MIT 6.02 **DRAFT** *Lecture Notes*
Fall 2011 (Last update: October 9, 2011)
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CHAPTER 8
Viterbi Decoding of Convolutional Codes

This chapter describes an elegant and efficient method to decode convolutional codes, whose construction and encoding we described in the previous chapter. This decoding method avoids explicitly enumerating the 2^N possible combinations of N-bit parity bit sequences. This method was invented by Andrew Viterbi '57 and bears his name.

8.1 The Problem

At the receiver, we have a sequence of voltage samples corresponding to the parity bits that the transmitter has sent. For simplicity, and without loss of generality, we will assume that the receiver picks a suitable sample for the bit, or better still, averages the set of samples corresponding to a bit, digitizes that value to a "0" or "1" by comparing to the threshold voltage (the demapping step), and propagates that bit decision to the decoder.

Thus, we have a *received bit sequence*, which for a convolutionally coded stream corresponds to the stream of parity bits. If we decode this received bit sequence with no other information from the receiver's sampling and demapper, then the decoding process is termed **hard decision decoding** ("hard decoding"). If, in addition, the decoder is given the actual voltage samples and uses that information in decoding the data, we term the process **soft decision decoding** ("soft decoding").

The Viterbi decoder can be used in either case. Intuitively, because hard decision decoding makes an early decision regarding whether a bit is 0 or 1, it throws away information in the digitizing process. It might make a wrong decision, especially for voltages near the threshold, introducing a greater number of bit errors in the received bit sequence. Although it still produces the most likely transmitted sequence *given* the received bit sequence, by introducing additional errors in the early digitization, the overall reduction in the probability of bit error will be smaller than with soft decision decoding. But it is conceptually easier to understand hard decoding, so we will start with that, before going on to soft decoding.

As mentioned in the previous chapter, the trellis provides a good framework for un-

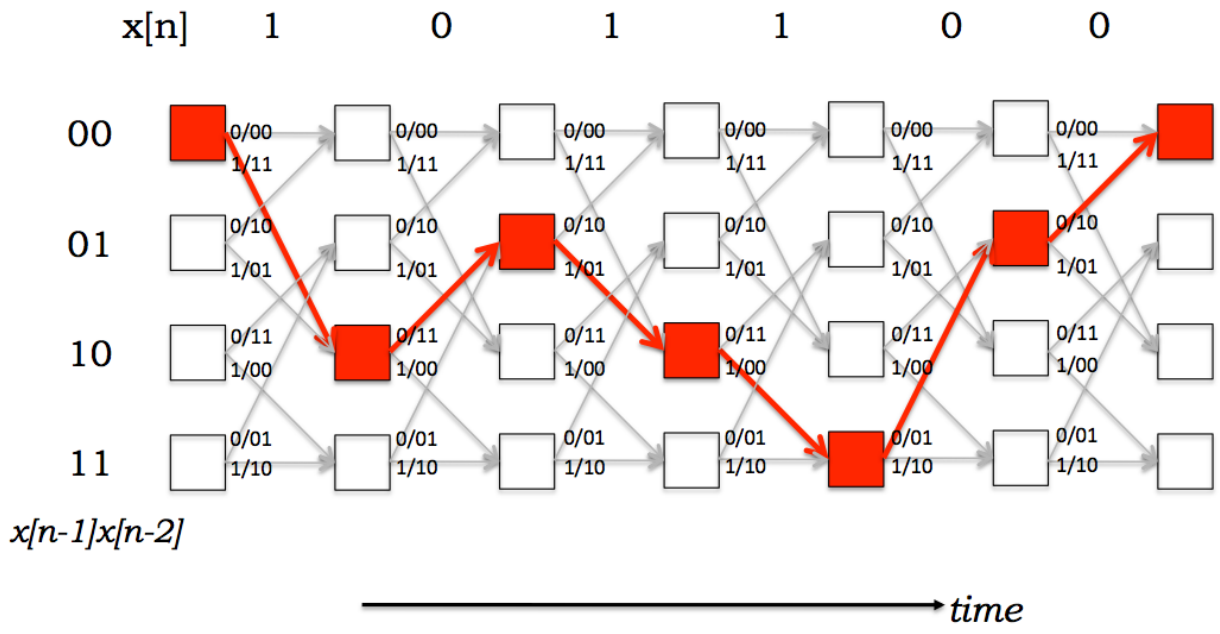


Figure 8-1: The trellis is a convenient way of viewing the decoding task and understanding the time evolution of the state machine.

derstanding the decoding procedure for convolutional codes (Figure 8-1). Suppose we have the entire trellis in front of us for a code, and now receive a sequence of digitized bits (or voltage samples). If there are no errors (i.e., the signal-to-noise ratio, SNR, is high enough), then there will be some path through the states of the trellis that would exactly match the received sequence. That path (specifically, the concatenation of the encoding of each state along the path) corresponds to the transmitted parity bits. From there, getting to the original message is easy because the top arc emanating from each node in the trellis corresponds to a "0" bit and the bottom arrow corresponds to a "1" bit.

When there are bit errors, what can we do? As explained earlier, finding the *most likely* transmitted message sequence is appealing because it minimizes the BER. If we can come up with a way to capture the errors introduced by going from one state to the next, then we can accumulate those errors along a path and come up with an estimate of the total number of errors along the path. Then, the path with the smallest such accumulation of errors is the path we want, and the transmitted message sequence can be easily determined by the concatenation of states explained above.

To solve this problem, we need a way to capture any errors that occur in going through the states of the trellis, and a way to navigate the trellis without actually materializing the entire trellis (i.e., without enumerating all possible paths through it and then finding the one with smallest accumulated error). The Viterbi decoder solves these problems. It is an example of a more general approach to solving optimization problems, called *dynamic programming*. Later in the course, we will apply similar concepts in network routing, an unrelated problem, to find good paths in multi-hop networks.

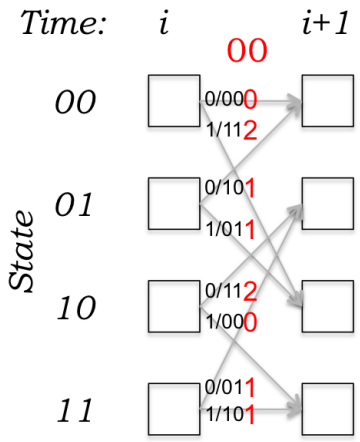


Figure 8-2: The branch metric for hard decision decoding. In this example, the receiver gets the parity bits 00.

8.2 The Viterbi Decoder

The decoding algorithm uses two metrics: the **branch metric** (BM) and the **path metric** (PM). The branch metric is a measure of the "distance" between what was transmitted and what was received, and is defined for each arc in the trellis. In hard decision decoding, where we are given a sequence of digitized parity bits, the branch metric is the *Hamming distance* between the expected parity bits and the received ones. An example is shown in Figure 8-2, where the received bits are 00. For each state transition, the number on the arc shows the branch metric for that transition. Two of the branch metrics are 0, corresponding to the only states and transitions where the corresponding Hamming distance is 0. The other non-zero branch metrics correspond to cases when there are bit errors.

The path metric is a value associated with a state in the trellis (i.e., a value associated with each node). For hard decision decoding, it corresponds to the Hamming distance over the most likely path from the initial state to the current state in the trellis. By "most likely", we mean the path with smallest Hamming distance between the initial state and the current state, measured over all possible paths between the two states. The path with the smallest Hamming distance minimizes the total number of bit errors, and is most likely when the BER is low.

The key insight in the Viterbi algorithm is that the receiver can compute the path metric for a (state, time) pair incrementally using the path metrics of previously computed states and the branch metrics.

8.2.1 Computing the Path Metric

Suppose the receiver has computed the path metric PM$[s, i]$ for each state s at time step i (recall that there are 2^{K-1} states, where K is the constraint length of the convolutional code). In hard decision decoding, the value of PM$[s, i]$ is the total number of bit errors detected when comparing the received parity bits to the most likely transmitted message, considering all messages that could have been sent by the transmitter until time step i (starting from state "00", which we will take to be the starting state always, by convention).

Among all the possible states at time step i, the most likely state is the one with the smallest path metric. If there is more than one such state, they are all equally good possibilities.

Now, how do we determine the path metric at time step $i+1$, PM$[s, i+1]$, for each state s? To answer this question, first observe that if the transmitter is at state s at time step $i+1$, then *it must have been in only one of two possible states at time step i*. These two *predecessor states*, labeled α and β, are always the same for a given state. In fact, they depend only on the constraint length of the code and not on the parity functions. Figure 8-2 shows the predecessor states for each state (the other end of each arrow). For instance, for state 00, $\alpha = 00$ and $\beta = 01$; for state 01, $\alpha = 10$ and $\beta = 11$.

Any message sequence that leaves the transmitter in state s at time $i+1$ *must have* left the transmitter in state α or state β at time i. For example, in Figure 8-2, to arrive in state '01' at time $i+1$, one of the following two properties *must hold*:

1. The transmitter was in state '10' at time i and the i^{th} message bit was a 0. If that is the case, then the transmitter sent '11' as the parity bits and there were two bit errors, because we received the bits 00. Then, the path metric of the new state, PM['01', $i+1$] is equal to PM['10', i] + 2, because the new state is '01' and the corresponding path metric is larger by 2 because there are 2 errors.
2. The other (mutually exclusive) possibility is that the transmitter was in state '11' at time i and the i^{th} message bit was a 0. If that is the case, then the transmitter sent 01 as the parity bits and tere was one bit error, because we received 00. The path metric of the new state, PM['01', $i+1$] is equal to PM['11', i] + 1.

Formalizing the above intuition, we can see that

$$\text{PM}[s, i+1] = \min(\text{PM}[\alpha, i] + \text{BM}[\alpha \rightarrow s], \text{PM}[\beta, i] + \text{BM}[\beta \rightarrow s]), \tag{8.1}$$

where α and β are the two predecessor states.

In the decoding algorithm, it is important to remember which arc corresponds to the minimum, because we need to traverse this path from the final state to the initial one keeping track of the arcs we used, and then finally *reverse* the order of the bits to produce the most likely message.

■ 8.2.2 Finding the Most Likely Path

We can now describe how the decoder finds the maximum-likelihood path. Initially, state '00' has a cost of 0 and the other $2^{k-1} - 1$ states have a cost of ∞.

The main loop of the algorithm consists of two main steps: first, calculating the branch metric for the next set of parity bits, and second, computing the path metric for the next column. The path metric computation may be thought of as an *add-compare-select* procedure:

1. *Add* the branch metric to the path metric for the old state.
2. *Compare* the sums for paths arriving at the new state (there are only two such paths to compare at each new state because there are only two incoming arcs from the previous column).
3. *Select* the path with the smallest value, breaking ties arbitrarily. This path corresponds to the one with fewest errors.

Figure 8-3 shows the decoding algorithm in action from one time step to the next. This example shows a received bit sequence of 11 10 11 00 01 10 and how the receiver processes it. The fourth picture from the top shows all four states with the same path metric. At this stage, any of these four states and the paths leading up to them are most likely transmitted bit sequences (they all have a Hamming distance of 2). The bottom-most picture shows the same situation with only the *survivor paths* shown. A survivor path is one that has a chance of being the maximum-likelihood path; there are many other paths that can be pruned away because there is no way in which they can be most likely. The reason why the Viterbi decoder is practical is that the number of survivor paths is much, much smaller than the total number of paths in the trellis.

Another important point about the Viterbi decoder is that *future knowledge* will help it break any ties, and in fact may even cause paths that were considered "most likely" at a certain time step to change. Figure 8-4 continues the example in Figure 8-3, proceeding until all the received parity bits are decoded to produce the most likely transmitted message, which has two bit errors.

■ 8.3 Soft Decision Decoding

Hard decision decoding digitizes the received voltage signals by comparing it to a threshold, *before* passing it to the decoder. As a result, we lose information: if the voltage was 0.500001, the confidence in the digitization is surely much lower than if the voltage was 0.999999. Both are treated as "1", and the decoder now treats them the same way, even though it is overwhelmingly more likely that 0.999999 is a "1" compared to the other value.

Soft decision decoding (also sometimes known as "soft input Viterbi decoding") builds on this observation. It *does not digitize the incoming samples prior to decoding*. Rather, it uses a continuous function of the analog sample as the input to the decoder. For example, if the expected parity bit is 0 and the received voltage is 0.3 V, we might use 0.3 (or 0.3^2, or some such function) as the value of the "bit" instead of digitizing it.

For technical reasons that will become apparent later, an attractive soft decision metric is the *square* of the difference between the received voltage and the expected one. If the convolutional code produces p parity bits, and the p corresponding analog samples are $v = v_1, v_2, \ldots, v_p$, one can construct a soft decision branch metric as follows

$$\text{BM}_{\text{soft}}[u, v] = \sum_{i=1}^{p} (u_i - v_i)^2, \tag{8.2}$$

where $u = u_1, u_2, \ldots, u_p$ are the *expected* p parity bits (each a 0 or 1). Figure 8-5 shows the soft decision branch metric for $p = 2$ when u is 00.

With soft decision decoding, the decoding algorithm is identical to the one previously described for hard decision decoding, except that the branch metric is no longer an integer Hamming distance but a positive real number (if the voltages are all between 0 and 1, then the branch metric is between 0 and 1 as well).

It turns out that this soft decision metric is closely related to the *probability of the decoding being correct* when the channel experiences additive Gaussian noise. First, let's look at the simple case of 1 parity bit (the more general case is a straightforward extension). Suppose

the receiver gets the i^{th} parity bit as v_i volts. (In hard decision decoding, it would decode $-$ as 0 or 1 depending on whether v_i was smaller or larger than 0.5.) What is the probability that v_i would have been received given that bit u_i (either 0 or 1) was sent? With zero-mean additive Gaussian noise, the PDF of this event is given by

$$f(v_i|u_i) = \frac{e^{-d_i^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}, \tag{8.3}$$

where $d_i = v_i^2$ if $u_i = 0$ and $d_i = (v_i - 1)^2$ if $u_i = 1$.

The log likelihood of this PDF is proportional to $-d_i^2$. Moreover, along a path, the PDF of the sequence $V = v_1, v_2, \ldots, v_p$ being received given that a code word $U = u_i, u_2, \ldots, u_p$ was sent, is given by the product of a number of terms each resembling Eq. (8.3). The logarithm of this PDF for the path is equal to the sum of the individual log likelihoods, and is proportional to $-\sum_i d_i^2$. But that's precisely the negative of the branch metric we defined in Eq. (8.2), which the Viterbi decoder minimizes along the different possible paths! Minimizing this path metric is identical to maximizing the log likelihood along the different paths, implying that the soft decision decoder produces the most likely path that is consistent with the received voltage sequence.

This direct relationship with the logarithm of the probability is the reason why we chose the sum of squares as the branch metric in Eq. (8.2). A different noise distribution (other than Gaussian) may entail a different soft decoding branch metric to obtain an analogous connection to the PDF of a correct decoding.

■ 8.4 Achieving Higher and Finer-Grained Rates: Puncturing

As described thus far, a convolutional code achieves a maximum rate of $1/r$, where r is the number of parity bit streams produced by the code. But what if we want a rate greater than $1/2$, or a rate between $1/r$ and $1/(r+1)$ for some r?

A general technique called **puncturing** gives us a way to do that. The idea is straightforward: the encoder does not send every parity bit produced on each stream, but "punctures" the stream sending only a subset of the bits that are agreed-upon between the encoder and decoder. For example, one might use a rate-$1/2$ code along with the puncturing schedule specified as a vector; for example, we might use the vector (101) on the first parity stream and (110) on the second. This notation means that the encoder sends the first and third bits but not the second bit on the first stream, and sends the first and second bits but not the third bit on the second stream. Thus, whereas the encoder would have sent two parity bits for every message bit without puncturing, it would now send four parity bits (instead of six) for every three message bits, giving a rate of $3/4$.

In this example, suppose the sender in the rate-$1/2$ code, without puncturing, emitted bits $p_0[0]p_1[0]p_0[1]p_1[1]p_0[2]p_1[2]\ldots$. Then, with the puncturing schedule given, the bits emitted would be $p_0[0]p_1[0] - p_1[1]p_0[2] - \ldots$, where each $-$ refers to an omitted bit.

At the decoder, when using a punctured code, missing parity bits don't participate in the calculation of branch metrics. Otherwise, the procedure is the same as before. We can think of each missing parity bit as a blank ('$-$') and run the decoder by just skipping over the blanks.

8.5 Performance Issues

There are three important performance metrics for convolutional coding and decoding:

1. How much state and space does the encoder need?
2. How much time does the decoder take?
3. What is the reduction in the bit error rate, and how does that compare with other codes?

8.5.1 Encoder and Decoder Complexity

The first question is the easiest: the amount of space is linear in K, the constraint length, and the encoder is much easier to implement than the Viterbi decoder. The decoding time depends mainly on K; as described, we need to process $O(2^K)$ transitions each bit time, so the time complexity is exponential in K. Moreover, as described, we can decode the first bits of the message only at the very end. A little thought will show that although a little future knowledge is useful, it is unlikely that what happens at bit time 1000 will change our decoding decision for bit 1, if the constraint length is, say, 6. In fact, in practice the decoder starts to decode bits once it has reached a time step that is a small multiple of the constraint length; experimental data suggests that $5 \cdot K$ message bit times (or thereabouts) is a reasonable decoding window, regardless of how long the parity bit stream corresponding to the message it.

8.5.2 Free Distance

The reduction in error probability and comparisons with other codes is a more involved and detailed issue. The answer depends on the constraint length (generally speaking, larger K has better error correction), the number of generators (larger this number, the lower the rate, and the better the error correction), and the amount of noise.

In fact, these factors are indirect proxies for the **free distance**, which largely determines how well a convolutional code corrects errors. Because convolutional codes are linear, everything we learned about linear codes applies here. In particular, the Hamming distance of a linear code, i.e., the minimum Hamming distance between any two valid codewords, is equal to the number of ones in the smallest non-zero codeword with minimum weight, where the weight of a codeword is the number of ones it contains.

In the context of convolutional codes, the smallest Hamming distance between any two valid codewords is called the *free distance*. Specifically, the free distance of a convolutional code is the difference in path metrics between the all-zeroes output and the path with the smallest non-zero path metric going from the initial 00 state to some future 00 state. Figure 8-6 illustrates this notion with an example. In this example, the free distance is 4, and it takes 8 output bits to get back to the correct state, so one would expect this code to be able to correct up to $\lfloor (4-1)/2 \rfloor = 1$ bit error in blocks of 8 bits, if the block starts at the first parity bit. In fact, this error correction power is essentially the same as an $(8, 4, 3)$ rectangular parity code. Note that the free distance in this example is 4, not 5: the smallest non-zero path metric between the initial 00 state and a future 00 state goes like this: $00 \rightarrow 10 \rightarrow 11 \rightarrow 01 \rightarrow 00$ and the corresponding path metrics increase as $0 \rightarrow 2 \rightarrow 2 \rightarrow 3 \rightarrow 4$.

Why do we define a "free distance", rather than just call it the Hamming distance, if it is defined the same way? The reason is that any code with Hamming distance D (whether linear or not) can correct all patterns of up to $\lfloor \frac{D-1}{2} \rfloor$ errors. If we just applied the same notion to convolutional codes, we will conclude that we can correct all single-bit errors in the example given, or in general, we can correct some fixed number of errors.

Now, convolutional coding produces an unbounded bit stream; these codes are markedly distinct from block codes in this regard. As a result, the $\lfloor \frac{D-1}{2} \rfloor$ formula is not too instructive because it doesn't capture the true error correction properties of the code. A convolutional code (with Viterbi decoding) can correct $t = \lfloor \frac{D-1}{2} \rfloor$ errors as long as these errors are "far enough apart". So the notion we use is the free distance because, in a sense, errors can keep occurring and as long as no more than t of them occur in a closely spaced burst, the decoder can correct them all.

■ 8.5.3 Comparing Codes: Simulation Results

In addition to the performance different hard decision convolutional codes, an important question is how much better soft decision decoding is compared to hard decision decoding. We address these questions by describing some simulation results here.

Figure 8-7 shows some representative performance results for a set of codes all of the same code rate $(1/2)$.[1] The top-most curve shows the uncoded probability of bit error, which may be modeled using the erfc function. The x axis plots the SNR on the dB scale, as defined in Chapter 5 (lower noise is toward the right). The y axis shows the probability of a decoding error on a log scale.

This figure shows the performance of three codes:

1. The $(8, 4, 3)$ rectangular parity code.

2. A convolutional code with generators $(111, 100)$ and constraint length $K = 3$, shown in the picture as "$K = 3$".

3. A convolutional code with generators $(1110, 1101)$ and constraint length $K = 4$, shown in the picture as "$K = 4$".

Some observations:

1. The probability of error is roughly the same for the rectangular parity code and hard decision decoding with $K = 3$. The free distance of the $K = 3$ convolutional code is 4, which means it can correct one bit error over blocks that are similar in length to the rectangular parity code we are comparing with. Intuitively, both schemes essentially produce parity bits that are built from similar amounts of history. In the rectangular parity case, the row parity bit comes from two successive message bits, while the column parity comes from two message bits with one skipped in between. But we also send the message bits, so we're mimicking a similar constraint length (amount of memory) to the $K = 3$ convolutional code.

[1] You will produce similar pictures in one of your lab tasks using your implementations of the Viterbi and rectangular parity code decoders.

2. The probability of error for a given amount of noise is noticeably lower for the $K = 4$ code compared to $K = 3$ code; the reason is that the free distance of this $K = 4$ code is 6, and it takes 7 trellis edges to achieve that ($000 \rightarrow 100 \rightarrow 010 \rightarrow 001 \rightarrow 000$), meaning that the code can correct up to 2 bit errors in sliding windows of length $2 \cdot 4 = 8$ bits.

3. The probability of error for a given amount of noise is dramatically lower with soft decision decoding than hard decision decoding. In fact, $K = 3$ and soft decoding beats $K = 4$ and hard decoding in these graphs. For a given error probability (and signal), the degree of noise that can be tolerated with soft decoding is much higher (about 2.5–3 dB, which is a good rule-of-thumb to apply in practice for the gain from soft decoding, all other things being equal).

Figure 8-8 shows a comparison of three different convolutional codes together with the uncoded case. Two of the codes are the same as in Figure 8-7, i.e., $(111, 110)$ and $(1110, 1101)$; these were picked because they were recommended by Bussgang's paper. The third code is $(111, 101)$, with parity equations

$$p_0[n] = x[n] + x[n-1] + x[n-2]$$
$$p_1[n] = x[n] + x[n-2].$$

The results of this comparison are shown in Figure 8-8. These graphs show the probability of decoding error (BER after decoding) for experiments that transmit messages of length 500,000 bits each. (Because the BER of the best codes in this set are on the order of 10^{-6}, we actually need to run the experiment over even longer messages when the SNR is higher than 3 dB; that's why we don't see results for one of the codes, where the experiment encountered no errors.)

Interestingly, these results show that the code $(111, 101)$ is stronger than the other two codes, even though its constraint length, 3, is smaller than that of $(1110, 1101)$. To understand why, we can calculate the free distance of this code, which turns out to be 5. This free distance is smaller than that of $(1110, 1101)$, whose free distance is 6, *but* the number of trellis edges to go from state 00 back to state 00 in the $(111, 101)$ case is only 3, corresponding to a 6-bit block. The relevant state transitions are $00 \rightarrow 10 \rightarrow 01 \rightarrow 00$ and the corresponding path metrics are $0 \rightarrow 2 \rightarrow 3 \rightarrow 5$. Hence, its error correcting power is marginally stronger than the $(1110, 1101)$ code.

■ 8.6 Summary

From its relatively modest, though hugely impactful, beginnings as a method to decode convolutional codes, Viterbi decoding has become one of the most widely used algorithms in a wide range of fields and engineering systems. Modern disk drives with "PRML" technology to speed-up accesses, speech recognition systems, natural language systems, and a variety of communication networks use this scheme or its variants.

In fact, a more modern view of the soft decision decoding technique described in this lecture is to think of the procedure as finding the most likely set of traversed states in a *Hidden Markov Model* (HMM). Some underlying phenomenon is modeled as a Markov state machine with probabilistic transitions between its states; we see noisy observations

from each state, and would like to piece together the observations to determine the most likely sequence of states traversed. It turns out that the Viterbi decoder is an excellent starting point to solve this class of problems (and sometimes the complete solution).

On the other hand, despite its undeniable success, Viterbi decoding isn't the only way to decode convolutional codes. For one thing, its computational complexity is exponential in the constraint length, K, because it does require each of these states to be enumerated. When K is large, one may use other decoding methods such as BCJR or Fano's sequential decoding scheme, for instance.

Convolutional codes themselves are very popular over both wired and wireless links. They are sometimes used as the "inner code" with an outer block error correcting code, but they may also be used with just an outer error detection code. They are also used as a component in more powerful codes line turbo codes, which are currently one of the highest-performing codes used in practice.

■ Problems and Exercises

1. Please check out and solve the online problems on error correction codes at `http://web.mit.edu/6.02/www/f2011/handouts/tutprobs/ecc.html`

2. Consider a convolutional code whose parity equations are

$$\begin{aligned} p_0[n] &= x[n] + x[n-1] + x[n-3] \\ p_1[n] &= x[n] + x[n-1] + x[n-2] \\ p_2[n] &= x[n] + x[n-2] + x[n-3] \end{aligned}$$

   (a) What is the rate of this code? How many states are in the state machine representation of this code?

   (b) Suppose the decoder reaches the state "110" during the forward pass of the Viterbi algorithm with this convolutional code.

      i. How many predecessor states (i.e., immediately preceding states) does state "110" have?

      ii. What are the bit-sequence representations of the predecessor states of state "110"?

      iii. What are the expected parity bits for the transitions from each of these predecessor states to state "110"? Specify each predecessor state and the expected parity bits associated with the corresponding transition below.

   (c) To increase the rate of the given code, Lem E. Tweakit punctures the p_0 parity stream using the vector (1 0 1 1 0), which means that every second and fifth bit produced on the stream are *not sent*. In addition, she punctures the p_1 parity stream using the vector (1 1 0 1 1). She sends the p_2 parity stream unchanged. What is the rate of the punctured code?

3. Let `conv_encode(x)` be the resulting bit-stream after encoding bit-string x with a convolutional code, C. Similarly, let `conv_decode(y)` be the result of decoding y to produce the maximum-likelihood estimate of the encoded message. Suppose we

send a message M using code C over some channel. Let P = `conv_encode(M)` and let R be the result of sending P over the channel and digitizing the received samples at the receiver (i.e., R is another bit-stream). Suppose we use Viterbi decoding on R, knowing C, and find that the maximum-likelihood estimate of M is $\hat{M}$. During the decoding, we find that the minimum path metric among all the states in the final stage of the trellis is D_{min}.

D_{min} is the Hamming distance between _____ and _____. Fill in the blanks, explaining your answer.

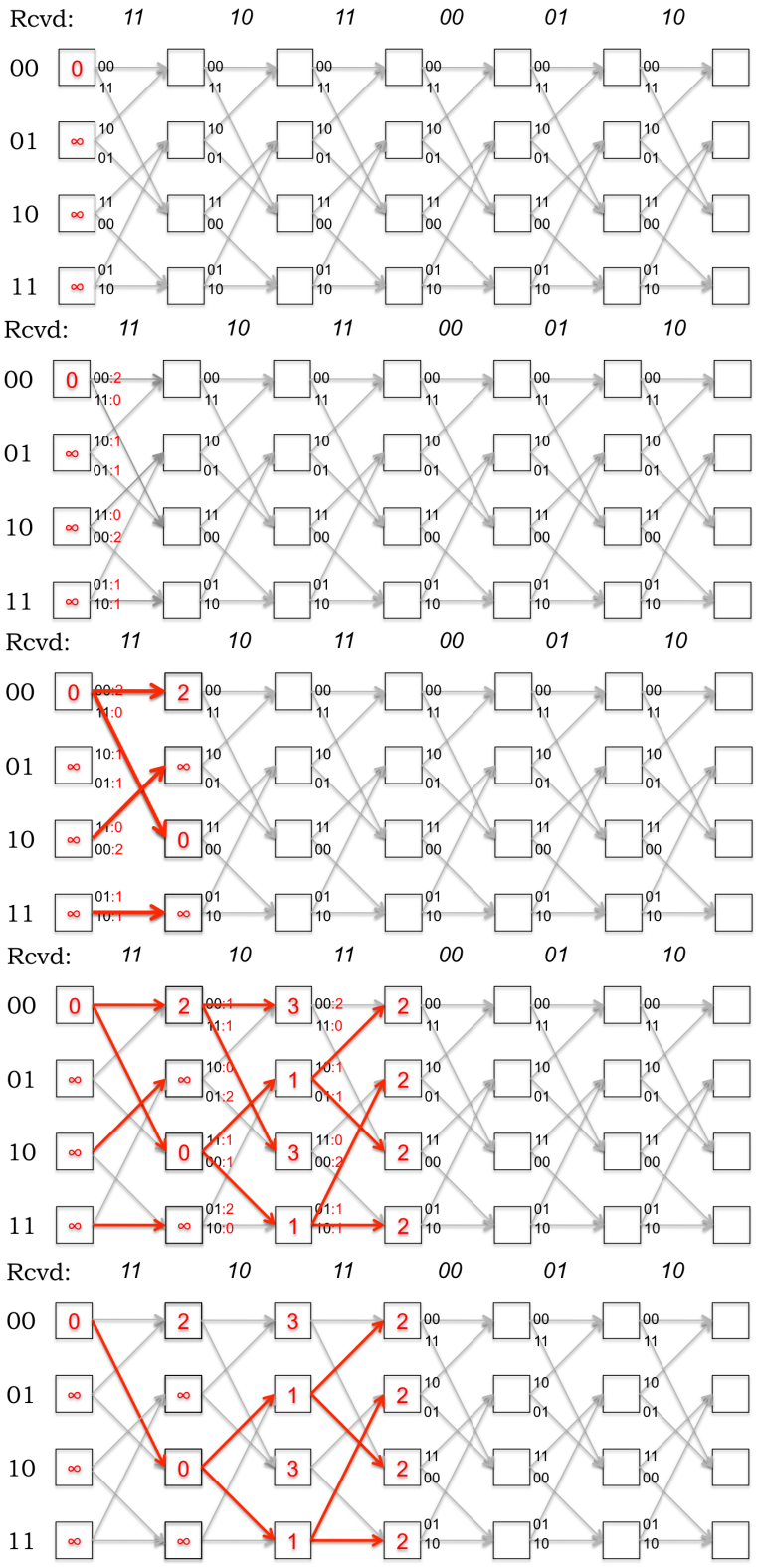


Figure 8-3: The Viterbi decoder in action. This picture shows four time steps. The bottom-most picture is the same as the one just before it, but with only the survivor paths shown.

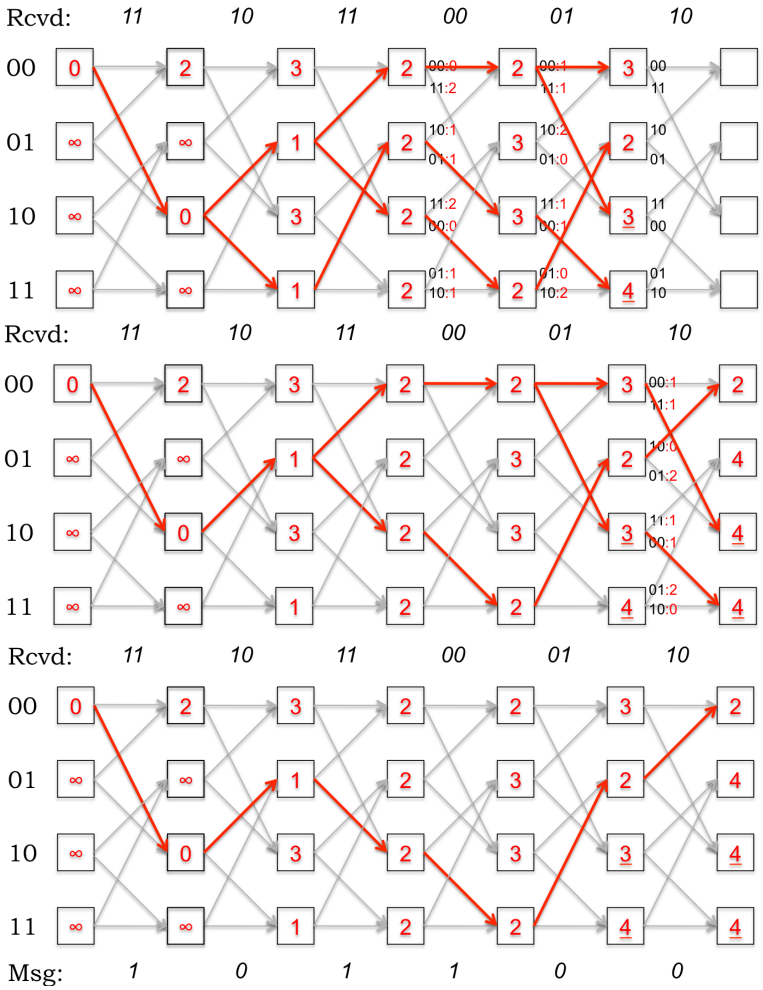


Figure 8-4: The Viterbi decoder in action (continued from Figure 8-3. The decoded message is shown. To produce this message, start from the final state with smallest path metric and work backwards, and then reverse the bits. At each state during the forward pass, it is important to remeber the arc that got us to this state, so that the backward pass can be done properly.

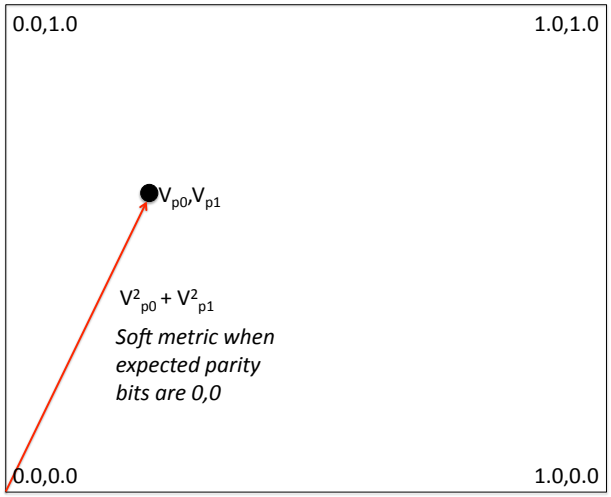
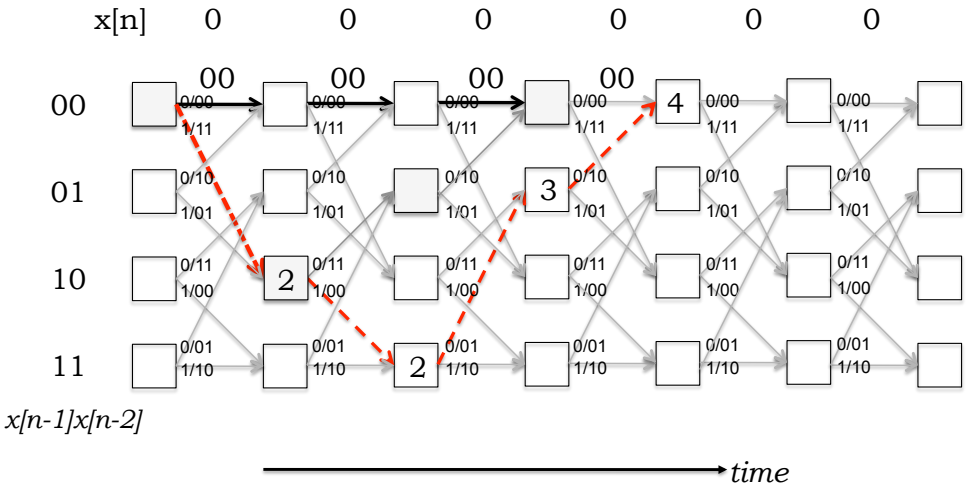


Figure 8-5: Branch metric for soft decision decoding.



The free distance is the difference in path metrics between the all-zeroes output and the path with the smallest non-zero path metric going from the initial 00 state to some future 00 state. It is 4 in this example. The path 00 → 10 →01 → 00 has a shorter length, but a higher path metric (of 5), so it is not the free distance.

Figure 8-6: The free distance of a convolutional code.

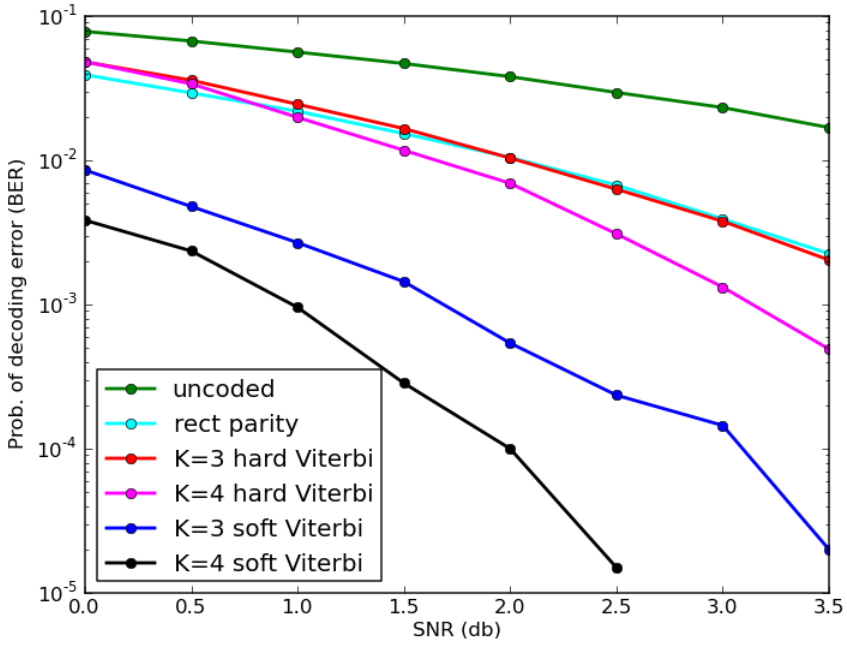


Figure 8-7: Error correcting performance results for different rate-1/2 codes.

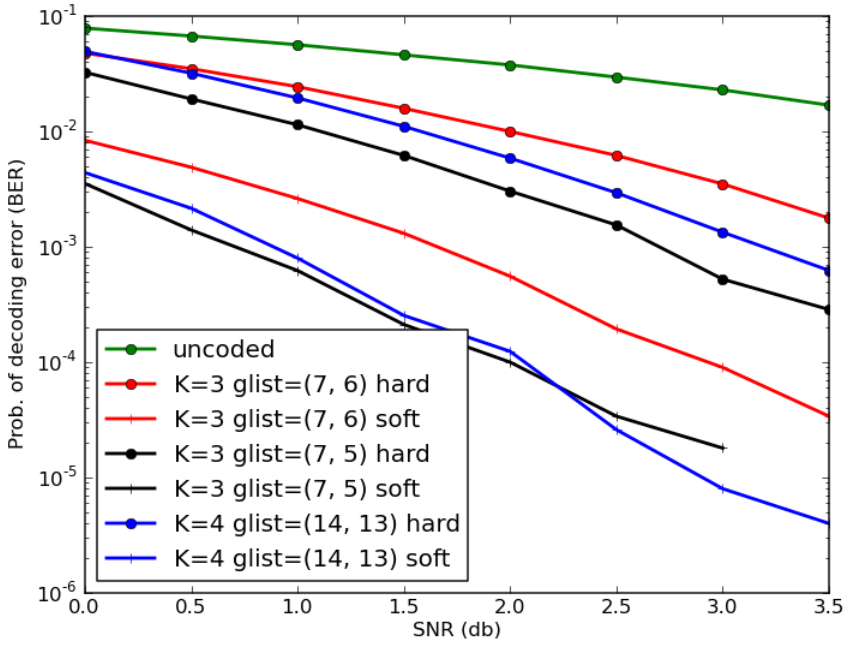


Figure 8-8: Error correcting performance results for three different rate-1/2 convolutional codes. The parameters of the three convolutional codes are $(111, 110)$ **(labeled "**$K = 3$ **glist=**$(7, 6)$**"),** $(1110, 1101)$ **(labeled "**$K = 4$ **glist=**$(14, 13)$**"), and** $(111, 101)$ **(labeled "**$K = 3$ **glist=**$(7, 5)$**"). The top three curves below the uncoded curve are for hard decision decoding; the bottom three curves are for soft decision decoding.**