GRADUATE TOPOLOGY NOTES

ISAAC

ABSTRACT. These are some of the notes I took during my graduate topology courses. They cover the basics for Point-set, Algebraic, and Differential Topology. They emphasizing the definitions, big theorems, and exercises for each theory. Hard/unintuitive exercises are marked with an asterisk.

CONTENTS

POINT-SET TOPOLOGY

Basic Topology.

Definition. In a space X, an **open neighborhood** of a point $x \in X$ is an open set U such that $x \in U \subseteq X$; a **neighborhood** of a point $x \in X$ is a set $V \subseteq X$ containing an open neighborhood; i.e. there exists an open U satisfying $x \in U \subseteq V$.

Quotient Spaces.

Theorem. (Characteristic Property of Quotient Maps) *If $p\colon X \to Y$ is a quotient map, a map $f\colon Y \to Z$ is continuous if and only if $f \circ p$ is continuous.*

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "fp"] & \\ Y \arrow[r, "f"'] & Z. \end{tikzcd}$$

Connectedness.

Definition. A space X is **connected** if there does not exist a **separation** of X, i.e. a pair of nonempty, disjoint, open sets $U, V \subset X$ covering X.

Definition. A space X is **locally connected** if every open neighborhood of every point contains a connected open neighborhood.

Definition. A space X is **path-connected** if for every pair of points $a, b \in X$, there exists a path $\gamma\colon I \to X$ from a to b, i.e. $\gamma(0) = a$ and $\gamma(1) = b$.

Definition. A space X is **locally path-connected** if every open neighborhood of every point contains a path-connected open neighborhood.

► ***Exercise***. *Prove $I = [0, 1]$ is connected. Conclude that path-connected implies connected.*

► ***Exercise***. *Give an example of a space which is path-connected, but not locally path-connected.*

► ***Exercise***. *Give an example of a space which is connected, but not path-connected.*

► ***Exercise***. *Prove that every connected, locally path-connected space is path-connected.*

Compactness.

Definition. A space X is **compact** if every open cover of X has a finite subcover.

Definition. A space X is **sequentially-compact** if every sequence has a convergent subsequence.

Theorem. (Heine-Borel) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem. (Bolzano-Weierstrass) *A bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

► ***Exercise***. *Show every closed subspace of a compact space is compact.*
► ***Exercise***. *Show every compact subspace of a Hausdorff space is closed.*
► ***Exercise***. *Show $I = [0, 1]$ is compact. Conclude that Heine-Borel holds.*
► ***Exercise***. *Prove Bolzano-Weierstrass; use this theorem to show a subset of $\mathbb{R}^n$ is sequentially-compact if and only if it is closed and bounded.**
► ***Exercise***. *Prove compact and sequential compactness are equivalent in metric spaces.**

Definition. A space X is **locally compact** if every point has a compact neighborhood.

Proper Maps.

Definition. A map between spaces is **proper** if the preimage of every compact subspace is compact.

► ***Exercise***. *Let Y be a locally compact, Hausdorff space. Prove a proper map $X \to Y$ is closed.*
► ***Exercise***. *Give an example of a proper map whose image is not closed.*

Metrizability.

Definition. A space with a metric is said to be **metrizable**.

Theorem. (Urysohn's Theorem) *A regular, Hausdorff, second-countable space is metrizable.*

► ***Exercise***. *Prove that manifolds are metrizable.*

Algebraic Topology

Homotopy Basics.

Definition. A map is **nullhomotopic** if it is homotopic to a constant map.

Definition. A **retract** of X onto $A \subseteq X$ is a map $r\colon X \to X$ with $r(X) = A$ and $r|_A = \mathbb{1}_A$.

Definition. A **deformation retract** of a space X onto a subspace A is a homotopy $f_t\colon X \to X$ relative to A with $f_0 = \mathbb{1}_X$ and $f_1(X) = A$.

► ***Exercise***. *Relate deformation retract and retract in terms of homotopic maps.*

Definition. A map $f\colon X \to Y$ is a **homotopy equivalence** if there is a map $g\colon Y \to X$ such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$. In such a case, X and Y are said to be **homotopy equivalent** or to have the same **homotopy type**.

Definition. A space having the homotopy type of a point is called **contractible**.

► ***Exercise***. *Prove that a space is contractible if its identity map is nullhomotopic.*
► ***Exercise***. *Prove if Y is contractible, $[X, Y]$ is trivial.*
► ***Exercise***. *Prove that if X is contractible and Y is path-connected, then $[X, Y]$ is trivial.*

Fundamental Group.

Definition. The **fundamental group** of X at $x \in X$ is the group $\pi_1(X, x) = [(S^1, s), (X, x)]$ under concatenation. Given a map $f\colon X \to Y$, the map $f_*\colon \pi_1(X, x) \to \pi_1(Y, y)$ defined by $f_*([\gamma]) = [f \circ \gamma]$ is called the **induced homomorphism** of f on the fundamental group.

Definition. A space is **simply connected** if its fundamental group is trivial.

- ***Exercise****. If X retracts onto A, prove the inclusion induced homomorphism is injective. If X deformation retracts onto A, prove this map is an isomorphism.*
- ***Exercise****. If $X \to Y$ is a homotopy equivalence, prove the induced map is an isomorphism.*

Theorem. (van Kampen) *Let X be a space; $A, B \subseteq X$ path-connected; $A \cap B$ nonempty and path-connected; $x \in A \cap B$. Then $\pi_1(X, x)$ is the* ***free product with amalgamation*** *of $\pi_1(A, x)$ with $\pi_1(B, x)$, written $\pi_1(A, x) *_{\pi_1(A\cap B,x)} \pi_1(B, x)$. That is, given group presentations*

$$\pi_1(A, x) = \langle a_1, \ldots, a_\ell \mid r_1, \ldots, r_p\rangle$$
$$\pi_1(B, x) = \langle b_1, \ldots, b_m \mid s_1, \ldots, s_q\rangle$$
$$\pi_1(A \cap B, x) = \langle c_1, \ldots, c_n \mid t_1, \ldots, t_r\rangle$$

and inclusion maps $i\colon A \to X$ and $j\colon B \to X$, we have a representation

$$\pi_1(X, x) \cong \langle a_1, \ldots, a_\ell, b_1, \ldots, b_m \mid r_1, \ldots, r_p, s_1, \ldots, s_q, i_*(c_k) = j_*(c_k)\rangle.$$

- ***Exercise****. Show S^n is simply connected for $n \geqslant 2$, and compute $\pi_1(P^n)$.*
- ***Exercise****. Find a presentation for the fundamental group of the Klein bottle. Find a double covering map $p\colon T \to K$ from the torus T to K. Describe the induced homomorphism on fundamental groups.*
- ***Exercise****. Compute $\pi_1(F_g)$, the genus g closed connected orientable surface.*

Covering Spaces.

Definition. A **covering space** of X is a space $\widetilde{X}$ with a map $p\colon \widetilde{X} \to X$ such that X is covered by open sets $\{U_i\}$ having the property $p^{-1}(U_i)$ is a disjoint union of open sets in $\widetilde{X}$, each mapped homeomorphically by p onto U_i.

Proposition. (Homotopy Lifting in Coverings) *Let $\widetilde{X} \to X$ be a covering space. Given a homotopy $f_t\colon Y \to X$ and a lift $\widetilde{f}_0\colon Y \to \widetilde{X}$, there exists a unique homotopy $\widetilde{f}_t\colon Y \to \widetilde{X}$ lifting f_t.*

Proposition. *The induced map of a cover space is injective; the image subgroup consists of homotopy classes of loops whose lifts are also loops.*

Proposition. *The number of sheets of a path-connected cover space $p\colon \widetilde{X} \to X$ is the index of $p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ in $\pi_1(X, x)$.*

Proposition. (Lifting Criterion) *Let $p\colon (\widetilde{X}, \widetilde{x}) \to (X, x)$ be a cover space, Y path-connected and locally path-connected, and $f\colon (Y, y) \to (X, x)$. Then a lift of f exists iff $f_*(\pi_1(Y, y)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}))$.*

- ***Exercise****. Calculate $\pi_1(P^n)$ again. You may assume $\pi_1(S^n) = 1$, $n > 1$.*
- ***Exercise****. Prove every map $P^3 \to T^2$ is nullhomotopic. Prove $\mathbb{1}\colon P^3 \to P^3$ is not nullhomotopic.*

Definition. A space is **semilocally simply-connected** if each point $x \in X$ is contained in a neighborhood U whose inclusion induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Theorem. (Classification of Coverings) *Suppose X is path-connected, locally path-connected, and semilocally simply-connected. Then there exists a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p\colon (\widetilde{X}, \widetilde{x}) \to (X, x)$ and the set of subgroups $\pi_1(X, x)$, obtained by associating the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ to the covering space $(\widetilde{X}, \widetilde{x})$.*

Definition. The **universal cover** of a space X is a simply-connected covering space $\widetilde{X}$.

- ► ***Exercise***. *Show homotopic spaces have the same universal covers.**
- ► ***Exercise***. *Show* T^2 *and* $S^1 \vee S^1 \vee S^2$ *are not homotopy equivalent (with(out) homotopy groups).*

Definition. For a covering space $p\colon \widetilde{X} \to X$, the homeomorphisms $\widetilde{X} \to \widetilde{X}$ are called **deck transformations**. These form a group $G(\widetilde{X})$ under composition.

- ► ***Exercise***. *Draw two lifts of the 2-torus, one regular and one irregular.*

Definition. A covering space $\widetilde{X} \to X$ is **normal** (or **regular**) if for any pair of lifts $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ of a point $x \in X$, there is a deck transformation taking $\widetilde{x}_1$ to $\widetilde{x}_2$.

Proposition. *Let* $p\colon (\widetilde{X}, \widetilde{x}) \to (X, x_0)$ *be a path-connected covering space of the path-connected, locally path-connected space* X*, and let* H *be the subgroup* $p_*(\pi_1(\widetilde{X}, \widetilde{x})) \subset \pi_1(X, x)$*. Then*
- *the covering space is normal iff* H *is a normal subgroup;*
- $G(\widetilde{X})$ *is isomorphic to* $N_H(\pi_1(X, x))/H$*.*

In particular, if $\widetilde{X}$ *is a normal cover,* $G(\widetilde{X}) \cong \pi_1(X, x)/H$*.*

Definition. Given a covering space $\widetilde{X} \to X$, the group $G(\widetilde{X})$ defines an action, called the **cover action**, on $\mathrm{Homeo}(Y)$ in the obvious way: $g \cdot Y = g(Y)$ for all $g \in G(\widetilde{X})$ and $Y \in \mathrm{Homeo}(Y)$.

- ► ***Exercise***. *Show that each* $y \in Y$ *has a neighborhood* U *such that all images* $g(U)$ *for varying* $g \in G(\widetilde{X})$ *are disjoint. In other words, the cover action is free* ($g_1(U) = g_2(U)$ *implies* $g_1 = g_2$)*.**

Homology Basics.

Definition. Given a sequence of homomorphisms of abelian groups and homomorphisms

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0$ for each n is called a **chain complex**. Note then $\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$; we define the ***n*-th homology group** of the chain complex to be the quotient group $H_n(C_n) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$. Elements of H_n are **homology classes**; elements of the kernel are **cycles**; elements of the image are **boundaries**; two cycles representing the same homology class are **homologous**.

Definition. Let X be a simplicial complex and $\Delta_n(X)$ be the free abelian group with basis the open n-simplices of X. Elements of $\Delta_n(X)$, called ***n*-chains**, are finite formal sums $\sum_\alpha n_\alpha \sigma_\alpha$, where $\sigma_\alpha\colon \Delta^n \to X$ and $n_\alpha \in \mathbb{Z}$. A **boundary homomorphism** $\partial_n\colon \Delta_n(X) \to \Delta_{n-1}(X)$ is defined on each simplex σ_α by

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \ldots, \widehat{v}_i, \ldots, v_n]}.$$

The chain complex (Δ_n, ∂_n) induces the ***n*-th simplicial homology group** of X in the usual way.

- ► ***Exercise***. *Read and define cellular and singular homology.*
- ► ***Exercise***. *Calculate the homology for the* n*-sphere,* n*-projective space,* n*-torus, genus* g *torus.*
- ► ***Exercise***. *Calculate the homology of the Klein bottle* K*. According to the Classification of Surfaces, all surfaces are given up to homeomorphism by*
$$S^2, T^2, \ldots, T^2 \#_m T^2, \ldots, P^2, \ldots, P^2 \#_m P^2, \ldots.$$
Determine the surface to which K *is homeomorphic.*
- ► ***Exercise***. *Let* X *be the space constructed from the torus* T^2 *and the real projective plane* P^2 *by identifying a meridinal circle on* T^2 *to a circle on* P^2 *representing the generator of* $\pi_1 P^2$*.*
  (a) *Compute the homology groups of* X*;*
  (b) *compute the fundamental group of* X*;*

(c) *show that X has exactly two inequivalent 2-fold covers;*
(d) *one cover has infinite cyclic fundamental group and is homotopy equivalent to a wedge of spheres. Give a geometric description of this cover and specify the wedge of spheres to which it is homotopy equivalent.*

- ***Exercise***. *Calculate the homology of the space obtained by identifying the circle $S^1 \times \{1\}$ on the torus $T^2 = S^1 \times S^1$ with a meridinal circle of the Klein bottle K.*
- ***Exercise***. *Let X have path components $\bigsqcup_i X_i$, prove that $H_n(X) \cong \oplus_i H_n(X_i)$.*
- ***Exercise***. *Show that $H_0(X) = \bigoplus \mathbb{Z}$, with one copy of $\mathbb{Z}$ for each path-component.**
- ***Exercise***. (Dimension) *Show that $H_0(\{x\}) = \mathbb{Z}$ and $H_n(\{x\}) = 0$ otherwise.*

Definition. The **reduced homology groups** $\widetilde{H}_n(X)$ are the homology groups of

$$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon(\sum_\alpha n_\alpha \sigma_\alpha) = \sum_\alpha n_\alpha$.

- ***Exercise***. *Prove that $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$ and $H_n(X) \cong \widetilde{H}_n(X)$ otherwise.*

Definition. A map $f\colon X \to Y$ induces a **chain homomorphism** $f_\#\colon C_n(X) \to C_n(Y)$ by extending $f_\#(\sigma) = f\sigma\colon \Delta^n \to Y$ linearly. If $f_\#\partial = \partial f_\#$, we say that $f_\#$ is a **chain map**.

- ***Exercise***. *Show that given a chain map $f_\#\colon C_n(X) \to C_n(Y)$ the induced $f_*\colon H_n(X) \to H_n(Y)$ given by $f_*[\sigma] = [f_\#\sigma] = [f\sigma]$ is a homomorphism.**
- ***Exercise***. (Homotopy) *Show that homotopic maps induce the same homomorphism.**
- ***Exercise***. (Functoriality) *Show that $\mathbb{1}_* = \mathbb{1}$ and $(fg)_* = f_*g_*$.*
- ***Exercise***. *Show homotopy equivalences induce isomorphisms on homology.*

Definition. For a pair (X, A), the **relative chain complex** of X relative to A is the chain complex $C_n(X, A) = C_n(X)/C_n(A)$ with boundary map $\partial\colon C_n(X, A) \to C_{n-1}(X, A)$ the induced quotient map. The resulting homology groups $H_n(X, A)$ are called the **relative homology groups** of X relative to A. We define **relative cycles** and **relative boundaries** similarly.

Definition. A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *splits on the left* if there is a homomorphism $\ell\colon B \to A$ such that $\ell f = \mathbb{1}_A$; it *splits on the right* if there is a homomorphism $r\colon C \to B$ such that $gr = \mathbb{1}_C$. In either case, we say the sequence **splits**.

Lemma. (SES Lemma) *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence:* (a) *it splits on the left iff it splits on the the right;* (b) *it splits if C is free abelian;* (c) *if it splits, $A \bigoplus C \cong B$.*

Lemma. (ES Lemma) *If $\ldots \xrightarrow{p} A \xrightarrow{q} B \xrightarrow{r} C \xrightarrow{s} \ldots$ is exact, there is a short exact sequence*

$$0 \to \operatorname{coker}(p) \to B \to \ker(s) \to 0$$

where $\operatorname{coker}(p) = B/\operatorname{im}(p)$.

Theorem. (Long Exact Sequence) *For any pair (X, A) we have a long exact sequence*

$$\cdots \xrightarrow{\partial_{n+1}} \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots \to \widetilde{H}_0(X, A) \to 0 \tag{1}$$

where $i\colon A \hookrightarrow X$ is an inclusion map and $j\colon X \to X/A$ is a quotient map.

- ***Exercise***. *Show that a retract $r\colon X \to A$ induces a split short exact sequence on homology. Conclude that $H_n(X) \cong H_n(A) \oplus H_n(X, A)$ (hint: use the Splitting Lemma).*
- ***Exercise***. *Calculate the homology groups of S^n with this theorem.*

Theorem. (Excision) *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms on homology.*

Definition. A pair (X, A) consisting of a space X with nonempty, closed subspace A that is a deformation retract of some neighborhood in X is called a **good pair**.

Proposition. *For good pairs (X, A), the quotient map $(X, A) \to (X/A, A/A)$ induces isomorphisms $H_n(X, A) \cong H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$ for all n.*

- ► ***Exercise***. *Prove that if nonempty, open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$ (hint: use excision with $Z = \mathbb{R}^n \setminus U$).*

Theorem. *The long exact sequence of a pair is* ***natural****; given $f\colon (X, A) \to (Y, B)$,*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X,A) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \cdots & \longrightarrow & H_n(B) & \xrightarrow{i_*} & H_n(Y) & \xrightarrow{j_*} & H_n(Y,B) & \xrightarrow{\partial} & H_{n-1}(B) & \longrightarrow & \cdots \end{array}$$

is a commutative diagram.

- ► ***Exercise***. (Naturality) *Show that $(f|_A)_*\partial = \partial f_*$. Conclude this theorem.*

More Homology.

Definition. For a finite CW complex X, the **Euler characteristic** of X is the integral value $\chi(X) = \sum(-1)^n c_n$, where c_n is the number of n-cells in X.

- ► ***Exercise***. *Show that $\chi(X) = \sum(-1)^n$ rank$(H_n(X))$.*
- ► ***Exercise***. *Recall that every closed connected orientable surface F is homeomorphic to a connected-sum of copies of the torus. The number of copies is called the genus of F.*
  - (a) *Find the genus of a surface in terms of Euler characteristic. No proof is required;*
  - (b) *Let F be a surface of genus g, and E be an n-fold covering space of F. Derive a formula for the genus h of E in terms of g and n.*
  - (c) *Draw pictures of two 3-fold covering spaces of a surface F of genus 2, one regular and one irregular. Indicate in your pictures the curve in F along which the connected-sum is performed, and its lifts in the covers.*

Theorem. (Mayer-Vietoris) *If A, B are subspaces of X whose interiors cover X, then*

$$\cdots \xrightarrow{\partial} H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0$$

is an exact sequence.

- ► ***Exercise***. *Calculate $H_k(S^n)$ and $H_k(T^n)$ using Mayer-Vietoris.*

Definition. Let G be an abelian group. There is no obstruction to repeating the construction for simplicial homology of X so that $n_i \in G$. The resulting homology groups $H_n(X; G)$ are called the **homology groups with coefficients in G**.

Definition. A **homology theory** is a pair of functors H and ∂ assigning

- each pair of spaces (X, A) a sequence $H_*(X, A)$ of abelian groups $H_n(X, A)$;
- each pair of spaces (X, A) a sequence ∂_* of homomorphisms $H_{n+1}(X, A) \xrightarrow{\partial_n} H_n(A)$;
- each map $f\colon (X, A) \to (Y, B)$ a sequence f_* of group homomorphisms $f_n\colon H_n(X, A) \to H_n(Y, B)$;

such that the following six axioms (referenced throughout this section) hold:

- (functoriality) $\mathbb{1}_* = \mathbb{1}$ and $(fg)_* = f_* g_*$;
- (naturality) $\partial f_* = (f|_A) * \partial$;
- (homotopy) $f \simeq g$ implies $f_* = g_*$;
- (exactness) for any pair (X, A), the sequence (1) is exact;

- (excision) If $\overline{U} \subset A^{\circ}$, then $(X - U, A - U) \hookrightarrow (X, A)$ is a homology equivalence;
- (dimension) $H_n(\mathrm{pt}) = 0$ for all $n \neq 0$.

► ***Exercise***. *Prove these axioms hold for simplicial, cellular, and singular homology.**

Cohomology.

Definition. For a homomorphism $\alpha\colon A \to B$ and abelian group G, the **dual homomorphism** $\alpha^*\colon \mathrm{Hom}(B, G) \to (A, G)$ is defined by $\alpha^*(\varphi) = \varphi\alpha$.

► ***Exercise***. *Show that $(\alpha\beta)^* = \beta^*\alpha^*$ for homomorphisms $A \xrightarrow{\beta} B \xrightarrow{\alpha} C$.*

Definition. Let (C, ∂) be a chain complex of free abelian groups, and let G be an abelian group. Consider the **dual cochain** (C^n, δ^n) consisting of cochain groups $C^n = (C_n)^* = \mathrm{Hom}(C_n, G)$ and coboundary map $\delta^n = (\partial_n)^*\colon C^{n-1} \to C^n$. The homology groups $H^n(C^n; G)$ of the resulting chain complex

$$\cdots \longleftarrow C^{n+1} \xleftarrow{\delta^{n+1}} C^n \xleftarrow{\delta^n} C^{n-1} \xleftarrow{\delta^{n-1}} \cdots \xleftarrow{\delta^1} C^0 \longleftarrow 0.$$

are called the **cohomology groups**.

► ***Exercise***. *Show that the chain complex for a space X induces a cochain complex, i.e. $\delta\delta = 0$.*

► ***Exercise***. *Compute cohomology of S^n, P^n, and F_g by constructing their cochains.*

Theorem. (Universal Coefficient Theorem for Cohomology) *If a chain complex (C, δ) of free abelian groups has homology groups $H_n(C)$, then there are split short exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0.$$

for all n and all G. In particular, when $G = \mathbb{Z}$ this gives

$$\boxed{H^n(C) \cong \mathrm{Free}(H_n(C)) \oplus \mathrm{Torsion}(H_{n-1}(C)).}$$

Theorem. (Universal Coefficient Theorem for Homology) *If C is a chain complex of free abelian groups, then there are split short exact sequences*

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0$$

for all n and all G.

Theorem. (Poincaré Duality) *If M is a closed, orientable n-manifold, then for all k*

$$H^k(M) \cong H_{n-k}(M).$$

Theorem. (Lefschetz Duality) *If M is a compact orientable n-manifold whose boundary is decomposed as the union of $(n-1)$-manifolds A and B sharing a common boundary (i.e. $A \cap B = \partial A = \partial B$), then for all k*

$$H^k(M, A) \cong H_{n-k}(M, B).$$

In particular, for $A = \varnothing$ and $B = \partial M$,

► ***Exercise***. *Let M be a compact, orientable 3-manifold with boundary, and assume $H_1(M) = 0$. Prove that the boundary of M is a disjoint union of 2-spheres. Hint: find $H_1(\partial M)$.*

► ***Exercise***. *Let M be a connected closed orientable 4-manifold with $H_1(M)$ finite. Show that the Euler characteristic $\chi(M) \geqslant 2$. Compute $H_k(M)$ and $H^k(M)$ in terms of $H_1(M)$ and $\chi(M)$.*

Higher Homotopy Groups.

Definition. The ***n*-th homotopy group** of X at $x \in X$ is the group $\pi_n(X,x) = [(I^n, \partial I^n), (X,x)]$ under concatenation.

- ***Exercise***. *Show that each $\pi_n(X,x)$ is abelian for $n > 1$.*
- ***Exercise***. *Show the induced map of a covering space is an isomorphism on $\pi_n(X,x)$ for $n > 1$.*

Definition. Let I^{n-1} denote the face of I^n with last coordinate 0, and let $J^{n-1} = \overline{\partial I^n - I^{n-1}}$. The ***n*-th relative homotopy groups** of X relative to A is the group

$$\pi_n(X,A,x) = [(I^n, \partial I^n, J^{n-1}), (X,A,x)].$$

- ***Exercise***. *Show $[f] \in \pi_n(X,A,x)$ is trivial if and only if it is homotopic* rel ∂I^{n-1} *to a map with image contained in A.*
- ***Exercise***. *Read and prove the long exact sequence of a pair (X,A) in Hatcher.*

Definition. A space is ***n*-connected** if its first n homotopy groups vanish.

Definition. For any space X and integer $k > 0$, there is a homomorphism $h_k \colon \pi_k(X) \to H_k(X)$ called the **Hurewicz map**: given a choice of generator $u_k \in H_k(S^k)$, a homotopy class of maps $[f] \in \pi_k(X)$ is mapped to $f_*(u_k) \in H_k(X)$.

Theorem. (Hurewicz) *If X is $(n-1)$-connected, the Hurewicz map $h_k \colon \pi_k(X) \to H_k(X)$ is an isomorphism for all $k \leqslant n$, when $n \geqslant 2$.*

Theorem. (Whitehead) *If a map between CW-complexes induces isomorphisms on all homotopy groups, then it is a homotopy equivalence. Moreover, if this map is an inclusion, the complex deformation retracts onto its subcomplex.*

- ***Exercise***. *Prove the Dunce Cap is contractible (hint: Hurewicz and Whitehead).*

Differential Topology

Smooth Manifolds.

Definition. A **topological *n*-manifold** is a Hausdorff, second-countable space that is locally Euclidean (i.e. each point has a neighborhood homeomorphic to a subset of $\mathbb{R}^n$.

Definition. A **coordinate chart** or M is a pair (U, φ) consisting of an open subset U and a homeomorphism $\varphi \colon U \to \widehat{U}$ to an open subset $\varphi(U) = \widehat{U} \subseteq \mathbb{R}^n$. We call U a **coordinate domain**. We call φ a **coordinate map**, and we denote its component functions as $\varphi(p) = (x^1(p), \ldots, x^n(p))$

- ***Exercise***. *Show that graphs of continuous functions, spheres, and projective spaces are topological manifolds.*
- ***Exercise***. *Show that the product of manifolds is a manifold.*
- ***Exercise***. *Show that a manifold is metrizable, with countably many components.*

Definition. Two coordinate charts (U, φ), (V, ψ) are **smoothly compatible** if either $U \cap V = \varnothing$ or the transition function $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is smooth.

Definition. An **atlas** for M is a collection $\mathcal{A}$ of charts whose coordinate domains cover M. A **smooth atlas** for M is an atlas whose coordinate charts are smoothly compatible.

Definition. A smooth atlas is **maximal** if it is not properly contained in any larger smooth atlas. A **smooth structure** or M is a maximal smooth atlas.

- ***Exercise***. *Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the* ***smooth structure determined by $\mathcal{A}$.***

- ***Exercise***. *Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*

- ***Exercise***. *Prove the aforementioned graphs, spheres, and projective spaces are smooth manifolds.*

- ***Exercise***. *Prove the smooth structure $\mathcal{A} = \{(\mathbb{R}, \mathbb{1}_{\mathbb{R}})\}$ on $\mathbb{R}$ is not unique.*

- ***Exercise***. *Prove the space $M(m \times n, \mathbb{R})$ of $m \times n$ matrices with real entries is a smooth manifold, and determine its dimension.*

- ***Exercise***. *Let K be the subspace of $M(m \times n, \mathbb{R})$ consisting of all rank 2 matrices. Show that K is a manifold and determine its dimension.*

Definition. Let $\mathcal{A}$ be a smooth structure on M, and let $U \subseteq M$ be open. Then U is an **open submanifold** of M with smooth structure defined by $\mathcal{A}_U = \{(V, \varphi) \in \mathcal{A} : V \subseteq U\}$.

- ***Exercise***. *Prove that open submanifolds are manifolds.*

- ***Exercise***. *Prove that the general linear group, i.e. the set of all invertible $n \times n$ matrices with real entries, is a smooth n^2-manifold.*

- ***Exercise***. *Let $m < n$. Prove that the subset $M_m(m \times n, \mathbb{R})$ consisting of $m \times n$ matrices with rank m is a smooth manifold.*

Smooth Maps.

Definition. Let M be a smooth n-manifold. A function $f\colon M \to \mathbb{R}^k$ is a **smooth function** at $p \in M$ if there is a chart (U, φ) containing p whose **coordinate representative** $\widehat{f} = f \circ \varphi^{-1}\colon \widehat{U} \to \mathbb{R}^k$ is smooth.

- ***Exercise***. *Given a smooth function, show its coordinate representation is smooth with respect to* any *chart (U, φ) for M.*

Definition. Let M and N be a smooth manifolds. A map $F\colon M \to N$ is a **smooth map** if for every $p \in M$ there are charts (U, φ) containing p and (V, ψ) containing $f(p)$ such that $F(U) \subseteq V$ and the **coordinate representative** $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U) \to \psi(V)$.

- ***Exercise***. *Prove every smooth map is continuous. Prove smoothness is a local condition (i.e. F is smooth iff each point $p \in M$ has a neighborhood U such that $F|_U$ is smooth).*

- ***Exercise***. *Prove that constant, identity, inclusion, and projection maps are smooth.*

Definition. A smooth map with smooth inverse is called a **diffeomorphism**.

Theorem. (Extension Lemma) *Suppose M is a smooth manifold, $A \subseteq M$ is a closed subset, and $f\colon A \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\widetilde{f}\colon M \to \mathbb{R}^k$ supported in U such that $\widetilde{f}|_A = f$.*

Theorem. (Level Sets) *Let M be a smooth manifold with closed subset K. There is a smooth nonnegative function $f\colon M \to \mathbb{R}$ such that $f^{-1}(0) = K$.*

Tangent Vectors.

Definition. A linear map $w\colon C^\infty(M) \to \mathbb{R}$ is a **derivation** at $p \in M$ if it satisfies the product rule $w(fg) = f(p)w(g) + g(p)w(f)$. The **tangent space** of M at p is the set of all derivations at p, denoted T_pM. An element of T_pM is called a **tangent vector** at p.

- ***Exercise***. *Show that $T_p(\mathbb{R}^n)$ is a vector space. Note that $T_p(\mathbb{R}^n) \cong \mathbb{R}^n$ (see Lee's text).*
- ***Exercise***. *Prove: if f is constant, then $vf = 0$; if $f(p) = g(p) = 0$, then $v(fg) = 0$.*

Definition. For smooth manifolds M and N and smooth map $F\colon M \to N$, the **differential** of F at $p \in M$ is the map

$$dF_p\colon T_pM \to T_{f(p)}N,$$

defined by $dF_p(v)(f) = v(f \circ F)$ for each $v \in T_p(M)$ and each $f \in C^\infty(N)$.

- ***Exercise***. *Prove the differential is linear, maps the identity on M to the identity on T_pM, distributes over composition, and works well with inverses.*
- ***Exercise***. *Let $U \subseteq M$ be open. Prove the inclusion induced differential $d\iota_p\colon T_pU \to T_pM$ is an isomorphism for all $p \in U$.**
- ***Exercise***. *Prove the dimension of the tangent space agrees with the dimension of the manifold.*

Proposition. *Let M be a smooth n-manifold and $p \in M$. Then T_pM is an n-dimensional vector space: for any smooth chart $(U, (x^i))$ containing p, the **coordinate vectors** $\partial/\partial x^i|_p$, defined by*

$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = (d\varphi^{-1})\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right),$$

*form a basis for T_pM, which we call a **coordinate basis**.*

Proposition. *The differential dF_p is represented in coordinate bases by the Jacobian matrix of the coordinate representative of F.*

Definition. The **tangent bundle** of M, denoted TM, is the space $TM = \bigsqcup_p T_pM$. A natural **projection map** $\pi\colon TM \to M$ defined by $(p, v) \mapsto p$ defines a fiber bundle.

Theorem. *The tangent bundle TM has a natural topology and smooth structure, making it a $2(\dim M)$-dimensional smooth manifold. With respect to this smooth structure, the projection map is smooth.*

The smooth structure on TM is defined as follows: given a chart (U, φ) for M, define a chart $(\pi^{-1}(U), \widetilde{\varphi})$ for TM with $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}\left(v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n).$$

Note that with respect to charts (U, φ) for M and $(\pi^{-1}(U), \widetilde{\varphi})$ for TM, the coordinate representative of the projection is $\pi(x, v) = x$.

Definition. The **global differential** of a smooth map $F\colon M \to N$ is the map $dF\colon TM \to TN$ whose restriction to each tangent space T_pM is the differential dF_p at $p \in M$.

- ***Exercise***. *Prove the global differential is a smooth map.**
- ***Exercise***. *Prove that the global differential induces a covariant functor; that is, the **tangent functor** is a covariant functor from the category of smooth manifolds to the category of smooth vector bundles: to each smooth manifold M it assigns the tangent bundle $TM \to M$, and to each smooth map $F\colon M \to N$ it assigns the pushforward $F^*\colon TM \to TN$.**

Immersions and Submersions.

Definition. The **rank** of a smooth map $F\colon M \to N$ at $p \in M$ is defined as the rank of the linear map dF_p. If F has the same rank at each point in M, we say it has **constant rank**.

Definition. A map $f\colon M \to N$ between smooth manifolds is an **immersion/submersion** if at each $p \in M$, the differential dF_p is injective/surjective, respectively.

► ***Exercise.*** *Prove that projections are submersions. Prove a smooth curve $\gamma\colon I \to M$ is an immersion if and only if $\gamma'(t) \neq 0$ for all $t \in I$.*

Theorem. (Inverse Function Theorem) *Suppose $F\colon M \to N$ is a smooth map with dF_p invertible. There are connected neighborhoods U and V of p and $F(p)$ with $F|_U$ a diffeomorphism onto V.*

► ***Exercise.*** *Prove this theorem; you may use the Inverse Function Theorem for $\mathbb{R}^n$.*

Theorem. (Rank Theorem) *Suppose $F\colon M^m \to N^n$ is a smooth map of constant rank. For each $p \in M$, there are smooth charts (U, φ) centered at p and (V, ψ) centered at $F(p)$ such that $F(U) \subseteq V$, in which F has coordinate representation*

$$\widehat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

In particular, if F is a smooth submersion, this becomes

$$\widehat{F}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n),$$

and if F is a smooth immersion, it is

$$\widehat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

► ***Exercise.*** *Prove the Rank Theorem (hint: for immersions and submersions construct functions on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, with which you can apply the Inverse Function Theorem).*

► ***Exercise.*** *Prove if X is compact and Y is connected, every submersion $X \to Y$ is surjective.*

► ***Exercise.*** *Show that there exist no submersions of compact manifolds into Euclidean spaces.*

Theorem. (Global Rank Theorem) *If $F\colon M \to N$ is a surjective/injective smooth map of constant rank, then it is a smooth submersion/immersion, respectively.*

► ***Exercise.*** *Prove the Global Rank Theorem for immersions (hint: use Rank Theorem).*

Embeddings.

Definition. A **smooth embedding** is a smooth immersion $F\colon M \to N$ that is also a topological embedding (i.e. a homeomorphism onto its image).

► ***Exercise.*** *Show that inclusions of open submanifolds $U \hookrightarrow M$ are smooth embeddings; show inclusions into products $M_j \hookrightarrow M_1 \times \cdots \times M_k$ are smooth embeddings ($1 \leqslant j \leqslant k$).*

► ***Exercise.*** *Give an example of a topological embedding that is not a smooth immersion; give an example of a smooth immersion that is not a topological embedding.*

Theorem. (Local Embedding Theorem) *A smooth map $F\colon M \to N$ is a smooth immersion if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U$ is a smooth embedding.*

Definition. A **section** of a continuous map $\pi\colon M \to N$ is a continuous right inverse. A **local section** is a section on some open subset $U \subseteq N$.

Theorem. (Local Section Theorem) *A smooth map $\pi\colon M \to N$ is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.*

► ***Exercise.*** *Show that smooth submersions are open maps, and conclude that surjective submersions are quotient maps (do this with and without the Local Section Theorem).*

Theorem. (Characteristic Property of Surjective Smooth Submersions) *If $\pi\colon M \to N$ is a surjective smooth submersion, a map $F\colon N \to P$ is smooth if and only if $F \circ \pi$ is smooth.*

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F\pi"] & \\ N \arrow[r, "F"'] & P. \end{tikzcd}$$

Submanifolds.

Definition. An **embedded submanifold** is a subset $S \subseteq M$ that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding. The difference of dimension $\dim M - \dim S$ is the **codimension** of S in M.

► ***Exercise***. *Show that open submanifolds are embedded submanifolds with codimension 0.*

► ***Exercise***. *Show that the image of an embedding is a submanifold.*

Definition. An embedded submanifold $S \subseteq M$ is **properly embedded** if the inclusion map $S \hookrightarrow M$ is a proper map.

► ***Exercise***. *If $S \subseteq M$ is an embedded submanifold, prove S is properly embedded if and only if it is a closed subset of M. Conclude every compact embedded submanifold is properly embedded.*

Definition. Given a map $\Phi\colon M \to N$ and a point $c \in N$, we call the set $\Phi^{-1}(c)$ a **level set** of Φ.

Theorem. (Constant-Rank Level Set Theorem) *If $\Phi\colon M \to N$ is smooth with constant rank r, each level set of Φ is a properly embedded submanifold of codimension r in M.*

Definition. Let $\Phi\colon M \to N$ be a smooth map. A point $p \in M$ is a **regular point** of Φ if the map $d\Phi_p\colon T_pM \to T_{\Phi(p)}N$ is surjective; it is a **critical point** otherwise.

Definition. Let $\Phi\colon M \to N$ be a smooth map. A point $c \in N$ is a **regular value** of Φ if every point of the level set $\Phi^{-1}(c)$ is a regular point; it is a **critical value** otherwise. The preimage of a regular value is called a **regular level set**.

Theorem. (Regular Level Set Theorem) *Every regular level set of a smooth map is a properly embedded submanifold with codimension equal to the dimension of the codomain.*

► ***Exercise***. *Show that the subset of rank 1 matrices in $M(2, \mathbb{R})$ is a smooth manifold.*

► ***Exercise***. *Show that the sphere is a smooth manifold.*

Definition. If $S \subseteq M$ is an embedded submanifold realized as a regular level set of a smooth map $\Phi\colon M \to N$, we call Φ a **defining map** for S. If $U \subseteq M$ is open and $\Phi\colon U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, then Φ is called a **local defining map** for S.

Definition. Let $\iota\colon S \to M$ be the inclusion map of a submanifold. The tangent space T_pS is defined as the subspace $d\iota_p(T_pS) \subseteq T_pM$.

► ***Exercise***. *Explore the equivalent definitions of T_pS using curves/derivations restricted to S.*

Theorem. *If $\Phi\colon U \to N$ is a local defining map for a submanifold $S \subseteq M$, then $T_pS = \ker(d\Phi_p)$ for each $p \in S \cap U$. If S is a level set of $\Phi\colon M \to N$, the same equality holds for each $p \in S$.*

Transversality.

Definition. Submanifolds X and Y of a manifold M are said to intersect **transversely** if their tangent spaces span the ambient tangent space, i.e. $T_x(X) + T_x(Y) = T_x(M)$ for each $x \in X \cap Y$.

► ***Exercise***. *Let $\mathcal{H} = \{(x, y, z) : x^2 + y^2 - z^2 = 1\}$ and $\mathcal{S}_a = \{(x, y, z) : x^2 + y^2 + z^2 = a\}$. Give equations for their intersection for varying values of $a \in \mathbb{R}_{\geqslant 0}$. For which values of a do they intersect transversely?*

Multilinear Algebra.

Definition. Suppose $V_1, \dots, V_k, W$ are vector spaces; $F\colon V_1 \times \cdots \times V_k \to W$ is **multilinear** if
$$F(v_1, \dots, v_j + av_j', \dots v_k) = F(v_1, \dots, v_j, \dots, v_n) + aF(v_1, \dots, v_j', \dots, v_k)$$
for all scalars a and all $1 \leqslant j \leqslant k$. We denote the space of all multilinear maps $V_1 \times \cdots \times V_k \to W$ by $L(V_1, \dots, V_k; W)$.

Definition. Let $F \in L(V_1, \dots, V_k; \mathbb{R})$ and $G \in L(W_1, \dots, W_\ell; \mathbb{R})$. The **tensor product** of F and G is the element $F \otimes G \in L(V_1, \dots, V_k, W_1, \dots, W_\ell; \mathbb{R})$ defined by
$$F \otimes G(v_1, \dots, v_k, w_1, \dots, w_\ell) = F(v_1, \dots, v_k)G(w_1, \dots, w_\ell).$$

► ***Exercise***. *Show the tensor product operation is bilinear and associative.*

Theorem. (Multilinear Functions Basis) *Let $V_1, \dots, V_k$ be real vector spaces of dimension $n_1, \dots, n_k$. For each $1 \leqslant j \leqslant k$, let $(e_1^{(j)}, \dots, e_{n_j}^{(j)})$ be a basis for V_j and let $(\varepsilon_{(j)}^1, \dots, \varepsilon_{(j)}^{n_j})$ be a basis for V_j^*;*
$$\mathcal{B} = \{\varepsilon_{(1)}^{i_1} \bigotimes \dots \bigotimes \varepsilon_{(k)}^{i_k} : 1 \leqslant i_1 \leqslant n_1, \dots, 1 \leqslant i_k \leqslant n_k\}$$
is a basis for $L(V_1, \dots, V_k; \mathbb{R})$.

Tensors.

Definition. For any set S, a **formal linear combination** of elements of S is a function $f\colon S \to \mathbb{R}$ such that $f(s) = 0$ for all but finitely many $s \in S$. The **free real vector space** on S, denoted $\mathcal{F}(S)$, is the set of all formal linear combinations of elements of S.

Proposition. (Characteristic Property of Free Vector Spaces) *For any set S and any vector space W, every map $f\colon S \to W$ has a unique extension to a linear map $F\colon \mathcal{F}(S) \to W$.*

Definition. Let $V_1, \dots, V_k$ be real vector spaces. Let $\mathcal{R}$ be the subspace of $\mathcal{F}(V_1 \times \cdots \times V_k)$ spanned by all elements of the following forms:
$$(v_1, \dots, av_i, \dots, v_k) = a(v_1, \dots, v_i, \dots v_k)$$
$$(v_1, \dots, v_i + v_i', \dots, v_k) - (v_1, \dots, v_i, \dots, v_k) - (v_1, \dots, v_i', \dots, v_j)$$
with $v_i, v_i' \in V_i$, $i \in \{1, \dots, k\}$, and $a \in \mathbb{R}$. We define the **tensor product of $\boldsymbol{V_1, \dots, V_k}$** to be the space
$$V_1 \bigotimes \dots \bigotimes V_k = \mathcal{F}(V_1 \times \cdots \times V_k)/\mathcal{R},$$
with natural projection Π. The equivalence class of an element $(v_1, \dots, v_k)$ is denoted
$$v_1 \bigotimes \dots \bigotimes v_k = \Pi(v_1, \dots, v_k),$$
and is referred to as the (abstract) **tensor product of $\boldsymbol{v_1, \dots, v_k}$**.

► ***Exercise***. *Show that abstract tensor products behave multilinearly, meaning*
$$v_1 \otimes \dots \otimes av_i \otimes \dots \otimes v_k = a(v_1 \otimes \dots \otimes v_i \otimes \dots \otimes v_k,$$
$$v_k \otimes \dots \otimes (v_i + v_i') \otimes \dots \otimes v_k = (v_1 \otimes \dots \otimes v_i \otimes \dots \otimes v_k) + (v_1 \otimes \dots \otimes v_i' \otimes \dots \otimes v_k).$$

Proposition. (Characteristic Property of Tensor Product Spaces) *Let $V_1, \dots, V_k$ be finite-dimensional real vector spaces. If $f\colon V_1 \times \cdots \times V_k \to X$ is any multilinear map into a vector space X, then there is a unique linear map $F\colon V_1 \otimes \dots \otimes V_k \to X$ such that the following diagram commutes:*

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "f"] \arrow[d, "\Pi"'] & X \\ V_1 \otimes \dots \otimes V_k \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

Proposition. (Basis of Tensor Product Space) *Let $V_1, \dots, V_k$ be real vector spaces of dimensions $n_1, \dots, n_k$. For each $j = 1, \dots, k$ suppose $(E_1^{(j)}, \dots, E_{n_j}^{(j)})$ is a basis for V_j. Then the set*

$$\mathcal{C} = \{E_{i_1}^{(1)} \otimes \dots \otimes E_{i_k}^{(k)} : 1 \leqslant i_1 \leqslant n_1, \dots, 1 \leqslant i_k \leqslant n_k\}$$

is a basis for $V_1 \otimes \dots \otimes V_k$.

Proposition. (Associativity of Tensor Product Space) *Let V_1, V_2, V_3 be finite-dimensional real vector spaces. There are unique isomorphisms*

$$V_1 \otimes (V_2 \otimes V_3) \cong V_1 \otimes V_2 \otimes V_3 \cong (V_1 \otimes V_2) \otimes V_3,$$

under which elements of the forms $v_1 \otimes (v_2 \otimes v_3)$, $v_1 \otimes v_2 \otimes v_3$, and $(v_1 \otimes v_2) \otimes v_3$ all correspond.

Proposition. *If $V_1, \dots, V_k$ are finite-dimensional vector spaces, there is a canonical isomorphism*

$$V_1^* \otimes \dots \otimes V_k^* \cong L(V_1, \dots, V_k; \mathbb{R}),$$

under which the abstract tensor product corresponds to the tensor product of covectors.

Vector Bundles.

Definition. Let $\pi \colon E \to M$ be a vector bundle. A **section** of E is a section of the map π, that is, a continuous right inverse $\sigma \colon M \to E$.

Definition. A **local section** of E is a continuous right inverse $\sigma \colon U \to E$ on some open $U \subseteq M$.

Definition. Let $\pi \colon E \to M$ be a vector bundle. If $U \subseteq M$ is open, a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ on E over U is **linearly independent** if their values $(\sigma_1(p), \dots, \sigma_k(p))$ form a linearly independent k-tuple in E_p for each $p \in U$. Similarly, the are said to **span** E if their values span E_p for each $p \in U$.

Definition. A **local frame** for E over U is an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ of linearly independent local sections over U that span E, that is, $(\sigma_1(p), \dots, \sigma_k(p))$ is a basis for E_p for each $p \in U$. This is a **frame** if $U = M$.

Orientation.

Definition. Let V be a real vector space of dimension $n \geqslant 1$. We say ordered basis $(e_1, \dots, e_n)$ and $(\widetilde{e}_1, \dots, \widetilde{e}_n)$ are **consistently oriented** if the transition matrix $e_i = B_i^j \widetilde{e}_j$ has positive determinant.

Definition. Let V be a real vector space of dimension $n \geqslant 1$. An **orientation** for V is an equivalence class of ordered basis, where two consistently oriented bases are equivalent. Any element in the orientation is said to be **positively oriented**; otherwise it is **negatively oriented**.

Proposition. *Let V be a real vector space of dimension $n \geqslant 1$. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines an orientation $\mathcal{O}_\omega$, the set of ordered bases $(e_1, \dots, e_n)$ such that $\omega(e_1, \dots, e_n) > 0$. We say that ω is a **positively oriented covector**.*

Definition. If (E_i) is a local frame for TM, we say that (E_i) is **positively oriented** if $(E_1|_p, \dots, E_n|_p)$ is a positively oriented basis for T_pM at each $p \in U$.

Definition. A pointwise orientation is **continuous** if every point of M is in the domain of an oriented local frame. An **orientation** for M is a continuous pointwise orientation.

Proposition. *Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.*

Tensors on Vector Spaces.

Definition. A ***p*-tensor** on a real vector space V is an element of the k-fold product $V^* \otimes \ldots \otimes V^*$; that is, a real-valued multilinear function $T\colon V \times \cdots \times V \to \mathbb{R}$. A 0-tensor is, by convention, a real number. The space of k-tensors is denoted $T^k(V^*)$.

Theorem. *If V^* has basis $\{\phi_1, \ldots, \phi_n\}$, then*
$$\{\phi_{i_1} \otimes \ldots \otimes \phi_{i_k} : 1 \leqslant i_1 \lneq \cdots \lneq i_k \leqslant n\}$$
forms a basis for $T^k(V^*)$. Consequently, $\dim T^k(V^*) = n^k$.

Definition. If T is a k-tensor and S is a ℓ-tensor, we define a $k+\ell$ tensor $T \otimes S$ by
$$T \otimes S(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = T(v_1, \ldots, v_k) \cdot S(v_{k+1}, \ldots, v_{k+\ell}).$$
The resulting tensor $T \otimes S$ is called the **tensor product** of T and S.

Definition. A k-tensor T is **alternating** if its sign is reversed whenever variables are transposed:
$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$
The alternating k-tensors form a vector subspace $\Lambda^k(V^*)$ of $\mathcal{T}^k(V^*)$.

- ***Exercise***. *Give two examples of k-tensors on $\mathbb{R}^k$ (one alternating, one not).*
- ***Exercise***. *Show that the tensor product is associative* and that it distributes over addition.*

Definition. Let $(-1)^\pi = \pm 1$ be the parity of $\pi \in S_k$. For a k-tensor T, define
$$T^\pi(v_1, \ldots, v_k) = T(v_{\pi(1)}, \ldots, v_{\pi(k)}), \qquad \text{and} \qquad \mathrm{Alt}(T) = \frac{1}{k!}\sum_{\pi \in S_k} (-1)^\pi T^\pi,$$
both of which are p-tensors.

- ***Exercise***. *Prove that T is alternating if $T^\pi = (-1)^\pi T$ for all $\pi \in S_k$.*
- ***Exercise***. *Prove that $(T^\pi)^\sigma = T^{\pi\sigma}$ for all $\pi, \sigma \in S_k$.*
- ***Exercise***. *Prove $\mathrm{Alt}(T)$ is alternating. Prove that T is alternating if and only if $\mathrm{Alt}(T) = T$.*

Definition. If $T \in \Lambda^k$, $S \in \Lambda^\ell$, we define their **wedge product** $T \wedge S = \mathrm{Alt}(T \otimes S) \in \Lambda^{k+\ell}$.

Theorem. *If V^* has basis $\{\phi_1, \ldots, \phi_n\}$, then*
$$\{\phi_I = \phi_{i_1} \wedge \cdots \wedge \phi_{i_k} : 1 \leqslant i_1 \lneq \cdots \lneq i_k \leqslant n\}$$
forms a basis for $\Lambda^k(V^*)$. Consequently, $\dim \Lambda^k(V^*) = \frac{n!}{k!(n-k)!}$.

- ***Exercise***. *Prove the wedge is associative* and distributes over addition and scalar multiplication.*
- ***Exercise***. *Prove that $\phi_I = \pm\phi_J$ when two index sequences I, J differ only in their orderings. Prove that $\phi_I = 0$ when any index of I is repeated.*
- ***Exercise***. *Prove $\phi_I \wedge \phi_J = (-1)^{k\ell}\phi_J \wedge \phi_I$ for any properly defined sequences I, J of length k, ℓ respectively. Conclude that $T \wedge S = (-1)^{k\ell} S \wedge T$ for any $T \in \Lambda^k$ and $S \in \Lambda^\ell$.*
- ***Exercise***. *By the previous theorem, $\dim \Lambda^k((\mathbb{R}^k)^*)$ is one dimensional when $k = \dim(\mathbb{R}^k)^*$. Why does this make sense? Consider the known alternating k-tensors on $(\mathbb{R}^k)^*$.*

Definition. We have, by convention, that $\Lambda^0(V^*) = \mathbb{R}$, and we extend the wedge product of any element in $\mathbb{R}$ with any tensor in $\Lambda^k(V^*)$ as the usual scalar multiplication. The wedge product then makes the direct sum
$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^n(V^*)$$
a noncommutative algebra, called the **exterior algebra of V^***, with identity element $1 \in \Lambda^0(V^*)$.

Proposition. *If $v^1, \ldots, v^k$ are covectors on V, then $v^1 \wedge \cdots \wedge v^k(v_1, \ldots, v_k) = \det(v^j(v_i))$.*

Differential Forms.

Definition. Let X be a smooth manifold, and recall that an element of the vector bundle

$$\Lambda^k(T^*X) = \coprod_{x \in X} \Lambda^k(T^*_x X)$$

is an alternating k-form ω that assigns to each point $x \in X$ an alternating p-tensor $\omega(x)$ on the tangent space of X at x. A ***p*-form** on X is a smooth section of the projection $\Lambda^k(T^*X) \to X$. The space of all k-forms on X is denoted $\Omega^k(X)$.

- ► ***Exercise***. *Characterize* 0*-forms.*
- ► ***Exercise***. *If* $\phi\colon X \to \mathbb{R}$ *is a smooth function,* $d\phi_x\colon T_x(X) \to \mathbb{R}$ *is a linear map, so the assignment* $x \mapsto d\phi_x$ *defines a 1-form* $d\phi$ *on* X*. In this manner, the coordinate functions* x^i *on* $\mathbb{R}^k$ *yield 1-forms* dx^i *on* $\mathbb{R}^k$*. At each* $z \in \mathbb{R}^k$*, what familiar objects are the* $dx^i(z)$*?*

Proposition. *Every* p*-form on an open set* $U \subseteq \mathbb{R}^k$ *may be uniquely expressed as a sum* $\sum_I f_I dx^I$ *over increasing index sequences* $I = (i_1 < \ldots, < i_p)$*, where each* f_I *is a function on* U*.*

Definition. If $f\colon X \to Y$ is smooth and ω is a smooth p-form on Y, we define the **pullback** of ω by f to be the p-tensor

$$f^*(\omega)(x) = (df_x)^*\omega[f(x)],$$

where $(df_x)^*$ is the dual (or transpose) map defined by $(df_x)^*\omega[f(x)] = \omega[f(x)] \circ df_x$.

- ► ***Exercise***. *Prove that* $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$ *and prove* $f^*(\omega \wedge \theta) = f^*\omega \wedge f^*\theta$ *and prove* $(fh)^*\omega = h^*f^*\omega$ *for* p*-forms* $\omega, \omega_1, \omega_2$ *and a* q*-form* θ*.*
- ► ***Exercise***. *Prove* $f^*dx^i = df^i$ *and conclude* $f^*(\omega) = \sum_I (f^*a_I) df^I$ *for* $\omega = \sum_I a_I dx^I$*.*
- ► ***Exercise***. *Prove that if* $f\colon X \to Y$ *smooth and* ϕ *a chart on* Y*, then* $f^*(d\phi) = d(f^*\phi)$*.*
- ► ***Exercise***. *Note: one of the reasons we want to look at an* n*-form* ω *on an* n*-manifold* M *is to be able to assign volume to a small region. The previous section showed these* ***volume forms*** *are essentially determinants on the tangent space. This makes sense because determinants (i) record volume of parallelepipeds and (ii) act like alternating* n*-tensors.*

Exterior Derivatives.

Definition. If $\omega = \sum_J \omega_J dx^J$ is a smooth k-form on an open subset $U \subseteq \mathbb{R}^n$, we define its **exterior derivative** $d\omega$ to be the following $(k+1)$-form:

$$d\left(\sum_J{}' \omega_J dk^J\right) = \sum_J{}' d\omega_J \wedge dx^J,$$

where $d\omega_J$ is just the differential of the function ω_J.

Integration on Manifolds.

Definition. Let $D \subseteq \mathbb{R}^n$ be a **domain of integration** (i.e. a bounded subset whose boundary has measure 0), and let $\omega = f dx^1 \wedge \cdots \wedge dx^n$ be an n-form on $\overline{D}$, where $f\colon \overline{D} \to \mathbb{R}$ is a continuous function. We define the **integral of $\boldsymbol{\omega}$** over D to be

$$\int_D f dx^1 \wedge \cdots \wedge dx^n = \int_D \omega = \int_D f dx^1 \ldots dx^n$$

Definition. Suppose M is an oriented smooth n-manifold and ω is an n-form, compactly supported in the domain of a chart (U, φ) that is either positively or negatively oriented. We define the **integral of $\boldsymbol{\omega}$ over $\boldsymbol{M}$** to be the value

$$\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^*\omega,$$

where the sign agrees with the positive/negative orientation.

Definition. Suppose M is an oriented smooth n-manifold and ω is a compactly supported n-form on M. Let $\{U_i\}$ be a finite open cover of $\operatorname{supp}(\omega)$ by domains of positively or negatively oriented charts, and let $\{\psi_i\}$ be a subordinate smooth partition of unity. Define the **integral of $\boldsymbol{\omega}$ over $\boldsymbol{M}$** to be the value

$$\int_M \omega = \sum_i \int_M \psi_i \, \omega$$

► ***Exercise***. *Show the above definition does not depend on choice of charts or partition of unity.*[*]

Theorem. (Stokes's Theorem) *Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$