(01-1)

Time Response Analysis & Steady State Error Analysis

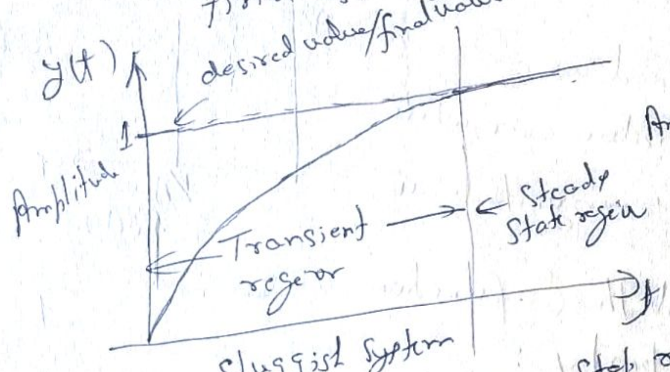
Def: The time response of a system is the output of the system as a function of time, when subjected to a given input.

* Time response of a system consists of two parts
Total response $(y(t))$ = Transient response $y_{tr}(t)$ + Steady state response $y_{ss}(t)$

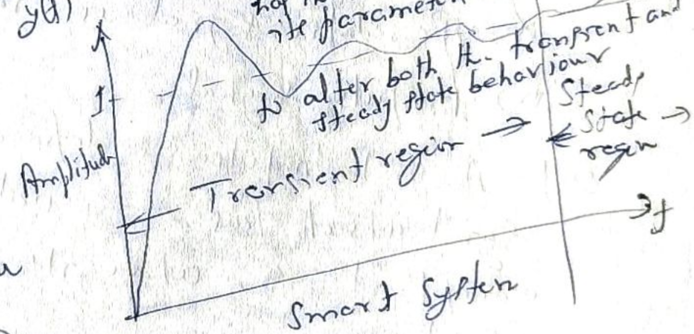
(i) **Transient response:** Transient response of the system is the portion of total time response during which the output changes from one value to another value.
Or It is the response before the output reaches the steady state value.

(ii) **Steady state response** of the system is the response of the system for a given input after a very long time. In steady state, the output response settles to its final steady state value or steady oscillations.

*A feedback control system has the inherent capability that its parameters can be adjusted to alter both the transient and steady state behaviour.



Sluggish System



Smart System

Fig. Step responses

Q. What are standard test signals? Discuss in detail.

Ans: Standard test signal or typical test signals: - The signal whose mathematical model is known.

(i) **Step Signal:** $u(t)$ & $u_s(t)$

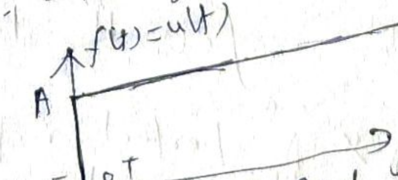


Fig. Step Signal

Def: The step signal is a signal whose value changes from one level (normally zero) to another level A in zero time. It is denoted by $u(t)$

$u(t) = 1$; for $t > 0$

It is $A\,u(t)$ when $u(t) = 0$, for $t < 0$

$f(t) = A\,u(t)$

& $\mathcal{L}[f(t)] = F(s) = \frac{A}{s}$

(02-1)

(ii) <u>Ramp Signal</u>: $r(t)$

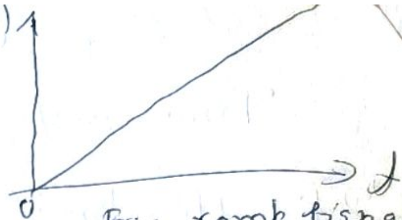


Fig: ramp signal

Def: The ramp is a signal which starts at a value of zero and increases linearly with time

<u>mathematically</u>: $f(t) = A\,r(t)$ for $t \geq 0$
$= 0$ for $t < 0$

$$\mathcal{L}[f(t)] = F(s) = \frac{A}{s^2}$$

If $A = 1$, then it is known as unit ramp

* Ramp signal is integral of a step function.

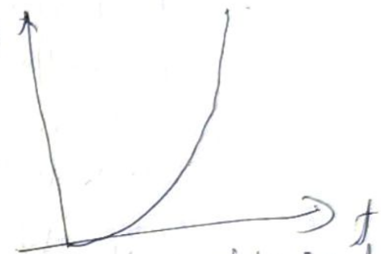


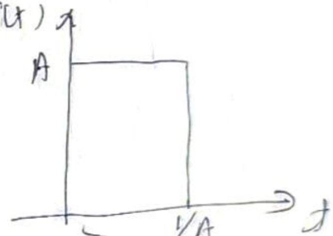
Fig. parabolic signal

(iii) <u>Parabolic Signal</u>

$f(t) = \frac{At^2}{2}$, for $t \geq 0$
$= 0$ for $t < 0$

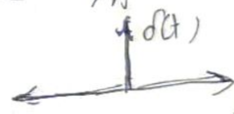
& $\mathcal{L}[f(t)] = F(s) = \frac{A}{s^3}$

* Parabolic signal is integral of a ramp signal



(iv) <u>Impulse Signal</u>

Def. A unit impulse is defined as a signal which has zero value everywhere except $t = 0$; where its magnitude is infinity.



It is generally called the δ-function and has the following property:

$\delta(t) = 0, \quad t \neq 0$

$\int_{-\epsilon}^{\epsilon} \delta(t)\,dt = 1$; where ϵ tends to zero

* A perfect impulse cannot be achieved in practice, it is usually approximated by a pulse of small width but unit area.

* Mathematically: an impulse function is the derivative of a step function i.e.

$$\delta(t) = \frac{du}{dt} = \dot{u}$$

& $\mathcal{L}[\delta(t)] = 1$

(03-1)

Q. What do you mean by Impulse Response Function? Discuss in details.

Ans: <u>Impulse Response Function</u>:

For a linear, time-invariant system, the transfer function $G(s)$ is

$$R(s) \xrightarrow{} \boxed{G(s)} \xrightarrow{} Y(s)$$

$R(s)$ input — L.T. of input

$Y(s) \rightarrow$ is L.T. of output (output)

$$G(s) = \frac{Y(s)}{R(s)}$$

output $= Y(s) = G(s) \cdot R(s)$ — (1)

<u>Convolution</u>: * multiplication in the complex domain (freq domain) is equivalent to convolution in the time domain, so inverse L.T. of eq. (1) is given by the following convolution integral:

$$y(t) = \int_0^t r(\tau)\, g(t-\tau)\, d\tau = \int_0^t g(\tau)\, r(t-\tau)\, d\tau$$

where $g(t) = r(t) = 0$ for $t < 0$

Now consider the output due to unit impulse input when all the initial conditions are zero,

Since $\mathcal{L}[\delta(t)] = 1 = R(s)$ (say) or $\Lambda(t)$

i.e. $R(s) = 1$

from eq (1) output $= Y(s) = G(s)$ — (2)

or $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)]$

or $y(t) = g(t) =$ impulse-response function

Def: The impulse response of a system, is the inverse Laplace transform of its transfer function.

* This function is also known as weighting function of the system.

* This function is also known as Green's function.

** Weighting function of a system can be used to find the system's response to any input $f(t)$, by means of the convolution integral.

Def: The impulse response function $g(t)$ is the response of a linear system to a unit impulse input when the initial conditions are zero. The Laplace transform of this function gives the transfer function. Therefore the transfer function and impulse response function of a linear time-invariant system contain the same information about the system dynamics.

(04-1)

Time Response of First order System.

Q.1 Obtain the impulse response of a 1st-order system and discuss it.

Ans

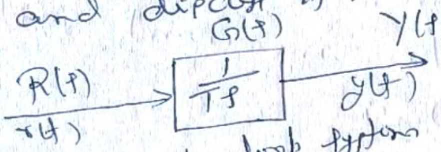


Fig. 1. open loop system

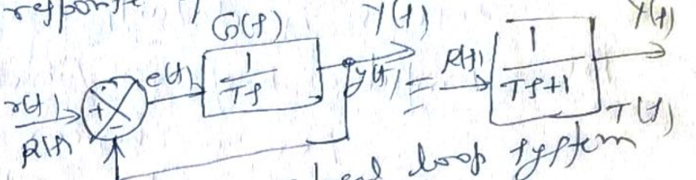


Fig 2. closed loop system

case(i) open loop : $Y(s) = G(s)\,R(s) = \frac{1}{Ts}\cdot R(s)$

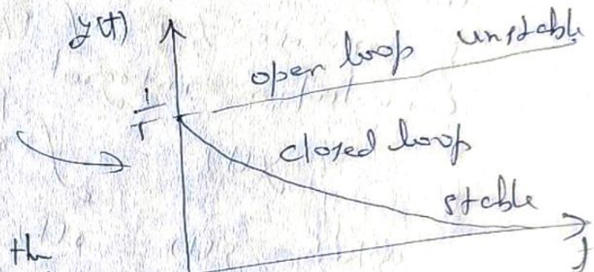
if $r(t) = \delta(t)$ = impulse, then $R(s) = 1$

$\therefore Y(s) = \frac{1}{Ts} \Rightarrow y(t) = \frac{1}{T}$

case(ii) closed loop: $Y(s) = T(s)\,R(s) = \frac{1}{Ts+1}\cdot R(s)$

for impulse $R(s)=1$, $\therefore Y(s) = \frac{1}{Ts+1} = \frac{1}{T(s+\frac{1}{T})}$

$$\therefore \boxed{y(t) = \frac{1}{T}e^{-\frac{t}{T}}}$$



Q2 Derive the expression for the Step-response of 1st order system. Also
or obtain the step-response of 1st order system.
define and discuss the concept of time constant.

Ans. case(i) open loop : from fig 1

$Y(s) = \frac{1}{Ts}\cdot R(s)$

for unit step - $R(s) = \frac{1}{s}$

$\therefore Y(s) = \frac{1}{Ts}\cdot\frac{1}{s} = \frac{1}{Ts^2}$

$y(t) = \frac{1}{T}t$

case(ii) closed loop system fig-2

$\frac{Y(s)}{R(s)} = \frac{1}{Ts+1}$

$Y(s) = \frac{1}{Ts+1}\cdot R(s) = \frac{1}{Ts+1}\cdot\frac{1}{s}$

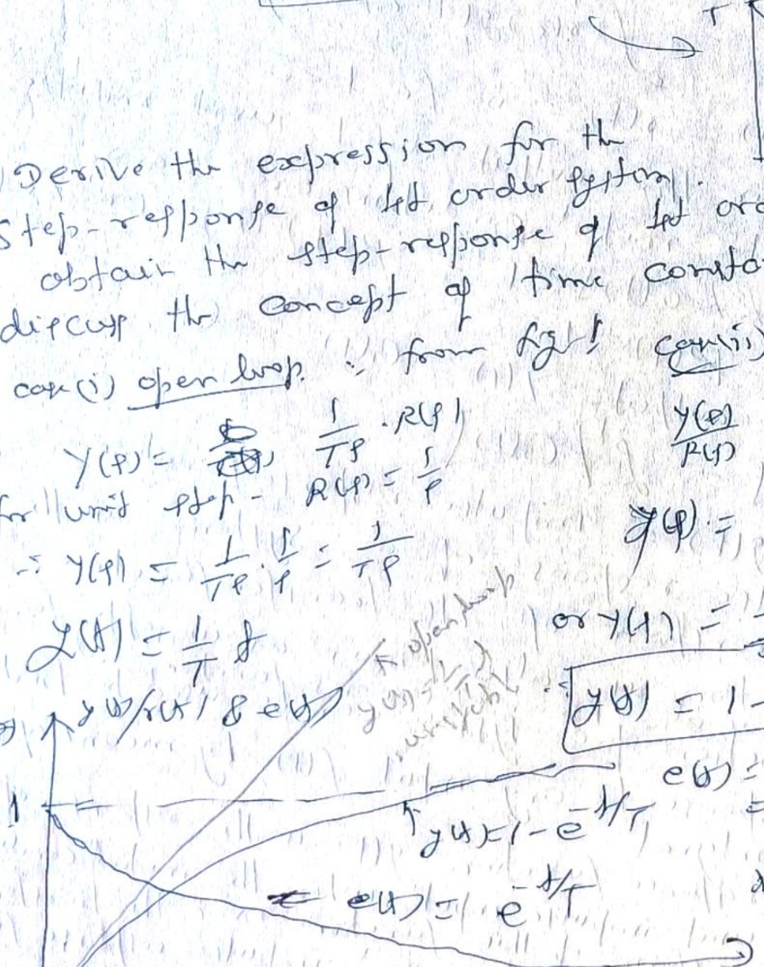
or $Y(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$

$$\boxed{y(t) = 1 - e^{-t/T}} \text{ for } t \geq 0$$

$e(t) = r(t) - y(t) = 1 - 1 + e^{-t/T} = e^{-t/T}$

$\lim_{t\to\infty} e(t) = e_{ss} = 0$

Thus the system with zero steady state error trace the unit step input.



unit step response of a 1st order sys

(05H)

Thus output rises exponentially from zero value to the final value of unity.

The initial slope of curve at $t = 0$ is given by

$$\left.\frac{dy}{dt}\right|_{t=0} = \frac{1}{T} \left. e^{-t/T} \right|_{t=0} = \frac{1}{T}$$

where T is known as the 'Time constant' of the system.

Time Constant (T)

At, $t = T$

$$y(t) = 1 - e^{-t/T} = 1 - e^{-1} = 1 - 0.368 = 0.632$$

Def: ① If the initial slope is maintained constant, then after how much time output reaches the final value, is the time constant of the system.

② Time constant is the time at which response reaches to 63.2% of its final value.

* Time constant is the indicative of how fast the system tends to reach the final value.
* Smaller the time constant, faster will be the response.
* A larger time constant corresponds to a sluggish system and a small time constant corresponds to a faster ~~system~~ response.
* The speed of response can be ~~quanti~~ quantitatively defined as the time for the output to become a particular percentage of its final value.

$t = 2T, \rightarrow y(t) = 0.865$

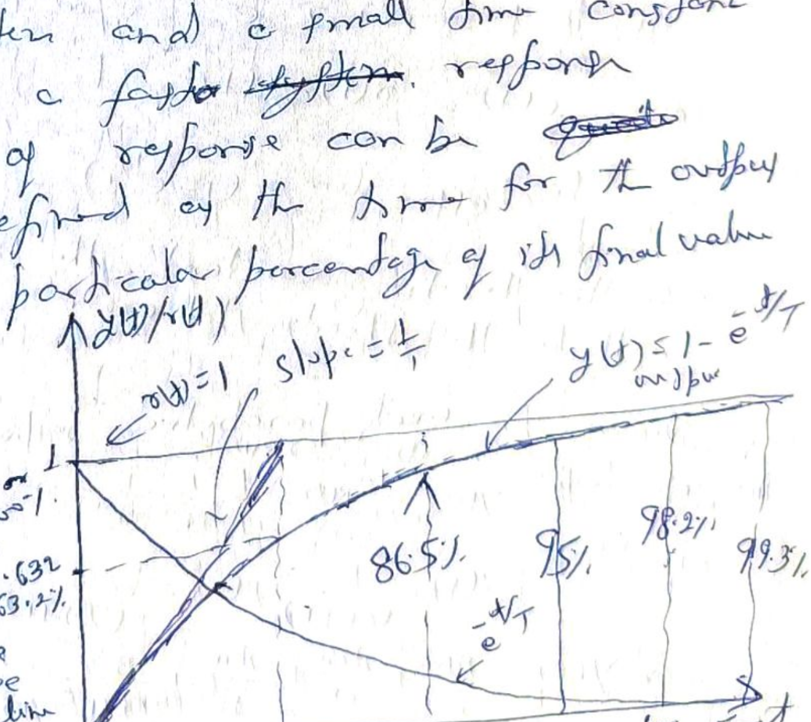
$t = 3T, \; y(t) = 0.95$

$t = 4T, \; y(t) = 0.982$

$t = 5T, \; y(t) = 0.993$

Thus for $t \geq 4T$ the response remains within 2% of the final value.

* In practice a reasonable estimate of the response time is the length of the time the response curve needs to reach the 2% line of the final value or 4T (4 times the Time constant).



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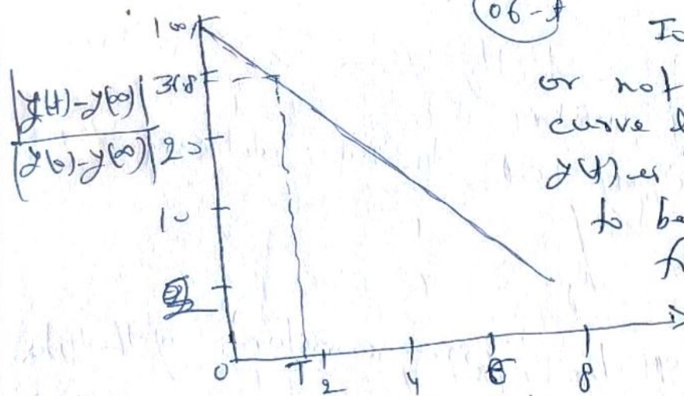


Plot on semilog paper

To determine experimentally whether or not the system is of first order, plot curve $\log|y(t)-y(\infty)|$ as a function of t where $y(t)$ is the output. If the curve turns out to be a straight line, the system is first order. The time constant T can be read from the graph as the time that satisfies the following eq

$$y(T)-y(\infty) = 0.368[y(0)-y(\infty)]$$

Note that instead of plotting $\log|y(t)-y(\infty)|$ vs t, it is convenient to plot $|y(t)-y(\infty)|/|y(0)-y(\infty)|$ vs t on semi log-paper

Q. Derive the unit-ramp response of 1st-order system and discuss it.

Ans case (i) open loop

$$R(s) \rightarrow \boxed{\frac{1}{Ts}} \rightarrow Y(s),\ y(t)$$

$r(t)=t \rightarrow R(s)=\frac{1}{s^2}$

$$Y(s) = \frac{1}{Ts}R(s) = \frac{1}{Ts}\cdot\frac{1}{s^2} = \frac{1}{Ts^3}$$

$$y(t) = \frac{1}{T}t^2$$

case (ii) closed loop

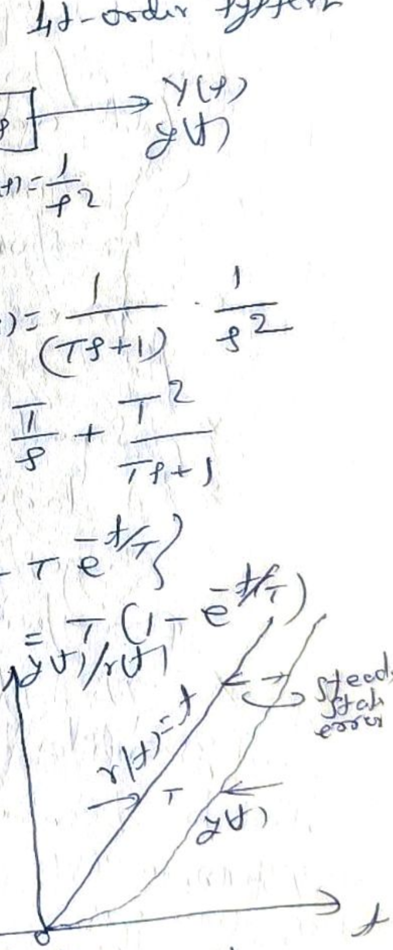
$$Y(s) = \frac{1}{(Ts+1)}\cdot R(s) = \frac{1}{(Ts+1)}\cdot\frac{1}{s^2}$$

$$\text{or } Y(s) = \frac{A_{11}}{s^2} + \frac{A_{12}}{s} + \frac{B}{Ts+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$\boxed{y(t) = t - T + T\cdot e^{-t/T}} \quad \text{for } t \geq 0$$

$$\text{error} = e(t) = r(t) - y(t) = t - \{t - T + Te^{-t/T}\}$$
$$= T - Te^{-t/T} = T(1-e^{-t/T})$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} T(1-e^{-t/T}) = T$$



* Smaller the time constant, the smaller will be the steady state error

Q. Derive the ~~unit~~ parabolic response and discuss it

$r(t)=\frac{t^2}{2} \Rightarrow R(s)=\frac{1}{s^3}$

Ans open loop.

$$Y(s) = \frac{1}{Ts}\cdot R(s) = \frac{1}{Ts}\cdot\frac{1}{s^3} = \frac{1}{Ts^4} \Rightarrow y(t) = \frac{1}{T}t^3$$

closed loop

$$Y(s) = \frac{1}{(Ts+1)}R(s) = \frac{1}{(Ts+1)}\cdot\frac{1}{s^3} = \frac{1}{s^3(Ts+1)}$$

$$\text{or } Y(s) = \frac{A_{13}}{s^3} + \frac{A_{12}}{s^2} + \frac{A_{11}}{s} + \frac{B}{Ts+1} = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{Ts+1}$$

$$y(t) = \frac{t^2}{2} - Tt + T^2 - T^2e^{-t/T}$$

$$e(t) = r(t) - y(t) = \frac{t^2}{2} - \left\{\frac{t^2}{2} - Tt + T^2 - T^2e^{-t/T}\right\} = T(t - T + Te^{-t/T})$$

& $e_{ss} = e(\infty) = \infty$

07-1

Important property of linear time-invariant system

For parabolic input, the output y(t) is $y(t) = \frac{t^2}{2} - Tt + T^2 - T^2 e^{-t/T}$ (for $t \geq 0$)

For unit ramp input " " " " $y(t) = t - T + Te^{-t/T}$ (for $t \geq 0$)

it is the derivative of parabolic input — which is the derivative of parabolic ~~input~~ response

For unit-step input, which is the derivative of unit-ramp input, the output $y(t) = 1 - e^{-t/T}$ → which is also the derivative of unit ramp response

For the unit impulse-input — which is the derivative of unit-step input, the output y(t) is

$y(t) = \frac{1}{T} e^{-t/T}$ for $t \geq 0$ → which is also the derivative of ~~step~~ step response

Conclusion

* Thus, the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal.
* Response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero output initial condition.

→ These are the property of linear-time invariant systems. Linear time-varying system and non-linear system do not possess this property.

Conclusion of 1st order system —

(1) ~~we~~ We have seen that if $t \to \infty$, the output always tends to a finite value. So, first order system can not be unstable.

(2) First order system is not quick acting as response is exponentially rising.

Step - Response of Second-Order System:

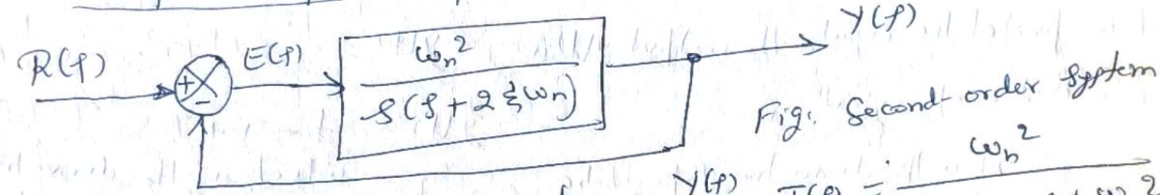


Fig: Second-order system

Closed loop transfer function $= \frac{Y(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

where ξ is the damping ratio and ω_n is the undamped natural frequency.

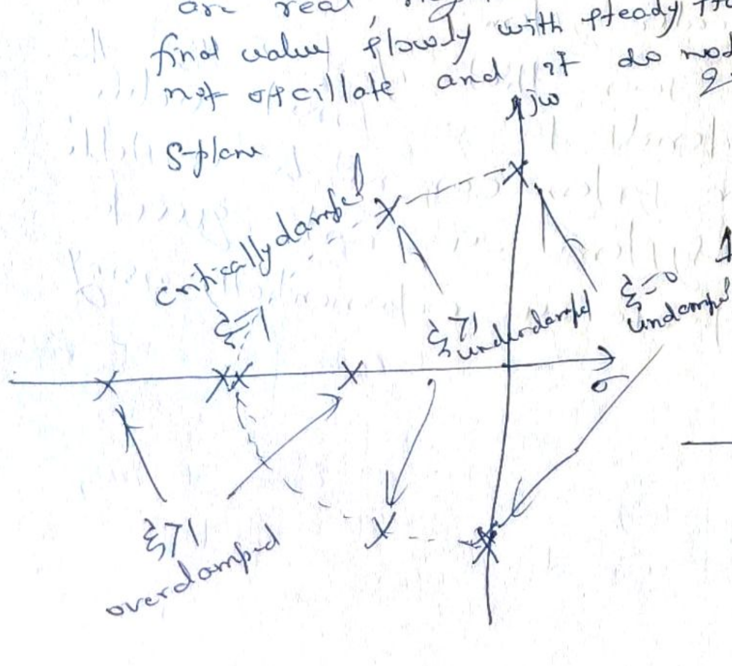
The dynamic behaviour of the 2nd-order system can be described in terms of two parameters ξ and ω_n.

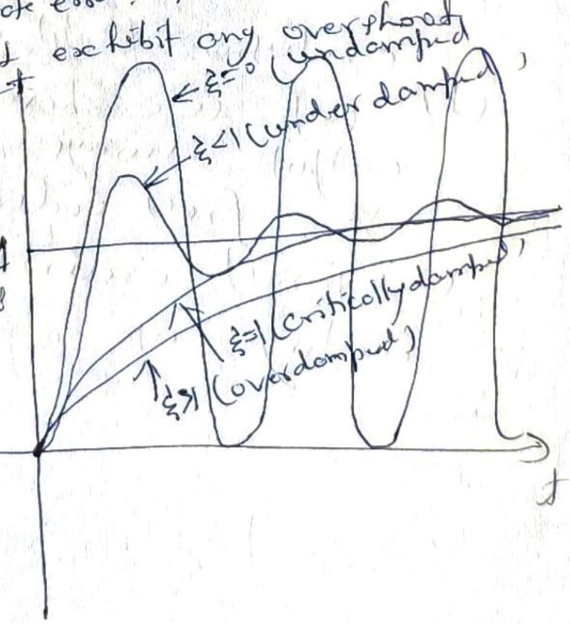
Case (i) If $\xi = 0$, the system is called undamped system and the transient response does not die out. The poles are pure imaginary and lie on the $j\omega$-axis. Response is purely oscillatory.

Case (ii) If $0 < \xi < 1$, the closed-loop poles are complex conjugates and lie in the left half of the s-plane. The system is called underdamped, and the transient response is oscillatory with damping.

Case (iii) If $\xi = 1$, the system is called critically damped. The poles are real, negative and equal. The response rises slowly and reaches the final value. The system do not exhibit any overshoot. Transient response do not oscillate.

Case (iv) If $\xi > 1$, the system is called overdamped system, the poles are real, negative and unequal. The output rises its final value slowly with steady state error. Transient response do not oscillate and it do not exhibit any overshoot.





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Underdamped case ($0 < \xi < 1$)

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{\omega_n^2}{\underbrace{(s+\xi\omega_n+j\omega_d)}\underbrace{(s+\xi\omega_n-j\omega_d)}}$$

For step input $R(s) = \frac{1}{s}$

$$Y(s) = \frac{\omega_n^2}{(s^2+2\xi\omega_n s+\omega_n^2)}\cdot\frac{1}{s}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s+2\xi\omega_n)}{(s+\xi\omega_n+j\omega_d)(s+\xi\omega_n-j\omega_d)}$$

$$= \frac{1}{s} - \frac{(s+2\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{(s+\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

using inverse L.T.

$$y(t) = 1 - \mathcal{L}^{-1}\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right] - \mathcal{L}^{-1}\left[\frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right]$$

$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \mathcal{L}^{-1}\left[\frac{\xi\omega_n}{\omega_d}\,\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]$$

$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}\,\mathcal{L}^{-1}\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]$$

$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}\,e^{-\xi\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\,e^{-\xi\omega_n t}\sin\omega_d t$$

$$\boxed{y(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]}$$

$$\text{or } y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\omega_d t+\theta\right] = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\omega_d t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

for $t \geq 0$

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= \frac{-2\xi\omega_n \pm 2\omega_n\sqrt{\xi^2-1}}{2}$$

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= -\xi\omega_n \pm j\omega_d$$

where $\omega_d = \omega_n\sqrt{1-\xi^2} \rightarrow$ damped natural freq

$F(s)$	$f(t)$
$\frac{\omega}{(s+a)^2+\omega^2}$	$\rightarrow e^{-at}\sin\omega t$
$\frac{s+a}{(s+a)^2+\omega^2}$	$\rightarrow e^{-at}\cos\omega t$
$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$	$\rightarrow \frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\omega_n\sqrt{1-\xi^2}\,t$
$\cancel{\frac{s}{s^2+2\xi\omega_n s+\omega_n^2}}$	$\rightarrow$

$$\mathcal{L}^{-1}\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right] = e^{-\xi\omega_n t}\cos\omega_d t$$

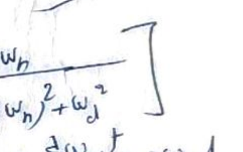
$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right] = e^{-\xi\omega_n t}\sin\omega_d t$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\omega_d^2 = \omega_n^2(1-\xi^2)$$

$$\omega_d^2 = \omega_n^2 - \omega_n^2\xi^2$$



$\cos\theta = \xi$

$\sin\theta = \sqrt{1-\xi^2}$

$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$

$\sin(A+B) = \sin A\cdot\cos B + \cos A\cdot\sin B$

(10-A)

error signal $= e(t) = r(t) - y(t) = 1 - y(t)$

$$= e^{-\xi \omega_n t}\left[\cos \omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t\right] \quad \text{for } t \geq 0$$

* The error signal exhibits a damped sinusoidal oscillation.
* At steady state or $t \to \infty$, no error exists between the input & output

ie $\lim_{t \to \infty} e(t) = e_{ss} = 0$

* If $\xi = 0$, $\omega_d = \omega_n\sqrt{1-\xi^2} = \omega_n \rightarrow$ the response becomes undamped, $\boxed{\omega_d = \omega_n}$ for $\xi = 0$ and oscillation will continue indefinitely
* For $\xi = 0$, response $\boxed{y(t) = 1 - \cos \omega_n t}$
* ω_d is always lower than the ω_n
* An increase in ξ would reduce the damped natural freq ω_d
* If ξ is increased beyond unity, the response becomes overdamped and will not oscillate.

$\omega_d = \omega_n$ for $\xi = 0$

$\omega_d < \omega_n$ for $\xi > 0$

Impulse Response of 2nd order system for $0<\xi<1$

$$Y(s) = \frac{\omega_n^2}{(s+\xi\omega_n - j\omega_d)(s+\xi\omega_n+j\omega_d)} \cdot R(s)$$

for impulse $R(s) = 1$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$= \frac{A}{s+\xi\omega_n - j\omega_d}\Bigg|_{s=-\xi\omega_n+j\omega_d} + \frac{B}{(s+\xi\omega_n+j\omega_d)}\Bigg|_{s=-\xi\omega_n-j\omega_d}$$

$$A = \frac{\omega_n^2}{s+\xi\omega_n+j\omega_d}\Bigg|_{s=-\xi\omega_n+j\omega_d} = \frac{\omega_n^2}{-\cancel{\xi\omega_n}+j\omega_d+j\omega_d\cancel{+\xi\omega_n}} = \frac{\omega_n^2}{2j\omega_d}$$

$$= \frac{\omega_n^2}{2j\omega_n\sqrt{1-\xi^2}} = \frac{\omega_n}{2j\sqrt{1-\xi^2}}$$

$$B = \frac{\omega_n^2}{s+\xi\omega_n-j\omega_d}\Bigg|_{s=-\xi\omega_n-j\omega_d} = \frac{\omega_n^2}{-\cancel{\xi\omega_n}-j\omega_d+\cancel{\xi\omega_n}-j\omega_d} = -\frac{\omega_n^2}{2j\omega_d}$$

$$= -\frac{\omega_n^2}{2j\omega_n\sqrt{1-\xi^2}} = -\frac{\omega_n}{2j\sqrt{1-\xi^2}}$$

$$Y(s) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left\{\frac{1}{s+\xi\omega_n-j\omega_d} - \frac{1}{s+\xi\omega_n+j\omega_d}\right\}$$

$$\cancel{= \frac{\omega_n}{2j\sqrt{1-\xi^2}}\cdot\frac{1}{\omega_d}\left\{\frac{\omega_d}{s+\xi\omega_n-j\omega_d} - \frac{\omega_d}{s+\xi\omega_n+j\omega_d}\right\}}$$

$$\cancel{= \frac{\omega_n}{2j\sqrt{1-\xi^2}}\cdot\frac{1}{\omega_n\sqrt{1-\xi^2}}\left\{\frac{\omega_d}{s+(\xi\omega_n-j\omega_d)} - \frac{\omega_d}{s+(\xi\omega_n+j\omega_d)}\right\}}$$

$$~~y(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left\{e^{-(\xi\omega_n+j\omega_d)t} - e^{-\xi\omega_n}\right.~~$$

$$y(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left\{e^{(-\xi\omega_n+j\omega_d)t} - e^{-(\xi\omega_n+j\omega_d)t}\right\}$$

$$= \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left\{e^{-\xi\omega_n t}\cdot e^{j\omega_d t} - e^{-\xi\omega_n t}\cdot e^{-j\omega_d t}\right\}$$

$$= \frac{\omega_n}{2j\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t}\left\{e^{j\omega_d t} - e^{-j\omega_d t}\right\}$$

$$= \frac{\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t}\left(\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right)$$

$$= \frac{\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t}\cdot \sin\omega_d t$$

or $\boxed{y(t) = \frac{\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t}\cdot \sin\left(\omega_n\sqrt{1-\xi^2}\right)t}$ for $t \geq 0$

Q: Find the impulse response of the second order system whose transfer function $G(s) = \frac{9}{s^2+4s+9}$

Ans: method 1. comparing $G(s) = \frac{9}{s^2+4s+9} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 9$

$\Rightarrow \omega_n = 3,\quad 2\xi\omega_n = 4 \Rightarrow \xi = \frac{4}{2\omega_n} = \frac{4}{2\times 3} = \frac{2}{3} = 0.667$

Impulse response $= y(t) = \frac{\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t} \sin(\omega_n\sqrt{1-\xi^2})t$

or $y(t) = \frac{3}{\sqrt{1-(0.667)^2}}\cdot e^{-\frac{2}{3}\times 3t}\cdot \sin\left(3\sqrt{1-(0.667)^2}\right)t$

$= 4.025\, e^{-2t}\cdot \sin 2.235\, t$

method 2 Roots of s^2+4s+9 are $s_1, s_2 = \frac{-4 \pm \sqrt{4^2-4\times 9}}{2} = -2 \pm j2.236$

$$G(s) = \frac{9}{\{s-(-2+j2.236)\}\{s-(-2-j2.236)\}}$$

$Y(s) = G(s)$ since $R(s) = 1$

$$Y(s) = \frac{A}{s-(-2+j2.236)} + \frac{A^*}{s-(-2-j2.236)}$$

$$= \frac{-j2.0125}{s-(-2-j2.236)} + \frac{j2.0125}{s-(-2-j2.236)}$$

$$\therefore y(t) = -j2.0125\, e^{(-2+j2.236)t} + j2.0125\, e^{(-2-j2.236)t}$$

$$= 4.025\, e^{-2t} \sin 2.236t$$

H.W Q2. Find the impulse response of a second order system whose transfer function is

(i) $G(s) = \dfrac{4}{s^2+4s+4}$ Ans $4t\, e^{-2t}$

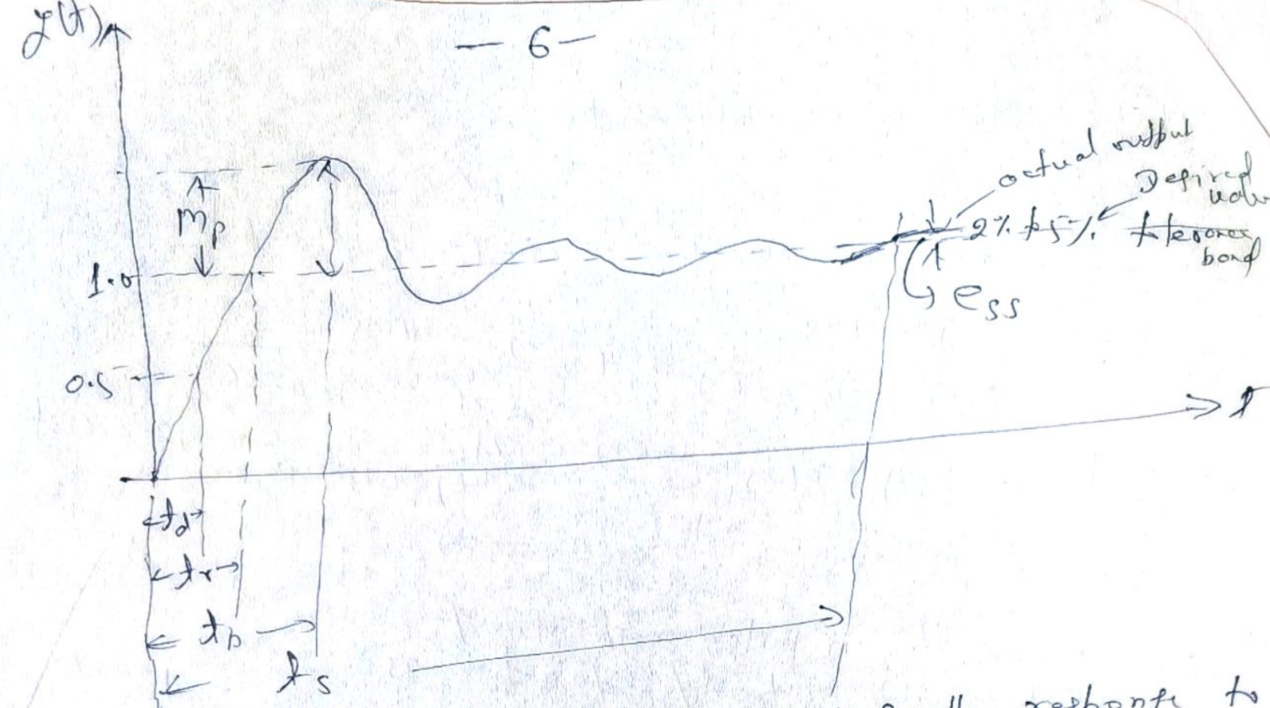
(ii) $G(s) = \dfrac{4}{s^2+2s+4}$ Ans $2.3\, e^{-t} \sin(1.732t)$

(iii) $G(s) = \dfrac{4}{s^2+8s+4}$ Ans $1.15\, e^{-4t} \sinh(2\sqrt{3}t)$

Q. Define and discuss transient response specifications of 2nd order system or time domain specification.

Ans. The performance of a control system are express in terms of the transient response to a unit step input because it is easy to generate. The transient response of a control system to a unit step input depends upon the initial conditions. Hence consider a 2nd order system with unit-step input and the system is initially at rest i.e. all initial conditions are zero. Common transient response specifications are

(1) Delay time (t_d)
(2) Rise time (t_r)
(3) Peak time (t_p)
(4) maximum overshoot (M_p)
(5) Settling time (t_s)
(6) Steady state error (e_{ss})



(1) Delay time, t_d → Time required for the response to reach 50% of the final value in first time

(2) Rise time, t_r → Time required for the response to rise from 0 to 100% of its final value in the first time for underdamped system and 10% to 90% for overdamped system.

(3) Peak time, t_p → Time required for the response to reach the first peak of the time response or first peak overshoot.

(4) Maximum Overshoot, M_p :— It is the normalized difference between the peak of the time response and steady state output. Maximum percentage overshoot is defined by $\frac{y(t_p) - y(\infty)}{y(\infty)} \times 100$

(5) Settling time (t_s) :— Time required for the response to reach and stay within the specified range (2% to 5%) of its final value.

(6) Steady state error (e_{ss}) → It is the difference between actual output and desired output as time 't' tends to infinity

$$e_{ss} = \lim_{t \to \infty} [r(t) - y(t)] = \lim_{s \to 0} s\,E(s)$$

Expression for time domain specification

(i) Rise time, t_r,

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega_d t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

Put $y(t) = 1$

$$\cancel{1} = \cancel{1} - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega_d t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

or

$$\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega_d t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right] = 0$$

since $e^{-\xi\omega_n t} \neq 0$

$$\therefore \sin\left[\omega_d t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right] = 0 = \sin n\pi$$

Put $n = 1$

$$\therefore \omega_d t_r + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \pi$$

$$\therefore t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_d\ (= \omega_n\sqrt{1-\xi^2})}$$

(ii) Peak time t_p

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[\omega_d t + \phi]$$

where $\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$

for maximum, $\frac{dy(t)}{dt} = 0$

$$\frac{dy}{dt} = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos\left[(\omega_n\sqrt{1-\xi^2})t + \phi\right]\omega_n\sqrt{1-\xi^2} + \sin\left[(\omega_n\sqrt{1-\xi^2})t + \phi\right]\frac{\xi\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} = 0$$

since $e^{-\xi\omega_n t} \neq 0$

on solving $\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$

First minimum (under shoot) occurs at $n = 2$

(iii) Expression for Maximum Overshoot: M_p

Maximum overshoot occurs at peak time i.e. $t = t_p$

Put $t = t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

$$y(t) = 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2} \cdot \frac{\pi}{\omega_n\sqrt{1-\xi^2}} + \phi\right]$$

~~when~~ Since $\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$ then $\sin\phi = \sqrt{1-\xi^2}$ & $\sin(\pi+\phi) = -\sin\phi$

$$= 1 - \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\pi+\phi)$$

$$\therefore y(t) = 1 + \frac{e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \sin\phi$$

$$y(t)_{max} = 1 + e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$\therefore M_p = y(t)_{max} - 1 = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$\boxed{\%\, M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100}$$

(V) Steady state error e_{ss}

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} [1 - y(t)]$$

$$= \lim_{t\to\infty}\left[\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)\right]$$

for a ramp input $R(s) = \frac{1}{s^2}$

$$e_{ss} = \frac{2\xi}{\omega_n} = \frac{1}{K_a}$$

$$e_{ss} = \lim_{t\to\infty} [r(t) - y(t)] = 0 \text{ for step input}$$

(IV) Settling time:

$t_s = \frac{4}{\xi\omega_n} = \frac{4}{\sigma}$ for 2% criteria (of reach its max for $\xi = 0.76$)

$t_s = \frac{3}{\sigma} = \frac{3}{\xi\omega_n}$ for 5% criteria (of reach its max for $\xi = 0.68$)

t_s is inversely proportional to ξ & ω_n

Q1 When a 2nd order control system is subjected to a unit step input, the value of $\xi = 0.5$ and $\omega_n = 6$ rad/sec. Determine the rise time, peak time, settling time and a peak overshoot.

Soln. $\xi = 0.5$, $\omega_n = 6$ rad/sec

(i) Rise time: $t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi - \tan^{-1}\frac{\sqrt{1-(0.5)^2}}{0.5}}{6\sqrt{1-(0.5)^2}} = \frac{3.14 - 1.047}{5.19}$

$= 0.403$ sec

(ii) Peak time, $t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{6\sqrt{1-(0.5)^2}} = 0.605$ sec

(iii) Settling time $t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.5 \times 6} = 1.33$ sec

(IV) Maximum overshoot: $M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100 = e^{-\frac{\pi \times 0.5}{\sqrt{1-(0.5)^2}}} \times 100$

$M_p = 0.163 \times 100 = 16.3\%$

Q. Obtain the unit-impulse response and the unit step response of a unity feedback system whose open-loop transfer function is

$$G(s) = \frac{2s+1}{s^2}$$

Soln. closed loop tr.fn $= \frac{Y(s)}{R(s)} = \frac{(2s+1)/s^2}{1+(2s+1)/s^2} = \frac{(2s+1)}{(s^2+2s+1)} = \frac{2s+1}{(s+1)^2}$

For unit impulse response $R(s) = 1$,

$$\therefore Y(s) = \frac{(2s+1)}{(s+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{(2s+1)}{(s+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{A}{s+1} + \frac{B}{(s+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{2}{s+1} - \frac{1}{(s+1)^2}\right]$$

or $y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = 2e^{-t} - te^{-t}$ Ans

For unit step response; $R(s) = \frac{1}{s}$

$$Y(s) = \frac{(2s+1)}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$= \frac{1}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$y(t) = 1 - e^{-t} + te^{-t}$$

Q. The open-loop transfer function of a system with unity feedback gain is given as $G(s) = \frac{20}{s^2+5s+6}$. Determine the damping ratio, maximum overshoot, rise time and peak time.

Ans $\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{20/(s^2+5s+6)}{1+\frac{20}{s^2+5s+6}} = \frac{20}{s^2+5s+26}$

comparing with $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \Rightarrow \omega_n^2 = 26$; $2\xi\omega_n = 5$

$\omega_n = \sqrt{26}$; $\xi = \frac{5}{2\sqrt{26}} = 0.49$

$\xi = 0.49$

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = e^{\frac{-\pi 0.49}{\sqrt{1-0.49^2}}} \times 100 = 17.1\%$

$t_r = \frac{\pi - \phi}{\omega_d}$ where $\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-(0.49)^2}}{0.49} = 1.059$ rad

$t_r = \frac{\pi - 1.059}{\omega_d} = \frac{\pi - 1.059}{\omega_n\sqrt{1-\xi^2}} = 468.53 \times 10^{-3}$ sec

$= 468.53$ msec

$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\sqrt{26}\sqrt{1-(0.49)^2}} = 706.78 \times 10^{-3}$ sec

$= 706.78$ msec

(01 - Error)

Error Analysis

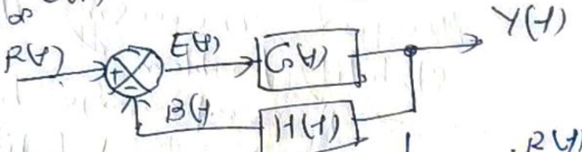
→ Steady State Error Analysis

→ Dynamic error Analysis or Generalized error Analysis

Steady State Error Analysis:

Steady State error: It is defined as the difference between output and input when $t \to \infty$. It is denoted by e_{ss} or $\lim_{t\to\infty} e(t)$ i.e. $e_{ss} = \lim_{t\to\infty} e(t)$

For a feedback system:



Error transfer function $= \dfrac{E(s)}{R(s)} = \dfrac{1}{1+G(s)H(s)} \Rightarrow E(s) = \dfrac{1}{1+G(s)H(s)} \cdot R(s)$

$\therefore e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\dfrac{R(s)}{1+G(s)H(s)}$ → using final value theorem

Static Error Constants: These are figures of merit of control systems. The higher the constants, the smaller is the steady-state error.

Order & type of the system.

Order → The highest power of the complex variable s in the denominator of the transfer function is called the order of the system.

$$G(s) = \frac{K}{s^2+4s+9}$$ → order is 2

Type: control system may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs and so on. The magnitude of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

The open loop tr. fn. of a unity feedback system can be written in two standard forms

ⓐ ~~Time constant form~~ Pole zero form

$$G(s)H(s) = \frac{K'(s+z_1)(s+z_2)\cdots(s+z_m)}{s^n(s+p_1)(s+p_2)\cdots(s+p_p)}$$

ⓑ Time constant form:

$$G(s)H(s) = \frac{\overset{\leftarrow \text{gain}}{K}(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^n(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

The gains of the two forms are related by $K = K' \dfrac{\prod_{i=1}^{m} z_i}{\prod_{j=1}^{p} p_j}$

(02-Er)

In the above two representation S^n in the denominator represents a pole of multiplicity 'n' at the origin.

Def of Type :- No. of integrations indicated by the open loop tr. fn.

If $n=0 \rightarrow$ type 0, $n=1 \rightarrow$ type-1, $n=2 \rightarrow$ type 2 ... and so on

* As the type is increased, accuracy is improved. However, increasing the type number aggravates the stability problem.
→ A compromise between steady-state accuracy and relative stability is always necessary.
→ In practical system, it is rare to have type-3 or higher system, because it is generally difficult to design stable systems having more than two integrations in the forward path.
* Open-loop gain K is directly related to steady-state error.

Static Error Constants :-

(i) Static Position Error Constants (K_p)
or Position error coefficients.

It is defined using step-input.

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+G(s)H(s)} R(s)$$

for step input $R(s) = \frac{1}{s}$

$$\therefore\ e_{ss} = \lim_{s\to 0} \cancel{s} \frac{1}{1+G(s)H(s)} \cdot \frac{1}{\cancel{s}} = \lim_{s\to 0} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\lim_{s\to 0} G(s)H(s)}$$

or $\boxed{e_{ss} = \frac{1}{1+K_p}}$

where $\boxed{K_p = \lim_{s\to 0} G(s)H(s)}$

$$K_p = \frac{1}{e_{ss}} - 1 = \frac{1-e_{ss}}{e_{ss}}$$

For a type-0 system:

$$K_p = \lim_{s\to 0} \frac{K(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = K$$

For a type-1 system:

$$K_p = \lim_{s\to 0} \frac{K(T_a s+1)(T_b s+1)\cdots}{s(T_1 s+1)(T_2 s+1)\cdots} = \infty$$

* For a type-0 system, the static position error constant K_p is finite, while for a type-1 or higher system K_p is infinite.

(03-Error)

For a unit step input e_{ss} may be of.

for type-0 system, $e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K}$

" type-1 or higher system, $e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$

* Thus, the response of a feedback control system to a step input involves a steady state error if there is no integration in the forward path.

→ If a small errors for step inputs can be tolerated, then a type-0 system may be permissible, provided that the gain K is sufficiently large.

→ If the gain K is too large, it is difficult to obtain reasonable relative stability.

→ If zero steady state error for a step input is desired, the type of the system may be one or higher.

ii) Static - Velocity Error Constant Or Coefficient (K_v)

It is defined using ramp input. For ramp input $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\,\frac{1}{1+G(s)H(s)}\,R(s) = \lim_{s\to 0} s\,\frac{1}{1+G(s)H(s)}\,\frac{1}{s^2}$$

or $$e_{ss} = \lim_{s\to 0} \frac{1}{s+sG(s)H(s)} = \frac{1}{\lim_{s\to 0} sG(s)H(s)} = \frac{1}{K_v}$$

i.e. $$\boxed{e_{ss} = \frac{1}{K_v}} \quad \text{where} \quad \boxed{K_v = \lim_{s\to 0} sG(s)H(s)}$$

* The term velocity error is used to express the steady-state error for a ramp input. The dimension of the velocity error is the same of the system error. That is, velocity error is not an error in velocity, but it is an error in position due to a ramp input.

For a type-0 system

$$K_v = \lim_{s\to 0} \frac{s\,K(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = 0 \implies e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

For a type-1 system

$$K_v = \lim_{s\to 0} s\,\frac{K(T_a s+1)(T_b s+1)\cdots}{s(T_1 s+1)(T_2 s+1)\cdots} = K \implies e_{ss} = \frac{1}{K_v} = \frac{1}{K} \to \text{finite value}$$

For a type-2 system

$$K_v = \lim_{s\to 0} s\,\frac{K(T_a s+1)(T_b s+1)\cdots}{s^2(T_1 s+1)(T_2 s+1)\cdots} = \infty \implies e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

(04 - Error)

* Thus, a type-0 system is incapable of following a ramp in the steady-state.
* Type-1 system with unity feedback can follow the ramp input with a finite error
* Type-2 or higher system can follow a ramp input with zero actuating error at steady state.

(iii) Static acceleration error constant or coefficient (K_a)

It is defined using parabolic input.

i.e. $r(t) = \frac{t^2}{2}$ for $t \geq 0 \implies R(s) = \frac{1}{s^3}$

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} s \frac{1}{1+G(s)H(s)} \cdot R(s) = \lim_{s \to 0} s \frac{1}{1+G(s)H(s)} \cdot \frac{1}{s^3}$$

$$\text{or } e_{ss} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)H(s)} = \frac{1}{\lim_{s \to 0} s^2 G(s)H(s)} = \frac{1}{K_a}$$

$$\boxed{e_{ss} = \frac{1}{K_a}} \quad \text{where} \quad \boxed{K_a = \lim_{s \to 0} s^2 G(s)H(s)}$$

* Thus, the acceleration error, is the steady state error in position error due to a parabolic input in an error in position

For a type-0 system:

$$K_a = \lim_{s \to 0} s^2 \frac{K(T_a s+1)(T_b s+1) - -}{(T_1 s+1)(T_2 s+1) - -} = 0 \implies e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

For a type-1 system:

$$K_a = \lim_{s \to 0} s^2 \frac{K(T_a s+1)(T_b s+1) - -}{s(T_1 s+1)(T_2 s+1) - -} = 0 \implies e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

For a type-2 system:

$$K_a = \lim_{s \to 0} s^2 \frac{K(T_a s+1)(T_b s+1) - -}{s^2(T_1 s+1)(T_2 s+1) - -} = K \implies e_{ss} = \frac{1}{K_a} = \frac{1}{K} = \text{finite}$$

For a type-3 system:

$$K_a = \lim_{s \to 0} s^2 \frac{K(T_a s+1)(T_b s+1) - -}{s^3(T_1 s+1)(T_2 s+1) - -} = \infty \implies e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

* Thus, type-0 and type-1 systems are incapable of following a parabolic input in the steady state.
* The type-2 system with unity feedback can follow a parabolic input with a finite actuating error signal

Table - Steady State error (SS-Error) in terms of Gain K

Type	step input $r(t) = 1$	ramp input $r(t) = t$	parabolic input $r(t) = \frac{t^2}{2}$
Type-0-System	$\frac{1}{1+K}$	∞	∞
Type-1 System	0	$\frac{1}{K}$	∞
Type-2 System	0	0	$\frac{1}{K}$

* In table, the finite values for steady-state error appears on the diagonal line. Above the diagonal the steady state errors are infinite, below the diagonal, they are zero

* The error constants K_p, K_v & K_a describes the ability of a system to reduce or eliminate steady-state error. Therefore, they are indicative of the steady-state performance.

* It is generally desirable to increase the error constants, while maintaining the transient response within an acceptable range

* If there is any conflict between the static velocity error constant and the static ~~error~~ acceleration error constant, then the latter may be considered less important than the former,

* To improve the steady state performance we can increase the type of the system by adding an integrator or integrators to the forward path → This introduces additional stability problem,

* The design of a satisfactory system with more than two integrators in series in the forward path is generally difficult.

Disadvantages of Error Constant method

(i) The error constant method does not apply to the system with inputs that is sinusoidal, since the final value theorem cannot apply to these cases.

(ii) Static error constants do not give any information on the steady state error when the inputs are other than three basic types i.e. step, ramp & parabolic

(iii) Static error constants fail to indicate the exact manner in which the error function changes with time.

(13)

overshoot when the system is subjected to a step-input.

Error Series Or Dynamic Error Co-efficient or Generalized Error co-efficient

In this, the error-constant concept is generalised to include inputs of almost any arbitrary function of time. Error-series is a more versatile representation of the steady-state-error.

Disadvantages of error constant method: — (1) When the steady-state error is infinite, which actually is due to the error increases with time; the error-constant method does not indicate how the error varies with time.

(2) The error constant method does not apply to systems with inputs that are sinusoidal, since the final value theorem cannot applied to these cases.

Error transfer function: Since $E(S) = \frac{R(S)}{1+G(S)H(S)}$ — (1)

Here the transform error function E(S) which could be of the form (1) or any function that is defined as the system error.

Let us express E(S) as:-

$$E(S) = W_e(S)\,R(S) \quad — (2)$$

where R(S) is the Laplace transform of the input r(t), and We(S) is the error-transfer function.

By using real convolution integral, the error signal e(t) is

$$e(t) = \int_{-\infty}^{t} W_e(\tau)\, r(t-\tau)\, d\tau \quad — (3)$$

where $w_e(\tau)$ is the inverse Laplace transform of $W_e(s)$.

If the first n derivatives of $r(t)$ exist for all values of t, the function $r(t-\tau)$ can be expanded into a Taylor series; that is,

$$r(t-\tau) = r(t) - \tau r^{(1)}(t) + \frac{\tau^2}{2!} r^{(2)}(t) - \frac{\tau^3}{3!} r^{(3)}(t) + \cdots \quad \text{--- (4)}$$

where $r^{(i)}(t)$ denotes the i^{th}-order derivative of $r(t)$ with respect to time.

Since $r(t)$ is considered to be zero for negative time, the limits of convolution integral in eq (3) may be taken from 0 to t.

Substituting in eq (3), we have

$$e(t) = \int_0^t w_e(\tau)\left[r(t) - \tau r^{(1)}(t) + \frac{\tau^2}{2!} r^{(2)}(t) - \frac{\tau^3}{3!} r^{(3)}(t) + \cdots\right] d\tau$$

$$= r(t)\int_0^t w_e(\tau)\, d\tau - r^{(1)}(t)\int_0^t \tau\, w_e(\tau)\, d\tau + r^{(2)}(t)\int_0^t \frac{\tau^2}{2!} w_e(\tau)\, d\tau - \cdots \quad \text{--- (5)}$$

Let $r_s(t)$ represents the steady-state part of the input $r(t)$, and $e_s(t)$ be the steady state part of ~~the~~ $e(t)$ that is due to $r_s(t)$. Then the steady state error function, $e_s(t)$, may be written as

$$e_s(t) = r_s(t)\int_0^\infty w_e(\tau)\, d\tau - r_s^{(1)}(t)\int_0^\infty \tau\, w_e(\tau)\, d\tau + r_s^{(2)}(t)\int_0^\infty \frac{\tau^2}{2!} w_e(\tau)\, d\tau - \cdots \quad \text{--- (6)}$$

Let us define:

$$\left.\begin{aligned} C_0 &= \int_0^\infty w_e(\tau)\, d\tau \\ C_1 &= -\int_0^\infty \tau\, w_e(\tau)\, d\tau \\ C_2 &= \int_0^\infty \tau^2 w_e(\tau)\, d\tau \\ &\vdots \\ C_n &= (-1)^n \int_0^\infty \tau^n w_e(\tau)\, d\tau \end{aligned}\right\} \quad \text{--- (7)}$$

(15)

eq. (6) can be written as.

$$e_s(t) = C_0 r_s(t) + C_1 r_s^{(1)}(t) + \frac{C_2}{2!} r_s^{(2)}(t) + \cdots + \frac{C_n}{n!} r_s^{(n)}(t) + \cdots \quad (8)$$

which is called the <u>error series</u>, and the coefficients $C_0, C_1, C_2, \ldots, C_n$ are called the <u>generalized error coefficients</u> or simply the <u>error coefficients</u>.

The error coefficients may be readily evaluated from the error transfer function $W_e(s)$. Since $W_e(s)$ and $w_e(\tau)$ are related through the Laplace transform, we have

$$W_e(s) = \int_0^\infty w_e(\tau)\, e^{-\tau s}\, d\tau \quad (9)$$

Taking limit on both sides of eq. (9) as s approaches zero, we have

$$\lim_{s \to 0} W_e(s) = \lim_{s \to 0} \int_0^\infty w_e(\tau)\, e^{-\tau s}\, d\tau \quad (10)$$

Taking the derivative of $W_e(s)$ of eq. (9) with respect to s, we get

$$\frac{dW_e(s)}{ds} = -\int_0^\infty \tau\, w_e(\tau)\, e^{-\tau s}\, d\tau \quad (11)$$

from which the error coefficient C_1 is determined as

$$C_1 = \lim_{s \to 0} \frac{dW_e(s)}{ds} \quad (12)$$

and others. —

$$C_2 = \lim_{s \to 0} \frac{d^2 W_e(s)}{ds^2} \quad (13)$$

$$C_3 = \lim_{s \to 0} \frac{d^3 W_e(s)}{ds^3} \quad (14)$$

$$\vdots$$

$$C_n = \lim_{s \to 0} \frac{d^n W_e(s)}{ds^n} \quad (15)$$

— o —

Alternatively:

Error-transfer function $W_e(S) = \frac{E(S)}{R(S)}$

$$= \frac{1}{K_1} + \frac{1}{K_2} s + \frac{1}{K_3} s^2 + \frac{1}{K_4} s^3 + \cdots$$

Arrange E & R in the ascending power of s and divide them, the coefficient which are get are the dynamic error ~~function coefficients~~ function.

Here K_1, K_2, K_3, K_4 --- are dynamic error coefficients

K_1 —— dynamic position error coefficient
K_2 -- " velocity " "
K_3 -- " acceleration " "

Q-1 Find the (a) static and (b) dynamic error-coefficient with unity feedback control system having the forward path tr. fn. of

(i) $G_1(S) = \frac{K}{TS+1}$ (ii) $G_2(S) = \frac{\omega_n^2}{S^2 + 2\xi\omega_n S}$

Soln (a) Static error coefficient

(i) $K_p = \lim_{S\to 0} G_1(S) = \lim_{S\to 0} \frac{K}{TS+1} = K$

$K_v = \lim_{S\to 0} S\,G_1(S) = \lim_{S\to 0} S \cdot \frac{K}{TS+1} = 0$

$K_a = \lim_{S\to 0} S^2 G_1(S) = \lim_{S\to 0} S^2 \frac{K}{TS+1} = 0$

(ii) $K_p = \lim_{S\to 0} G_2(S) = \lim_{S\to 0} \frac{\omega_n^2}{S(S+2\xi\omega_n)} = \infty$

$K_v = \lim_{S\to 0} S\,G_2(S) = \lim_{S\to 0} \frac{\cancel{S}\cdot\omega_n^2}{\cancel{S}(S+2\xi\omega_n)} = \frac{\omega_n}{2\xi}$

$K_a = \lim_{S\to 0} S\,G_2(S) = \lim_{S\to 0} \frac{S^2\,\omega_n^2}{S(S+2\xi\omega_n)} = 0$

(b) Dynamic error coefficient:

(i) $G_1(S) = \frac{K}{TS+1}$

Since $W_e(S) = \frac{E(S)}{R(S)} = \frac{1}{1+G(S)H(S)}$

Here $H(S) = 1$

$\Rightarrow W_e(S) = \frac{E(S)}{R(S)} = \frac{1}{1+G_1(S)} = \frac{1}{1+\frac{K}{TS+1}}$

$= \frac{1+TS}{1+K+TS}$

$= \frac{1}{1+K} + \frac{TK}{(1+K)^2}S + \cdots$

$$(1+K)+TS \,)\, 1+TS \,(\, \frac{1}{1+K} + \frac{TK}{(1+K)^2}S + \cdots$$
$$\cancel{1} + \frac{TS}{1+K}$$
$$TS\left(1-\frac{1}{1+K}\right)$$
$$\frac{TKS}{1+K}$$
$$\cancel{\frac{TKS}{1+K}} + \frac{T^2KS^2}{(1+K)^2}$$
$$-\frac{T^2KS^2}{1+K^2}$$

$\therefore \frac{1}{K_1} = \frac{1}{1+K} \Rightarrow K_1 = 1+K_p$

$\frac{1}{K_2} = \frac{TK}{(1+K)^2}$

$\vdots$

(ii) $W_e(S) = \frac{E(S)}{R(S)} = \frac{1}{1+G_2(S)} = \frac{1}{1+\frac{\omega_n^2}{S^2+2\xi\omega_n S}}$

$= \frac{2\xi\omega_n S + S^2}{\omega_n^2 + 2\xi\omega_n S + S^2}$

$= \frac{2\xi}{\omega} S + \frac{1-4\xi^2}{\omega_n^2} S^2 + \cdots$

$\therefore \frac{1}{K_1} = 0 \longrightarrow K_1 = \infty$

$\frac{1}{K_2} = \frac{2\xi}{\omega_n} \longrightarrow K_2 = \frac{\omega_n}{2\xi}$

$\frac{1}{K_3} = \frac{1-4\xi^2}{\omega_n^2} \longrightarrow K_3 = \frac{\omega_n^2}{1-4\xi^2}$

(18)

Q2 Find the static error coefficient, dynamic error - coefficient of a unity feedback control system with forward path transfer function

$$G(S) = \frac{10}{S(S+1)}$$

due to an input described by $r(t) = a_0 + a_1 t + a_2 t^2$. Find also the steady state error, if

Soln Steady state error

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{S\to 0} S E(S)$$

Step I Evaluate $\frac{E(S)}{R(S)}$ for the given problem.

$$\frac{E(S)}{R(S)} = \frac{1}{1+G(S)} \quad \text{for unity feedback system}$$

put G

$$= \frac{1}{1+\frac{10}{S(S+1)}} = \frac{S(S+1)}{S(S+1)+10}$$

Step II Arrange in ascending order of S

$$\frac{E(S)}{R(S)} = \frac{S+S^2}{10+S+S^2}$$

$$= 0.1S + 0.09S^2 - 0.01S^3 + \cdots$$

Compare the coefficient of S in this problem to the standard form which is given below.

$$\frac{E(S)}{R(S)} = \frac{1}{K_1} + \frac{1}{K_2}S + \frac{1}{K_3}S^2 + \frac{1}{K_4}S^3 + \cdots$$

$$\therefore \frac{1}{K_1} = 0 \rightarrow K_1 = \infty$$

$$\frac{1}{K_2} = 0.1 \rightarrow K_2 = 10$$

$$\frac{1}{K_3} = 0.09 \rightarrow K_3 = 11.11$$

$$\frac{1}{K_4} = 0.01 \rightarrow K_4 = 100$$

$$\frac{E(S)}{R(S)} = \frac{1}{K_1} + \frac{1}{K_2}S + \frac{1}{K_3}S^2 + \frac{1}{K_4}S^3 + \cdots$$

Suppose: $E(S) = \frac{1}{K_1}R(S) + \frac{1}{K_2}S R(S) + \frac{1}{K_3}S^2 R(S) + \cdots$

Taking inverse Laplace transform of both sides

$$e(t) = \frac{1}{K_1}r(t) + \frac{1}{K_2}\frac{dr(t)}{dt} + \frac{1}{K_3}\frac{d^2 r(t)}{dt^2} + \cdots$$

$\therefore$ Steady state error in terms of dynamic error coefficients

(19)

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left\{ \frac{1}{K_1} r(t) + \frac{1}{K_2}\frac{dr(t)}{dt} + \frac{1}{K_3}\frac{d^2 r(t)}{dt^2} \right\}$$

$$r(t) = a_0 + a_1 t + a_2 t^2$$

$$\therefore \frac{dr(t)}{dt} = a_1 + 2a_2 t$$

$$\frac{d^2 r(t)}{dt^2} = 2a_2$$

Since $\frac{1}{K_1} = 0$, $\frac{1}{K_2} = 0.1$, $\frac{1}{K_3} = 0.09$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left\{ \frac{1}{K_1}(a_0 + a_1 t + a_2 t^2) + \frac{1}{K_2}(a_1 + 2a_2 t) + \frac{1}{K_3} 2a_2 + \dots \right\}$$

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left\{ 0 + 0.1(a_1 + 2a_2 t) + 0.09(2a_2) \dots \right\}$$

$$= \infty$$

$$\therefore e_{ss} = \lim_{t\to\infty} e(t) = \infty$$

So error is infinity.

For $e_{ss} \neq \infty$, we remove $a_2 t^2$ from the general input.

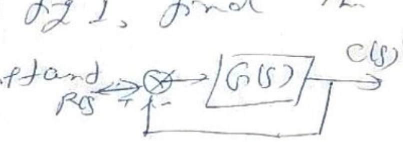
So input can only be of the type $a_0 + a_1 t$ of $a_2 t^2$

Q. Find the static & dynamic error constants, or Evaluate the steady-state error of a feedback control system by the use of the error-constants and the error-series methods,

$$G(s) = \frac{K}{s+1}$$

Soln.

Q: For the stable system shown in fig 1, find the position, velocity and error constants when $G(s) = \dfrac{4(s+2)}{s(s+1)(s+4)}$



Soln: It is a unity feedback system for which,

$$G(s) = \frac{4(s+2)}{s(s+1)(s+4)}$$

Hence $K_p = \lim_{s \to 0} G(s) = \infty$

$$K_v = \lim_{s \to 0} sG(s) = 2$$

$$K_a = \lim_{s \to 0} s^2 G(s) = 0$$

Q: For a unity feedback system, the open loop transfer function is $G(s) = \dfrac{K(s+S_1)}{s^2(s+S_2)(s+S_3)}$

(a) State the type of the system

(b) Find K_p, K_v and K_a.

(c) Find the steady state error due to an input which is described as

$$r(t) = (R_1 + R_2 t + R_3 t^2)\, u_{-1}(t)$$

Soln: (a) It is a type 2 system.

(b) $K_p = \lim_{s \to 0} G(s) = \infty$; $K_v = \lim_{s \to 0} s\,G(s) = \infty$,

$$K_a = \lim_{s \to 0} s^2 G(s) = \frac{K S_1}{S_2 S_3} = K' \text{ (constant)}$$

(c) The input can be considered as a summation of step, ramp and parabolic inputs. Using superposition theorem,

For the input $R_1 u_{-1}(t)$; $e_{ss1} = \dfrac{R_1}{1+K_p} = 0$

" " input $R_2 t\, u_{-1}(t)$; $e_{ss2} = \dfrac{R_2}{K_v} = 0$

" " " $R_3 t^2 u_{-1}(t)$, $e_{ss3} = \dfrac{2R_3}{K_a}$ $\left(\text{for } t^2 \Rightarrow \dfrac{2}{s^3}\right)$

$\therefore$ for the given input,

$$e_{ss} = \frac{2R_3}{K_a} = \frac{2R_3 S_2 S_3}{S_1 K}$$

Performance Index (P.I.)

Def. A Performance index is a number which indicates the "goodness" of System performance. A control System is considered optimal if the values of the parameters are ~~selected~~ chosen/selected so that the selected performance index is minimum or maximum. The optimal values of the parameters depends directly upon the performance index selected.

Or The performance index is a function whose value indicates how well the actual performance of the System matches with the desired performance.

Parameters or commonly used specifications: Peak overshoot, settling time, gain margin, phase margin, steady state error.

Requirements of Performance Indexes:

(1) The Performance specification consists of a single performance index, i.e. It must yields a single positive number or zero

(2) A P.I must offer selectivity i.e. its power to clearly distinguish between an optimum and non-optimum system.

(3) A P.I must be a function of the parameters of the system and it must exhibits a minimum or maximum.

(4) Finally, to be practical a P.I must be easily computed, analytically or experimentally.

Various error performance indexes are:

(1) <u>Integral Square Error (I.S.E.)</u>

$$I.S.E = \int_0^\infty e^2(t)\,dt \quad — (1)$$

This is the most commonly

let $x(t) \to$ desired output & $y(t) \to$ actual output

$\therefore$ error $e(t) = x(t) - y(t)$

unless, $\lim_{t\to\infty} e(t) = 0$, the performance index approaches infinity.

If $\lim_{t\to\infty} e(t)$ does not approach zero, we may define $\boxed{e(t) = y(\infty) - y(t)}$ $\to$ with this definition of the error, the performance indexes will yield finite numbers.

According to the integral square error (ISE) criterion, the quality of system performance is evaluated by the following integral $\int_0^\infty e^2(t)\,dt$ where the upper limit ∞ may be replaced by T which is chosen sufficiently large so that e(t) for $T < t$ is negligible.

The optimal system is the one which minimizes this integral.

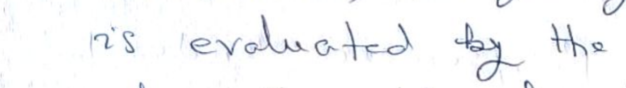


Fig 1. Desired output x(t) vs t.

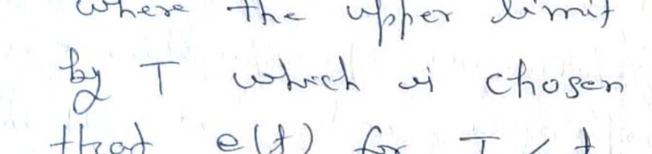


Fig 2. actual output y(t) vs t

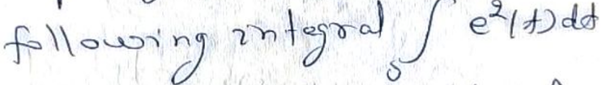


Fig 3. error e(t) vs t

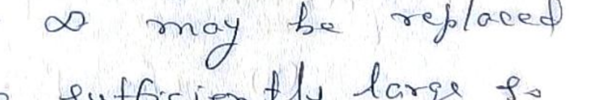


Fig 4 $e^2(t)$ vs t

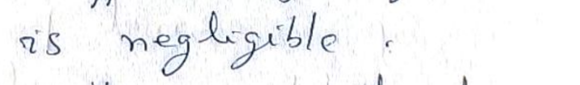


Fig 5. $\int e^2(t)\,dt$ vs t

* The characteristics of the P.I. is that ~~the~~ it weights large errors heavily and small errors lightly.
* This criteria is not very selective.
* A system designed by this criterion tends to show a rapid decrease in a large initial error. Hence the response is fast and oscillatory. Thus the system has poor relative stability.
* It has practical significance between the minimization of P.I results in the minimization of power consumption for some systems, such as spacecraft system.

(2) <u>Integral of time-multiplied square error criterion</u> (ITSE) :-

$$ITSE = \int_0^\infty t\, e^2(t)\, dt \quad - (2)$$

The optimal system is one which minimizes the above integral. This criterion has a better selectivity than the integral-square-error criterion.

Initial error is weighted lightly while errors occurring late in the transient response penalized heavily.

(3) <u>Integral-Absolute error criterion (IAE)</u>

The PI defined by IAE is as

$$IAE = \int_0^\infty |e(t)|\, dt \quad - (3)$$

An optimum system based on this criterion is a system which has reasonable damping and a

satisfactory transient response characteristics.
This is one of most easily applied P.I.

* minimization of the integral absolute error is directly related to the minimization of fuel consumption of spacecraft system.
* selectivity is not too good. This P.I cannot (5) be easily evaluated by analytical means. It is practically convenient for analog computer.

(4) Integral-of-time-multiplied absolute-error criterion (ITAE):

The optimum system is one which minimizes the following performance index, $\int_0^\infty t\,|e(t)|\,dt$ i.e. $ITAE = \int_0^\infty t\,|e(t)|\,dt$ —(4)

A system designed by use of this criterion has a characteristic that the overshoot in the transient response is small and oscillations are well damped. This criterion possesses good selectivity and is important over the integral-absolute-error criterion.

However it is very difficult to evaluate analytically, although it can be easily measured experimentally.

Q1. Consider the system whose closed-loop tranfer function is $\frac{C(s)}{R(s)} = \frac{1}{s^2+2\xi s+1}$ $(0<\xi<1)$

compute $\int_0^\infty c^2(t)\,dt$ for a unit-impulse input

Soln. For a unit impulse input, $R(s)=1$, Hence

$$C(s) = \frac{1}{s^2+2\xi s+1} = \frac{1}{\sqrt{1-\xi^2}} \cdot \frac{\sqrt{1-\xi^2}}{(s+\xi)^2+1-\xi^2}$$

Taking inverse Laplace transform.

$$c(t) = \frac{1}{\sqrt{1-\xi^2}} e^{-\xi t} \sin\sqrt{1-\xi^2}\,t \quad (t \geq 0)$$

$$c^2(t) = \frac{1}{1-\xi^2} e^{-2\xi t} \sin^2\sqrt{1-\xi^2}\,t$$

$$= \frac{1}{1-\xi^2} e^{-2\xi t} \cdot \frac{1}{2}\left(1-\cos 2\sqrt{1-\xi^2}\,t\right)$$

$$\mathcal{L}[c^2(t)] = \frac{1}{2(1-\xi^2)}\left[\frac{1}{s+2\xi} - \frac{s+2\xi}{(s+2\xi)^2+4(1-\xi^2)}\right]$$

Then $\int_0^\infty c^2(t)\,dt = \lim_{s\to 0} \mathcal{L}[c^2(t)]$

$$= \lim_{s\to 0} \frac{1}{2(1-\xi^2)}\left[\frac{1}{(s+2\xi)} - \frac{(s+2\xi)}{(s+2\xi)^2+4(1-\xi^2)}\right]$$

$$= \frac{1}{4\xi}$$

Q2. Compute the following performance indices

$\int_0^\infty |e(t)|\,dt$; $\int_0^\infty t\,|e(t)|\,dt$; $\int_0^\infty t\left[|e(t)| + |e(t)|\right]dt$

for the system $\frac{C(s)}{R(s)} = \frac{1}{s^2+2\xi s+1}$ $(\xi \geq 1)$

$$E(s) = \mathcal{L}[e(t)] = R(s) - C(s)$$

Assume that the system is initially at rest and is subjected to a unit step input

Soln. For a unit-step input ; $R(s) = \frac{1}{s}$ and

$$E(s) = R(s) - C(s) = \frac{1}{s} - \frac{1}{s^2+2\xi s+1} \cdot \frac{1}{s}$$

$$= \frac{1}{s} \cdot \frac{s^2+2\xi s}{s^2+2\xi s+1}$$

$$= \frac{-\xi + \sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s+\xi+\sqrt{\xi^2-1}} + \frac{\xi+\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s+\xi-\sqrt{\xi^2-1}}$$

(1) $$\int_0^\infty |e(t)|\,dt = \int_0^\infty e(t)\,dt = \lim_{s\to 0} E(s)$$

$$= \frac{-\xi + \sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} \cdot \frac{1}{\xi+\sqrt{\xi^2-1}} + \frac{\xi+\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} \cdot \frac{1}{\xi-\sqrt{\xi^2-1}}$$

$$= 2\xi \quad (\xi \geq 1)$$

For $\xi \geq 1$, the system does not overshoot

Hence $|x(t)| = x(t)$ for all $t \geq 0$

(2) $$\int_0^\infty t\,|e(t)|\,dt = \int_0^\infty t\,e(t)\,dt = \lim_{s\to 0}\left[-\frac{d}{ds}E(s)\right]$$

$$= \lim_{s\to 0}\left[\frac{-\xi + \sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} \cdot \frac{1}{(s+\xi+\sqrt{\xi^2-1})^2} + \frac{\xi+\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}} \cdot \frac{1}{(s+\xi-\sqrt{\xi^2-1})^2}\right]$$

$$= 4\xi^2 - 1 \quad \text{for } \xi \geq 1$$

Time-Domain Analysis.

Various forms of first order transfer functions.:

(i) $G(s) = \dfrac{K}{Ts+1} \rightarrow$ Gain-time constant form — ①

where K is the gain and T is the time constant of the system.

(ii) $G(s) = \dfrac{K'}{s+a} \rightarrow$ Pole-zero form — ②

where K' is the gain of the system.

Eq. ② can be obtained from eq. ① of — ③

$$G(s) = \frac{K}{Ts+1} = \frac{\frac{K}{T}}{s+\frac{1}{T}} = \frac{K'}{s+a}$$

where $K' = \dfrac{K}{T}$ & $a = \dfrac{1}{T}$

Q1. Find the open loop gain and the time constants for the following first order system.

(i) $G(s) = \dfrac{5}{2s+1}$ (ii) $G(s) = \dfrac{5}{s+2}$ (iii) $G(s) = \dfrac{2}{3s+4}$

Ans. (i) gain = 5
Time const = T = 2

(ii) Convert it into ~~transfer~~ Time constant fn. form

$G(s) = \dfrac{5/2}{\frac{1}{2}s+1}$

So gain = 5/2 = 2.5
& T = 1/2 = 0.5

(iii) $G(s) = \dfrac{2/4}{\frac{3}{4}s+1}$

$K = \frac{2}{4} = 0.5$

$T = \frac{3}{4} = 0.75$

Q2. Find the impulse response of the 2nd order system whose transfer function $G(s) = \dfrac{9}{s^2+4s+9}$

Soln. Method-1.
Comparing $G(s) = \dfrac{9}{s^2+4s+9}$ with $G(s) = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\therefore \omega_n^2 = 9 \Rightarrow \omega_n = 3$ & $2\xi\omega_n = 4 \Rightarrow \xi = \dfrac{4}{2\omega_n} = \dfrac{4}{2\times 3} = \dfrac{2}{3} = 0.667$

Impulse response for $\xi < 1$ is given by

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n\sqrt{1-\xi^2})t$$

$$= \frac{3}{\sqrt{1-(0.667)^2}} e^{-\frac{2}{3}\cdot 3t} \cdot \sin(3\sqrt{1-(0.667)^2})t$$

$$= 4.025 e^{-2t} \sin 2.235t$$

For step input $y(t) = 1 - \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \times \sin(\omega_d t + \theta)$

Method 2

for $s^2+4s+9 \xrightarrow{\text{root are}} s_1, s_2 = \dfrac{-4 \pm \sqrt{(4)^2-4\times 9}}{2}$

$= \dfrac{-4 \pm \sqrt{16-36}}{2} = \dfrac{-4\pm\sqrt{-20}}{2} = \dfrac{-4\pm j\sqrt{20}}{2} = \dfrac{-4\pm j2\sqrt{5}}{2}$

$= -2 \pm j2.236$

$$G(s) = \frac{9}{[s-(-2+j2.236)][s-(-2-j2.236)]}$$

$C(s) = G(s)$ since $R(s) = 1$ for impulse fn

$\therefore C(t) = \mathcal{L}^{-1}[G(s)]$

$$C(s) = \frac{A}{s-(-2+j2.236)} + \frac{A^*}{s-(-2-j2.236)}$$

$$= \frac{-j2.0125}{s-(-2+j2.236)} + \frac{j2.0125}{s-(-2-j2.236)}$$

$$c(t) = -j2.0125\, e^{(-2+j2.236)} + j2.0125\, e^{(-2-j2.236)t}$$

$$= 4.025\, e^{-2t} \cdot \sin 2.236t$$

Time domain specifications

(1) $t_r = \dfrac{\pi - \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}} = \dfrac{\pi-\phi}{\omega_d}$ for $0<\xi<1$

(2) $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$

(3) $M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$ & $\% M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}\times 100$

$M_p = c(t_p)-1$

(4) Settling time $= t_s = -\ln\dfrac{[0.02\sqrt{1-\xi^2}]}{\xi\omega_n}$

$\boxed{t_s = \dfrac{4}{\xi\omega_n}}$ approx. for 2% criteria

$\boxed{t_s = \dfrac{3}{\xi\omega_n}}$ → approx for 5% criterion

(4) $t_s = \dfrac{4}{\xi\omega_n} = 4T$ & time constant $= \dfrac{1}{\xi\omega_n}$ → for 2% tolerance band

$t_s = \dfrac{3}{\xi\omega_n} = 3T$ for 5% tolerance band

(5) $e_{ss} = \lim_{t\to\infty}[1-c(t)] = 0$ i.e. zero for unit step input

$\dfrac{2\xi}{\omega_n} = \dfrac{1}{K_v}$ for ramp input

for ramp input $e_{ss} = \lim_{t\to\infty}[t-c(t)] = \dfrac{2\xi}{\omega_n} = \dfrac{1}{K_v}$

Q2 When a second order control system is subjected to a unit step input, the values of $\xi = 0.5$ & $\omega_n = 6$ rad/sec. Determine the rise time, peak time, settling time and peak overshoot.

Ans. given $\xi = 0.5$ & $\omega_n = 6$ rad/sec

(i) $t_r = \dfrac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}} = \dfrac{\pi - \tan^{-1}\frac{\sqrt{1-(0.5)^2}}{0.5}}{6\sqrt{1-(0.5)^2}} = \dfrac{3.14 - 1.047}{5.19} = 0.403$ sec

(ii) $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}} = \dfrac{\pi}{6\sqrt{1-(0.5)^2}} = 0.605$ sec.

(iii) $t_s = \dfrac{4}{\xi\omega_n} = \dfrac{4}{0.5 \times 6} = 1.33$ sec

(iv) $M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100 = e^{-\frac{\pi \times 0.5}{\sqrt{1-(0.5)^2}}} \times 100 = 0.163 \times 100 = 16.3\%$

Q3. Obtain the unit step and unit impulse response of the following. $\dfrac{C(s)}{R(s)} = \dfrac{10}{s^2 + 2s + 10}$

Ans. $\dfrac{C(s)}{R(s)} = \dfrac{10}{s^2 + 2s + 10}$; for unit step $R(s) = \dfrac{1}{s}$

$\therefore C(s) = \dfrac{10}{s(s^2 + 2s + 10)} = \dfrac{10}{s(s+1+j3)(s+1-j3)}$

$\therefore c(t) = \mathcal{L}^{-1}\left[\dfrac{1}{s} + \dfrac{-0.5 - j0.166}{s+1+j3} + \dfrac{-0.5 + j0.166}{s+1-j3}\right]$

$= 1 + (-0.5 - j0.166)\,e^{(-1-j3)t} + (-0.5 + j0.166)\,e^{(-1+j3)t}$

$= 1 - e^{-t}[0.5(e^{j3t} + e^{-j3t}) + j0.166(e^{-j3t} - e^{j3t})]$

$= 1 - e^{-t}[\cos 3t + j0.33 \sin 3t]$

$= 1 - 1.053\,e^{-t}.\sin(3t + 71.736^\circ)$

Q 4. The closed loop tr. fr. is given by $G(s) = \dfrac{s(s^2 + 9s + 19)}{s^3 + 7s^2 + 14s + 8}$. Determine the response of the system when a unit step is applied as input.

Ans. $C(s) = G(s).R(s) = \dfrac{s(s^2 + 9s + 19)}{(s^3 + 7s^2 + 14s + 8)} \cdot \dfrac{1}{s} = \dfrac{s^2 + 9s + 19}{s^3 + 7s^2 + 14s + 8}$

$C(s) = \dfrac{11}{3(s+1)} - \dfrac{5}{2(s+2)} - \dfrac{1}{6(s+4)}$

$\therefore c(t) = \dfrac{11}{3}e^{-t} - \dfrac{5}{2}e^{-2t} - \dfrac{1}{6}e^{-4t}$

Q.1. A control system having $\omega_n = 5$ rad/sec and $\xi = 0.6$. Determine rise time, peak time, maximum overshoot and settling time when the system is subjected to unit step input.

Ans given undamped natural angular freq. $= \omega_n = 5$, damping ratio $\xi = 0.6$

(i) Damped natural angular velocity $= \omega_d = \omega_n\sqrt{1-\xi^2} = 5\sqrt{1-0.36}$
$= 4$

(ii) real part of the roots of the ch. eq. $= \sigma = \xi\omega_n = 5 \times 0.6 = 3$

$T = \frac{1}{\sigma}$

(iii) Rise time $= t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}} = \frac{3.14 - 0.93}{4}$

$= 0.55$ sec

$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-(0.6)^2}}{0.6} = 0.93$

(iv) Peak time $= t_p = \frac{\pi}{\omega_d} = \tan^{-1}\frac{3.14}{4} = 0.785$

(v) Maximum overshoot, $M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 0.095$
$= 9.5\%$

(vi) Settling time for 2% tolerance band $= 4T = \frac{4}{\sigma} = \frac{4}{3} = 1.33$ sec

" " " 5% " " $= 3T = \frac{3}{\sigma} = \frac{3}{3} = 1$ sec

Q2. Determine the approximate value of K_1 and calculate the steady state error for a unit step input for the system shown in Fig 1.

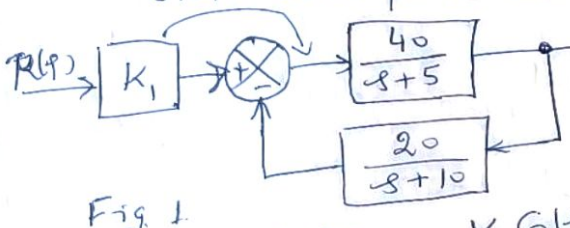


Fig. 1

Soln. $H(s) = \frac{20}{s+10} = \frac{K_1}{0.1s+1} = \frac{2}{0.1s+1}$

$K_1 = 2$

$\frac{Y(s)}{R(s)} = \frac{K_1 G(s)}{1 + K_1 G(s) H(s)}$ & $E(s) = \frac{R(s)}{1+K_1 G(s) H(s)}$

Steady state error $= e_{ss} = \lim_{s\to 0} s \cdot E(s) = \lim_{s \to 0} \frac{s}{1+K_1G(s)H(s)} \cdot \frac{1}{s}$

$= \lim_{s\to 0} \frac{1}{1 + K_1 \frac{40}{(s+5)} \cdot \frac{20}{(s+10)}} = \frac{1}{1 + 2 \cdot \frac{40}{5} \cdot \frac{20}{10}} = \frac{1}{1+2\times 8\times 2}$

$= \frac{1}{33} = 0.03$
$= 3\%$

Time Domain Analysis

(1) $t_r = \dfrac{\pi - \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$

(2) $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$

(3) $\%\, M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100$

(4) $t_s = \dfrac{4}{\xi\omega_n}$

(5) $e_{ss} = \dfrac{2\xi}{\omega_n}$ for ramp input $= \dfrac{1}{K_v}$

$= 0$ for step input

Q1 The open loop transfer function of a servo system with unity feedback is given by

$$G(s) = \frac{10}{(s+2)(s+5)}$$

Determine the damping ratio, undamped natural frequency of oscillation. What is the percentage over shoot to a unit step input.

Sol: $G(s) = \dfrac{10}{(s+2)(s+5)}$, $H(s) = 1$

Ch. eq. $1 + G(s)H(s) = 1 + \dfrac{10}{(s+2)(s+5)} = 0$

or $s^2 + 7s + 20 = 0$

Comparing with $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$\omega_n^2 = 20 \longrightarrow \omega_n = 4.472$ rad/sec

$2\xi\omega_n = 7 \Longrightarrow \xi = 0.7826$

or $2 \times \xi \times 4.472 = 7$

$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100 \Longrightarrow M_p = 1.92\%$

Q. A unity feedback system is characterised by an open loop tr. fn. $G(s) = \frac{K}{s(s+10)}$. Determine the gain K so that the system will have a damping ratio of 0.5. For this value of K, determine settling time, peak over shoot and peak time for a unit step input.

Soln. Given $G(s) = \frac{K}{s(s+10)}$, $H(s) = 1$ & $\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1 + \frac{K}{s(s+10)}} = \frac{K}{s^2+10s+K}$

(i) comparing with the std. one $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$

$\omega_n^2 = K$ & $2\xi\omega_n = 10$

Given $\xi = 0.5$, or $2\times 0.5\,\omega_n = 10 \Rightarrow \omega_n = \frac{10}{1} = 10$

$\therefore K = (\omega_n^2) = (10)^2 = 100$

(ii) $t_s = \frac{4}{\xi\omega_n} = \frac{4}{(0.5)(10)} = \frac{4}{5} = 0.8$ sec

(iii) $M_p = 100\, e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 100\, e^{\frac{-0.5\times 3.14}{\sqrt{1-(0.5)^2}}} = 16.3\,\%$

(iv) $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{10\sqrt{1-(0.5)^2}} = \frac{\pi}{10(0.866)}$

$= \frac{3.14}{8.66} = 0.362$ sec

Q. For a closed loop tr. fn given by $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

Determine the value of ξ and ω_n so that the system responds to a step input with approximately 5% overshoot and with a settling time of 2 sec.

Ans t_s (for 2% criteria) $= \frac{4}{\xi\omega_n} = 2$ sec $\Rightarrow \xi\omega_n = 2$.

~~ts (for 5% criteria)~~ Peak over shoot $= M_p = 5\,\%$

$\Rightarrow e^{-\xi\pi/\sqrt{1-\xi^2}} = 5\,\%$

or $e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.05$

using natural log on both side

$\frac{\xi\pi}{\sqrt{1-\xi^2}} = 3$

$\Rightarrow \xi = 0.69$

$\xi\omega_n = 2 \Rightarrow \omega_n = \frac{2}{\xi} = \frac{2}{0.69} = 2.896$ rad/sec

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{(2.896)^2}{s^2+2(2)s+(2.896)^2} = \frac{8.396}{s^2+4s+8.396}$$

Q. For the ~~given~~ system described by a differential equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 8x$ where x is the input & y is the output.
Determine the undamped natural freq, damping ratio, damped natural freq, peak time and settling time.

Ans overall tr. fn. $= \frac{Y(s)}{X(s)} = \frac{8}{s^2+4s+8}$

Comparing $\frac{8}{s^2+4s+8}$ with $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 8$ & $2\xi\omega_n = 4$

undamped natural freq $= \omega_n = \sqrt{8}$ rad/sec

Damping ratio $= \xi = \frac{4}{2\omega_n} = \frac{4}{2\sqrt{8}} = \frac{1}{\sqrt{2}} = 0.707$

Damped natural freq $= \omega_d = \omega_n\sqrt{1-\xi^2}$

$= \sqrt{8}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{8}\sqrt{0.5} = 2$ rad/sec

Peak overshoot $= M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100$

$= e^{-\pi\times(0.707)/\sqrt{1-(0.707)^2}} \times 100 = 4.33\%$

Settling time $= t_s = \frac{4}{\xi\omega_n} = \frac{4}{(0.707)\sqrt{8}} = 2$ sec

Q. ~~A unity feedback system is charact~~
The closed loop transfer function is given by

$$G(s) = \frac{s(s^2+9s+19)}{s^3+7s^2+14s+8}$$

Determine the response of the system when a unit step is applied as input

Ans. $Y(s) = G(s).R(s) = \frac{1}{s}.\frac{s(s^2+9s+19)}{(s^3+7s^2+14s+8)} = \frac{11}{3(s+1)} - \frac{5}{2}\frac{1}{s+2} - \frac{1}{6(s+4)}$

$\therefore y(t) = \frac{11}{3}e^{-t} - \frac{5}{2}e^{-2t} - \frac{1}{6}e^{-4t}$ Ans

Q. Obtain the unit step response and unit impulse response of the following system.

$$\frac{Y(s)}{R(s)} = \frac{10}{s^2+2s+10}$$

Soln. $\frac{Y(s)}{R(s)} = \frac{10}{s^2+2s+10}$; For unit step input $R(s) = \frac{1}{s}$

$$\Rightarrow Y(s) = \frac{10}{s(s^2+2s+10)} = \frac{A}{s} + \cancel{\frac{B}{(s+0.5+j0.166)}} + \cancel{\frac{B^*}{(s+0.5-j0.166)}}$$

$$= \frac{A}{s} + \frac{B}{s+1+3j} + \frac{B^*}{s+1-3j}$$

where $A = \cancel{-0.5-j0.166}$ 1, $B = -0.5 - j0.166$, $B^* = -0.5 + j0.166$

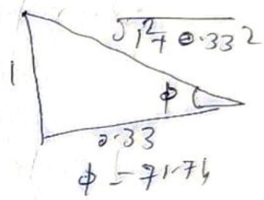
$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{-0.5-j0.166}{s+1+j3} + \frac{-0.5+j0.166}{s+1-3j}\right]$$

$$= 1 + (-0.5-j0.166)e^{(-1-3j)t} + (-0.5-j0.166)e^{(-1+j3)t}$$

$$= 1 - e^{-t}\left[(0.5+j0.166)e^{-j3t} + (0.5-j0.166)e^{j3t}\right]$$

$$= 1 - e^{-t}\left[0.5(e^{j3t}+e^{-j3t}) + j0.166(e^{-j3t} - e^{j3t})\right]$$

$$= 1 - e^{-t}[\cos 3t + j\, 0.33 \sin 3t]$$



$\phi = 71.74$

$$= 1 - e^{-t}\left(\sqrt{1^2+0.33^2}\right)\left(\frac{1}{\sqrt{1^2+0.33^2}}\cos 3t + j\frac{0.33}{\sqrt{1^2+0.33^2}}\sin 3t\right)$$

$$= 1 - 1.053\, e^{-t}[0.9496 \cos 3t + 0.3134 \sin 3t)]$$

$$= 1 - 1.053\, e^{-t}[\sin 71.74 \cos 3t + \cos 71.74^\circ \sin 3t]$$

$$= 1 - 1.053\, e^{-t}[\sin(3t + 71.736^\circ)$$

$$y(t) = 1 - 1.053\, e^{-t}\sin(3t + 71.736^\circ)$$

Alternate method. on comparing with standard 2nd order system

$$\frac{10}{s^2+2s+10} = \frac{\omega_n^2}{s^2+2\xi\omega_n t+\omega_n^2} \Rightarrow \omega_n = \sqrt{10} \;\&\; \xi = \frac{1}{\sqrt{10}}$$

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_n t + \phi) = 1 - \frac{e^{-t}}{\sqrt{1-\frac{1}{10}}}\sin\left(\sqrt{10}\sqrt{1-\frac{1}{10}}\, t + \phi\right)$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\frac{1}{10}}}{\frac{1}{\sqrt{10}}} = 71.736^\circ$$

$$= 1 - 1.1053\, e^{-t}\sin(3t + 71.736^\circ)$$

For unit impulse: — $R(s) = 1$

$$Y(s) = \frac{10}{s^2+2s+10} = \frac{10}{(s+1)^2+3^2}$$

since $\mathcal{L}^{-1}\left[\frac{b}{(s+a)^2+b^2}\right] = e^{-at}\sin bt$

$$y(t) = 3.33\, e^{-t}\sin 3t.$$