Partial Order & Cardinality

Summer 2016

Definitions

Definition 1. A relation on a set S is a subset R of $S \times S$.

Definition 2. A relation $\leq$ on a set S which is reflexive, transitive, and antisymmetric is called a partial order; the pair $(S, \leq)$ is called a partially ordered set (poset).

i. *reflexive*: $x \leq x$ for all $x \in S$

ii. *transitive*: if $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x, y, z \in S$

iii. *antisymmetric*: if $x \leq y$ and $y \leq x$ then $x = y$ for all $x, y \in S$

Definition 3. A partial order $\leq$ on S is a total order if it satisfies trichotomy.

iv. *trichotomy*: $x \leq y$ or $y \leq x$ for all $x, y \in S$

Definition 4. Let A and B be sets. Define the cardinality relation $\leq$ on sets such that $|A| \leq |B|$ if there exists an injective function from A to B. We write $|A| = |B|$ if there exists a bijection between A and B.

Schröder-Bernstein Theorem

Theorem 1. *(Schröder-Bernstein) If $f : A \to B$ and $g : B \to A$ are injective functions, there exists a bijection $h : A \to B$.*

Propositions

Proposition 1. *The relation of cardinality forms a partial order on sets.*

i. *reflexive*: The identity forms a bijection from any set to itself.

ii. *transitive*: Supposing $|A| \leq |B|$ and $|B| \leq |C|$, compose the resulting injective functions to form another injective function between A and C.

iii. *antisymmetric*: From Schröder-Bernstein, the definition for the cardinality relation implies transitivity.

Proposition 2. *The relation of cardinality forms a total order on sets.*

Proof. From Proposition 1, it suffices to show that trichotomy holds under this relation. $\square$