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Linear Regression Models

Segment 3 – Other Considerations in the MLRM
Topic 2 – Nonlinearity of Data: Residual Plots, Heteroskedasticity: Non-constant Variance of Error & Weighted Least Squares

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Topics

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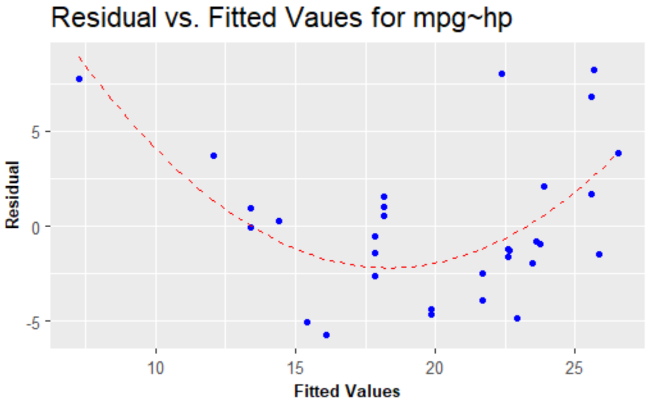
A residual plot shows the relationship between the residuals and the fitted values.



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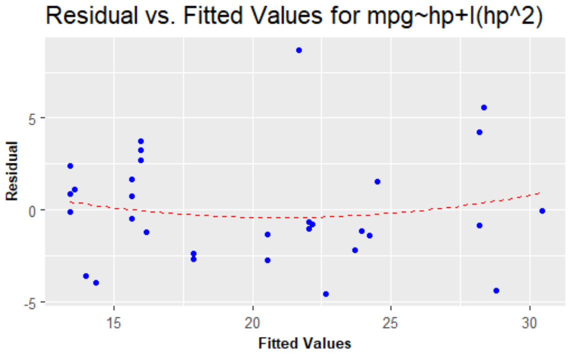

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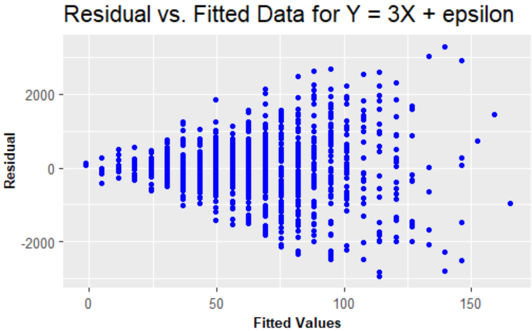
A funnel-shaped residual plot is an indication of heteroskedasticity, which means that the random error term does not have a constant variance impacting the *standard error*, *confidence interval*, and hypothesis test calculations.



Heteroskedasticity: Non-constant Variance of Error & Weighted Least Squares

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- The weights are chosen such that samples with large error variances contribute less to the summation above,

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- Describe methods for dealing with non-constant variance random error in linear regression