

Linear Regression Models

Segment 4 – Model Diagnostics

Topic 3 – Bias and Variance

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Topics



1. The Bias-Variance Decomposition









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• For this lecture, we will consider the true population relationship $Y = f(X) + \varepsilon$ with the assumptions:

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$$\sigma^2 + \left(\hat{f}(X) - f(X)\right)^2.$$





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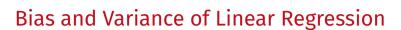


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prediction error = irreducible error $\sigma^2 + bias^2 + variance$.







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- Which model will have a higher bias?
- Which model will have a higher variance?

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- Describe the general equation for bias-variance decomposition.
- Describe bias and variance in the context of linear regression