THINGS TO REMEMBER
ANALYSIS

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Abstract. These are definitions, facts, theorems, or likewise that I feel are particularly important or difficult to remember. Organization is broken into sections of the exam.

1. Metric Spaces

Topics: completeness, compactness, connectedness, Baire category theorem, spaces of continuous functions, contraction mapping theorem, Weierstrass approximation theorem

Definition. An element x of a metric space X is a **limit point** of a subset $A \subseteq X$ if any neighborhood of x contains an element of $A \setminus x$.

Definition. A metric space is **compact** if every open cover has a finite subcover. A metric space is **sequentially compact** if every sequence has a convergent subsequence. A metric space is **limit point compact** if every infinite set of points has a limit point.

Proposition. *A metric space is compact iff limit point compact iff sequential compact.*

Definition. Let (X, d) be a metric space. A map $T \colon X \to X$ is a **contraction map** if there exists some $c \in [0, 1)$ such that $d(x, y) \leqslant cd(Tx, Ty)$ for all $x, y \in X$.

Theorem. (Contraction Mapping Thm) *Let (X, d) be a nonempty, complete metric space. Then T has a unique fixed point, obtained by considering the sequence $x_n = T(x_{n-1})$ with arbitrary $x_0 \in X$.*

Theorem. (Weierstrass Approximation) *Suppose f is a continuous real-valued function. Then for any $\varepsilon > 0$, there is a polynomial $p(x)$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$.*

2. Analytic Functions

Topics: Analytic functions, Cauchy's theorem and integral formula; harmonic functions and the maximum principle; Laurent series; isolated singularities, residues, and applications to evaluation of real integrals; analytic continuation; the argument principle, Rouché's theorem; conformal maps and the Riemann mapping theorem (know statement)

Definition. A **region** is a nonempty, open, connected subset of the complex plane.

Theorem. *Given an open set $A \subseteq \mathbb{C}$ and a function $f\colon A \to \mathbb{C}$, $f(z) = u(x,y) + iv(x,y)$, the* ***Cauchy-Riemann equations***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied if and only if $f'(z_0)$ exists at $z_0 = (x_0, y_0)$ in A.

Definition. Given a smooth curve $\gamma\colon I \to \mathbb{C}$ with $\gamma = u + iv$, define

$$\int_0^1 \gamma(t)\ dt = \int_0^1 u(t)\ dt + i\int_0^1 v(t)\ dt.$$

We call such a function γ a **contour**. Moreover, if $A \subseteq \mathbb{C}$ is open with $\gamma(I) \subseteq A$ and $f\colon A \to \mathbb{C}$ is continuous, the value

$$\int_\gamma f(z)\ dz = \int_0^1 f(\gamma(t))\gamma'(t)\ dt$$

is called the **contour integral** of a continuous function $f\colon A \to \mathbb{C}$ along γ.

Theorem. (Fundamental Theorem of Contour Integrals) *Suppose $\gamma\colon I \to \mathbb{C}$ is a smooth curve and F is analytic on some open $A \subseteq \mathbb{C}$ containing $\gamma(I)$. Assume F' is continuous (unnecessary). Then*

$$\int_\gamma F'(z)\ dz = F(\gamma(1)) - F(\gamma(0)),$$

and in particular, if γ is closed, $\int_\gamma F' = 0$.

Theorem. *Suppose f is continuous on an open set $A \subseteq \mathbb{C}$. The following are equivalent:*

(i) *if γ_1 and γ_2 are smooth curves in A with common endpoints, then $\int_{\gamma_1} f = \int_{\gamma_2} f$;*
(ii) *there exists an analytic function $F\colon A \to \mathbb{C}$ with $F' = f$;*
(iii) *if Γ is a closed curve in A, then $\int_\Gamma f = 0$.*

Theorem. (Cauchy's Theorem) *Let $A \subseteq \mathbb{C}$ be open, $f\colon A \to \mathbb{C}$ analytic, and γ a closed curve which is nullhomotopic in A. Then $\int_\gamma f = 0$.*

Definition. Let $\gamma\colon A \to \mathbb{C}$ be a curve in $\mathbb{C}$ with z_0 a point not in the image of γ. The value

$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}$$

is called the **winding number** of γ around z_0.

Theorem. (Cauchy's Integral Formula) *Let $f\colon A \to \mathbb{C}$ be analytic. Then for any closed loop γ which is nullhomotopic in A and any $z_0 \in A$ not in the image of γ, we have*

$$f^{(k)}(z_0) I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{k+1}} dz$$

for all $k \in \mathbb{N}$. In particular, all derivatives of f exist.

Theorem. (Cauchy's Inequalities) *Let f be analytic on a region A, containing the disk bound by a circle γ in A of radius R centered at z_0. If f is bounded on γ by some $M > 0$, then for all $k \in \mathbb{N}$*
$$f^{(k)}(z_0) \leqslant \frac{k!M}{R^k}.$$

Theorem. (Liouville's Theorem) *A bounded, entire function is constant.*

Theorem. (Fundamental Theorem of Algebra) *A polynomial of degree $n \geqslant 1$ has a root.*

Theorem. (Morera's Theorem) *Let f be continuous on a region A, and suppose $\int_\gamma f = 0$ for all closed curves γ in A. Then f is analytic on A with analytic antiderivative.*

Theorem. (Maximum Modulus Principle) *Suppose $A \subset \mathbb{C}$ is open, connected, and bounded. Let $u\colon \overline{A} \to \mathbb{R}$ be analytic on A and continuous on $\overline{A}$. Then $|f|$ has a finite maximum value on $\overline{A}$, attained at some point on ∂A. If this value is attained on the interior, the function is constant.*

Theorem. (Schwartz Lemma) *Let $f\colon D \to D$ be analytic on the open unit disk D with $f(0) = 0$. Then $|f'(z)| \leqslant 1$ and $|f(z)| \leqslant |z|$ for all $z \in D$. If $|f'(0)| = 1$ or if there is a point $z_0 \neq 0$ for which $|f(z_0)| = |z_0|$, then there is a constant c , $|c| = 1$ such that $f(z) = cz$ for all $z \in D$.*

Definition. A twice differentiable function $u\colon A \to \mathbb{R}$ on an open set A is **harmonic** if its Laplacian vanishes, that is, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$.

Theorem. (Maximum Principle) *Suppose $A \subset \mathbb{C}$ is open, connected, and bounded. Let $u\colon \overline{A} \to \mathbb{R}$ be continuous and harmonic on A and M be the maximum of u on ∂A. Then*

(i) *$u(x, y) \leqslant M$ for all $(x, y) \in A$;*
(ii) *if $u(x, y) = M$ for some $(x, y) \in A$, then u is constant on A.*

A corresponding statement holds for the minimum, obtained by applying the above to $-u$.

Theorem. (Analytic Convergence Theorem) *Let (f_n) be a sequence of analytic functions on an open set $A \subseteq \mathbb{C}$. Then the following hold:*

(i) *if $f_n \to f$ uniformly on closed disks in A, then f is analytic. Moreover, $f_k' \to f'$ pointwise on A and uniformly on closed disks in A;*
(ii) *if $\sum_n f_n \to f$ uniformly on closed disks in A, then f is analytic. Moreover, $f' = \sum_n f_n'$ pointwise on A and uniformly on closed disks.*

Theorem. (Taylor's Theorem) *The **Taylor series** of a function f, analytic on a disk $B(z_0, r)$ around some $z_0 \in \mathbb{C}$, is given by*
$$f(z) = \sum_n \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$
This series converges pointwise to $f(z)$ on $B(z_0, r)$ and converges uniformly on closed disks in $B(z_0, r)$. Furthermore, the Taylor series diverges on $\mathbb{C} \setminus \overline{B(z_0, r)}$.

Examples:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ for } |z| < 1; \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ everywhere};$$
$$\sin(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{2n-1!} \text{ everywhere}; \qquad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n!)} \text{ everywhere};$$

Theorem. (Laurent's Theorem) *Let f be analytic on an annulus A about $z_0 \in \mathbb{C}$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},$$

*where both series converge absolutely on A and uniformly on radially dilated annuli contained in A. This series is called the **Laurent series** of f about z_0.*

Definition. If f is analytic on some ε-neighborhood of z_0, then z_0 is an **isolated singularity**:

(i) if all but finitely many b_n are zero, then z_0 is a **pole**, with the lowest index of a nonzero b_k referred to as the **order** of the pole;
(ii) if all the b_k's are nonzero, then z_0 is an **essential singularity**;
(iii) if all the b_k'z are zero, then z_0 is a **removable singularity**;

In general, a pole of order 1 is a **simple pole** and $b_1 = \mathrm{Res}(f; z_0)$ is the **residue** of f at z_0.

Properties:

- z_0 is a removable singularity iff $\lim_{z\to z_0}(z-z_0)f(z) = 0$;
- z_0 is a simple pole iff $\lim_{z\to z_0}(z-z_0)f(z)$ exists and is nonzero, with value the residue;
- z_0 is a pole of order $k \geqslant 0$ iff there is a function ϕ analytic near z_0 such that $f(z) = \phi(z)/(z-z_0)^k$;
- if z_0 is a simple pole of $g(z)/h(z)$ with $g(z_0) = h(z_0) = 0$ and $h'(z_0) \neq 0$, then $b_1 = g(z_0)/h'(z_0)$.

Theorem. (Residue Theorem) *Let γ be a curve nullhomotopic in A, and let f be analytic on the region $A \subseteq \mathbb{C}$ except for finitely many isolated singularities $\{z_1, \ldots, z_n\}$, none lying on γ. Then*

$$\int_\gamma f(z)\ dz = 2\pi i \sum_{i=1}^{n} Res(f; z_i) I(\gamma; z_i).$$

In general, we consider circles γ oriented counterclockwise, simplifying the calculation.

Theorem. (Jordan's Lemma) *Let f be analytic on a semicircle S_r of radius $r > 0$ centered at 0 contained in the upper-half plane. If $f(z) = g(z)e^{aiz}$ for some $a > 0$, then*

$$\left| \int_{S_r} f(z) dz \right| \leqslant \frac{\pi}{a} M_r,$$

where M_r is the max of $|g|$ on S_r.

Definition. A map $f\colon A \to B$ is **conformal** if for each $z_0 \in A$, f rotates tangent vectors to curves at z_0 by a definite angle θ and stretches them by a definite factor r.

Theorem. (Conformal Mapping Theorem) *Let $f\colon A \to B$ be analytic, $f' \neq 0$. Then f is conformal.*

Theorem. *If $f\colon A \to B$ is conformal and bijective, then f^{-1} is conformal.*

Theorem. (Riemann Mapping Theorem) *Let A be a connected, simply connected region, except $\mathbb{C}$. Then for any $z_0 \in A$, there exists a unique bijective, conformal map $f\colon A \to D$ to the open unit disk such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Definition. A **fractional linear transformation** is a conformal map T of the form

$$T(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are fixed complex numbers satisfying $ad - bc \neq 0$ (to avoid T constant).

Theorem. (Cross Ratio) *Given distinct triples (w_1, w_2, w_3) and (z_1, z_2, z_3) of complex numbers, there exists a unique fractional lineal transformation T taking $z_i \mapsto w_i$, satisfying*

$$\frac{Tz - w_1}{Tz - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}.$$

Theorem. (Analytic Continuation) *If f and g are analytic on A and agree on a sequence (z_n) converging to $z_0 \in A$, then $f \equiv g$ on A.*

Theorem. (Argument Principle) *Let γ be a contour, and let f be analytic inside and along γ, except at finitely many poles. Then*

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum 2\pi i(Z - P),$$

where Z and P denote the number of zeros and poles of f inside γ, respectively.

Theorem. (Rouché's Theorem) *Let f and g be analytic on a region A with ∂A a simple closed curve. If $|g| \leqslant |f|$ on ∂A, then f and $f+g$ have the same number of roots inside A, with multiplicity.*

3. Measure Theory

Topics: Riemann-Stieltjes Integral, measures, measurable functions and Lebesgue integral, Lebesgue measure, Fubini's theorem, Borel measures, absolute continuity, Lebesgue and Radon-Nikodym theorems, L^p spaces, Riesze representation theorem, differentiation of measures

Riemann-Steiltjes Integral.

Definition. Given a bounded function $g\colon I \to \mathbb{R}$ and a partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $I = [a, b]$, let $I_k = [x_k, x_{k+1}]$ and $m_k = \inf_{I_k} g$ and $M_k = \sup_{I_k} g$. If f is a nondecreasing function on I, let $\Delta_k f = f(x_{k+1}) - f(x_k)$. The **lower sum** and **upper sum** of g corresponding to $\mathcal{P}$ with respect to f are, respectively,

$$s(g, f, \mathcal{P}) = \sum_{k=0}^{n-1} m_k \Delta_k f, \qquad S(g, f, \mathcal{P}) = \sum_{k=0}^{n-1} M_k \Delta_k f.$$

The **lower Riemann-Stieltjes** and **upper Riemann-Stieltjes** of g w.r.t. f are, respectively,

$$L(g, f, I) = \sup_{\mathcal{P}} s(g, f, \mathcal{P}), \qquad U(g, f, I) = \inf_{\mathcal{P}} S(g, f, \mathcal{P}).$$

If $L(g, f, I) = U(g, f, I)$, we say that g is **Riemann-Stieltjes integrable** w.r.t. f over I and write $g \in \mathcal{R}(f, I)$; this common value is denoted $\int_a^b g\ df$. Setting $f(x) = x$ gives the Riemann integral. The class of Riemann-Stieltjes integrable functions on I is denoted $\mathcal{R}(I)$.

Properties:

- if g is bounded on I, f nondecreasing on I, then $g \in \mathcal{R}(f, I)$ if and only if given $\varepsilon > 0$ there exists a partition $\mathcal{P}$ such that $S(g, f, \mathcal{P}) - s(g, f, \mathcal{P}) < \varepsilon$;
- if g is continuous and f is BV, then $g \in \mathcal{R}(f, I)$;
- if f BV on I, $\{g_n\}$ a sequence of bounded functions $g_n \rightrightarrows g$ on I, and $g_n, g \in \mathcal{R}(f, I)$, then $\lim_{n\to\infty} \int_a^b g_n\ df = \int_a^b g\ df$.

Algebras.

Definition. A class $\mathcal{A}$ of subsets of X is an **algebra** if

(i) $\mathcal{A}$ is nonempty;
(ii) $E \in \mathcal{A}$ implies $X \setminus E \in \mathcal{A}$;
(iii) $\{E_k\}_1^n \subseteq \mathcal{A}$ implies $\bigcup_1^n E_k \in \mathcal{A}$.

Definition. An algebra $\mathcal{A}$ of subsets of X is a **$\boldsymbol{\sigma}$-algebra** if $\{E_k\}_1^\infty \subseteq \mathcal{A}$ implies $\bigcup_1^\infty E_k \in \mathcal{A}$.

Definition. Given a sequence $\{A_n\}$ of sets, the sets **$\limsup A_n$** and **$\liminf A_n$** are, respectively

$$\limsup A_n = \bigcap_{m=1}^\infty \left(\bigcup_{n=m}^\infty A_n \right), \qquad \liminf A_n = \bigcup_{m=1}^\infty \left(\bigcap_{n=m}^\infty A_n \right).$$

In words, $\limsup A_n$ is the set of elements belonging to countably many A_n's, whereas $\liminf A_n$ are those belonging to all but finitely many.

Properties:

- if $\mathcal{A}$ is an algebra on X and $E \subset X$, then $\mathcal{A}_E = \{E \cap A : A \in \mathcal{A}\}$ is an algebra;
- common algebras: $\mathcal{A} = \{\varnothing, X\}$, $\mathcal{A} = \mathcal{P}(X)$, $\mathcal{E} = \{$ finite unions of $(a, b] : a, b \in \mathbb{R}\}$;
- $\{\varnothing, X\}$ and $\mathcal{P}(x)$ are σ-algebras, whereas $\mathcal{E}$ is not;
- if E_1, E_2 belong to an algebra $\mathcal{A}$, then $E_1 \cap E_2$ and $E_1 \setminus E_2 \in \mathcal{A}$;
- if $\{A_n\} \subseteq \mathcal{A}$, a σ-algebra, then $\limsup A_n, \liminf A_n \in \mathcal{A}$.

Definition. If $\mathcal{C}$ is a collection of subsets of X, the intersection $\mathcal{S}(\mathcal{C})$ of all σ-algebras containing $\mathcal{C}$ is a σ-algebra on X called the ***σ*-algebra generated by *𝒞***.

Definition. Let $\mathcal{O}$ be the set of all open subsets of $\mathbb{R}^n$. The σ-algebra $\mathcal{S}(\mathcal{O})$ is called the **Borel *σ*-algebra** on $\mathbb{R}^n$.

Definition. Given a set X and a σ-algebra $\mathcal{M}$ on X, we say that μ is a **measure** provided

(i) $\mu\colon \mathcal{M} \to [0, \infty]$ and $\mu(\varnothing) = 0$;
(ii) if $\{E_k\}_1^\infty \subseteq \mathcal{M}$ a sequence of disjoint sets, then
$$\mu\Big(\bigcup_{k=1}^\infty E_k\Big) = \sum_{k=1}^\infty \mu(E_k).$$

We say $(X, \mathcal{M}, \mu)$ is a measure space; the sets in $\mathcal{M}$ are measurable sets.

Definition. A measure space is **complete** if any subset of a nullset is also measurable and null.

Definition. A measure is ***σ*-finite** if X is the countable union of μ-finite measurable sets.

Properties:

- (monotone) if E, F are measurable and $E \subseteq F$, then $\mu(E) \leqslant \mu(F)$;
- (subtractive) if in addition to the first bullet, $\mu(E) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$;
- (σ-additive) if $\{E_k\}$ a sequence of measurable sets, then $\mu\big(\bigcup E_k\big) \leqslant \sum \mu(E_k)$;
- (continuity from below) if $\{E_k\}$ sequence of nondecreasing measurable sets, $\mu\big(\bigcup E_k\big) = \lim_k \mu(E_k)$;
- (continuity from above) if $\{E_k\}$ sequence of nonincreasing measurable sets and $\mu(E_k) < \infty$ for some k, then $\mu\big(\bigcap E_k\big) = \lim_k \mu(E_k)$;
- (Borel-Cantelli) if $\{E_n\}$ are measurable with $\sum_k \mu(E_k) < \infty$, then $\mu(\limsup E_k) = 0$.

Lebesgue Measure.

Definition. The **volume** of a parallelepiped, i.e. a compact set
$$I^k = \{(x_1, \ldots, x_n) : a_k \leqslant x_k \leqslant b_x, 1 \leqslant k \leqslant n\} \subset \mathbb{R}^n$$
is given by $v(I^k) = \prod_{k=1}^n (b_k - a_k)$.

Definition. The **outer measure** of a subset $A \subseteq \mathbb{R}^n$ is the quantity
$$|A|_e = \inf\Big\{ \sum_k v(I_k) : A \subseteq \bigcup_k I_k \Big\},$$
where the infimum is taken over all countable coverings of A by closed intervals.

Definition. A subset of $\mathbb{R}^n$ is a ***G***$\boldsymbol{G_\delta}$ **set** if it is the intersection of an at most countable family of open sets. The complement of a G_δ set is an $\boldsymbol{F_\sigma}$ **set**, i.e. an at most countable union of closed sets.

Properties:

- (monotone) if $A \subseteq B$, then $|A|_e \leqslant |B|_e$;
- outer measure agrees with volume on open/closed intervals, i.e. $|I^k|_e = v(I^k)$;
- (σ-subadditive) any sequence $\{E_k\}$ of subsets of $\mathbb{R}^n$ satisfy $|\bigcup_k E_u|_e \leqslant \sum_k |E_k|_e$;
- for any $E \subseteq \mathbb{R}^n$, we have $|E|_e = \inf\{|\mathcal{O}|_e : \mathcal{O} \text{ open}, E \subseteq \mathcal{O}\}$;
- the outer measure of $E \subseteq \mathbb{R}^n$ is exactly approximated by a G_δ set H, i.e. $E \subseteq H$ and $|H|_e = |E|_e$.

Definition. We say $E \subseteq \mathbb{R}^n$ is **Lebesgue measurable** if for any $\varepsilon > 0$, there exists an open set $\mathcal{O} \supseteq E$ such that $|\mathcal{O} \setminus E| < \varepsilon$. The class of all Lebesgue measurable sets is denoted by $\mathcal{L}$.

Properties:

- $\mathcal{L}$ is a σ-algebra;
- $|\cdot|_e$ restricted to $\mathcal{L}$ is a measure, called the **Lebesgue measure**.

Measurable Functions.

Definition. Let $\mathcal{M}$ be a σ-algebra on a set X. We say that an extended real-valued function f on X is **measurable** if for any real number λ, the set

$$\{f > \lambda\} := \{x \in X : f(x) < \lambda\}$$

is $\mathcal{M}$-measurable in X; that is, the level sets of f are measurable.

Properties:

- the following statements are equivalent:
  - f is measurable;
  - for any real λ, $\{f \geqslant \lambda\}$ is measurable;
  - for any real λ, $\{f < \lambda\}$ is measurable;
  - for any real λ, $\{f \leqslant \lambda\}$ is measurable;
- f is measurable iff $\{f = -\infty\}$ and $\{\lambda < f < \infty\}$ are measurable for each real λ;
- f is measurable iff $\{f = -\infty\}$ and $f^{-1}(\mathcal{O})$ are measurable for each open $\mathcal{O} \subseteq \mathbb{R}$.

Definition. Given a measure space $(X, \mathcal{M}, \mu)$, we say that a property $P(x)$ is true ***μ*-almost everywhere**, or μ-a.e., on a measurable subset E of X if $\mu(\{x \in E : \ P(x)\}) = 0$.

Properties:

- a function is finite μ-a.e. on E if $\mu(\{x \in E : f(x) = \pm\infty\}) = 0$;
- in a complete μ-measure space, if f, g are extended real-valued functions with f measurable and $g = f$ μ-a.e., then g is also measurable with $\mu(\{g > \lambda\}) = \mu(\{f > \lambda\})$;
- in general, we work with equivalence classes of functions which are equal μ-a.e.;
- if f and g are extended real-valued measurable, then $f + g$ and $\{f > g\}$ are measurable;
- if f and g are measurable, finite μ-a.e., then fg is measurable, and if $g \neq 0$ μ-a.e., also f/g is measurable;
- if $\{f_n\}$ is a sequence of extended real-valued measurable functions which pointwise converge to some f, then f is measurable;

Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and f be an extended real-valued function defined on X. Then there is a sequence $\{f_n\}$ of simple real-valued functions defined on X, i.e.*

$$f_n(x) = \sum_{i=1}^{k_n} c_{i,n}\chi_{E_{i,n}}(x), c_{i,n} \text{ real}, E_{i,n} \text{ disjoint},$$

converging to f pointwise. Furthermore,

(i) *if f is measurable, so are the f_n's;*
(ii) *if f is nonnegative, the sequence $\{f_n\}$ is nondecreasing with $f_n(x) \leqslant f(x)$ all x, n;*
(iii) *if f is bounded, then the f_n's converge uniformly.*

Integration.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let ϕ be a nonnegative simple function

$$\phi(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x), a_k \in \mathbb{R}$$

where the A_k's form a measurable pairwise disjoint partition of X. The **integral** of ϕ over X with respect to μ is defined as the quantity

$$\int_X \phi \ d\mu = \sum_{k=1}^{n} a_k \mu(A_k).$$

Properties:

- the integral is well-defined with respect to the definition of ϕ;
- the integral is positively linear;
- the integral is monotone;
- the set function $\nu(E) = \int_X \phi\chi_E \; d\mu = \int_E \phi \; d\mu$ is a measure on $(X, \mathcal{M}, \mu)$.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a nonnegative measurable function on X. Define the set

$$\mathcal{F}_f = \{\phi \colon \phi \text{ simple, and } 0 \leqslant \phi \leqslant f\}.$$

The **integral** of f over X with respect to μ is the quantity

$$\int_X f \; d\mu = \sup \left\{ \int_X \phi \; d\mu : \phi \in \mathcal{F}_f \right\}.$$

Properties:

- if f is simple, the above definitions of the integral agree;
- if $f = g$ μ-a.e. then the integrals are equal;
- the integral is monotone with respect to functions and measures.

Theorem. (Monotone Convergence Theorem) *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a nondecreasing sequence of nonnegative finite μ-a.e. measurable functions defined on X. Then $\lim_n f_n(x) = f(x)$ exists everywhere, $f(x)$ is nonnegative and measurable, and*

$$\int_X f \; d\mu = \lim_{n \to \infty} \int_X f_n \; d\mu.$$

Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of nonnegative extended real-valued measurable functions defined on X. Then $f = \sum_n f_n$ is nonnegative, extended real-valued and measurable, and*

$$\int_X f \; d\mu = \sum_n \int_X f_n \; d\mu.$$

Theorem. *Let $(X, \mathcal{M}, \; d\mu)$ be a measure space, and let f be a nonnegative extended real-valued measurable function defined on X. Then the set function*

$$\nu(E) = \int_E f \; d\mu, E \in \mathcal{M},$$

is a measure on $(X, \mathcal{M})$.

Theorem. (Fatou's Lemma) *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of nonnegative extended real-valued measurable functions defined on X. Then*

$$\int_X \liminf f_n \; d\mu \leqslant \liminf \int_X f_n \; d\mu.$$

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let F be an extended real-valued measurable function defined on X; we can write $f = f^+ - f^-$ as the difference of two nonnegative functions. In particular, the integrals of $f^\pm$ exist, and if either is finite, we define the **integral** of f over X with respect to μ as the value

$$\int_X f \; d\mu = \int_X f^+ \; d\mu - \int_X f^- \; d\mu.$$

The class of **integrable** functions f

Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be an extended real-valued function defined on X for which the integral over X with respect to μ is defined. Then*

$$\left| \int_X f \; d\mu \right| \leqslant \int_X |f| \; d\mu.$$

Theorem. (Chebychev's Inequality) *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be an extended real-valued function defined on X. Then for any real $\lambda > 0$ we have*

$$\lambda\mu(\{|f| > \lambda\}) \leqslant \int_X |f| \ d\mu.$$

In particular, if $f \in L(\mu)$ is nonnegative with $\int_X f \ d\mu = 0$, then $f = 0$ μ-a.e.

Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f, g \in L(\mu)$. Then the integral of $f + g$ over X with respect to μ is defined and*

$$\int_X (f+g) \ d\mu = \int_X f \ d\mu + \int_X g \ d\mu.$$

Theorem. (Fatou's Lemma) *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of extended real-valued measurable functions defined on X. If there is an integrable function g such that $g \leqslant f_n$ for all n. Then $\liminf f_n$ and f_n are in $L(\mu)$ with*

$$\int_X \liminf f_n \ d\mu \leqslant \liminf \int_X f_n \ d\mu.$$

Conversely, if there exists an integrable function g such that $f_n \leqslant g$ for all n, then $\limsup f_n$ and f_n are in $L(\mu)$ with

$$\limsup \int_X f_n \ d\mu \leqslant \int_X \limsup f_n \ d\mu.$$

Theorem. (LDCT) *Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of extended real-valued measurable functions defined on X such that $\lim_n f_n = f$ exists μ-a.e. and there is an integrable function g such that $|f_n| \leqslant g$ μ-a.e. Then f is integrable and*

$$\int_X f \ d\mu = \lim_{n\to\infty} \int_X f_n \ d\mu.$$

Theorem. *Let g be a bounded real-valued function defined on $I = [a, b]$ and suppose $g \in \mathcal{R}(I)$. Then $g \in L(I)$ and*

$$\int_a^b g(x) \ dx = \int_I g \ dx.$$

Theorem. *Suppose that the nonnegative function g is finite on $I = (a, b]$ and that*

$$\int_{a^+}^b g(x) \ dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b g(x) \ dx$$

exists. Then $g \in L([a, b])$ and

$$\int_I g \ dx = \int_{a^+}^b g(x) \ dx.$$

Theorem. *Suppose g is a real-valued bounded function defined on $I = [a, b]$. Then $g \in \mathcal{R}(I)$ if and only if g is continuous a.e. on I.*

More about L^1.

Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $||\cdot||_1 = \int_X (\cdot) \ d\mu$ is a complete metric on $L(\mu)$.*

Theorem. *The space $C_0\mathbb{R}^n$ of continuous functions vanishing off a compact set is dense in $L(\mathbb{R}^n)$.*

Definition. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$. Denote the open interval of side length $2r$ by

$$I(x, r) = \{(y_1, \ldots, y_n) : |x_i - y_i| < r, i = 1, \ldots, n\}.$$

Definition. Suppose f is an integrable function which vanishes off a compact set. For $x \in \mathbb{R}^n$, define the **Hardy-Littlewood maximal function** of f as

$$M(f) = \sup_{r>0} \frac{1}{|I(x,r)|} \int_{I(x,r)} |f| \ dy.$$

Theorem. (Hardy-Littlewood) *Suppose f is an integrable function vanishing off $I(0,2)$. Then for any $\lambda > 0$ we have*

$$\lambda|\{Mf > \lambda\}| \leqslant 3^n \int_{\mathbb{R}^n} |f| \; dy.$$

Theorem. (Lebesgue Differentiation Theorem) *Suppose f is an integrable function which vanishes off $I(0,2)$. Then*

$$\lim_{r \to 0} \frac{1}{|I(x,r)|} \int_{I(x,r)} f(y) \; dy = f(x) \quad \text{a.e. on } I(0,1).$$

Borel Measures.

Definition. A measure μ on $(\mathbb{R}^n, \mathcal{B}_n)$ is called a **Borel measure**.

Definition. A Borel measure μ is **regular** if for any $E \in \mathcal{B}_n$, the value $\mu(E)$ can be computed by

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\},$$
$$\mu(E) = \inf\{\mu(\mathcal{O}) : \mathcal{O} \supseteq E \text{ open}\}.$$

Roughly, μ is determined by compact/open sets in $\mathbb{R}^n$.

Theorem. *A Borel measure that is finite on bounded subsets of $\mathbb{R}^n$ is regular.*

Definition. A **distribution function** induced by a measure μ is a function $F_y \colon \mathbb{R} \to \mathbb{R}$ given by $F_y(x) = \mu((y,x])$, for any extended real y.

Properties:

- distribution functions with $y = \infty$ are nondecreasing;
- distribution functions with $y = \infty$ are right-continuous.

Theorem. *Let $\mathcal{BB} = \{\mu : \mu \in$ is a Borel measure on the line, finite on bounded sets$\}$, and let $\mathcal{D} = \{F : F$ is nondecreasing and right continuous$\}/\{f - g \equiv c \in \mathbb{R}\}$. Then there is an injective mapping T from $\mathcal{BB}$ to $\mathcal{D}$ which satisfies the following: if $T\mu = F$ and c is an arbitrary constant, we have*

$$F(x) = \begin{cases} c + \mu((0,x]) & \text{if } x > 0 \\ c & \text{if } x = 0 \\ c - \mu((x,0]) & \text{if } x < 0 \end{cases}$$

or equivalently, $F(y) - F(x) = \mu((x,y])$. We denote $\mu = \mu_F$.

Absolute Continuity.

Theorem. *Let I e an open subinterval of the line and suppose f is a monotone real-valued function defined on I. Then f' exists a.e. on I.*

Theorem. *Suppose f is a nondecreasing real-valued function defined on $I = (a,b)$ such that $f(b^-) - f(a^+) < \infty$. Then $f' \in L(I)$ and*

$$\int_I f' \; dx \leqslant f(b^-) - f(a^+).$$

Theorem. *Suppose f is BV on a bounded interval $I = [a,b]$. Then $f' \in L(I)$ and*

$$\int_I |f'| \; dx \leqslant V(f; a, b).$$

Definition. A function $f \in L([a,b])$ is **absolutely continuous** if given $\varepsilon > 0$, there is a $\delta > 0$ such that for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a,b]$, we have

$$\sum_i |F(b_i) - F(a_i)| < \varepsilon, \text{ whenever } \sum_i (b_i - a_i) < \delta.$$

Theorem. *Let $I = [a,b]$ and suppose f is AC on I. Then f is BV on I, and consequently, f' exists a.e. and it is integrable there.*

Theorem. *Suppose f is continuous, BV, real-valued on $I = [a,b]$. Then f is AC on I iff f maps null sets into null sets.*

Definition. An a.e. differentiable function f on I is **singular** if $f' = 0$ a.e. on I.

Theorem. *Suppose f is an AC singular function defined on an interval I. Then f is constant.*

Theorem. *Suppose f is a real-valued function defined on $I = [a,b]$. Then f is AC on I if and only if f' exists a.e. in (a,b), it is integrable there, and*

$$f(x) - f(a) = \int_{[a,x]} f'(t)\, dt, \quad a \leqslant x \leqslant b.$$

Theorem. *Suppose f is BV on $I = [a,b]$. Then there exist an AC function g and a singular function h such that $f = g + h$. Up to constants, the decomposition is unique.*

Signed Measures.

Definition. Given a set X and σ-algebra $\mathcal{M}$ on X, we say that a set function ν on $\mathcal{M}$ is a **signed measure** provided the following hold:

- $\nu\colon \mathcal{M} \to [-\infty, \infty]$, with ν obtaining at most one of $\pm\infty$ and $\nu(\varnothing) = 0$;
- if $\{E_k\} \subseteq \mathcal{M}$ is a sequence of pairwise disjoint sets, then

$$\nu\left(\bigcup_1^\infty E_k\right) = \sum_1^\infty \nu(E_k).$$

Definition. Let μ be a measure and ν a signed measure on $(X, \mathcal{M})$. Then ν is **absolutely continuous** with respect to a measure μ, denoted $\nu \ll \mu$, if $\nu(A) = 0$ for any $A \in \mathcal{M}$ with $\mu(A) = 0$.

Theorem. *Let μ_F be a Borel measure. Then μ_F is absolutely continuous with respect to the Lebesgue measure if and only if F is AC on every bounded interval of $\mathbb{R}$.*

Theorem. *Suppose μ is a measure and ν is a signed measure on $(X, \mathcal{M})$ so that every μ-finite set is ν-finite. Then $\nu \ll \mu$ if and only if given $\varepsilon < 0$ there is a $\delta > 0$ such that*

$$|\nu(E)| < \varepsilon \quad \text{whenever} \quad \mu(E) < \delta.$$

Theorem. *Suppose $(X, \mathcal{M}, \mu)$ is a probability measure space and ν is a signed measure on $(X, \mathcal{M})$ such that*

$$|\nu(E)| \leqslant \mu(E)$$

all $E \in \mathcal{M}$. Then there exists a unique measurable function $f\colon X \to [-1, 1]$ such that

$$\nu(E) = \int_E f\, d\mu$$

all $E \in \mathcal{M}$. Uniqueness is up to equality μ-a.e.

Theorem. *Let ν be a signed measure on $(X, \mathcal{M})$ and suppose that its variation $|\nu|$ is a probability measure on $(X, \mathcal{M})$. Then there exist two disjoint, measurable sets A and B whose union is X so that $\nu(E \cap A) \geqslant 0$ and $\nu(E \cap B) \leqslant 0$ for all $E \in \mathcal{M}$.*

Theorem. *Suppose that λ, μ are σ-finite measures on $(X, \mathcal{M})$ with $\lambda(E) \leqslant \mu(E)$ for all $E \in \mathcal{M}$. Then there exists a unique nonnegative measurable function $f \colon X \to I$ such that*
$$\lambda(E) = \int_E f \, d\mu$$
for all $E \in \mathcal{M}$. Furthermore, if g is a measurable extended real-valued function defined on X, then
$$\int_X g \, d\lambda = \int_X gf \, d\mu.$$

Definition. Two signed measures μ and ν on $(X, \mathcal{M})$ are **mutually singular**, denoted $\mu \perp \nu$, if there exists a disjoint partition A, B of X such that $|\mu|(A) = 0 = |\nu|(B)$.

Proposition. *Suppose μ_F is a finite Borel measure. Then μ_F is singular with respect to the Lebesgue measure if and only if F is singular.*

Theorem. *Suppose μ is a σ-finite measure and ν is a signed measure on $(X, \mathcal{M})$. If $|\nu|$ is σ-finite, then there exist unique signed measures ν_a and ν_s which satisfy $\nu = \nu_a + \nu_s$ and $\nu_a \ll \mu$ and $\nu_s \perp \mu$.*

Theorem. (Radon-Nikodým) *Let μ be a σ-finite measure and ν a signed measure on $(X, \mathcal{M})$. If $|\nu|$ is σ-finite and $\nu \ll \mu$, then there exists an extended real-valued measurable function h defined on X such that if $E \in \mathcal{M}$ and $|\nu|(E) < \infty$*
$$\nu(E) = \int_E h \, d\mu.$$
We call h the Radon-Nikodým derivative of ν with respect to μ and one writes
$$h = \frac{d\nu}{d\mu}.$$
Also h is unique in the μ-a.e. sense.

L^p Spaces.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space and f an extended real-valued measurable function defined on X. Then for $1 \leqslant p < \infty$, $|f|^p$ is also measurable and the expression
$$||f||_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$
for $0 < p < \infty$ is well-defined, and is called the ***p*-norm** of f. The space of measurable functions with finite p-norm is denoted $L^p(X, \mu)$.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space and f an extended real-valued measurable function defined on X. Then the expression
$$||f||_\infty = \inf\{\lambda > 0 : \mu(\{|f| > \lambda\}) = 0\}$$
is well-defined and is called the **∞-norm** of f. The space of measurable functions with finite ∞-norm is denoted $L^\infty(X, \mu)$.

Theorem. (Hölder's Inequality) *Suppose $1 \leqslant p < q \leqslant \infty$, with p, q conjugate transpose, and let $f \in L^p(\mu)$ and $g \in L^q(\mu)$. Then fg is integrable, and*
$$\int_X |fg| \, d\mu \leqslant ||f||_p ||g||_q.$$

Theorem. (Minkowsky's Inequality) *Suppose $f, g \in L^p(\mu)$, $1 \leqslant p < \infty$. Then*
$$||f + g||_p \leqslant ||f||_p + ||g||_p.$$

Theorem. (Riesz-Fischer) *The p-norm induces a complete metric on $L^p(\mu)$.*

Theorem. (Riesz Representation) *Let $(X, \mathcal{M}, \mu)$ be a measure space and p, q conjugate transpose. Then if μ is σ-finite, to each continuous linear functional L on L^p, there corresponds a unique $g \in L^q$ such that $||L|| = ||g||_q$ and*

$$Lf = \int_X fg \; d\mu.$$

<u>**Fubini's Theorem**</u>.

Definition. Given measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, a **measurable rectangle** in the σ-algebra $\mathcal{M} \times \mathcal{N}$ is any subset of $X \times Y$ of the form $A \times B$, for $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Finite unions of pairwise disjoint measurable rectangles are called **elementary sets**.

Definition. If $E \subseteq X \times Y$, we define a **section** of E as the set

$$E_x = \{y \in Y : (x, y) \in E\}, x \in X;$$
$$E^y = \{x \in X : (x, y) \in E\}, y \in Y.$$

Definition. Let f be a measurable function on $X \times Y$. The ***X*-section** at $x \in X$ of f is

$$f_x(y) = f(x, y), x \in X;$$

similarly, the ***Y*-section** at $y \in Y$ is

$$f^y(x) = f(x, y), y \in Y.$$

Proposition. *Every section of a measurable set is measurable. Every X-section and Y-section of a measurable function is measurable.*

Theorem. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and suppose $E \in \mathcal{M} \times \mathcal{N}$. Then for each $x \in X$ and $y \in Y$, the functions $\nu(E_x)$ and $\mu(E^y)$ are measurable. Furthermore,*

$$\int_X \nu(E_x) \; d\mu = \int_Y \mu(E^y) \; d\nu.$$

Theorem. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu$ be σ-finite measure spaces and f be a nonnegative extended real-valued measurable function defined on $(X \times Y, \mathcal{M} \times \mathcal{N})$. Then $\int_Y f_x(y) d\nu$ is a measurable function on $(X, \mathcal{M})$ and $\int_X f^y(x) \; d\mu$ is a measurable function on $(Y, \mathcal{N})$ and*

$$\int_{X \times Y} f \; d(\mu \times \nu) = \int_X \int_Y f_x(y) \; d\nu \; d\mu = \int_Y \int_X f^y \; d\mu d\nu.$$

Corollary. *Under the assumptions of the previous theorem, if*

$$\int_X \int_Y |f|_x(y) \; d\nu d\mu < \infty,$$

then $f \in L(X \times Y, \mu \times \nu)$.

Theorem. (Fubini) *Under the assumptions of the previous theorem, if $f \in L(X \times Y, \mu \times \nu)$, then $f_x \in L(X, \mu)$ μ-a.e. on X and $f^y \in L(Y, \nu)$ ν-a.e. on Y and*

$$\int_Y f_x(y) \; d\nu \in L(X, \mu), \int_X f^y(x) \; d\mu \in L(Y, \nu)$$

and the result of the previous theorem holds.

4. Functional Analysis

Topics: Banach spaces, Hilbert spaces, linear transformations and functionals, Riesz representation theorem (duality), Hahn-Banach theorem, open mapping theorem, closed graph theorem, uniform boundedness theorem

Normed Linear Spaces.

Definition. A **norm** on on a vector space X is a nonnegative functional $||\cdot||\colon X \to \mathbb{R}$ satisfying
- (i) (triangle inequality) $||x+y|| \leqslant ||x|| + ||y||$ all $x, y \in X$;
- (ii) (absolute homogeneity) $||\lambda x|| = \lambda ||x||$ all $x \in X$, $\lambda \in \mathbb{R}$;
- (iii) (uniqueness) $||x|| = 0$ implies $x = 0$.

The pair $(X, ||\cdot||)$ is called a **normed linear space**. A nonnegative functional satisfying (i) and (ii) is called a **semi-norm**.

Definition. Let (x_n) be a sequence in X. We say (x_n) **converges** to some $x \in X$ if for every $\varepsilon > 0$ there is an index $N \in \mathbb{N}$ such that $||x_n - x|| < \varepsilon$ for all $n \geqslant N$. We say (x_n) is **Cauchy** if for every $\varepsilon > 0$ there is an index $N \in \mathbb{N}$ such that $||x_n - x_m|| < \varepsilon$ for all $n, m \geqslant N$.

Definition. A normed linear space is equipped with a metric $d(x, y) = ||x - y||$. If the space is complete (i.e. Cauchy sequences converge) with respect to this metric, we say it is a **Banach space**.

Definition. Let (x_n) be a sequence in X and $s \in X$. We say $\sum x_n$ is **convergent** if the sequence (s_n) of partial sums $s_n = x_1 + \cdots + x_n$ converges in X. We say $\sum x_n$ is **absolutely convergent** if the numerical sequence $\sum ||x_n||$ converges.

Theorem. *Let X be a normed linear space. Then X is a Banach space if and only if every absolutely convergent series converges.*

Definition. A **linear functional** on a vector space X is a functional $L\colon X \to \mathbb{R}$ such that
$$L(x + \lambda y) = L(x) + \lambda L(y)$$
for all $x, y \in X$ and $\lambda \in \mathbb{R}$.

Definition. A linear functional L on X is **bounded** if there is a constant $c \in \mathbb{R}$ such that $|Lx| \leqslant c||x||$ for all $x \in X$.

Theorem. (Hahn-Banach) *Suppose X is a real linear space with a semi-norm. Let X_0 be a linear subspace of X and L_0 a linear functional on X_0 such that $L_0 x \leqslant ||x||$ for all $x \in X_0$. Then there is a linear functional L on X extending L_0 so that $Lx \leqslant ||x||$ for all $x \in X$.*

Definition. A functional L on a normed linear space X is **continuous** if the image of any convergent sequence is convergent.

Proposition. *A linear functional on a normed linear space is bounded if and only if it is continuous.*

Definition. The **dual space** to a normed linear space X is the space X^* of all bounded linear functionals on X.

Proposition. *Suppose X is a normed space. Then X^* is a Banach space with respect to the **functional norm**,*
$$||L|| = \sup_{x \neq 0} \frac{|Lx|}{||x||}.$$

Theorem. (Hahn-Banach) *Suppose X is a normed linear space, and let L_0 be a bounded linear functional defined on a subspace X_0 of X. Then there exists a bounded linear functional L defined on X extending L_0 and satisfying $||L|| = ||L_0||$.*

Basic Principles.

Definition. Let (X, d) be a metric space. A set $E \subseteq X$ is **nowhere dense** if its closure $\overline{E}$ has empty interior. A subset of X is of **first category** if it is a countable union of nowhere dense sets; otherwise, it is of **second category**.

Theorem. (Baire Category) *A complete metric space is of second category in itself.*

Definition. Let X and Y be normed linear spaces over the same field of scalars. An **operator** is a map $T\colon X \to Y$. We say T is a **linear operator** if $T(x_1 + \lambda x_2) = Tx_1 + \lambda Tx_2$ for all $x_1, x_2 \in X$.

Definition. An operator $T\colon X \to Y$ is **continuous** if for every $x_0 \in X$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $||Tx - Tx_0|| < \varepsilon$ whenever $||x - x_0|| < \delta$.

Definition. An operator $T\colon X \to Y$ is **bounded** if its **operator norm** is finite:

$$||T|| = \sup_{||x|| \neq 0} \frac{||Tx||}{||x||} < \infty.$$

Proposition. *Let $T\colon X \to Y$ be a linear operator. Then the following are equivalent*

(i) *T is continuous at a point $x \in X$;*
(ii) *T is uniformly continuous on X;*
(iii) *T is bounded.*

The space of all bounded linear operators $X \to Y$ is denoted $\mathcal{B}(X, Y)$.

Proposition. *Let X, Y be normed linear spaces over the same field. Then $\mathcal{B}(X, Y)$ is a normed linear space under the operator norm. Moreover, $\mathcal{B}(X, Y)$ is a Banach space if and only if Y is a Banach space.*

Proposition. *Let $T \in \mathcal{B}(X, Y)$. Then T^{-1} exists and is continuous if and only if there exists a constant $c > 0$ such that $||Tx|| \geqslant c||x||$ for all $x \in X$.*

Definition. A family $\mathcal{F} \subseteq \mathcal{B}(X, Y)$ is **norm bounded** if $\sup_{T \in \mathcal{F}} ||T||$ is finite. Similarly, it is **pointwise bounded** if $\sup_{T \in \mathcal{F}} ||Tx||$ is finite for each $x \in X$.

Theorem. (Uniform Boundedness) *Let X be a Banach space and Y a normed linear space. Then a collection $\mathcal{F} \subseteq \mathcal{B}(X, Y)$ is norm bounded if and only if it is pointwise bounded.*

Definition. We say $T \in \mathcal{B}(X, Y)$ is **open** if the image of every open set $U \subseteq X$ is open in Y.

Theorem. (Open Mapping) *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is onto, T is open.*

Corollary. *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is injective, T has a well-defined and bounded inverse $T^{-1} \in \mathcal{B}(Y, X)$.*

Definition. Let X and Y be normed spaces, and let $A \subseteq X$. We say $T\colon A \to Y$ is **closed** in X if whenever a sequence $(x_n) \subseteq A$ converging to $x \in X$ and whose image sequence $(Tx_n) \subseteq Y$ converges to y, we have $x \in A$ and $Tx = y$.

Definition. Let X, Y be normed spaces, and define a norm on $X \times Y$ by $||(x, y)|| = ||x|| + ||y||$. Given a linear map $T\colon A \subseteq X \to Y$, the **graph** of T is the set

$$G(T) = \{(x, Tx) : x \in A\} \subseteq X \times Y.$$

Since T is linear, $G(T)$ is a linear subspace of $X \times Y$.

Proposition. *When T is closed, $G(T)$ is a closed subspace of $X \times Y$.*

Proposition. *If $A \subseteq X$ is a closed subspace and T is continuous, then T is closed in X.*

Theorem. (Closed Graph) *Let X, Y be Banach spaces and $T\colon X \to Y$ a linear operator. If T is closed in X, then T is continuous in X.*

Hilbert Spaces.

Definition. A complex vector space X is said to be an **inner product space** provided it has an **inner product**, i.e. a complex valued function $\langle \cdot, \cdot \rangle$ on $X \times X$ satisfying

(i) (linearity) $\langle x_1 + \lambda x_2, y \rangle = \langle x_1, y \rangle + \lambda \langle x_2, y \rangle$ all $x_1, x_2 \in X$ and $\lambda \in \mathbb{C}$;
(ii) (conjugate) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ all $x, y \in X$;
(iii) (absolute homogeneity) $\langle x, x \rangle \geqslant 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

Definition. In an inner product space X, the **induced inner product norm** is defined as $||x|| = \sqrt{\langle x, x \rangle}$, under which X is a normed linear space. If X is complete with respect to this norm, it is called a **Hilbert space**.

Properties:

- (i) and (ii) imply conjugate linearity: $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$;
- $\langle x, 0 \rangle = \langle 0, x \rangle = 0$ for all $x \in X$;
- (Cauchy-Schwarz Inequality) $|\langle x, y \rangle|^2 \leqslant \langle x, x \rangle \langle y, y \rangle$;
- the given norm is a norm (by Cauchy-Schwarz);
- the inner product is continuous (by Cauchy Schwarz): $x_n \to x$, $y_n \to y$ implies $\langle x_n, y_n \rangle \to \langle x, y \rangle$;

Definition. An onto linear mapping $T \colon X \to Y$ between inner product spaces over the same field of scalars is an **isomorphism** if it preserves inner products: $\langle Tx, Ty \rangle = \langle x, y \rangle$ all $x, y \in X$.

Proposition. *Suppose X is an inner product space. Then there exists a Hilbert space Y and an isomorphism T of X onto a dense subspace of Y. The space Y is unique up to isomorphism.*

Proposition. *A normed linear space X is an inner product space if and only if the* ***parallelogram law*** *holds: for any $x, y \in X$*

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2).$$

Definition. Elements x, y in an inner product space X are said to be orthogonal, written $x \perp y$, if $\langle x, y \rangle = 0$. If $x \in X$ is orthogonal to each element of $A \subseteq X$, we write $x \perp A$.

Proposition. (Pythagorean Thm) *If $(x_i)_1^n$ is a collection of pairwise orthogonal elements, then*

$$\left|\left| \sum_1^n x_i \right|\right| = \sum_1^n ||x_i||^2.$$

Definition. A subset C of a normed linear space X is **convex** if for every $x, y \in C$ the set $\{\eta x + (1 - \eta) y : 0 \leqslant \eta \leqslant 1\}$ is contained in C.

Proposition. (Existence of Minimizing Element) *Let X be an inner product space and $M \subseteq X$ nonempty, complete, and convex. Then for every $x \in X$ there exists a unique $y \in M$ such that*

$$d(x, M) := \inf_{x' \in M} ||x' - x|| = ||x - y||.$$

Definition. The **orthogonal compliment** of a subset A of an inner product space X is the set

$$A^{\perp} = \{x \in X : x \perp y \text{ for all } y \in A\}.$$

Proposition. *The subspace $A^{\perp}$ is a closed subspace of X.*

Theorem. *Let X be a Hlibert space and M a complete subspace of X. Then $X = M + M^{\perp}$, where the representation $x = x_1 + x_2$ of any $x \in X$ (by $x_1 \in M$ and $x_2 \in M^{\perp}$) is unique.*

Definition. Let M be a complete subspace of a Hilbert space, and let $x = x_1 + x_2 \in X$ for $x_1 \in M$ and $x_2 \in M^{\perp}$. Then x_1 and x_2 are called the **projection of x** onto M and $M^{\perp}$, respectively. The map sending x onto either of its projections is called the **projection operator**.

Theorem. (Riesz) *Let X be a Hilbert space, and suppose L is a bounded linear functional on X. Then there exists a unique $y \in X$ such that*

$$Lx = \langle x, y \rangle, \text{ all } x \in X.$$

Moreover, $||L|| = ||y||$.

Proposition. *If X is a Hilbert space, then X^* is also a Hilbert space.*

Definition. A **orthonormal system** is a subset $\{x_1, \ldots, x_n\}$ of a vector space such that $||x_i|| = 1$ for all $1 \leqslant i \leqslant n$ and $x_j \perp x_k$ for all $1 \leqslant j \neq k \leqslant n$.

Proposition. *If M is a closed subspace of a normed space X and $\{x_1, \ldots, x_n\} \subseteq X$, then the span $\{M, x_1, \ldots, x_n\}$ is a closed subspace of X.*

Proposition. (Bessel's Inequality) *Suppose $\{x_\alpha\}_{\alpha \in A}$ is an ONS in a Hilbert space X. Then*

$$\sum_{\alpha \in A} |\langle x, x_\alpha \rangle|^2 \leqslant ||x||^2, \text{ all } x \in X.$$

In particular, for each $x \in X$, all but an at most countable number of the ***Fourier coefficients*** *$\langle x, x_\alpha \rangle$ of x with respect to the ONS $\{x_\alpha\}$ vanish.*

Definition. An ONS $\{x_\alpha\}_{\alpha \in A}$ in a Hilbert space X is **maximal**, or complete, if no nonzero element can be added to it so that the resulting collection of elements is still an ONS in X.

Theorem. *Suppose $\{x_\alpha\}_{\alpha \in A}$ is an ONS in a Hilbert space X. Then the following are equivalent*

(i) *$\{x_\alpha\}$ is a maximal ONS in X;*
(ii) *the collectein of all finite linear combinations of $\{x_\alpha\}$ is dense in X;*
(iii) (Plancherel's Equality) *Equality holds in Bassel's inequality;*
(iv) (Paseval's Identity) *For all $x, y \in X$, we have*

$$\langle x, y \rangle = \sum_{\alpha \in A} \langle x, x_\alpha \rangle \overline{\langle y, x_\alpha \rangle}.$$