(01-R)

Root Locus Techniques

* The contruction of root loci is basically a graphical problem.
* It is developed by E.V. Evan.
* It is a Graphical method
* In the design of control system, it is often necessary to investigate the performance of a system when one or more parameters of the system varies over a given range.
* The root locus technique is not confined to the inclusive study of control systems.
* The equation under investigation does not necessarily have to be the ch. eq. of a linear system.

Let us consider the ch. eq. for one variable parameter:

$$F(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n + K(s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m) = 0 \quad - (1)$$

where K is the parameter considered to vary between $-\infty$ to ∞. The coefficients $a_1, a_2, \cdots a_n$; $b_1, b_2, \cdots b_{m-1}, b_m$ are assumed to be fixed.

These coefficients may be real or complex.

Our main interest is in real coefficients

Locus: Locus is a path of point something varying.

~~Root loci~~ * The loci or path of the roots of eq. (1) when K varies between $-\infty$ to ∞ are generally referred to as the complete root loci in control literature

Root loci: The portion of the root loci when K assumes positive values i.e $0 \leq K < \infty$

Complementary root loci: The portion of the root loci when K assumes negative values i.e. $-\infty < K \leq 0$

Root Contours : Loci of roots when more than one parameter varies.

Complete root loci ; The complete root loci refers to the combination of the root loci and the complementary root loci.

Basic Conditions of the root loci :

Let us consider the eq. ① in the ch. eq of a linear control System that has the closed - loop transfer function.

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \quad \text{--- ②}$$

Ch - eq. $1+G(s)H(s) = 0$

* In other words the left side of eq. ① should corresponds to the numerator of $1+G(s)H(s)$, ~~or the~~ or the roots of the ch. eq. must also satisfy :

$$1+G(s)H(s) = 0 \quad \text{--- ③}$$

Relationship between the ch. eq. [i.e. $1+G(s)H(s)=0$] and the loop tr. fn. $G(s)H(s)$:

Divide both side of eq. ① by the terms that do not contain K, we get

$$1+\frac{K(s^m+b_1 s^{m-1}+b_2 s^{m-2}+ \cdots + b_{m-1}s+b_m)}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+ \cdots +a_{n-1}s+a_n} = 0 \quad \text{--- ④}$$

comparing eq. ③ & eq. ④

$$G(s)H(s) = \frac{K(s^m+b_1 s^{m-1}+b_2 s^{m-2}+ \cdots + b_{m-1}s+b_m)}{s^n+a_1 s^{n-1}+ \cdots + a_{n-1}s+a_n} \quad \text{---} ($$

where $G(s)H(s)$ is known as the loop tr. fn of the control System

Golden Rule : The procedure of dividing both side of the ch. eq by the terms that do not contain K as the Golden Rule (

$$s(s+2)(s+3)+K(s+1)=0 \Rightarrow 1+\frac{K(s+1)}{s(s+2)(s+3)}=0$$

Complete root loci are the loci of the points in the s-plane that satisfy eq. (3) when K is varied between $-\infty$ and ∞

Let us express $G(s)H(s)$ as

$$G(s)H(s) = K\,G_1(s)H_1(s) \quad -- (6)$$

where $G_1(s)H_1(s)$ no longer contains the variable parameter K.

The eq. (3) can be written as

$$1 + K\,G_1(s)H_1(s) = 0$$

$$\therefore G_1(s)H_1(s) = -\frac{1}{K} \quad -- (7)$$

Since this is a complex eq., hence the following conditions must be satisfied simultaneously:

(i) magnitude condition: $$|G_1(s)H_1(s)| = \frac{1}{|K|} \quad \text{for } -\infty < K < \infty \quad -- (8)$$

and (ii) phase condition: $$\angle G_1(s)H_1(s) = (2q+1)\pi \quad \text{for } K \geq 0 \quad -- (9)$$

or $$\angle G_1(s)H_1(s) = 2q\pi \quad \text{for } K \leq 0 \quad -- (10)$$

where $q = 0, \pm 1, \pm 2, ---$ all integers

In practice, the complete root loci are constructed by finding all points in the s-plane that satisfy eq. (9) and eq. (10), and then the values of K along the loci are determined using eq. (8)

eq (5) must be written as

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2) ---- (s+z_m)}{(s+p_1)(s+p_2) --- (s+p_n)} = K\,G_1(s)H_1(s) \quad -- (11)$$

mag. or $$|G_1(s)H_1(s)| = \frac{\prod_{i=1}^{m} |s+z_i|}{\prod_{j=1}^{n} |s+p_j|} = \frac{1}{|K|} \quad \text{for } -\infty < K < \infty \quad -- (12)$$

phase $$\angle G_1(s)H_1(s) = \sum_{i=1}^{m} \angle s+z_i - \sum_{j=1}^{n} \angle s+p_j = (2q+1)\pi \quad \text{for } 0 \leq K < \infty \quad -- (13)$$

and $$\angle G_1(s)H_1(s) = \sum_{i=1}^{m} \angle s+z_i - \sum_{j=1}^{n} \angle s+p_j = 2q\pi \quad \text{for } -\infty < K \leq 0 \quad -- (14)$$

for $q = 0, \pm 1, \pm 2 ---$

04/12

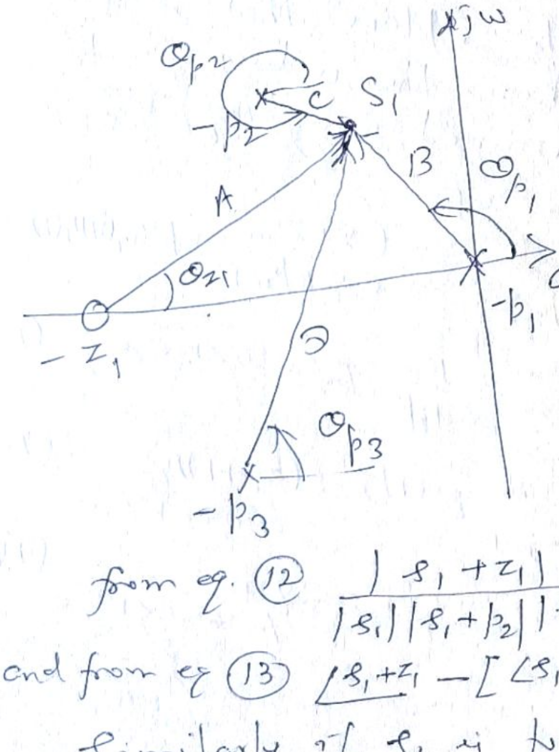
* eq (13) implies that for any positive value of K, a point (eg s_1) in the s-plane is a point on the root loci if the difference between the sums of the angles of the vectors drawn from the zeros and the poles of G(s)H(s) to s_1 is an odd multiple of 180°

* Similarly for negative values of K eq. (14) shows that any point on the complementary root loci must satisfy the condition that the differences between the sums of the angle of the vectors drawn from the zeros and the poles to the point is an even multiple of 180° or '0' degree

Let us consider: (to use of eq (13) & (14) for the construction of the root loci)

$$G(s)H(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)} \quad \text{— (15)}$$

location of the poles and the zeros of G(s)H(s) are arbitrarily assumed.



Now select the arbitrary point, s_1, in the s-plane and ~~the zeros of G(s)H(s) to the point s_1~~ draw the vectors directing from the poles and the zeros of G(s) H(s) to the point s_1. If s_1 is indeed a point on the root loci($0 \le K < \infty$) it must satisfy the following two conditions simultaneously

from eq. (12) $$\frac{|s_1+z_1|}{|s_1||s_1+p_2||s_1+p_3|} = \frac{1}{|K|} \quad \text{— (16)}$$

and from eq (13) $$\angle s_1+z_1 - [\angle s_1 + \angle s_1+p_2 + \angle s_1+p_3] = (2q+1)\pi \quad \text{— (17)}$$ for $q = 0, \pm1, \pm2$ —

Similarly if s_1 is to be a point on the complementary root loci ($-\infty < K \le 0$), then it must satisfy eq. (14)

05/12

i.e. $\angle s+z_1 - [\angle s_1 + \angle s_1+p_2 + \angle s_1+p_3] = 2q\pi$ — (18)

for $q = 0, \pm 1, \pm 2 \ldots$

* Root loci represents the loci of zeros of $[1+G(s)H(s)]$

* The angles $\theta_{z_1}, \theta_{p_1}, \theta_{p_2}$ & θ_{p_3} are the angles of the vectors measured with the positive real axis as zero reference.

eq. (17) & (18) becomes

$$\theta_{z_1} - (\theta_{p_1} + \theta_{p_2} + \theta_{p_3}) = (2q+1)\pi \quad \text{for } 0 \le K < \infty \quad -(19)$$

and $\theta_{z_1} - (\theta_{p_1} + \theta_{p_2} + \theta_{p_3}) = 2q\pi$ for $-\infty < K \le 0$ — (20)

If s_1 is found to satisfy either of eq. (19) or eq (20) then eq (16) is used to determine the value of K at that point

$$\therefore K = \frac{|s_1||s_1+p_2||s_1+p_3|}{|s_1+z_1|} \quad —— (21)$$

where $|s_1+z_1|$ is the length of the vector drawn from the zero z_1 to the point s_1

$$\therefore |K| = \frac{BCD}{A}$$

The sign of K, depends on whether s_1 is on the root loci or on the complementary root loci.

Spirule → A special graphical aid can be used to help the root locus plot.

06/12

Construction/steps of the Root Locus (Loci):

Step 1 Find out poles and zeros of $G(s)H(s)$ and plot in the s-plane

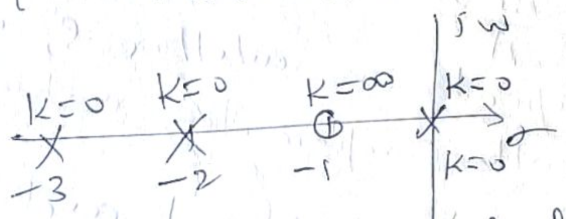
Step 2 The no. of root loci or the no. of root locus or branches is equal to the order of the ch. eq or the no. of poles or zeros whichever is greater

Step 3 The root locus starts from the open loop poles and it terminates at open loop zeros

$$\left|G_1(s)H_1(s)\right| = \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{j=1}|s+p_j|} = \frac{1}{|K|}$$

or $1+KG_1(s)H_1(s)=0 \Rightarrow |G_1(s)H_1(s)| = \left|-\frac{1}{K}\right| = \frac{1}{K}$

for eq. $s(s+2)(s+3)+K(s+1)=0 \Rightarrow 1+\underbrace{\frac{K(s+1)}{s(s+2)(s+3)}}_{\to G(s)H(s)}=0$



Step 4 The complete root locus are symmetrical with respect to the axis of the s-plane, or with respect to the axis of symmetry of the poles & zeros of $G(s)H(s)$ & the axis of symmetry of the poles & zeros of $G(s)H(s)$ located at infinity (∞).

Step 5 Those zeros which are located at infinity(∞), the root locus will approach asymptotically α.

(i) No. of asymptotes = No. of poles − No. of zeros $=(p-z)$

(ii) Angle of asymptotes with +ve direction of σ-axis is given by the formula $\theta_q = \frac{(2q+1)\pi}{p-z}$ for $q=0,1,2,\ldots\{|p-z|-1\}$ for $K\geq 0$

and $\theta_q = \frac{2q\pi}{p-z}$ for $K\leq 0$

(iii) The asymptotes will cross on the real axis and this cross-over point is called as Centroid.

(07/12)

<u>Centroid</u> : The point at which the asymptotes intersect is known as centroid.

$$\text{Centroid} = \sigma = \frac{\sum p_i - \sum z_i}{p - z}$$

$$= \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{p - z}$$

<u>Step 6</u> A point on the real axis is a part of the locus if the no. of poles and zeros to the right of the point concerned is odd.

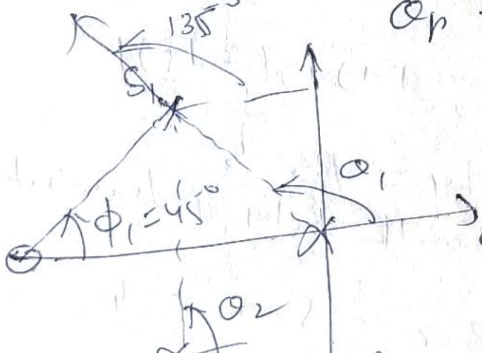
<u>Step 7</u> <u>Angle of departure (from the poles) and the angle arrival (at zeros)</u>

$$\text{Angle of departure} = (2k+1)\pi + \theta_z - \theta_p$$

where θ_z = sum of the angles by zeros

θ_p = sum of the angles by poles

and $k = 0, \pm 1, \pm 2$ ---



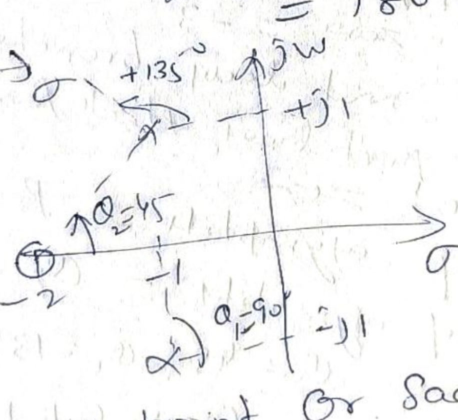
Angle of departure at point P_1

$$= 180 + \phi_1 - (\theta_1 + \theta_2)$$

$\theta_P = 180 + 45 - (135 + 90)$

$= 180 + 45 - 225$

$= 225 - 225 = 0$



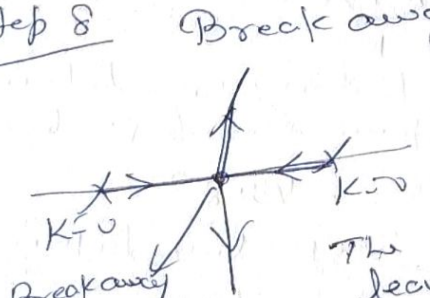
Angle of departure

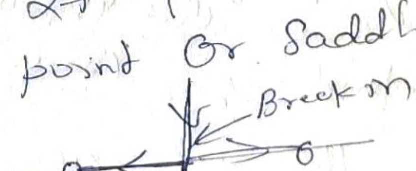
$= 180 + \theta_2 - \theta_1$

$= 180 + 45° - 90°$

$= +135°$

<u>Step 8</u> Breakaway point Or Saddle point





The points where the root locus leave are called <u>breakaway point or saddle point</u>

$$\frac{dK}{ds} = 0 \quad \text{Or} \quad \frac{d\,G(s)H(s)}{ds} = 0$$

(08/12)

Step 9 Intersection with the $j\omega$ - axis $\rightarrow$ obtained by R-H method

Step 10 Calculation of the values of K on the root loci

$$|K| = \frac{1}{|G_1(s)H_1(s)|}$$

$$= \frac{\text{product of the length of vectors drawn from the poles of } G(s)H(s) \text{ to } s_1}{\text{product of the length of vectors drawn from the zeros of } G(s)H(s) \text{ to } s_1}$$

Example 1 Draw the root locus of the given system. $s(s+5)(s+6)(s^2+2s+2)+K(s+3)=0$

Ans

Step 1 using Golden rule $\longrightarrow$ Find loop t.f.

$$G(s)H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$$

Step 2 $P = 5$ & $Z = 1$

location of poles $= 0, -5, -6, -1+j1$ & $-1-j1$

" " zero $= -3$

Step 2 no. of root locus branches $= P = 5$

Step 3 Root locus starts from the poles & terminates at zero

Step 4 no of asymptotes $= P-Z = 5-1 = 4$

Angle of asymptotes $= \theta_q = \frac{(2q+1)\pi}{P-Z}$

$\theta_0 = 45$, $\theta_1 = 135$, $\theta_2 = 225$ or $-45°$ & $\theta_3 = 315°$ or $-135°$

Centroid $= \sigma_c = \frac{(0-5-6-1-1)-(-3)}{5-1} = -\frac{10}{4} = -2.5$

Step 5 On the real axis root locus are between $s=0$ and $s=-3$ & between $s=-5$ & $s=-6$.

Step 6 Angle of departure $= \theta = -43.8°$

Step 7 Break away point — It lies between -5 & -6

$s = -5.53$ - by trial & error

(09/B)

Step 8 Intersection with the jω-axis – by R-H method

Ch. eq $= s^5 + 13s^4 + 54s^3 + 82s^2 + (60+K)s + 3K = 0$

s^5	1	54	$(60+K)$
s^4	13	82	$3K$
s^3	47.7	$60+0.769K$	0
s^2	$65.6-0.212K$	$3K$	0
s^1	$\dfrac{3940-105K-0.163K^2}{65.6-0.212K}$	0	0
s^0	$3K$	0	

$\rightarrow K < 309$

$\rightarrow K < 35$

$\rightarrow K > 0$

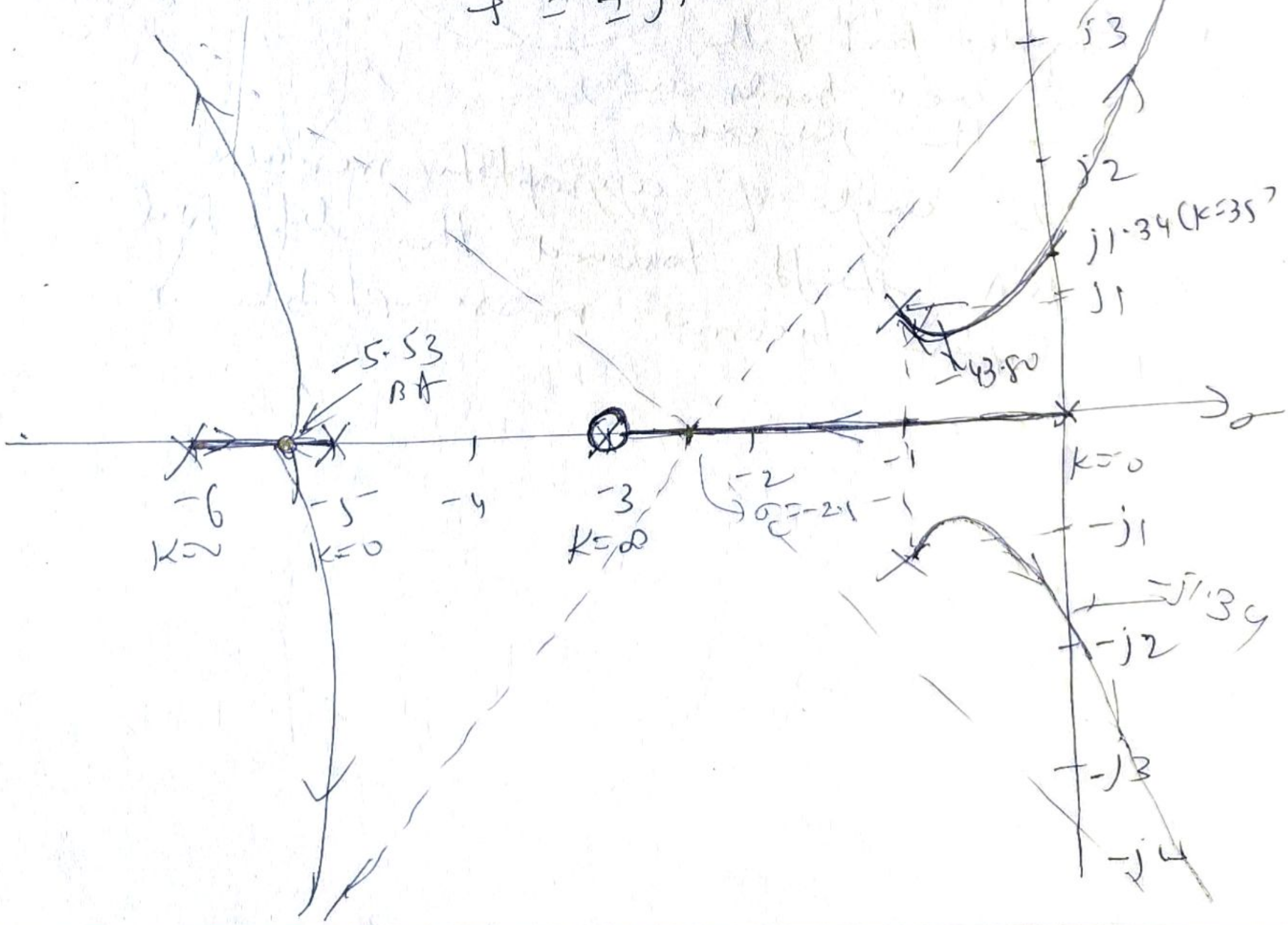
Here $0 < K < 35$

$A(s) = (65.6 - 0.212K)s^2 + 3K = 0$

substituting $K = 35$

$58.2s^2 + 105 = 0$

$s = \pm j1.34$



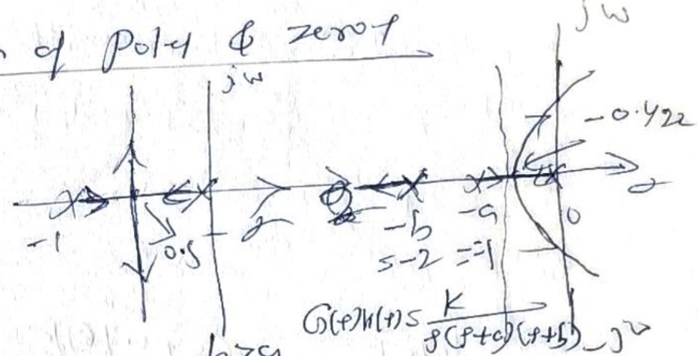
(10/12)

Effect of addition of Poles & zeros

~~Addition of pole~~

Ex- $G(s)H(s) = \frac{K}{s(s+a)}$; $a > 0$

let $a = 1$



$G(s)H(s) = \frac{K}{s(s+a)(s+b)}$

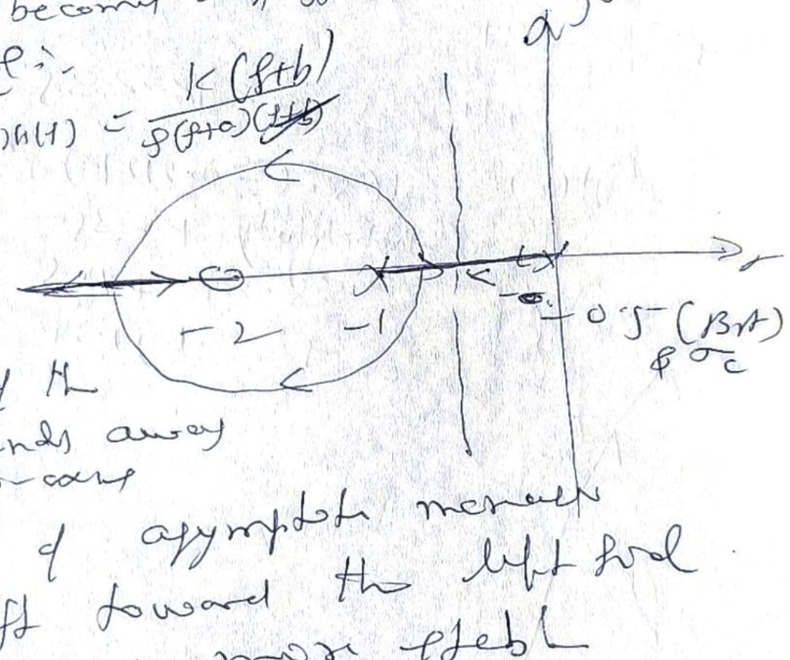
$b > a$

1. Complex part of the root loci bends towards the right half of s-plane
2. The angle of asymptotes changes from ±90° to ±60°
3. The break away point is also move to the right side i.e. for -0.5 to -0.422
4. The system becomes less stable

Addition of zero:-

$G(s)H(s) = \frac{K}{s(s+c)} \rightarrow G(s)H(s) = \frac{K(s+b)}{s(s+a)\cancel{(s+b)}}$

$b > c$



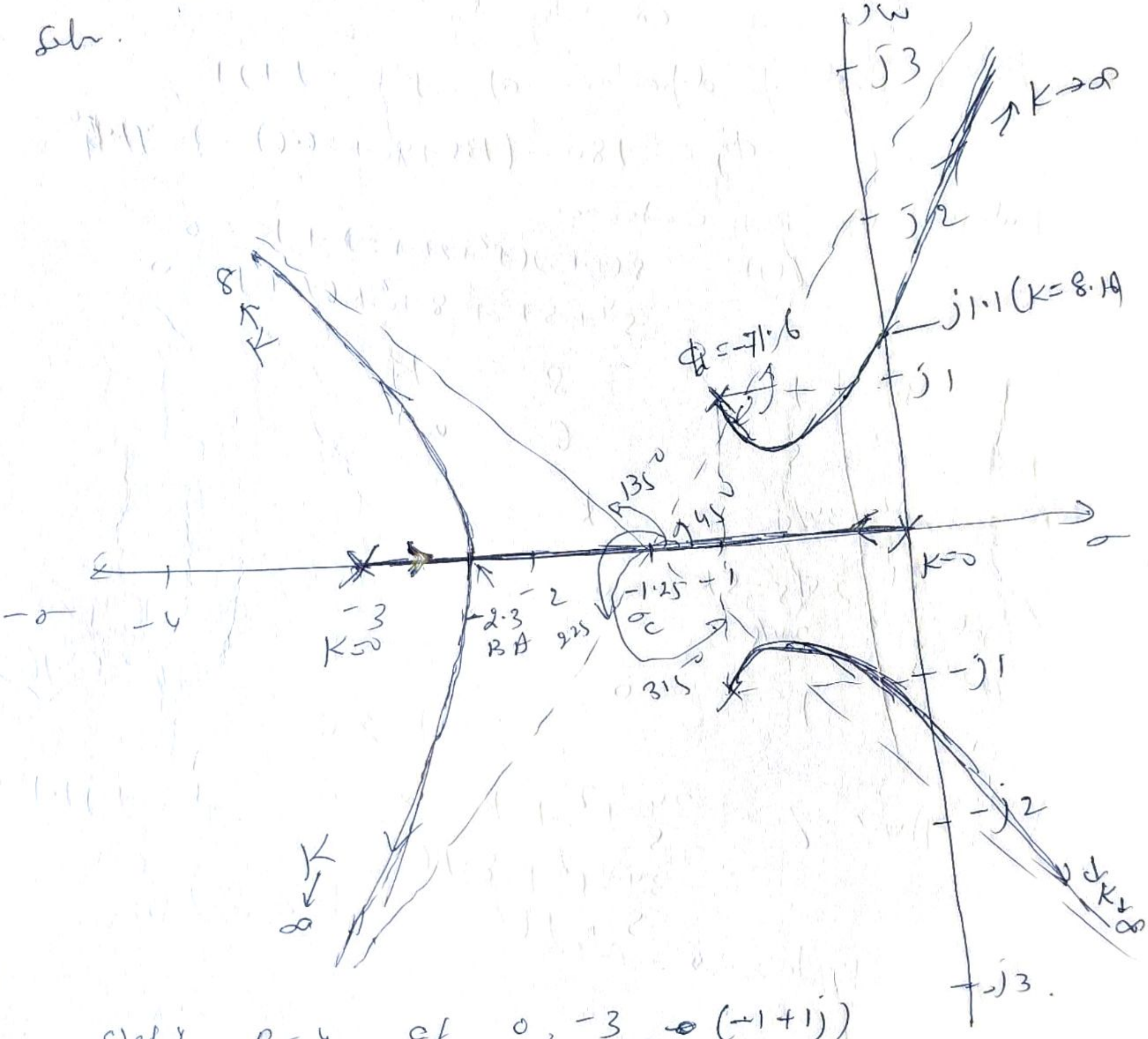
1. Complex part of the root loci bends away from the jw-axis
2. The angle of asymptote moves towards the left side.
3. BA shift toward the left
4. System becomes more stable

(11-R)

Q A feedback control system has an open-loop transfer function $G(s)H(s) = \frac{K}{s(s+3)(s^2+2s+2)}$

Sketch the root locus as K is varied from 0 to ∞

Soln.



Step 1 $P = 4$, at $0, -3, (-1+1j)$ & $(-1-1j)$

$Z = 0$, no finite zero

Step 2 no of root locus branches = 4

Step 3 Root locus exists between $-3 \le s \le 0$ on real axis

Step 4 no. of asymptotes $= P - Z = 4 - 0 = 4$

Angle of asymptotes $= \theta_q = \frac{(2q+1)\pi}{P-Z} \Rightarrow \theta_0 = 45, \theta_1 = 135, \theta_2 = 225$ & $\theta_3 = 315°$

σ_c (centroid) $= \frac{-3-1-1}{4} = -1.25$

(1Q-R)

Rule 6 BA $K = -s(s+3)(s^2+2s+2)$

$= -(s^4+5s^3+8s^2+6s)$

$\frac{dK}{ds} = -4(s^3+3.75s^2+4s+1.5) = 0$

on solving $s = -2.3$ ✓, $-0.725 \pm j0.365$ ✗

Rule 7 Angle of departure at $s = -1+j1$

$\phi_d = 180 - (135+90+26.6) = -71.6^0$

Rule 8 use R-H criterion

$s(s+3)(s^2+2s+2) + K = 0$

$q(s) = s(s+3)(s^2+2s+2) + K = 0$

$= s^4 + 5s^3 + 8s^2 + 6s + K = 0$

s^4	1	8	K
s^3	5	6	0
s^2	34/5	K	
s^1	$\frac{204/5 - 5K}{34/5}$	0	
s^0	K	0	

$\frac{204}{5} - 5K > 0 \rightarrow K = 8.16$

$K > 0$

Aux eq: $\frac{34}{5}s^2 + K = 0 \rightarrow s = \pm j1.1$

$\frac{34}{5}s^2 + 8.16 = 0$

System is stable for $K < 8.16$

④ ①

Q Plot the root locus for a unity feedback system having loop tr. fn $G(s)H(s) = \frac{K}{s(s+1)(s+2)}$. when K varies from 0 to ∞. Also determine the gain if damping ratio is 0.5.

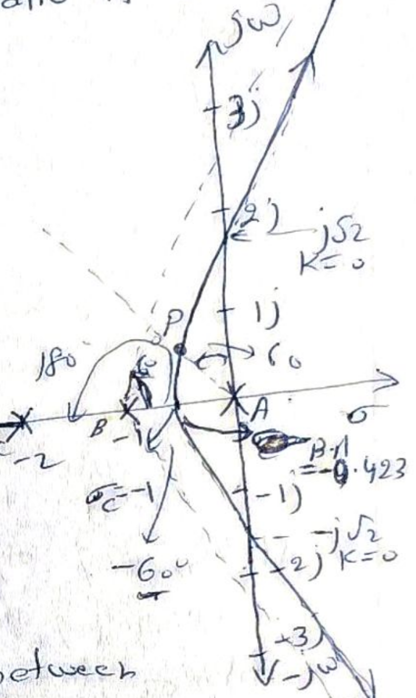
Ans Given $G(s)H(s) = \frac{K}{s(s+1)(s+2)}$

1) P = No. of Poles = 3 at 0, −1 & −2

Z = no. of zero = 0

2) No. of root locus branches = 3

3) Since there is no open loop zero in the finite region, hence all the branches terminate at infinity along the asymptote.



(4) Angle of asymptote $= \theta_A = \frac{(2k+1)}{P-Z}\pi$

−σ −3 C −2

for k = 0, 1, 2, 3 ---

$= 60^\circ, 180^\circ \& 300^\circ$

σ_c (centroid) $= \frac{-1-2}{3} = -1 = \frac{\Sigma p_i - \Sigma z_i}{P-Z}$

(5) On the σ-axis root locus lies between 0 & −1 and −2 & $-\infty$

(6) Breakaway point (B·A) ie $\frac{dK}{ds} = 0$ Or $\frac{dG(s)H(s)}{ds} = 0$

$K = -s(s+1)(s+2) = -(s^3+3s^2+2s) \Rightarrow \frac{dK}{ds} = -(3s^2+6s+2) = 0$

$s_{1,2} = -\frac{6 \pm \sqrt{36-24}}{6} = -0.423, \cancel{-1.577}$

BA = −0.423

(7) Intersection with jω-axis → ch. eq. $= 1+G(s) = 1+\frac{K}{s(s+1)(s+2)} = 0$

$\Rightarrow s(s+1)(s+2)+K = 0$ or $s^3+3s^2+2s+K = 0$

use R-H criteria

s^3	1	2	0
s^2	3	K	0
s^1	$\frac{6-K}{3}$	0	
s^0	K		

→ K > 0

→ K ≤ 6

Critical value of K = 6

Aux eq $A(s) = 3s^2 + 6 = 0$

$s^2 = -\frac{6}{3} = -2 = (2j)^2$

$s = \pm j\sqrt{2}$

(8) Draw a line from the origin that makes an angle of $\cos^{-1}\xi = \cos^{-1}0.5 = 60^\circ$ with the real axis

Co-ordinate of P = −0.33 + j0.58

K = 0.667 × 0.908 × 1.768 = 1.07

At s = −2.5, K = 0.5 × 1.5 × 2.5 = 1.875

③ ②

Q Sketch the root locus of the unity feedback system having

$$G(s) = \frac{K}{s(s+2)(s+4)}$$

where K is varied from 0 to ∞. Hence obtain the value of K for which the system is unstable

Solution: $G(s)H(s) = \frac{K}{s(s+2)(s+4)}$

Step 1 $P = 3$ at $0, -2, -4$

$Z = 0$.

Step 2 No of root locus branch = $P = 3$

Step 3 The three branches of the root locus start at the open-loop poles at $s = 0$, $s = -2$ and $s = -4$ when $K = 0$. The three branches terminated at the zero at infinity when $K = \infty$

Step 4 (i) No of asymptote $= P - Z = 3 - 0 = 3$

(ii) Angle of asymptotes $= \frac{(2q+1)\pi}{P-Z}$; $\theta_0 = \frac{(2\times0+1)\pi}{3} = \frac{\pi}{3} = 60°$

$\theta_1 = 180°$ & $\theta_2 = 300°$

(iii) $\sigma_c = \frac{(0-2-4)-0}{3} = -2$

Step 5 ~~Some the~~ on the real axis root are (i) between $s = 0$ & $s = -2$

(ii) between $s = -4$ & $s = -\infty$

Step 6 Since there is no complex pole so there is no requirement of angle of departure

Step 7 Break away point

$\frac{dK}{ds} = 0$

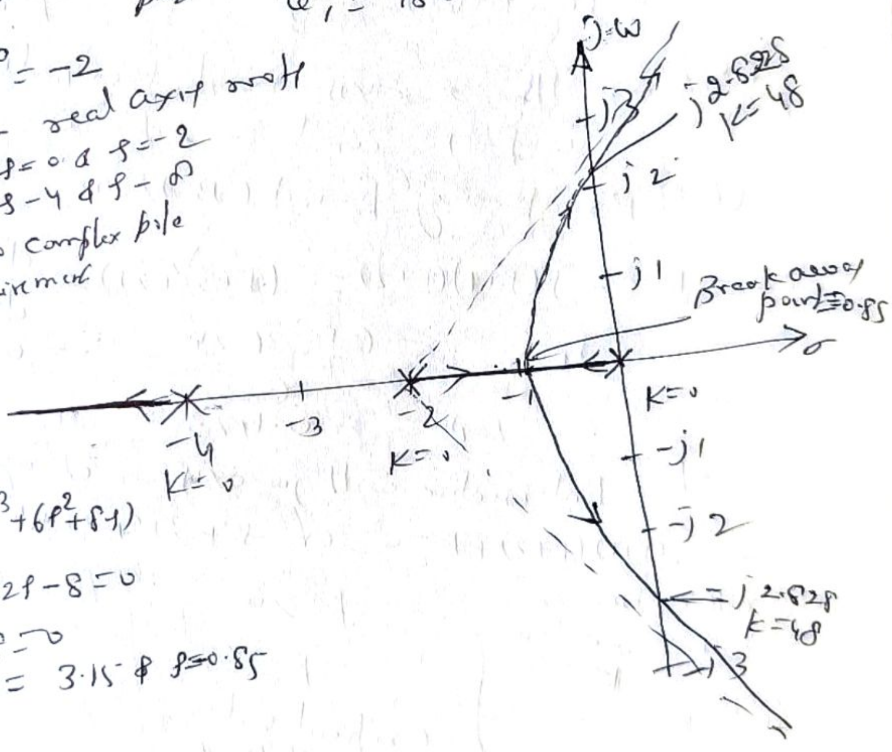
$s(s+2)(s+4) + K = 0$

$K = -s(s+2)(s+4) = -(s^3 + 6s^2 + 8s)$

$\frac{d\{-s^3 - 6s^2 - 8s\}}{ds} = -3s^2 - 12s - 8 = 0$

or $3s^2 + 12s + 8 = 0$

$\therefore s = -2 \pm 1.15 = 3.15$ & $s = 0.85$



Step 8 $1 + G(s)H(s) = 0$

ch eq $s^3 + 6s^2 + 8s + K = 0$

s^3	1	8
s^2	6	K
s^1	$\frac{48-K}{6}$	0
s^0	K	

$K > 0$

$48 - K > 0$

$K < 48$

$0 < K < 48$

$6s^2 + \cancel{K} = 0$

$6s^2 + 48 = 0$

$\therefore s = \pm\sqrt{-8} = \pm j2.828$

Root Locus Technique

Gain and phase margin from root locus plot:

Gain Margin (G.M) and Phase Margin (PM) of a control system can be determined by root locus plot.

Gain Margin: G.M: Gain margin is the factor by which
Def:
we should multiply the design value of gain K before the system becomes unstable.

A system becomes on the verge of loosing stability when root locus intersects the imaginary axis.

$$\text{Gain Margin} = \frac{\text{Value of gain at the intersection point of root locus and imaginary axis}}{\text{Design value of gain } K}$$

When the root locus does not cross imaginary axis, gain margin becomes infinite as the system will not loose its stability for any value of gain K.

Phase Margin:

Find out the point $j\omega_1$ on the imaginary axis such that $|G(j\omega_1)H(j\omega_1)| = 1$ for the design value of K. From this, we determine ω_1. Use trial and error method to find out ω_1. After finding ω_1, find out $\angle G(j\omega_1)H(j\omega_1)$.

Phase margin is given by —

$$\text{Phase margin} = 180^\circ + \angle G(j\omega_1)H(j\omega_1)$$

Q1. For unity feedback system $G(s) = \dfrac{K}{s(s+1)(s+3)}$ plot the root locus. Also comment on the stability.

Soln-
$$G(s)H(s) = \frac{K}{s(s+1)(s+3)}$$

$P = 3,\ Z = 0,\quad$ 3 no. of branches

Asymptotes $\quad \sigma = \dfrac{-1-3}{3} = -1.33$

$$\theta_q = \frac{(2q+1)}{P-Z}\pi \rightarrow \theta_1 = 60^\circ,\ \theta_2 = 180^\circ,\ \theta_3 = 300^\circ$$

~~Intersection with the jω axis~~; Break away point, $\quad$ ch. eq. $1 + G(s)H(s) = 0$

or $1 + \dfrac{K}{s(s+1)(s+3)} = 0$

or $s^3 + 4s^2 + 3s + K = 0$

$K = -(s^3 + 4s^2 + 3s)$

$\therefore \frac{dK}{ds} = \frac{d}{ds}\left\{-(s^3+4s^2+3s)\right\} = 0$

or $-(3s^2+8s+3) = 0$

$\therefore 3s^2+8s+3 = 0$

$s = \frac{-8 \pm \sqrt{64-36}}{6} = -2.21, -0.88$

So breakaway point is at $s = -0.88$ as -2.21 lies in the region where root locus is not existing.

<u>Intersection with jω-axis & critical gain :</u>

ch. eq. is $s^3+4s^2+3s+K = 0$

Use R-H criterion:

s^3	1	3
s^2	4	K
s^1	$\frac{12-K}{4}$	0
s^0	K	

Stable range is $K > 0$ & $\frac{12-K}{4} > 0$

$0 < K < 12$

At $K=12$, the system will be marginally stable.

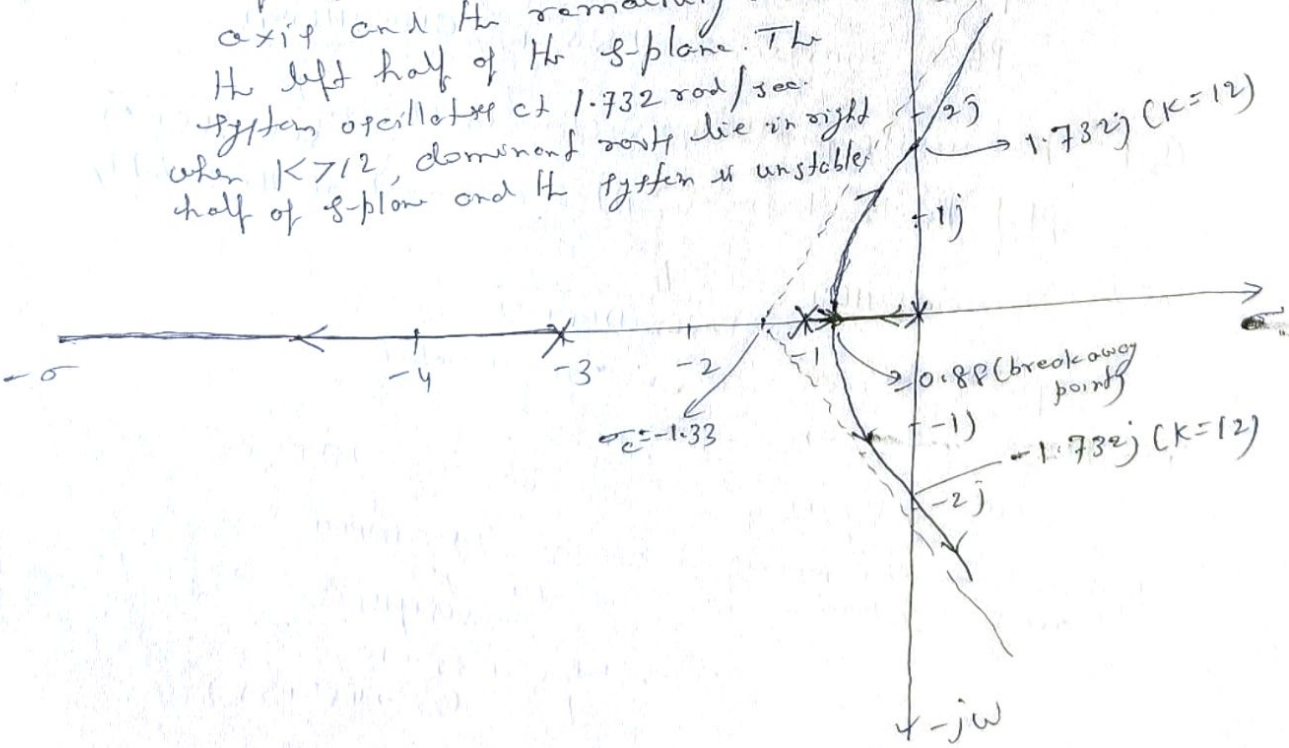
At this point, the freq. is given by

$4s^2+12 = 0$ so $s = \pm j1.732$ rad/sec

System is stable for $0 < K < 12$.

At $K=12$, the system is marginally stable. At this value of K, a pair of dominant roots lies on the imaginary axis and the remaining roots is the left half of the s-plane. The system oscillates at 1.732 rad/sec.

When $K>12$, dominant roots lie in right half of s-plane and the system is unstable.



3

Q2. The open loop tr. fn. of unity feedback control system is given by $\frac{K}{s(s+1)(s+2)}$. Plot root locus with K as variable parameter from zero to infinity. Determine whether it is possible to obtain damping factor 0.5 and if so the values of K to obtain this damping factor.

Soln

$$G(s) = \frac{K}{s(s+1)(s+2)} \;;\; H(s) = 1 \;;\; G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

$P = 3,\ Z = 0$, P's are at $0, -1, -2$ and no explicit zero.

There will be three nos. of branches and these will terminate at infinity.

centroid $\sigma_c = \frac{-1+(-2)}{P-Z} = -\frac{3}{3} = -1$

Angles of asymptotes with real axis

$$\theta_q = \frac{(2q+1)\pi}{P-Z} \rightarrow \theta_0 = 60^\circ,\ \theta_1 = 180^\circ \ \& \ \theta_2 = 300^\circ$$

Breakaway points $\quad 1 + G(s)H(s) = 0$

or $1 + \frac{K}{s(s+1)(s+2)} = 0$

$\Rightarrow s^3 + 3s^2 + 2s + K = 0$

$K = -(s^3 + 3s^2 + 2s)$

$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$

so, $s_1 = -1.573$, & $s_2 = -0.422$

Breakaway point is -0.422.

Intersection with imaginary axis: use R-H criterion

ch. eq. $s^3 + 3s^2 + 2s + K = 0$

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	0
s^0	K	

$K > 0$

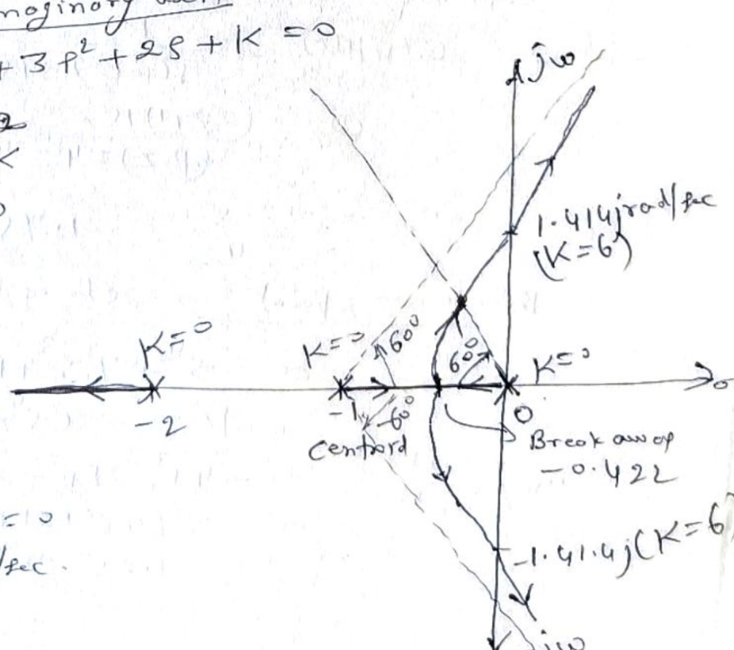
& $\frac{6-K}{3} > 0$

stable range is $0 < K < 6$

critical value $K \leq 6$

$A(s) = 3s^2 + K = 3s^2 + 6 = 0$

$s = j1.414$ rad/sec.



Since $\xi = 0.5$ (4)

draw a line on the root locus equal to an angle $\theta = \cos^{-1} 0.5 = 60$ with the real axis

The line intersects root locus in stable region when $s = -0.34 + j0.56$.

we know that on root locus

$$|G(j\omega)H(j\omega)| = 1$$

i.e $\left| \frac{K}{s(s+1)(s+2)} \right| = 1$

At $s = j\omega$

i.e $\left| \frac{K}{j\omega(j\omega+1)(j\omega+2)} \right| = 1$

At $s = -0.34 + j0.56$

$$\left| \frac{K}{(-0.34+j0.56)(-0.34+j0.56+1)(-0.34+j0.56+2)} \right| = 1$$

solving this we get $K = 0.99$

Q3 A unity feedback control system has the open loop transfer function $G(s) = \frac{K}{s^2(s+2)(s+5)}$.

Sketch the root locus diagram indicating clearly the break-away points, asymptotes and their centroid. Find the value of K and ω at the imaginary axis intersection. For what values of K is the closed loop system stable?

Soln $G(s)H(s) = \frac{K}{s^2(s+2)(s+5)}$; $P = 4$, $Z = 0$, root locus branches $= 4$

$\theta_q = \frac{(2q+1)180}{(P-Z)=4}$; $\theta_0 = 45^\circ$, $\theta_1 = 135^\circ$, $\theta_2 = 225^\circ$ & $\theta_3 = 315^\circ$

$\sigma_c = \frac{-2-5}{4} = -1.75$

Break away point : ch. eq. $1 + \frac{K}{s^2(s+2)(s+5)} = 0$

or $s^4 + 7s^3 + 10s^2 + K = 0$

$K = -(s^4 + 7s^3 + 10s^2)$

using $\frac{dK}{ds} = 0$, $-4s^3 - 21s^2 - 20s = 0$

or $s(4s^2 + 21s + 20) = 0$

$\therefore s = 0, -1.25$ and -4.0

Root Locus

(1)

Q1 Sketch the root locus (loci) for the system shown in fig 1. and show that the system is stable for all $K > 0$.

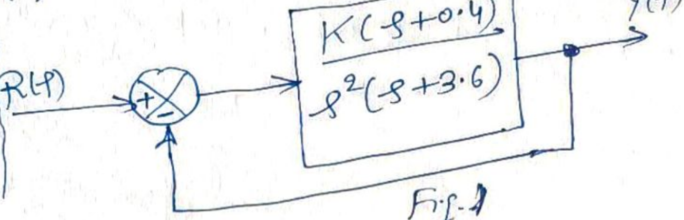


Fig-1

Soln. $G(s)H(s) = \frac{K(s+0.4)}{s^2(s+3.6)}$

$P = 3,\ Z = 1$

$P_1 = 0,\ P_2 = 0,\ P_3 = -3.6$

$Z_1 = -0.4$

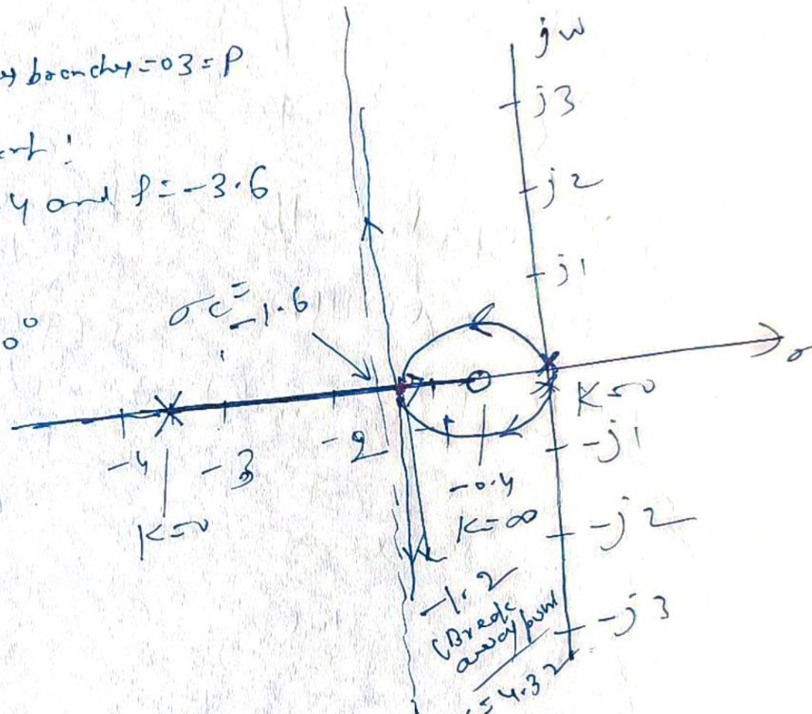
no. of root locus branches $= 03 = P$

real axis locus segment:

between $s = -0.4$ and $s = -3.6$

$\sigma_c = -1.6$

Angle of asymptotes $= \pm 90^\circ$



Break away points

$1 + G(s)H(s) = 0$

$1 + \frac{K(s+0.4)}{s^2(s+3.6)} = 0$

$-K = \frac{s^2(s+3.6)}{(s+0.4)}$

$-\frac{dK}{ds} = \frac{(3s^2+7.2s)(s+0.4) - (s^3+3.6s^2)}{(s+0.4)^2}$

use $\frac{dK}{ds} = 0$

$\Rightarrow s^3 + 2.4s^2 + 1.44s = 0$

or $s(s+1.2)^2 = 0 \Rightarrow s = 0, -1.2$

Thus $s = 0$ is break away point and $s = -1.2$ is break in point. Note that $s = -1.2$ is repeated root

value of gain K at $s = -1.2$ is $= 4.32$

root loci meet at point $s = -1.2$ for $K = 4.32$

Break away angle $= \pm 180^\circ / r = \pm \frac{180^\circ}{3} = \pm 60^\circ$

Imaginary axis crossing point :

Ch-eq $\quad 1+\dfrac{K(s+0.4)}{s^2(s+3.6)}=0$

$s^3+3.6s^2+Ks+0.4K=0$

s^3	1	K
		$0.4K$
s^2	3.6	
		0
s^1	$\dfrac{3.2K}{3.6}$	
s^0	$0.4K$	$K>0.$

Q2. Sketch the root locus for the loop tr-f

$$G(s)H(s)=\frac{K}{s(s+3)(s^2+3s+11.25)}$$

and comment on stability.

Soln: $P=4$, $Z=0$, location $s=0,-3,-1.5\pm j3$

root locus branch $=4$

$\sigma_c=-1.5$, $\theta_A=45^\circ, 135^\circ, 225^\circ$ & 315°

or. $\pm 45^\circ$ & $\pm 135^\circ$

Break away points :

$$1+\frac{K}{s(s+3)(s^2+3s+11.25)}=0$$

$$-K=s(s+3)(s^2+3s+11.25)$$

$$=s^4+6s^3+20.25s^2+33.75s$$

$$-\frac{dK}{ds}=\frac{d}{ds}[s^4+6s^3+20.25s^2+33.75s]=0$$

$$=4s^3+18s^2+40.5s+33.75$$

$s=-1.5$ and $-1.5\pm j1.84$

The point $s=-1.5$ lies on real axis locus segment.

So, this is an actual break away point.

To test whether or not, the points $s=-1.5\pm j1.84$, are actual break away/entry point, we angle criterion

$$\angle \frac{K}{s(s+3)(s+1.5+j3)(s+1.5-j3)} \Bigg|_{s=-1.5+j1.84}$$

$$= -\tan^{-1}\left(\frac{1.84}{-1.5}\right) - \tan^{-1}\left(\frac{1.84}{1.5}\right) - \tan^{-1}(\infty) - \tan^{-1}(-\infty)$$

$$= -180^{\circ}$$

Thus the angle criterion is satisfied and points $s=-1.5 \pm j1.84$ are actual break away/entry points.

Use magnitude criterion i.e. $|G(s)H(s)| = 1$, to obtain the corresponding values of K as follows

$$K\Big|_{s=-1.5} = \left| s(s+3)(s^2+3s+11.25) \right|_{s=-1.5}$$

$$= 20.25$$

and $$K\Big|_{s=-1.5+j1.84} = \left| s(s+3)(s^2+3s+11.25) \right|_{s=-1.5+j1.84} = 31.64$$

Intersection of root locus with imaginary axis:

The ch. eq. $s(s+3)(s^2+3s+11.25)+K=0$

or $s^4+6s^3+20.25s^2+33.75s+K=0$

s^4	1	20.25	K
s^3	6	33.75	
s^2	14.625	K	
s^1	$\frac{493.59-6K}{14.625}$	0	
s^0	K		

$\rightarrow K = 82.27$

$A(s) = 14.625 s^2 + K = 0$

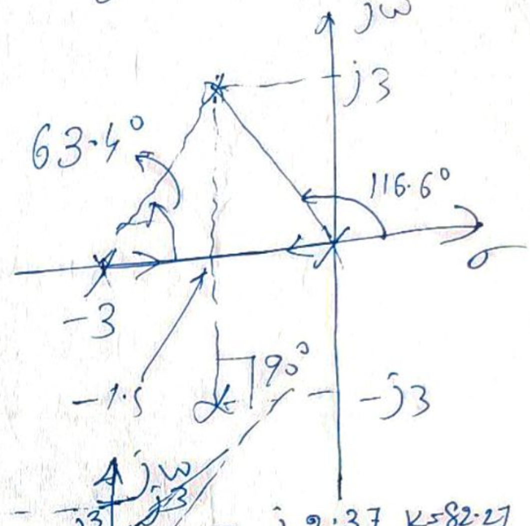
$$s^2 = -\frac{82.27}{14.625}$$

$$s = \pm j2.37$$

So the locus branch crosses the imaginary axis at $s = \pm j2.37$.

Angle of departure from top complex pole:

$$\phi_d = 180^\circ - (90^\circ + 63.4^\circ + 116.6^\circ) = -90^\circ$$



Comment on stability

For $0 < K < 82.27$ system is stable

For $K = 82.27$ system is marginally stable

For $K > 82.27$ system is unstable

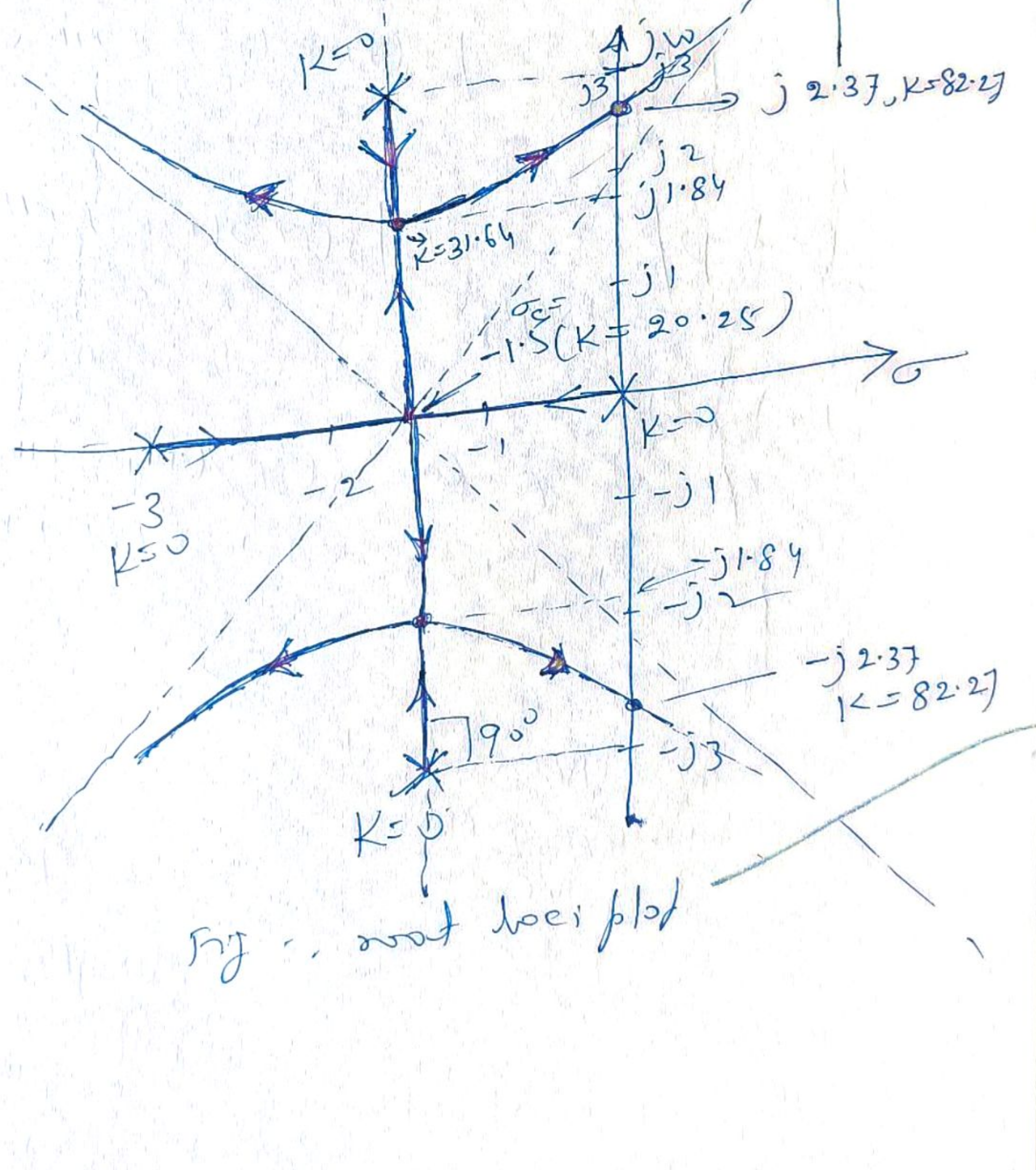


Fig. : root loci plot

Q1 Sketch the root locus for a unity feedback system with open loop transfer function

$$G(s) = \frac{K(s+0.5)}{s^2(s+4.5)}$$

Sol. Angle of asymptotes = $90°$ & $270°$

centroid $\sigma_c = -2$

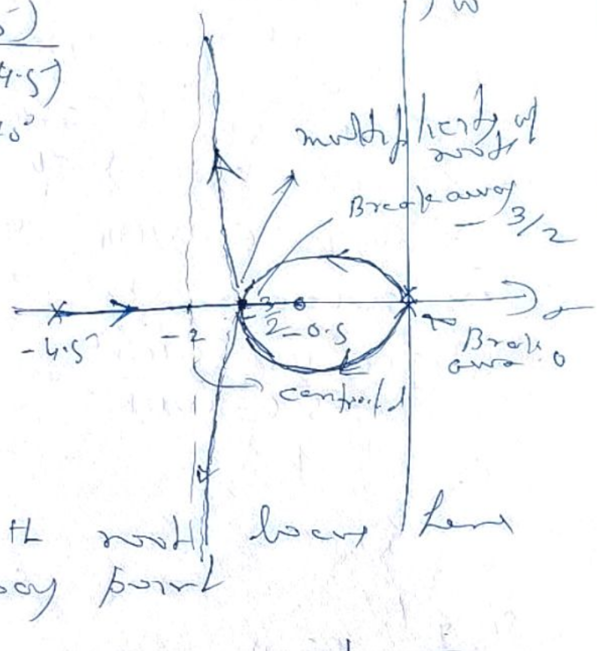
Break away point $\frac{dK}{ds} = 0$

$$K = -\frac{s^2(s+4.5)}{s+0.5}$$

$$\frac{dK}{ds} = -\frac{2s(s+\frac{3}{2})^2}{(s+0.5)^2}$$

$\therefore s = 0, -\frac{3}{2}, -\frac{3}{2}$

Both 0 & $-\frac{3}{2}$ are on the root locus hence both are break away point



Ch. eq: $s^3 + 4.5s^2 + Ks + 0.5K = 0$

s^3	1	K	0
s^2	4.5	0.5K	0
s^1	$\frac{4K}{4.5}$	0	
s^0	0.5K		

$0.5K > 0 \Rightarrow K > 0$

$\frac{4K}{4.5} > 0 \Rightarrow K > 0$

Stable range = $0 < K < \infty$

Q. Draw the root locus of the given system

$$s(s+5)(s+6)(s^2+2s+2) + K(s+3) = 0$$

Ans $$G(s)H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$$

Step 1: $P = 5$, at $0, -5, -6, -1-j, -1+j$

$Z = 1$, at -3.

Step 2 No. of root locus branches $= P = 5$

Step 3 No. of asymptotes $= P - Z = 5 - 1 = 4$

Angle of asymptotes $= \theta_0 = \frac{(2q+1)\pi}{P-Z} = \frac{180}{4} = 45°$

$\theta_1 = 135°$, $\theta_2 = 225°$ or $-45°$ & $\theta_3 = 315°$ or $-135°$

$$\sigma_c = \frac{(0-5-6-1-1)-(-3)}{5-1} = \frac{-10}{4} = -2.5$$

Step 4 Root locus on the real axis between $s=0$ & -3, $s=-5$ & -6

Step 5 Angle of departure is $\theta = -43.8°$

Step 6 Intersection with $j\omega$-axis.

ch. eq. $q(s) = s^5 + 13s^4 + 54s^3 + 82s^2 + (60+K)s + 3K = 0$

using R-H crit

s^5	1	54	$60+K$
s^4	13	82	$3K$
s^3	47.7	$60+0.769K$	0
s^2	$65.6-0.212K$	$3K$	0
s^1	$\dfrac{3940-105K-0.163K^2}{65.6-0.212K}$	0	
s^0	$3K$		

$65.6 - 0.212K > 0$

$\Rightarrow K < 309$

$3940 - 105K - 0.163K^2 > 0$

$\Rightarrow K < 35$

$K > 0$

Critical value $K = 35$

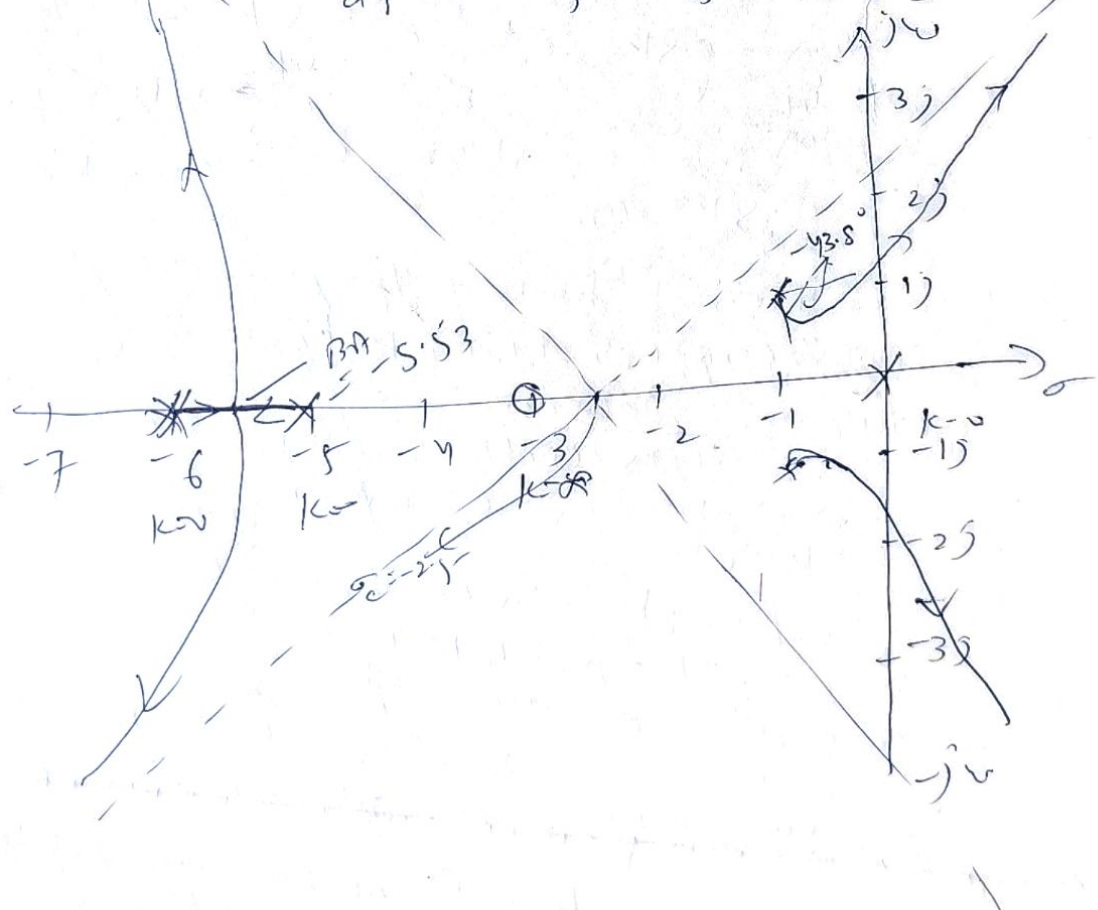
$A(s) = (65.6 - 0.212K)s^2 + 3K = 0$

or $58.2s^2 + 105 = 0$

$s = \pm j1.34$

Step 7 Break away point

w.r. $\frac{dK}{ds} = 0$, $s = -5.53$



(2) Lin's Method :- (1) (6)

Consider algebraic eq.

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_2 s^2 + a_1 s + a_0 = 0 \quad -(1)$$

Find trial factor using the three lowest order terms of the original equation from eq (1), find trial factor is $a_2 s^2 + a_1 s + a_0$ or in the form

$$s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}.$$

This trial factor is divided by original eq.

$$
\begin{array}{r|l}
 & s^{n-2} + \dots \\
s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2} & s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_2 s^2 + a_1 s + a_0 \\
 & \qquad \vdots \\
 & \qquad b_2 s^2 + b_1 s + b_0 \\
 & \qquad c_2 s^2 + c_1 s + c_0 \\
\hline
 & \qquad \text{Remainder}
\end{array}
$$

If the remainder is too large, the next trial used is

$$s^2 + \frac{b_1}{b_2}s + \frac{b_0}{b_2}$$

This procedure is continued until the remainder is negligible. The last trial factor is a quadratic factor of the original equation. Complex roots occur in complex conjugate pairs in the original equation.

* If highest power of the original equation is odd, then there must be at least one real root. To determine this root, the trial divisor is chosen from the two lowest order terms of the original eq. If n is odd, the first trial divisor

$$s + \frac{a_0}{a_1}$$

The original equation is divided by this divisor and process is continued.

(2) Ex apply Lin's method to the following (9)

eq $s^4 + 9s^3 + 30s^2 + 42s + 20 = 0$ (1)

First trial divisor is

$$s^2 + \frac{42}{30}s + \frac{20}{30} = s^2 + 1.4s + 0.67.$$

division is performed as follows:-

$$
\begin{array}{r|l}
 & s^2 + 7.6s + 18.69 \\
\hline
s^2 + 1.4s + 0.67 & s^4 + 9s^3 + 30s^2 + 42s + 20 \\
 & \underline{s^4 + 1.4s^3 + 0.67s^2} \\
 & 7.6s^3 + 29.33s^2 + 42s \\
 & \underline{7.6s^3 + 10.64s^2 + 5.09s} \\
 & 18.69s^2 + 36.9s + 20 \\
 & \underline{18.69s^2 + 26.16s + 12.52} \\
 & 10.74s + 7.48
\end{array}
$$

comparing the coefficients of the remainder with those of the first divisor, it is found that the remainder is not negligible. The second trial divisor is

$$s^2 + \frac{36.9}{18.69}s + \frac{20}{18.69} \approx s^2 + 2s + 1.1$$

division by the second trial factor is performed as follows:

$$
\begin{array}{r|l}
 & s^2 + 7s + 14.9 \\
\hline
s^2 + 2s + 1.1 & s^4 + 9s^3 + 30s^2 + 42s + 20 \\
 & \underline{s^4 + 2s^3 + 1.1s^2} \\
 & 7s^3 + 28.9s^2 + 42s \\
 & \underline{7s^3 + 14s^2 + 7.7s} \\
 & 14.9s^2 + 34.3s + 20 \\
 & \underline{14.9s^2 + 29.8s + 16.4} \\
 & 4.5s - 3.6
\end{array}
$$

Third trial factor is

$$s^2 + \frac{34.3}{14.9}s + \frac{20}{14.9} = s^2 + 2.3s + 1.34$$

$s^2 + 2.5s + 1.5$ yield a final trial factor of $(s^2 + 2.5s + 1.5)$ and a quotient of $(s^2 + 6.5s + 12.2)$ with a remainder of $1.7s + 1.7$ which can be considered as negligible.

Thus equation (1) may be written as

$$(s^2 + 2.5s + 1.5)(s^2 + 6.5s + 12.2) = 0$$

or $(s+1)(s+1.5)(s + 3.25 + j\sqrt{1.6})(s + 3.25 - j\sqrt{1.6}) = 0$