Math4CS
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1 Induction

1. **Courtyard Tiling**

   The problem is to tile a courtyard with dimensions $2^n \times 2^n$. We are required to install a statue of a wealthy donor in one of the central square, and only special L-shaped tiles can be used. We need to prove this is feasible.

   Theorem For all $n \geq 0$ there exists a tiling of a $2^n \times 2^n$ courtyard with the donor in a central square.

   Proof. Prove by induction. Let $P(n)$ be the proposition that there exists a tiling of a $2^n \times 2^n$ courtyard with the donor placed in any location.

   Base case: $P(0)$ is true because the donor fills the whole courtyard.

   Inductive step: Suppose $P(n)$ is true, we need to prove that $P(n) \to P(n+1)$. A $2^{n+1} \times 2^{n+1}$ courtyard consists of four $2^n \times 2^n$ quadrants, each of them can be tiled with the donor placed in any location. Let the donor be in one of the four central squares, and the remaining three central squares can fit a L-shaped tile. Now we can tile each of the four quadrants by the induction hypothesis. This proves that $P(n) \to P(n+1)$. The theorem follows as a special case. □

2 Graph

1. **Theorem**. Every graph $G = (V, E)$ has at least $|V| - |E|$ connected components.

   Proof. We use induction on the number of edges. Let $P(n)$ be the proposition that every graph $G = (V, E)$ has at least $|V| - n$ connected components where $|E| = n$.

   Base case: In a graph where $|E| = 0$, every vertex is a connected component itself, thus the graph has exactly $|V| - 0 = |V|$ connected components.

   Inductive step: Assume that $P(n)$ holds for $n \geq 0$, that is, a graph with $|E| = n$ has at least $|V| - n$ connected components. Consider a graph with $n+1$ edges. Remove an edge (u, v) to create a n-edge graph G', which has at least $|V| - n$ connected components. Now add (u, v) to obtain the original graph G. If u and v were in the same connected component of G', then G has the same number of connected components as G'. If u and v were in the different connected components of G', then adding (u, v) would merge these two components of G' into one in G, but all other components remain. In both cases, the number of connected components in G is at least $|V| - n - 1 = |V| - (n+1)$.

   The theorem follows by induction. □

2. **Theorem**. Let G be a digraph(possible with self-loops) with vertices $v_1, \cdots, v_n$. Let M be the adjacency matrix of G. Then M_{ij}^k is equal to the number of length-k walk from v_i to v_j.

Proof. We use induction on k. Let $P(k)$ be the proposition that the number of length-k walk from v_i to v_j is M_{ij}^k, for all i, j.

Base case: for $k = 0$, a vertex only has length-0 walk to itself. Since M^0 is the identity matrix, $P(0)$ holds.

Inductive step: Assume $P(k)$ holds, we will prove that M_{ij}^{k+1} is equal to the number of length-$(k+1)$ walk from v_i to v_j. A length-$(k+1)$ walk from v_i to v_j consists of a walk of length-k from v_i to some intermediate vertex v_m followed by an edge (v_m, v_j). Therefore, the number of M_{ij}^{k+1} is equal to:

$$M_{iv_1}^k M_{v_1 j} + M_{iv_2}^k M_{v_2 j} + \cdots + M_{iv_n}^k M_{v_n j} \tag{1}$$

This is exactly M_{ij}^{k+1}, thus $P(k+1)$ also holds. The theorem follows by induction. □

3. **Theorem**. Every tree $T = (V, E)$ has the following properties:

   (a) There is a unique path between every pair of vertices.

   Proof. There is at least one path between every pair of vertices since the graph is connected. Suppose that there are two different paths between vertices u and v. Beginning at u, let x be the first vertex where paths diverge, and let y be the next vertex they share. Then there are two paths between x and y with no common edges, which creates a cycle. This is a contradiction since the graph is acyclic. Therefore there is exactly one path between every pair of vertices. □

   (b) Adding any edge creates a cycle.

   Proof. Suppose an edge was added between vertices u and v. Since there is already a path between u and v, this edge creates another path from u to v. This contradicts the fact that there can be only one path between every pair of vertices. □

   (c) Removing any edge disconnect the graph.

   Proof. Suppose that removing an edge between u and v doesn't disconnect the graph, then there is another path between u and v exists. This contradicts the fact that there can be only one path between every pair of vertices. □

   (d) Every tree with at least two vertices has at least two leaves.

   Proof. Suppose the sequence $v_1 v_2 \cdots v_m$ is the longest path in T. Then $m \geq 2$, since a tree with two vertices must contain at least one edge. There cannot be an edge between v_1 and v_i for $2 < i \leq m$, because otherwise $v_1 \cdots v_i$ would form a cycle. There cannot also be an edge between v_1 and u for some vertex u that is not on the path, because otherwise the path would be longer. Therefore the only edge incident to v_1 is (v_1, v_2), which means that v_1 is a leaf. By a symmetric argument, v_m is also a leaf. □

   (e) $|V| = |E| + 1$.

   Proof. We use induction on $|V|$. For the tree with a single vertex, the claim holds because $|E| + 1 = 0 + 1 = 1 = |V|$. Suppose that the claim holds for all n-vertex tree and consider an $(n+1)$-vertex tree T. Let v be a leaf of the tree. Deleting v and its incident edge gives us a n-vertex tree where $|V| = |E| + 1$ holds. Now add the vertex v and its incident edge back, then the equation still holds because both the number of vertices and edges increase by 1. Thus the claim holds for T and, by induction, for all trees. □