Cryptanalysis on ECC-based Algorithms

NT219.P22.ANTT

NGUYEN NGOC TU

Our Speakers

Phan Nguyễn Việt Bắc

23520087

Trần Gia Bảo

23520139

Châu Hoàng Phúc

23521191

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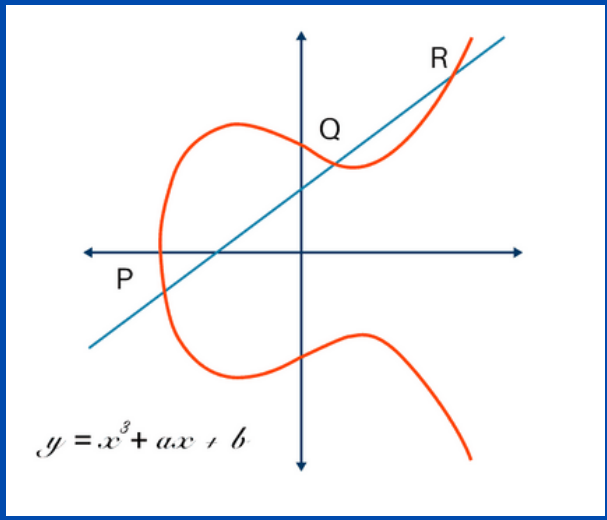
ECC quick review

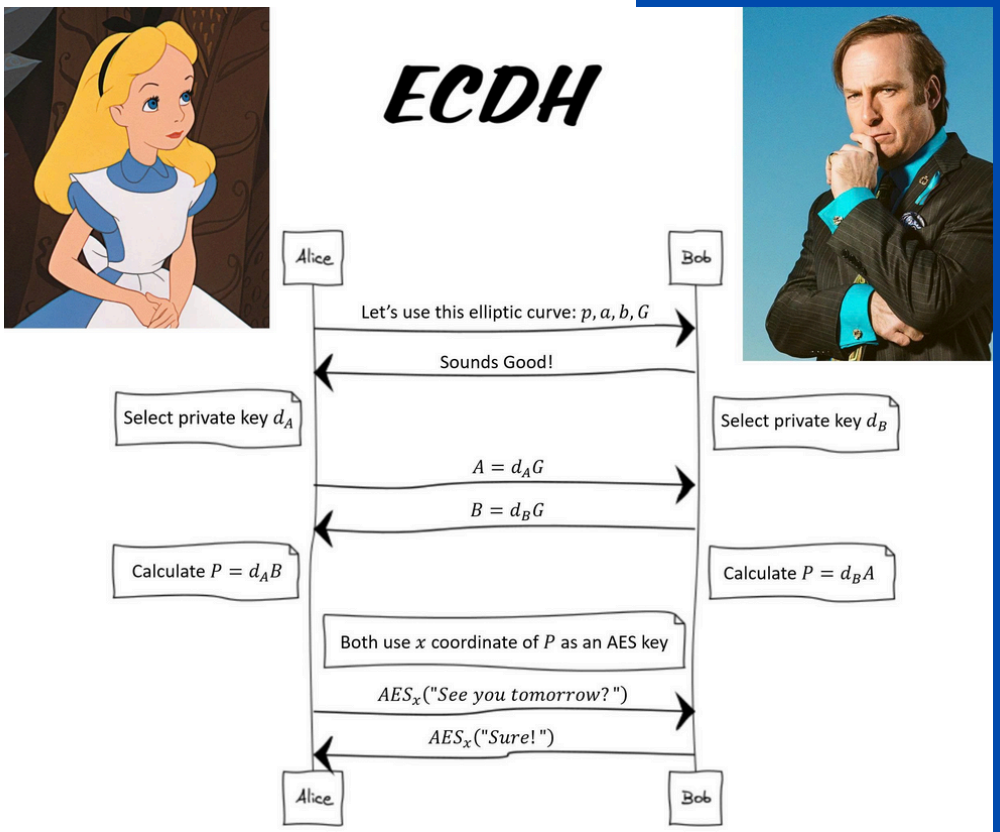
An elliptic curve in **Weierstrass form**

$$y^2 = x^3 + Ax + B, \quad \text{where } 4A^3 + 27B^2 \neq 0$$



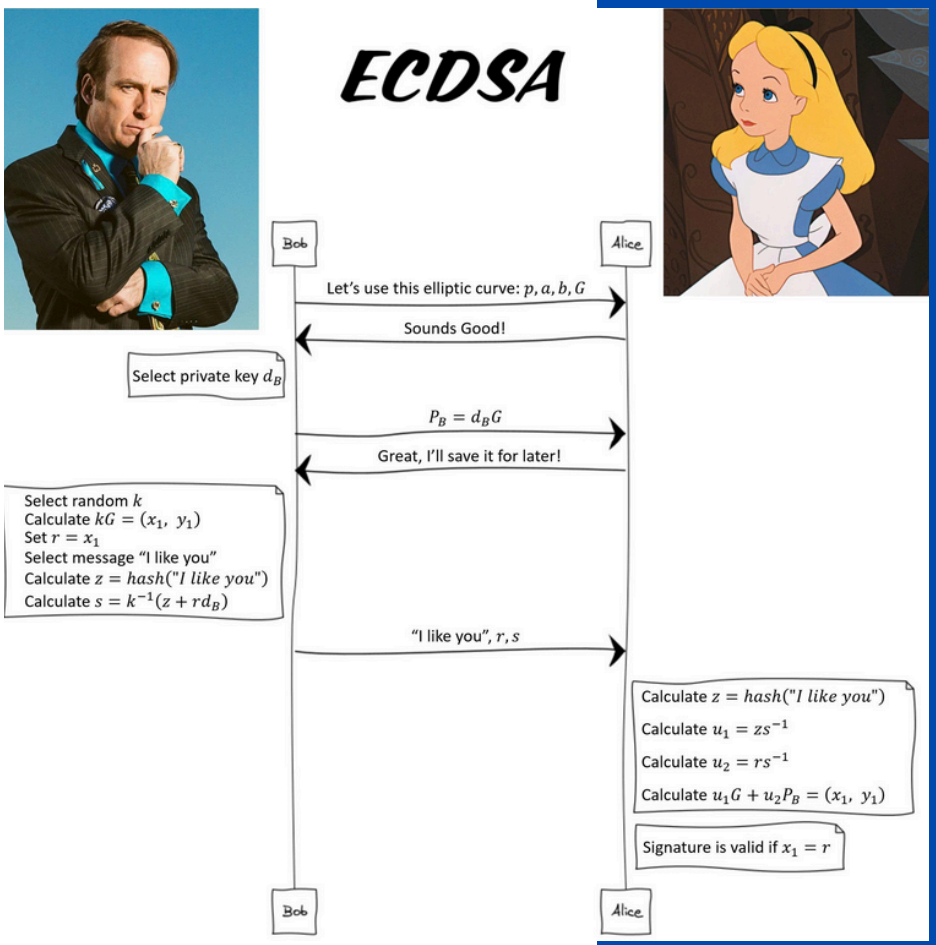
The security foundation of ECC lies in the computational difficulty of the **Elliptic Curve Discrete Logarithm Problem (ECDLP).**

ECDH



ECDH is a variant of the Diffie-Hellman key exchange protocol, allowing two parties to establish a shared secret key over an insecure channel.
The security of ECDH relies on the hardness of the Elliptic Curve Discrete Logarithm Problem (ECDLP).

ECC quick review

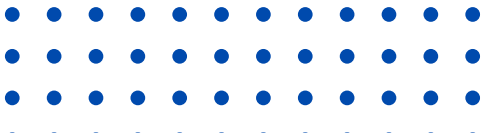


ECDSA is a widely used digital signature algorithm that verifies the authenticity and integrity of messages.
It's a variant of the Digital Signature Algorithm (DSA) that employs elliptic curve operations.

Baby-step Giant-step

The system uses elliptic curves of small orders n

Order of P (n)	$\sqrt{n}$ (BSGS steps)	Memory and computation required
2^{80}	2^{40}	Attackable with supercomputers
2^{160}	2^{80}	Too large for current practical attacks
2^{256} (modern ECC standard)	2^{128}	**Infeasible to attack** using BSGS



Baby-step Giant-step

$$Q = xP$$

Giả sử bạn viết x dưới dạng:

$$x = im + j \Rightarrow Q = xP = imP + jP$$

Chuyển vế:

$$Q - imP = jP$$

For each

Baby-step $j \in [0, m-1]$,

compute:

$$B_j = jP$$

Compute:

Giant-step $S = mP$

This is the "giant step" increment.

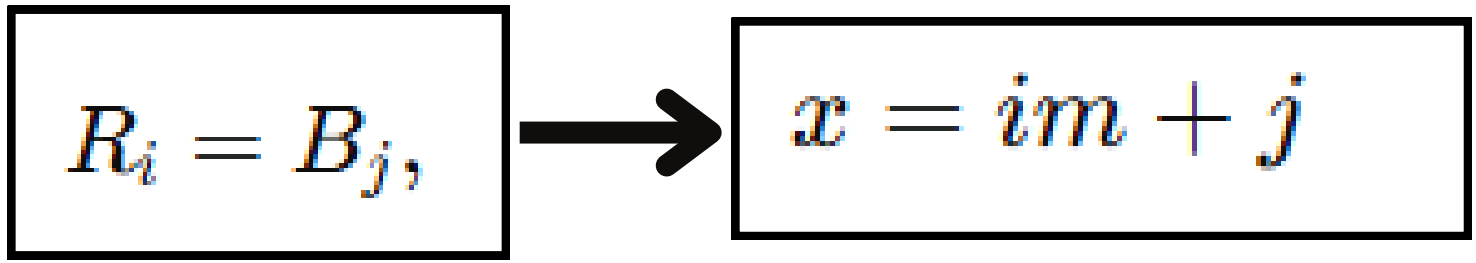
Then, for each

$$i \in [0, m-1],$$

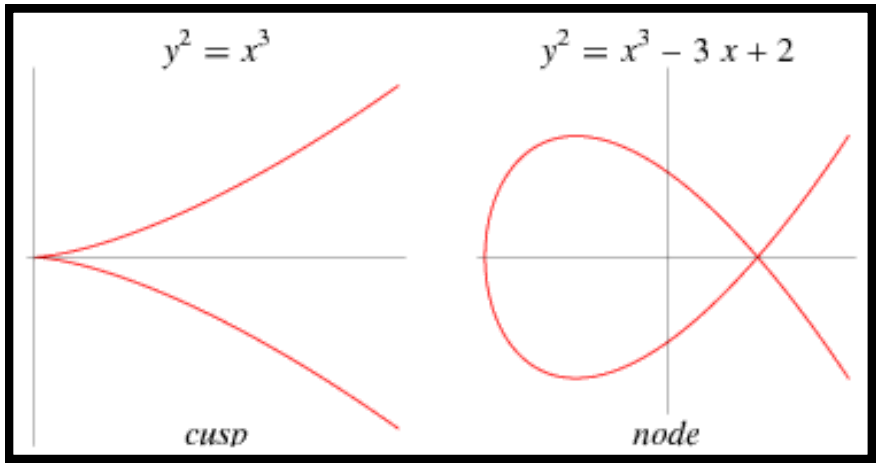
compute:

$$R_i = Q - iS = Q - i(mP)$$

Baby-step Giant-step



Singular Curve



If the discriminant is zero ($\Delta=0$), then **the curve is singular** and **the DLP is easy**.

$$4A^3 + 27B^2 \equiv 0 \pmod{p}$$

There are 2 type of Singular Curve
So there are 2 differents way to attack

Node

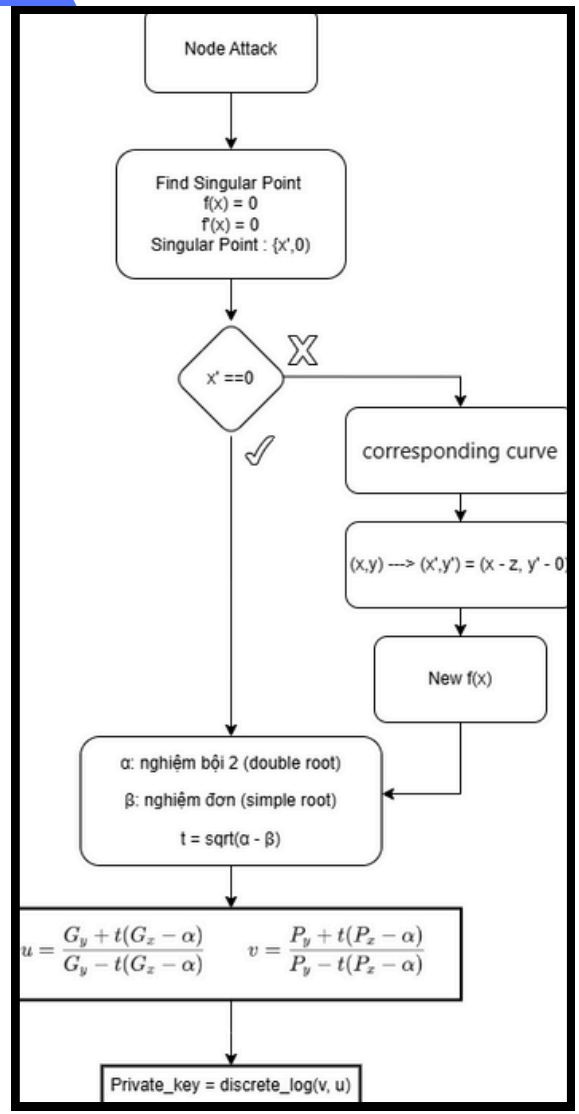
F(x), **Δ=0 and A <> 0, B = 0**

$$y^2 = (x-\alpha)^2(x-\beta)$$

$$t = \sqrt{\alpha - \beta}$$

$$u = \frac{G_y + t(G_x - \alpha)}{G_y - t(G_x - \alpha)}, \quad v = \frac{P_y + t(P_x - \alpha)}{P_y - t(P_x - \alpha)}$$

$$k = \log_u v \mod p$$



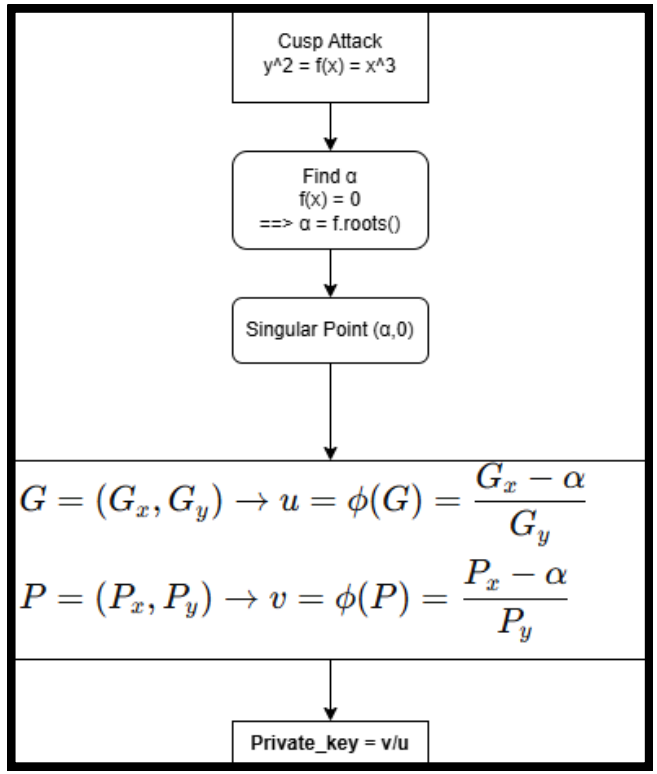
Cusp

F(x), **Δ=0 and A = 0, B = 0**

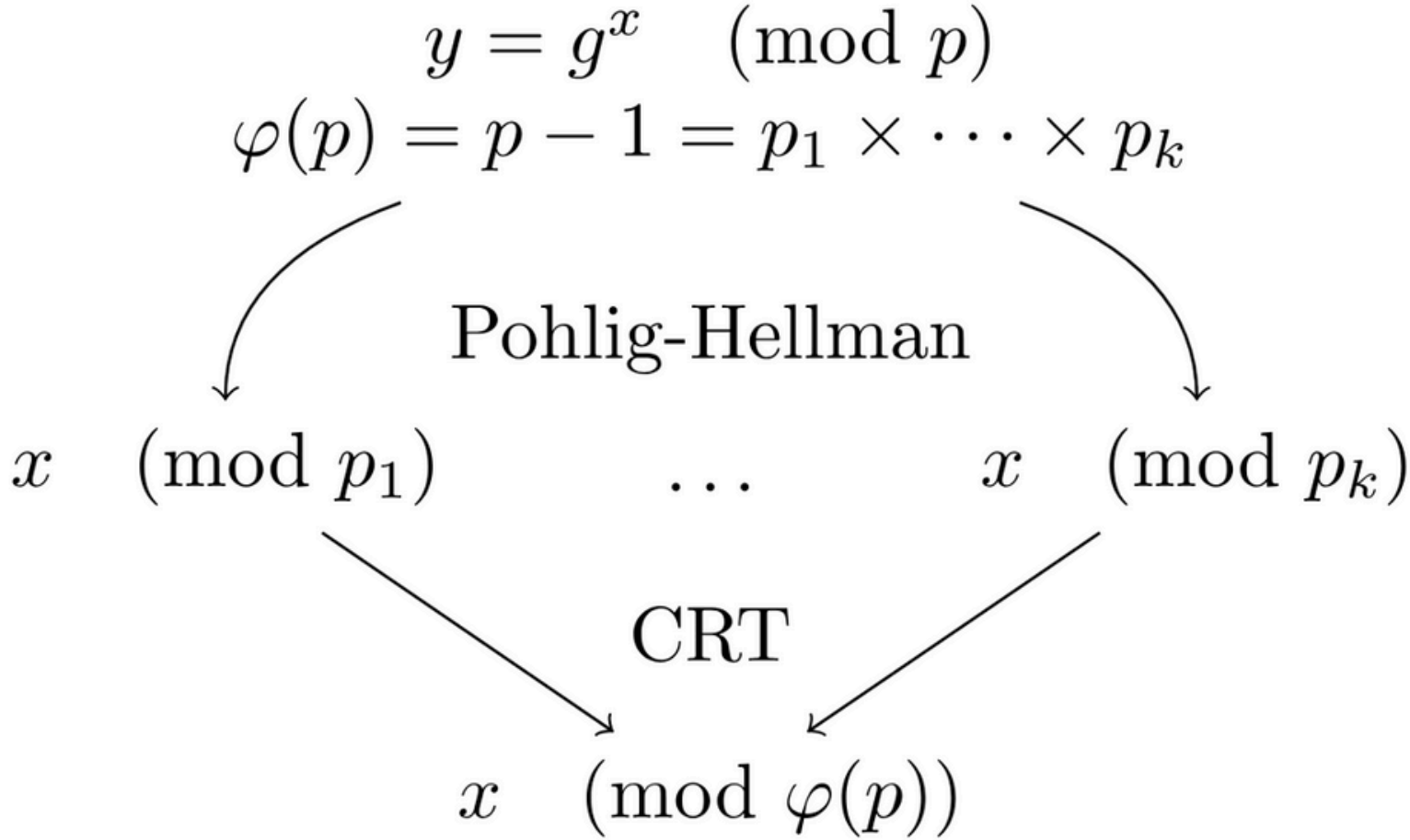
$$y^2 = x^3 \pmod{p}$$

$$y^2 = (x - \alpha)^3$$

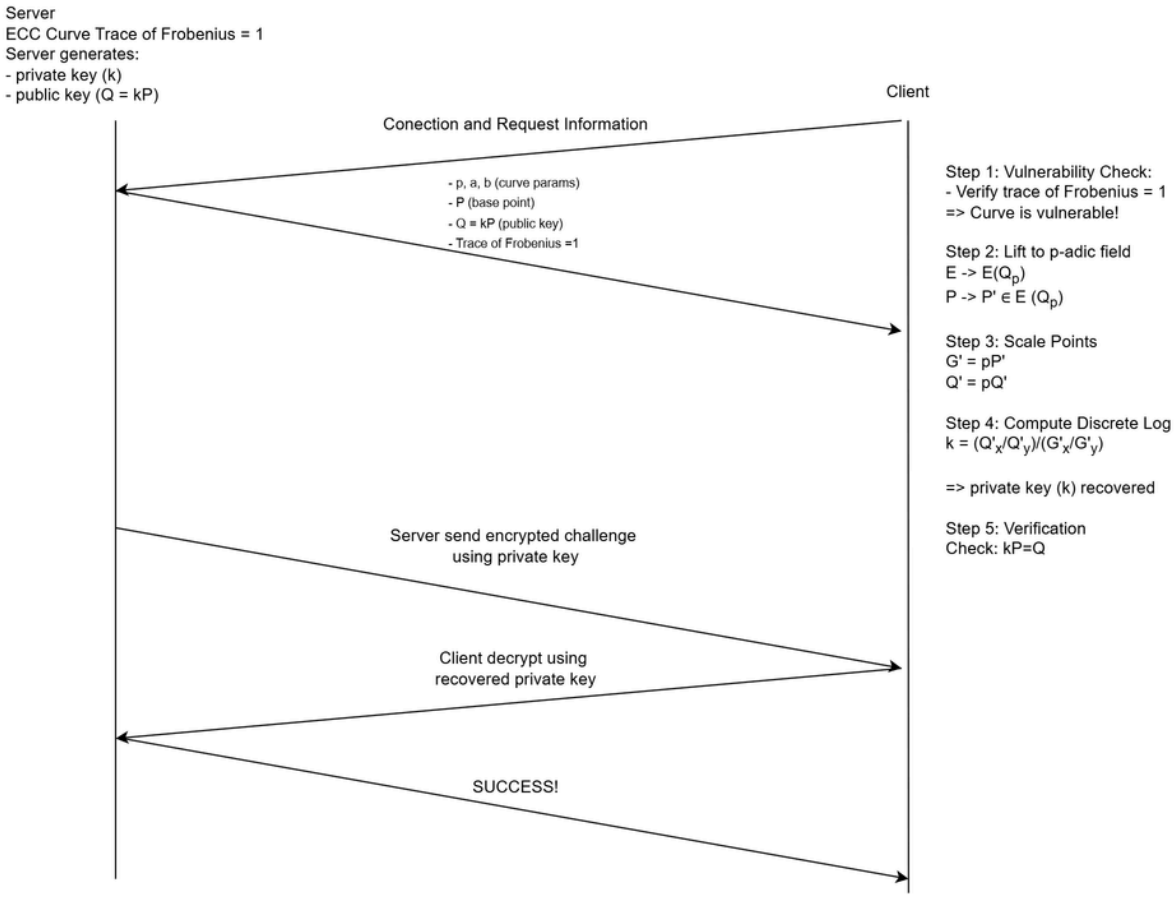
$$\phi(P) = k \cdot \phi(G) \Rightarrow k = \frac{\phi(P)}{\phi(G)} = \frac{x_2 - \alpha}{y_2} \cdot \frac{y_1}{x_1 - \alpha}$$



Pohlig-Hellman



Smart Attack



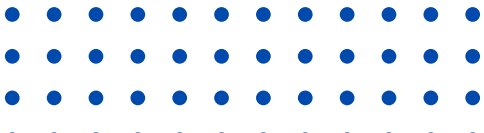
MOV Attack

MOV attack - (1)

- Choose m , such that $E[N] \subset E(F_{q^m})$
  By Corollary 3.11, $\mu_N \subset F_{q^m}$
- MOV attack (Want to solve $Q = kP$ for k.)
  1. Choose a random point $T \in E(F_{q^m})$
  2. Compute $M = ord(T)$
  3. Let $d = \gcd(M, N)$, and let $T_1 = (M/d)T$
     Then $d = ord(T_1), \quad d \mid N$, so $T_1 \in E[N]$
  4. Compute $\zeta_1 = e_N(P, T_1)$ and $\zeta_2 = e_N(Q, T_1)$
     Then both $\zeta_1, \zeta_2 \in \mu_d \subseteq F^*_{q^m}$
  5. Solve discrete logarithm problem

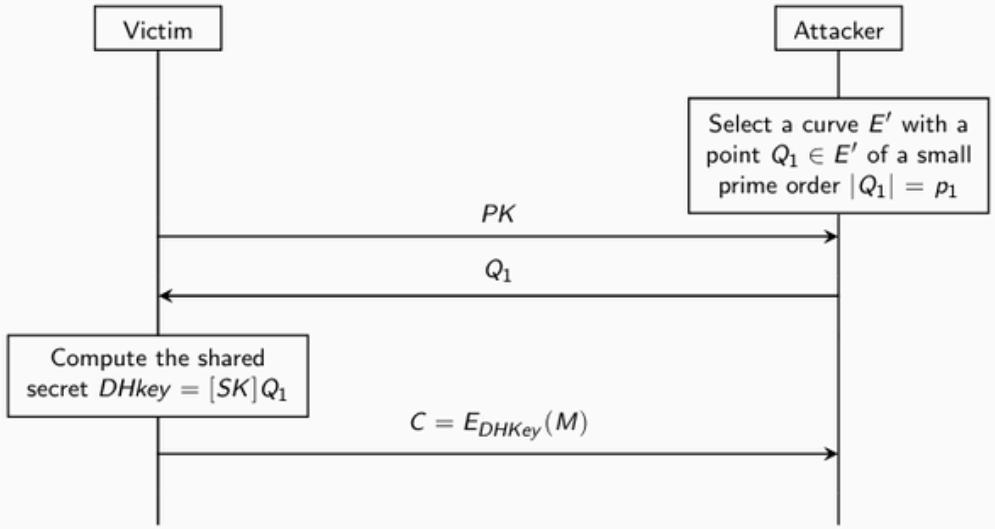
     $$\zeta_2 = \zeta_1^k \quad \text{in } F^*_{q^m}$$

     This will give $k \pmod d$
  6. Repeat with random points T until the lcm of d's is N .
     This determines $k \pmod N$

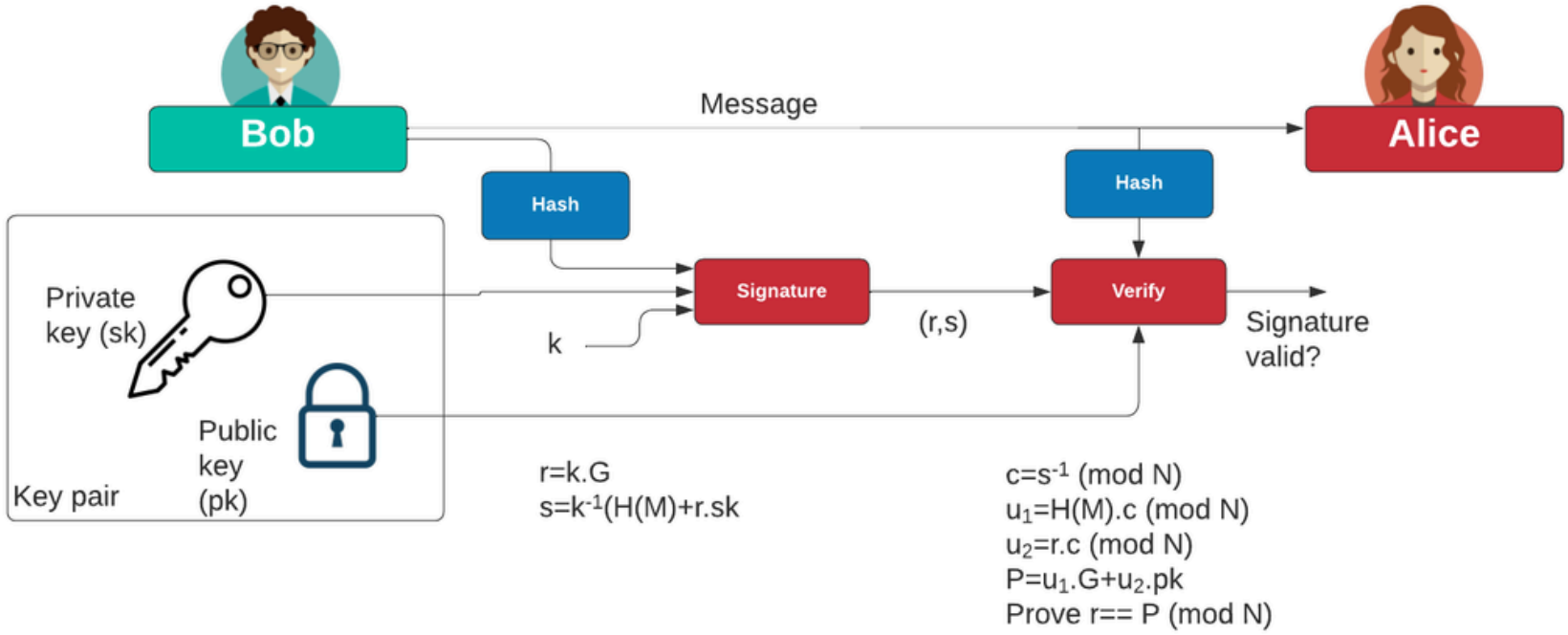


Invalid Curve Attack

- Let SK be the secret key of the victim device and let $PK = [SK]P$ its public key.
- Let E' be a different group defined by the curve equation $y^2 = x^3 + ax + b'$ with the same a and a different b' parameter.



Not Hashing Message



Nonce Reuse Attack

$$s = k^{-1}(z + r \cdot d_A) \pmod{n}$$

REUSE NONCE K

$$\Rightarrow$$

$$s_1 = k^{-1}(z_1 + r_1 \cdot d_A) \pmod{n}$$

$$s_2 = k^{-1}(z_2 + r_2 \cdot d_A) \pmod{n}$$

Nonce Reuse Attack

By rearranging the equations to solve for k^{-1}:

1. $k^{-1} = s_1^{-1}(z_1 + r_1 \cdot d_A) \pmod{n}$
2. $k^{-1} = s_2^{-1}(z_2 + r_2 \cdot d_A) \pmod{n}$

Since k^{-1} is the same in both cases, we can set them equal:

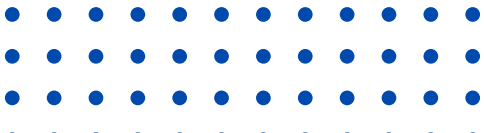
$$s_1^{-1}(z_1 + r_1 \cdot d_A) = s_2^{-1}(z_2 + r_2 \cdot d_A) \pmod{n}$$

This equation can then be rearranged to solve for d_A, the private key:

$$s_2 z_1 + s_2 r_1 d_A = s_1 z_2 + s_1 r_2 d_A \pmod{n}$$
$$s_2 z_1 - s_1 z_2 = s_1 r_2 d_A - s_2 r_1 d_A \pmod{n}$$
$$s_2 z_1 - s_1 z_2 = d_A(s_1 r_2 - s_2 r_1) \pmod{n}$$

Finally, the private key d_A can be found:

$$d_A = (s_2 z_1 - s_1 z_2)(s_1 r_2 - s_2 r_1)^{-1} \pmod{n}$$



Biased Nonces attack

Formulating ECDSA as a hidden number problem

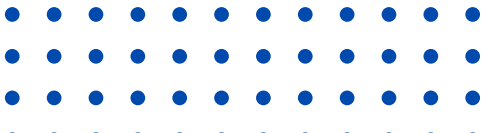
[Howgrave-Graham Smart 2001], [Nguyen Shparlinski 2003]

We have a system of equations in unknowns $k_1, \ldots, k_m, d$:

$$\begin{aligned} k_1 - t_1 d - a_1 &\equiv 0 \bmod n \\ k_2 - t_2 d - a_2 &\equiv 0 \bmod n \\ &\vdots \\ k_m - t_m d - a_m &\equiv 0 \bmod n \end{aligned}$$

We assume the k_i are small.

(Instance of the *hidden number problem* [Boneh Venkatesan 96].)



Biased Nonces attack

Solving the hidden number problem with CVP

Input:

$$k_1 - t_1 d - a_1 \equiv 0 \bmod n$$
$$\vdots$$
$$k_m - t_m d - a_m \equiv 0 \bmod n$$

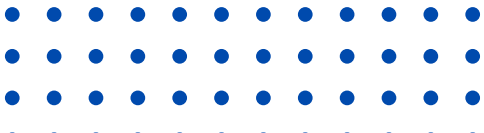
in unknowns $k_1, \ldots, k_m, d$, where $|k_i| < B$.

Construct the lattice basis

$$M = \begin{bmatrix} n & & & \\ & n & & \\ & & \ddots & \\ & & & n \\ t_1 & t_2 & \ldots & t_m \end{bmatrix}$$

Solve CVP with target vector $v_t = (a_1, a_2, \ldots, a_m)$.

$v_k = (k_1, k_2, \ldots, k_m)$ will be the distance.



Biased Nonces attack

Solving the hidden number problem with CVP embedding

Construct the lattice

$$M = \begin{bmatrix} n & & & & & \\ & n & & & & \\ & & \ddots & & & \\ & & & n & & \\ t_1 & t_2 & \dots & t_m & B/n & \\ a_1 & a_2 & \dots & a_m & & B \end{bmatrix}$$

Want vector
$v_k = (k_1, k_2, \dots, k_m, Bd/n, B)$

We have:

- $\dim L = m + 2$ $\qquad$ $\det L = B^2 n^{m-1}$
- Ignoring approximation factors, LLL or BKZ will find a vector

$$|v| \leq (\det L)^{1/\dim L}$$

- We are searching for a vector with length $|v_k| \leq \sqrt{m+2}B$.
- Thus we expect to find v_k when

$$\log B \leq \lfloor \log n(m-1)/m - (\log m)/2 \rfloor$$

If you have any questions, you are welcome to ask

Thank You