Matrices

Singular and Non-Singular Matrices :-

A square matrix A is said to be singular, if $|A| = 0$, and non-singular if $|A| \neq 0$.

Properties of Matrix Addition and Multiplication :-

(i) $A + B = B + A$ (Commutative)

(ii) $(A + B) + C = A + (B + C)$ (Associative)

(iii) $AB \neq BA$ (Not Commutative)

(iv) $(AB)C = A(BC)$ (Associative)

(v) $A(B + C) = AB + AC$ (Distributive)

Transpose of a Matrix :-

If $A = [a_{ij}]_{m \times n}$, then $A' = [b_{ij}]_{n \times m}$ where, $b_{ij} = a_{ji}$

Properties of Transpose of a Matrix :-

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$

(iii) $(AB)' = B'A'$

Symmetric Matrix :-

$$A' = A$$

Skew-Symmetric Matrix (or Anti-Symmetric Matrix) :-

$$A' = -A$$

(If A and B are symmetric matrices then $(AB - BA)$ is a skew-symmetric matrix.)

Orthogonal Matrix :-

$$AA' = A'A = I$$

Idempotent Matrix :-

$$A^2 = A$$

Involutary Matrix :-

$$A^2 = I$$

Every square matrix can uniquely be expressed as the sum of a Symmetric Matrix and a Skew-Symmetric Matrix.

$$A \Rightarrow \frac{1}{2}(A + A')[\text{Symmetric}] + \frac{1}{2}(A - A')[\text{Skew-Symmetric}]$$

Adjoint of a Square Matrix :-

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then :-

Co-factor of a_1

$$\text{Adj } A = \text{transpose of } \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

Properties of Adjoint :-

(i) $A(\text{Adj } A) = (\text{Adj } A)A = |A| I_n$

(ii) $\text{Adj}(AB) = (\text{Adj } B).(\text{Adj } A)$

Inverse of a Square Matrix :-

$$A^{-1} = \frac{\text{Adj } A}{|A|} ; \quad |A| \neq 0$$

Properties of Inverse :-

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

(iii) $(A')^{-1} = (A^{-1})'$

(iv) Only a non-singular square matrix can have an inverse.

Example :- If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then find A^{-1}.

Solution :-

$$\text{Adj } A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$|A| = 3(-3+4) + 3(2) - 4(-2) = 1$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Elementary Matrices :-

The matrix obtained from a unit matrix I by subjecting it to one of E-operations is called an elementary matrix.

Rank of a Matrix :-

Let A be any $m \times n$ matrix. It has square sub-matrices of different orders. The determinants of these square sub-matrices are called minors of A.

A matrix is said to be of rank r, if :-

(i) It has atleast one non-zero minor of order r.

(ii) All the minors of order $(r+1)$ or higher than r are zero.

\# Rank of A $= r$ is written as $\rho(A) = r$.

\# If A is a non-singular $n \times n$ matrix, then $\rho(A) = n$

Echelon form method of finding rank :-

In this form of the matrix, each of the first r elements of the leading diagonal is 1 and every element below the ~~dig~~ diagonal/r^{th} row is zero.

The rank of the matrix is equal to the no. of non-zero diagonal elements when it has been reduced to Echelon form.

Solution of a System of Linear Equations :-

A system of equations having no-solution is called an inconsistent system of equations.

A system of equations having one or more solution is called a consistent system of equations.

For a system of non-homogeneous linear equations $AX = B$:-

(i) if $\rho[A:B] \neq \rho(A)$, the system is inconsistent.
(ii) if $\rho[A:B] = \rho(A) =$ number of unknowns, the system has a unique solution.
(iii) if $\rho[A:B] = \rho(A) <$ number of unknowns, the system has an infinite number of solutions.

For a system of homogeneous linear equations $AX = 0$:-

(i) $X = 0$ is always a solution (Trivial Solution).
(ii) if $\rho(A) =$ number of unknowns, the system has only the trivial solution.
(iii) if $\rho(A) <$ number of unknowns, the system has an infinite number of non-trivial solutions.

Homogeneous System is always consistent.

Linear Dependence and Linear Independence of Vectors :-

A set of r n-vectors $X_1, X_2, \ldots X_r$ is said to be linearly dependent if there exist r scalars (numbers) $k_1, k_2, \ldots, k_r$, not all zero, such that :-

$$k_1 X_1 + k_2 X_2 + \ldots + k_r X_r = 0$$

It is called linearly independent if every relation of the type :-

$$k_1 X_1 + k_2 X_2 + \ldots + k_r X_r = 0$$

implies $k_1 = k_2 = \ldots = k_r = 0$

Characteristic Equation :-

If A is a square matrix of order n, we can form the matrix $(A - \lambda I)$, where λ is a scalar and I is the unit matrix of order n.

The determinant of this matrix equated to zero i.e.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the characteristic equation of A.

The roots of this equation are called the characteristic roots or eigen values of A.

Eigen Vectors and Eigen Values :-

Consider a square matrix A of size $(n \times n)$, then a column vector X of size $(n \times 1)$ is called the Eigen Vector of A, if :-

$$AX = \lambda X$$

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow (A - \lambda I)X = 0 \qquad \text{--- (1)}$$

where, λ is a nonzero scalar.

The characteristic roots of equation-1 are called the Eigen Values.

Example :- Find the eigen values and eigen vectors of the matrix :-

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution :- The characteristic equation of the given matrix is :-

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0 \Rightarrow \boxed{\lambda = -3, -3, 5}$$

Eigen Value

Corresponding to $\lambda = -3$, the eigen vectors are given by :-

$$(A + 3I)X_1 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Here, we get only one independent equation :-

$$x_1 + 2x_2 - 3x_3 = 0$$

Let $x_3 = k_1$ and $x_2 = k_2$, then $x_1 = 3k_1 - 2k_2$

$$X_1 = \begin{bmatrix} 3k_1 - 2k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Ans

Corresponding to $\lambda = 5$, the eigen vectors are given by :-

$$(A - 5I)X_2 = 0$$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = k_3, \quad x_2 = 2k_3, \quad x_3 = -k_3$$

$$\Rightarrow X_2 = k_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Ans

Properties of Eigen Vectors and Eigen Values :-

(1) The sum of the eigen values of a matrix A is equal to trace of A.

(2) The product of the eigen values of a matrix A is equal to its determinant.

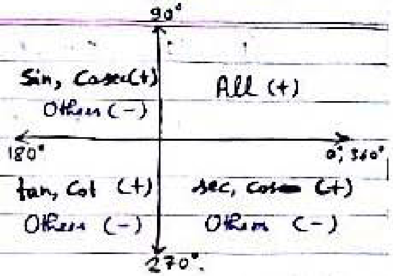
(3) If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value.

(4) The eigen values of an idempotent matrix are either zero or unity.

The trace or spur of a square matrix is the sum of its diagonal elements.

Limit, Continuity & Differentiability

Trigonometry :-



(1) $\sin^2\theta + \cos^2\theta = 1$

(2) $1 + \tan^2\theta = \sec^2\theta$

(3) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

(4) $\sec\theta = 1/\cos\theta$ (5) $\operatorname{cosec}\theta = 1/\sin\theta$

(6) $\tan\theta = \sin\theta/\cos\theta$ (7) $\cot\theta = \cos\theta/\sin\theta$

(7) (a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(b) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(c) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(d) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(e) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$, (f) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(8) (a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(b) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(d) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(9) (a) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(b) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(c) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(d) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(10) (a) $\sin 2x = 2\sin x\cos x$

(b) $\cos 2x = (\cos^2 x - \sin^2 x) = (2\cos^2 x - 1) = (1 - 2\sin^2 x)$

(c) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

(II) (a) $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

(b) $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$

Limits :-

Some important expensions to be used :-

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$

(ii) $\left(\frac{x^n - a^n}{x-a}\right) = (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})$

(iii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(iv) $a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \ldots$

(v) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

(vi) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(vii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(viii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

Some Important Theorems on Limits :-

(i) $\lim_{x\to a}\left(\frac{x^n - a^n}{x-a}\right) = na^{n-1}$, where $a > 0$

(ii) $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = 1$, (iii) $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \log_e a$

(iv) $\lim_{x\to 0}(1+x)^{1/x} = e$ | $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right) = 0$

(v) $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$ | $\lim_{x\to\infty}\left(\frac{\cos x}{x}\right) = 0$

(vi) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$, (vii) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1$

Continuity and Differentiability :-

\# A function is continuous, if its graph is a single unbroken curve with no holes or jumps.

\# A function is differentiable, if its graph is relatively smooth, and does not contain any breaks, or bends.

\# A differentiable function is always continuous, but a continuous function need not be differentiable.

Differentiation :-

Some important formulaes :-

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$, (ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x\log_e a$, (iv) $\frac{d}{dx}(\sin x) = \cos x$

(v) $\frac{d}{dx}\cos x = -\sin x$, (vi) $\frac{d}{dx}(\tan x) = \sec^2 x$

(vii) $\frac{d}{dx}\sec x = \sec x\tan x$, (viii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x$

(ix) $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$, (x) $\frac{d}{dx}(\log_e x) = 1/x$

(xi) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, (xii) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

(xiii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$, (xiv) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

(xv) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$, (xvi) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

(xvii) $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$

(xviii) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

(xix) $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$.

(xx) $\frac{d}{dx}\log_a x = \frac{1}{x\log_e a}$

Integration

Fundamental Integration Formulas :-

(i) $\int x^n \, dx = \frac{x^{n+1}}{(n+1)}$ (ii) $\int \frac{1}{x} \, dx = \log x + c$

(iii) $\int e^x \, dx = e^x + c$ (iv) $\int a^x \, dx = \frac{a^x}{\log_e a} + c$

(v) $\int \sin x \, dx = -\cos x + c$ (vi) $\int \cos x \, dx = \sin x + c$

(vii) $\int \sec^2 x \, dx = \tan x + c$ (viii) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

(ix) $\int \sec x \tan x \, dx = \sec x + c$ (x) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

(xi) $\int \cot x \, dx = \log(\sin x) + c$ (xii) $\int \tan x \, dx = -\log(\cos x) + c$

(xiii) $\int \sec x \, dx = \log(\sec x + \tan x) + c$

(xiv) $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + c$

(xv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$ (xvi) $\int \frac{-1}{\sqrt{a^2 - x^2}} \, dx = \cos^{-1}\left(\frac{x}{a}\right) + c$

(xvii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$, (xviii) $\int \frac{-1}{a^2 + x^2} \, dx = \frac{1}{a} \cot^{-1}\left(\frac{x}{a}\right) + c$

(xix) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$, (xx) $\int \frac{-dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + c$

Integration by Substitution :-

$$I = \int f(g(x)) \cdot g'(x) \, dx,$$

let $g(x) = t$

$\Rightarrow g'(x) \, dx = dt$

$$\Rightarrow I = \int f(t) \, dt$$

* Note :-

If $\int f(x) \, dx = g(x) + c$, then :-

$$\int f(ax+b) \, dx = \frac{1}{a} g(ax+b) + c$$

Integration by Parts :-

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Numerical Methods

Numerical Solution of Algebraic Equations :-

(1) Bisection Method :-

$f(x)=0$

Let the function $f(x)$ be continuous b/w a & b and let $f(a)$ be (−)ve and $f(b)$ be (+)ve, then the first approximation to the root is :-

$$x_1 = \frac{1}{2}(a+b)$$

If $f(x_1)=0$, then x_1 is a root of $f(x)=0$ otherwise, if $f(x_1)$ is negative :- $x_2 = \frac{1}{2}(x_1+b)$

if $f(x_1)$ is positive :- $x_2 = \frac{1}{2}(a+x_1)$ and so on.

(2) Secant Method :-

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n)-f(x_{n-1})}\right] f(x_n)$$

(3) Newton-Raphson Method :-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(4) Regula-Falsi Method :- (Always Converges)

$$x_{n+1} = x_0 - \left[\frac{x_n - x_0}{f(x_n)-f(x_0)}\right] f(x_0)$$

Convergence Rate :-

Newton Raphson > Secant > Regula-Falsi
(Quadratic) (Linear)

\# Newton's method does not always converge.

\# Order of convergence of secant method = 1.618
\# Order of convergence of newton's method = 2
\# Order of convergence of Regula-Falsi Method = 1
\# If the initial values are not close enough to the root, then there is no guarantee that the secant method converges.

Numerical Integration :-

(1) Trapezoidal Rule :-

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\right]$$

(2) Simpson's One-Third Rule :-

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\dots+y_{n-1}) + 2(y_2+y_4+\dots+y_{n-2})\right]$$

(3) Simpson's Three-Eighth Rule :-

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots+y_{n-2}+y_{n-1}) + 2(y_3+y_6+\dots+y_{n-3})\right]$$

Here h is the interval, and n is the no. of intervals.

$y_0 = f(x_0)$, $y_1 = f(x_0+h)$, $y_2 = f(x_0+2h)$ ... and so on.

For Simpson's One-third rule, n is even.

For Simpson's three-eighth rule, n should be multiple of 3.

Error in Trapezoidal Rule :-

$$\text{Error} \le \frac{(b-a)^3}{12\,n^2}\max|f''(x)|$$

or $$\text{Error} \le \frac{(b-a)}{12} h^2 \max|f''(x)|$$

Error in Simpson's Rule :-

$$\text{Error} \le \frac{(b-a)}{180} h^4 \max|f^{(4)}(x)|$$

Here, $a = x_0$, $b = x_0 + nh$,

$$h = \frac{b-a}{n}$$

and $f''(x)$ & $f^{(4)}(x)$ refers to the values taken on $[a, b]$

Simpson's rule provides exact results for any polynomial f of degree three or less.

Trapezoidal rule gives exact results for any polynomial f of degree one