Linear Regression Models

Segment 1 – Simple Linear Regression Model

Topic 2 – Simple Linear Regression Model (SLRM) and its Assumptions

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Topics

1. The Geometric Idea Behind Simple Linear Regression Model (SLRM)

2. Population & Sample - Revisited in the Context of SLRM

3. Ordinary Least Squares Estimation for an SLRM

4. Assumptions in SLRM

The Geometric Idea Behind Simple Linear Regression Model (SLRM)

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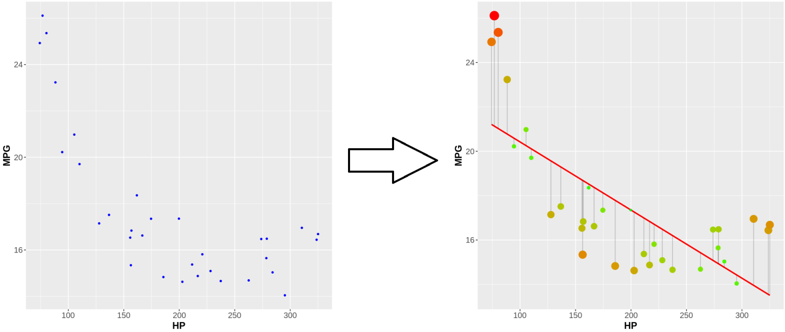
Given a dataset (*random samples from the population*), find a straight line that *fits* the data (*response variable and a single predictor*) well in an *average sense*:



Population & Sample - Revisited in the Context of SLRM

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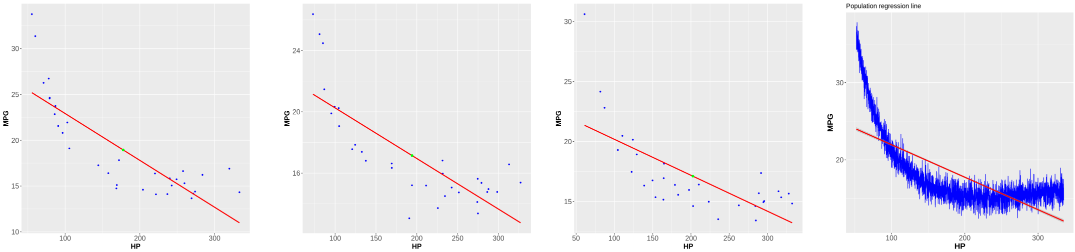
Note that the straight line of best fit will depend on the dataset but there is <u>only one</u> unique straight line of best fit for the entire population data:



Ordinary Least Squares Estimation for an SLRM

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- Note that in the SLRM above, we use the same symbol ε for the random error term which now additionally includes the effect of missing out a possibly nonlinear relationship between Y and X_1.

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$$\min \sum_{i=1}^{n} \left(r^{(i)}\right)^2 = \sum_{i=1}^{n} \left(y^{(i)} - \left(\beta_0 + \beta_1 x_1^{(i)}\right)\right)^2.$$

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$$\begin{cases} \dfrac{\partial(\text{RSS})}{\partial \beta_0} = 0 \Rightarrow -2\sum_{i=1}^{n}\left(y^{(i)} - \left(\beta_0 + \beta_1 x_1^{(i)}\right)\right) = 0, \\ \dfrac{\partial(\text{RSS})}{\partial \beta_1} = 0 \Rightarrow -2\sum_{i=1}^{n}\left(y^{(i)} - \left(\beta_0 + \beta_1 x_1^{(i)}\right)\right) x_1^{(i)} = 0. \end{cases}$$

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- Solving this results in the estimates

$$\hat{\beta}_0 = \bar{y}_n - \hat{\beta}_1 \bar{x}_n,$$
$$\hat{\beta}_1 = \frac{s_{xy}}{s_{xx}},$$

Ordinary Least Squares Estimation for an SLRM - Continued

where

$$s_{xy} = \underbrace{\sum_{i=1}^{n} \left(x_1^{(i)} - \bar{x}_n\right)\left(y^{(i)} - \bar{y}_n\right)}_{\text{sample covariance-like measure}}$$

$$s_{xx} = \underbrace{\sum_{i=1}^{n} \left(x_1^{(i)} - \bar{x}_n\right)^2}_{\text{sample variance-like measure in the predictor}},$$

$$\bar{x}_n = \underbrace{\frac{1}{n}\sum_{i=1}^{n} x_1^{(i)}}_{\text{sample mean of predictors}} \quad \text{and } \bar{y}_n = \underbrace{\frac{1}{n}\sum_{i=1}^{n} y^{(i)}}_{\text{sample mean of responses}}.$$

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- For a random ith sample, note that in the linear approximation $Y^{(i)} = \beta_0 + \beta_1 X_1^{(i)} + \varepsilon^{(i)}$, the random error term $\varepsilon^{(i)}$ for the ith sample is the same as its residual $R^{(i)}$.

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- Later, for the purpose of constructing hypotheses tests and confidence intervals for the least squares estimates, we will also assume that $\varepsilon^{(i)}$ is normally distributed.