# Problem Set 1 - Solution

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1. Prove by contradiction. Suppose that $\sqrt[3]{2}$ is sensible. Then by definition, there exist positive integers $a$ and $b$ such that $\sqrt{a/b} = \sqrt[3]{2}$. Squaring both sides of the equation gives us $a/b = \sqrt[3]{4}$, which implies that $\sqrt[3]{4}$ is rational.

   This means there exists positive integers $x$ and $y$ such that $x/y = \sqrt[3]{4}$ and $x/y$ is in lowest term. Therefore, we have:

   $$\begin{aligned} x/y &= \sqrt[3]{4} && (1)\\ x^3/y^3 &= 4 && (2)\\ x^3 &= 4y^3 && (3) \end{aligned}$$

   In the last equation, the right side is even, so is the left side. Since $x^3$ is even, $x$ must be even. Therefore $x^3$ is a multiple of 8, this implies that $y^3$ is also a even number, thus $y$ is even, too. $x$ and $y$ are both even, this contradicts the assumption that $x/y$ is in lowest term.

   Thus $\sqrt[3]{4}$ is irrational, the original assumption must not be true.

2. **A Wrong Attempt:**

   $$\exists x.\ (E(x,y) \land E(x,z) \land x \neq y \land x \neq z \land y \neq z) \tag{4}$$

   This doesn't say "$x$ emailed exactly two other people in the class.". It also doesn't existentially quantify $y$ and $z$.

   **Right Attempt:**
   First, we modify the above predicate to express that there exist students $x$, $y$ and $z$ such that $x$ have emailed $y$ and $z$.

   $$\exists x \exists y \exists z.\ (E(x,y) \land E(x,z) \land x \neq y \land x \neq z \land y \neq z) \tag{5}$$

   The following predicate restricts that $x$ emailed exactly $y$ and $y$, besides herself.

   $$\forall s,\ E(x,s) \implies s = x \lor s = y \lor s = z \tag{6}$$

   Combining these two predicates, we can say that there exists some student $x$ who has emailed to exactly two other students $y$ and $z$, besides possibly herself.

   $$\begin{aligned} \exists x \exists y \exists z.\ (E(x,y) \land E(x,z) \land && (7)\\ x \neq y \land x \neq z \land y \neq z \land && (8)\\ \forall s,\ E(x,s) \implies s = x \lor s = y \lor s = z) && (9) \end{aligned}$$

3. (a) $\exists a, b, c.\ (n = a \cdot a + b \cdot b + c \cdot c)$

(b) We can express $x = 1$ as: $\forall y.\ (xy = y)$. Further we can express $x > 1$ as: $\exists y.\ (y = 1 \wedge x > y)$. Replacing $y = 1$ with the previous predicate gives us: $\exists y.\ (\forall z.\ (yz = z) \wedge x > y)$.

(c)

$$\neg(\exists x.\ (x > 1 \wedge x < n \wedge \exists y.\ (y > 1 \wedge y < n \wedge xy = n))) \tag{10}$$

**A better version:**

$$IS-PRIME(n) \equiv (n > 1) \wedge \neg(\exists x \exists y.\ (x > 1 \wedge y > 1 \wedge x \cdot y = n)) \tag{11}$$

(d)

$$\exists n \exists p \exists q.\ \text{IS-PRIME}(p) \wedge\ \text{IS-PRIME}(q) \wedge (n = p \cdot q) \wedge (p \neq q) \tag{12}$$

(e) **My Attempt:**

$$\neg(\exists n.\ \text{IS-PRIME}(n) \wedge\ \text{IS-PRIME}(p) \wedge\ \forall p, n > p) \tag{13}$$

**Right Solution:**

$$\neg(\exists n.\ \text{IS-PRIME}(n) \wedge (\forall p,\ \text{IS-PRIME}(p) \implies n \geq p)) \tag{14}$$

(f) We can express $n > 2$ as:

$$\exists k.\ (k = 1) \wedge (n > k + k). \tag{15}$$

So the predicate is:

$$\forall (n), (n > 2 \wedge \exists k.n = k + k) \implies \exists p \exists q.\ \text{IS-PRIME}(p) \wedge\ \text{IS-PRIME}(q) \wedge (n = p + q) \tag{16}$$

(g)

$$\forall (n), (n > 1 \implies \exists p.\ \text{IS-PRIME}(p) \wedge (n < p) \wedge (p < n + n)) \tag{17}$$

4. *Proof.* We prove by contradiction. Assume that there is a surjection $f$ from set $A$ to its powerset for some $A$ which is infinite. Let the set $W$ be $W = \{x \in A | x \notin f(x)\}$. So by definition,

$$x \in W \iff x \notin f(x) \tag{18}$$

$W$ is a member of $\mathcal{P}(A)$ since $W$ is a subset of $A$. Because there is a surjection $f$ from $A$ to $\mathcal{P}(A)$, we know that there must exist an element $a \in A$ such that $W = f(a)$. So from the above equation, we have,

$$x \in f(a) \iff x \notin f(x) \tag{19}$$

for all $x \in A$. Substituting $a$ for $x$ yields a contradiction, proving that there cannot be such a $f$. $\square$

5. (a) *Proof.* Let $D$ be the domain for the variables and $P, Q$ be some binary predicates on $D$. We need to show that if $\exists z.[P(z) \wedge Q(z)]$ holds under this interpretation, then so does $[\exists x.P(x) \wedge \exists y.Q(y)]$. So suppose $\exists z.[P(z) \wedge Q(z)]$. So some element $z_0 \in D$ such that $P(z) \wedge Q(z)$ is true. So there exists some $x \in D$ and some $y \in D$ such that $P(z) \wedge Q(z)$ is true. Namely, $x = z_0$ and $y = z_0$. That is, $[\exists x.P(x) \wedge \exists y.Q(y)]$ holds under this interpretation, as required. $\square$

(b) *Proof.* We can prove by describing an counter model. Let the domain be the integers and $P(x)$ be $x > 100$, and $Q(y)$ be $y < 50$. $[\exists x.P(x) \wedge \exists y.Q(y)]$ would be true because we can let $x = 101$ and $y = 49$. But $\exists z.[P(z) \wedge Q(z)]$ asserts that there exists an integer $z$ such that $z > 100$ while $z < 50$, which is certainly false. $\square$

6. (a) Let $A = 0$, $B = 1$, $C = 2$ and $D = 3$. Then $L = (A \cup C) \times (B \cup D) = \{(0, 1), (0, 3), (2, 1), (2, 3)\}$ and $R = (A \times B) \cup (C \times D) = \{(0, 1), (2, 3)\}$. Thus $L \neq R$.

(b) The mistake lies in the third iff. The claim:
"either $x \in A$ or $x \in C$, and either $y \in B$ and $y \in D$ **iff**
($x \in A$ and $y \in B$) or else ($x \in C$ and $y \in D$)"
is not true. There are 4 possible combinations in total. Two others are ($x \in A$ and $y \in D$) and ($x \in C$ and $y \in B$).

(c) Replacing the third "iff" with "which is true when" which yields a correct prove that $(x, y) \in L$ is true when $(x, y) \in R$, which implies that $R \subseteq L$.