MMÜ753 – Vehicle Control Systems

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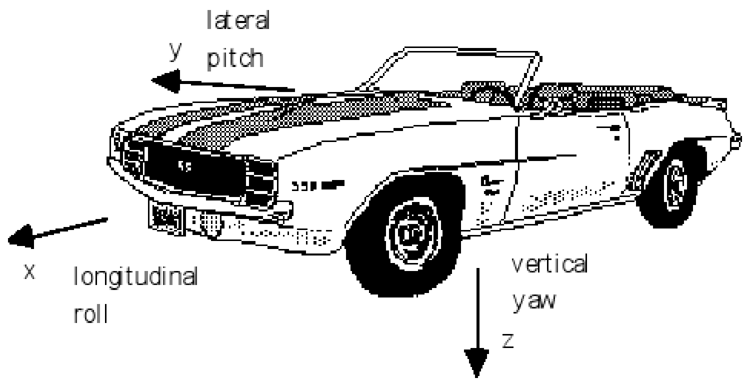
CHAPTER – III

VEHICLE LATERAL DYNAMICS

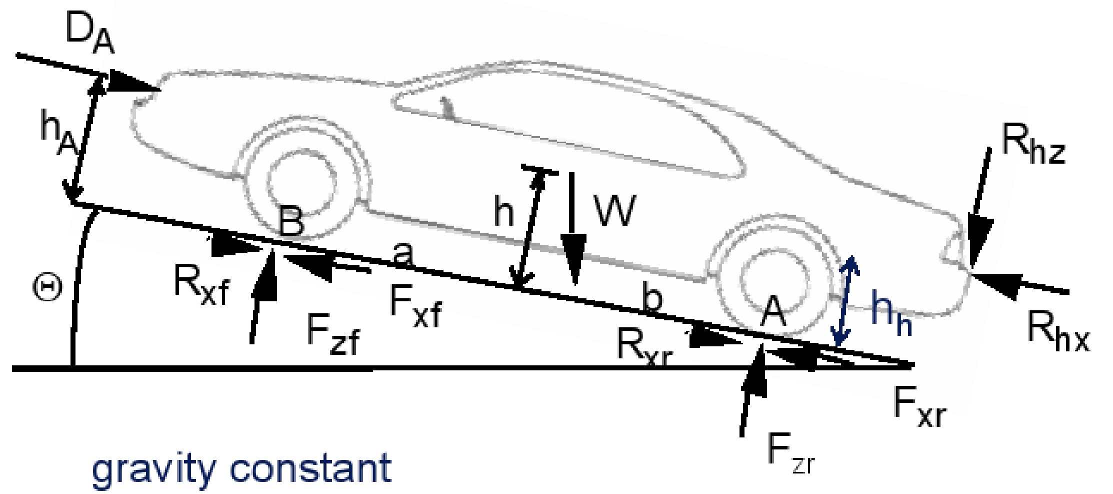
SAE Vehicle-Fixed Coordinate System: Symbols and Definitions



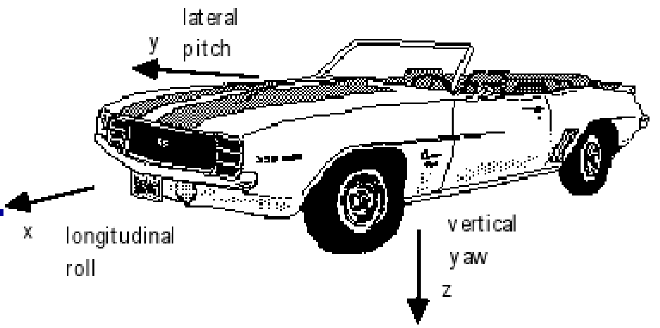
- Pitch angle: The angle between x-axis and the horizontal plane
- Roll angle: The angle between y-axis and the horizontal plane.
- Yaw angle: The angle between x-axis and the X-axis of inertial frame

Axis	Translational Velocity	Angular Displace-ment	Angular Velocity	Force Component	Moment Component
x	u (forward)	ϕ	p or $\dot{\phi}$ (roll)	Fx	Mx
y	v (lateral)	θ	q or $\dot{\theta}$ (pitch)	Fy	My
z	w (vertical)	ψ	r or $\dot{\psi}$ (yaw)	Fz	Mz

Longitudinal Vehicle Motion



g: gravity constant
D_A : aerodynamic drag force
R_h : drawbar force
W : weight of the vehicle (= mg)
F_x : tractive force
F_z : tire normal force
R_x : rolling resistance force
ma_x: inertial force



Vehicle Lateral Models

$$\sum F_x = m\dot{u}'$$

Lateral $\quad \sum F_y = m(\dot{v} + r u_o)$

$$\sum F_z = m(\dot{w} - q u_o)$$

Roll $\quad \sum M_x = I_{xx}\dot{p} - I_{xz}\dot{r}$

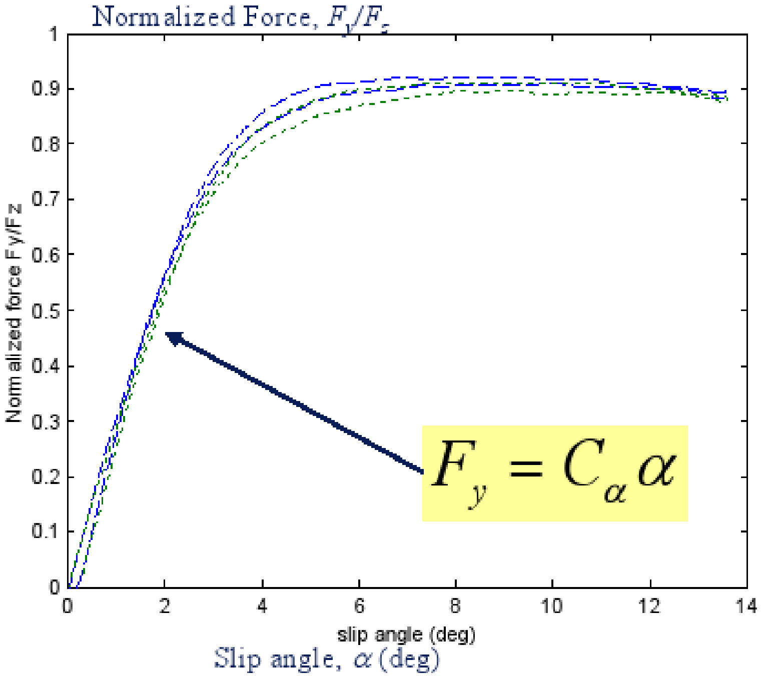
$$\sum M_y = I_{yy}\dot{q}$$

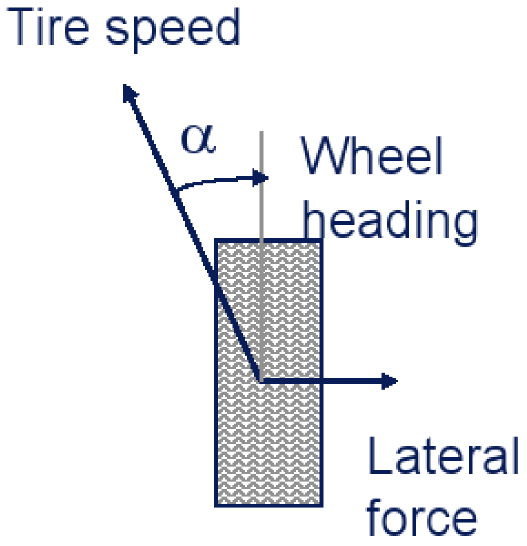
Yaw $\quad \sum M_z = -I_{xz}\dot{p} + I_{zz}\dot{r}$

- Two popular types of models:
  - Lateral/Yaw (2DOF Handling, Lateral/Yaw)
  - Lateral/Yaw/Roll (3DOF Lateral/Yaw/Roll)

Vehicle Lateral Dynamics

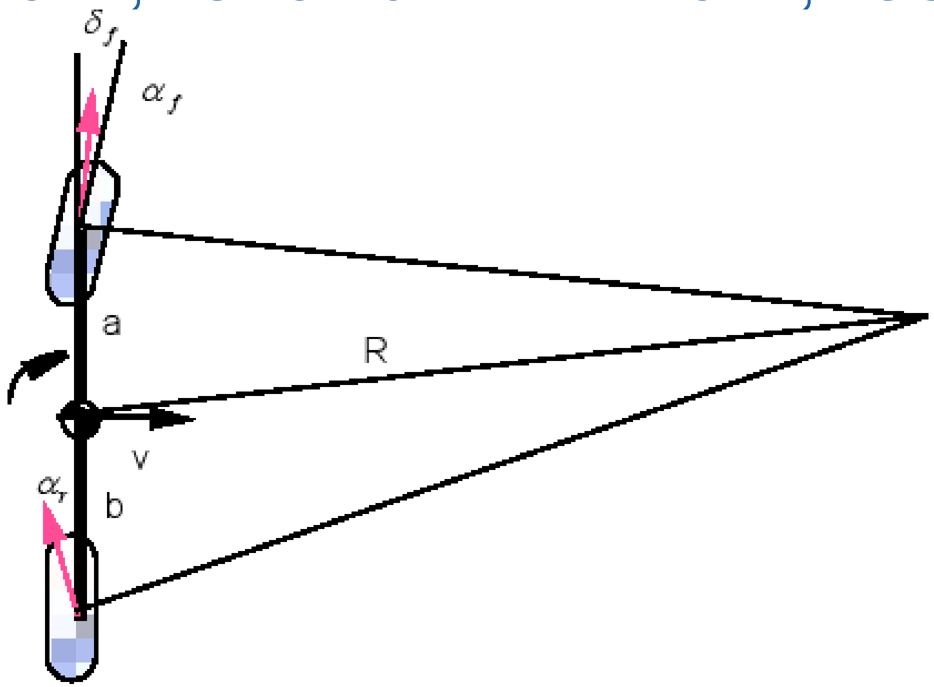
- Lateral and yaw (i.e., need to find equations for $\dot{v}$ and $\dot{r}$)
- Critical step is to link input (steering) with tire lateral f… lateral dynamics





The 2 DOF “Bicycle Model”

- Relationship between (front axle) steering angle and slip angles



$$\alpha_f = \delta_f - \tan^{-1}\left(\frac{v + ar}{u}\right) \approx \delta_f - \frac{v + ar}{u}$$

$$\alpha_r = \delta_r - \tan^{-1}\left(\frac{v - br}{u}\right) \approx -\frac{v - br}{u}$$

The 2 DOF "Bicycle Model"

- u: forward speed
- v: lateral speed
- r: yaw rate
- R: Turning radius (u/r)
- L = a + b: Wheel base

- Ackerman Angle:

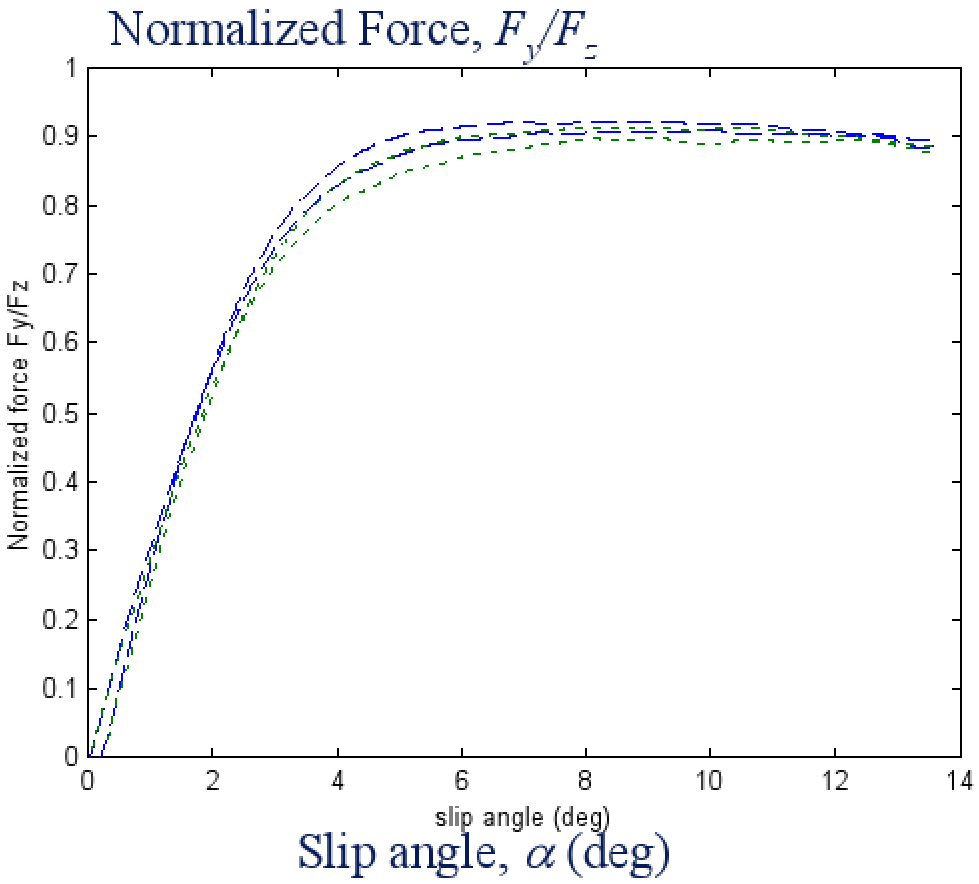
$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

Oversteer/Understeer/Neutral Steer

- <u>If the front slip angle is larger than the rear slip angle, this condition is termed "understeer"</u>
- This implies that the steering angle must be larger than the Ackerman angle to maintain a constant radius turn at non-zero speed
- Roughly, this means if the front tires are more deformable or soft (e.g., low tire pressure, cheap tires), the vehicl rsteer

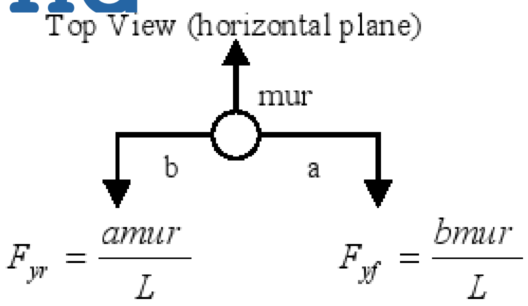
$\alpha_f > \alpha_r$	understeer
$\alpha_f < \alpha_r$	oversteer
$\alpha_f = \alpha_r$	neutral steer

Tire Lateral Forces



- Tire lateral forces depend on normal force, are a nonlinear function of tire slip, and depend on tire design (e.g., radial, bias-ply) and inflation pressure
- Small s $F_y = C_\alpha \alpha$ e:

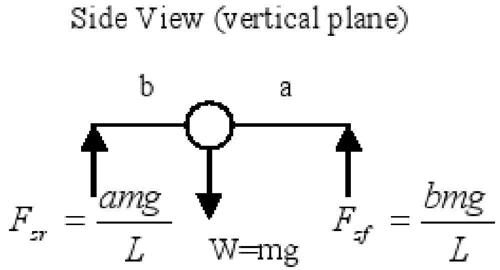
Force Balance Under Steady-State Turning



$$\dot{v} = \dot{r} = 0$$

$$mur = F_{yf} + F_{yr}$$
$$0 = F_{yf} \cdot a - F_{yr} \cdot b$$

$$\rightarrow \quad F_{yf} = \frac{b}{L} mur \qquad F_{yr} = \frac{a}{L} mur$$



$$\boxed{\frac{F_{yf}}{F_{zf}} = \frac{F_{yr}}{F_{zr}} = \frac{ur}{g}}$$

$$mg = F_{zf} + F_{zr}$$
$$0 = F_{zf} \cdot a - F_{zr} \cdot b$$

$$\rightarrow \quad F_{zf} = \frac{b}{L} mg \qquad F_{zr} = \frac{a}{L} mg$$

Understeer Coefficient

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

$$= \frac{L}{R} + mur \cdot \frac{b}{L}\frac{1}{C_{\alpha f}} - mur \cdot \frac{a}{L}\frac{1}{C_{\alpha r}}$$

$$= \frac{L}{R} + \underbrace{\left(\frac{mb}{LC_{\alpha f}} - \frac{ma}{LC_{\alpha r}}\right)}_{K_{us}}\frac{u^2}{R}$$

K_{us} rad/(m/s^2)

$$= \frac{L}{R} + \underbrace{\left(\frac{F_{zf}}{C_{\alpha f}} - \frac{F_{zr}}{C_{\alpha r}}\right)}_{K'_{us}}\frac{u^2}{Rg}$$

K'_{us} rad/g

Nondimensional lateral acceleration in g's

$$\delta_f = \frac{L}{R} + K_{us} a_y$$

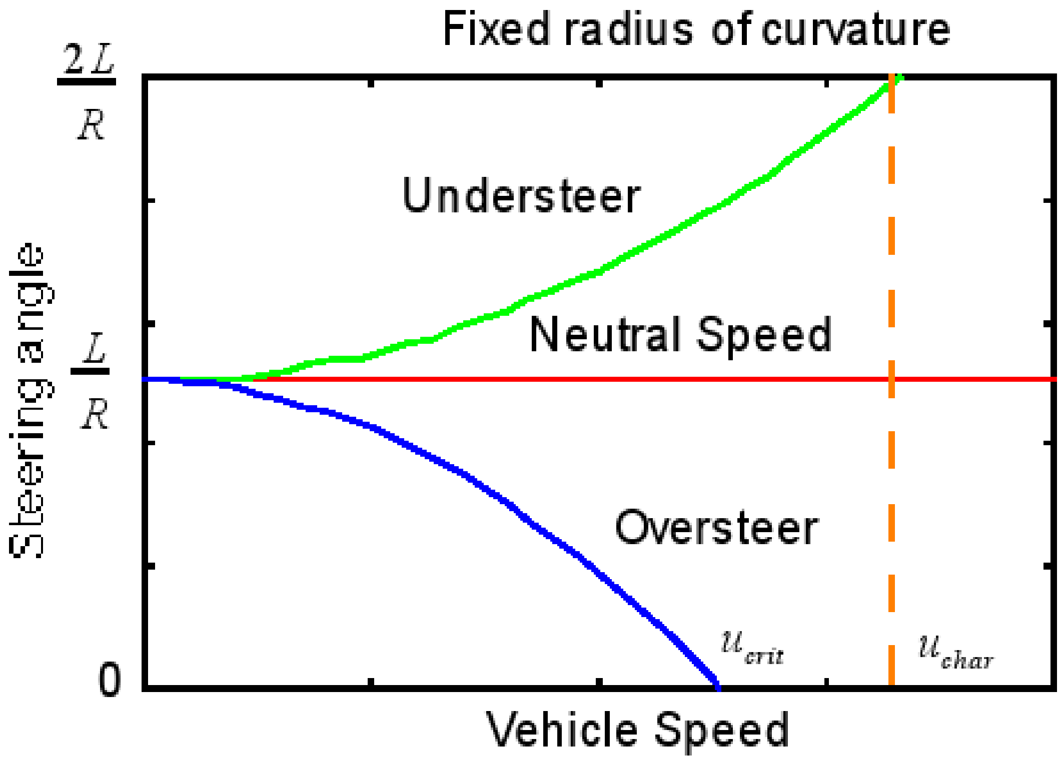
Characteristic Speed and Critical Speed

- The characteristic speed (of an understeer vehicle) is defined as the speed at which the steer angle is twice the Ackerman a

$$u_{char} = \sqrt{\frac{L}{K_{us}}}$$

- The critical speed (of an oversteer vehicle) is defined as the speed at which the steer ang es zero.

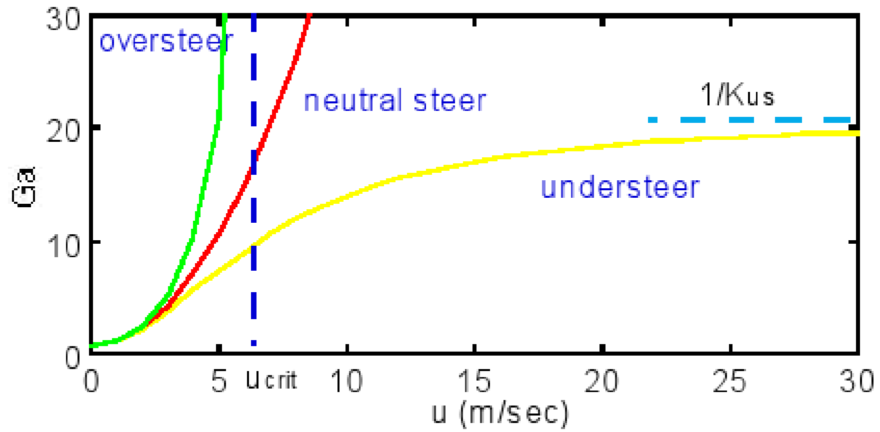
$$u_{crit} = \sqrt{\frac{-L}{K_{us}}}$$



Vehicle Handling Gains

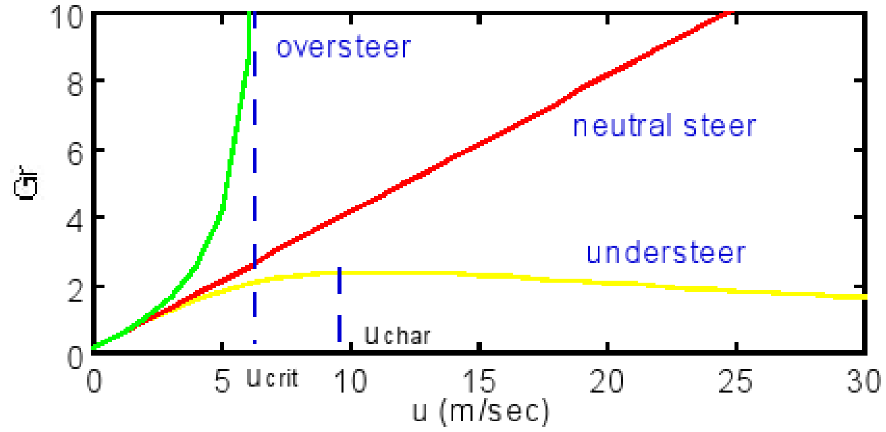
- Lateral Acceleration Gain

$$G_a = \frac{a_y}{\delta_f} = \frac{\frac{u^2}{R}}{\delta_f} = \frac{\frac{u^2}{L}}{\delta_f \cdot \frac{R}{L}} = \frac{\frac{u^2}{L}}{1 + K_{us}\frac{u^2}{L}} = \frac{u^2}{L + K_{us}u^2}$$



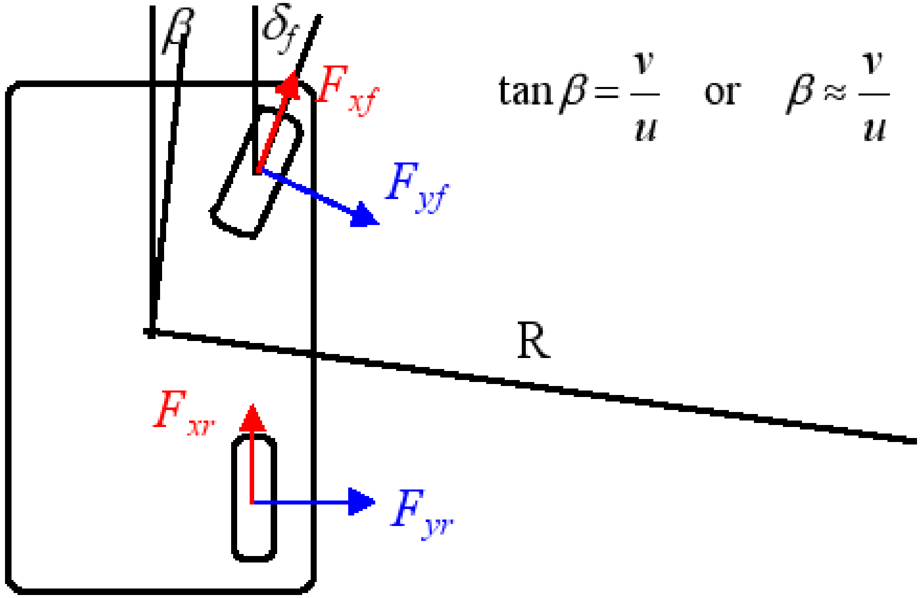
- Yaw rate gain

$$G_r = \frac{r}{\delta_f} = \frac{\frac{u}{R}}{\delta_f} = \frac{u}{L + K_{us}u^2}$$



Effect of Tractive Forces on K_{us}

- For the vehicle in a steady turn, apply Newton's 2nd law in the y direction at each



$$\tan\beta = \frac{v}{u} \quad \text{or} \quad \beta \approx \frac{v}{u}$$

$$F_{yf}\cos(\delta_f-\beta)+F_{xf}\sin(\delta_f-\beta)=F_{yf}\cos\left(\frac{a}{R}+\alpha_f\right)+F_{xf}\sin\left(\frac{a}{R}+\alpha_f\right)=m\frac{b}{L}\frac{u^2}{R}$$

$$F_{yr}\cos(\beta)-F_{xr}\sin(\beta)=F_{yr}\cos\left(\frac{b}{R}-\alpha_r\right)-F_{xr}\sin\left(\frac{b}{R}-\alpha_r\right)=m\frac{a}{L}\frac{u^2}{R}$$

Small angle assumption: $\sin(\theta)\approx\theta,\ \cos(\theta)\approx 1$

Linear tire: $F_{yf}\approx C_{\alpha f}\alpha_f,\ \ F_{yr}\approx C_{\alpha r}\alpha_r$

Effect of Tractive Forces on K_{us} (cont.)

$$F_{yf}\cos\left(\frac{a}{R}+\alpha_f\right)+F_{xf}\sin\left(\frac{a}{R}+\alpha_f\right)\approx F_{yf}+F_{xf}\left(\frac{a}{R}+\alpha_f\right)=m\frac{b}{L}\frac{u^2}{R}$$

$$\Downarrow$$

$$C_{\alpha f}\alpha_f+F_{xf}\left(\frac{a}{R}+\alpha_f\right)$$

$$F_{yr}\cos\left(\frac{b}{R}-\alpha_r\right)-F_{xr}\sin\left(\frac{b}{R}-\alpha_r\right)\approx F_{yr}-F_{xr}\left(\frac{b}{R}-\alpha_r\right)=m\frac{a}{L}\frac{u^2}{R}$$

$$\Downarrow$$

$$C_{\alpha r}\alpha_r-F_{xr}\left(\frac{b}{R}-\alpha_r\right)$$

$$\Rightarrow \boxed{\alpha_f=\frac{m\frac{b}{L}\frac{u_o^2}{R}-F_{xf}\frac{a}{R}}{C_{\alpha f}+F_{xf}}} \qquad \boxed{\alpha_r=\frac{m\frac{a}{L}\frac{u_o^2}{R}+F_{xr}\frac{b}{R}}{C_{\alpha r}+F_{xr}}}$$

Effect of Tractive Forces on K_{us}

$$\delta_f = \frac{L}{R} + (\alpha_f - \alpha_r) = \frac{L}{R} + \frac{m\frac{b}{L}\frac{u^2}{R} - F_{xf}\frac{a}{R}}{C_{\alpha f} + F_{xf}} - \frac{m\frac{a}{L}\frac{u^2}{R} + F_{xr}\frac{b}{R}}{C_{\alpha r} + F_{xr}}$$

$$= \underbrace{\frac{L}{R} - \frac{F_{xf}}{C_{\alpha f} + F_{xf}}\frac{a}{R} - \frac{F_{xr}}{C_{\alpha r} + F_{xr}}\frac{b}{R}}_{\text{The new Ackermann angle}} + \underbrace{\left[\frac{mb}{L(C_{\alpha r} + F_{xr})} - \frac{ma}{L(C_{\alpha f} + F_{xf})}\right]}_{\text{The new Kus}}\frac{u^2}{R}$$

- Front wheel drive: More understeer? More oversteer?

2DOF Vehicle Transient Model

$$\sum F_x = m\dot{u}'$$

$$\sum F_y = m(\dot{v} + ru_o)$$

$$\sum F_z = m(\dot{w} - qu_o)$$

$$\sum M_x = I_{xx}\dot{p} - I_{xz}\dot{r}$$

$$\sum M_y = I_{yy}\dot{q}$$

$$\sum M_z = -I_{xz}\dot{p} + I_{zz}\dot{r}$$

Figure out the tire forces
and corresponding yaw moment

Keep the model linear

- small angle assumption
- linear tire approximation

2DOF Vehicle Transient Model

- Slip Angles:

$$\alpha_f = \delta_f - \tan^{-1}\left(\frac{v+ar}{u}\right) \approx \delta_f - \frac{v+ar}{u}$$

$$\alpha_r = \delta_r - \tan^{-1}\left(\frac{v-br}{u}\right) \approx -\frac{v-br}{u}$$

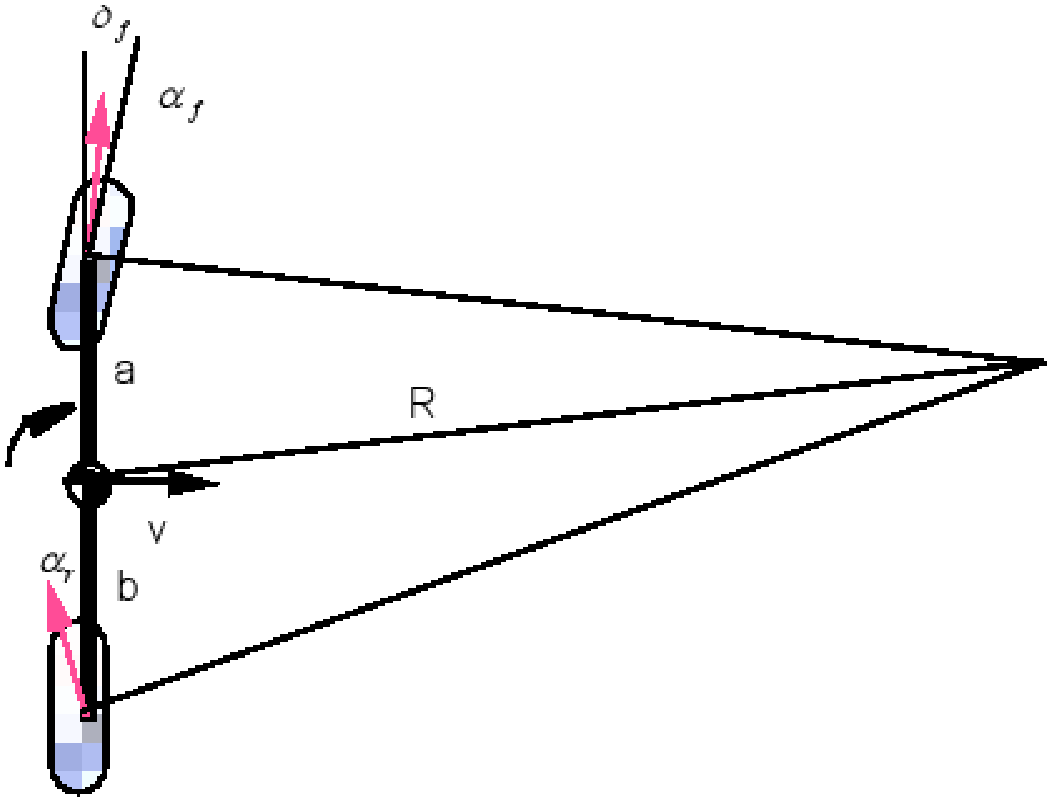
- Lateral Tire F $F_y = C_\alpha \alpha$

$\Sigma F_y = m(\dot{v} + u_0 r):$

$$C_{\alpha f}\left[\delta_f - \frac{v+ar}{u_0}\right] + C_{\alpha r}\left[-\frac{v-br}{u_0}\right] = m(\dot{v} + u_0 r)$$

$\Sigma M_z = I_z \dot{r}:$

$$C_{\alpha f}\left[\delta_f - \frac{v+ar}{u_0}\right]a - C_{\alpha r}\left[-\frac{v-br}{u_0}\right]b = I_z \dot{r}$$



State Space Form of the 2DOF Model

$$\frac{d}{dt}\begin{bmatrix} v \\ r \end{bmatrix} = \begin{bmatrix} \dfrac{-(C_{\alpha f}+C_{\alpha r})}{mu_o} & \dfrac{bC_{\alpha r}-aC_{\alpha f}}{mu_o}-u_o \\ \dfrac{bC_{\alpha r}-aC_{\alpha f}}{I_z u_o} & \dfrac{-(a^2 C_{\alpha f}+b^2 C_{\alpha r})}{I_z u_o} \end{bmatrix}\begin{bmatrix} v \\ r \end{bmatrix} + \begin{bmatrix} \dfrac{C_{\alpha f}}{m} \\ \dfrac{aC_{\alpha f}}{I_z} \end{bmatrix}\delta_f$$

Or equivalently, since $\tan\beta = \dfrac{v}{u_0}$ or $v \approx \beta u_0$

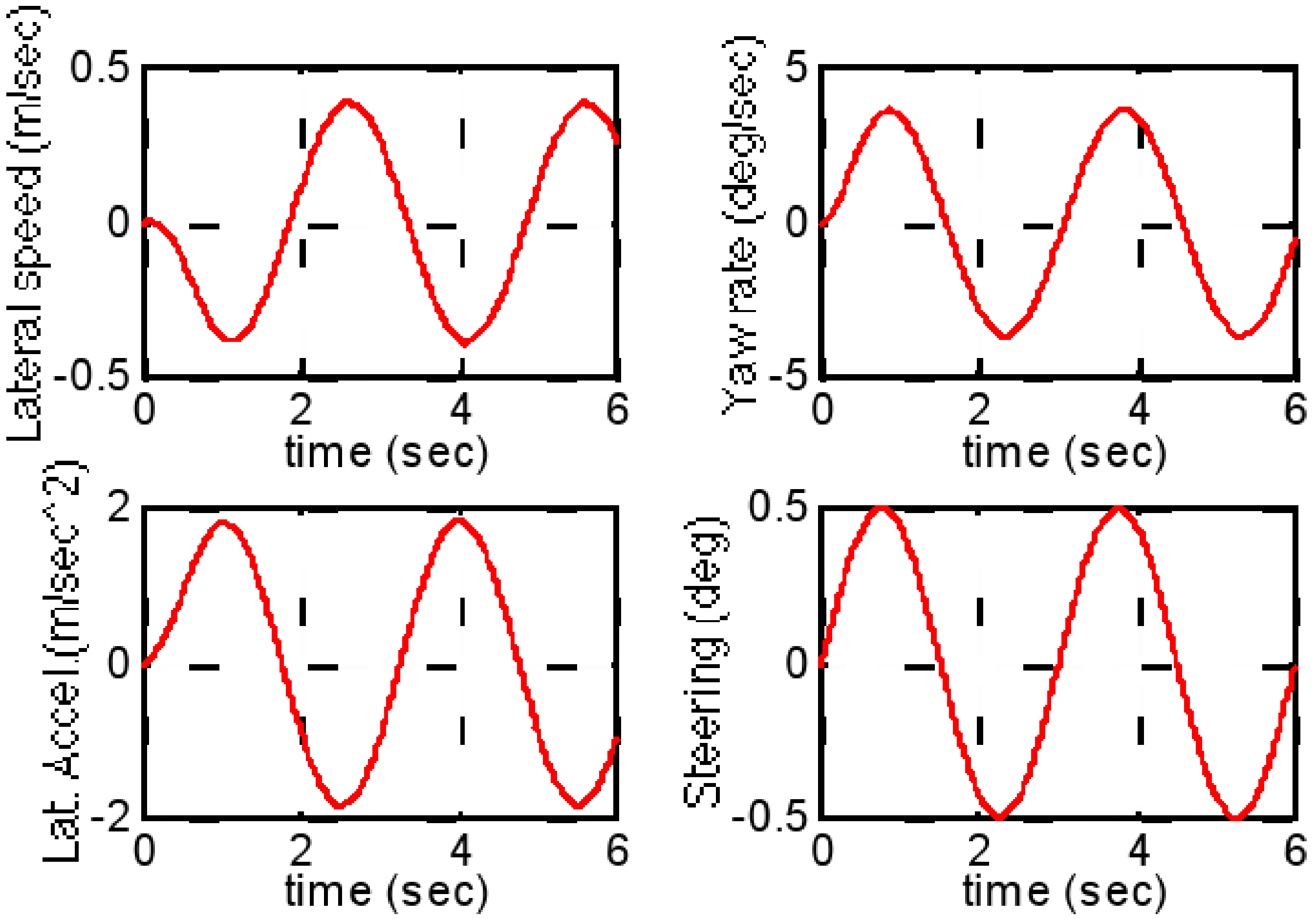
$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -\left(\dfrac{C_{\alpha f}+C_{\alpha r}}{mu_o}\right) & -\left(\dfrac{aC_{\alpha f}-bC_{\alpha r}}{mu_o^2}\right)-1 \\ -\left(\dfrac{aC_{\alpha f}-bC_{\alpha r}}{I_z}\right) & -\left(\dfrac{C_{\alpha f}a^2+C_{\alpha r}b^2}{I_z u_o}\right) \end{bmatrix}\begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \dfrac{C_{\alpha f}}{mu_o} \\ \dfrac{aC_{\alpha f}}{I_z} \end{bmatrix}\delta_f$$

State Space Form of the 2DOF Model

```
% Ex3_6a.m
% 2DOF model vehicle parameters
a   =  1.14;                    % distance c.g. to front axle (m)
L   =  2.54;                    % wheel base (m)
m   =  1500;                    % mass (kg)
Iz  =  2420.0; % yaw moment of inertia (kg-m^2)
Caf =  44000*2;                 % cornering stiffness--front axle (N/rad)
Car =  47000*2;                 % cornering stiffness-- rear axle (N/rad)
b=L-a;     g=9.81;
Kus = m*b/(L*Caf) - m*a/(L*Car);  % (rad/(m/sec^2))
u=20.0;                         % forward speed in m/sec
A=[-(Caf+Car)/(m*u),          (b*Car-a*Caf)/(m*u)-u
   (b*Car-a*Caf)/(Iz*u), -(a^2*Caf+b^2*Car)/(Iz*u)];
B=[Caf/m;      a*Caf/Iz];
C_lat = [1 0]; D_lat = 0;                   % Lateral speed
C_yaw = [0 1]; D_yaw = 0; % Yaw rate
C_acc=A(1,:) + u*[0,1];
D_acc = B(1);                               % Lateral acceleration
C = [C_lat; C_yaw; C_acc];
D = [D_lat; D_yaw; D_acc];

t=[0:0.01:6];
U=0.5*pi/180*sin(1/3*2*pi*t);
```

```
Y=lsim(A,B,C,D,U,t);


Subplot, subplot(221)
plot(t,Y(:,1),'r'); grid
xlabel('time (sec)')
ylabel('Lateral speed (m/sec)')

subplot(222)
plot(t,Y(:,2)*180/pi,'r'); grid
xlabel('time (sec)')
ylabel('Yaw rate (deg/sec)')

subplot(223)
plot(t,Y(:,3),'r'); grid
xlabel('time (sec)')
ylabel('Lat. Accel.(m/sec^2)')

subplot(224)
plot(t,U*180/pi,'r'); grid
xlabel('time (sec)')
ylabel('Steering (deg)')
```

State Space Form of the 2DOF Model




```
Subplot, subplot(221)
plot(t,Y(:,1),'r'); grid
xlabel('time (sec)')
ylabel('Lateral speed (m/sec)')

subplot(222)
plot(t,Y(:,2)*180/pi,'r'); grid
xlabel('time (sec)')
ylabel('Yaw rate (deg/sec)')

subplot(223)
plot(t,Y(:,3),'r'); grid
xlabel('time (sec)')
ylabel('Lat. Accel.(m/sec^2)')

subplot(224)
plot(t,U*180/pi,'r'); grid
xlabel('time (sec)')
ylabel('Steering (deg)')
```

Three Degree of Freedom Model

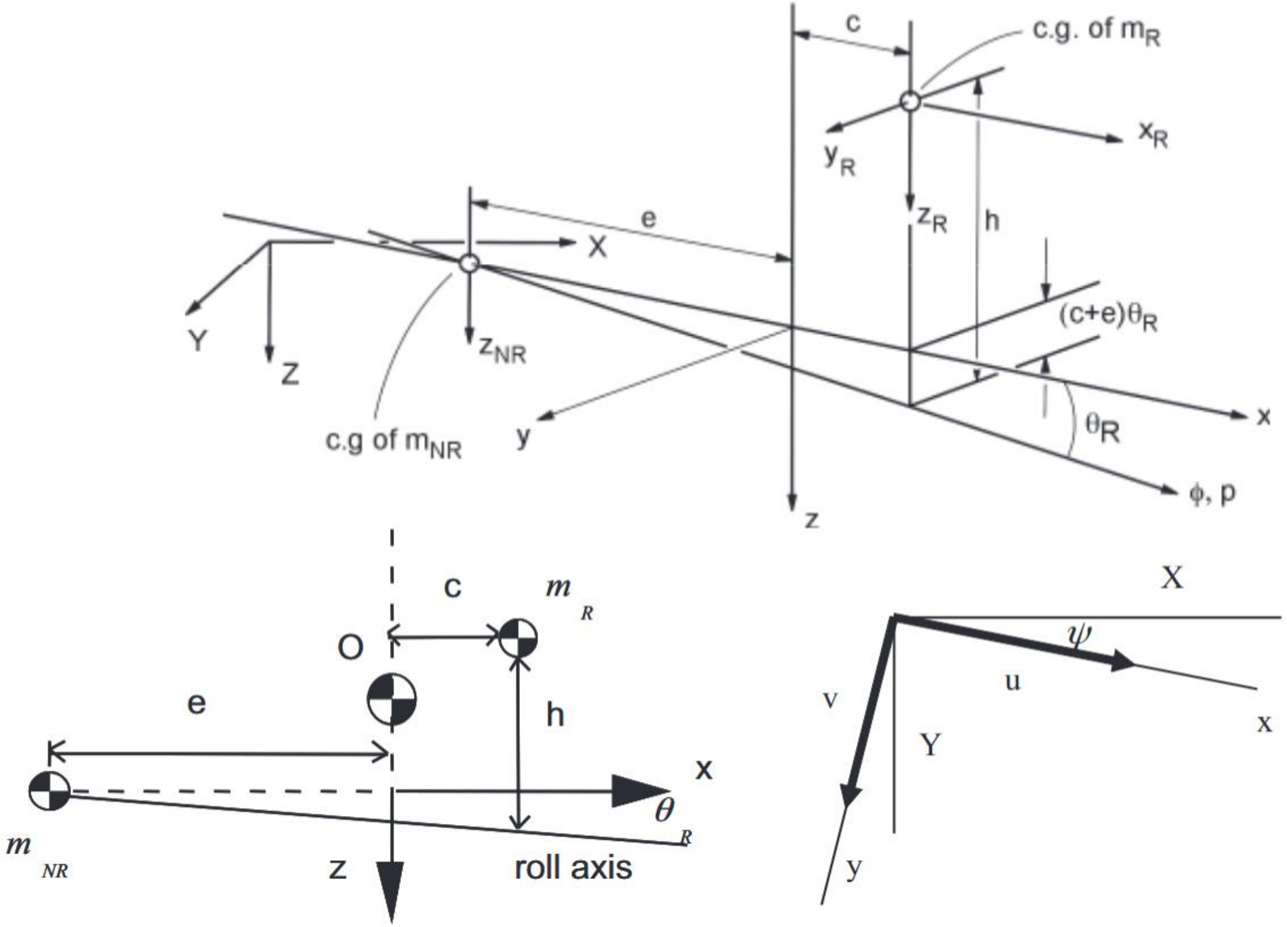


Figure 4.12. Schematic of the three-DOF model.

Three Degree of Freedom Model

- The 3DOF model includes the roll mode (p and φ)
- 3DOF requires 4 states (β or v, r, p and φ)
- Steering compliance and camber effects included
- The idea is to make the handling model more realistic
- The coefficients $Y_\beta, Y_r, Y_\varphi, Y_\delta, N_\beta, N_r, N_\varphi, N_\delta, L_r, L_\varphi$ are termed stability derivatives

$$\begin{bmatrix} mu_o & 0 & m_R h & 0 \\ 0 & I_z & I_{xz} & 0 \\ m_R h u_o & I_{xz} & I_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{p} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} -Y_\beta & mu_o - Y_r & 0 & -Y_\phi \\ -N_\beta & -N_r & 0 & -N_\phi \\ 0 & m_R h u_o & -L_p & -L_\phi \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \phi \end{bmatrix} = \begin{bmatrix} Y_\delta \\ N_\delta \\ 0 \\ 0 \end{bmatrix} \delta_f$$

MATLAB/SIMULINK Revisited (Yaw_roll.m)

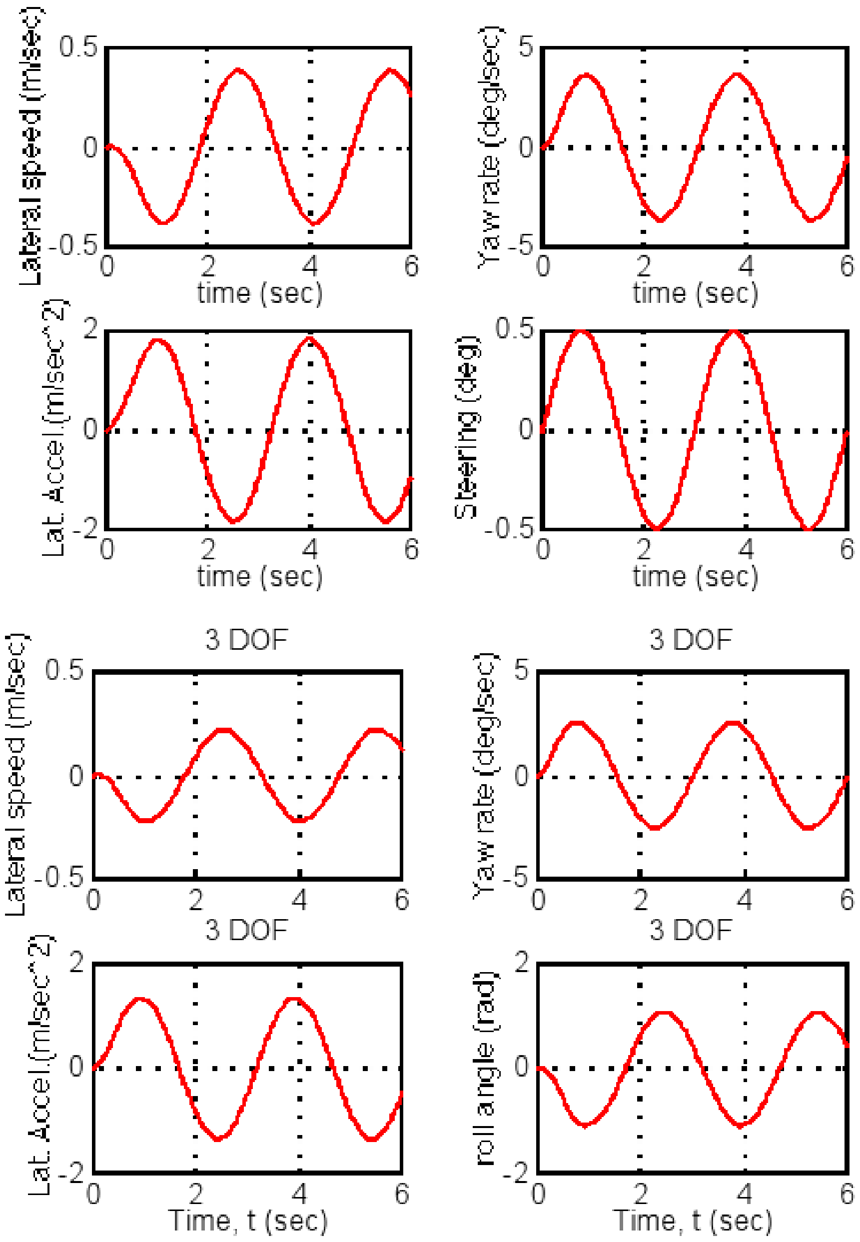
```
% Example MATLAB/SIMULINK simulation programs
% Yaw/Roll Model example
% Work with the following two files:
%  Yaw_roll_model_sfun.m   (the S- function)
%  example_3dof.mdl (the MATLAB function)
u=20;                          % forward speed (m/sec)
t=[0:0.01:6];
steer=30*sin(1/3*2*pi*t);  % in deg!!!, not in rad

OPTIONS = simset('Solver','ode45','MaxStep',1e-2, ...
                 'RelTol',1e-3,'AbsTol',1e-3);
sim('example_3dof',[0:0.01:6],OPTIONS);

Subplot, subplot(221)
plot(t,v,'r'); grid
xlabel('time (sec)')
ylabel('Lateral speed (m/sec)')
```

Ex. 3.6 – 2 DOF vs 3 DOF Simulation

3DOF model also includes roll degree of freedom (p and φ)

3DOF requires at least 4 states (β or v, r, p and φ) or 6 including y and ψ

Steering compliance and camber effects included

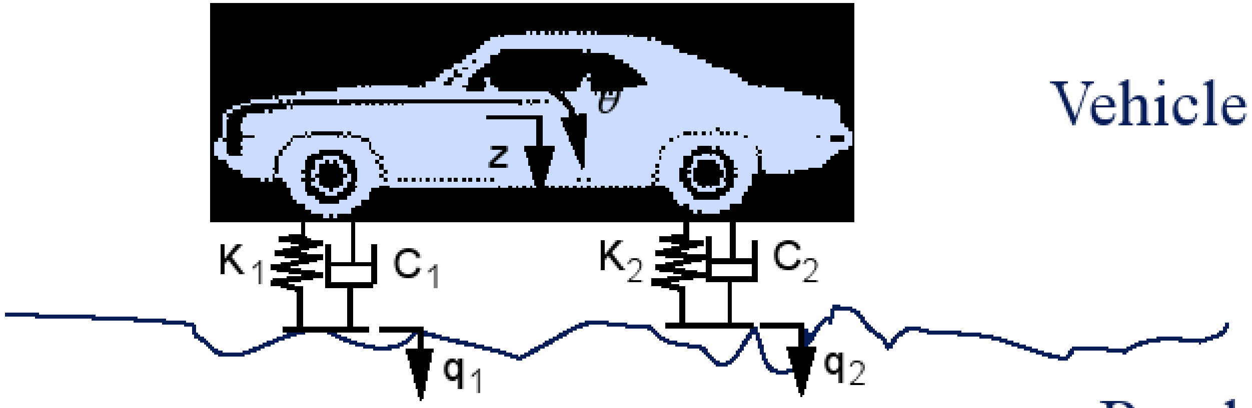


CHAPTER – IV

VEHICLE VERTICAL DYNAMICS

Vehicle Vertical Dynamics

Human ride perception



Vehicle dynamics

Road excitation

Road Excitation Generation

Alternative 1: Measure road file and use it directly.

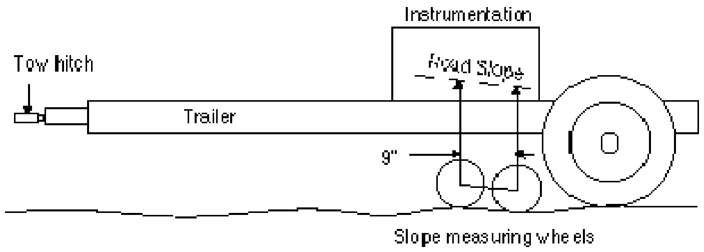
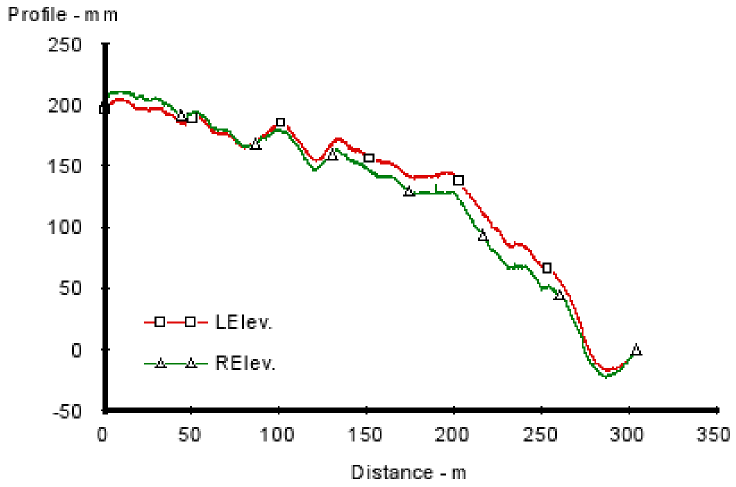


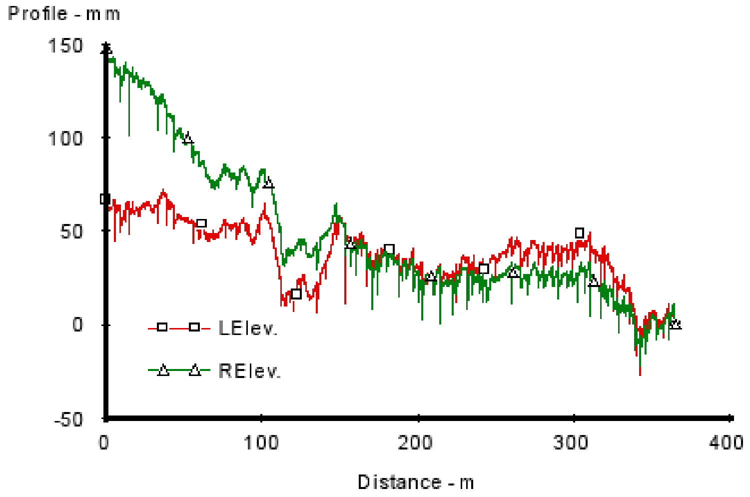
Fig. 6 Illustration of the CHLOE

http://www.umtri.umich.edu/erd/roughness/

Example road profile files:



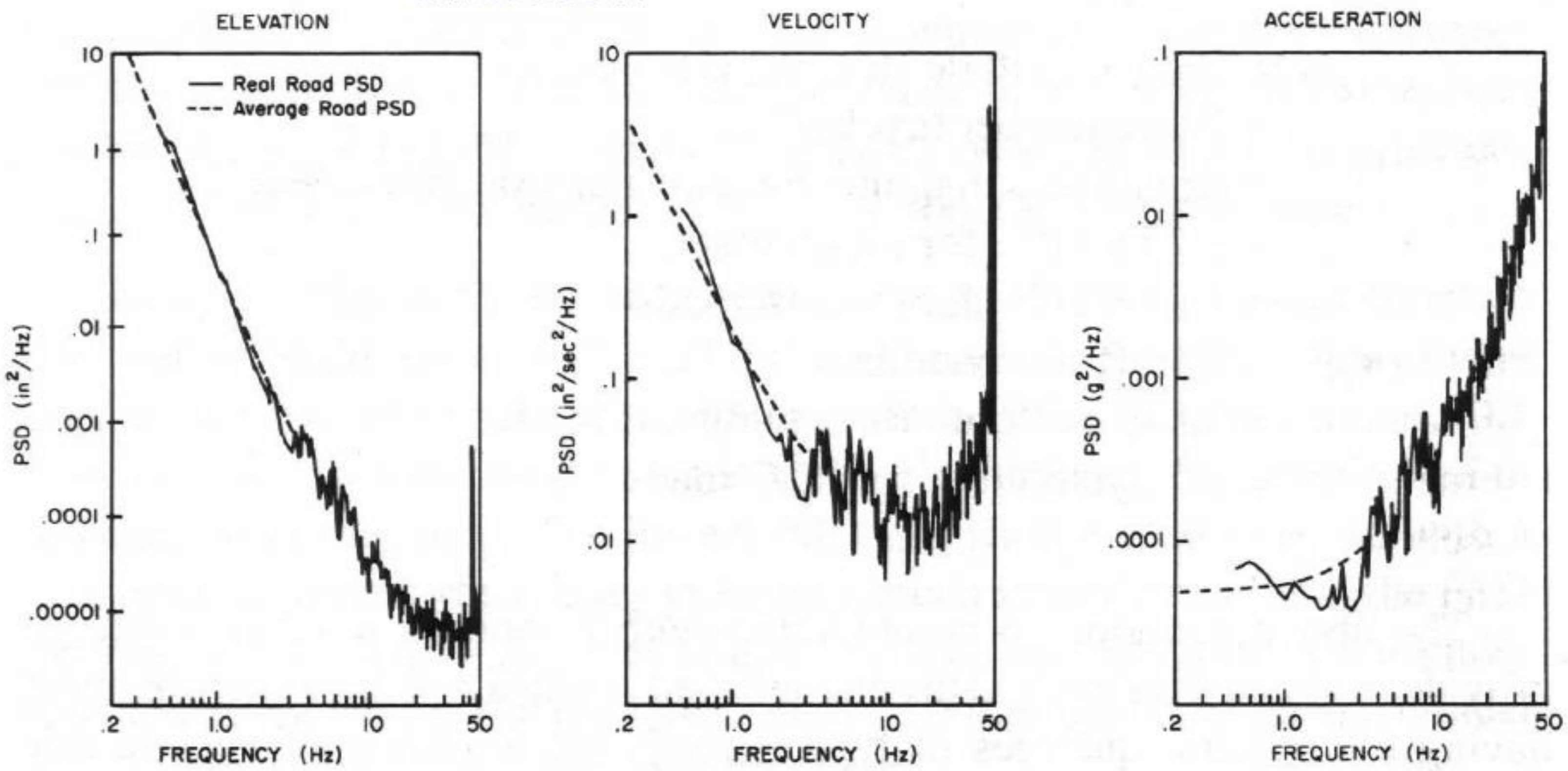
M14 East, just east of US 23. New asphalt. Outside lane.



M14 East at Mile Post 11. Badly Faulted PCC. Outside lane.

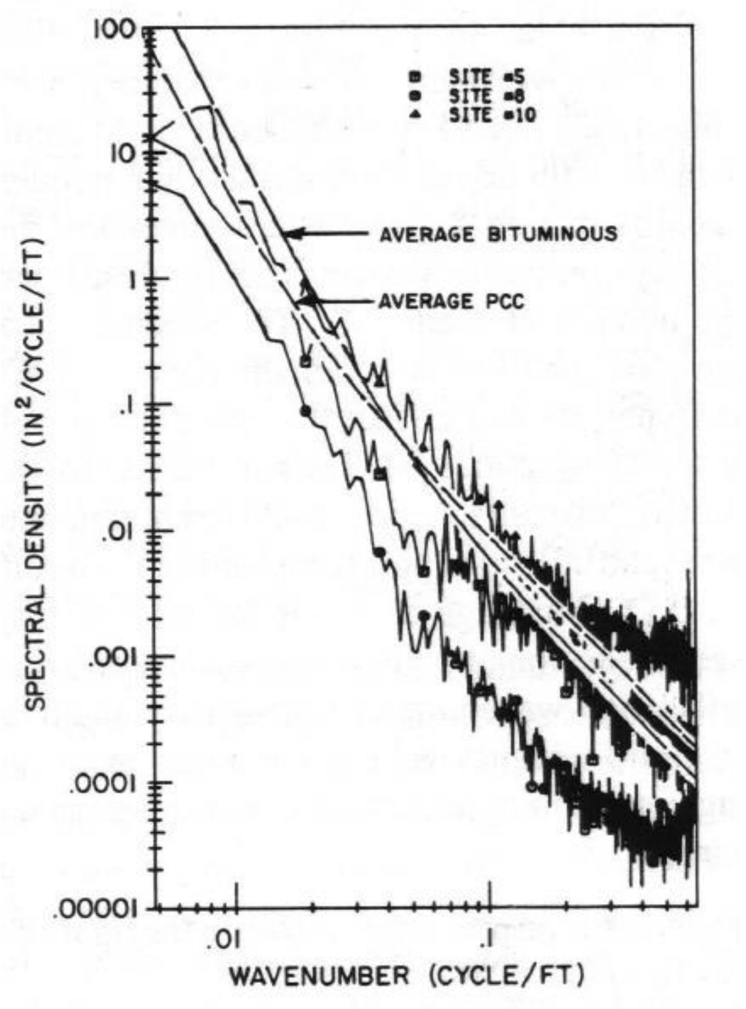
Road Excitation Generation (cont.)

<u>Alternative 2:</u> Use a statistical model to represent road excitation.



Model Road Input as a Random Variable

- The road unevenness is stochastic in nature:
  - Described as a random process
  - Characterized by its statistical features

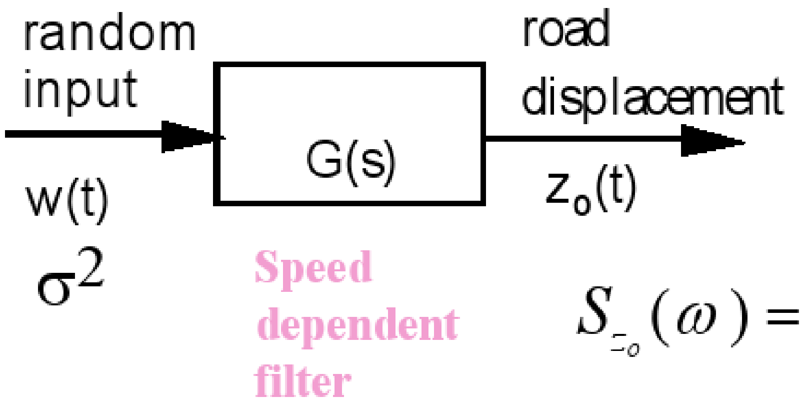


Two Common Road Models

- Model the road velocity profile $\dot{z}_o(t) = w(t)$ as white noise
  - Use MATLAB randn() command for normal

```
sigma = 0.3;                  % standard deviation in (in/sec)
t = 0:0.001:1;                % time horizon
zo_dot = sigma * randn(size(t));          % (in/sec)
```

- Model the road displacement $z_o(t)$ as colored noise



$$S_{z_o}(\omega) = |G(\omega)|^2 \frac{\sigma^2}{2\pi}$$

Colored-Noise Road Displacement Model

- Assume a simple first order f

$$G(s) = \frac{\omega_o}{s + \omega_o}$$

$$|G(j\omega)|^2 = \frac{\omega_o^2}{\omega^2 + \omega_o^2} = \frac{(\omega_o / \omega)^2}{(\omega_o / \omega)^2 + 1}$$

- Define spatial frequency for constant u_o

$$\omega' = \frac{\omega}{u_o} \quad \begin{matrix}(\text{rad/m})\\(\text{rad/ft})\end{matrix} \qquad f' = \frac{f}{u_o} \quad \begin{matrix}(\text{cycle/m})\\(\text{cycle/ft})\end{matrix}$$

$$S_{z_o}(f') = 2\pi(\sigma f'_o)^2 \frac{\frac{1}{2\pi f'^2}}{1 + \left(\frac{f'_o}{f'}\right)^2}$$

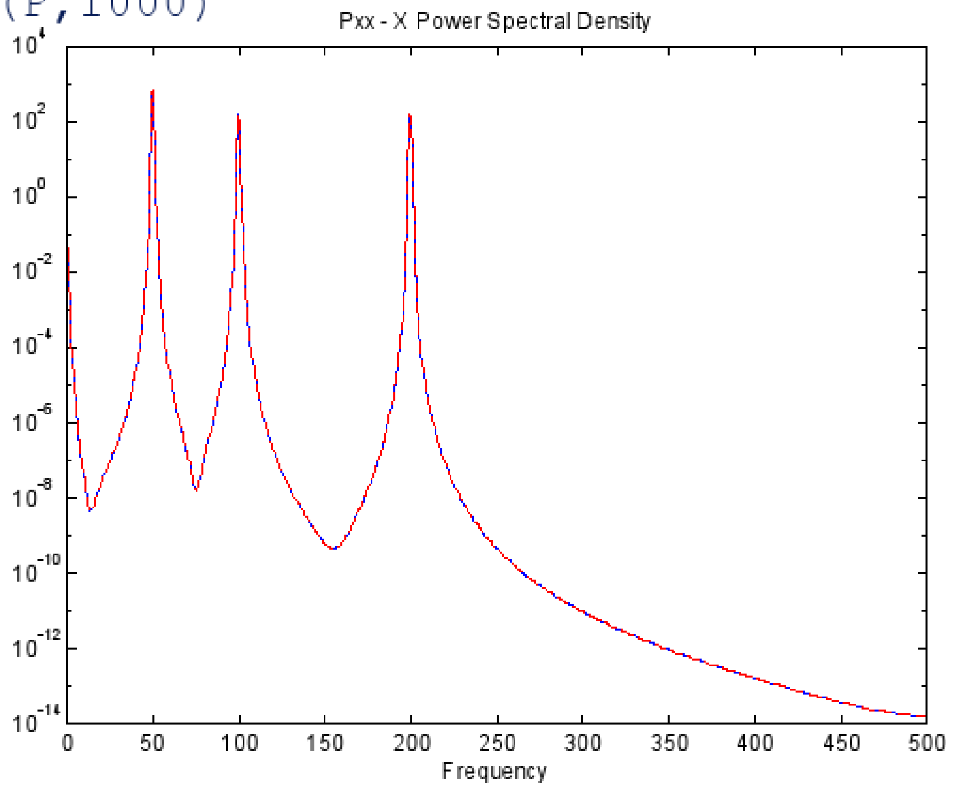
S_o

S_0 : roughness magnitude parameter
(1.25E-5 ft for rough roads
1.25E-6 ft for smooth roads)

f'_0 : cutoff spatial frequency
(0.05 cycles/ft for bituminous roads
0.02 cycles/ft for Portland cement concrete roads)

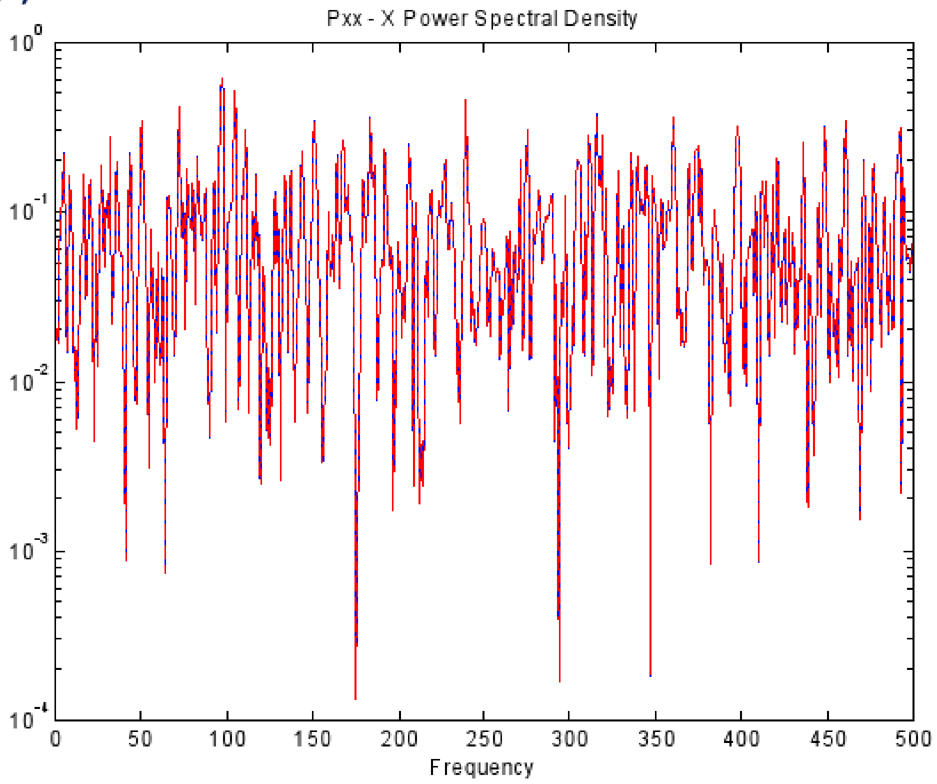
Spectrum() & specplot()

```
t = 0:0.001:1.023;
y = 2*sin(100*pi*t) + sin(200*pi*t)+ sin(400*pi*t);
P=spectrum(y,1024);
specplot(P,1000)
```



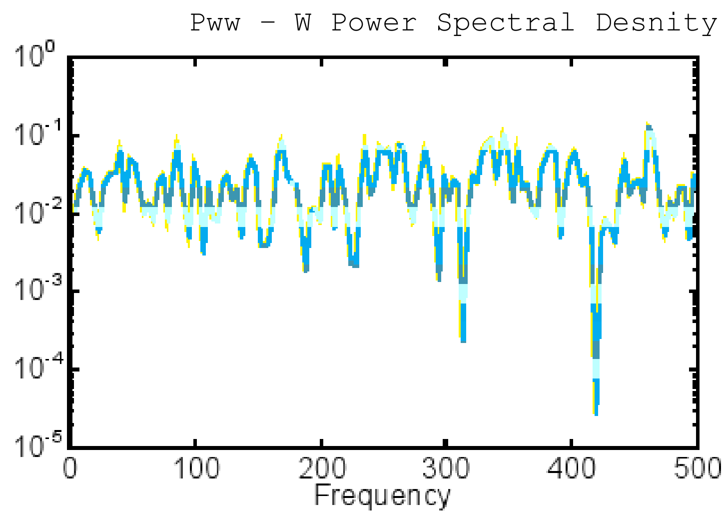
Spectrum() & specplot() – Random Data

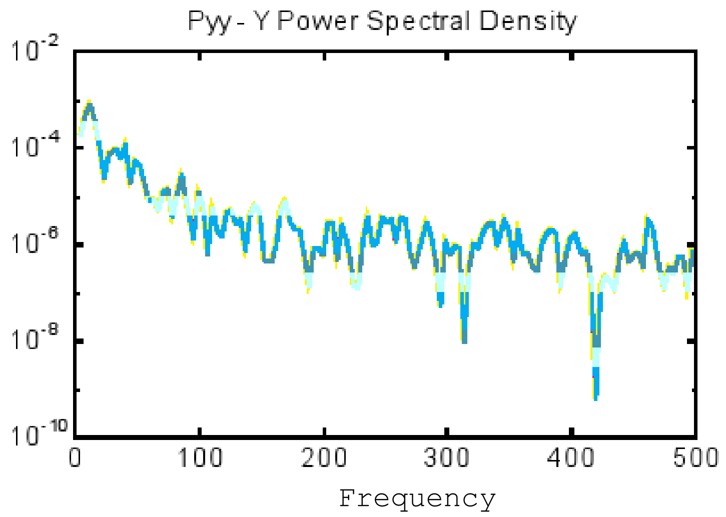
```
t = 0:0.001:1.023;
y2=rand(size(t));
P2=spectrum(y2,1024);
specplot(P2,1000)
```



Ex 3.7 Road Profile Generation

```
% Ex3_7.m
w=zeros(256,1);
Vel=80;      % ft/sec
w0 = 2.*pi*Vel*0.02;
% PCC road surface
S0 = 1.25e-5;    % rough road
sigma = (2*pi*Vel*sqrt(S0))/w0;
w = sigma*randn(size(w));
% Define the filter
T = 0.001; Fs = 1/T;
B = [1-exp(-w0*T)];
A = [1 -exp(-w0*T)];
y=filter(B,A,w);
% Determine the power spectrum
% for y and w
P = spectrum(w,y,256);
specplot(P,Fs)
```





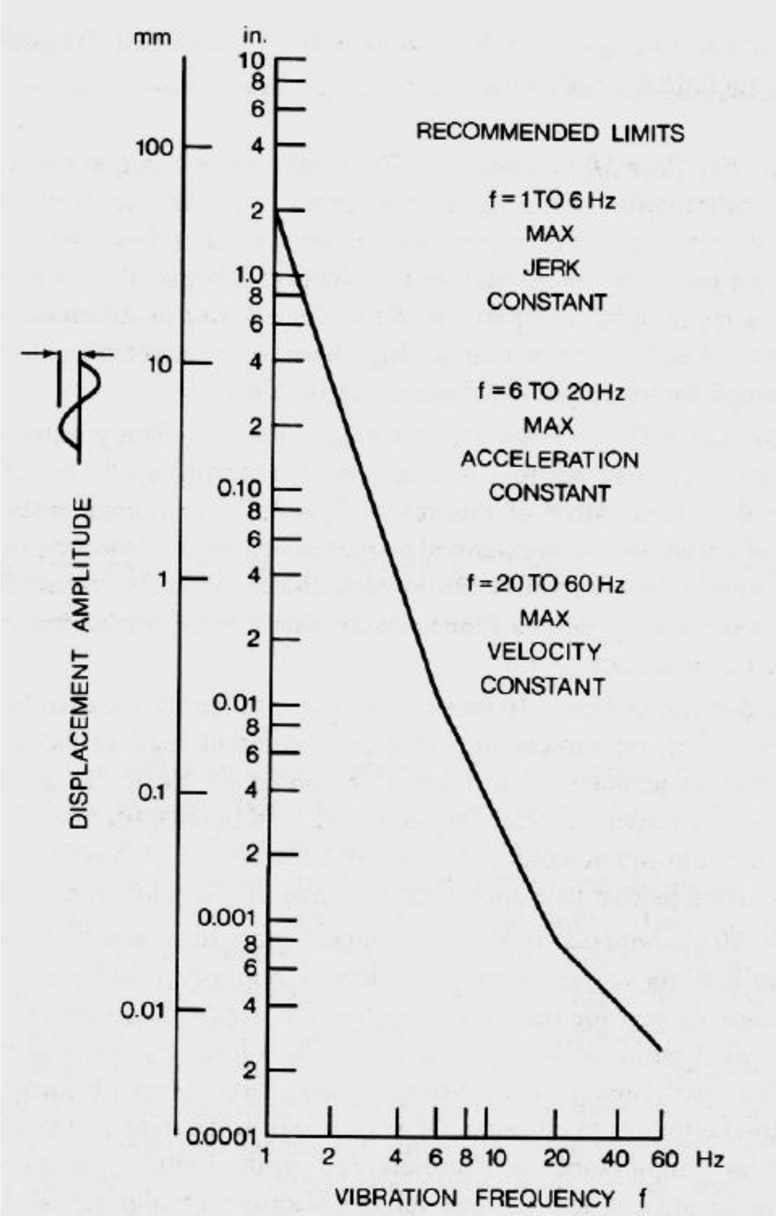
Ride Quality

Subjective in nature and this is difficult to have a universal standard.

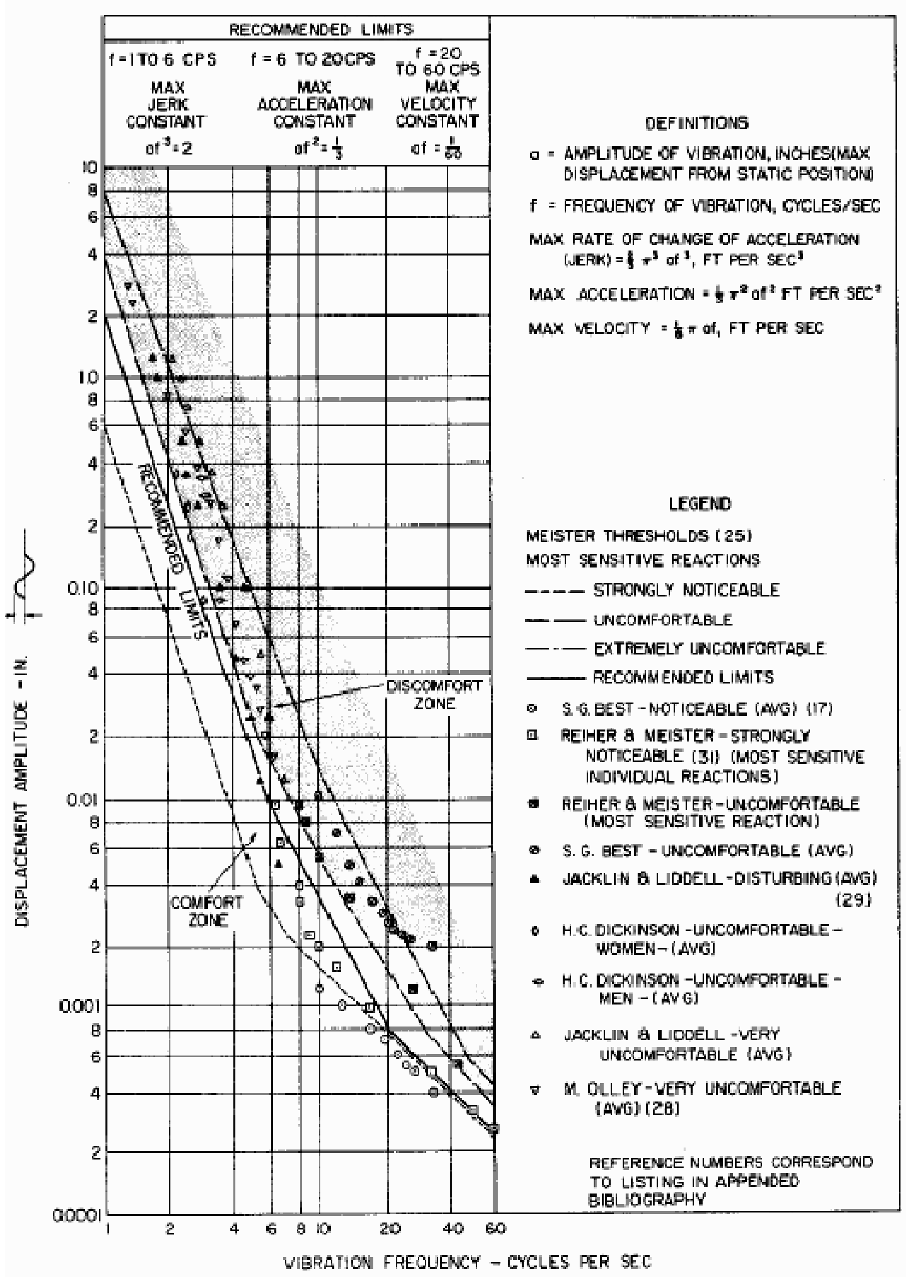
To quantify ride quality -> correlate physical signals (e.g., suspension acceleration) with subjective passenger perception.

Janeway's comfort criterion (SAE Ride and Vibration Data Manual J6a) is a popular formula

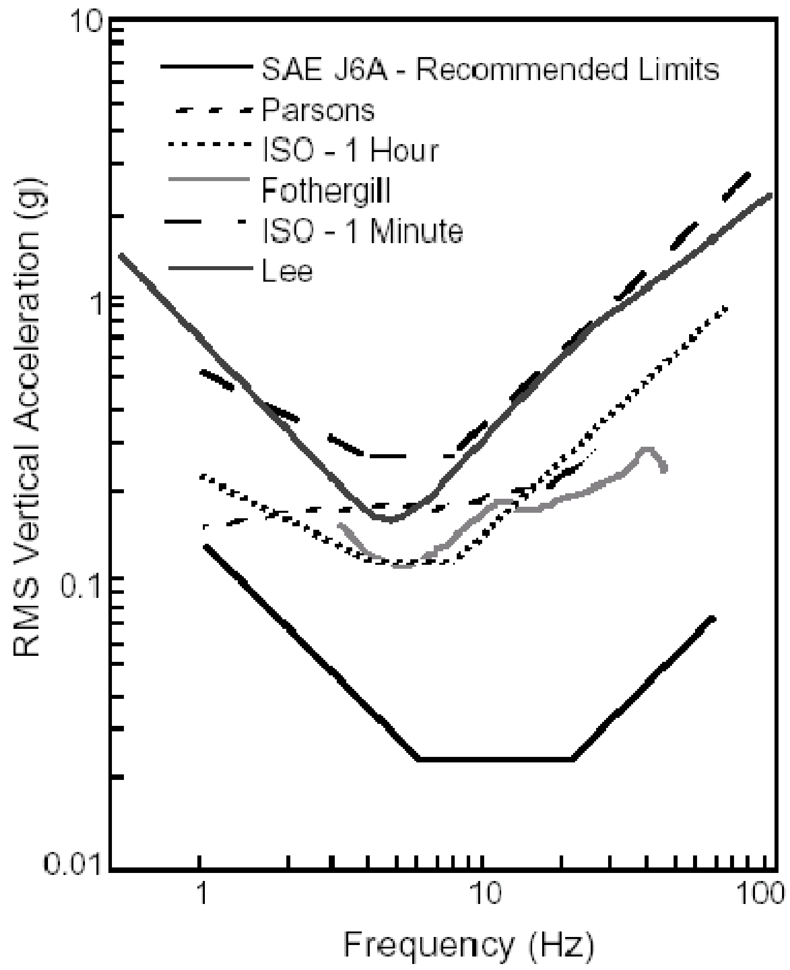
Maximum Allowable road displacement magnitude for single frequency excitations with frequency between 1 and 60 Hz

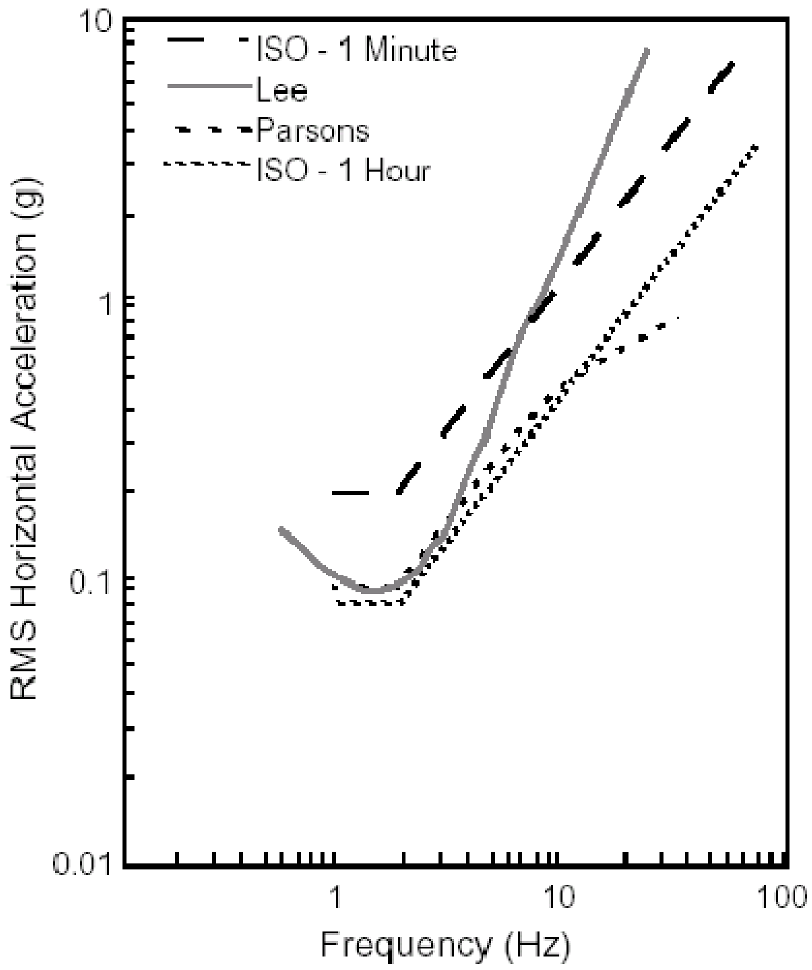


Ride Quality (cont.)



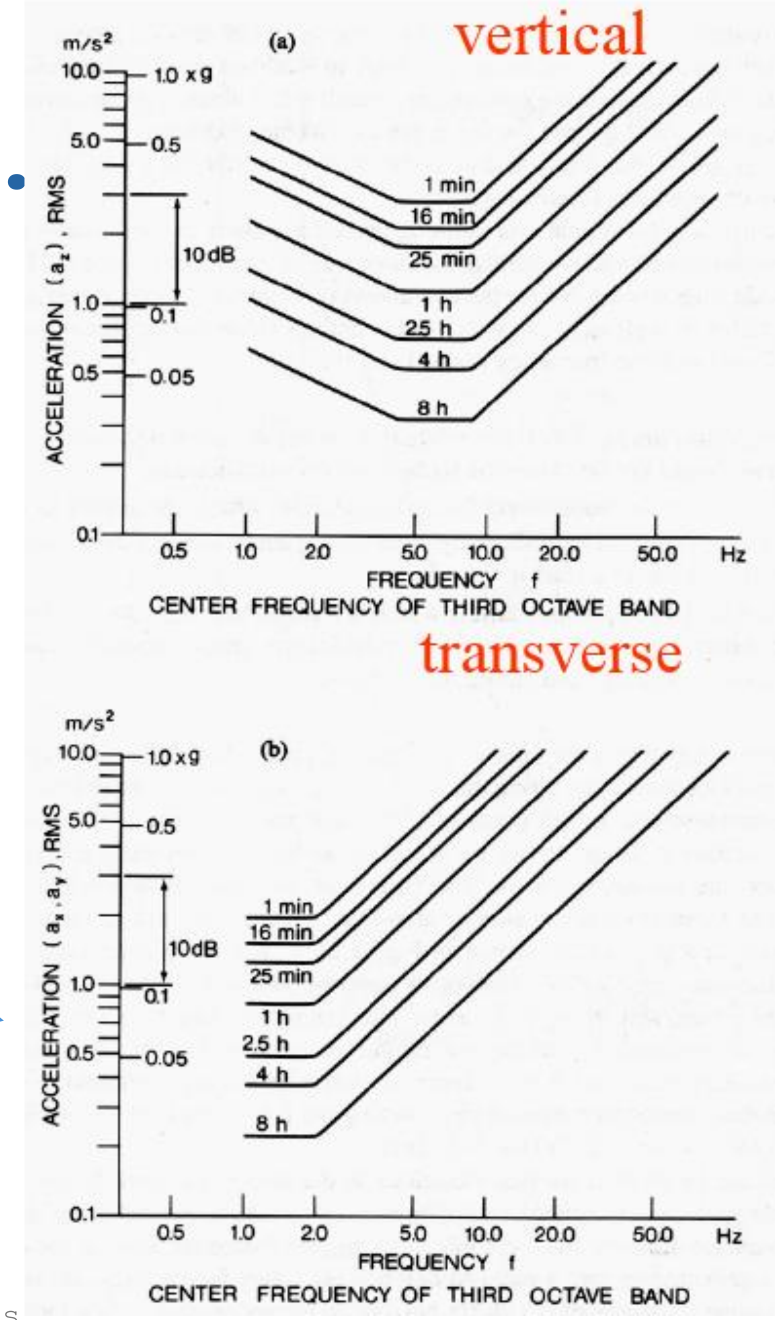
Other Standards





Ride Quality (cont.)

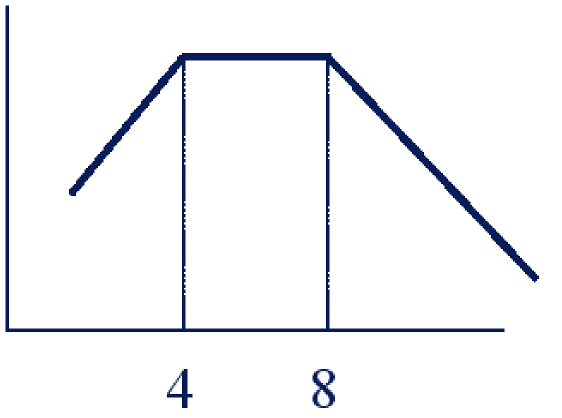
- ISO 2631 Criterion
  - Presents three limits for vibrations (both vertical and transverse):
    - Safety Limit, Fatigue Limit, and Comfort Limit
- Fatigue (decreased proficiency) limit
  - The safety limit is 6 dB higher, and comfort limit is 10 dB lower than the fatigue limit

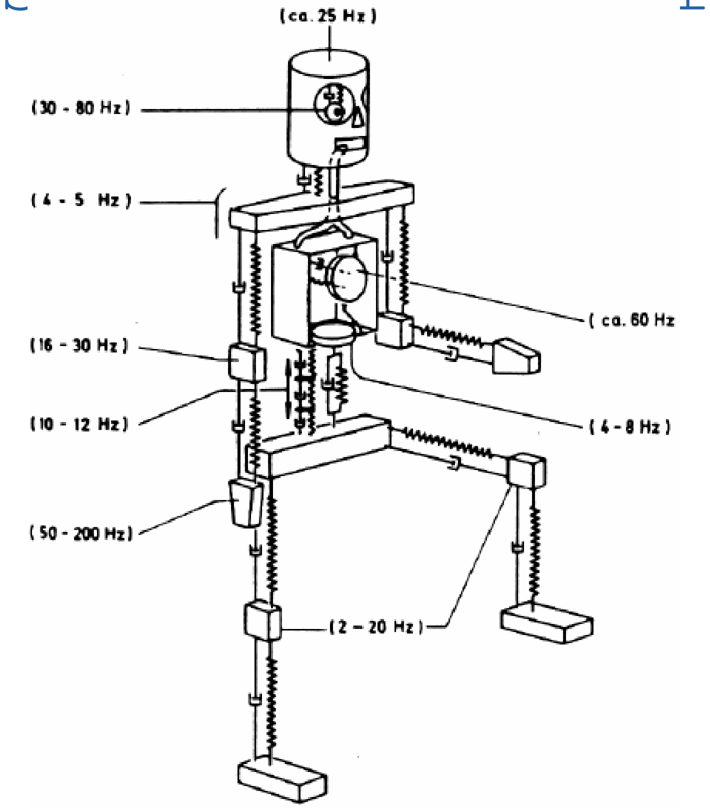


Ride Quality – ISO 2631 Weighting Criterion

- A frequency dependent weighting criterion for acceleration signal to represent the discomfort level.

$$H_{iso}(f) = \begin{cases} 0.5\sqrt{f} & 1 \le f < 4 \\ 1.0 & 4 \le f < 8 \\ 8/f & 8 \le f \le 80 \end{cases}$$



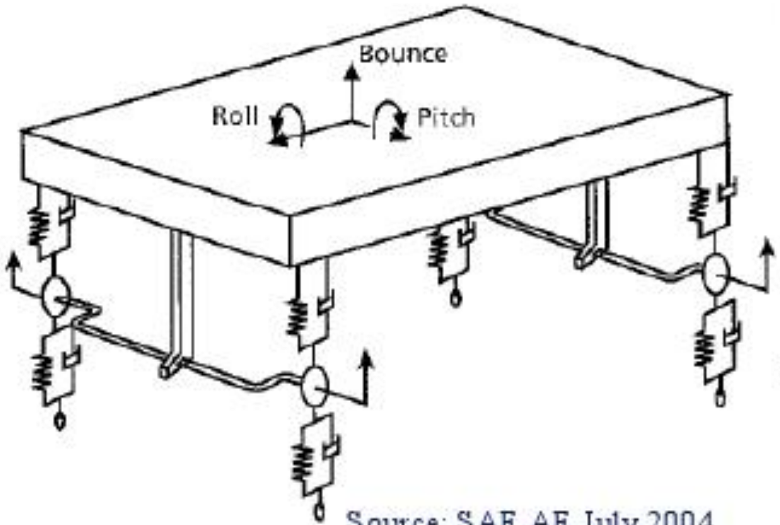


Ride Suspension Models

- Various models have been developed to study passenger car ride quality.
  - For a full-car model, we usually have 7DOF.
- Simplified models are more commonly used.
  - Half-car models only include vertical and pitch (bounce and pitch) motions (4DOF)
  - Quarter-car models only represent the motions of a corner of the car and could be 1DOF (sprung mass only) or 2DOF (spring and unsprung mass vertical motions).

Ride Suspension Models

Full-car



Source: SAE AE July 2004

Half-car

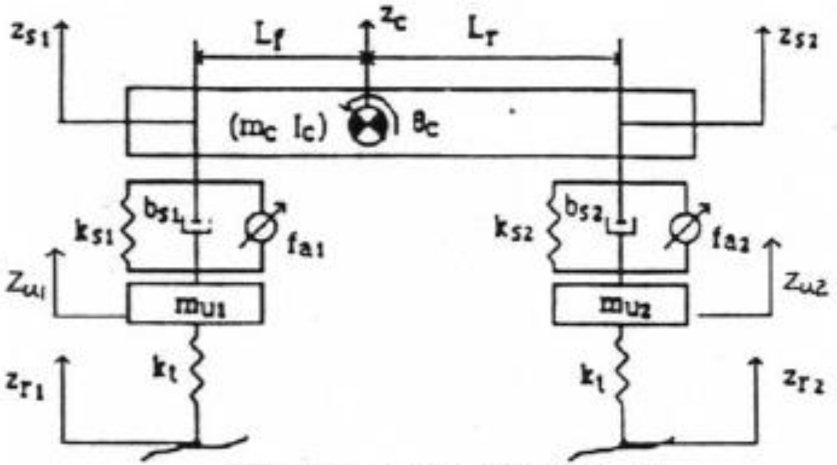
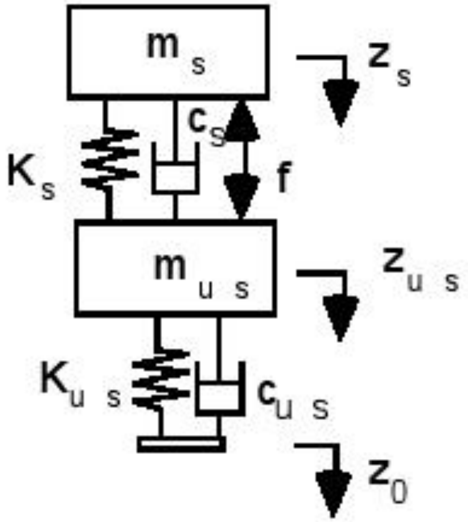


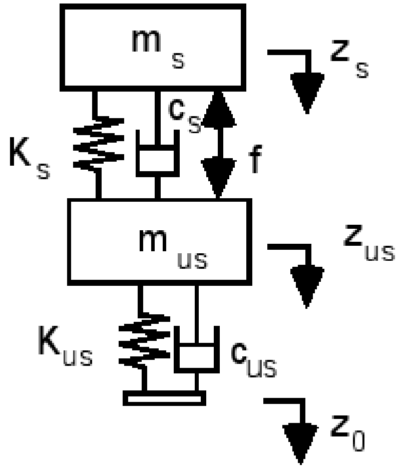
FIG.16 STANDARD HALF CAR

Source: Rajamani

Quarter-car



Quarter-Car Suspension Models (2DOF)



m_s	sprung mass
m_{us}	unsprung mass
K_s	suspension spring
c_s	susp. damping
K_{us}	tire spring
z_0	road profile
c_{us}	tire damping

$$m_s\ddot{z}_s + c_s(\dot{z}_s - \dot{z}_{us}) + k_s(z_s - z_{us}) = -f$$

$$m_{us}\ddot{z}_{us} + c_s(\dot{z}_{us} - \dot{z}_s) + k_s(z_{us} - z_s) + c_{us}(\dot{z}_{us} - \dot{z}_0) + k_{us}(z_{us} - z_0) = f$$

$$\frac{d}{dt}\begin{bmatrix} z_{us}-z_0 \\ \dot{z}_{us} \\ z_s - z_{us} \\ \dot{z}_s \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_{us}}{m_{us}} & -\frac{(c_s+c_{us})}{m_{us}} & \frac{k_s}{m_{us}} & \frac{c_s}{m_{us}} \\ 0 & -1 & 0 & 1 \\ 0 & \frac{c_s}{m_s} & -\frac{k_s}{m_s} & -\frac{c_s}{m_s} \end{bmatrix}\begin{bmatrix} z_{us}-z_0 \\ \dot{z}_{us} \\ z_s - z_{us} \\ \dot{z}_s \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{m_s}{m_{us}} \\ 0 \\ -1 \end{bmatrix}\frac{f}{m_s} + \begin{bmatrix} -1 \\ \frac{c_{us}}{m_{us}} \\ 0 \\ 0 \end{bmatrix}\dot{z}_0 \Rightarrow \dot{\underline{x}} = A\underline{x} + Bu + Gw$$

Normalized State Equation

$$\frac{d}{dt}\begin{bmatrix} z_{us}-z_0 \\ \dot{z}_{us} \\ z_s-z_{us} \\ \dot{z}_s \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_{us}}{m_{us}} & -\frac{(c_s+c_{us})}{m_{us}} & \frac{k_s}{m_{us}} & \frac{c_s}{m_{us}} \\ 0 & -1 & 0 & 1 \\ 0 & \frac{c_s}{m_s} & -\frac{k_s}{m_s} & -\frac{c_s}{m_s} \end{bmatrix} \begin{bmatrix} z_{us}-z_0 \\ \dot{z}_{us} \\ z_s-z_{us} \\ \dot{z}_s \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{m_s}{m_{us}} \\ 0 \\ -1 \end{bmatrix} \frac{f}{m_s} + \begin{bmatrix} -1 \\ \frac{c_{us}}{m_{us}} \\ 0 \\ 0 \end{bmatrix} \dot{z}_0$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2(\rho\zeta_2\omega_2+\zeta_1\omega_1) & \rho\omega_2^2 & 2\rho\zeta_2\omega_2 \\ 0 & -1 & 0 & 1 \\ 0 & 2\zeta_2\omega_2 & -\omega_2^2 & -2\zeta_2\omega_2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \rho \\ 0 \\ -1 \end{bmatrix} \quad G = \begin{bmatrix} -1 \\ 2\zeta_1\omega_1 \\ 0 \\ 0 \end{bmatrix}$$

Wheel hop mode

Sprung mass mode

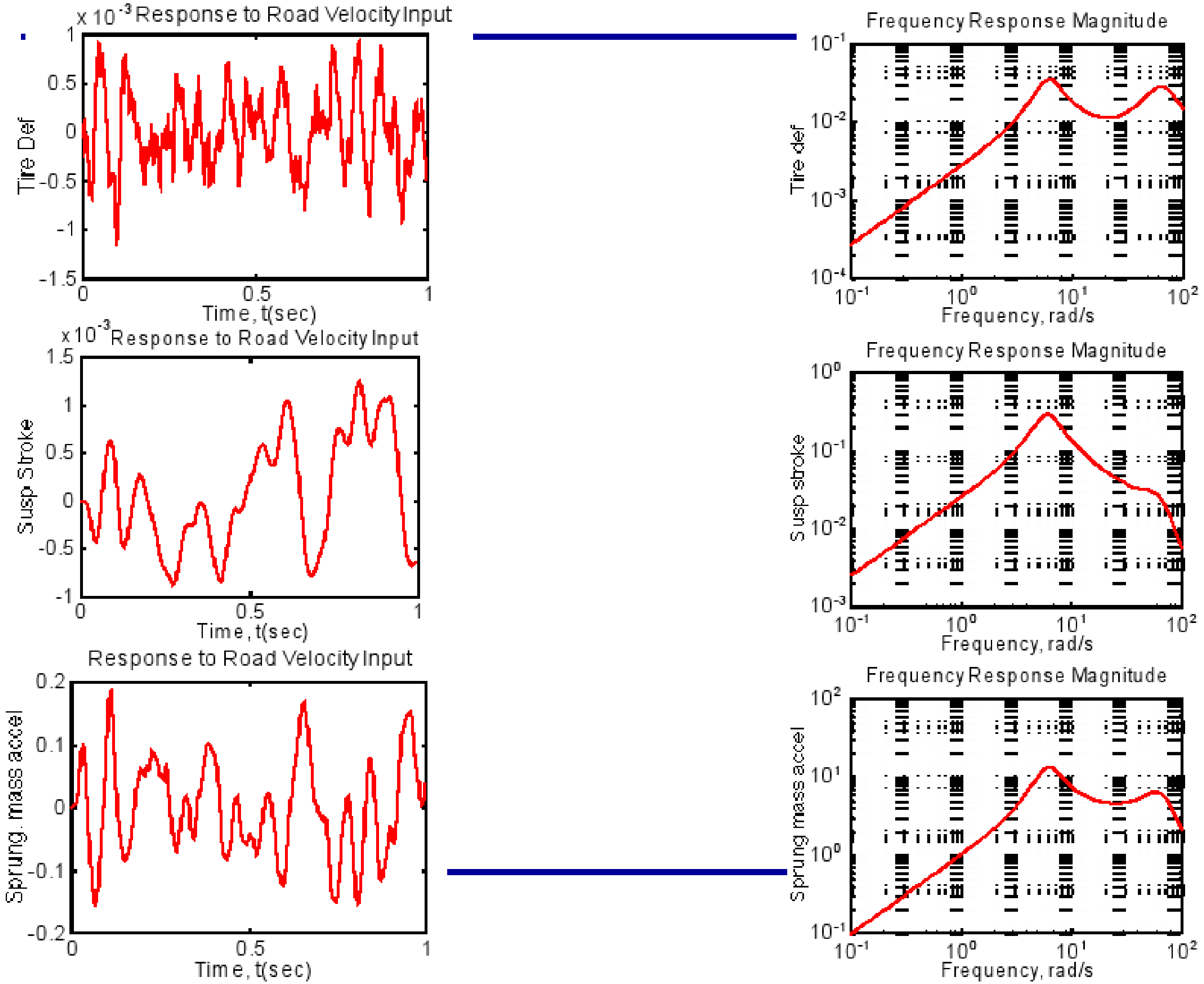
$$\rho = \frac{m_s}{m_{us}} \qquad \omega_1 = \sqrt{\frac{k_{us}}{m_{us}}} \quad \zeta_1 = \frac{c_{us}}{2m_{us}\omega_1} \qquad \omega_2 = \sqrt{\frac{k_s}{m_s}} \quad \zeta_2 = \frac{c_s}{2m_s\omega_2}$$

Ex 3.8 Quarter-Car Model

```
% Ex3_8.m
w1 = 20*pi; % w1 = sqrt(kus/mus)
w2 = 2.0*pi; % w2 = sqrt(ks/ms)
z1 = 0.0; % z1 = cus/(2*mus*w1)
z2 = 0.3; % z2 = cs/(2*ms*w2)
rho = 10.; % rho = ms/mus
 % Open loop system equations:
 A = [0 1 0 0
    -w1^2 -2*(z2*w2*rho+z1*w1)
      rho*w2^2     2*z2*w2*rho
    0 -1 0 1
    0 2*z2*w2 -w2^2 -2*z2*w2];
 B = [0 rho 0 -1]';
 G = [-1 2*z1*w1 0 0]';
 %
 % Define outputs of interest:
C1=[1 0 0 0]; D1= 0.0;
% output=tire deflection
[num1, den1]=ss2tf(A,G,C1,D1,1);
C2=[0 0 1 0]; D2= 0.0;
% output=suspension stroke
[num2, den2]=ss2tf(A,G,C2,D2,1);
C3=[A(4,:)]; D3= 0.0;
% output=sprung mass accel.
[num3, den3]=ss2tf(A,G,C3,D3,1);
```

```
% Generate the white noise input w(t)
t=[0:0.001:1];
Amp=1.65E-5;
Vel=80; sigma=sqrt(2.*pi*Amp*Vel);
w=sigma*randn(size(t));
%
% Simulate the response of interest:
y1=lsim(num1,den1,w,t);
y2=lsim(num2,den2,w,t);
y3=lsim(num3,den3,w,t);

freq=logspace(-1,2,100);
[mag1, phase1]=bode(num1,den1,freq);
loglog(freq,mag1,'r');
title('Frequency Response Magnitude');
xlabel('Frequency, rad/s');
ylabel('Tire def'); grid; pause
[mag2, phase2]=bode(num2,den2,freq);
loglog(freq,mag2,'r');
title('Frequency Response Magnitude');
xlabel('Frequency, rad/s');
ylabel('Susp stroke');  grid; pause
[mag3, phase3]=bode(num3,den3,freq);
loglog(freq,mag3,'r');
```

Ex 3.8 Quarter-Car Model



Invariant Properties of Quarter-Car Models

- By assuming zero tire damping, and summing the two eqn

$$m_s\ddot{z}_s + m_{us}\ddot{z}_{us} + k_{us}(z_{us} - z_0) = 0 \longrightarrow \text{invariant equation}$$

- Independent of suspension designs (passive, activ s:

$$H_A(s) \equiv \frac{\ddot{z}_s(s)}{\dot{z}_0(s)} \qquad H_{RS}(s) \equiv \frac{z_s(s) - z_{us}(s)}{\dot{z}_0(s)} \qquad H_{TD}(s) \equiv \frac{z_{us}(s) - z_0(s)}{\dot{z}_0(s)}$$

- In their paper Hedrick and Butsuen [1988] have shown that because of this invariance property, a force (from a passive or active suspension) acting between the spring and unsprung masses cannot independently affect all the three transfer functions above.

What Invariant Properties Mean

- Specifically, Hedrick and Butsuen (1988) show that (at certain frequencies near the wheel hop frequency) only one of these transfer function can be independently specified.
- However, Levitt and Zorka [1991] include non-zero tire damping (C_{us}) and point out that this invariance property can be significantly altered, even for small tire damping values.
- They argue that C_{us} should be included in suspension design

What Invariant Properties Mean

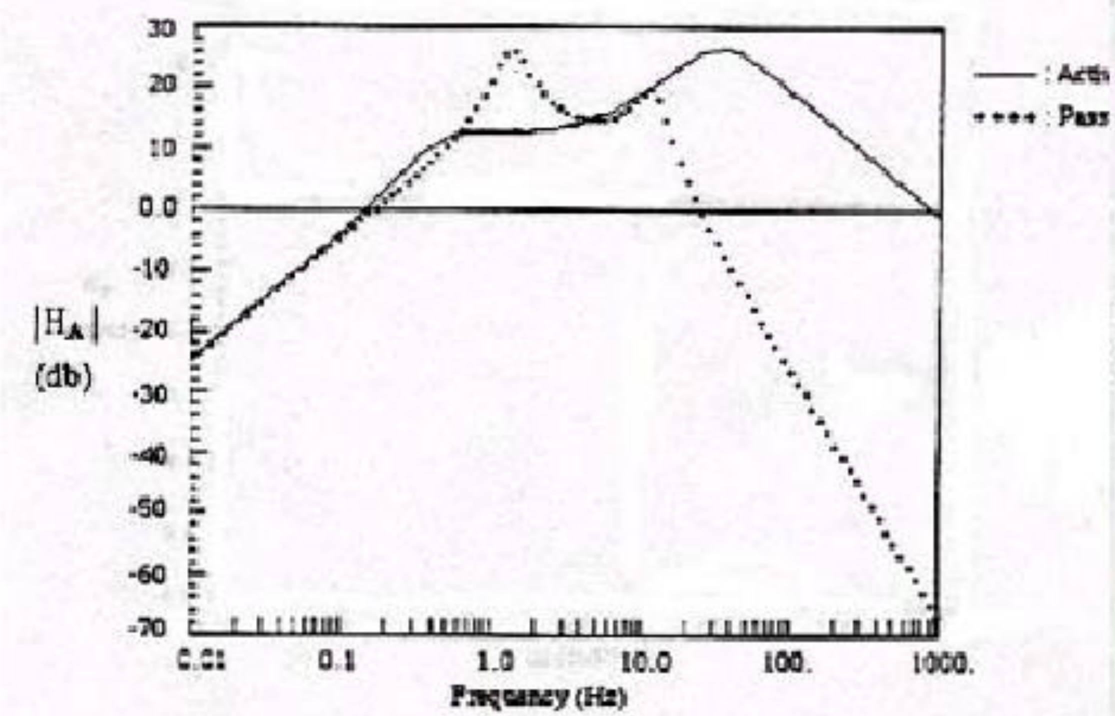


Fig 3a Acceleration transfer functions

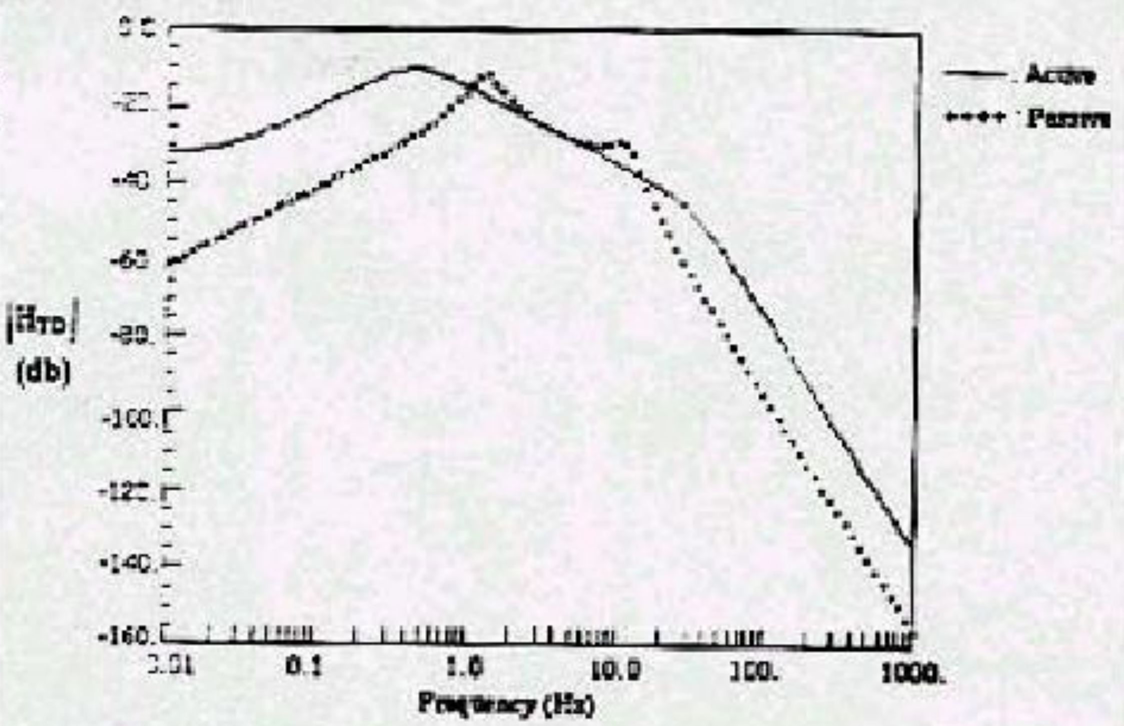


Fig 6 Rattle space transfer function