

Talk: PACT IV - Ye Ol' Khovanov Homology

September 9, 2021 1:35 PM

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I MOTIVATION

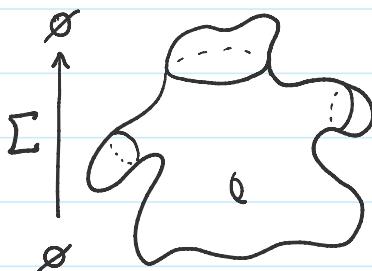
II KHovanov Homology of Links

III KHovanov Homology of Surfaces

IV. Khovanov homology of closed surfaces

TODAY D means disk

Given knotted surface in B^4 , represent it as a link cobordism $\Sigma : \emptyset \rightarrow \emptyset$



$$\begin{array}{ccc} \emptyset & \rightsquigarrow & \text{Kh}(\emptyset) = \mathbb{Z} \\ \downarrow \Sigma & & \downarrow \text{Kh}(\Sigma) \\ \emptyset & \rightsquigarrow & \text{Kh}(\emptyset) = \mathbb{Z} \ni 1 \\ & & n_\Sigma := |\text{Kh}(\Sigma)(1)| \in \mathbb{Z} \end{array}$$

push into $S^3 \times [0,1]$

Khovanov-Jacobson number of Σ

NOTE invariant of ambient (no \cong) iso. class of Σ

Ex We calculated:

(a) a std S^2 has $n_{S^2} = 0$

(b) a std T^2 has $n_{T^2} = 2$

Lemma $n_\Sigma \in \text{Kh}^{0, \chi(\Sigma)}(\emptyset) = \begin{cases} \mathbb{Z} & \text{if } \chi(\Sigma) = 0 \\ 0 & \text{if } \chi(\Sigma) \neq 0 \end{cases}$

$\Rightarrow n_\Sigma = 0$ for Σ with $\chi(\Sigma) \neq 0$ (ie. only knotted tori have nontrivial n_Σ)

② Does n_Σ distinguish knotted tori?

A: no!

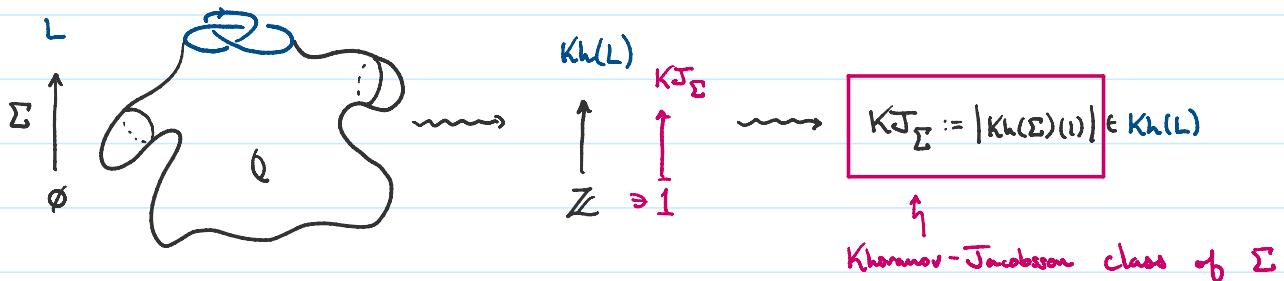
Thus (Rasmussen '05, Tomova '05) Khovanov-Jacobson numbers are determined by genus: $n_\Sigma = \pm 2$ for $\chi(\Sigma) = 0$ and Σ connected.

acts multiplicatively over multiple components

IV

Khovanov homology of slice disks

Given a surface* $\Sigma \subset B^4$, represent as a link cob $\Sigma : \emptyset \rightarrow L$



NOTE link of bdry-pres. into class of Σ

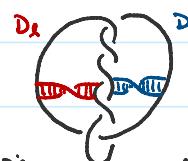
Note: this is a relative version of n_Σ (if $L = \emptyset$ then $n_\Sigma = \text{KJ}_\Sigma$)

(?) Does KJ_Σ distinguish any Σ from any Σ' ?
(with $\partial\Sigma = L = \partial\Sigma'$ and $g(\Sigma) = g(\Sigma')$)



Theorem 1 (Swann '10, S. '20) The ^{slice} disks D_L and D_R induce distinct maps on Khovanov homology and hence are not isotopic rel bdry.

Theorem 2 (S. '20) The ^{slice} disks D'_L and D'_R for $\#_n(9_{46})$ induce distinct maps on Khovanov homology and hence are not isotopic rel bdry.



induce distinct
rel bdry.

Theorem 3 (S.-Swann '21) The 2^n slices for $\#_n(9_{46})$ induce distinct maps.

Proof look @ KJ classes ($\text{KJ}_{D_L} \neq \pm \text{KJ}_{D_R}$, etc)

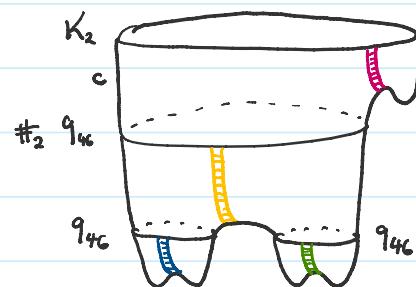
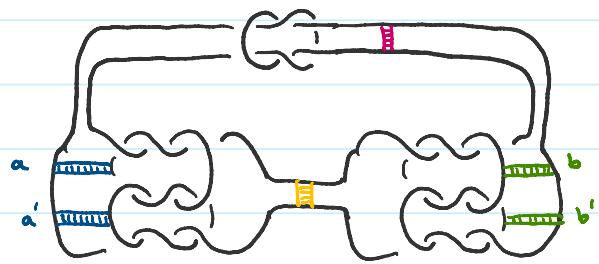
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Ex. $\#_2(9_{46}) = 9_{46} \# 9_{46}$ has 4 slices, given by

1. either a or a'
2. either b or b'
3. c

(ie pick one of two slices for each copy of 9_{46} and bndy-connect sum)

SIDE BOARD



Note Can also be done for $\#_n(6_i)$ and a-la-carte-sums of 6_i and 9_{46}

Thm (S.-Swann '21) There are prime knots K_n with 2^n slices that each idmokhatanirb.

Proof Thm 3 + Thm (L-Z) + Thm (K-L)

- apply C to $\#_n(9_{46})$ to get K_n
- attach C to any slice D of $\#_n(9_{46})$ to get slice CD of K_n
- KT classes will remain distinct after attaching C

can compose link
not by stacking

Thm (Levine-Zemke '19) Ribbon-concordances induce injections on Kh.

(Recall: ribbon-concordances are link colorings $\Sigma \cong S^1 \times [0,1]$ w/o maxima)

Thm (Kirby-Lickorish '79) Every knot is (ribbon) concordant to a prime knot.

(?) How does one show $KJ_{\Sigma} \neq \pm KJ_{\Sigma'}$?

(4)

Note: KJ_{Σ} is the homology class $\pm [eKJ_{\Sigma}]$ of
a cycle $eKJ_{\Sigma} = e(\Sigma)(1) \in Kh(L)$

$$So \quad 0 = KJ_{\Sigma} \pm KJ_{\Sigma'} = [eKJ_{\Sigma} \pm eKJ_{\Sigma'}]$$

if $c := eKJ_{\Sigma} \pm eKJ_{\Sigma'}$ is a boundary (ie $c = d(c)$ for some $c \in eKh^{k+1}(L)$)

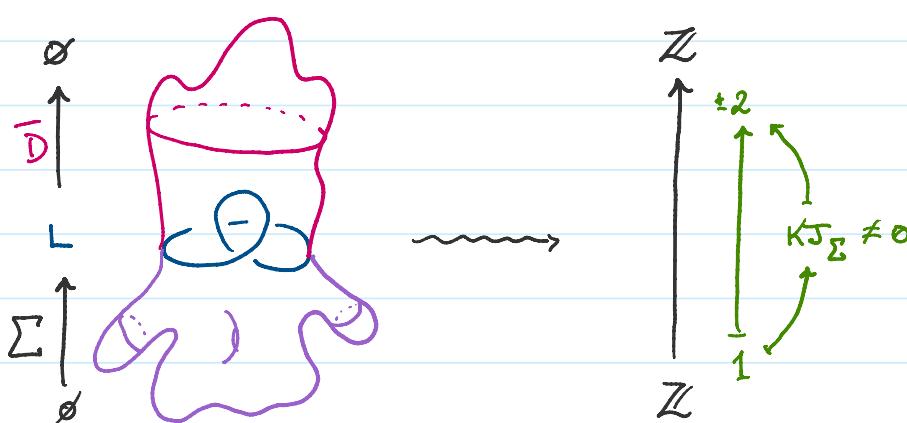
This can be done using a computer!

(?) How do the relative and absolute cases relate?
 $(L \neq \emptyset) \qquad (L = \emptyset)$

Thm (Scharlemann '10) Suppose K is slice w/ slice disk D .

Then any $\Sigma: \emptyset \rightarrow K$ with $g(\Sigma) = 0$ or 1 has nontrivial KJ_{Σ} .

Proof ($g=1$)

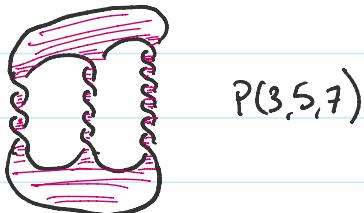


$(g=0)$ Stack on $\overline{D} \# T^2$ to D (put genus on top)
Some factorization occurs

(5)

Converse If K bounds a genus 1 surface Σ with $K\mathbb{J}_\Sigma = 0$ then
 K is not slice.

There (Swann '10) The pretzel knots $P(p,q,r)$ with ^{pos.} odd p,q,r have
genus 1 Seifert surface w/ trivial KJ class and thus are not slice.



Conjecture Sliceness of odd 3-stranded pretzel knots is fully determined by KJ-classes.

$$(\alpha' \rightarrow)\alpha' \rightarrow \beta \text{ (} \rightarrow X\text{)}$$

IDEA The Conway knot has a genus 1 surface (unknotting # 1)
" nontrivial KJ class