The Khovanov homology of slice disks

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Bryn Mawr College

Princeton Topology Seminar

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Motivation for a slice disk

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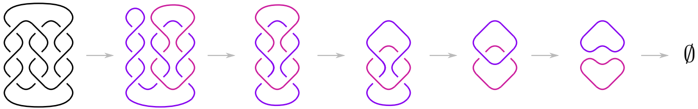

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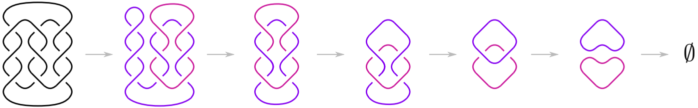

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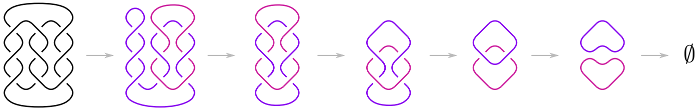

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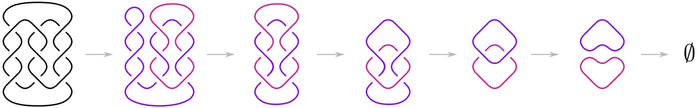

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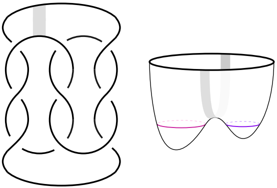

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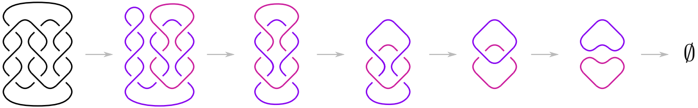

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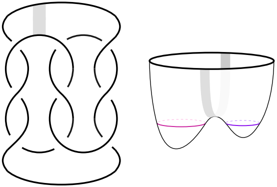

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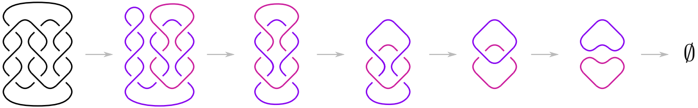

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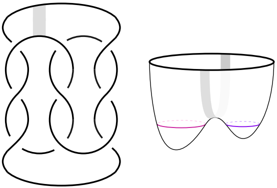
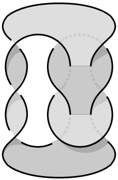
(a) 

(b) 

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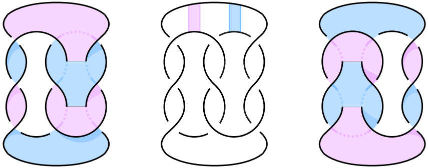

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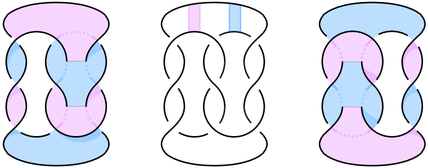

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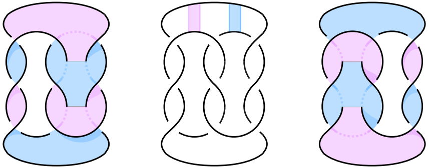

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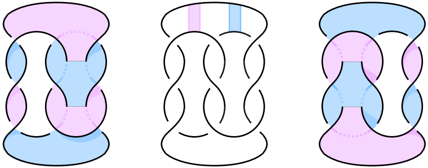

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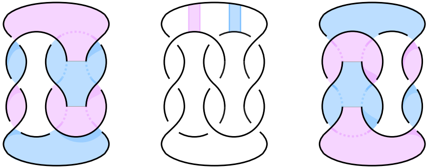

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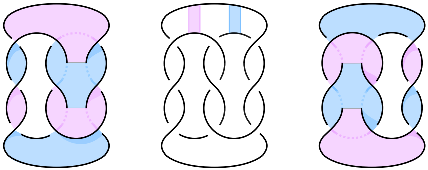

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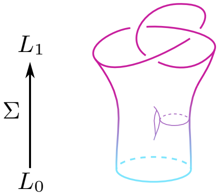

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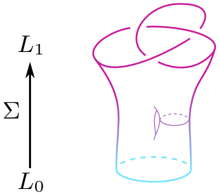

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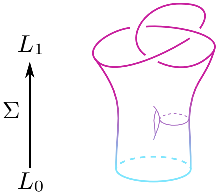

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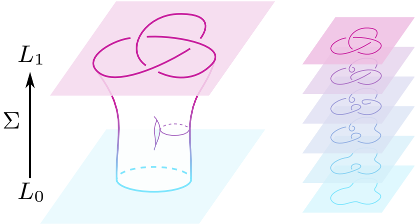

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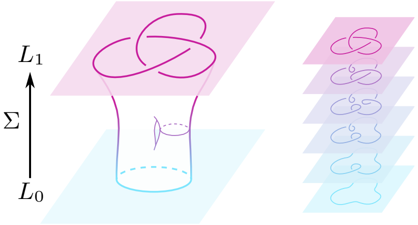

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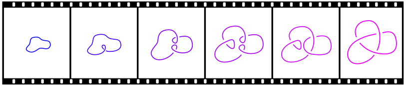

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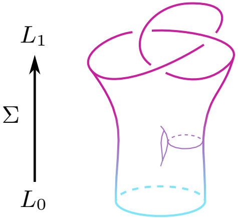			

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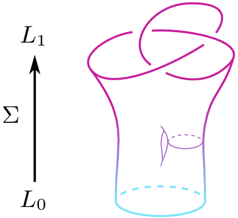			

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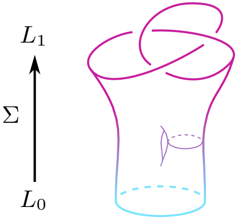		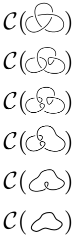	

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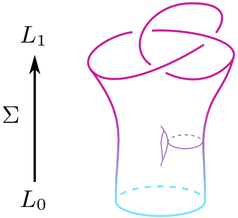		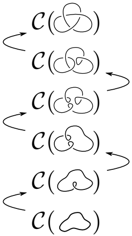	

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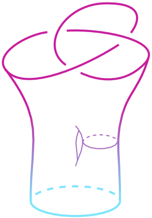
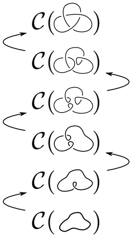
L_1 Σ L_0 			$\mathcal{C}(D_1)$ $\mathcal{C}(\Sigma)$ $\mathcal{C}(D_0)$

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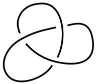

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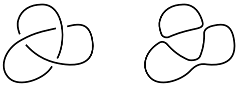

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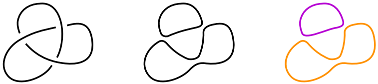

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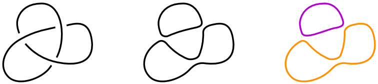

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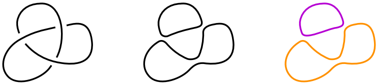

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Let's take a quick look at $\mathcal{C}(3_1)$

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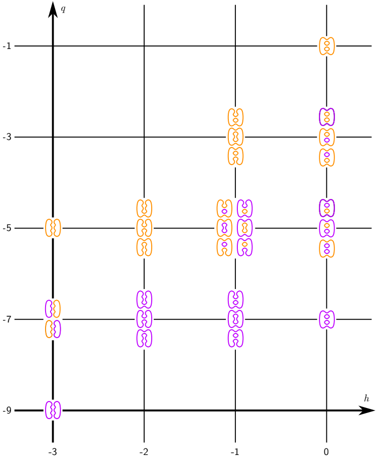

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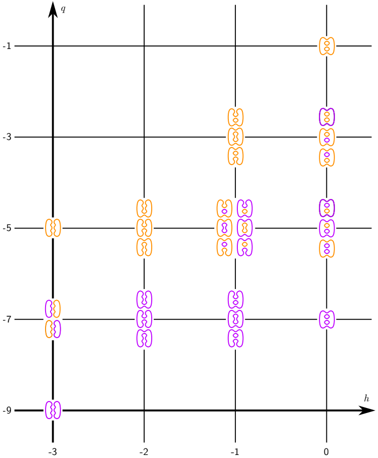

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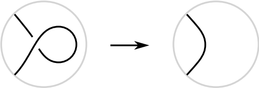

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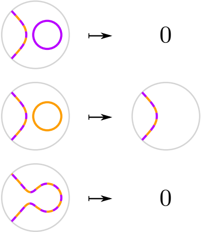

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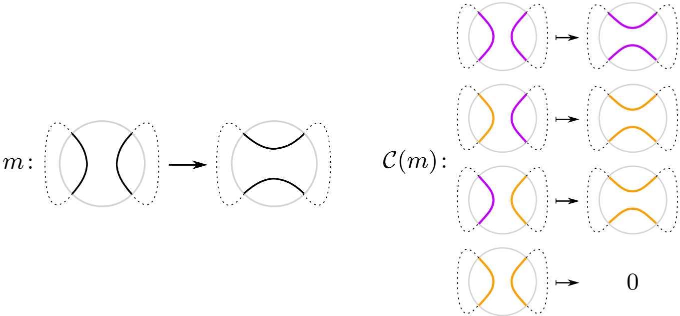

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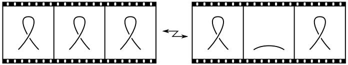

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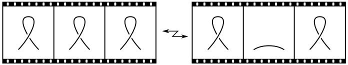

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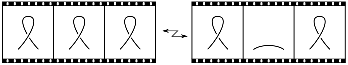

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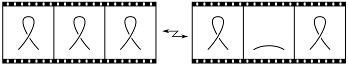

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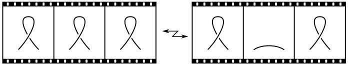

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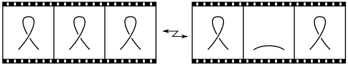

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- conclude Σ, Σ' are not isotopic rel boundary

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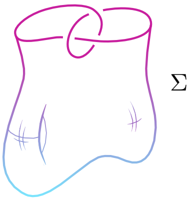

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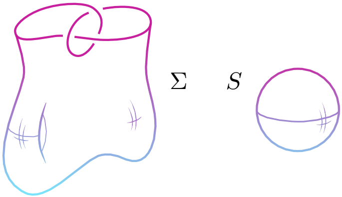

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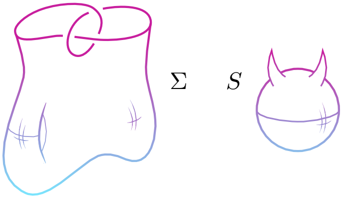

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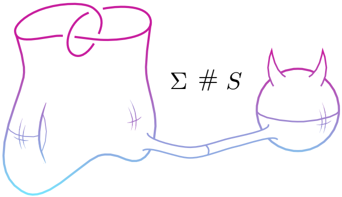

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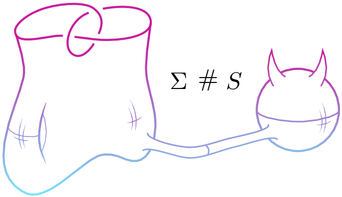

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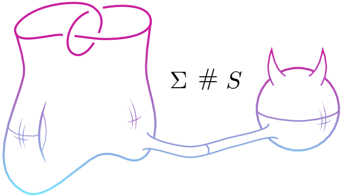


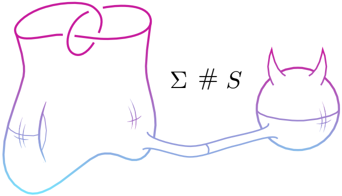
Theorem (Swann, Hayden-Sundberg)

The map on Khovanov homology induced by a link cobordism is invariant under connected sums with knotted 2-spheres.

A brief remark on local knottedness

In general, it is (perhaps too) easy to build such link cobordisms:

- Given $\Sigma\colon L_0 \to L_1$, we create a new (unique) link cobordism Σ'
- Choose your favorite knotted 2-sphere S and connect-sum with Σ
- Then Σ and $\Sigma' := \Sigma \# S$ are (generally) not isotopic rel boundary.



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Takeaway: maps on Khovanov homology detect more than local knotting

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Question:

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$$\begin{tikzcd} \emptyset \\ \emptyset \arrow[u, "\Sigma"'] \end{tikzcd} \longrightarrow \begin{tikzcd} \mathbb{Z} \\ \mathbb{Z} \arrow[u, "\mathcal{H}(\Sigma)"'] \end{tikzcd}$$

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*For a link cobordism $\Sigma\colon \emptyset \to \emptyset$, the φ-**number** of Σ*

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Theorem (Rasmussen, Tanaka)

The φ-numbers associated to connected $\Sigma \subset B^4$ are determined by genus:

- *if $g(\Sigma) = 1$, then $\varphi(\Sigma) = \pm\,2$*
- *if $g(\Sigma) \neq 1$, then $\varphi(\Sigma) = \ \ 0$*

Cases

Idea:

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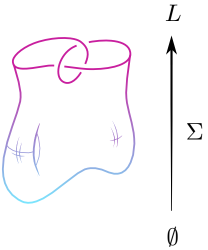

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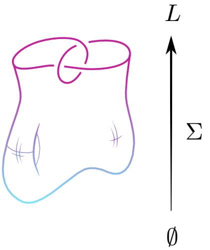

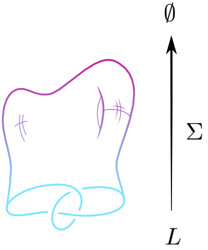

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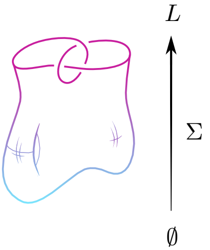
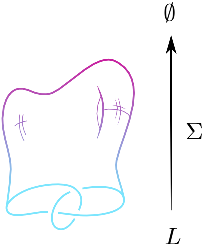
We consider these cases separately in the next two sections.





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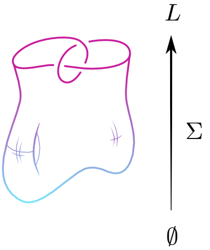

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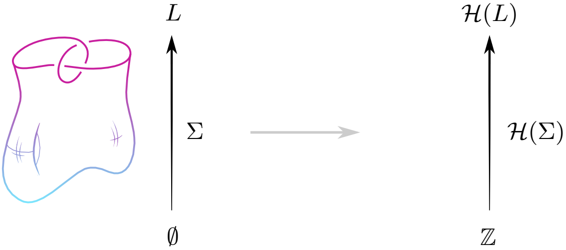

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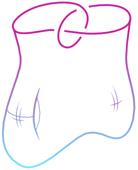

$$\emptyset \xrightarrow{\Sigma} L \qquad \longrightarrow \qquad 1 \in \mathbb{Z} \xrightarrow{\mathcal{H}(\Sigma)} \mathcal{H}(\Sigma)(1) \in \mathcal{H}(L)$$

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If so, we say $\Sigma_{0,1}$ are $\boldsymbol{\varphi}$-**distinguished**.

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Theorem (Swann, Sundberg)

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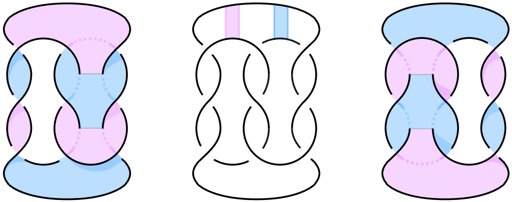

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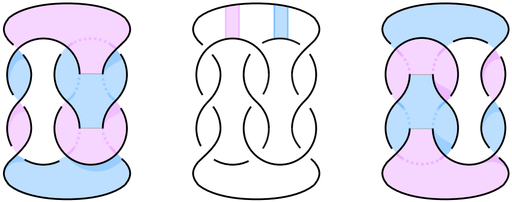
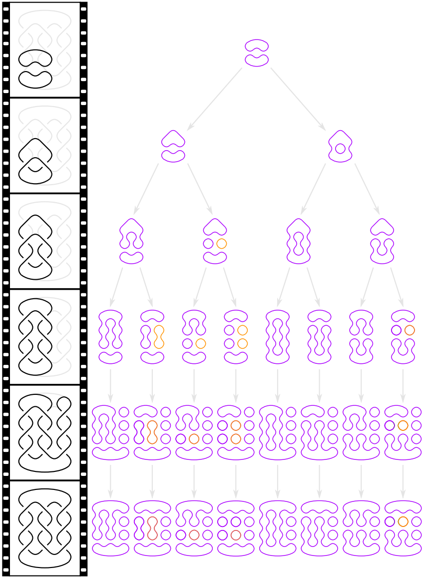


What do $\varphi(D_\ell)$ and $\varphi(D_r)$ look like?

Applications of φ-classes



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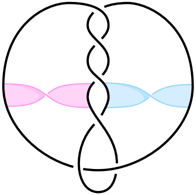

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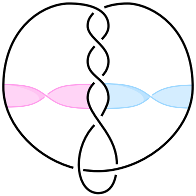

Are there knots with more than 2 unique slice disks?

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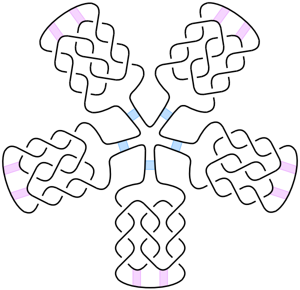
Slice disks are obtained by boundary-summing copies of D_ℓ and D_r.



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- So, extend the 2^n slice disks for $K = \#_n(9_{46})$ by a ribbon-concordance $C\colon K \to K_n$ to a prime knot K_n
- These slice disks are pairwise φ-distinguished using injectivity and functoriality of the induced maps on Khovanov homology:

$$\varphi(C \circ D) = \mathcal{H}(C)(\varphi(D)) \neq \pm\mathcal{H}(C)(\varphi(D')) = \varphi(C \circ D')$$

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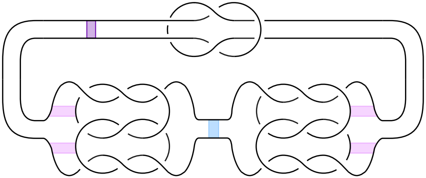

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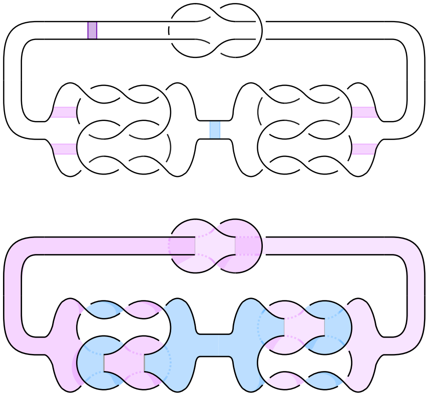

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If $\Sigma\colon \emptyset \to K$ *has genus* $g(\Sigma) = 1$ *and* $\varphi(\Sigma) = 0$ *then* K *is not smoothly slice.*

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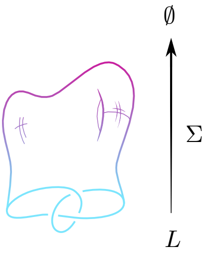

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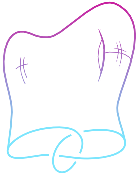

$$\begin{tikzcd} \emptyset & & & \mathbb{Z} \\ L \arrow[u, "\Sigma"'] & \arrow[r, gray] & & \mathcal{H}(L) \arrow[u, "\mathcal{H}(\Sigma)"'] \end{tikzcd}$$

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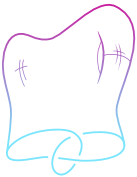
Let's look at the second case: the dual link cobordism $\Sigma\colon L \to \emptyset$.



$$\begin{tikzcd} \emptyset & & \mathcal{H}(\Sigma)(\varphi) \in \mathbb{Z} \\ L \arrow[u, "\Sigma"'] \arrow[r, gray] & & \varphi \in \mathcal{H}(L) \arrow[u, "\mathcal{H}(\Sigma)"'] \arrow[u, bend left, dotted] \end{tikzcd}$$

Method:

- a surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ induces a link cobordism $\Sigma\colon L \to \emptyset$
- it induces a map $\mathcal{H}(\Sigma)\colon \mathcal{H}(L) \to \mathbb{Z}$
- choose a class $\varphi \in \mathcal{H}(L)$, and note that $\mathcal{H}(\Sigma)(\varphi) \in \mathbb{Z}$ is an up-to-sign invariant of the isotopy class of Σ.

Defining φ^*-numbers

Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ and a class $\varphi \in \mathcal{H}(L)$, the **$\varphi^*$-number**

$$\varphi^*(\Sigma) := \mathcal{H}(\Sigma)(\varphi) \in \mathbb{Z}$$

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Can we find $\Sigma_{0,1} \subset B^4$ bounding a common $L \subset S^3$ *and* a class $\varphi \in \mathcal{H}(L)$ such that $\varphi^*(\Sigma_0) \neq \pm\, \varphi^*(\Sigma_1)$?

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If so, we say $\Sigma_{0,1}$ are $\boldsymbol{\varphi^*}$**-distinguished**.

Applications of φ^*-numbers

Theorem (Hayden-Sundberg)

The pair of slice disks D_ℓ and D_r for the knot K (below) are φ^*-distinguished by the given class $\varphi \in \mathcal{H}(K)$, and therefore, are not isotopic rel boundary.

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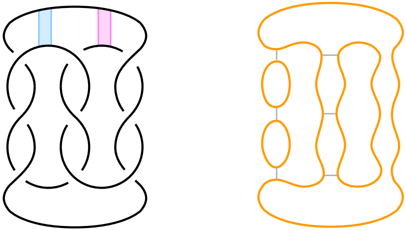

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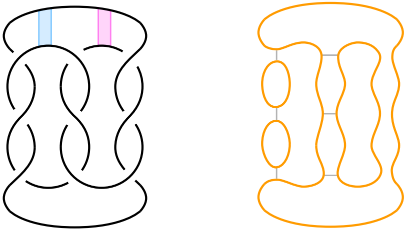

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Proof idea: show $\varphi^*(D_\ell) = 1$ and $\varphi^*(D_r) = 0$

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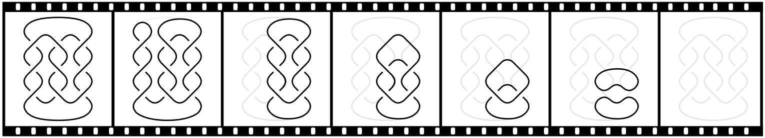

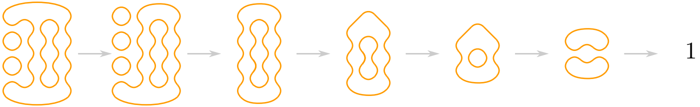

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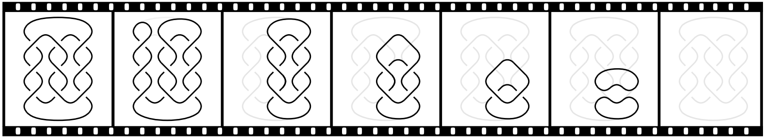

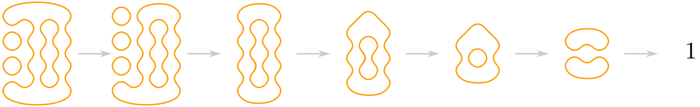

So $\varphi^*(D_\ell) = 1$ and $\varphi^*(D_r) = 0$, as desired.

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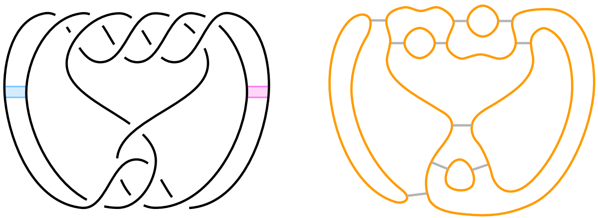

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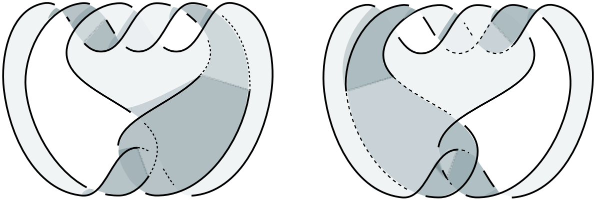

Slice disks for $K = 15n_{103488}$ (image by Kyle Hayden).

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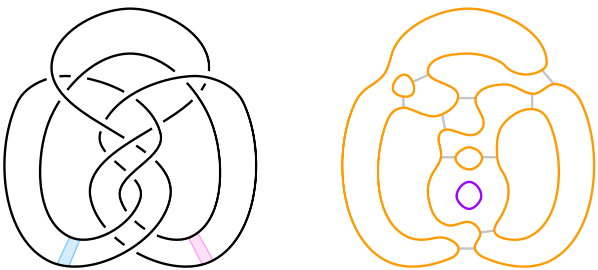

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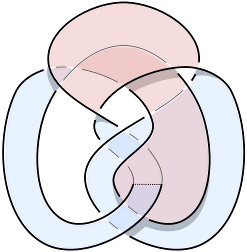

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Thank You!

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Bibliography I

- D Bar-Natan, *Khovanov's homology for tangles and cobordisms*, **Geom. Topol.**, 9:1443-1499, 2005.
- Anthony Conway and Mark Powell, *Characterisation of homotopy ribbon discs*, **Adv. Math.**, 391:Paper No. 107960, 2021.
- Kyle Hayden, *Corks, covers, and complex curves*, arXiv:2107.06856, 2021.
- Kyle Hayden and Isaac Sundberg, *Khovanov homology and exotic surfaces in the 4-ball*, arXiv:2108.04810, 2021.
- Magnus Jacobsson, *An invariant of link cobordisms from Khovanov homology*, **Algebr. Geom. Topol.**, 4:1211-1251, 2004.
- András Juhász and Ian Zemke, *Distinguishing slice disks using knot floer homology*, *Seceta Math. (N.S.)*,20(1), 2020.
- Mikhail Khovanov, *A categorification of the Jones polynomial*, **Duke Math. J.**, 101(3):359-426, 2000.

Bibliography II

- Mikhail Khovanov, *An invariant of tangle cobordisms*, **Transactions of the American Mathematical Society**, 358(1):315-327, 2006.
- Adam S. Levine and Ian Zemke, *Khovanov homology and ribbon concordances*, **Bull. Lond. Math. Soc.**, 51(6):1099-1103, 2019.
- Allison N. Miller and Mark Powell, Stabilization distance between surfaces, *Enseign. Math.*,**65**:397-440, 2020.
- Lisa Piccirillo, *The Conway knot is not slice*, **Ann. of Math.** (2), 191(2):581-591, 2020.
- Jacob Rasmussen, *Khovanov's invariant for closed surfaces*, arXiv:math/0502527, 2005.
- Isaac Sundberg and Jonah Swann, *Relative Khovanov-Jacobsson classes*, arXiv:2103.01438, 2021.
- Jonah Swann, *Relative Khovanov-Jacobsson classes of spanning surfaces*, *Ph.D. Thesis, Bryn Mawr College*, 2010.

Bibliography III

Kokoro Tanaka, *Khovanov-Jacobsson numbers and invariants of surface-knots derived from Bar-Natan's theory*, **Proc. Amer. Math. Soc.**, 134(12):3685–3689, 2005.