Stability

External stability | Internal stability

Defi: A small changes in the system input, or in initial conditions or in system parameters do not result in large changes in system output.

A linear time-invariant system is stable if the following two notions of system stability are satisfied.

(1) **BIBO stability**: A system is stable if its output is bounded for any bounded input.

(2) **Asymptotic stability**: In the ~~presence~~ absence of the input, the output tends towards zero.

or

A system is stable if its response to a bounded disturbing signal vanishes ultimately as time "t" approaches infinity (zero-input response)

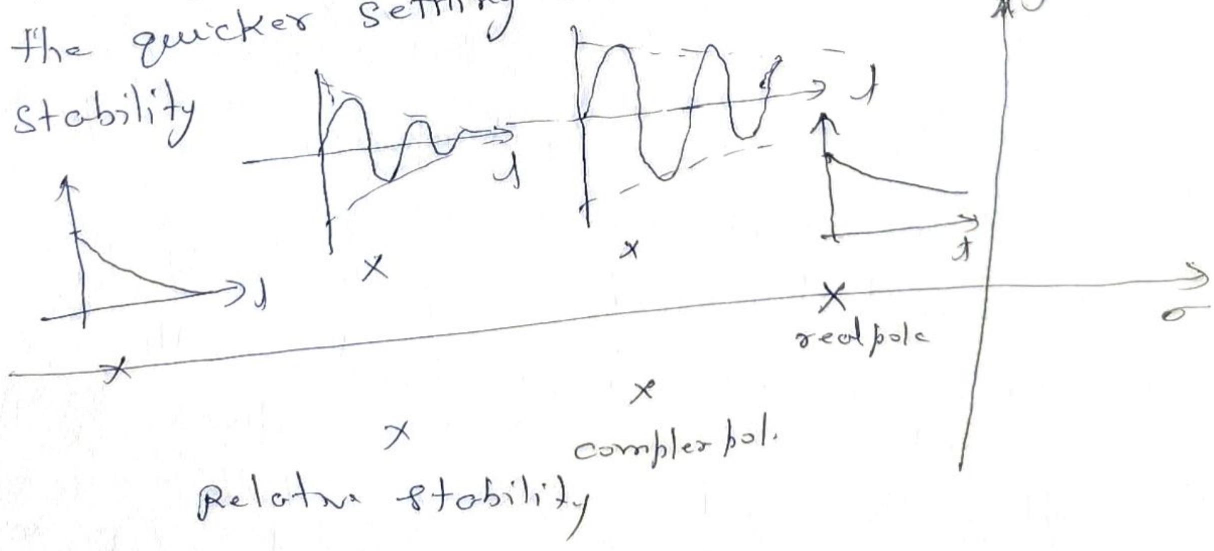
Unstable: A system is unstable if its response to a bounded ~~disturbing~~ signal results in an output of infinite amplitude or an oscillatory signal.

Limitedly Stable: If the ~~output~~ response to a bounded input signal results in constant amplitude or a constant amplitude oscillation, then the system may be stable or unstable under some limited constraints, such a system is called limitedly stable.

Conditionally stable: If a system response is stable for a limited range of variation of its parameters, it is called conditionally stable system.

Absolutely ~~stability~~ stable or Absolute stability: If a system response for all variation of its parameters is stable for all variation of its parameters then it is called absolutely stable system. It is the quality of stable or unstable performance, i.e. answer in terms of yes or NO.

Relative stability: It is the quantitative study of stability. The stability expressed in quantitative terms is called relative stability. It is the answer to the question, how much stability? It is measured by relative settling times of each root or pair of roots. Since the settling time is inversely proportional to the negative part of the root, the larger value of negative real part means the quicker settling time and improved relative stability.



Relative stability

Characteristic equation: The stability study is based on the properties of transfer function

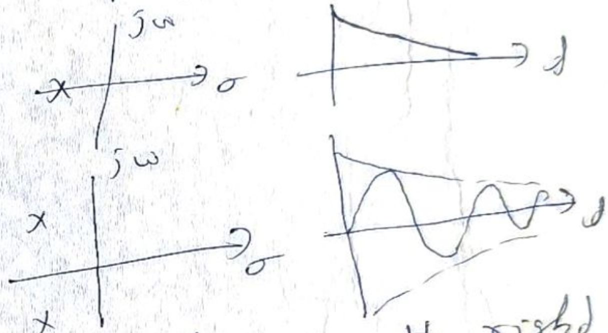
$$\text{Tr. fn.} = G(s) = \frac{C(s)}{R(s)} = \frac{p(s)}{q(s)}$$

The denominator polynomial $q(s)$ is called the characteristic polynomial and $q(s) = 0$ is called the characteristic equation. (Since it characterises the stability of the characteristic equation or control system irrespective of the input.)

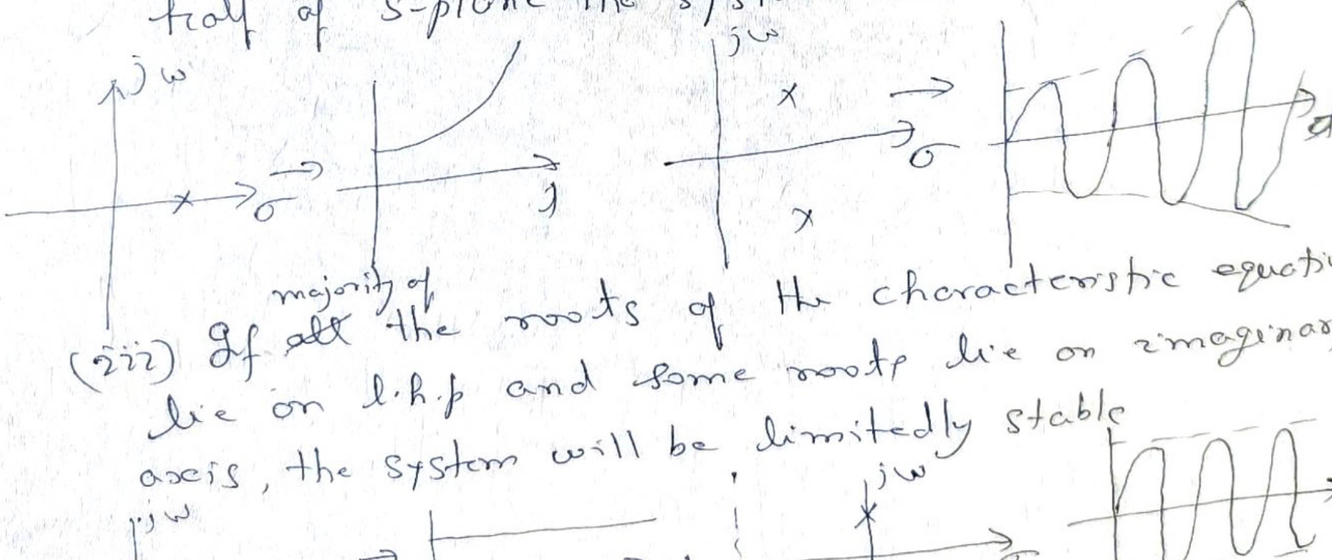
In terms of location of ch. roots of ch. eq.

(i) The system is stable, if all the roots of the ch. eq. lie on left half of s-plane.

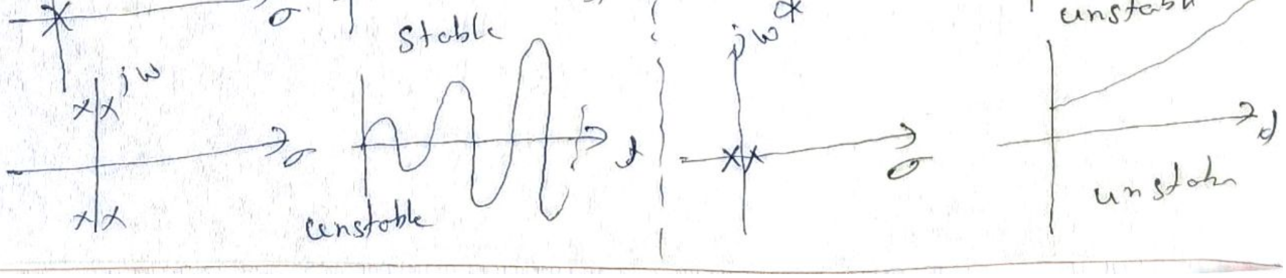
(ii)



(ii) If any root of ch. eq. lie on the right half of s-plane the system will become unstable



(iii) If majority of the roots of the characteristic equation lie on l.h.p and some roots lie on imaginary axis, the system will be limitedly stable



Necessary conditions for stability:

ch. eq. is written as

$$q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0 \quad \text{--- (1)}$$

$$a_0 > 0$$

(1) All coefficients of eq. (1) be real and have the same sign

(2) None of the coefficient of eq. (1) should be zero (i.e. no missing terms).

1st order: $a_0 s + a_1 = 0 \Rightarrow s = -\frac{a_1}{a_0}$ → for all +ve value of a_1 & a_0 the system is stable

2nd order: $a_0 s^2 + a_1 s + a_2 = 0$

$$\therefore s_1, s_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$$

for all +ve value of a_0, a_1 & a_2, the system is stable

3rd-order: $s^3 + s^2 + 4s + 30 = 0$

$$(s+3)(s-1+j3)(s-1-j3) = 0$$

Time response (consists of two parts)

↓ Transient response or transient solution or natural response or complementary function

By transient response, we mean that the part which goes from the initial state to the final state.

or

Transient response is that part of the total response that approaches zero as time approaches infinity.

↓ Steady state response or forced response or particular integral

Steady state response is that part of the total response that does not approach zero as time approaches infinity

or

The manner in which the system output behaves as t approaches infinity

A. Hurwitz & E. J. Routh independently published the method of investigating the sufficient conditions of stability of a system

Hurwitz Criterion — in terms of determinants

Routh Criterion — in terms of array formation

Both are equivalent.

Hurwitz Stability Criterion:

Let us consider the nth order system of

$q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$

For the stability of this system, it is necessary and sufficient that the 'n' determinant formed from the coefficients $a_0, a_1, \ldots, a_n$ of the ch. eq. be positive. These determinants are taken as the principal minors of the following arrangement (called the Hurwitz determinant)

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \cdots & & & \cdots & a_{n+1} & a_n \end{vmatrix} \quad \text{—(I)}$$

Coefficients with indices larger than n or with negative indices are replaced by zeros.

The necessary and sufficient conditions for stability are $\Delta_1 = a_1 > 0$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0 \; ; \quad \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0$$

Δ_n = entire arrangement of eq. (I) > 0

where $\Delta_{n-1} = 0$; the system is limitedly stable.

Example 1 : Consider 4th order system with the ch eq.

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

as. $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$

Test the stability using Hurwitz arrangement.

Soln.: Hurwitz arrangement

$$\begin{array}{c} a_1 \\ a_3 \\ a_5 \\ a_7 \end{array} \begin{vmatrix} 8 & 1 & 0 & 0 \\ 16 & 18 & 8 & 1 \\ 0 & 5 & 16 & 18 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$\Delta_1 = 8 > 0$

$\Delta_2 = \begin{vmatrix} 8 & 1 \\ 16 & 18 \end{vmatrix} = 128 > 0$

$\Delta_3 = \begin{vmatrix} 8 & 1 & 0 \\ 16 & 18 & 8 \\ 0 & 5 & 16 \end{vmatrix} = 1728 > 0$

$\Delta_4 = 5\Delta_3 > 0$

Hence the system is stable.

(i) Algebraic method
(2) Absolute stability

Routh-Hurwitz (R-H) criterion.

This is an algebraic method which gives answer to the absolute stability of the linear time-invariant system. This criterion tests whether any roots of the ch. eq. lie in the R.H.P

Let us consider the ch. eq. is $1 + G(s)H(s) = 0$

i.e. $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$ — (1)

First inspect this equation using necessary conditions. R-H criterion is used only when all the coefficients are real positive and there is no missing term.

Step 1. Form Routh array as:

(i) The first row consists of even coefficients i.e. $a_0, a_2, a_4, a_6 \cdots$, starting from the highest term and write the alternate coefficient in row.

(ii) Second row consists of odd coefficients, i.e. $a_1, a_3, a_5, a_7 \cdots$ i.e. starts from the next highest power and keep writing in rows the alternate coefficients

s^n	a_0	a_2	a_4 a_6	$\cdots a_n$
s^{n-1}	a_1	a_3	a_5 a_7	$\cdots 0$
s^{n-2}	$\frac{a_1 a_2 - a_0 a_3}{a_1} = b_1$	$\frac{a_1 a_4 - a_0 a_5}{a_1} = b_2$	$\frac{a_1 a_6 - a_0 a_7}{a_1} = b_3$	$\cdots$
or s^{n-2}	b_1	b_2	b_3	$\cdots 0$
s^{n-3}	$\frac{b_1 a_3 - a_1 b_2}{b_1}$	$\frac{b_1 a_5 - a_1 b_3}{b_1}$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
s^0	$\cdots$			

Step II : This process is continued. The absolute stability is determined by inspecting the first column.

Routh stability criterion : " All the roots of the polynomials are in the left half of the s-plane if all the elements of the first column are of same sign. If there are changes of signs in the elements of the first column, the number of sign changes in the elements of the first column indicates the nos. of roots with positive real part.

Special cases of R-H criterion

Case 1. When the first term in any row of the Routh array is zero while rest of the row has at least one non-zero term.

Because of this zero term, the term in the next row become infinity and Routh's test break down.

Remmeddy: (i) Substitute a small +ve number ϵ for the zero and proceed to evaluate the rest of the Routh array.

(ii) Modify the original ch. eq. replacing s by $\frac{1}{z}$. Now apply the Routh's test on the modified equation in terms of z

Case 2. When all the elements in any one row of the Routh array are zero.

This condition indicates that there are symmetrically located roots in the s-plane i.e. pair of real roots with opposite signs and/or pair of conjugate roots on the imaginary axis and/or complex conjugate roots forming quadrate in the s-plane.

(i) pairs of real roots with opposite sign (ii) pairs of imaginary roots (iii) pairs of complex-conjugate roots forming symmetry about the origin of S-plane

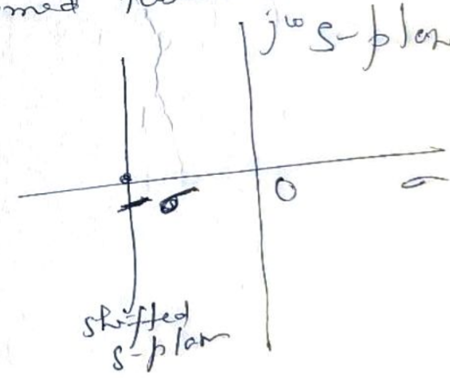
<u>Auxiliary Equation</u>: The equation that is formed by using the coefficients of the row just above the row of zeros is called the auxiliary equation. The order of auxiliary equation is always even and it indicate the number of root pairs that are equal in magnitude but opposite in sign.

<u>Remmedy</u>: (i) Take the derivative of the auxiliary equation with respect to s.

(ii) Replace the row of zeros with the coefficients of the resultant equation obtained by taking the derivative of the auxiliary equation

(iii) carry on the Routh test in the usual manner with the newely formed tabulation

<u>Relative Stability Analysis</u>:



Shift the S-plane axis and apply the Routh's stability criterion

Substitute $\boxed{S = Z - \sigma}$

in the ch. eq., now poly is in terms of Z & apply R-H criterion

Q1. Test the stability of a given system using R-H criterion

$$S^5 + 6S^4 + 3S^3 + 2S^2 + S + 1 = 0$$

Soln

S^5	1	3	1
S^4	6	2	1
S^3	$16/6$	$5/6$	0
S^2	$\frac{1}{8}$	1	0
S^1	$-\frac{123}{6}$	0	
S^0	1		

The system is unstable

Since 2 no. of sign change the 2 roots lie in RHP

Q2. Test the stability using R-H criterion

$q(s) = (s-1)^2(s+2) = s^3 - 3s + 2 = 0$

Soln.

s^3	1	-3
s^2	0	2
s^1	∞	
s^0	--	

method I
replace '0' element by $\in$ (a +ve no)

s^3	1	-3
s^2	$\in$	2
s^1	$\frac{-3\in-2}{\in}$	0
s^0	2	

Two sign changes, 2 root lies in RHP, unstable system

or (method 2)

Q. replace s by $\frac{1}{z}$

$\frac{1}{z^3} - \frac{3}{z} + 2 = 0 \implies 2z^3 - 3z^2 + 1 = 0$

z^3	2	0
z^2	-3	1
z^1	$\frac{2}{3}$	0
z^0	1	

2 changes in sign.
so 2 roots lie in the r.h.p, unstable system

Q 3. Test the stability of the given system using RH criterion

$s^4 + s^3 - 3s^2 - s + 2 = 0$

Soln

s^4	1	-3	2
s^3	1	-1	0
s^2	-2	2	
s^1	0	0	

Routh Formation breaks

obtain auxiliary equation
i.e. $A(s) = -2s^2 + 2 = 0$
take the derivative of $A(s)$ with respect to s
i.e. $\frac{dA(s)}{ds} = -4s$
now the row of zeros in the Routh tabulation is replaced by the coefficient of above eq

s^4	1	-3	2
s^3	1	-1	0
s^2	-2	2 -- coeff of aux. eq.	
s^1	-4	0 -- coeff of $\frac{dA(s)}{ds}$	
s^0	2	0	

two changes in sign
→ two roots in the R.H.P
solving the aux. eq.

$-2s^2 + 2 = 0$

$s^2 = 1$

or $s = \pm 1$

Q4. Test the stability of the given system using R-H criterion

$$q(s) = s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = 0$$

Soln.

s^6	1	-2	-7	-4
s^5	1	-3	-4	
s^4	1	-3	-4	
s^3	0	0	0	

$$A(s) = s^4 - 3s^2 - 4 = 0$$

$$\frac{dA(s)}{ds} = 4s^3 - 6s = 0$$

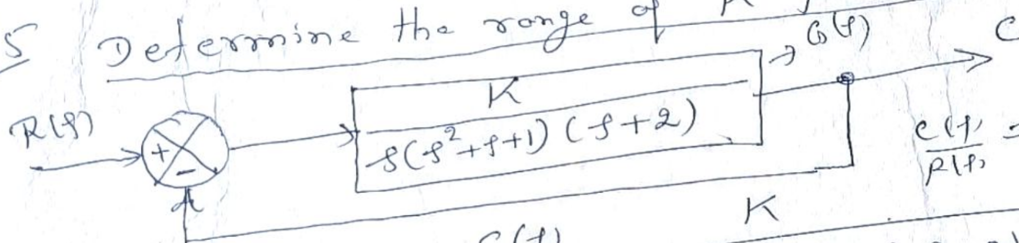
modified part

s^6	1	-2	-7	-4
s^5	1	-3	-4	
s^4	1	-3	-4	
s^3	4	-6	0	
s^2	-1.5	-4	0	
s^1	-16.7	0		
s^0	-4	0		

one sign change so one root lie in the R.H.P and their location are $A(s) = s^4 - 3s^2 - 4 = 0$

$\therefore s = +2, -2, +j$ and $-j$

Q5 Determine the range of K for stability:



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Soln: closed loop tr fn $= \dfrac{C(s)}{R(s)} = \dfrac{K}{s(s^2+s+1)(s+2)+K} = \dfrac{p(s)}{q(s)}$

$$q(s) = s(s^2+s+1)(s+2) + K = 0$$

ie. $s^4 + 3s^3 + 3s^2 + 2s + K = 0$

s^4	1	3	K
s^3	3	2	0
s^2	$\frac{7}{3}$	K	
s^1	$2 - \frac{9}{7}K$		
s^0	K		

For stability K must be +ve and all the coefficients in the first column must be positive

$\frac{14}{9} > K > 0$

when $K = \frac{14}{9}$ the system becomes oscillator and mathematically, the oscillation is sustained at constant amplitude

$K > 0$

$2 - \frac{9}{7}K > 0$ Or $2 > \frac{9}{7}K \Rightarrow \frac{9}{7}K < 2 \Rightarrow K < \frac{14}{9}$

Q6. Show that all roots of the following ch. eq lie to the left of the line s or $\sigma = -1$

(i) $s^3 + 7s^2 + 25s + 39 = 0$ (ii) $s^3 + 2s^2 + 28s + 24 = 0$

Let us check if all the roots of this equation have real parts more negative than $\sigma = -1$

Ans. (i) Shift the origin to $s = -1$ by substituting $s = z - 1$

$\therefore$ ch. eq is $z^3 + 4z^2 + 14z + 20 = 0$

z^3	1	14
z^2	4	20
z^1	9	
z^0	20	

All the roots of the original ch. eq. in s-plane lie to the left of $s = -1$.

Q7 Test the stability using R-H criterion

$Q(s) = s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$

Sol:

s^4	1	18	5
s^3	8	16	0
s^2	16	5	
s^1	$\frac{27}{2}$	0	
s^0	5		

Stable

Q8. $Q(s) = 3s^4 + 10s^3 + 5s^2 + 5s + 2 = 0$

s^4	3	5	2
s^3	10 / 2	5 / 1	0 / 0
s^2	$\frac{7}{2}$	2	
s^1	$-\frac{1}{7}$		
s^0	2		

Two sign change → unstable

Q9. $\Delta(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4 = 0$

s^6	1	3	1	4
s^5	4 / 2	2 / 1	4 / 2	0 / 0
s^4	$5/2$	0	4	
s^3	1	$-5/6$	0	
s^2	3	4		
s^1	$-38/15$			
s^0	4			

unstable

Stability Analysis

Q.1. A system has $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)(s+8)}$ when K is positive. Determine the range of K for stability.

Soln. Ch. eq. $= 1 + G(s)H(s) = 0$

$$1 + \frac{K}{s(s+2)(s+4)(s+8)} = 0$$

or $s(s+2)(s+4)(s+8) + K = 0$

or $s^4 + 14s^3 + 56s^2 + 64s + K = 0$

Routh table are

s^4	1	56	K
s^3	14	64	0
s^2	51.4285	K	0
s^1	$\frac{3291.42 - 14K}{51.4285}$	0	0
s^0	K		

For stability $K > 0$ & $3291.42 - 14K > 0$

$$\therefore K < \frac{3291.42}{14} = 235.102$$

$$0 < K < 235.102$$

Q2. The output $c(t)$ of a control system is related to the input by

$$[s^4 + 2s^3 + 2s^2 + (K+3)s + K]\,C(s) = K(s+1)R(s)$$

when K is positive gain of an amplifier

(i) with $K = 6$, will the output response be stable?

(ii) determine the limiting positive values of K for stability.

Soln. $$\frac{C(s)}{R(s)} = \frac{K(s+1)}{s^4 + 2s^3 + 2s^2 + (K+3)s + K}$$

Ch. eq. is $s^4 + 2s^3 + 2s^2 + (K+3)s + K = 0$

obtain the Routh's array as:

~~(2)~~

$$\begin{array}{c|ccc}
s^4 & 1 & 2 & K \\
s^3 & 2 & (K+3) & 0 \\
s^2 & \frac{(1-K)}{2} & K & 0 \\
s^1 & \frac{(1-K)(K+3)-4K}{(1-K)} & 0 & 0 \\
s^0 & K & &
\end{array}$$

For stability ; from s^2 - row

$$\frac{1-K}{2} > 0 \implies K < 1$$

Thus the system will be unstable for $K = 6$

Also from s^0 - row, $K > 0$ for stability

From s^1 - row

$$\frac{(1-K)(K+3)-4K}{(1-K)} > 0 \text{ for stability}$$

$\therefore \quad (1-K)(K+3) - 4K > 0$

or $\quad 3 + K - 3K - K^2 - 4K > 0$

or $\quad 3 - 6K - K^2 > 0$

or $\quad K^2 + 6K - 3 < 0$

or $\quad (K - 0.464)(K + 6.464) < 0$

or $\quad K < 0.464$

or $\quad K < -6.464 \quad \times$

K can not be less than one.

$\therefore$ range of K for stability is $0 < K < 0.464$

Q 3. Examine the stability by Routh's criterion of the system whose ch. eq. is

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$$

Soln. Routh's array are as.

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 15 \\
s^3 & 0 & -12 & 0
\end{array}$$

replacing zero by small +ve no. $+\epsilon$

s^5	1	2	3
s^4	1	2	15
s^3	ϵ	-12	0
s^2	$\frac{2\epsilon+12}{\epsilon}$	15	0
s^1	$\frac{\frac{-12(2\epsilon+12)}{\epsilon}-15\epsilon}{\frac{2\epsilon+12}{\epsilon}}$	0	0
s^0	15	0	0

$$\lim_{\epsilon\to 0}\frac{2\epsilon+12}{\epsilon}=\infty$$

$$=\lim_{\epsilon\to 0}\frac{\frac{-12(2\epsilon+12)}{\epsilon}-15\epsilon}{\frac{2\epsilon+12}{\epsilon}}$$

$$=\lim_{\epsilon\to 0}\frac{-24\epsilon-144-\epsilon^2}{2\epsilon+12}=-12$$

Thus there are two sign change, hence the system is unstable.

Disadvantages of Hurwitz's criterion:

(i) It is very complicated and time consuming for solving higher order systems.

(ii) This method is unable to give the exact number of poles located in the R.H.P of s-plane

(iii) It is very tough to predict marginal stability

Limitations of the Routh-Hurwitz Method:

Although ~~RH~~ simple and straight forward to use for the determination of absolute stability the R-H method has certain limitations. Such as

(a) The method is valid only if the characteristic equation is algebraic and that all the coefficients are real.

(b) The method offers information only on the absolute stability of the system but does not give any indication of the relative stability, only by successive hit and trial, we may obtain some information about the location of the roots of the ch. eq.

(c) The R-H method gives us the no. of roots of ch. eq. in the right-half-plane, left half plane or jw-axis.

About the poles in the right or left half plane, the method normally does not indicate the nature of roots as real or complex

(d) If the method shows that the system is unstable, it provides absolutely no information regarding the approach to be used for stabilization.