.)

Exercise 9.5. 🤨 Compute the free part of the Khovanov homology of trefoil diagram D in Figure 11. You may use the fact that $\mathrm{Kh}(D) \cong \mathbb{Z}^4 \bigoplus \mathbb{Z}_2$.

(*Note*: this is a **green** problem, but because it may take some time, it is marked **orange**.)

Exercise 9.6. 🤨 Let L, L_1, L_2 be links. Prove the following

a. $\mathrm{Kh}(L_1 \sqcup L_2) \cong \mathrm{Kh}(L_1) \otimes \mathrm{Kh}(L_2)$
b. $\mathrm{Kh}(L) \cong \mathrm{Kh}(rL)$
c. $\mathrm{Kh}^{h,q}(L) \cong \mathrm{Kh}^{-h,-q}(\overline{L})$
d. $\mathrm{Kh}^{h,\mathrm{even/odd}}(L)$ is trivial if L has an odd/even number of components.

Lecture 10. Lee homology and the-s invariant

Exercise 10.1. 🥰 Prove that the s-invariant is a concordance invariant. You may use without proof that $s(K_1 \# K_2) = s(K_1) + s(K_2)$ and $s(\overline{K}) = -s(K)$ (*Hint*: recall the proof that the signature σ of a knot is a concordance invariant.)

Exercise 10.2. 🥰 Let K be a positive knot. Recall from class that $s(K) = 2g_4(K) = 2g_3(K)$. In the proof of this fact, we explicitly calculated the s-invariant of K as the quantity

$$s(K) = n - k + 1$$

where n is the number of crossings in K and k is the number of components in the orientation-induced smoothing of K. Use these facts to reprove the Milnor conjecture (c.f., Exercise 8.4).

Exercise 10.3. 🤨 Show that the $(-3, 5, 7)$-pretzel knot K is topologically slice but not smoothly slice (*Hint*: calculate the Alexander polynomial and s-invariant.)

Lecture 11. Alexander polynomial one knots (part II) and obstructions from Donaldson's theorem

Exercise 11.1. 🥰 Compute the Arf invariant for the twist knots.

Exercise 11.2. 🥰 Give an example of a topologically slice knot which has Alexander polynomial not equal to one, and which does not seem obviously to be smoothly slice.

Exercise 11.3. 🥰 Complete the Venn diagram of Exercise 4.1 with examples in all the regions of the diagram. Write down which invariants/constructions could be used to show that your given knots lie in the relevant spot in the diagram.

Exercise 11.4. 🤨 Let $E8$ denote the positive definite bilinear form corresponding to the $E8$ lattice. Show that the forms $E8 \oplus n\langle +1\rangle$ and $(8+n)\langle +1\rangle$ are not equivalent, for any $n \geq 0$. Here $n\langle +1\rangle$ denotes the $n \times n$ identity form.

Hint: For $n = 0$, note that the $E8$ form is *even*. For other n, count the number of vectors in $\mathbb{Z}^{8+n}$ with square one.

Exercise 11.5. 🥺 Follow the outline of Exercise 6.6 to show that the double branched cover of S^3 branched along the positive Whitehead double $\mathrm{Wh}^+(K)$ (see Exercise 3.5) of a knot K is given by the 1/2-framed Dehn surgery on S^3 along $K\#r(K)$, where $r(K)$ denotes the reverse of K.

Hint: Following the argument from the previous exercise, you should reach a diagram consisting of two parallel copies of $K\#r(K)$, each with framing $+1$. To finish the argument, slide one strand over the other, to get a 2-component link, consisting of $K\#r(K)$, with framing $+1$, and a meridian, with framing $+2$. The final step is called a 'slam dunk' move on Dehn surgery diagrams – look this up (and possibly also handle slides, for the previous step).

Lecture 12. Special topics

Exercise 12.1. 🥰 Let K and J be knots such that the 0-traces $X_0(K)$ and $X(J)$ are diffeomorphic. Suppose that $s(K) = 2$. Show that J is not smoothly slice.

Remark: This was the strategy used by Lisa Piccirillo to show that the Conway knot is not slice. See `https://arxiv.org/abs/1808.02923` and `https://www.quantamagazine.org/graduate-student-solves-decades-old-conway-knot-problem-20200519/` for more details.

Exercise 12.2. 🤨 Prove that the Poincaré conjecture implies the Schoenflies conjecture, in your favourite dimension and category.

Hint: You could use Exercise 7.3.

Exercise 12.3. 🥺 Let A be an arc in $\mathbb{R}^3_{\geq 0}$ with endpoints in $\mathbb{R}^2$, like in the lecture. Let L be a straight arc joining the two endpoints of A in $\mathbb{R}^2$. Let K denote the knot $A \cup L$. Let $spin(A)$ denote the 2-knot constructed by spinning the arc A. Show that $\pi_1(\mathbb{R}^4 \setminus spin(A)) \cong \pi_1(\mathbb{R}^3 \setminus K)$.

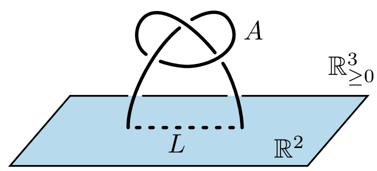

Figure 12. Setting up the spin of the knot K, with $K = 3_1$ illustrated.

Exam Questions

The final exam will include problems from the following list, which is a proper subset of the **green** problems from above, as well as definitions. We will provide some solutions to these problems before the date of the exams, however, we highly recommend that you also produce proofs that you are happy with and encourage you to work with others via the TA sessions and discord.

(1) 1.1
(2) 2.1
(3) 3.2, 3.5
(4) 4.1, 4.2
(5) 5.2, 5.4
(6) 6.1
(7) 7.1, 7.2
(8) 8.3, 8.4
(9) 9.1, 9.2
(10) 10.1
(11) 11.1, 11.3
(12) 12.1