CS545 Spring, 2015

Homework Assignment #3
Due: March 26
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1. (a) Suppose we have 6 activities, for activity i, s_i is the start time, f_i is the finish time and d_i is the duration. And we sort them in order of increasing duration:

i	1	2	3	4	5	6
s_i	8	3	1	4	4	0
f_i	9	5	4	8	9	6
d_i	1	2	3	4	5	6

The greedy procedure when the activities are considered in order of increasing duration would be: 1) select activity 1 since it is the local best; 2) select activity 2 since it is compatible with activity 1; 3) no other activities can be selected since they are not compatible with either activity 1 or 2. So the procedure yields the solution $\{1, 2\}$, but the optimal solution should be $\{3, 4, 1\}$. Therefore the greedy procedure is not correct.

(b) With the same input, now let us sort them in order of increasing start-time:

i	1	2	3	4	5	6
s_i	0	1	3	4	4	8
f_i	6	4	5	8	9	9

The greedy procedure when the activities are considered in order of increasing duration is:

Function $GreedyActivitySelection(S, F, n)$
$A := \{1\}$ // activity with the earliest start time
$j := 1$ // indicates the activity with greatest start time in A
for $i := 2$ to n
  if $S[i] \geq F[j]$
    $A := A \cup \{i\}$
    $j := i$
return A

Running the above procedure with the same input yields the solution $\{1, 6\}$ which is certainly not an optimal solution. Therefore the greedy procedure is not correct.

(c) Consider the following 9 activities ordered by the number of overlaps which is denoted by o_i:

i	1	2	3	4	5	6	7	8	9
s_i	4	0	7	1	2	3	5	6	6
f_i	6	3	10	4	4	5	7	8	9
o_i	2	2	2	3	3	3	3	3	3

When considering the order of increasing number of overlaps, the greedy procedure would first pick activity 1, then pick activity 2, and finally pick activity 3. Thus the solution derived by the greedy procedure is $\{1, 2, 3\}$, but the optimal solution should be $\{2, 6, 7, 3\}$. Therefore the greedy procedure is not correct.

2. Label n gas stations as $g_1, g_2, \cdots, g_n$ and let d_i be the distance between gas stations g_i and g_{i+1}, specially, d_0 denotes the distance between g_1 and city A, and d_n denotes the distance between city B and g_n. Let D be the total distance between city A and B, where $D = \sum_{i=0}^{n} d_i$.

Greedy algorithm

The greedy strategy is to refuel at the gas station whose distance from the current stop is closest to m but less than m.

Function $GreedyTripRefuel()$
$A := \emptyset$ // the set of refueling stops
$i := 0$ // indicates the farthest gas station one can make.
$T_1 := 0$ // distance traveled before making a refuel
$T_2 := 0$ // total distance traveled
while $T_2 < D$
if $(T_2 - T_1) \leq m$
$T_2 := T_2 + d_i$
$i := i + 1$
else
$i := i - 1$ // cannot make it to station g_i, so must refuel at station g_{i-1}
$A := A \cup \{g_i\}$
$T_2 := T_2 - d_i$
$T_1 := T_2$
return A

The algorithm runs in $O(n)$ time since there are at most n gas stations along the way.

Correctness

Lemma: Suppose A is a subset of an optimal solution where the latest refuel stop is station g_k. Let g_i be the gas station after g_k whose distance from g_k is closest to m but less than m. Then $A \cup \{g_i\}$ is also a subset of an optimal solution.

Proof. Since A is a subset of an optimal solution, let A^* be an optimal solution with $A \subseteq A^*$. Let g_j be the gas station in $A^* - A$ that is next to g_k. We have two cases:

- case-1: $i = j$. Then $A \cup \{g_i\} \subseteq A^*$, the lemma holds.
- case-2: $i \neq j$. Since the distance from g_k to g_i is closest to m but less than m, this distance must be greater than the distance from g_k to g_j. Thus we can replace g_j with g_i and still yields an optimal solution since g_i is closer to the future stops than g_j.

□

Theorem: $GreedyTripRefuel()$ finds an optimal solution.

Proof. Initially, A is an empty set, which is a subset of an optimal solution and g_k in this case would be city A. By the lemma, $A \cup g_i$ is a subset of an optimal solution where g_i is the gas station whose distance from city A is closest to m but less than m.

By induction on the number of iterations, when the function terminates, A is a subset of an optimal solution.

Since from the starting point, the greedy algorithm makes the least possible stops, there is no better solution with fewer stops than the one produced by the greedy algorithm. Therefore A is optimal. □

3. (a) **Greedy algorithm**

Let the set of n tasks be $\{a_i | 1 \leq i \leq n\}$. The greedy strategy is to schedule the shortest task first.

i. Merge sort the n tasks in the order of increasing execution time such that $t_1 \leq t_2 \leq \cdots \leq t_n$.

ii. Schedule the tasks in the order of $\{a_1, a_2, \cdots, a_n\}$.

Step (a) takes $O(n \log n)$ time and step (b) takes $O(n)$ time, thus the overall runtime is $O(n \log n)$.

Correctness

First, we define containment relationship.
We define $\{a_1, a_2, \cdots, a_m\}$ as a schedule sequence such that a_i is scheduled after a_{i-1} where $1 < i \leq m$. For a schedule sequence $\{a_1, a_2, \cdots, a_n\}$, we define $\{a_1, a_2, \cdots, a_k\}$ as a prefix-subsequence of it where $k \leq n$.

Lemma: Suppose A is a prefix-subsequence of an optimal schedule sequence. Let a_i be the task that has the shortest execution time among those tasks that are not in A. Let A' be the prefix-subsequence by appending a_i to the end of A. Then A' is also a prefix-subsequence of an optimal schedule sequence.

Proof. Since A is a prefix-subsequence of an optimal schedule sequence, let A^* be the optimal schedule sequence that contains A as a prefix-subsequence. Let a_i be the task that has the shortest execution time among those tasks in $A^* - A$. By appending a_i to the end of the A, we get another prefix-subsequence A'. There are two cases:

i. A' is a prefix-subsequence of A^*, the lemma holds.

ii. A' is not a prefix-subsequence of A^*, Let a_j be the task that appears first in the subsequence $A^* - A$. In this case we must have: $t_i = t_j$. We prove this by contradiction:
Suppose $t_i \neq t_j$, since a_i is the task with the shortest execution time in $A^* - A$, we must have: $t_j > t_i$. Let the last task in A be a_m. Without loss of generality, assume $A^* - A$ is ordered as $a_j, a_{x_1}, \cdots, a_{x_s}, a_i, a_{y_1}, \cdots, a_{y_r}$ such that there are s tasks scheduled after a_j and before a_i, and there are r tasks scheduled after a_i. Let c_{x_k} denotes the completion time for the tasks scheduled in between a_j and a_i, and let c_{y_k} the completion time for the tasks scheduled after a_i. We have:

$$c_j = c_m + t_j \tag{1}$$

$$c_{x_k} = c_j + \sum_{1 \leq i \leq k}^{k \leq s} t_{x_i} \tag{2}$$

$$c_i = c_j + \sum_{1 \leq i \leq s} t_{x_i} + t_i \tag{3}$$

$$c_{y_k} = c_i + \sum_{1 \leq i \leq k}^{k \leq r} t_{y_i} \tag{4}$$

Now suppose we exchange the positions of a_i and a_j to produce another subsequence $(A^* - A)'$
$a_i, a_{x_1}, \cdots, a_{x_s}, a_j, a_{y_1}, \cdots, a_{y_r}$ and together with A, this produces another schedule sequence $A^{*'}$. In this case, let c'_j be the completion time for a_j, c'_i be the completion time for a_i, c'_{x_k} be the completion time for the tasks scheduled in between a_j and a_i, and let c'_{y_k} be

the completion time for the tasks scheduled after a_j. We have:

$$c'_i = c_m + t_i \tag{5}$$

$$c'_{x_k} = c'_i + \sum_{1\le i\le k}^{k\le s} t_{x_i} \tag{6}$$

$$c'_j = c'_i + \sum_{1\le i\le s} t_{x_i} + t_j \tag{7}$$

$$c'_{y_k} = c'_j + \sum_{1\le i\le k}^{k\le r} t_{y_i} \tag{8}$$

Now we compare completion times for the two subsequences:

$$c_j - c'_i = (c_m + t_j) - (c_m + t_i) \tag{9}$$
$$= t_j - t_i \tag{10}$$
$$> 0 \tag{11}$$

$$c_{x_k} - c'_{x_k} = (c_j + \sum_{1\le i\le k}^{k\le s} t_{x_i}) - (c'_i + \sum_{1\le i\le k}^{k\le s} t_{x_i}) \tag{12}$$
$$= c_j - c'_i \tag{13}$$
$$> 0 \tag{14}$$

$$c_i - c'_j = (c_j + \sum_{1\le i\le s} t_{x_i} + t_i) - (c'_i + \sum_{1\le i\le s} t_{x_i} + t_j) \tag{15}$$
$$= c_j - c'_i + t_i - t_j \tag{16}$$
$$= t_j - t'_i + t_i - t_j \tag{17}$$
$$= 0 \tag{18}$$

$$c_{y_k} - c'_{y_k} = (c_i + \sum_{1\le i\le k}^{k\le r} t_{y_i}) - (c'_j + \sum_{1\le i\le k}^{k\le r} t_{y_i}) \tag{19}$$
$$= c_i - c'_j \tag{20}$$
$$= 0 \tag{21}$$

Therefore, the sum of the completion time for $A^* - A$ is absolutely greater than the sum of the completion time for $(A^* - A)'$. So the average completion time of A^* must be greater than that of $A^{*'}$, we have found a better solution than A^*. This contradicts the fact that A^* is an optimal solution. Thus we must have $t_i = t_j$.
So we can get another optimal schedule sequence by exchanging the order of a_i and a_j because since $t_i = t_j$, the exchange will not affect the overall average completion time. And this new optimal schedule sequence has A' as a prefix-subsequence. The lemma still holds.

□

Theorem: The greedy algorithm finds an optimal schedule sequence.

Proof. Initially, A is a empty sequence, which is always a prefix-subsequence of any schedule sequence. Suppose all n tasks $a_1, a_2, \cdots, a_n$ have been sorted in the order of increasing execution time. By the lemma, $\{a_1\}$ is a prefix-subsequence of an optimal schedule sequence.

By induction on the number of iterations, when the algorithm terminates, A is still a prefix-subsequence of an optimal schedule sequence. And when the algorithm terminates, A would contain all n tasks, no other schedule sequence can properly contain A, thus A is the optimal schedule sequence. □

(b) **Algorithm**

The algorithm will use the shortest-remaining-time policy to find an optimal schedule. It uses a min-heap to organize the released but not completed tasks where heap ordering is based on their remaining execution times. A scheduling decision is made when a task is completed or when a new task is released.

Assume that the heap has the following operations:

- $heap.insert(a, r)$: insert task a with the remaining time r.
- $heap.pop()$: remove the task a with the smallest remaining time r and return the pair (a, r).
- $heap.findmin()$: find the task a with the smallest remaining time r and return the pair (a, r).
- $heap.notempty()$: return true if the heap is not empty, otherwise return false.

Let A be the set of n tasks: $\{a_i | 1 \leq i \leq n\}$, T be the set of execution times: $\{t_i | 1 \leq i \leq n\}$, and R be the set of release times: $\{r_i | 1 \leq i \leq n\}$. The scheduling algorithm is implemented as follows:

Function $SRTFschedule(heap, A, R, T, r)$

sort tasks in the order of increasing releasing time using quick sort
$S := \emptyset$ // set of scheduled tasks
$i := 1$
$minr_1 := r_i$ // start time for the next scheduled task
$heap.insert(a_i, t_i)$
while $r_{i+1} == r_i$
    $i := i + 1$
    $heap.insert(a_i, t_i)$
$i := i + 1$
while $i \leq n$
    $minr_2 := r_i$ // next release time
    $(a, r) := heap.pop()$
    $S := S \cup \{a\}$ // schedule task a
    if $minr_2 - minr_1 \geq r$ //task a can finish before the next batch of released tasks
        $minr_1 := minr_1 + r$ // update the start time for the next scheduled task
        $(a, r) = heap.pop()$ // schedule the next task on heap before next batch of release
        $S := S \cup \{a\}$
    else // task a runs for $(minr_2 - minr_1)$ time, then the next batch of release comes
        $heap.insert(a_i, t_i)$
        while $r_{i+1} == r_i$
            $i := i + 1$
            $heap.insert(a_i, t_i)$
        $i := i + 1$
        $(a', r') := heap.findmin()$
        $r := r - (minr_2 - minr_1)$ // update the remaining time for task a
        $minr_1 = minr_2$ // update $minr_1$
        if $r' \geq r$ // continue to run task a
            continue
        else // current running task a is preempted
            $heap.insert(a, r)$ // put task a back to the heap
            $(a, r) := heap.pop()$ // schedule the task with the smallest remaining time
            $S := S \cup \{a\}$

while heap.notempty() // no more tasks to release
    $(a, r) := heap.pop()$
    $S := S \cup \{a\}$

Runtime

Quick sort takes $O(n \log n)$ time;
$heap.insert()$ and $heap.pop()$ take $O(\log n)$ time;
$heap.findmain()$ takes $O(1)$ time;
The first outer-most **while** loop takes $O(n \log n)$ time;
The second outer-most **while** loop takes $O(n \log n)$ time; note that although there is another **while** loop inside the outer-loop, this inner-loop will increment the value of i which decrement the number of iterations of the outer-loop, thus the amortized cost for each iteration of outer-loop is still $O(\log n)$.
The third outer-most **while** loop also takes $O(n \log n)$ time;
Thus the total run time of the algorithm is $O(n \log n)$.

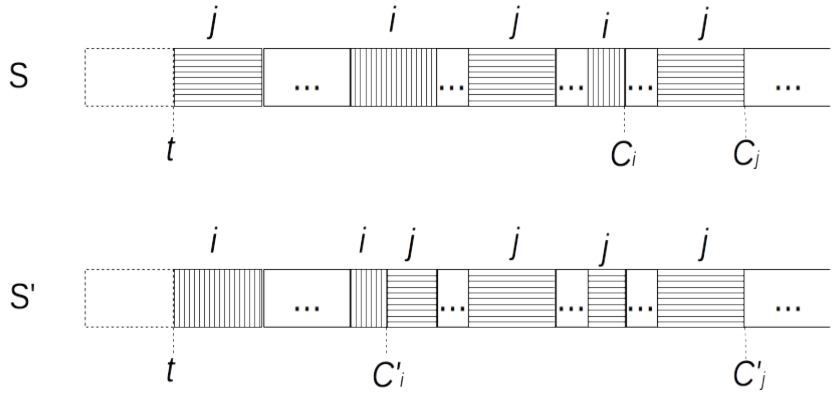
Correctness

Theorem: Shortest-remaining-time-first schedule is optimal.

Proof. Prove by contradiction.
Assume that Shortest-remaining-time-first schedule is not optimal, then there exists an optimal schedule S that does not use shortest-remaining-time-first policy.
Suppose at time t, instead of scheduling task i with the shortest-remaining-time at the time being, S schedules task k with longer remaining-time. Let R_i and R_j be the remaining time for tasks i and j, and C_i and C_j be the completion time for tasks i and j. We have: $R_i < R_j$. In total, $R_i + R_j$ is spent on tasks i and j after time t. We assume that $C_i < C_j$.
We can get another schedule S' by interchanging schedule sequence of tasks i and j:

i. devote the first R_i units of time that were distributed to either task i or j after t completely to task i until its completion.
ii. devote the remaining R_j units of time for task j since task i has completed.

The two schedules are shown in the figure below.



Let C'_i and C'_j be the completion time for tasks i and j in the new schedule S'.
By doing this interchanging, we get a better solution, because:

$$C_i > C'_i \tag{22}$$
$$C_j = C'_j \tag{23}$$

Since in the new schedule, other tasks other than task i have the same completion time as before, thus the total sum of completion of the new schedule S' is less than that of S. This contradicts the fact that S' is optimal. Thus shortest-remaining-time-first schedule is optimal. □

4. (a) **Greedy algorithm**

The greedy strategy is to always choose the largest denomination that is not greater than the remaining amount.
The function $GreedyCoinChange(n)$ implements this strategy.

Function *GreedyCoinChange*(n)

$A := \emptyset$ // the set of choose coins

denominations $:= \{25, 10, 5, 1\}$ // set of available denominations

while $n > 0$

$x :=$ the largest value in *denominations* that is $\leq n$

$A := A \cup \{x\}$ // pick x

$n := n - x$

return A

The runtime of the algorithm is $O(n)$.

Correctness

First, we make the following claims:

- Claim-1: There cannot be more than 4 pennies in an optimal solution because 5 pennies can be replaced by 1 nickle which yields a better solution with 4 coins less.
- Claim-2: There cannot be more than 1 nickle in an optimal solution because 2 nickles can be replaced by 1 dime which yields a better solution with 1 coin less.
- Claim-3: There cannot be more than 2 dimes in an optimal solution because 3 dimes can be replaced by 1 quarter and 1 nickle which yield a better solution with 1 coin less.
- Claim-4: There cannot be 2 dimes and 1 nickle both in an optimal solution because 2 dimes and 1 nickle can be replaced by 1 quarter which yield a better solution with 2 coins less.

Lemma: Suppose A is a subset of an optimal solution A^*. Let m be the amount of the remaining cents, that is, total cents in $A^* - A$. Let x be the largest value among $\{25, 10, 5, 1\}$ that is not greater than m. Then $A' = A \cup \{x\}$ is also a subset of an optimal solution.

Proof. We prove by cases. There are total 4 cases.

- Case-1: x = 25.
  In this case, we have $m \geq 25$. If $x \in A^* - A$, the lemma holds. If $x \notin A^* - A$, then $A^* - A$ cannot contain quarters. We have two sub-cases: 1) $25 \leq m < 30$, then the best possible solution is to pick 2 dimes and 1 nickle, and $m - 25$ pennies. But this violates the *Claim-4*; 2) $m \geq 30$, then the best possible solution must contain at least 3 dimes, but this violates the *Claim-3*. Since in both sub-cases, the best possible solutions violate some claims, we know that they must not be subset of an optimal solution. Therefore x must be in $A^* - A$, the lemma holds.
- Case-2: x = 10.
  In this case, we have $10 \leq m < 25$. If $x \in A^* - A$, the lemma holds. If $x \notin A^* - A$, then $A^* - A$ cannot contain quarters and dimes. The best possible solution is to pick at least 2 nickles, but this violates the *Claim-2*. So x must be in $A^* - A$, the lemma holds.
- Case-3: x = 5.
  In this case, we have $5 \leq m < 10$. If $x \in A^* - A$, the lemma holds. If $x \notin A^* - A$, then $A^* - A$ cannot contain quarters, dimes and nickles. The best possible solution is to pick m pennies, but this violates the *Claim-1*. So x must be in $A^* - A$, the lemma holds.
- Case-4: x = 1.
  In this case, we have $1 \leq m < 5$ and x must be in $A^* - A$ because pennies are the only choices, thus the lemma holds.

Since for all 4 cases, the lemma holds, it must be true. □

Theorem: Greedy algorithm *GreedyCoinChange* finds an optimal solution.

Proof. Initially, A is an empty set which is always a subset of any optimal solutions.By the lemma, $\{x\}$ is a subset of an optimal solution where x is the largest denomination that is not greater than n. By induction on the number of iterations, when the algorithm terminates, A is still a subset of an optimal solution. When the algorithm terminates, A would contain coins with the total amount equals n and satisfy the 4 claims, thus no other solutions can properly contain A, thus A is optimal. □

(b) *Proof.* prove by giving an counter example. Suppose the denomination system is the set: $\{1, 3, 4\}$ and $n = 6$. The greedy algorithm *GreedyCoinChange* yields the solution $\{4, 1, 1\}$. But the optimal solution is $\{3, 3\}$. This proves that greedy algorithm from Part (a) does not make optimal change for all systems of coins. □

(c) **Lemma:** There cannot be more than $(a - 1)$ coins with the value a^i in an optimal solution where $0 \leq i < k$.

Proof. Prove by contradiction. Suppose there is an optimal solution that contains more than $(a - 1)$ coins of the value a^i for some i where $0 \leq i < k$. But a coins of the value a^i can be replaced with 1 coin with the value of a^{i+1}. This yields a better solution with $a - 1$ coins less since $a > 1$. This is a contradiction. Therefore the lemma must always hold. □

Theorem: The greedy algorithm in part-a makes optimal change for the system of coins with denominations $\{a^0, a^1, \cdots, a^k\}$ where $a > 1$ and $k \geq 1$.

Proof. Prove by contradiction.
Let A be the solution found by the greedy algorithm. Suppose A is not optimal, then let A' be an optimal solution. Define $C = \{c_i | 0 \leq i \leq k\}$ as the set of coin counts for A where c_i is the number of coins with the value of a^i in A. And define $C' = \{c'_i | 0 \leq i \leq k\}$ as the set of coin counts for A' where c'_i is the number of coins with the value of a^i in A'.
Let j be the first highest index for which $c_j \neq c'_j$. Then we must have: $c_j > c'_j$ because our algorithm makes greedy choices locally and picks as much a^j as possible and this is the first value on which they disagree. Also $j > 0$ because if only c_0 and c'_0 are different, then the total amount in A and A' would be different.
Since $c_j > c'_j$ and $c'_j \geq 0$, the partial amount $B = \sum_{i=0}^{j-1} c'_i a^i$ in A' must satisfy the condition that $B \geq a^j$.
Also, by the lemma, the largest possible value for the partial amount B is:

$$\sum_{i=0}^{j-1} c'_i a^i = (a-1)a^0 + (a-1)a^1 + \cdots + (a-1)a^{j-1} \tag{24}$$

$$= (a-1)\sum_{i=0}^{j-1} a^i \tag{25}$$

$$= (a-1)\frac{1-a^j}{1-a} \tag{26}$$

$$= a^j - 1 \tag{27}$$

$$< a^j \tag{28}$$

This contradicts the fact that $B \geq a^j$. Thus A must be an optimal solution. □

(d) **Recursive structure of an optimal change**

Give a set of denomination $D = \{d_1, d_2, \cdots, d_k\}$ where $d_1 = 1$ and an amount of cents n, an optimal change must start with one of coins in D, say it is d_i where $1 \leq i \leq k$. Then the optimal change must be 1+ (optimal change for $n - d_i$).

Recurrence equation

Define $C(i)$ be the optimal number of changes made for amount of i cents. The goal value is $C(n)$.

$$C(i) := \begin{cases} \min\limits_{1 \leq j \leq k} (C(i - d_j)) + 1 & i > 0 \\ 0 & i = 0 \\ \infty & i < 0 \end{cases} \tag{29}$$

Evaluate the recurrence equation

We fill the table up from $C(0)$ until $C(n)$.

The size of table is $O(n)$ and to fill each cell, we need $O(k)$ comparison, so the total run time of filling the table is $O(kn)$.

Recover the optimal solution

For the optimal number of coins, we simply look up the value of $C(n)$.

For the number of each denomination in the optimal solution, we can trace back to see how many coins are needed to make change for $(n - d_i)$ cents for each i where $1 \leq i \leq k$, and choose the smallest number. Repeat this process until the remaining amount is ≤ 0.