Linear Regression Models

Segment 2 – Multiple Linear Regression Model

Topic 1 – Matrix Approach for Multiple Linear Regression Model

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Topics

1. Matrix Notations for Data: Design Matrix
2. Dealing with Categorical Covariates
3. Multiple Linear Regression Models (MLRM) and assumptions
4. Ordinary Least Squares (OLS) Solution: Intuition, Geometry, & Algebraic Proof
5. Residual Vector and its Properties

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$$\Rightarrow \mathbf{r} = \mathbf{y} - \mathbf{X}\boldsymbol{\beta}.$$

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	heatinghot air	heatinghot water/steam
electric	0	0
hot air	1	0
hot water/steam	0	1

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- Note that $\sum_{i=1}^{n} \left(r^{(i)}\right)^2 = \underbrace{\left\|\begin{bmatrix} r^{(1)} \\ \vdots \\ r^{(n)} \end{bmatrix}\right\|^2}_{\text{norm of vector squared}} = \|\mathbf{r}\|^2.$

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- The resulting solution is the OLS solution: $\hat{\beta} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}.$

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- Full rank of the design matrix $\mathbf{X}$ ensures the existence of $\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}$.

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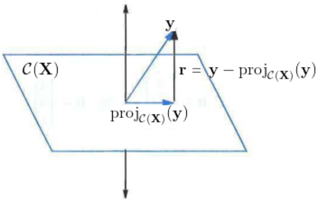
Minimizing $\|\mathbf{r}\|^2 = \|\mathbf{y} - \mathbf{X}\beta\|^2 = \left\| \mathbf{y} - \left(\underbrace{\beta_0 \mathbf{x}_1 + \beta_1 \mathbf{x}_2 + \cdots + \beta_p \mathbf{x}_{p+1}}_{\text{linear combination of columns of } \mathbf{X}} \right) \right\|^2$ is equivalent to solving the equation $\mathbf{X}\hat{\beta} = \text{proj}_{\mathcal{C}(\mathbf{X})}(\mathbf{y})$ which represents the *orthogonal projection* of $\mathbf{y}$ on to the column space of the design matrix $\mathcal{C}(\mathbf{X})$ (set of all possible linear combinations of the columns of $\mathbf{X}$):



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- Using the fact that the design matrix $\mathbf{X}$ has full rank (that is, its columns are linearly independent), we arrive at the unique OLS solution $\hat{\beta} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$.

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- This is a reiteration of the fact that linear regression works best on an average.