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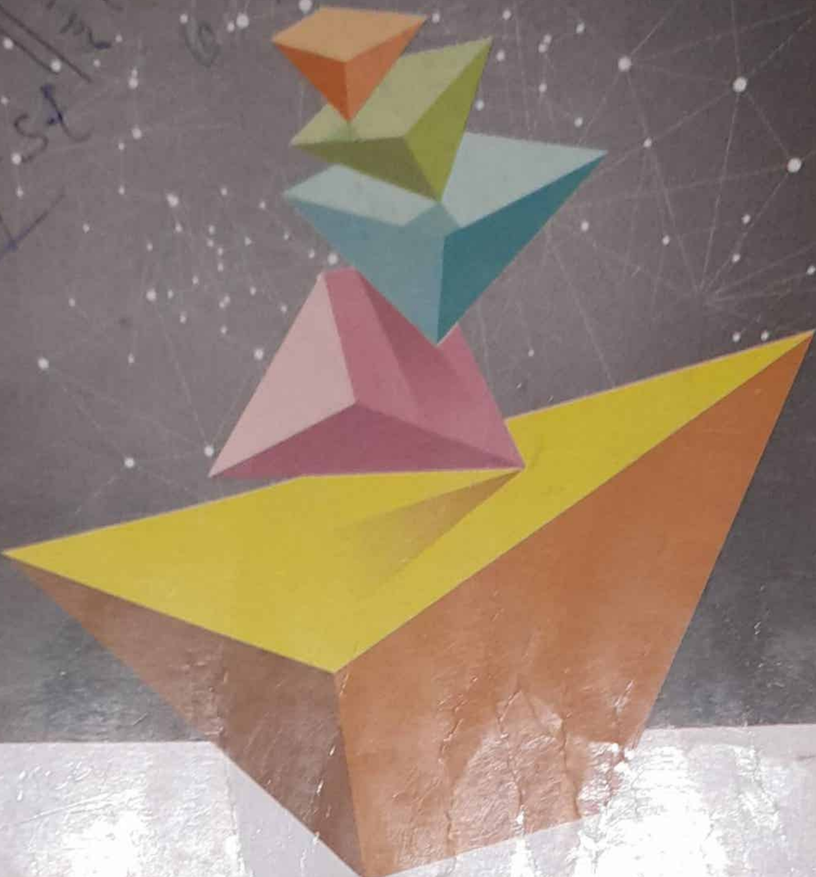


RAJASTHAN TECHNICAL UNIVERSITY, KOTA
A SERIES TO MAKE
ENGINEERING EASY
UNITWISE ARRANGEMENT

B.TECH.
I SEMESTER

ENGINEERING SOLVED PAPERS

- Engineering Mathematics-I
- Engineering Physics
- Engineering Chemistry
- Communication Skill
- Human Values
- Programming for Problem Solving
- Basic Mechanical Engineering
- Basic Electrical Engineering
- Basic Civil Engineering

SALIENT FEATURES

- According to the Latest Syllabus
- Solutions According to the Marks
- Chapter in a Nutshell
- Lucid Explanations
- Model Test Papers for Practice

SOLVED BY
EXPERIENCED FACULTIES

VICKY KUMAR
(CSE) (124)

B.TECH.

I Semester

M1 *

Solved Papers

[According to the Latest Syllabus of RTU, Kota]

✓① ✱② sem.

CONTENTS

2nd Semester

for part-2

read 12th class Maths (differencial eqn). & (Matrix)

Engineering Mathematics-I

Calculus :

Improper integrals (Beta and Gamma functions) and their properties; Applications of definite integrals to evaluate surface areas and volumes of revolutions.

Sequences and Series :

Convergence of sequence and series, tests for convergence; Power series, Taylor's series, series for exponential, trigonometric and logarithm functions.

Fourier Series :

Periodic functions, Fourier series, Euler's formula, Change of intervals,Half range sine and cosine series, Parseval's theorem.

Multivariable Calculus (Differentiation) :

Limit, continuity and partial derivatives, directional derivatives, total derivative; Tangent plane and normal line; Maxima, minima and saddle points; Method of Lagrange multipliers; Gradient, curl and divergence.

Multivariable Calculus (Integration) : Multiple Integration: Double integrals (Cartesian), change of order of integration in double integrals, Change of variables (Cartesian to polar), Applications: areas and volumes, Centre of mass and Gravity (constant and variable densities); Triple integrals (Cartesian), Simple applications involving cubes, sphere and rectangular parallelepipeds; Scalar line integrals, vector line integrals, scalar surface integrals, vector surface integrals, Theorems of Green, Gauss and Stokes.

1E2201

1E2201

B.Tech. I Sem. (Main) Examination, Dec.-2017
MA-101 Engineering Mathematics-I

Time : 3 Hours

Maximum Marks : 80
Min. Passing Marks : 28

Instructions to Candidates :
Attempt any five questions, including question no. 1 which is compulsory. All questions carry equal marks. Schematic diagrams must be shown wherever necessary. Any data you feel missing may suitably be assumed and stated clearly. Units of quantities used/calculated must be stated clearly.

Q.1 Compulsory, Answer for each sub-question be given in about 25 words- **(8×2=16)**

(a) Define concave upward and Concave downward.

(b) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that $x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \sin 2u$

(c) Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$, where $u = e^x \sin y$, $v = x + \log \sin y$

(d) Change the order of integration only in $\int_0^1\int_{e^x}^{e} \frac{dy\,dx}{\log y}$.

(e) Find the area, by double integration, bounded by parabola $y^2 = 4ax$ and its latus rectum.

(f) Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point P (2, 1, 3) in the direction of the vector $\vec{a} = \hat{i} - 2\hat{k}$.

(g) Prove that $\vec{F} = (y^2\cos x + z^2)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$ is a conservative field.

(h) Write the Cartesian formula of Gauss divergence theorem.

Q.2 **(a)** Find the asymptotes of the curve- **(8)**
$4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 - 7 = 0.$

(b) Transform the integral $\int_0^a\int_0^{\sqrt{a^2-x^2}} y^2\sqrt{(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates, and, hence evaluate it. **(8)**

Q.3 **(a)** Trace the curve $x^3 + y^3 = 3axy$. **(8)**

(b) Evaluate $\iiint_V x^2\,dx\,dy\,dz$ over the region V enclosed by the planes. **(8)**

$x = 0, y = 0, z = 0$ and $x + y + z = a$

Q.4 **(a)** Let when $f(x,y) = \frac{x^3 - y^3}{x^2 + y^2}$, when $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that the function f is continuous but not differentiable at the origin. **(8)**

(b) Prove that $\int_0^2 (8-x)^{-1/3}dx = \frac{2\pi}{3\sqrt{3}}$. **(8)**

Q.5 (a) If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$ (8)

(b) Prove that $\operatorname{div}(r^n \vec{r}) = (n+3)r^n$ (8)

Q.6 (a) Use Taylor's theorem to expand sin xy in powers of $(x-1)$ and $(y-\pi/2)$ up to second degree terms. (8)

(b) Verify Green's theorem in the plane for $\int_c (xy+y^2)dx + x^2 dy$, where C is the closed curve of the region bounded by x and $y = x^2$. (8)

Q.7 (a) Use Lagrange's method of multipliers to find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid. (8)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(b) Verify stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)i + 2xy\,j$, integrated around the rectangle $z = 0$ and bounded by the lines $x = 0, y = 0, x = a$ and $y = b$. (8)

□□□

ENGINEERING MATHEMATICS – I

CALCULUS 1

PREVIOUS YEARS QUESTIONS

I. SHORT ANSWER TYPE QUESTIONS

Prob.1 *Prove that –*

$$\overline{|(n+1)} = n\overline{|(n)}, n > 0$$

Sol. $\overline{|(n+1)} = \int_0^\infty x^n e^{-x}\, dx = \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty nx^{n-1}(-e^{-x})\, dx$

$$= n\int_0^\infty x^{n-1}e^{-x}\, dx \qquad \left[\because \lim_{x\to\infty} x^n e^{-x} = 0\right]$$

$$= n\overline{|(n)} \qquad \text{(by definition)}$$

$$\therefore \quad \overline{|(n+1)} = n\overline{|(n)}$$

Prob.2 *Prove that –*

$$\overline{|(1)} = 1$$

Sol. $\overline{|(1)} = \int_0^\infty e^{-x}\, dx = [-e^{-x}]_0^\infty = 1$

$$\therefore \quad \overline{|(1)} = 1$$

Prob.3 *Evaluate* $\int_0^{\pi/2} \sin^5 x \cos^6 x\, dx$

Sol. Here, $\int_0^{\pi/2} \sin^5 x \cos^6 x\, dx = \dfrac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{6+5+2}{2}\right)}$

$$= \frac{\Gamma(3)\Gamma(7/2)}{2\Gamma(13/2)}$$

$$\frac{(2\cdot 1)\left(\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\right)}{2\left(\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\right)} = \frac{8}{693} \qquad \textbf{Ans.}$$

Prob.4 *Evaluate* $\int_0^{\pi/2} \sin^6 x\, dx$

Sol. $\therefore \int_0^{\pi/2} \sin^6 x\, dx = \int_0^{\pi/2} \sin^6 x \cos^0 x\, dx$

$$= \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)} = \frac{\Gamma(7/2)\Gamma(1/2)}{2\Gamma(4)}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot 3\cdot 2\cdot 1} = \frac{5\pi}{32} \qquad \textbf{Ans.}$$

Prob.5 *Evaluate* $\int_0^a x^2(a^2 - x^2)^{3/2} dx$

Sol. Suppose $x = a\sin\theta$, when $dx = a\cos\theta d\theta$ and, when $x = 0, \theta = 0$ and when $x = a,\ \theta = \dfrac{\pi}{2}$

$$\therefore \int_0^a x^2\left(a^2-x^2\right)^{3/2} dx$$

$$=\int_0^{\pi/2} a^2\sin^2\theta(a^2-a^2\sin^2\theta)^{3/2} a\cos\theta\, d\theta$$

$$=\int_0^{\pi/2} a^2\sin^2\theta\, a^3\cos^3\theta\, a\cos\theta\, d\theta$$

$$=a^6\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta$$

$$=a^6\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{2+4+2}{2}\right)}$$

$$=a^6\frac{\Gamma(3/2)\,\Gamma(5/2)}{2\Gamma(4)}$$

$$=a^6\frac{\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot3\cdot2\cdot1}=\frac{\pi a^6}{32}$$ **Ans.**

Prob.6 ***Evaluate*** $\int_0^1 x^6(1-x^2)^{1/2}\,dx$

Sol. Suppose $x=\sin\theta$, then $dx=\cos\theta\, d\theta$ and when $x=0,\theta=0$ and when $x=1,\ \theta=\frac{\pi}{2}$.

$$\therefore \int_0^1 x^6(1-x^2)^{1/2}\,dx=\int_0^{\pi/2}\sin^6\theta(1-\sin^2\theta)^{1/2}\cos\theta\, d\theta$$

$$=\int_0^{\pi/2}\sin^6\theta\cos^2\theta\, d\theta$$

$$=\frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{6+2+2}{2}\right)}=\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(5)}$$

$$=\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot4\cdot3\cdot2\cdot1}=\frac{5\pi}{256}$$ **Ans.**

Prob.7 ***Show that*** $\Gamma\left(\frac{s+1}{r}\right)=ra\frac{(s+1)}{r}\int_0^\infty x^s e^{-ax^r}dx$; s, r, a ***are positive constants.***

Sol. Proof : Put $y=a\,x^r$ then $dy=arx^{r-1}dx$

$$\text{Now}\quad \int_0^\infty x^s e^{-ax^r}dx=\int_0^\infty\left[\left(\frac{y}{a}\right)^{\frac{1}{r}}\right]^s e^{-y}\cdot\frac{1}{a^r x^{r-1}}dy$$

$$=\left[ra\frac{(s+1)}{2}\right]^{-1}\int_0^\infty y\frac{(s+1-r)}{r}e^{-y}dy$$

$$=\frac{\Gamma\left(\frac{s+1}{r}\right)}{ra\frac{(s+1)}{r}}$$

$$\text{Hence,}\ \Gamma\frac{(s+1)}{r}=ra\frac{(s+1)}{r}\int_0^\infty x^s e^{-ax^r}dx$$ **Proved**

Prob.8 ***Prove that –***

$$\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}\cdot 0<n<1$$

Sol. $\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}\cdot 0<n<1$

We know that $B(m,n)=\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}dx$

$$\Gamma(n)\Gamma(m)=\Gamma(m+n)\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}dx$$

putting $m+n=1$ or $m=1-n$ in the above equation

$$\Gamma(n)\Gamma(1-n)=(1)\int_0^\infty\frac{x^{n-1}}{(1+x)}dx=(1)\frac{\pi}{\sin n\pi}$$

$$\left[\because \int_0^\infty\frac{x^{n-1}}{1+x}dx=\frac{\pi}{\sin n\pi}\ \text{when}\ 0<n<1\right.$$

$$\therefore\quad \Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}$$

Prob.9 ***Prove that –***

$$B(m, n) = B(m+1, n) + B(m, n+1)$$

Sol. $B(m, n) = B(m+1, n) + B(m, n+1)$

We know that $B(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

putting $m+1$ in place of m, we have

$$B(m+1, n) = \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}$$

like that $\quad B(m, n+1) = \dfrac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$

$$\therefore B(m+1, n) + B(m, n+1) = \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{m\Gamma(m)\Gamma(n) + \Gamma(m)n\Gamma(n)}{(m+n)\,\Gamma(m+n)} = \frac{(m+n)\,\Gamma(m)\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m, n)$$

Proved.

II. LONG ANSWER TYPE QUESTIONS

Prob.1 ***Prove that*** $\displaystyle\int_0^2 (8-x^3)^{-1/3}\,dx = \frac{2\pi}{3\sqrt{3}}$.

[R.T.U. 2017]

Sol. Put $\quad x^3 = 8\sin^2\theta$

$$\therefore \quad x = 2\sin^{2/3}\theta$$

$$dx = 2.\frac{2}{3}\sin^{-1/3}\theta\cos\theta$$

$$\therefore \quad I = \int_0^{\pi/2} (8-8\sin^2\theta)^{-1/3} . \frac{4}{3}\sin^{-1/3}\theta\cos\theta d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \cos^{-1/3}\theta\sin^{-1/3}\theta\cos\theta d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \sin^{-1/3}\theta\ \cos^{1/3}\theta\, d\theta$$

$$= \frac{2}{3}.\frac{\Gamma\left(\frac{-\frac{1}{3}+1}{2}\right)\Gamma\left(\frac{\frac{1}{3}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{3}+\frac{1}{3}+2}{2}\right)} = \frac{2}{3}\frac{\Gamma\left(\frac{2}{3}\right).\Gamma\left(\frac{4}{3}\right)}{2\Gamma(1)}$$

$$= \frac{1}{3}\Gamma\left(\frac{2}{3}\right).\frac{1}{3}.\Gamma\left(\frac{1}{3}\right)$$

$$= \frac{1}{9}.\Gamma\left(\frac{1}{3}\right).\Gamma\left(1-\frac{1}{3}\right) = \frac{1}{9}.\frac{\pi}{\sin\frac{\pi}{3}}$$

$$= \frac{1}{9}.\frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{9\sqrt{3}}$$

Prob.2 ***Prove that :*** $\;B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m > 0, n > 0.$

[R.T.U. 2014]

OR

Show that :

$$B(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

[R.T.U. 2011]

OR

Show that $\;B(m,n) = \displaystyle\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$

[R.T.U. 2015]

Sol. In the given integral,

Taking $a^m b^n \displaystyle\int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}$

Now put ax = by

i.e. $x = \left(\dfrac{b}{a}\right)y$ so that $dx = \left(\dfrac{b}{a}\right)dy$

when $x = 0$, $y = 0$ and when $x \to \infty, y \to \infty$,

$$\therefore a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,dx = a^m b^n \int_0^\infty \frac{\left(\frac{b}{a}y\right)^{m-1}\left(\frac{b}{a}\right)dy}{(b)^{m+n}(1+y)^{m+n}}$$

$$= a^m b^n \int_0^\infty \frac{(b)^{m-1+1-m-n}(a)^{1-m-1}(y)^{m-1}\,y^{m-1}\,dy}{(1+y)^{m+n}}.$$

$$= a^m b^n \int_0^\infty \frac{(b)^{-n}(a)^{-m}\,dy}{(1+y)^{m+n}} = \int_0^\infty \frac{(y)^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= B\,(m,n)$$

Now to prove

$$B(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

we have $\dfrac{\Gamma m}{z^m}=\int_0^\infty e^{-zx}x^{m-1}dx$

$$\Gamma m=z^m\int_0^\infty e^{-zx}x^{m-1}dx=\int_0^\infty z^m\,e^{-zx}x^{m-1}dx$$

Multiplying both side by $e^{-z}z^{n-1}$, we get

$$\Gamma(m)\,e^{-z}z^{n-1}=\int_0^\infty e^{-z(1+x)}z^{m+n-1}x^{m-1}dx$$

Now integrating both sides of equation (1) with respect to z from 0 to ∞, we get

$$\Gamma(m)\int_0^\infty e^{-z}z^{n-1}dz=\int_0^\infty\left[\int_0^\infty e^{-z(1+x)}z^{m+n-1}x^{m-1}dx\right]dz$$

$$\Gamma m\,\Gamma n=\int_0^\infty\left[\int_0^\infty e^{-z(1+x)}z^{m+n-1}dz\right]x^{m-1}dx$$

$$=\int_0^\infty\frac{\Gamma(m+n)}{(1+x)^{m+n}}x^{m-1}dx$$

$$=\Gamma(m+n)\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$=\Gamma(m+n)\,B(m,n)\Rightarrow B(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$ **Ans.**

Prob.3 *Prove that the surface and volume of the solid generated by revolving the loop of the curve $x=t^2$, $y=t-t^3/3$ about the X-axis are respectively 3π and $3\pi/4$.* *[R.T.U. 2012]*

Sol. The given problem can be traced as follow :

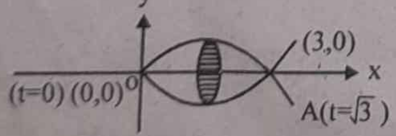


Fig.

The equation of the curve are

$$x=t^2,\ y=t-\frac{1}{3}t^3$$

$$\therefore\quad \frac{dx}{dt}=2t \text{ and } \frac{dy}{dt}=(1-t^2)$$

$$\text{Hence } \frac{dS}{dt}=\sqrt{\left\{4t^2+(1-t^2)^2\right\}}=\sqrt{(1+t^2)^2}$$

or $\quad \dfrac{dS}{dt}=1+t^2$

Also for the loop (putting y = 0) t varies from 0 to $\sqrt{3}$

$\therefore$ The required surface

$$=2\pi\int_{t=0}^{\sqrt{3}}y\,dS=2\pi\int_{t=0}^{\sqrt{3}}y\frac{dS}{dt}dt$$

$$=2\pi\int_0^{\sqrt{3}}\left(t-\frac{1}{3}t^3\right)(1+t^2)\,dt$$

$$=\frac{2\pi}{3}\int_0^{\sqrt{3}}(2t+2t^3-t^5)\,dt$$

$$=\frac{2\pi}{3}\left[\frac{3}{2}t^2+\frac{1}{2}t^4-\frac{1}{6}t^5\right]_0^{\sqrt{3}}$$

$$=\frac{2\pi}{3}\left[\frac{9}{2}+\frac{9}{2}-\frac{9}{2}\right]=3\pi$$

Required volume

$$V=\int_0^{\sqrt{3}}\pi y^2dx=\pi\int_0^{\sqrt{3}}\left(t-\frac{t^3}{3}\right)^2\frac{dx}{dt}dt$$

$$=\pi\int_0^{\sqrt{3}}\left(t^2+\frac{t^6}{9}-\frac{2t^4}{3}\right)2t\,dt$$

$$=2\pi\left[\frac{t^4}{4}+\frac{t^8}{72}-\frac{2}{3}\frac{t^6}{6}\right]_0^{\sqrt{3}}=2\pi\left[\frac{9}{4}+\frac{9}{8}-3\right]=\frac{3\pi}{4}$$

Ans.

Prob.4 *Prove that* $\displaystyle\int_0^{\pi/2}\cos^m\theta\,\sin^n\theta\,d\theta=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$

[R.T.U. 20

Sol. Since $\displaystyle\Gamma m=\int_0^\infty e^{-x}x^{m-1}dx$ ...(i)

And $\displaystyle\Gamma n=\int_0^\infty e^{-y}y^{n-1}dy$ ...(ii)

Put $x=p^2$ in (i) and $y=q^2$ in (ii), we get

$$\Gamma m=2\int_0^\infty e^{-p^2}p^{2m-1}dp \quad ...(iii)$$

$$\Gamma n=2\int_0^\infty e^{-q^2}q^{2n-1}dq \quad ...(iv)$$

$$\Rightarrow \overline{|m}\,\overline{|n} = 4\int_0^\infty\int_0^\infty e^{-(p^2+q^2)}p^{2m-1}q^{2n-1}dpdq$$

Put $p = r\cos\theta,\ q = r\sin\theta$
and $dpdq = rd\theta dr$

$$\Rightarrow \overline{|m}\,\overline{|n} = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}r^{2(m+n)-1}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta dr$$

$$= \left[2\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta\right]\left[2\int_0^\infty e^{-r^2}r^{2(m+n)-1}dr\right]$$

$$\Rightarrow 2\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta = \frac{\overline{|m}\,\overline{|n}}{\overline{|(m+n)}}$$

Put $m = \frac{m+1}{2}$

and $n = \frac{n+1}{2}$, we get

$$\int_0^{\pi/2}\cos^m\theta\sin^n\theta d\theta = \frac{\overline{\left|\frac{m+1}{2}\right.}\,\overline{\left|\frac{n+1}{2}\right.}}{2\overline{\left|\frac{m+n+2}{2}\right.}}$$ **Hence proved.**

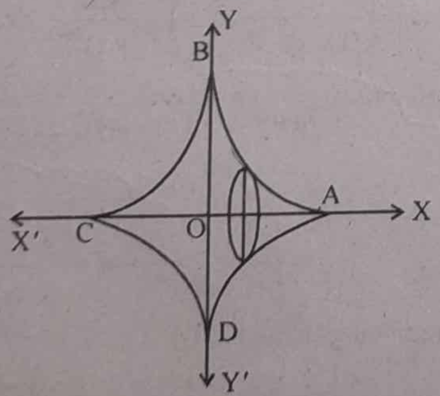
Prob.5 ***Find the whole length of Astroid $x = a\cos^3 t,\ y = a\sin^3 t$. Also find the volume when it revolves about the axis of x.*** ***[R.T.U. 2009, 07, Raj.Univ. 2006]***

OR

Find the volume of the solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$. ***[R.T.U. 2008]***

Sol. Since the curve is symmetrical about both the axes

$\therefore$ Whole length = 4 × length of arc AB



$$= 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

$$= 4\int_0^{\pi/2}\sqrt{(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2}dt$$

$$= 4\cdot 3\cdot a\int_0^{\pi/2}\cos t\sin t\sqrt{(\cos^2 t + \sin^2 t)}dt$$

$$= 12a\int_0^{\pi/2}\sin t\cos t\,dt$$

$$= 12a\left(\frac{\sin^2 t}{2}\right)_0^{\pi/2}$$

$$= 12a\cdot\frac{1}{2} = 6a \text{ units}$$

$$\text{Volume } = 2\int_0^{\pi/2}\pi y^2\frac{dx}{dt}.dt$$

$$= 2\int_0^{\pi/2}\pi a^2\sin^6 t(-3a\cos^2 t)\sin t\,dt.$$

$$= -6\pi a^3\int_0^{\pi/2}\sin^7 t\cos^2 t dt$$

$$= -6\pi a^3\frac{\overline{\left|\frac{8}{2}\right.}\,\overline{\left|\frac{3}{2}\right.}}{2\overline{\left|\frac{7+2+2}{2}\right.}}$$

$$= -6\pi a^3\frac{3\cdot 2\cdot 1\cdot\frac{1}{2}.\sqrt{\pi}}{2\cdot\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}$$

$$= \frac{-32\pi a^3}{105} = \frac{32\pi a^3}{105} \text{ cubic units}$$ **Ans.**

Prob.6 ***Evaluate*** $\int_0^a x^3(2ax - x^2)^{3/2}dx$

Sol. Suppose $x = 2a\sin^2\theta$ then $dx = 4a\sin\theta\cos\theta d\theta$ and when $x = 0, \theta = 0$ and when $x = a, \theta = \frac{\pi}{4}$

$$\therefore \int_0^a x^3(2ax - x^2)^{3/2}dx$$

$$= \int_0^{\pi/4} a^3\sin^6\theta\left(4a^2\sin^2\theta - 4a^2\sin^4\theta\right)^{3/2}4a\sin\theta\cos\theta d\theta$$

$$= \int_0^{\pi/4} a^3\sin^6\theta\ 8a^3\sin^3\theta\cos^3\theta\cdot 4a\sin\theta\cos\theta d\theta$$

$$= 2^8 a^7 \int_0^{\pi/4} \sin^{10}\theta \cos^4\theta \, d\theta$$

$$= 2a^7 \int_0^{\pi/4} (2\sin^2\theta)^3 (2\sin\theta\cos\theta)^4 \, d\theta$$

$$= 2a^7 \int_0^{\pi/4} (1-\cos 2\theta)^3 \sin^4 2\theta \, d\theta$$

$$= 2a^7 \int_0^{\pi/2} (1-\cos t)^3 \sin^4 t \left(\frac{dt}{2}\right) \qquad \text{where } 2\theta = t$$

$$= a^7 \int_0^{\pi/2} (1 - 3\cos t + 3\cos^2 t - \cos^3 t) \sin^4 t \, dt$$

$$= a^7 \left[\int_0^{\pi/2} \sin^4 t \, dt - 3\int_0^{\pi/2} \sin^4 t \cos t \, dt + 3\int_0^{\pi/2} \sin^4 t \cos^2 dt - \int_0^{\pi/2} \sin^4 t \cos^3 t \, dt \right]$$

$$= a^7 \left[\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} - 3\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(1)}{2\Gamma\left(\frac{7}{2}\right)} + 3\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)} - \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{2\Gamma\left(\frac{9}{2}\right)} \right]$$

$$= a^7 \left[\frac{3\pi}{16} - \frac{3}{5} + \frac{3\pi}{32} - \frac{2}{35} \right] = a^7 \left[\frac{9\pi}{32} - \frac{23}{35} \right]$$

Prob.7 ***Prove that :***

(i) $$\int_0^\infty e^{-ax} \sin bx \, x^{m-1} dx = \frac{\overline{|m}}{(a^2+b^2)^{m/2}} \sin m\theta$$

(ii) $$\int_0^\infty e^{-ax} \cos bx \, x^{m-1} dx = \frac{\overline{|m}}{(a^2+b^2)^{m/2}} \cos m\theta$$

[R.T.U. 2007]

Sol. To solve $\int_0^\infty x^{m-1} e^{-ax} \sin bx dx$ and $\int_0^\infty x^{m-1} e^{-ax} \cos bx dx$

substitute

$x = zx$ in Gamma Function

$$\int_0^\infty x^{m-1} e^{-zx} dx = \frac{\Gamma(m)}{z^m}$$

Put $z = a - ib$

$$\int_0^\infty x^{m-1} e^{-(a-ib)x} dx = \frac{\Gamma(m)}{(a-ib)^m} = \frac{\Gamma(m)(1+ib)^m}{(a-ib)^m (a+ib)^m}$$

$$= \frac{\Gamma(m)(a+ib)^m}{(a^2+b^2)^m}$$

Suppose $a = r\cos\theta$ & $b = r\sin\theta$ then $r = \sqrt{a^2+b^2}$

and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore \int_0^\infty x^{m-1} e^{-ax} e^{ibx} dx = \frac{\Gamma(m) r^m (\cos\theta + i\sin\theta)^m}{r^{2m}}$$

or $$\int_0^\infty x^{m-1} e^{-ax} (\cos bx + i\sin bx) dx = \frac{\Gamma(m)(\cos m\theta + i\sin m\theta)}{(a^2+b^2)^{m/2}}$$

now comparing the real and imaginary part of both sides

$$\int_0^\infty x^{m-1} e^{-ax} \cos bx \, dx = \frac{\Gamma(m)}{(a^2+b^2)^{n/2}} \cos m\theta$$

and $$\int_0^\infty x^{m-1} e^{-ax} \sin bx \, dx = \frac{\Gamma(m)}{(a^2+b^2)^{m/2}} \sin m\theta$$

Deductions : By putting $a = 0 \left(\theta = \frac{\pi}{2}\right)$ in above function we get

$$\int_0^\infty x^{m-1} \cos bx \, dx = \frac{\Gamma(m)}{b^m} \cos\left(\frac{m\pi}{2}\right)$$

$$\int_0^\infty x^{m-1} \sin bx \, dx = \frac{\Gamma(m)}{b^m} \sin\left(\frac{n\pi}{2}\right)$$

Prob.8 ***Solve :***

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = B(m_1 n)$$

[Raj. Univ. 2...]

Sol. Given : $$I = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

[Put $x = \frac{1}{t}$; in first part of integral]

$$= \int_\infty^1 \frac{\left(\frac{1}{t}\right)^{m-1}}{(1+t)^{m+n}} + m + n\left(\frac{1}{-t^2}\right) dt + I_2$$

[by replacing 'x' by t in I_2]

$$= \int_1^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt + \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt$$

$$= \int_1^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt = B(m, n) \qquad \text{Pro...}$$

Prob.9 $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)....\Gamma\left(\frac{n-1}{n}\right)=\frac{2\pi^{(n-1)/2}}{n^{1/2}}$

Where n is positive integer greater than one.

Sol. Suppose

$$\rho=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)....\Gamma\left(\frac{n-2}{n}\right)\Gamma\left(\frac{n-1}{n}\right)$$

$$=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)....\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{1}{n}\right) \quad ...(1)$$

the above equation can be written in reverse order

$$\rho=\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right).....\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right) \quad ...(2)$$

Multiplying (1) and (2)

$$\rho^2=\left\{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right\}\left\{\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\right\}$$

$$\cdots\left\{\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(\frac{2}{n}\right)\right\}\left\{\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)\right\}$$

we know that

$$\Gamma\left(\frac{r}{n}\right)\Gamma\left(1-\frac{r}{n}\right)=\frac{\pi}{\sin\left(\frac{r\pi}{n}\right)}$$

$$\therefore \quad \rho^2=\frac{\pi}{\sin\left(\frac{\pi}{n}\right)}\cdot\frac{\pi}{\sin\left(\frac{2\pi}{n}\right)}\cdot\frac{\pi}{\sin\left(\frac{3\pi}{n}\right)}....\frac{\pi}{\sin\left\{(n-1)\ \frac{\pi}{n}\right\}}$$

$$=\frac{\pi^{n-1}}{\left(\sin\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)...\sin\left\{(n-1)\frac{\pi}{n}\right\}} \quad ...(3)$$

by trigonometry, we know that

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\sin\left[\theta+\frac{\pi}{n}\right]\sin\left[\theta+\frac{2\pi}{n}\right]....\sin\left[\theta+\frac{n-1}{n}\pi\right] \quad ...(4)$$

taking limit $\theta\to 0$

Left side $=\lim\limits_{\theta\to 0}\frac{\sin n\theta}{\sin\theta}=\lim\limits_{\theta\to 0}\left\{n\cdot\frac{\sin n\theta}{n\theta}\cdot\frac{\theta}{\sin\theta}\right\}=n$

Right side

$$=\lim_{\theta\to 0}2^{n-1}\sin\left[\theta+\frac{\pi}{n}\right]\sin\left[\theta+\frac{2\pi}{n}\right]....\sin\left[\theta+\frac{n-1}{n}\pi\right]$$

$$=2^{n-1}\sin\left[\frac{\pi}{n}\right]\sin\left[\frac{2\pi}{n}\right].....\sin\left[\frac{n-1}{n}\pi\right]$$

$$\therefore \quad n=2^{n-1}\sin\left[\frac{\pi}{n}\right]\sin\left[\frac{2\pi}{n}\right]....\sin\left[\frac{n-1}{n}\pi\right] \quad ...(5)$$

now from (3) and (5)

$$\rho^2=\pi^{n-1}\left[\frac{2^{n-1}}{n}\right]=\frac{(2\pi)^{n-1}}{n} \qquad \rho=\frac{(2\pi)^{(n-1)/2}}{n^{1/2}}$$

$$\therefore \quad \Gamma\left[\frac{1}{n}\right]\Gamma\left[\frac{2}{n}\right]\Gamma\left[\frac{3}{n}\right]...\Gamma\left[\frac{n-1}{n}\right]=\frac{(2\pi)^{(n-1)/2}}{n^{1/2}}$$

Prob.10 ***Prove that***

$$\int_0^1\frac{dx}{\sqrt{1-x^3}}=\frac{[\Gamma(1/3)]^3}{2^{\pi/3}\sqrt{3\pi}}$$

Sol. Suppose $x^3=\sin^2\theta$ or $x=\sin^{2/3}\theta$

then $dx=\frac{2}{3}\sin^{-1/3}\theta\cos\theta d\theta$

$$\therefore \quad \int_0^1\frac{dx}{\sqrt{1-x^3}}=\int_0^{\pi/2}\frac{\frac{2}{3}\sin^{-1/3}\theta\cos\theta d\theta}{\sqrt{1-\sin^2\theta}}$$

$$=\frac{2}{3}\int_0^{\pi/2}\sin^{-1/3}\theta\cos\theta d\theta$$

$$=\frac{2}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{6}\right)}=\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{5}{6}\right)} \quad ...(1)$$

We know that $2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\sqrt{\pi}(2m)$

if we put $m=\frac{1}{3}$

$$2^{-1/3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)=\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right)$$

$$\frac{1}{\Gamma\left(\frac{5}{6}\right)}=\frac{2^{-1/3}\left(\frac{1}{3}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right)} \quad ...(2)$$

now from (1) and (2)

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{3} \cdot \frac{2^{-1/3}\Gamma\left(\frac{1}{3}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right)}$$

$$= \frac{1}{3\cdot 2^{1/3}} \cdot \frac{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^3}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}$$

$$= \frac{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^3}{\frac{1}{3\cdot 2^{1/3}}} \cdot \frac{\sqrt{3}}{2\pi} = \frac{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^3}{2^{1/3}\sqrt{3}\pi}$$ **Ans.**

Prob.11 (a) ***Prove that the volume of a sphere of radius 'a' is $\frac{4}{3}\pi a^3$***

(b) Prove that the volume generated by revolving the cardioid $r = a(1+\cos\theta)$ about the initial line is $\frac{8}{3}\pi a^3$.

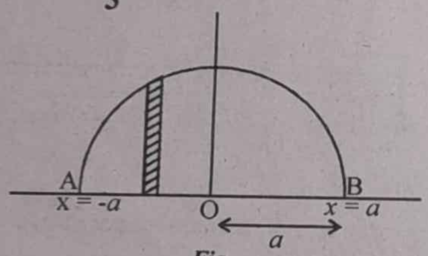


Fig.

Sol.(a) The sphere is the solid of revolution generated by revolving a semi-circular area about its diameter. Let us take the semicircular region of the circle $x^2 + y^2 = a^2$. The volume of solid obtained by revolving it about its diameter AB *i.e.* from $x = -a$ to $x = a$.

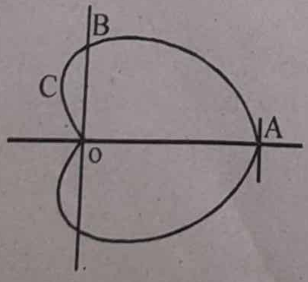


Fig.

$$V = \int_{-a}^{a} \pi y^2 dx = \int_{-a}^{a} \pi(a^2 - x^2)dx$$

$$= \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3$$ Proved

(b) Revolving the upper half of the cardioid $r = a(1+\cos\theta)$, *i.e.*, from $\theta = 0$ to $\theta = \pi$ about the initial line gives the solid required as curve is symmetrical about the initial line. The volume of the solid is

$$V = \int_0^{\pi} \pi y^2 \frac{dx}{d\theta} d\theta = \int_0^{\pi} \pi(r\sin\theta)^2 \frac{d}{d\theta}(r\cos\theta)d\theta$$

$$= \int_0^{\pi} \pi[a(1+\cos\theta)\sin\theta]^2 \frac{d}{d\theta}[a(1+\cos\theta)\cos\theta]d\theta$$

[Since $r = a(1+\cos\theta)$]

$$= \pi a^3 \int_0^{\pi} \sin^2\theta(1+\cos\theta)^2[-\sin\theta - 2\cos\theta\sin\theta]d\theta$$

$$= -\pi a^3 \int_0^{\pi} (1+\cos\theta)^2 \sin^3\theta.(1+2\cos\theta)d\theta$$

Let $\cos\theta = t \Rightarrow -\sin\theta d\theta = dt$

when $\theta = 0 : t = \cos 0 = 1$ and

when $\theta = \pi : t = \cos\pi = -1$

then $V = -\pi a^3 \int_1^{-1} (1+t)^2(1-t^2)(1+2t)dt$

$$= \pi a^3 \int_{-1}^{1} (1+t^2+2t)(1-t^2)(1+2t)dt$$

$$= \pi a^3 \int_{-1}^{1} (1-t^4+2t-2t^3)(1+2t)dt$$

$$= \pi a^3 \int_{-1}^{1} (1+4t+4t^2-2t^3-5t^4-2t^5)dt$$

$$= \pi a^3\left[\int_{-1}^{1} dt + 4\int_{-1}^{1} t\,dt + 4\int_{-1}^{1} t^2 dt - 2\int_{-1}^{1} t^3 dt - 5\int_{-1}^{1} t^4 dt - 2\int_{-1}^{1} t^5 dt\right]$$

$$= \pi a^3\left[2.4\int_0^1 t^2 dt - 2.5\int_0^1 t^4 dt + 2.\int_0^1 dt\right]$$

$$\left[\begin{aligned}\int_{-a}^{a} f(x)dx &= 0, \text{if } f(x) \text{ is odd, } i.e., f(-x) = -f(x)\\ &= 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even, } i.e., f(-x) = f(x)\end{aligned}\right]$$

$$= 2\pi a^3\left[4\left(\frac{t^3}{3}\right)_0^1 - 5\left(\frac{t^5}{5}\right)_0^1 + (t)_0^1\right]$$

$$= 2\pi a^3\left[\frac{4}{3} - 1 + 1\right] = \frac{8}{3}\pi a^3$$ **Proved.**

Prob.12 ***The part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cutoff by a latus rectum revolves about the tangent at the nearer vertex. Find the volume of the solid thus generated.***

Sol. PQP′ be the latus rectum and the nearer vertex is A(a, 0). The latus rectum cuts the axis at $Q(ae, 0)$ and the length of latus rectum $2l$(say) then PQ = l

Since P(ae, l) lies on the ellipse

$$\frac{a^2e^2}{a^2} + \frac{l^2}{b^2} = 1 \Rightarrow l^2 = b^2(1 - e^2)$$

But, $b^2 = a^2(1 - e^2)$

$$\Rightarrow \quad l^2 = \frac{b^4}{a^2}, \text{ i.e. } l = \frac{b^2}{a}$$

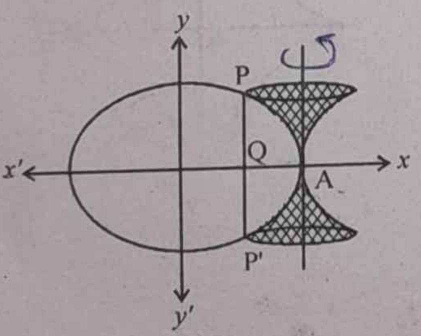


Fig.

The solid is obtained by revolving the part of the ellipse cut off by the latus ractum *i.e.* $y = -l$ to $y = l$ about $x = a$ (line parallel to y-axis) then the volume of the solid of revolution:

$$V = \pi\int_{-l}^{l}(a - x)^2\,dy = 2\pi\int_0^l(a^2 + x^2 - 2ax)\,dy$$

$$= 2\pi\int_0^l\left[a^2 + \frac{a^2}{b^2}(b^2 - y^2) - 2a.\frac{a}{b}\sqrt{b^2 - y^2}\right]dy$$

$$= 2\pi\int_0^l\left[a^2 + a^2 - \frac{y^2a^2}{b^2} - \frac{2a^2}{b}\sqrt{b^2 - y^2}\right]dy$$

$$= \frac{2\pi a^2}{b^2}\int_0^l\left[2b^2 - y^2 - 2b\sqrt{b^2 - y^2}\right]dy$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2y - 2b\left\{\frac{y}{2}\sqrt{b^2 - y^2} + \frac{b^2}{2}\sin^{-1}\frac{y}{b}\right\} - \frac{y^3}{3}\right]_0^l$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2l - 2b\frac{l}{2}\sqrt{b^2 - l^2} - 2b.\frac{b^2}{2}\sin^{-1}\frac{l}{b} - \frac{l^3}{3}\right]$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2.\frac{b^2}{a} - b.\frac{b^2}{a}\sqrt{b^2 - \frac{b^4}{a^2}} - b^3\sin^{-1}\frac{b^2}{ab} - \frac{1}{3}\left(\frac{b^2}{a}\right)^3\right]$$

$$= \frac{2\pi a^2}{b^2}\left[\frac{2b^4}{a} - \frac{b^4}{a}\sqrt{1 - \frac{b^2}{a^2}} - b^3\sin^{-1}\frac{b}{a} - \frac{b^6}{3a^3}\right]$$

$$= \frac{2\pi a^2}{b^2}\left[\frac{2b^4}{a} - \frac{b^4}{a}\sqrt{e^2} - b^3\sin^{-1}\frac{b}{a} - \frac{b^6}{3a^3}\right]$$

$$= \frac{2\pi b}{a}\left[2ba^2 - ba^2e - a^3\sin^{-1}\frac{b}{a} - \frac{b^3}{3}\right]$$

$$= \frac{2\pi b}{3a}\left[6a^2b - 3a^2be - b^3 - 3a^3\sin^{-1}\frac{b}{a}\right]$$ **Ans.**

Prob.13 ***Find the area of the surface formed by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis.*** ***[Raj. Univ. 2005]***

Sol. The ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$

Major axis of which is ØA = 4 and Minor axis is OB = 2

Area of surface formed by the revolution of the ellipse about x-axis

$$S = 2.2\pi\int_0^4 y\frac{ds}{dx}dx$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore \quad 4y^2 = 16 - x^2 = \sqrt{1 + \frac{x^2}{16y^2}};$$

$$8y\frac{dy}{dx} = -2x; \quad \frac{dy}{dx} = \frac{-x}{4y}$$

$$=\frac{1}{y}\sqrt{y^2+\frac{x^2}{16}}$$

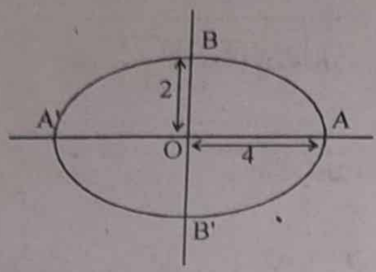


Fig.

then $S=4\pi\int_0^4 y.\frac{1}{y}\sqrt{y^2+\frac{x^2}{16}}\,dx$

$$=4\pi\int_0^4\sqrt{y^2+\frac{x^2}{16}}\,dx$$

Since $\frac{x^2}{16}+\frac{y^2}{4}=1$

$$=4\pi\int_0^4\sqrt{4\left(1-\frac{x^2}{16}\right)+\frac{x^2}{16}}\,dx \quad \Rightarrow y^2=4\left[1-\frac{x^2}{16}\right]$$

$$=4\pi\int_0^4\sqrt{4-\frac{3x^2}{16}}\,dx \;=4\pi\int_0^4\frac{1}{4}\sqrt{64-3x^2}\,dx$$

$$=\pi\int_0^4\sqrt{64-3x^2}\,dx$$

$$=\pi\int_0^{\pi/3}\sqrt{64-64\sin^2\theta}.\frac{8\cos\theta}{\sqrt{3}}\,d\theta$$

Let $\sqrt{3}\,x=8\sin\theta$, $\sqrt{3}\,dx=8\cos\theta\,d\theta$

$x=0$, $8\sin\theta=0\Rightarrow\theta=0$

$x=4$, $8\sin\theta=4\sqrt{3}$

$$\Rightarrow \quad \sin\theta=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{\pi}{3} \quad =\frac{\pi}{\sqrt{3}}\int_0^{\frac{\pi}{3}}8\cos\theta.8\cos\theta\,d\theta$$

$$=\frac{64\pi}{\sqrt{3}}\int_0^{\pi/3}\cos^2\theta\,d\theta=\frac{64\pi}{\sqrt{3}}\int_0^{\pi/3}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=\frac{32\pi}{\sqrt{3}}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/3}$$

$$=\frac{32\pi}{\sqrt{3}}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right]$$

$$=8\pi\left[\frac{4\pi}{3\sqrt{3}}+1\right]$$

Prob.14 ***Find the surface of the solid generated by the revolution of the astroid***

$x=a\cos^3 t, y=a\sin^3 t$ about the x-axis.

[R.T.U. 2011,

Sol. The shape of the astroid is given below, the required solid can be obtained by revolution of upper half of the astroid about x-axis.

The astroid can be written in parametric form $x=a\cos^3 t, y=a\sin^3 t$ where t varies from 0 to 2π. The surface of the solid after revolution of upper half of the astroid

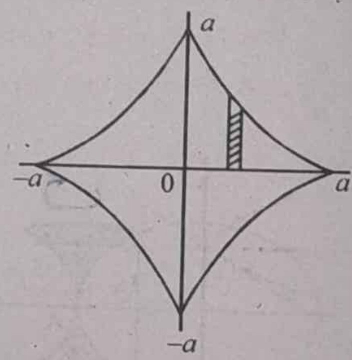


Fig.

$$s=2.\int_0^{\pi/2}2\pi y\frac{ds}{dt}\,dt$$

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$=\sqrt{\left(-3a\cos^2 t\sin t\right)^2+\left(3a\sin^2 t\cos t\right)^2}$$

$$=3a\sin t\cos t$$

$$s=4\pi\int_0^{\pi/2}\left(a\sin^3 t\right).\left(3a\sin t\cos t\right)dt$$

$$=12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t\,dt$$

$$= 12\pi a^2 . \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{2}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)}$$

$$= 12\pi a^2 \frac{\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{12\pi a^2}{5}$$ **Ans.**

Prob.15 ***Find the surface of the solid generated by the revolution of the laminscate $r^2 = a^2 \cos 2\theta$ about the initial line.***

Sol. The curve is symmetric about initial line $\theta = \frac{\pi}{2}$ and the pole. The curve is made of four loops where the loop in the first quadrant varies from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

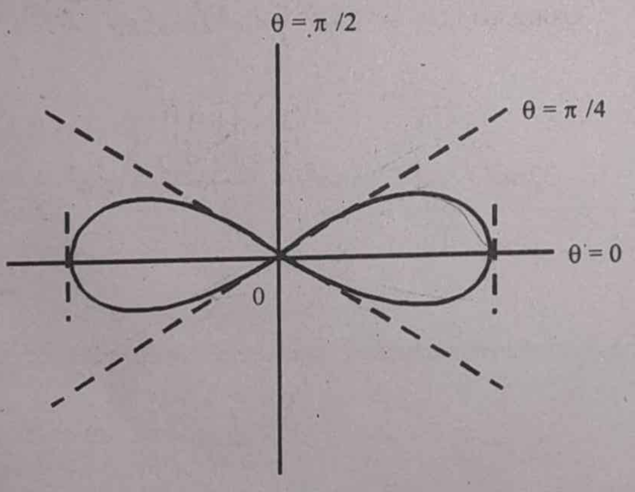


Fig.

The surface of the solid of revolution of the upper half of the curve about initial line

$$S = 2\int_0^{\pi/4} 2\pi y \frac{ds}{d\theta} d\theta = 4\pi \int_0^{\pi/4} (r\sin\theta)\frac{ds}{d\theta} d\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$r^2 = a^2 \cos 2\theta \Rightarrow 2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta,$$

$$\frac{dr}{d\theta} = \frac{-a^2 \sin 2\theta}{r}$$

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4 \sin^2 2\theta}{r^2} = \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta} = \frac{a^2 \sin^2 2\theta}{\cos 2\theta}$$

$$\frac{ds}{d\theta} = \sqrt{a^2 \cos 2\theta + \frac{a^2 \sin^2 2\theta}{\cos 2\theta}}$$

$$= \frac{a}{\sqrt{\cos 2\theta}}\sqrt{\cos^2 2\theta + \sin^2 2\theta}$$

$$= \frac{a}{\sqrt{\cos 2\theta}}$$

$$S = 4\pi \int_0^{\pi/4} r\sin\theta . \frac{a}{\sqrt{\cos 2\theta}} d\theta$$

$$= 4\pi \int_0^{\pi/4} a\sqrt{\cos 2\theta}.\sin\theta.\frac{a}{\sqrt{\cos 2\theta}} d\theta$$

[Since $r^2 = a^2 \cos 2\theta$]

$$S = 4\pi a^2 \int_0^{\pi/4} \sin\theta d\theta = -4\pi a^2 \left[\cos\theta\right]_0^{\pi/4}$$

$$= \frac{4\pi a^2}{\sqrt{2}}\left(\sqrt{2} - 1\right)$$ **Ans.**

Prob.16 ***Find the volume of the following :***

(a) ***$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ about its base.***

(b) ***$x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ about the tangent at the vertex.***

Sol.(a) The shape of the cycloid

$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$$

is given in the figure the volume of the solid generated when it is revolved about its base i.e. x-axis is

$$V = \int_0^{2\pi} \pi y^2 \frac{dx}{d\theta} d\theta = \int_0^{2\pi} \pi . a^2 (1 - \cos\theta)^2 . a(1 - \cos\theta) d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1 - \cos\theta)^3 d\theta = \pi a^3 \cdot \int_0^{2\pi} \left(2\sin^2 \frac{\theta}{2}\right)^3 d\theta$$

Let $\frac{\theta}{2} = t : d\theta = 2dt$

$\theta = 2\pi : t = \pi$ and $\theta = 0 : t = 0$

$$V = 16a^3 . \pi \int_0^{\pi} \sin^6 t dt$$

$$= 32a^3.\pi\int_0^{\pi/2}\sin^6 t\,dt = 32\pi a^3 \frac{\left|\frac{7}{2}\right|\frac{1}{2}}{2\left|\frac{8}{2}\right.}$$

$$= 32\pi a^3 \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\left|\frac{1}{2}\right|\frac{1}{2}}{2.3.2.1} = \pi a^3.5\pi = 5\pi^2 a^3$$ **Ans.**

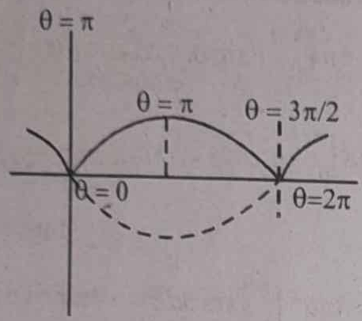


Fig.

(b) The shape of the given cycloid is given in the figure where vertex is the origin and the tangent at the vertex is x-axis and θ varies from 0 to π.

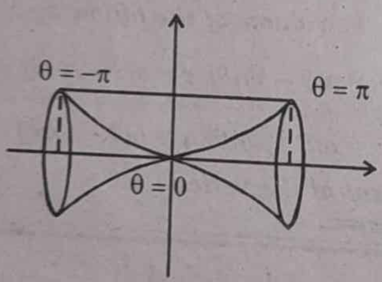


Fig.

Volume of the solid when curve is revolved abo tangent at the vertex, *i.e.*, x-axis

$$V = 2\int_0^{\pi}\pi y^2 \frac{dx}{d\theta}d\theta$$

$$= 2\pi\int_0^{\pi} a^2(1-\cos\theta)^2.\frac{d}{d\theta}\{a(\theta+\sin\theta)\}\,d\theta$$

$$= 2\pi a^3\int_0^{\pi}(1-\cos\theta)^2(1+\cos\theta)d\theta$$

$$= 2\pi a^3\int_0^{\pi}\left(2\sin^2\frac{\theta}{2}\right)^2\left(2\cos^2\frac{\theta}{2}\right)d\theta$$

$$= 2\pi a^3\int_0^{\pi} 8\sin^4\frac{\theta}{2}\cos^2\frac{\theta}{2}d\theta$$

Let $\frac{\theta}{2} = t, d\theta = 2dt$, $\theta = \pi$, $t = \frac{\pi}{2}$ and $\theta = 0, t = 0$

Using in (1), $V = 32\pi a^3\int_0^{\pi/2}\sin^4 t\cos^2 t\,dt$

$$= 32\pi a^3.\frac{\left|\frac{5}{2}\right|\frac{3}{2}}{2\left|\frac{8}{2}\right.} = 16\pi a^3\frac{\frac{3}{2}.\frac{1}{2}\left|\frac{1}{2}\right.\frac{1}{2}\left|\frac{1}{2}\right.}{3.2.1} = \pi^2 a^3$$ □

SEQUENCES AND SERIES 2

PREVIOUS YEARS QUESTIONS

I. SHORT ANSWER TYPE QUESTIONS

Prob.1 *Show that the series*
$1^2 + 2^2 + 3^2 + \ldots + n^2 + \ldots$, diverges to ∞.

Sol. Given, $u_n = n^2$

$$S_n = \sum u_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 + \ldots$$

Then, $$s_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

and $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6}$$

$$= \infty$$

$\Rightarrow$ Sequence diverges to $+\infty$

Prob.2 *Test the series* $\sum_{n=1}^{\infty} (-1)^{n-1}$

Sol. Given that

$$\sum u_n = \sum_{n=1}^{\infty} (-1)^{n-1}$$

and $$s_n = 1 - 1 + 1 - 1 + 1 - 1 + \ldots \text{ n terms}$$

$$= \begin{cases} 0 \text{ for even terms} \\ 1 \text{ for odd terms} \end{cases}$$

Hence, $\sum u_n$ converges to 0 and 1, depends on number of terms whether even or odd. Therefore, series oscillate finitely.

Prob.3 *Test the series*
$5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots \infty$ terms.

Sol. Given, the series

$$\sum u_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots \infty \text{ terms}$$

Then, $$s_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots \text{ n terms}$$

Here, addition of n terms depends on the value of n,

$$s_n = \begin{cases} 0 \text{ if } n = 3k \\ 5 \text{ if } n = 3k+1 \\ 1 \text{ if } n = 3k+2 \end{cases}$$

Hence, given series oscillate finitely.

Prob.4 *Test the series for convergence* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Sol. The given alternating series, $\sum (-1)^{n-1} u_n$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

$$u_n = \frac{1}{n}$$

and $$u_{n+1} = \frac{1}{n+1}$$

$$= \frac{1}{n} > \frac{1}{n+1}, \forall n$$

or $u_n > u_{n+1}, \forall n$

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence, both condition satisfied.

$\sum u_n$ is convergent.

Prob.5 ***Prove that every absolute convergent series is convergent always.***

Sol. Let us consider $\sum u_n$ is an absolutely convergent series.

Then, $\sum_{n=1}^{\infty} |u_n|$ is convergent

Considering Cauchy's general principle of convergence, for a small positive number $\in > 0, \exists a$ +ve integer k is such away.

$$= \left||u_{k+1}| + |u_{k+2}| + \ldots + |u_k|\right| < \in \forall n > k$$

or $|u_{k+1}| + |u_{k+2}| + \ldots + |u_k| < \in \forall n > k$

The given series $\sum_{n=1}^{\infty} u_n$ is convergent

Then $\sum |u_n|$ is convergent.

$\sum u_n$ is convergent.

Prob.6 ***Prove that the series*** $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots\infty$ ***convergent absolutely.***

Sol. The given alternating series is

$$\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$$

or $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n^3}$

$$|u_n| = \frac{|\sin nx|}{n^3} \le \frac{1}{n^3}, \forall n$$

$$v_n = \frac{1}{n^3}$$

but by comparision text

$\sum |u_n|$ converges

The given series converges absolutely.

II. Long Answer Type Questions

Prob.1 ***Discuss the convergence of the sequence*** $\{b_n\}$ ***where***

(i) $b_n = \frac{n}{1+n^2}$

(ii) $b_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^n}$

(iii) $b_n = \frac{1+n}{n}$

Sol. (i) Given,

$$b_n = \frac{n}{1+n^2}$$

Therefore,

$$b_{n+1} = \frac{1+n}{1+(1+n)^2}$$

Hence,

$$b_{n+1} - b_n = \frac{1+n}{1+(1+n)^2} - \frac{n}{1+n^2}$$

$$= \frac{(1+n)(1+n^2) - n\left[1+(1+n)^2\right]}{\left[1+(1+n)^2\right]\left[1+n^2\right]}$$

$$= \frac{1-n-n^2}{(2+2n+n^2)(1+n^2)} < 0, \forall n$$

$\Rightarrow \quad a_{n+1} < a_n, \forall n$

$\Rightarrow \{a_n\}$ is a decreasing sequence

and $b_n = \frac{n}{1+n^2} > 0 \ \forall n$

$\Rightarrow \{b_n\}$ is bounded below.

Since, given sequence is decreasing and bounded below, therefore, it is convergent sequence.

$$\text{Now} \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{1+n^2}$$

$$= \lim_{n\to\infty} \frac{1/n}{1+1/n^2}$$

$$= 0$$

Hence, sequence converges to zero.

(ii) Given sequence $\{b_n\}$

$$b_n = 1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}$$

$$= \frac{1\cdot\left(1+\frac{1}{3^{n+1}}\right)}{1-\frac{1}{3}}$$

$$\left[\begin{array}{l}\text{G.P. with first term 1, n+1} \\ \text{terms and common ratio 1/3.}\end{array}\right]$$

$$\left[S_n = \frac{a\cdot(1-r^n)}{1-r}\right]$$

$$= \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right)$$

$$\text{and} \qquad b_{n+1} = 1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}+\frac{1}{3^{n+1}}$$

$$\Rightarrow b_{n+1} - b_n = \frac{1}{3^{n+1}} > 0$$

$\Rightarrow$ $\{b_n\}$ is an increasing sequence.

$$\text{and} \quad b_n = \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) < \frac{3}{2}\ \forall n$$

$\Rightarrow$ $\{b_n\}$ is an increasing sequence, bounded above, therefore convergent.

$$\lim_{n\to\infty} = \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) = \frac{3}{2}$$

$$= \frac{3}{2}$$

Hence, the sequence is convergent and converges to $\frac{3}{2}$.

(iii) Given sequence is $\{b_n\}$

General term,

$$b_n = \frac{1+n}{n}$$

$$= 1+\frac{1}{n}$$

$$\text{and,} \quad b_{n+1} - b_n = \frac{-1}{n(n+1)} < 0, \forall n$$

$$b_{n+1} < b_n, \forall n$$

$\Rightarrow$ $\{b_n\}$ is a decreasing sequence,

$$\text{and,} \qquad b_n = 1+\frac{1}{n} > 1\ \forall n$$

Hence, $\{b_n\}$ is bounded below.

$$\text{Now,} \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= 1$$

$\therefore$ The given sequence $\{b_n\}$ is convergent and converges to 1.

Prob.2 ***Test the series for convergence.***

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots$$

infinite

Sol. Deletion or addition of terms in ~~a~~ finite series don't change the nature of series, therefore, avoid first term,

We have

$$u_n = \frac{n^n}{(n+1)^{n+1}}$$

$$= \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}}$$

$$= \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$$

Assuming $v_n = \dfrac{1}{n}$

Now, $$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e}\cdot\frac{1}{1}$$

$$= \frac{1}{e} \text{ (which is finite and non-zero)}$$

$\Rightarrow$ Σu_n and Σv_n diverges or converges together

But

$$\Sigma v_n = \sum\frac{1}{n}$$

On comparision with p-series test

$$p = 1$$

$\Rightarrow$ $\sum\frac{1}{n}$ is convergent

$\Rightarrow$ Σv_n is convergent

$\Rightarrow$ Σu_n is convergent **Ans.**

Prob.3 *Test the convergence or divergence of the following series if $x > 0$.*

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots$$

Sol. Given that

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

and so, $$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

and $$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}}\cdot\frac{1}{x^2}$$

$$= \frac{1+\frac{2}{n}}{1+\frac{1}{n}}\cdot\sqrt{1+\frac{1}{n}}\cdot\frac{1}{x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By D' Alembert's Ratio Test,

Σu_n is convergent, if $\frac{1}{x^2} > 1$

or $x^2 < 1$

and, divergent if $\frac{1}{x^2} < 1$ or $x^2 > 1$

For $x^2 = 1$,

$$u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

and $$v_n = \frac{1}{n^{3/2}}$$

Then $$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

Hence, by comparison test, Σv_n is convergent

Therefore, Σu_n is convergent for $x^2 \le 1$ and dive if $x^2 > 1$.

Prob.4 *Test the convergence of the series,*

$$\sum\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n, x>0$$

Sol. Given that

$$u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}}x^n$$

Therefore,

$$u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}}x^{n+1}$$

$$= \sqrt{\frac{n}{n^2+1}\cdot\frac{(n+1)^2+1}{n+1}}\cdot\frac{1}{x}$$

$$= \sqrt{\frac{n}{n+1}\cdot\frac{n^2+2n+2}{n^2+1}\cdot\frac{1}{x}}$$

Now, $$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{1}{x}}$$

$$= \frac{1}{x}$$

Therefore, by D'Alembert's Ratio Test, $\sum u_n$ diverges if $\frac{1}{x} < 1 \Rightarrow x > 1$; converges if $\frac{1}{x} > 1 \Rightarrow x < 1$.

But, if $x = 1$, The ratio test fails

and $$u_n = \sqrt{\frac{n}{n^2+1}}$$

$$= \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}}$$

$$= \frac{1}{\sqrt{n}}\frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

Assuming, $v_n = \frac{1}{\sqrt{n}}$

Thus, $$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

which is finite and non-zero. On p-series test $\sum v_n$ is divergent as $p = \frac{1}{2} < 1$

$\sum u_n$ is a divergent series.

Prob.5 ***Test for convergence the given positive term series,***

$$1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\ldots$$

Sol. Avoiding the first term,

$$u_n = \frac{(\alpha+1)(2\alpha+1)\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots(n\beta+1)}$$

and $$u_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots(n\alpha+1)\left[(n+1)\alpha+1\right]}{(\beta+1)(2\beta+1)\ldots(n\beta+1)\left[(n+1)\beta+1\right]}$$

Therefore, $$\frac{u_n}{u_{n+1}} = \frac{(n+1)\beta+1}{(n+1)\alpha+1} = \frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}$$

Now, $$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{u_n}{u_{n+1}}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}$$

$$= \frac{\beta}{\alpha}$$

By, D' Alembert's Ratio Test, Series is convergent if

$$\frac{\beta}{\alpha} > 1 \Rightarrow \beta > \alpha > 0,$$

series is divergent if

$$\frac{\beta}{\alpha} < 1 \Rightarrow \alpha > \beta > 0.$$

and, if $\alpha = \beta$, then $u_n = 1 \neq 0$

$\sum u_n$ is divergent on a series of positive term.

Prob.6 ***Examine the following series for convergence or divergence.***

$$\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots, x>0$$

Sol. Given that a series,

$$\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots, x>0$$

Leaving first term, we have

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

Then $(u_n)^{1/n} = \frac{n+1}{n+2}x$

$$= \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x$$

and $\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x$

$$= x$$

Therefore, by Cauchy's root test, the given series is convergent if $x < 1$ and divergent if $x = 1$, but on x = 1,

$$u_n = \left(\frac{n+1}{n+2}\right)^n = \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n$$

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n = \frac{e}{e^2} = \frac{1}{e}$

which is non-zero and finite

and $\sum u_n$ is divergent, since, the series is of positive terms.

Therefore, the given series is divergent if $x \geq 1$ and convergent if x < 1

Prob.7 ***Test the convergence of the series,***

$$1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\ldots$$

Sol. Given series,

$$1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\ldots$$

Avoiding first term, we have general term,

$$u_n = \frac{n!}{(n+1)^n}x^n$$

Therefore,

$$u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$$

Now, $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n!}{(n+1)^n}.x^n.\frac{(n+2)^{n+1}}{(n+1)!}.\frac{1}{x^{n+1}}$

$$= \lim_{n\to\infty}\frac{n!}{(n+1)!}.\frac{(n+2)^{n+1}}{(n+1)^n}.\frac{1}{x}$$

$$= \lim_{n\to\infty}\frac{1}{n^n\left(1+\frac{1}{n}\right)^n}.\frac{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}{n+1}.\frac{1}{x}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}.\frac{1}{x}$$

$$= \frac{e^2.1}{e.1}.\frac{1}{x} = \frac{e^2}{e}.\frac{1}{x} = \frac{e}{x}$$

By D'Alembert's ratio test, the given series is diverges if $\frac{e}{x} < 1$ or $x > e$; and converges if $\frac{e}{x} > 1$ or $x < e$.

Let us consider, if $x = e \Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$

Then, the Ratio test fails,

Now,

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}}.\frac{1}{e}$$

As we know, if expression involves the number 'e' then we should apply Logarithmic test,

$$\log\frac{u_n}{u_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right)-(n+1)\log\left(1+\frac{1}{n}\right)-\log e$$

$$= (n+1)\left[\log\left(1+\frac{2}{n}\right)-\log\left(1+\frac{1}{n}\right)\right]-1$$

$$= (n+1)\left[\left(\frac{2}{n}-\frac{1}{2}.\frac{4}{n^2}+\frac{1}{3}.\frac{8}{n^3}...\right)-\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-...\right)\right]-1$$

$$= (n+1)\left[\frac{1}{n}-\frac{3}{2n^2}+...\right]-1$$

$$= 1-\frac{3}{2n}+\frac{1}{n}-\frac{3}{2n^2}+...-1$$

$$= -\frac{1}{2n+}-\frac{3}{2n^2}+...$$

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\left[-\frac{1}{2n}-\frac{3}{2n^2}+...\right]$$

$$= -\frac{1}{2} < 1$$

By the logarith mic test, the given series is divergent.

Prob.8 ***Test the convergence of the series :***

$$\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}, p>0$$

Sol. Given the series,

$$\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}, p>0$$

Then $u_n = \dfrac{1}{n(\log n)^p}$

$= f(n)$

$$f(x) = \frac{1}{x(\log x)^p}$$

Where $p>0, x\geq 2$ and $f(x)$ is positive and monotonic decreasing.

By Cauchy's Intergral Test,

$\int_2^{\infty} f(x)dx$ and $\sum_{n=2}^{\infty} u_n$ diverges or converges together.

(a) If $p>1, p-1$ is positive,

$$\text{Then, } \int_2^{\infty} f(x)dx = -\frac{1}{p-1}\left[\frac{1}{(\log x)^{p-1}}\right]_2^{\infty}$$

$$= -\frac{1}{p-1}\left[0-\frac{1}{(\log 2)^{p-1}}\right]$$

$$= \frac{1}{(p-1)(\log 2)^{p-1}}$$

= finite number

Therefore, $\int_2^{\infty} f(x)dx$ converges

$\sum_{n=2}^{\infty} u_n$ converges

(b) If $p<1$, $1-p$ is positive,

Then,

$$\int_2^{\infty} f(x)dx = \frac{1}{1-p}\left[(\log x)^{1-p}\right]_2^{\infty}$$

$$= \infty$$

Therefore, $\int_2^{\infty} f(x)dx$ diverges,

Hence, $\sum_{n=2}^{\infty} u_n$ diverges

(c) If $p \neq 1$

Now, $\int_2^{\infty} f(x)dx$

$$= \int_2^{\infty}(\log x)^{-p}\frac{1}{x}dx$$

$$= \left[\frac{(\log x)^{1-p}}{1-p}\right]_2^{\infty}$$

(d) If $p=1$, $f(x) = \dfrac{1}{x(\log x)}$

$$\text{Then, } \int_2^{\infty} f(x)dx = \int_2^{\infty}\frac{dx}{x\log x} = \int_2^{\infty}\frac{1/x}{\log x}dx = \infty$$

Therefore, $\int_2^{\infty} f(x)dx$ diverges

$\sum_{n=2}^{\infty} u_n$ divergent

$\sum u_n$ diverges if $0<p\leq 1$ and converges if $p>1$.

Prob.9 ***Test the convergence and absolute convergence of the given series :***

$$\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\frac{1}{4.6}+\ldots\infty$$

Sol. The given series,

$$\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\frac{1}{4.6}+\ldots$$

$$=\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n.(n+2)}$$

Therefore, $u_n=\dfrac{1}{n(n+2)}>0\forall n$

$$u_{n+1}=\frac{1}{(n+1)(n+3)}$$

But $\quad u_n-u_{n+1}=\dfrac{1}{n(n+2)}-\dfrac{1}{(n+1)(n+3)}$

$$=\frac{2n+3}{n(n+1)(n+2)(n+3)}>0,\forall n$$

$$=u_n>u_{n+1}\forall n$$

Now, $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\dfrac{1}{n(n+2)}$

and $\quad |u_n|=\dfrac{1}{n(n+2)}=\dfrac{1}{n^2\left(1+\dfrac{2}{n}\right)}$

$\therefore \quad v_n=\dfrac{1}{n^2}$

Therefore,

$$\lim_{n\to\infty}\frac{|u_n|}{v_n}=\lim_{n\to\infty}\frac{1}{1+\dfrac{2}{n^2}}=1$$

By comparision test $|u_n|$ and v_n converges or diverges together.

But, as per p-series test $\sum v_n$ is converges.

$\sum u_n$ is an absolutely convergent.

Prob.10 $\dfrac{1}{2(\log 2)^p}-\dfrac{1}{3(\log 3)^p}+\dfrac{1}{4(\log 4)^p}-\ldots\infty,(p$

Sol. The given series,

$$\frac{1}{2(\log 2)^p}-\frac{1}{3(\log 3)^p}+\frac{1}{4(\log 4)^p}-\ldots$$

and $\quad \sum_{n=2}^{\infty}u_n=\sum_{n=2}^{\infty}(-1)^n\dfrac{1}{n(\log n)^p}$

Let $\quad |u_n|=\dfrac{1}{n(\log n)^p} \quad \forall n\geq 2$

and $\quad \sum_{n=2}^{\infty}|u_n|$ is convergent, if $p>1$

and divergent if $0<p<1$

Therefore, the given series is absolutely converge $p>1$.

Here, $\quad u_n=\dfrac{1}{n(\log n)^p}$

$$u_{n+1}=\frac{1}{(n+1)(\log(n+1))^p}$$

Here, $n<n+1,\forall n\geq 2$

$\therefore \quad n(\log n)^p<(n+1)(\log(n+1))^p,\forall n\geq 2$

Taking reciprocal,

$$=\frac{1}{n(\log n)^p}>\frac{1}{(n+1)(\log(n+1))^p}\forall n\geq 2$$

$$=u_n>u_{n+1},\forall n\geq 2$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n(\log n)^p}=0$$

Therefore, $\sum u_n$ is convergent.

Therefore, the given series is absolutely converge if $p>1$ and conditionally convergent for $0<p\leq 1$.

Prob.11 *Find the Taylor's series expression about the point $\frac{\pi}{3}$ of the function $f(x) = \log\cos x$*

Sol. For the function $f(x)$ about point a, Taylor's series expression can be written as

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots$$

Here, $a=\frac{\pi}{3}$

$$f(x) = \log\cos x \Rightarrow f\left(\frac{\pi}{3}\right)=\log\frac{1}{2}$$

$$f'(x) = \frac{1}{\cos x}(-\sin x)=-\tan x$$

and $f'(a)=f'\left(\frac{\pi}{3}\right)=-\tan\frac{\pi}{3}=-\sqrt{3}$

$$f''(x)=-\sec^2 x$$

$$f''\left(\frac{\pi}{3}\right)=-\sec^2\frac{\pi}{3}=-4$$

$$f'''(x) = -2\tan x-2\tan^3 x$$

$$f'''\left(\frac{\pi}{3}\right) = -8\sqrt{3}$$

Now,

$$f(x)= f\left(\frac{\pi}{3}\right)+\left(x-\frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right)+\frac{x-\frac{\pi}{3}}{2!}f''\left(\frac{\pi}{3}\right)+\ldots$$

On putting the values of various terms

$$\log\cos x= \log\frac{1}{2}-\sqrt{3}\left(x-\frac{\pi}{3}\right)-4\frac{\left(x-\frac{\pi}{3}\right)^2}{2!}$$

$$-8\sqrt{3}\frac{\left(x-\frac{\pi}{3}\right)^3}{3!}+\ldots$$

Prob.12 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$

$$u_n = \frac{1}{2n-1}$$

$$= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Sol. The given series is clearly alternating series and term going decreasing numerically.

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{2n-1}=0$

As per Leibnitz Test, The given series is convergent.

But when all terms making positive,

$$\sum|u_n| = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$$

$$|u_n| = \frac{1}{2n-1}$$

$$v_n = \frac{1}{n}$$

and $\lim_{n\to\infty}\frac{|u_n|}{v_n} = \lim_{n\to\infty}\frac{n}{2n-1}=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}$

$$=\frac{1}{2}=\text{finite} \neq 0$$

by comparision test,

$\sum|u_n|$ and $\sum v_n$ converges or diverges together.

But $v_n = \frac{1}{n}$

P = 1

$\therefore$ $\sum v_n$ is divergent

$\sum|u_n|$ also divergent.

Therefore, given series is conditionally convergent.

Prob.13 *Test the convergence of the series,*

$$\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}+\ldots$$

Sol. Given series is,

$$\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}+\ldots$$

$$u_n=\frac{1^2.3^2.5^2\ldots(2n-1)^2}{2^2.4^2.6^2\ldots(2n)^2}$$

and $u_{n+1}=\dfrac{1^2.3^2.5^2\ldots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\ldots(2n)^2(2n+2)^2}$

Therefore,

$$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}=\frac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2}$$

$$=\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

and $\lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}}=1$

$\Rightarrow$ Ratio test fails

Then, $n\left(\dfrac{u_n}{u_{n+1}}-1\right)=n\left(\dfrac{(2n+2)^2}{(2n+1)^2}-1\right)$

$$=n\frac{(4n+3)}{(2n+1)^2}$$

$$=\frac{4n^2+3n}{(2n+1)^2}$$

$$=\frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2}$$

Now, $\lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}}-1\right)=\lim\limits_{n\to\infty}\left(\dfrac{1+\frac{3}{4n}}{1+\frac{1}{2n}}\right)$

$$=1$$

Raabe's Test Fails

Now, Gauss Test

$$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}=\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$$=\left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$=\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{1}{n}+\frac{3}{4n^2}-\ldots\right)$$

$$=1+\frac{1}{n}-\frac{1}{4n^2}+\ldots$$

$$=1+\frac{1}{n}+0\left(\frac{1}{n^2}\right)$$

Comparing with

$$\frac{u_n}{u_{n+1}}=1+\frac{k}{n}+0\left(\frac{1}{n^2}\right)$$

$$k=1$$

$\sum u_n$ is divergent.

FOURIER SERIES 3

PREVIOUS YEARS QUESTIONS

I. SHORT ANSWER TYPE QUESTIONS

Prob.1 ***Define periodic function with example.***

Sol. A function $f(t)$ is known as periodic if $f(t \pm p) = f(t)\ \forall$ t, and for some positive number p. Here p is known as period of $f(t)$.

For example, $f(t) = \sin 2t$ is periodic function with period $p = \pi$ for all values of t.

i.e. $f(t + \pi) = \sin [2(t + \pi)] = \sin (2t + 2\pi)$
$= \sin 2t = f(t)$

Prob.2 ***Define orthogonal function.***

Sol. Consider two functions which are piecewise continuous on some interval $[a, b]$ and their inner product defined as

$$(f_1, f_2) = \int_a^b f_1 f_2\, dx$$

Then f_1, f_2 are orthogonal function if $(f_1, f_2) = 0$

Ex. $\int_{-\pi}^{\pi} \sin 3x \cos 3x = 0$

$\Rightarrow$ $\sin 3x$ and $\cos 3x$ are orthogonal functions.

Prob.3 ***Find the formulation of fourier series for odd function in the interval of (–c, c) of length 2c.***

Sol. If $f(x)$ is an odd function defined in the interval $-c$ to c. Then the Fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

where $b_n = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\, dx$

Prob.4 ***Defind even function with suitable examples.***

Sol. A function $f(x)$ in known as even function if $f(-x) = f(x)$ for all value of x.

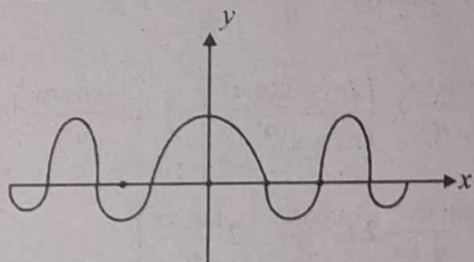


Fig. : Even function

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even function.

For examples x^2, $\cos x$, $e^x + e^{-x}$, $x^4 + \cos 2x + 2$ etc.

II. LONG ANSWER TYPE QUESTIONS

Prob.1 ***Find the Fourier series for the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$. Hence show that***

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.$$

[R.T.U. 2012, CE 09, EC 09, EE 09, ME 05]

Sol. The Fourier series is given by

$$f(x) = x + x^2$$

$$= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), -\pi < x < \pi \quad \ldots (i)$$

Then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x + x^2)dx$

$$a_0 = \frac{1}{2\pi}\left(\frac{x^2}{2} + \frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^3}{3}+\frac{\pi^3}{3}\right] = \frac{\pi^2}{3} \qquad \text{... (ii)}$$

Also $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\, dx$$

$$\left[\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd function} \\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even function} \end{cases}\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\, dx$$

[$\because$ $x \cos nx$ is an odd function]

$$a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \int 2x\frac{\sin nx}{n}dx\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \left\{2x\left(\frac{-\cos nx}{n^2}\right) - \int 2\left(\frac{-\cos nx}{n^2}\right)dx\right\}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\left\{\pi\frac{\sin n\pi}{n} - 0\right\} + 2\left\{\pi\frac{\cos n\pi}{n^2} - 0\right\} - 2\left\{\frac{\sin n\pi}{n^3} - 0\right\}\right]$$

$$a_n = \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right] = \frac{4}{n^2}(-1)^n \qquad \text{... (iii)}$$

[$\because$ $\sin n\pi = 0$, $\cos n\pi = (-1)^n$]

Again $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx$$

[$\because$ $x^2 \sin nx$ is an odd function]

$$b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \int\left(\frac{-\cos nx}{n}\right)dx\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

[$\because$ $\sin n\pi = 0$, $\cos n\pi = (-1)^n$]

$$b_n = \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n}\right] = \frac{-2}{n}(-1)^n \qquad \text{... (iv)}$$

Substituting the value of a_0, a_n and b_n from eq. (ii), (iii) and (iv) respectively in eq. (i), we have

$$f(x) = x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + \sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n\sin nx$$

$$f(x) = x + x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right]$$

$$+2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right] \qquad \text{... (v)}$$

The given series on the R.H.S of eq. (v) represents $x + x^2$ for all the values of x lie in the given interval $-\pi < x < \pi$ except the end points $-\pi$ and π.

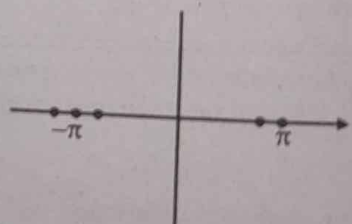
For $x = -\pi$, the sum of the series on the right of eq. (v)

$$= \frac{1}{2}[f(-\pi+0) + f(-\pi-0)]$$

$$= \frac{1}{2}[f(-\pi+0) + f(\pi-0)]$$

$$= \frac{1}{2}[-\pi + \pi^2 + \pi + \pi^2]$$

$$= \pi^2$$



Now from eq. (v), we put $x = -\pi$, then

$$\pi^2 = \frac{\pi^2}{3} - 4\left(\cos\pi - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} \dots\right) + 0$$

or $\quad 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$

or $\quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$ **Ans.**

Prob.2 *Find the Fourier series expansion for the function if*

$$f(x) = 0 \qquad \text{for } -\pi \le x < 0$$
$$= \sin x \qquad \text{for } 0 < x \le \pi$$

Hence show that

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...=\frac{1}{2} \text{ and}$$

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-....=\frac{\pi-2}{4}$$

[R.U. 2002]

Sol. The Fourier series is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \quad ...(i)$$

Then $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0dx+\int_{0}^{\pi}\sin nxdx\right]$$

$$a_0=\frac{1}{2\pi}(-\cos x)_0^{\pi}=\frac{1}{2\pi}[1-\cos\pi]$$

$$\therefore \quad a_0=\frac{2}{2\pi}=\frac{1}{\pi} \quad ...(ii)$$

Also $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right]$$

$$a_n=\frac{1}{\pi}\left[\int_{0}^{\pi}\sin x\cos nxdx\right]$$

$$a_n=\frac{1}{2\pi}\left[\int_{0}^{\pi}2\sin x\cos nxdx\right]$$

[2sinA cos B = sin (A+B) – sin (B – A)]

$$a_n=\int_0^{\pi}[\sin(1+n)x-\sin(n-1)x]dx$$

$$a_n=\frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)}+\frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$a_n=\frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{(n-1)}+\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]$$

$$a_n=\frac{1}{2\pi}\left[-\frac{(\cos n\pi\cos\pi-\sin n\pi\sin\pi)}{(n+1)}+\frac{(\cos n\pi\cos\pi+\sin n\pi\sin\pi)}{(n-1)}+\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]$$

[$\because$ cos (A + B) = cos A cos B – sin Asin B]

$$a_n=\frac{1}{2\pi}\left[+\frac{\cos n\pi}{(n+1)}-\frac{\cos n\pi}{(n-1)}-\frac{2}{n^2-1}\right]$$

$$a_n=\frac{1}{2\pi}\left[\frac{(n-1)\cos n\pi-(n+1)\cos n\pi-2}{(n^2-1)}\right]$$

$$a_n=\frac{1}{2\pi}\left[\frac{-2\cos n\pi-2}{n^2-1}\right]$$

$$\therefore \quad a_n=\frac{(1+\cos n\pi)}{\pi(1-n^2)}, n\neq 1$$

This value of a_n for $n \neq 1$, so a_1 can not be found from a_n.

Now $a_1=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos xdx$

$$a_1=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cos xdx+\int_{0}^{\pi}\sin x\cos xdx\right]$$

$$a_1=\frac{1}{2\pi}\int_0^{\pi}\sin 2xdx=\frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$a_1=0$$

Again $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\sin nxdx+\int_{0}^{\pi}\sin x\sin nxdx\right]$$

$$b_n=\frac{1}{2\pi}\left[\int_{0}^{\pi}2\sin x\sin nxdx\right]$$

[2sinA sin B = cos (A–B) – cos (A + B)]

$$b_n=\frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)x-\cos(n+1)x]dx$$

$$b_n=\frac{1}{2\pi}\left[\frac{\sin(n-1)x}{(n-1)}-\frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi}$$

$$b_n=\frac{1}{2\pi}\left[\frac{\sin nx\cos x-\cos nx\sin x}{(n-1)}-\frac{\sin nx\cos x+\cos nx\sin x}{(n+1)}\right]_0^{\pi}$$

$$b_n = 0 \; ; n \neq 1$$

Now $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x dx$

$$b_1 \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \sin x dx + \int_{0}^{\pi} \sin x \sin x dx\right]$$

$$b_1 = \frac{1}{2\pi}\int_{0}^{\pi} 2\sin^2 x dx$$

$$b_1 = \frac{1}{2\pi}\int_{0}^{\pi}(1-\cos 2x)dx$$

$$b_1 = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[(\pi - 0)\left(\frac{\sin 2\pi}{2} - 0\right)\right] = \frac{1}{2}$$

Substituting the value of a_0, a_n, b_n, a_1 and b_1 in eq. (i), we have

$$f(x) = \frac{1}{\pi} + 0 + \sum_{n=2}^{\infty}\frac{(\cos n\pi + 1)}{\pi(1-n^2)}\cos nx + \frac{1}{2}\sin x + 0$$

$$f(x) = \frac{1}{\pi} + \left[\frac{2}{\pi(1-2^2)}\cos 2x + \frac{2}{\pi(1-4^2)}\cos 4x\right.$$

$$\left. + \frac{2}{\pi(1-6^2)}\cos 6x + ... \frac{1}{2}\sin x\right] \quad ... \text{(iii)}$$

The given series on the R.H.S. of (iii) represents $f(x)$ for all the values of x lie in the interval $-\pi \leq x < 0$ and $0 < x \leq \pi$ except the value at $x = 0$.

Now $f(x)$ is discontinuous at $x = 0$.

hence for $x = 0$, the sum of the series on the R.H.S of eq. (iii)

$$= \frac{1}{2}[f(0+0) + f(0-0)]$$

$$= \frac{1}{2}[\sin 0 + 0] = 0$$

Now from eq. (iii), on putting $x = 0$, we have

$$0 = \frac{1}{\pi} + \frac{2}{\pi}\left[\frac{1}{(1+2)(1-2)} + \frac{1}{(1-4)(1+4)} + \frac{1}{(1-6)(1+6)} + ...\right]$$

or $$\frac{2}{\pi}\left[\frac{1}{3.1} + \frac{1}{5.3} + \frac{1}{7.5} + ...\right] = \frac{1}{\pi}$$

or $$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... = \frac{1}{2}$$

Put $x = \frac{\pi}{2}$ in eq. (iii), we have

$$\sin\frac{\pi}{2} = \frac{1}{\pi} + \frac{2}{\pi}\left[\frac{1}{(1+2)(1-2)}(-1) + \frac{1}{(1-4)(1+4)}\right.$$

$$\left. + \frac{1}{(1-6)(1+6)}(-1)\right] + \frac{1}{2}$$

$$1 - \frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ...\right]$$

$$\Rightarrow \frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ....\right]$$

$$\Rightarrow \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - .... = \frac{\pi - 2}{4}$$

Prob.3 ***Find the Fourier series expansion for the function given as***

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi \leq x < 0 \\ 0 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad 0 < x \leq \pi \end{cases}$$

Hence prove that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

[R.U. 2004, 0

Sol. The Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_{0}^{\pi} 1 dx\right]$$

$$= \frac{1}{2\pi}\left\{(-x)_{-\pi}^{0} + (x)_0^{\pi}\right\}$$

$$a_0 = \frac{1}{2\pi}[-\pi + \pi] = 0$$

Also $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nx dx + \int_{0}^{\pi}(1)\cos nx dx\right]$$

$$a_n = \frac{1}{\pi}\left[\left(-\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right]$$

$$a_n = 0$$

Again $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nx dx + \int_{0}^{\pi}(1)\sin nx dx\right]$$

$$b_n = \frac{1}{\pi}\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(\frac{-\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{1}{n}\{\cos n0 - \cos n(-\pi)\} + \frac{1}{n}\{-\cos n\pi + \cos n0\}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{1}{n}(1-\cos n\pi) + \frac{1}{n}(1-\cos n\pi)\right]$$

$$b_n = \frac{2}{n\pi}[1-(-1)^n] \qquad [\cos n\pi = (-1)^n]$$

Substituting the value of a_0, a_n and b_n in eq. (i), the required Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-(-1)^n\right]\sin nx$$

$$= \frac{2}{\pi}\left[2\sin x + \frac{2\sin 3x}{3} + \frac{2\sin 5x}{5} + \dots\right]$$

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right] \qquad \dots\text{(ii)}$$

Put $x = \frac{\pi}{2}$ in (ii), then

$$1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Proved

Prob.4 ***Obtain the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$. Hence show that***

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

and $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$ *[R.U. 2006, 04, 02]*

Sol. Since x^2 is an even function, therefore,

$$b_n = 0$$

Hence $f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \dots\text{(i)}$

Now $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$a_0 = \frac{1}{3}\pi^2$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$

on integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \int 2x\frac{\sin nx}{n}dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nt}{n} - \left\{2x\left(-\frac{\cos nx}{n^2}\right) - \int 2\left(-\frac{\cos nx}{n^2}\right)dx\right\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi\cos n\pi}{n^2} + 0\right] = \frac{4}{n^2}\cos n\pi$$

$$\Rightarrow a_n = \frac{4}{n^2}(-1)^n \qquad [\because \cos n\pi = (-1)^n]$$

Subsituting the value of a_0 and a_n in eq. (i), we have

$$f(x)=x^2=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$$\Rightarrow \quad x^2=\frac{\pi^2}{3}+4\left[-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\frac{\cos 4x}{4^2}-\ldots\right]$$

$$\Rightarrow \quad x^2=\frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\ldots\right] \quad \ldots \text{(ii)}$$

Put $x=\pi$ in eq. (ii), we have

$$\pi^2=\frac{\pi^2}{3}-4\left[-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\ldots\right]$$

$$\Rightarrow \quad \pi^2-\frac{\pi^2}{3}=4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right]$$

$$\Rightarrow \quad \frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots \quad \ldots \text{(iii)}$$

Proved

Put $x=0$ in eq. (ii), we have

$$0=\frac{\pi^2}{3}-4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\right]$$

$$\Rightarrow \quad \frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}-\ldots \quad \ldots \text{(iv)}$$

Proved

Now adding eq. (iii) and eq. (iv), we get

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

$$\Rightarrow \quad \frac{\pi^2}{4}=2\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right]$$

$$\Rightarrow \quad \frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots$$

Proved

Prob.5 ***Find the Fourier series to represent $f(x)=|x|$ for $-\pi\le x<\pi$.*** **[R.U. 2002]**

Sol. Since $f(-x)=|-x|=|x|=f(x)$

$\therefore$ $f(x)=|x|$ is an even function, therefore, $b_n=0$

Hence $f(x)=|x|=a_0+\sum_{n=1}^{\infty}a_n\cos nx \quad \ldots$ (i)

Now $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$

$$a_0=\frac{1}{\pi}\int_0^{\pi}f(x)dx=\frac{1}{\pi}\int_0^{\pi}|x|\,dx=\frac{1}{\pi}\int_0^{\pi}x\,dx$$

$$a_0=\frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}=\frac{\pi}{2}$$

and $a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$

$$=\frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

On integrating by parts,

$$a_n=\frac{2}{\pi}\left[x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[0+\frac{\cos n\pi-\cos n0}{n^2}\right]$$

$$=\frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

Subsituting the value of a_0 and a_n in eq. (i), we have

$$f(x)=|x|=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n-1\right]\cos nx$$

$$|x|=\frac{\pi}{2}+\frac{2}{\pi}\left[-\frac{2}{1^2}\cos x-\frac{2}{3^2}\cos 3x-\frac{2}{5^2}\cos 5x+\ldots\right]$$

$$|x|=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right]$$

Ans.

Prob.6 ***If $f(x)=|\cos x|$, expand $f(x)$ as a Fourier series in the interval $(-\pi,\pi)$.*** **[R.T.U. 2015]**

Sol. As $f(-x)=|\cos(-x)|=|\cos x|=f(x)$,

$\therefore$ $|\cos x|$ is an even function, therefore, $b_n=0$

Hence $f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx \quad \ldots$ (i)

where $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_0^{\pi}f(x)dx$

$$=\frac{1}{\pi}\int_0^{\pi}|\cos x|\,dx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}(-\cos x)dx\right]$$

$$\left[\because |\cos x| = \begin{cases} \cos x & 0 < x < \dfrac{\pi}{2} \\ -\cos x & \dfrac{\pi}{2} < x < \pi \end{cases}\right]$$

$$= \frac{1}{\pi}\left[\left\{(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin \pi - \sin\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad a_0 = \frac{1}{\pi}[(1-0)-(0-1)] = \frac{2}{\pi}$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx\, dx + \int_{\pi/2}^{\pi} (-\cos x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos x \cos nx\, dx + \int_{\pi/2}^{\pi} -2\cos x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \{\cos(n+1)x + \cos(n-1)x\}\, dx - \int_{\pi/2}^{\pi} \{\cos(n+1)x + \cos(n-1)x\}\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_0^{\pi/2} - \left(\frac{\sin(n+1)x}{(n+1)} + \frac{\sin(n-1)x}{(n-1)}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{(n-1)}\right\} + \left(\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{(n-1)}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{(n+1)} - \frac{\cos\frac{n\pi}{2}}{(n-1)} + \frac{\cos\frac{n\pi}{2}}{(n+1)} - \frac{\cos\frac{n\pi}{2}}{(n-1)}\right]$$

[Using sin(A+B) = sinAcosB+cosAsinB]

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{(n+1)} - \frac{\cos\frac{n\pi}{2}}{(n-1)}\right] = -\frac{4\cos\frac{n\pi}{2}}{\pi(n^2-1)}; \quad [n \neq 1]$$

$$\Rightarrow \quad a_n = -\frac{4\cos\frac{n\pi}{2}}{\pi(n^2-1)}; \quad [n \neq 1]$$

Now $$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos x dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\cos x\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x\, dx - \int_{\pi/2}^{\pi} \cos^2 x\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{(1+\cos 2x)}{2} dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right) dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)_0^{\pi/2} - \left(\frac{x}{2} + \frac{\sin 2x}{4}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi}{4} + \frac{\sin\pi}{4}\right\} - 0 - \left\{\frac{\pi}{2} + \frac{\sin 2\pi}{4}\right\} + \left\{\frac{\pi}{4} + \frac{\sin\pi}{4}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{\pi}{2} + \frac{\pi}{4}\right]$$

$$\Rightarrow \quad a_1 = 0$$

Substituting the values of a_0, a_1 and a_n in eq. (i), the required Fourier series is

$$f(x) = |\cos x| = \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)}\cos nx$$

$$\therefore \quad |\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{2^2-1} - \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} - \dots\right]$$

$$\Rightarrow \quad |\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \frac{\cos 6x}{35} - \dots\right]$$

Ans.

Prob.7 *Find the Fourier series to represent $f(x) = |\sin x|$ in the interval $-\pi < x < \pi$* *[R.T.U. 2010]*

Sol. $|\sin x|$ is always an even function

$[\because f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)]$

so that $b_n = 0$.

Hence $f(x) = |\sin x| = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ ... (i)

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{\pi} |\sin x|\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\, dx \; [\because \sin x \text{ is +ive for } 0 < x < \pi]$$

$$\Rightarrow \quad a_0 = \frac{1}{\pi}(-\cos x)_0^{\pi} = \frac{2}{\pi}$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi};$$

where $n \neq 1$

$$= \frac{1}{\pi}\left[-\frac{(\cos nx \cos x - \sin nx \sin x)}{(n+1)} + \frac{(\cos nx \cos x + \sin nx \sin x)}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-\cos n\pi - 1)}{(n+1)} + \frac{(-\cos n\pi - 1)}{(n-1)}\right]$$

$$= \frac{(\cos n\pi + 1)}{\pi}\left[\frac{1}{(n+1)} - \frac{1}{(n-1)}\right]; \quad n \neq 1$$

$$= \frac{(1 + \cos n\pi)}{\pi}\frac{(-2)}{(n^2 - 1)};$$

$$\Rightarrow \quad a_n = \frac{1}{\pi}\frac{2}{(1-n^2)}\left[1 + (-1)^n\right];$$

$$\Rightarrow \quad a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{4}{\pi(1-n^2)} & \text{if } n \text{ is even} \end{cases};$$

Now $$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x\, dx = \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi}$$

$$\Rightarrow \quad a_1 = \frac{1}{\pi}\left(\frac{-1+1}{2}\right) = 0$$

Substituting the value of a_0, a_n and a_1 in eq. (i), we have

$$f(x) = |\sin x|$$

$$= \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{4}{\pi(1-n^2)}\cos nx \quad \text{[for even}$$

$$f(x) = |\sin x|$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right] \quad \text{Ans.}$$

Prob.8 *Find the Fourier series to represent $f(x) = x - x^2$ in the interval $x = -1$ to $x = 1$.* *[M.N.I.T. 2007, R.U. 2007, 19...]*

Sol. Here the length of interval $2c = 2$

i.e. $c = 1$

Hence the Fourier series is

$$f(x) = x - x^2$$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{1} + b_n \sin\frac{n\pi x}{1}\right) \quad \ldots$$

where $a_0 = \dfrac{1}{2.1}\int_{-1}^{1} f(x)dx$

$$= \frac{1}{2}\int_{-1}^{1}(x - x^2)dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2} - \frac{1}{3} - \frac{1}{3}\right]$$

$$\Rightarrow \quad a_0 = -\frac{1}{3} \qquad \text{... (ii)}$$

Also $a_n = \dfrac{1}{1}\int_{-1}^{1} f(x)\cos\dfrac{n\pi x}{1}dx$

$$= \int_{-1}^{1}(x - x^2)\cos n\pi x\, dx$$

$$= \int_{-1}^{1} x\cos n\pi x\, dx - \int_{-1}^{1} x^2 \cos n\pi x\, dx$$

$$= -2\int_0^1 x^2 \cos n\pi x\, dx$$

($\because \int_{-1}^{1} x\cos n\pi x\, dx = 0$; integrand being odd function and $\int_{-1}^{1} x^2 \cos n\pi x\, dx = 2\int_0^1 x^2 \cos n\pi x\, dx$; integrand being even function)

now integrating by parts, we have

$$a_n = -2\left[\left(x^2\frac{\sin n\pi x}{n\pi}\right) - \int 2x\frac{\sin n\pi x}{n\pi}dx\right]_0^1$$

$$a_n = -2\left[\left(x^2\frac{\sin n\pi x}{n\pi}\right) - 2\left\{x\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right.\right.$$

$$\left.\left. - \int 1.\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)dx\right\}\right]_0^1$$

$$= -2\left[\left(x^2\frac{\sin n\pi x}{n\pi}\right) - 2x\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right.$$

$$\left. -2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= -2\left[\frac{\sin n\pi}{n\pi} + \frac{2\cos n\pi}{n^2\pi^2} - \frac{2\sin n\pi}{n^3\pi^3}\right]$$

$$\Rightarrow \quad a_n = -\frac{4}{n^2\pi^2}(-1)^n = \frac{4(-1)^{n+1}}{n^2\pi^2} \qquad \text{... (iii)}$$

[$\because \cos n\pi = (-1)^n$ and $\sin n\pi = 0$]

Again $b_n = \dfrac{1}{1}\int_{-1}^{1} f(x)\sin\dfrac{n\pi x}{1}dx$

$$= \int_{-1}^{1}(x - x^2)\sin n\pi x\, dx$$

$$= \int_{-1}^{1} x\sin n\pi x\, dx - \int_{-1}^{1} x^2 \sin n\pi x\, dx$$

$$= 2\int_0^1 x\sin n\pi x\, dx$$

[$\because \int_{-1}^{1} x^2 \sin n\pi x\, dx = 0$; integrand being odd function and $\int_{-1}^{1} x\sin n\pi x\, dx = 2\int_0^1 x\sin n\pi x\, dx$; integrand being even function]

Now integrating by parts, we get

$$b_n = 2\left[\left(-x\frac{\cos n\pi x}{n\pi}\right) + \int\frac{\cos n\pi x}{n\pi}dx\right]_0^1$$

$$= 2\left[\left(-x\frac{\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$= 2\left[-1\frac{\cos n\pi}{n\pi}\right] = \frac{-2}{n\pi}(-1)^n$$

$$\therefore \quad b_n = \frac{2}{n\pi}(-1)^{n+1} \qquad \text{... (iv)}$$

Substituting the value of a_0, a_n and b_n from eq.(ii), (iii) and (iv) respectively in eq.(i), the required Fourier series is

$$f(x) = x - x^2 = -\frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2\pi^2}\cos n\pi x$$

$$+ \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin n\pi x$$

$$\Rightarrow x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left[\frac{\cos\pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - ...\right]$$

$$+ \frac{2}{\pi}\left[\frac{\sin\pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - ...\right]$$

Ans.

Prob.9 ***State and prove Parseval's Theorem.***

Sol. Statement : Let f(t) be a periodic function with period 2c, defined in the interval (–c,c), Then

$$\int_{-c}^{c} |f(t)|^2\, dx = c\left[a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

where a_0, a_n, b_n are fourier coefficients.

Proof : The fourier series for f(t) defined in the interval, (–c, c) is given by

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{c} + b_n \sin\frac{n\pi t}{c}\right)$$

where, a_0, a_n and b_n are fourier constants. On multiplying both sides by f(t) and then intergrating term by term from –c to c, we have

$$\int_{-c}^{c} |f(t)|^2\, dt = a_0 \int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty}\left[a_n \int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt\right] + \sum_{n=1}^{\infty}\left[b_n \int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt\right]$$

$$\int_{-c}^{c} |f(t)|\, dt = a_0 .c. a_0 + \sum_{n=1}^{\infty} a_n (ca_n) + \sum_{n=1}^{\infty} b_n (cb_n)$$

$$\int_{-c}^{c} |f(t)|^2\, dt = c\left[a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

Note : Assume f(t) be a function, which is defined in the interval (a, b), then

$$\left[\left(\frac{1}{b-a}\right)\int_0^b |f(t)|^2\, dt\right]^{1/2}$$

is called the root mean square or effective value of f(t) and denoted by the symbol, $|f(t)|_{rms}$

$$|f(t)|_{rms}^2 = \frac{1}{b-a}\int_a^b [f(t)]^2\, dt$$

Prob.10 ***Express f(x) = x as a half range sine series in the interval 0 < x < 2.*** ***[R.T.U. (ME 2009), R.U. 2000]***

Sol. Draw the graph of $f(x) = x$ in $0 < x < 2$, then we have the line OA. Now we extend the function $f(x)$ in the interval $-2 < x < 0$, given the line OA′, such that the new function is symmetrical about the origin *i.e.* in opposite quadrant, therefore represents an odd function in the interval (–2, 2)

$$g(x) = x \qquad 0 < x < 2$$
$$= x \qquad -2 < x < 0$$

Thus g(x) is an odd periodic function in the interval (–2, 2) with period 4

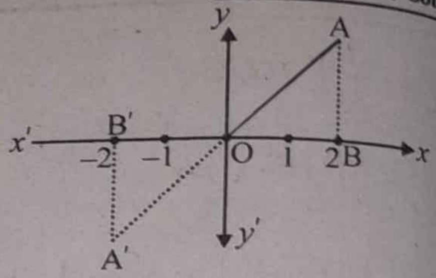


Fig.

Then the Fourier series for the function g(x) in interval (–2, 2) is given by

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \qquad [\because 2c = 4, c = 2]$$

where $b_n = \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$

$$\Rightarrow \quad b_n = \int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

on integrating by parts, we get

$$b_n = \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_0^2$$

$$\Rightarrow \quad b_n = \frac{2.2}{n\pi}(0 - \cos n\pi)$$

$$\Rightarrow \quad b_n = \frac{-4}{n\pi}(-1)^n$$

Substituting the value of b_n from eq. (ii) in eq. (i) we have

$$g(x) = \sum_{n=1}^{\infty} \frac{-4}{n\pi}(-1)^n \sin\frac{n\pi x}{2}$$

$$g(x) = \frac{-4}{\pi}\left[-\sin\frac{\pi x}{2} + \frac{\sin\frac{2\pi x}{2}}{2} - \frac{\sin\frac{3\pi x}{2}}{3} + \frac{\sin\frac{4\pi x}{2}}{4} - \ldots\right]$$

$$g(x) = \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin\frac{4\pi x}{2} \ldots\right]$$

Then the half range sine series of $f(x)$ in (0, 2)

$$f(x)=\frac{4}{\pi}\left[\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}-\frac{1}{4}\sin\frac{4\pi x}{2}+\ldots\right]$$

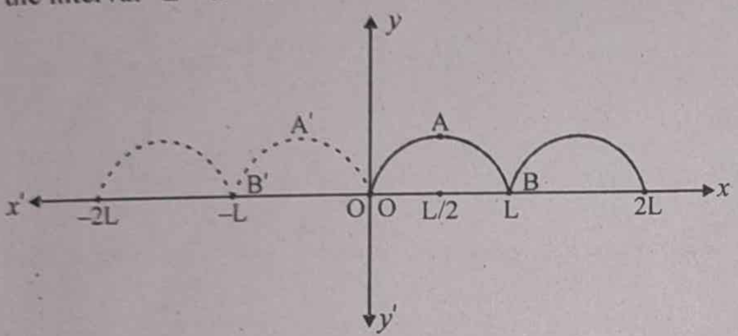
[since $f(x)$ equals to $g(x)$ in the interval $(0, 2)$]

Ans.

Prob.11 ***Find the half range cosine series for the function*** $f(x)=\frac{\sin \pi x}{L}$; $0 < x < L$. *[R.U. 2002]*

Sol. Draw the graph of the function $f(x)=\frac{\sin\pi x}{L}$ in the interval $0 < x < L$ is the line OAB.

Let us extend the function in the interval $-L < x < 0$ shown by the line O A'B' such that new function is symmetrical about the y-axis and therefore represents an even function in the interval $-L < x < L$



$$g(x)=\frac{\sin\pi x}{L}, \qquad 0 < x < L$$

$$g(x)=-\frac{\sin\pi x}{L}, \qquad -L < x < 0$$

Hence $g(x)$ is an even periodic function with period $2L$.

Now the Fourier series expansion of $g(x)$ in the interval $(-L, L)$ is

$$g(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L} \qquad \text{... (i)}$$

where $a_0=\frac{1}{L}\int_0^L f(x)dx$

$$=\frac{1}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)dx$$

$$=\frac{1}{L}\cdot\frac{L}{\pi}\left(-\cos\frac{\pi x}{L}\right)_0^L$$

$$\Rightarrow \quad a_0=\frac{1}{\pi}(1+1)=\frac{2}{\pi} \qquad \text{... (ii)}$$

and $a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$=\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{1}{L}\int_0^L\left[\sin(1+n)\frac{\pi x}{L}+\sin(1-n)\frac{\pi x}{L}\right]dx$$

$$=\frac{1}{L}\left[\left(-\frac{\cos(1+n)\frac{\pi x}{L}}{\frac{(1+n)\pi}{L}}\right)+\left(-\frac{\cos(1-n)\frac{\pi x}{L}}{\frac{(1-n)\pi}{L}}\right)\right]_0^L$$

$$=-\left[\frac{\cos\frac{\pi x}{L}\cos\frac{n\pi x}{L}-\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}}{(1+n)\pi}+\frac{\cos\frac{\pi x}{L}\cos\frac{n\pi x}{L}+\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}}{(1-n)\pi}\right]_0^L ;n\neq 1$$

$$=-\left[-\frac{\cos n\pi}{(1+n)\pi}-\frac{\cos n\pi}{(1-n)\pi}\right]+\left[\frac{1}{(1+n)\pi}+\frac{1}{(1-n)\pi}\right];\ n\neq 1$$

$$=\frac{1}{\pi}\left[\frac{1}{(n+1)}[(-1)^n+1]-\frac{1}{(n-1)}[(-1)^n+1]\right];\ n\neq 1$$

$$=\frac{1}{\pi}\left[\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]\left[(-1)^n+1\right];n\neq 1$$

$$\Rightarrow \quad a_n=\frac{-2[(-1)^n+1]}{\pi(n^2-1)};\ n\neq 1 \qquad \text{... (iii)}$$

Now $a_1=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi x}{L}\right)dx$

$$=\frac{2}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)dx$$

$$=\frac{1}{L}\int_0^L\sin\left(\frac{2\pi x}{L}\right)dx$$

$$=\frac{1}{L}\left(-\frac{\cos\left(\frac{2\pi x}{L}\right)}{\frac{2\pi}{L}}\right)_0^L$$

$$=-\frac{1}{L}\frac{L}{2\pi}(1-1)=0$$

$$\Rightarrow \quad a_1=0 \qquad \text{... (iv)}$$

Using eq.(ii), (iii) and eq.(iv) in eq.(i), we have

$$g(x)=\frac{2}{\pi}+0-\sum_{n=2}^{\infty}\frac{2[(-1)^n+1]}{\pi(n^2-1)}\cos\left(\frac{n\pi x}{L}\right)$$

$$g(x)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{(2^2-1)}\cos\frac{2\pi x}{L}+\frac{1}{(4^2-1)}\cos\frac{4\pi x}{L}+\ldots\right]$$

$$g(x)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{1.3}\cos\frac{2\pi x}{L}+\frac{1}{3.5}\cos\frac{4\pi x}{L}+\frac{1}{5.7}\cos\frac{6\pi x}{L}+\ldots\right]$$

Hence the half range cosine series of $f(x)$ in (0, L) is

$$f(x)=g(x)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{1.3}\cos\frac{2\pi x}{L}+\frac{1}{3.5}\cos\frac{4\pi x}{L}+\frac{1}{5.7}\cos\frac{6\pi x}{L}+\ldots\right]$$ **Ans.**

□□□

MULTIVARIABLE CALCULUS (DIFFERENTIATION) 4

PREVIOUS YEARS QUESTIONS

I. SHORT ANSWER TYPE QUESTIONS

Prob.1 *If* $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, *prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$ *[R.T.U. 2017]*

Sol. Here $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\Rightarrow \quad \tan u = \frac{x^3+y^3}{x+y} = V \text{ (say)} \quad ...(i)$$

$$\Rightarrow \quad V = x^2\left[\frac{1+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)}\right] = x^2\phi\left(\frac{y}{x}\right)$$

$\therefore$ V is a homogeneous function of degree 2. Thus, on applying Euler's theorem

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 2V$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$\Rightarrow \quad x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u\cos^2 u$$

$$= 2\sin u\cos u = \sin 2u \quad ...(ii)$$

Prob.2 *Find the directional derivative of* $f(x, y, z) = 2x^2 + 3y^2 + z^2$ *at the point P (2, 1, 3) in the direction of the vector* $\vec{a} = \hat{i} - 2\hat{k}$. *[R.T.U. 2017]*

Sol. Here, $f(x, y, z) = 2x^2 + 3y^2 + z^2$

$$\Rightarrow \quad \nabla f = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](2x^2+3y^2+z^2)$$

$$\Rightarrow \quad \nabla f = 4x\,\hat{i} + 6y\,\hat{j} + 2z\,\hat{k}$$

Then

$$(\nabla f)_{(2,1,3)} = 8\hat{i} + 6\hat{j} + 6\hat{k}$$

Let $\vec{a} = \hat{i} - 2\hat{k}$

Then unit vector in the direction of $\hat{i} - 2\hat{k}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}-2\hat{k}}{\sqrt{(1)^2+(-2)^2}} = \frac{\hat{i}-2\hat{k}}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{k}$$

Thus the required directional derivative is

$$(\nabla f)_{(2,1,3)}\,\hat{a} = (8\hat{i}+6\hat{j}+6\hat{k}).\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{k}\right)$$

$$= \frac{8}{\sqrt{5}} - \frac{12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

Prob.3 *Find* $\frac{dy}{dx}$, *if* $x^y + y^x = c$

Sol. $f(x,y)=x^y+y^x$

then $\dfrac{dy}{dx}=\dfrac{-\partial f/\partial x}{\partial f/\partial y}$

$\therefore \dfrac{\partial f}{\partial x}=yx^{y-1}+y^x\log y$ and $\dfrac{\partial f}{\partial y}=xy^{x-1}+x^y\log x$

$\therefore \dfrac{dy}{dx}=-\left(\dfrac{\partial f}{\partial x}\right)/\left(\dfrac{\partial f}{\partial y}\right)=-\left[\dfrac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}\right]$

Prob.4 ***Give the test for maxima and minima.***

Sol. $f(x)$ has a maximum value at $x=a$ if $f'(a)=0$ and $f''(a)$ is negative. Similarly $f(x)$ has a minimum value at $x=a$ if $f'(a)=0$ and $f''(a)$ is positive.

In case if $f'(a)=f''(a)=\ldots\ldots=f^{n-1}(a)=0$ and $f^n(a)\neq 0$ then $f(x)$ has maximum or minimum if n is even. Also for a maximum $f^{(n)}(a)$ must be negative and for a minimum $f^{(n)}(a)$ must be positive.

Prob.5 ***State the "Euler's theorem".***

Sol. Euler's Theorem :

If $f(x,y)$ be a homogeneous function in x and y of degree n, then

(i) $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf$

(ii) $x^2\dfrac{\partial^2 f}{\partial x^2}+2xy\dfrac{\partial^2 f}{\partial x\partial y}+y^2\dfrac{\partial^2 f}{\partial y^2}=n(n-1)f.$

Prob.6 ***If $f(x,y)=0$, $\phi(y,z)=0$,***

show that $\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial \phi}{\partial z}\dfrac{dz}{dx}=\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial \phi}{\partial y}$

Sol. $\because f(x,y)=0$ therefore $\dfrac{dy}{dx}=-\left(\dfrac{\partial f}{\partial x}\right)/\left(\dfrac{\partial f}{\partial y}\right)$ ...(1)

similarly $\phi(y,z)=0\Rightarrow \dfrac{dz}{dy}=-\left(\dfrac{\partial \phi}{\partial y}\right)/\left(\dfrac{\partial \phi}{\partial z}\right)$ ...(2)

Multiplying (1) and (2), we have

$$\frac{dy}{dx}\cdot\frac{dz}{dy}=\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)/\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial \phi}{\partial z}\right)$$

or $\left(\dfrac{\partial f}{\partial y}\right)\left(\dfrac{\partial \phi}{\partial z}\right)\dfrac{dz}{dx}=\left(\dfrac{\partial f}{\partial x}\right)\left(\dfrac{\partial \phi}{\partial y}\right)$

Prob.7 ***What is meant by homogeneous function?***

Sol. An expression of the form

$$a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots\ldots+a_{n-1}xy^{n-1}+a_ny^n$$

in which every term is of degree n, is called homogeneous function of degree n.

This homogeneous function can also be written as

$$x^n\left[a_0+a_1\left(\frac{y}{x}\right)+a_2\left(\frac{y}{x}\right)^2+\ldots\ldots+a_{n-1}\left(\frac{y}{x}\right)^{n-1}+a_n\left(\frac{y}{x}\right)^n\right]$$

$=x^nf$

Prob.8 ***If $u=e^{xyz}$, then find the value of $\dfrac{\partial^3 u}{\partial x\partial y\partial z}$.*** *[Raj. Univ.*

Sol. $u=e^{xyz}$.

then $\dfrac{\partial u}{\partial z}=e^{xyz}\,xy$

again differentiating (2) w.r.t. y we have

$$\frac{\partial^2 u}{\partial y\partial z}=xe^{xyz}+xy\,e^{xyz}xz$$

$$=xe^{xyz}+x^2yz\,e^{xyz}$$

differentiating (3) partially w.r.t.x.

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=e^{xyz}+2xyz\,e^{xyz}+x^2yz\,e^{xyz}\,yz+xe^{xy}\,yz$$

$$=e^{xyz}+3xyz\,e^{xyz}+x^2y^2z^2\,e^{xyz}$$

$$=\left(1+3xyz+x^2y^2z^2\right)e^{xyz}.$$

II. Long Answer Type Questions

Prob.1 ***Let $f(x,y)=\dfrac{x^3-y^3}{x^2+y^2}$, when $(x,y)\neq(0,0)$ and $f(0,0)=0$. Show that the function f is continuous but not differentiable at the origin.*** *[R.T.U.*

Sol. Given $f(x, y) = \dfrac{x^3 - y^3}{x^2 + y^2}$

Clearly the given function is always defined for all values of x and y.

So, it is continuous

$$\text{Again, } f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h-0}$$

$$= \lim_{h\to 0}\frac{\frac{h^3-0}{h^2+0}-0}{h-0}$$

$$= \lim_{h\to 0}\frac{h^3}{h^3} = 1$$

$$\text{and } \quad f_y(0,0) = \lim_{h\to 0}\frac{(0,h)-f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{0-h^3}{0+h^2}-0}{h}$$

$$= \lim_{h\to 0}\left(\frac{-h^3}{h^3}\right)$$

$$= -1$$

Since $f_x(0, 0) \neq f_y(0, 0)$, So f is not differentiable at the origin.

Prob.2 ***If $u = f(y - z, z - x, x - y)$, prove that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$.*** *[R.T.U. 2017]*

Sol. Given u

$= f(y - z, z - x, x - y)$

Let $y - z = t_1$,

$z - x = t_2$,

and $x - y = t_3$

where t_1, t_2 and t_3 are functions of x, y and z.

$$\text{Now } \frac{\partial t_1}{\partial x} = 0, \frac{\partial t_1}{\partial y} = 1, \frac{\partial t_1}{\partial z} = -1$$

$$\frac{\partial t_2}{\partial x} = -1, \frac{\partial t_2}{\partial y} = 0, \frac{\partial t_2}{\partial z} = 1$$

$$\frac{\partial t_3}{\partial x} = 1, \frac{\partial t_3}{\partial y} = -1, \frac{\partial t_3}{\partial z} = 0$$

$$\text{So } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial x}+\frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial x}+\frac{\partial u}{dt_3}.\frac{\partial t_3}{\partial x}$$

$$= \frac{\partial u}{\partial t_1}(0)+\frac{\partial u}{\partial t_2}(-1)+\frac{\partial u}{\partial t_3}(1)$$

$$= -\frac{\partial u}{\partial t_2}+\frac{\partial u}{\partial t_3} \quad \text{...(1)}$$

Similarly

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial t_2}+\frac{\partial u}{\partial t_1} \quad \text{...(2)}$$

$$\text{and } \quad \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial t_1}+\frac{\partial u}{\partial t_2} \quad \text{...(3)}$$

Now add eq. (1), (2) and (3)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = 0$$

Hence Proved.

Prob. 3 ***Prove that div $(r^n \vec{r}) = (n+3)r^n$*** *[R.T.U. 2017]*

Sol. r^n is scalar and $\vec{r}$ is a vector

we known that $\text{div}(f\vec{A}) = f(\nabla.\vec{A})+(\Delta f).\vec{A}$

$$\text{div}(r^n\vec{r}) = r^n(\nabla.\vec{r})+(\nabla r^n).\vec{r}$$

$$= r^n\left[\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(x\hat{i}+y\hat{j}+z\hat{k}) + \left(\hat{i}\frac{\partial r^n}{\partial x}+\hat{j}\frac{\partial r^n}{\partial y}+\hat{k}\frac{\partial r^n}{\partial z}\right)\right]$$

$$= r^n(1+1+1) + \left[\hat{i}(nr^{n-1})\frac{\partial r}{\partial x}+\hat{j}(nr^{n-1})\frac{\partial r}{\partial y}+\hat{k}(nr^{n-1})\frac{\partial r}{\partial z}\right]\vec{r}$$

$$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence

$$\text{div}(r^n\vec{r}) = 3r^n + nr^{n-1}\left(\hat{i}\frac{x}{r}+\hat{j}\frac{y}{r}+\hat{k}\frac{z}{r}\right).(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= 3r^n + nr^{n-1}\left(\frac{x^2+y^2+z^2}{r}\right)$$

$$= 3r^n + nr^{n-1}\left(\frac{r^2}{r}\right)$$

$$= 3r^n + nr^n$$

$$= r^n(n+3)$$

Prob.4 *Use Lagrange's method of multipliers to find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid.*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ *[R.T.U. 2017]*

Sol. Three coterminus arms of parallelopiped inscribed in the ellipsoid are $2x$, $2y$ and $2z$, therefore its volume u (say)is

$$u = 2x.2y.2z = 8xyz \quad ...(1)$$

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ ...(2)

Differentiating (1) & (2)

$$du = 8yz\,dx + 8zx\,dy + 8xy\,dz = 0 \quad ...(3)$$

and $\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz=0$ ...(4)

Multiplying equation (4) by 4λ and adding in (3),

$$du = 8\left(yz+\frac{\lambda x}{a^2}\right)dx+8\left(zx+\frac{\lambda y}{b^2}\right)+8\left(xy+\frac{\lambda z}{c^2}\right)dz=0$$

From Lagrange's method,

$$yz+\frac{\lambda x}{a^2}=0, zx+\frac{\lambda y}{b^2}=0 \text{ and } xy+\frac{\lambda z}{c^2}=0 \quad ...(5)$$

Adding these equation after multiplying by x, y and z respectively and then using (1) and (2),

$$\frac{3u}{8}+\lambda(1)=0 \Rightarrow \lambda=-\frac{3u}{8}$$

Substituting the value of λ in (5) and using (1),

Since $yz=-\frac{\lambda x}{a^2}$, therefore, $\frac{u}{8x}=\frac{3ux}{8a^2} \Rightarrow \frac{x^2}{a^2}=\frac{1}{3}$

Similarly $\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3}$

Therefore the volume is $u=8xyz=\frac{8abc}{3\sqrt{3}}$. **Ans.**

Prob.5(a) *If $x = t_1 \cos\alpha - t_2 \sin\alpha$, $y = t_1 \sin\alpha, + t_2 \cos\alpha$,*

Show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial t_1^2}+\frac{\partial^2 u}{\partial t_2^2}$ *[R.T.U. Dec. 2016]*

(b) *Discuss the maxima and minima of the function:* $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ *[R.T.U. Dec. 2016]*

OR

Find the Maxima and Minima of the function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

[R.T.U. 201

Sol.(a) $x = t_1 \cos\alpha - t_2 \sin\alpha$

and $y = t_1 \sin\alpha + t_2 \cos x$

from equation (i), $\frac{\partial x}{\partial t_1}=\cos\alpha, \frac{\partial x}{\partial t_2}=-\sin\alpha$

from equation (ii), $\frac{\partial y}{\partial t_1}=\sin\alpha, \frac{\partial y}{\partial t_2}=\cos\alpha$

$$\therefore \frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_1}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1}$$

$$=\frac{\partial u}{\partial x}.\cos\alpha+\frac{\partial u}{\partial y}.\sin\alpha$$

$$\therefore \frac{\partial^2 u}{\partial t_1^2}=\frac{\partial}{\partial t_1}\left(\frac{\partial u}{\partial t_1}\right)=\cos\alpha\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t_1}\right)+\sin\alpha\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t_1}\right)$$

$$=\cos\alpha\frac{\partial}{\partial x}\left[\cos\alpha.\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}\right]$$

$$+\sin\alpha\frac{\partial}{\partial y}\left[\cos\alpha.\frac{\partial u}{\partial x}+\sin\alpha.\frac{\partial u}{\partial y}\right]$$

(Substituting values of $\frac{\partial u}{\partial t_1}$ from (iii))

$$=\cos\alpha\left[\cos\alpha\cdot\frac{\partial^2 u}{\partial x^2}+\sin\alpha\cdot\frac{\partial^2 u}{\partial x\partial y}\right]$$

$$+\sin\alpha\left[\cos\alpha\cdot\frac{\partial^2 u}{\partial y\cdot\partial x}+\sin\alpha\cdot\frac{\partial^2 u}{\partial y^2}\right]$$

Or

$$\frac{\partial^2 u}{\partial t_1^2}=\cos^2\alpha\frac{\partial^2 u}{\partial x^2}+2\sin\alpha\cos\alpha.\frac{\partial^2 u}{\partial x.\partial y}+\sin^2\alpha.\frac{\partial^2 u}{\partial y^2}$$

Similarly, $\frac{\partial^2 u}{\partial t_2^2}=$

$$\sin^2\alpha.\frac{\partial^2 u}{\partial x^2}-2\sin\alpha.\cos\alpha.\frac{\partial^2 u}{\partial x.\partial y}+\cos^2\alpha.\frac{\partial^2 u}{\partial y^2}$$

Adding equation (iv) and (v), we have

$$\frac{\partial^2 u}{\partial t_1^2}+\frac{\partial^2 u}{\partial t_2^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$ A

Sol.(b) Let $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Then $\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y$

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y$

The stationary points are given by

$\frac{\partial u}{\partial x} = 0$

$4x^3 - 4x + 4y = 0$ ... (i)

$\frac{\partial u}{\partial y} = 0$

$4y^3 + 4x - 4y = 0$ ...(ii)

Now we shall find the points (x,y) satisfying the simultaneous equations (i) and (ii)

Adding (i) and (ii), we get

$4x^3 + 4y^3 = 0$

$x^3 + y^3 = 0$

$(x+y)(x^2 + y^2 - xy) = 0$

$\therefore$ either $x + y = 0$ ....(iii)

or $x^2 + y^2 - xy = 0$ ....(iv)

First we solve the simultaneous equations (i) and (iii).

From equation (iii), we have

$y = -x$

putting $y = -x$ in equation (i), we get

$4x^3 - 8x = 0$

$x^3 - 2x = 0$

$x(x^2 - 2) = 0$

$x = 0, \sqrt{2}, -\sqrt{2}$

The corresponding values of y are

$y = 0, \sqrt{2}, -\sqrt{2}$

Thus the points $(0,0), (\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ satisfy equation (i) and equation (ii)

If we solve the equations (i) and (iv), we get (0,0) as the only real solution. Hence the function is stationary at the points.

$(0,0), (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, -\sqrt{2})$

we have

$r = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4$

$s = \frac{\partial^2 u}{\partial x \partial y} = 4$

$t = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

At (0,0)

$r = -4$

$s = 4$

$t = -4$

so that $rt - s^2 = 16 - 16 = 0$

Thus at the point (0,0) the case is doubtful and further investigation is needed.

At $(\sqrt{2}, -\sqrt{2})$

$r = 20$

$s = 4$

$t = 20$

so that $rt - s^2 = 400 - 16 = +ve$

$\therefore$ u has an extreme value at this point, since r is positive, therefore u has a minimum at this point.

At $(-\sqrt{2}, \sqrt{2})$

$r = 20$

$s = 4$

$t = 20$

so that $rt - s^2$ is positive since r is positive, therefore u has a minimum at this point also.

Prob.6 ***Find the minimum/extreme value of $x^2 + y^2 + z^2$, subject to the condition $ax + by + cz = p$.*** ***[R.T.U. 2014, 10]***

OR

Find the minimum value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$. ***[R.T.U. Jan. 2016]***

Sol. Let $f(x, y, z) = x^2 + y^2 + z^2$. From the relation $ax + by + cz = p$, we get $z = \frac{p - ax - by}{c}$. Putting this value of z in f(x, y, z), we get

$$f(x,y,z)=x^2+y^2+\left(\frac{p-ax-by}{c}\right)^2$$

as a function of two variables x and y. Then,

$$f_x=2x-\frac{2a}{c^2}(p-ax-by)$$

and $$f_y=2y-\frac{2b}{c^2}(p-ax-by)$$

For extreme points, we must have $f_x=f_y=0$. Thus,

$$2x-\frac{2a}{c^2}(p-ax-by)=0$$

and $$2y-\frac{2b}{c^2}(p-ax-by)=0.$$

Solving these equations, we get

$$x=\frac{ap}{a^2+b^2+c^2}$$

and $$y=\frac{bp}{a^2+b^2+c^2}$$

Now, $$f_{xx}=2+\frac{2a^2}{c^2}$$

$$f_{xy}=\frac{2ab}{c^2}$$

and $$f_{yy}=2+\frac{2b^2}{c^2},$$

so that $$rt-s^2=4\left(1+\frac{a^2}{c^2}\right)\left(1+\frac{b^2}{c^2}\right)-\frac{4a^2b^2}{c^4}$$

$$=4\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)\text{(Positive)}.$$

Also $r=f_{xx}$ is positive. Therefore, f(x, y) has a minimum at $\left(\frac{ap}{a^2+b^2+c^2},\frac{bp}{a^2+b^2+c^2}\right)$ and the minimum value is

$$\text{Min. } f(x,y,z)=\frac{p^2}{a^2+b^2+c^2}.$$

Prob.7 (a) ***If $u=f(x,y), x=r\cos\theta,\ y=r\sin\theta$ Show that***

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(b) ***Find points on the surface $z^2=xy+1$ wh[ose] distances from the origin are minimum.*** [R.T.U. 2...]

Sol. (a) $\because u=f(x,y)$ and $x=r\cos\theta, y=r\sin\theta$

So, $u=f(r,\theta)$

$\because x=r\cos\theta;\ y=r\sin\theta$

So that $r^2=x^2+y^2$ and $\theta=\tan^{-1}\left(y/x\right)$

So, $$\frac{\partial r}{\partial x}=\frac{x}{r} \text{ and } \frac{\partial r}{\partial y}=\frac{y}{r}$$

and $$\frac{\partial\theta}{\partial x}=\frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)=-\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y}=\frac{x^2}{x^2+y^2}\cdot\frac{1}{x}=\frac{\cos\theta}{r}$$

Now $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}$$

$$\frac{\partial u}{\partial x}=\frac{x}{r}\cdot\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

and $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}$$

$$=\frac{y}{r}\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$$

Now,

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{x^2+y^2}{r^2}\right)\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\sin^2\theta+\cos^2\theta}{r^2}\right)$$

$$\left(\frac{\partial u}{\partial\theta}\right)^2+\left(-\frac{2x\sin\theta}{r^2}+\frac{2y\cos\theta}{r^2}\right)\frac{\partial u}{\partial r}\frac{\partial u}{\partial\theta}$$

or $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\frac{r^2}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2$$

$$+\left(-\frac{2\sin\theta\cos\theta}{r}+\frac{2\sin\theta\cos\theta}{r}\right)\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial\theta}\right)$$

$$=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2$$

Sol. (b) Let the point is (x,y,z) so the distance from th[e] origin is given by

$$D=\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

$$D^2=x^2+y^2+z^2=f(x,y,z)$$

So we have to minimize

$$f(x,y,z,)=x^2+y^2+z^2 \qquad ...(1)$$

Subject to the condition

$$z^2=xy+1 \qquad ...(2)$$

Use (2) in (1) we get

$$f=x^2+y^2+xy+1$$

For extreme point

$$\frac{\partial f}{\partial x}=2x+y=0 \qquad ...(3)$$

and $\frac{\partial f}{\partial y}=2y+x=0 \qquad ...(4)$

eq. (3) – eq. (4) gives $x-y=0$

So $x=y$

So from eq. (3) we get $x=0;\ y=0$

Now $r=\frac{\partial^2 f}{\partial x^2}=2;\ s=\frac{\partial^2 f}{\partial x\partial y}=1;\ t=\frac{\partial^2 f}{\partial y^2}=2$

$\because rt-s^2>0$ and $r>0$ at $(0,0)$

So this is the point of minima, so the function f is minimum at

$x=0, y=0$ So $z^2=1$ (Using eq. (2))

$z=\pm 1$

than the points are $(0,0,1)$ and $(0,0,-1)$.

Prob.8 *If* $u=\sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$, *prove that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{20}\tan u. \qquad \textit{[R.T.U. 2015, 2011]}$$

OR

State Euler's theorem. If : $u=\sin^{-1}\left\{\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}\right\}$ *find the value of :* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ ***[R.T.U. 2010, Raj. Univ. 2001]***

Sol. Euler's Theorem : If (x, y) be a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

We have $\sin u=\frac{x^{\frac{1}{4}}}{x^{\frac{1}{5}}}\left\{\frac{1+\left(\frac{y}{x}\right)^{\frac{1}{4}}}{1+\left(\frac{y}{x}\right)^{\frac{1}{5}}}\right\}=x^{\frac{1}{20}}f\left(\frac{y}{x}\right)$

sin u is a homogeneous function of degree 1/20.

So, $x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\frac{1}{20}\sin u$

or $\cos u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{1}{20}\sin u$

Dividing both sides by cos u $\Rightarrow \left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{1}{20}\tan u$

Prob.9 (a) ***If*** $z(x+y)=x^2+y^2$, ***show that***

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) \qquad \textit{[R.T.U. 2014]}$$

(b) ***If*** $u=f(r)$, ***where*** $r=\sqrt{x^2+y^2}$, ***prove that***

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$$

[R.T.U. 2014, Raj.Univ. 2006]

Sol. (a) We have,

$$z=\frac{x^2+y^2}{x+y}$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{(x+y)(2x)-1(x^2+y^2)}{(x+y)^2}=\frac{x^2+2xy-y^2}{(x+y)^2} \qquad ...(i)$$

and, $$\frac{\partial z}{\partial y}=\frac{(x+y)(2y)-1(x^2+y^2)}{(x+y)^2}=\frac{y^2+2xy-x^2}{(x+y)^2} \qquad ...(ii)$$

Now, $$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=\left[\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{y^2+2xy-x^2}{(x+y)^2}\right]^2$$

$$=\left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2=\left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad ...(iii)$$

Also,

$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)=4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$=4\left[1-\frac{4xy}{(x+y)^2}\right]=4\left[\left(\frac{x-y}{x+y}\right)^2\right] \qquad ...(iv)$$

From (iii) & (iv),

$$\Rightarrow \left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) \qquad \textbf{Proved.}$$

Sol.(b) $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x};\ r^2=x^2+y^2$

Partially differentiating w.r.t. x and w.r.t. y

$$\frac{\partial r}{\partial x}=\frac{x}{r};\ \frac{\partial r}{\partial y}=\frac{y}{r};\ \frac{\partial u}{\partial x}=\frac{x}{r}f'(r)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r}f'(r) + x\frac{\partial}{\partial r}\left(\frac{1}{r}f'(r)\right)\frac{\partial r}{\partial x}$$

$$= \frac{1}{r}f'(r) + x\frac{1}{r}f''(r)\frac{x}{r} - \frac{x}{r^2}f'(r)\frac{x}{r}$$

$$= \frac{1}{r}f'(r) + \frac{x^2}{r^2}f''(r) - \frac{x^2}{r^3}f'(r) \quad ...(i)$$

Similarly $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} = \frac{y}{r}f'(r)$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r}f'(r) + \frac{y^2}{r^2}f''(r) - \frac{y^2}{r^3}f'(r) \quad ...(ii)$$

adding equation (i) and (ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r}f'(r) + \frac{(x^2+y^2)}{r^2}f''(r) - \frac{(x^2+y^2)}{r^3}f'(r)$$

$$= \frac{2}{r}f'(r) + \frac{r^2}{r^2}f''(r) - \frac{r^2}{r^3}f'(r)$$

$$= \frac{1}{r}f'(r) + f''(r).$$ **Hence Proved.**

Prob.10 ***Find the points where the function***

$$x^3y^2(1-x-y)$$

has maximum or minimum value and also find the value of the function at these points. *[R.T.U. 2014, 09]*

Sol. For the given function

$$u = x^3y^2(1-x-y)$$

$$\frac{\partial u}{\partial x} = 3x^2y^2(1-x-y) - x^3y^2$$

$$\frac{\partial u}{\partial y} = 2x^3y(1-x-y) - x^3y^2$$

Now, for maximum or mimimum, $\frac{\partial u}{\partial x} = 0$ & $\frac{\partial u}{\partial y} = 0$

$$3x^2y^2(1-x-y) - x^3y^2 = 0 \quad ...(i)$$

$$2x^3y\,(1-x-y) - x^3y^2 = 0 \quad ...(ii)$$

Subtracting (ii) from (i), gives

$$x^2y(1-x-y) - (3y-2x) = 0 \Rightarrow y = \frac{2}{3}x$$

using this, we get $x = \frac{1}{2}$. From $x = \frac{1}{2} \Rightarrow y = \frac{1}{3}$ (Using (i) & (ii))

$\left(\frac{1}{2}, \frac{1}{3}\right)$ is the stationary point.

*[**NOTE:** However, there are many more stationary points like (0, 2); (0, 5); (0, 0) etc. for which we are not concerned. At these points, $rt - s^2 = 0$ (failure)]*

$$r = \frac{\partial^2 u}{\partial x^2} = (6xy^2 - 12x^2y^2 - 6xy^3)\Big|_{\left(\frac{1}{2},\frac{1}{3}\right)} = -\frac{1}{9}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2\Big|_{\left(\frac{1}{2},\frac{1}{3}\right)} = ...$$

$$t = \frac{\partial^2 u}{\partial y^2}(2x^3 - 2x^4 - 6x^3y)\Big|_{\left(\frac{1}{2},\frac{1}{3}\right)} = \frac{-1}{8}$$

Now, $rt - s^2 = \left(\frac{-1}{9}\right)\left(\frac{-1}{8}\right) - \left(\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = +\frac{1}{144}$

$r = \frac{-1}{9} < 0$ is negative.

$$r = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$= 6\left(\frac{1}{2}\right)\left(\frac{1}{9}\right) - 12\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) - 6\left(\frac{1}{2}\right)\left(\frac{1}{27}\right)$$

$$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{9} < 0$$

$\therefore$ $u(x, y)$ has a maximum value at $\left(\frac{1}{2}, \frac{1}{3}\right)$

Maximum value $= \frac{1}{8}\cdot\frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{8}\cdot\frac{1}{9}\cdot\frac{1}{6} = \frac{1}{432}$

Prob.11 ***State Euler's theorem on homogeneous functi... Verify Euler's theorem for the function.***

$$f(x,y) = \frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}$$

[R.T.U. ...]

Sol. Statement: The statement of Euler's Theorem... Homogenous functions is as follows :

If f(x, y) is homogenous function in x, y of degree n...

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof: Since f(x, y) is a homogenous function, there...

$$f(x,y) = x^n F\left(\frac{y}{x}\right)$$

Differentiating partially w.r.t x and y, we get

$$\frac{\partial f}{\partial x} = nx^{n-1}F + x^n F'\left(\frac{y}{x}\right),\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial f}{\partial y} = x^n F'\left(\frac{y}{x}\right).\frac{1}{x}$$

Multiplying eq.(3) by x and eq.(4) by y and adding, we...

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n F - x^{n-1}F'\left(\frac{y}{x}\right) + yx^{n-1}F'\left(\frac{y}{x}\right)$$

$$= nx^n F\left(\frac{y}{x}\right)$$
$$= nf(x, y)$$

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}$$

Given that

$$U = f(x,y) = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

$$= \frac{x^{1/4}}{x^{1/5}}\left(\frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}\right)$$

$$= x^{1/20}\frac{\left[1+\left(\frac{y}{x}\right)^{1/4}\right]}{\left[1+\left(\frac{y}{x}\right)^{1/5}\right]} = x^{1/20}\phi\left(\frac{y}{x}\right)$$

$\therefore$ U is a homogenous function of degree $\frac{1}{20}$

$\therefore$ By Euler's Theorem, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{20}f \quad ...(5)$$

To verify differentiating f(x, y) partially w.r.t. x

$$f(x, y) = \left(x^{1/4} + y^{1/4}\right)\left(x^{1/5} + y^{1/5}\right)^{-1}$$

$$\frac{\partial f}{\partial x} = \left(\frac{1}{4}x^{-3/4}\right)\left(x^{1/5} + y^{1/5}\right)^{-1}$$
$$-\left(x^{1/4} + y^{1/4}\right)\left(x^{1/5} + y^{1/5}\right)^{-2}\frac{1}{5}x^{-4/5}$$

$$x\frac{\partial f}{\partial x} = \frac{1}{4}x^{1/4}\left(x^{1/5} + y^{1/5}\right)^{-1} - \frac{(x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2}\frac{1}{5}x^{1/5} \quad ...(6)$$

Similarly,

$$y\frac{\partial f}{\partial y} = \frac{1}{4}y^{1/4}\left(x^{1/5} + y^{1/5}\right)^{-1} - \frac{x^{1/4} + y^{1/4}}{(x^{1/5} + y^{1/5})^2}\left(\frac{1}{5}\right)y^{1/5} \quad ...(7)$$

Adding eq. (6) and (7) we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{4}\left(x^{1/4} + y^{1/4}\right)\left(x^{1/5} + y^{1/5}\right)^{-1}$$
$$-\frac{1}{5}\frac{\left(x^{1/4} + y^{1/4}\right)\left(x^{1/5} + y^{1/5}\right)}{\left(x^{1/5} + y^{1/5}\right)^2}$$

$$= \frac{1}{4}\frac{\left(x^{1/4} + y^{1/4}\right)}{\left(x^{1/5} + y^{1/5}\right)} - \frac{1}{5}\frac{x^{1/4} + y^{1/4}}{\left(x^{1/5} + y^{1/5}\right)}$$

$$= \frac{1}{20}\frac{\left(x^{1/4} + y^{1/4}\right)}{\left(x^{1/5} + y^{1/5}\right)} = \frac{1}{20}f(x,y)$$

Which verify the result (1) i.e. Euler's Theorem.

Prob.12 ***Find the maximum and minimum values of***
$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$ ***[R.T.U. 2010]***

Sol. Let $u = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$\therefore \quad \frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$

and $$\frac{\partial u}{\partial y} = 6xy - 30y$$

Also $$r = \frac{\partial^2 u}{\partial x^2} = 6x - 30$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 u}{\partial y^2} = 6x - 30$$

For maxima or minima of u, we have

$$\frac{\partial u}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0 \quad ...(i)$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow 6xy - 30y = 0 \quad ...(ii)$$

From equation (i) $(3x^2 - 30x + 72) + 3y^2 = 0$

$$\Rightarrow \quad x^2 - 10x + 24 + y^2 = 0$$
$$\Rightarrow \quad x^2 - 6x - 4x + 24 + y^2 = 0$$
$$\Rightarrow \quad x(x-6) - 4(x-6) + y^2 = 0$$
$$\Rightarrow \quad (x-6)(x-4) + y^2 = 0 \quad ...(iii)$$

From equation (ii), $6y(x - 5) = 0$

$$\Rightarrow \quad y = 0 \text{ and } x = 5$$

From equation (iii), for $y=0$; $x=6, 4$

And for $x=5$; $x^2-10x+24+y^2=0$

$$25-50+24+y^2=0$$
$$y^2=1$$
$$\Rightarrow \quad y=\pm 1$$

$\Rightarrow$ x is imaginary

i.e. we have two points (5,1) and (5, –1)

Now for (6, 0)

$$r=36-30=6,\ s=0,\ t=36$$
$$\Rightarrow \quad rt-s^2=216$$
and $\quad r=6>0$

i.e. at the point (6, 0) function is minima and value of the function

$$=(6)^3+0-15(6)^2-0+72\times 6$$
$$=216-540+432$$
$$=108$$

And at the point (4, 0)

$$r=24-30=-6$$
And $\quad s=0$
$$r=-6$$
$$\Rightarrow \quad rt-s^2=(-6)\times(-6)-0=36>0$$
and $\quad r=-6<0$

i.e. u is maxima at (4, 0)

and maximum value $=(4)^3+0-15\times 16-0+72\times 4$

$$=64-240+288$$
$$=112$$

(5,1) and (5,–1) are the saddle points as at these points,

$r=0$; $s=30$; $t=0$ $\qquad \therefore\ rt-s^2<0$

Hence, at the point (6, 0) minimum value is 108 and at point (4, 0) maximum value is 112.

Prob.13 ***If $u=\log(x^3+y^3+z^3-3xyz)$, then show that***

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-3(x+y+z)^{-2}$$

[R.T.U. 2008, 2000]

Sol. Given $u=\log(x^3+y^3+z^3-3xyz)$

if 1, ω, ω^2 are cube roots of unity, then

$x^3+y^3+z^3-3xyz$

$$=(x+y+z).(x+y\omega+z\omega^2).(x+y\omega^2+$$

So, the given relation can be written as

$$u=\log(x+y+z)+\log(x+y\omega+z\omega^2)$$
$$+\log(x+y\omega^2+$$

Differentiating u with respect to x (partially)

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{1}{x+y+z}+\frac{1}{x+y\omega+z\omega^2}+\frac{1}{x+y\omega^2+z\omega}$$

Now,

$$\frac{\partial^2 u}{\partial x^2}=\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y\omega+z\omega^2)^2}+\frac{-1}{(x+y\omega^2+z\omega)^2}$$

Also,

$$\frac{\partial u}{\partial y}=\frac{1}{x+y+z}+\frac{\omega}{x+y\omega+z\omega^2}+\frac{\omega^2}{x+y\omega^2+z\omega}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{-1}{(x+y+z)^2}+\frac{-\omega^2}{(x+y\omega+z\omega^2)^2}+\frac{-\omega^4}{(x+y\omega^2+z\omega)^2}$$

$$\text{And,}\quad \frac{\partial u}{\partial z}=\frac{1}{x+y+z}+\frac{\omega^2}{x+y\omega+z\omega^2}+\frac{\omega}{x+y\omega^2+z\omega}$$

Differentiating,

$$\frac{\partial^2 u}{\partial z^2}=\frac{-1}{(x+y+z)^2}+\frac{-\omega^4}{(x+y\omega+z\omega^2)^2}+\frac{-\omega^2}{(x+y\omega^2+z\omega)^2}$$

Adding equation (1), (2), (3)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{-3}{(x+y+z)^2}-\frac{(1+\omega^2+\omega^4)}{(x+y\omega+z\omega^2)^2}-\frac{(1+\omega^4+}{(x+y\omega^2+}$$

$$=\frac{-3}{(x+y+z)^2}-0-0$$

$$[\because \omega^4=\omega^3.\omega=\omega \text{ and } 1+\omega+\omega^2=$$

$$=\frac{-3}{(x+y+z)^2}=-3(x+y+z)^{-2}$$

Hence pro

Prob.14 ***In a triangle ABC, the angles and sides a and are made to vary in such a way that the a remains constant and the side c also rem***

constant. Show that if a and b vary by small amounts δa and δb respectively then cos Aδa + cos Bδb *[R.T.U. 2001]*

Sol. Let Δ denote the area of the triangle ABC

$$\Delta = \frac{1}{2} ac \sin B, \quad ...(1)$$

Where Δ is constant. Also we know that

$$b^2 = c^2 + a^2 - 2ac\cos B \quad ...(2)$$

Differentiating (1), we get

$$0 = \frac{1}{2}c\left[a\cos B\delta B + \sin B\delta a\right]$$

or $a\cos B\delta b = -\sin B\,\delta a$ (3)

Again from (2) differentiating, we get

$$2b\delta b = 2a - 2c\left\{a(-\sin B)\delta B + \cos B.\delta a\right\}$$

or $b\delta b = (a - c\cos B)\delta a + ac\sin B\delta B$

$$= (a - c\cos B)\delta a + ac\sin B\left[\frac{-\sin B\delta a}{a\cos B}\right],$$

from (3) $= \left[\dfrac{(a - c\cos B)\cos B - c\sin^2 B}{\cos B}\right]\delta a$

or $b\delta b = \left(\dfrac{a\cos B - c}{\cos B}\right)\delta a$

$$= \left\{\frac{a\cos B - (a\cos B + b\cos A)}{\cos B}\right\}\delta a,$$

or $b\delta b = -\left(\dfrac{b\cos A}{\cos B}\right)\delta a$ or $\cos A\delta a + \cos B\delta b = 0.$

$[\because c = a\cos B + b\cos A]$ **Proved.**

Prob.15 ***Find the maxima and minima of***

$$u = x^2 + y^2 + z^2$$

subject to the conditions

$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + mx + nz = 0.$$

[R.T.U. 2014, 10]

Sol. Let $u = x^2 + y^2 + z^2$

then $du = 2xdx + 2ydy + 2zdz$ ...(1)

Since it is given that

$$ax^2 + by^2 + cz^2 = 1 \quad ...(2)$$

$$lx + my + nz = 0 \quad ...(3)$$

Differentiating (2) and (3) gives

$$2axdx + 2bydy + 2czdz = 0 \quad ...(4)$$

$$ldx + mdy + ndz = 0 \quad ...(5)$$

Multiplying equation (4) by λ and (5) by 2μ and then adding in (1)

$$du = 2(x + \lambda ax + \mu l)dx + 2(y + \lambda by + \mu m)dy + 2(2 + \lambda cz + \mu n)dz$$

$$= 0$$

by Lagrange method for stationary values

$$x + \lambda ax + \mu l = 0 \quad ...(6)$$

$$y + \lambda by + \mu m = 0 \quad ...(7)$$

$$z + \lambda cz + \mu n = 0 \quad ...(8)$$

Multiplying the equations (6), (7) & (8) by x, y, z respectively and then adding we get

$$(x^2 + y^2 + z^2) + \lambda(ax^2 + by^2 + cz^2) + \mu(lx + my + nz) = 0$$

Using (2), (3) and the given function u,

$$u + \lambda + 0 = 0$$

Put the value of λ in (6), (7) & (8)

$$x(1 - au) + \mu l = 0,$$

$$y(1 - bu) + \mu m = 0,$$

$$z(1 - cu) + \mu n = 0$$

Put the value of x, y & z in $lx + my + nz = 0$

$$\frac{l^2\mu}{(au - 1)} + \frac{m^2\mu}{(bu - 1)} + \frac{n^2\mu}{(cu - 1)} = 0 \quad ...(9)$$

Eqn. (9) gives two stationary values of u. **Ans.**

Prob.16 ***Find*** $\nabla\phi$ ***if***

(a) $\phi = \log|\vec{r}|$

(b) $\phi = \dfrac{1}{r}$

(c) $\phi = r^n$.

Sol.(a) $\because$ $\vec{r} = x\hat{i} + y\hat{j} + 3\hat{k}$ then $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Given

$$\phi = \log|\vec{r}| = \log\sqrt{x^2 + y^2 + z^2}$$

$$= \frac{1}{2}\log(x^2 + y^2 + z^2)$$

$$\nabla\phi = \frac{1}{2}\nabla \log(x^2+y^2+z^2)$$

$$= \frac{1}{2}\left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]\log(x^2+y^2+z^2)$$

$$= \frac{1}{2}\left\{\hat{i}\frac{\partial}{\partial x}\log(x^2+y^2+z^2) + \hat{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \hat{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)\right\}$$

$$= \frac{1}{2}\left\{\hat{i}\frac{2x}{x^2+y^2+z^2} + \hat{j}\frac{2y}{x^2+y^2+z^2} + \hat{k}\frac{2z}{x^2+y^2+z^2}\right\}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$$

$$= \frac{\vec{r}}{r^2}$$

(b) $$\nabla\phi = \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \nabla\{(x^2+y^2+z^2)^{-1/2}\}$$

$$= \left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]\{(x^2+y^2+z^2)\}^{-1/2}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= \hat{i}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x\right\} + \hat{j}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2y\right\} + \hat{k}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2z\right\}$$

$$= \frac{-x\hat{i}-y\hat{j}-z\hat{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\vec{r}}{r^3}$$

(c) $$\nabla\phi = \nabla r^n = \nabla\left(\sqrt{x^2+y^2+z^2}\right)^n$$

$$= \nabla(x^2+y^2+z^2)^{n/2}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}$$

$$= \hat{i}\left\{\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}2x\right\} + \hat{j}\left\{\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}2y\right\} + \hat{k}\left\{\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}2z\right\}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= n(r^2)^{\frac{n}{2}-1}\vec{r} = nr^{n-2}\vec{r}$$

Prob.17 *If $\vec{v_1}$ and $\vec{v_2}$ are vectors joining the fixed p[oints] (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively [to a] variable point (x, y, z) prove that [...] $(\vec{v_1}\cdot\vec{v_2}) = \vec{v_1}+\vec{v_2}$.* *[R.U. 2004, 2001, 1996]*

Sol. Let $\overline{AP}$ and $\overline{BP}$ be two vectors joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ to a variable point $P(x, y, z)$ respect[ively].

Then $$\vec{v_1} = \overline{AP} = (x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}$$

$$\vec{v_2} = \overline{BP} = (x-x_2)\hat{i}+(y-y_2)\hat{j}+(z-z_2)\hat{k}$$

$$\vec{v_1}\cdot\vec{v_2} = (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)$$

$$= \phi \text{ (say)}$$

Now

$$\text{grad}\left(\vec{v_1}\cdot\vec{v_2}\right)=\nabla\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

$$\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial x}[(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)]$$

$$=(x-x_1)+(x-x_2)=2x-(x_1+x_2),$$

$$\frac{\partial\phi}{\partial y}=\frac{\partial}{\partial y}[(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)]$$

$$=(y-y_1)+(y-y_2)=2y-(y_1+y_2),$$

$$\frac{\partial\phi}{\partial z}=\frac{\partial}{\partial z}[(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)]$$

$$=(z-z_1)+(z-z_2)=2z-(z_1+z_2)$$

$$\therefore\quad \text{grad}\left(\vec{v_1}\cdot\vec{v_2}\right)=[2x-(x_1+x_2)]\hat{i}+[2y-(y_1+y_2)]\hat{j}+[2z-(z_1+z_2)]\hat{k}$$

$$=[(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}]+[(x-x_2)\hat{i}+(y-y_2)\hat{j}+(z-z_2)\hat{k}]$$

$$\Rightarrow\ \text{grad}\left(\vec{v_1}\cdot\vec{v_2}\right)=\vec{v_1}+\vec{v_2}$$

Prob.18 ***If $\vec{a}$ is a vector function and u is a scalar function, then prove that***

(i) $div\ (u\vec{a})=u\ div\ \vec{a}+\vec{a}\ .\ grad\ u$

i.e. $\nabla.(u\vec{a})=u(\nabla.\vec{a})+\vec{a}.(\nabla u)$

(ii) $curl\ (u\vec{a})=(grad\ u)\times\vec{a}+u\ curl\ \vec{a}$

i.e. $\nabla\times(u\vec{a})=(\nabla u)\times\vec{a}+u(\nabla\times\vec{a})$

Sol. (i) $div\ (u\vec{a})=\nabla.(u\vec{a})$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(u\vec{a})$$

$$=\hat{i}.\frac{\partial}{\partial x}\left(u\vec{a}\right)+\hat{j}.\frac{\partial}{\partial y}\left(u\vec{a}\right)+\hat{k}.\frac{\partial}{\partial z}\left(u\vec{a}\right)$$

$$=\sum\left[\hat{i}.\frac{\partial}{\partial x}(u\vec{a})\right]$$

$$=\sum\left[\hat{i}.\left(\frac{\partial u}{\partial x}\vec{a}+u\frac{\partial\vec{a}}{\partial x}\right)\right]$$

$$=\sum\left[\hat{i}.\left(\frac{\partial u}{\partial x}\vec{a}\right)\right]+\sum\left[\hat{i}.\left(u\frac{\partial\vec{a}}{\partial x}\right)\right]$$

Since $\vec{a}.(n\vec{b})=(n\vec{a}).\vec{b}=n(\vec{a}.\vec{b})$

$$\therefore\quad div(u\vec{a})=\sum\left[\left(\frac{\partial u}{\partial x}\hat{i}\right).\vec{a}\right]+\sum\left[u\left(\hat{i}.\frac{\partial\vec{a}}{\partial x}\right)\right]$$

$$=\left(\sum\frac{\partial u}{\partial x}\hat{i}\right).\vec{a}+u\sum\left(\hat{i}.\frac{\partial\vec{a}}{\partial x}\right)$$

$$=(\nabla u).\vec{a}+u\left(\nabla.\vec{a}\right)$$

$$=u\left(\nabla.\vec{a}\right)+\vec{a}.(\nabla u)$$

(ii) $curl\ (u\vec{a})=\nabla\times(u\vec{a})$

$$=\left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]\times(u\vec{a})$$

$$=\sum\left[\hat{i}\times\frac{\partial}{\partial x}(u\vec{a})\right]$$

$$=\sum\left[\hat{i}\times\left(\frac{\partial u}{\partial x}\vec{a}+u\frac{\partial\vec{a}}{\partial x}\right)\right]$$

$$=\sum\left[\hat{i}\times\left(\frac{\partial u}{\partial x}\vec{a}\right)\right]+\sum\left[\hat{i}\times\left(u\frac{\partial\vec{a}}{\partial x}\right)\right]$$

Since $\vec{a}\times(n\vec{b})=(n\vec{a})\times\vec{b}=n(\vec{a}\times\vec{b})$

$$\therefore\quad curl(u\vec{a})=\sum\left[\left(\hat{i}\frac{\partial u}{\partial x}\right)\times\vec{a}\right]+\sum\left[u\left(\hat{i}\times\frac{\partial\vec{a}}{\partial x}\right)\right]$$

$$=\left(\sum\hat{i}\frac{\partial u}{\partial x}\right)\times\vec{a}+u\left[\sum\left(\hat{i}\times\frac{\partial\vec{a}}{\partial x}\right)\right]$$

$$=(\nabla u)\times\vec{a}+u(\nabla\times\vec{a})$$

$$=u(\nabla\times\vec{a})+(\nabla u)\times\vec{a}$$

Prob.19 ***If r and $\vec{r}$ have their usual meanings then***

(i) Show that $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$

[R.T.U. 2011, 10; R. U. 2001]

(ii) Find div (r grad r^3)

[R. T. U. 2008]

Sol. Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r^2 = x^2 + y^2 + z^2$

(i) $$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$$

$$= \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right)$$

$$= \sum \frac{\partial}{\partial x}\left[\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x}\right] = \sum \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right]$$

$$= \sum \left[f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}\frac{\partial}{\partial x} f'(r)\right]$$

$$= \sum \left[f'(r)\left\{\frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}\right\} + \frac{x}{r}\frac{\partial f'(r)}{\partial r}\frac{\partial r}{\partial x}\right]$$

$$= \sum \left[f'(r)\left\{\frac{1}{r} - \frac{x^2}{r^3}\right\} + \frac{x^2}{r^2} f''(r)\right] \quad \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

$$= f'(r)\left[\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \left(\frac{1}{r} - \frac{y^2}{r^3}\right) + \left(\frac{1}{r} - \frac{z^2}{r^3}\right)\right] + \frac{f''(r)}{r^2}\left(x^2 + y^2 + z^2\right)$$

$$= f'(r)\left[\frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3}\right] + f''(r)\left(\frac{x^2 + y^2 + z^2}{r^2}\right)$$

$$\nabla^2 f(r) = \left(\frac{3}{r} - \frac{1}{r}\right) f'(r) + f''(r)$$

$$\Rightarrow \quad \nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r)$$

(ii) $$div\ (\text{r grad } r^{-3}) = \nabla.\left(\vec{r}\,\nabla\frac{1}{r^3}\right)$$

Now $$\nabla\frac{1}{r^3} = -\frac{3\vec{r}}{r^5}$$

$$\therefore\ div\left(\vec{r}\,\nabla\frac{1}{r^3}\right) = div\left(\frac{-3}{r^4}\vec{r}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{-3}{r^4}(x\hat{i} + y\hat{j} + z\hat{k})\right)$$

$$= -3\left[\frac{\partial}{\partial x}\left(\frac{x}{r^4}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^4}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^4}\right)\right]$$

$$= -3\sum \frac{\partial}{\partial x}\left(\frac{x}{r^4}\right)$$

$$= -3\sum\left[\frac{1}{r^4} - \frac{4x}{r^5}\cdot\frac{\partial r}{\partial x}\right] = -3\sum\left[\frac{1}{r^4} - \frac{4x^2}{r^6}\right]$$

$$= -3\left[\left(\frac{1}{r^4} - \frac{4x^2}{r^6}\right) + \left(\frac{1}{r^4} - \frac{4y^2}{r^6}\right) + \left(\frac{1}{r^4} - \frac{4z^2}{r^6}\right)\right]$$

$$= -3\left[\frac{3}{r^4} - \frac{4(x^2 + y^2 + z^2)}{r^6}\right]$$

$$= -3\left[\frac{3}{r^4} - \frac{4}{r^4}\right] = \frac{3}{r^4}$$

Prob.20 ***Prove that*** $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi.\nabla\psi + \psi\nabla^2\phi$

Sol. $\because \nabla(\phi\psi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi\psi)$

$$= \hat{i}\frac{\partial}{\partial x}(\phi\psi) + \hat{j}\frac{\partial}{\partial y}(\phi\psi) + \hat{k}\frac{\partial}{\partial z}(\phi\psi)$$

$$= \hat{i}\left[\frac{\partial\phi}{\partial x}\psi + \phi\frac{\partial\psi}{\partial x}\right] + \hat{j}\left[\frac{\partial\phi}{\partial y}\psi + \phi\frac{\partial\psi}{\partial y}\right] + \hat{k}\left[\frac{\partial\phi}{\partial z}\psi + \phi\frac{\partial\psi}{\partial z}\right]$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\psi + \left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right)\phi$$

$$= \psi(\nabla\phi) + \phi(\nabla\psi)$$

Again $\nabla^2(\phi\psi) = \nabla\cdot[\nabla(\phi\psi)]$

$$= \nabla\cdot[\psi(\nabla\phi) + \phi(\nabla\psi)]$$

$$= \nabla\cdot[\psi(\nabla\phi)] + \nabla\cdot[\phi(\nabla\psi)]$$

$$= [(\nabla\psi)\cdot(\nabla\phi) + \psi\nabla\cdot(\nabla\phi)] + [(\nabla\phi)\cdot(\nabla\psi) + \phi\nabla\cdot(\nabla\psi)]$$

$$= (\nabla\psi)\cdot(\nabla\phi) + \psi\nabla^2\phi + (\nabla\phi)\cdot(\nabla\psi) + \phi\nabla^2\psi$$

$$= \phi\nabla^2\psi + \psi\nabla^2\phi + 2(\nabla\phi)\cdot(\nabla\psi)$$

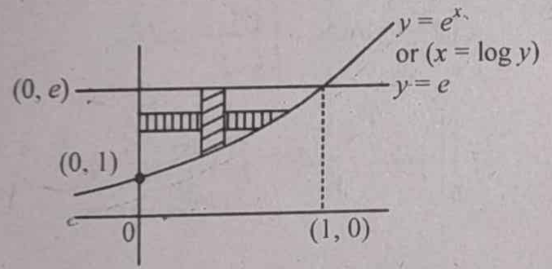
MULTIVARIABLE CALCULUS (INTEGRATION) 5

PREVIOUS YEARS QUESTIONS

I. SHORT ANSWER TYPE QUESTIONS

Prob.1 *Change the order of integration only in* $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$. **[R.T.U. 2017]**

Sol. Tracing the given limits gives the shaded region as follows:



When we convert the limits into horizontal strip, it gives

$$= \int_1^e \int_0^{\log y} \frac{1}{\log y} dx dy = \int_1^e \left[\int_0^{\log y} \frac{1}{\log y} dx \right] dy$$

constant.

$$= \int_1^e \frac{1}{\log y} [x]_0^{\log y} dy = \int_1^e dy = e - 1$$ **Ans.**

Prob.2 *Find the area, by double integration, bounded by parabola $y^2 = 4\,ax$ and its latus rectum.* **[R.T.U. 2017]**

Sol. Area bounded by $y^2 = 4ax$ and latus rectum $= 2 \times$ area of first quadrant

$$A = 2\int_0^a y dx$$

$$= 2\int_0^a 2\sqrt{a} x^{1/2} dx$$

$$= 4\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^a$$

$$= \frac{8}{3}a^{1/2}.a^{3/2}$$

$$= \frac{8}{3}a^2$$

Prob.3 *Write the Cartesian formula of Gauss divergence theorem.* **[R.T.U. 2017]**

Sol. Gauss's Divergence Theorem

Let E be a simple solid region and S be the boundary surface of E with positive orientation . Let $\vec{F}$ be vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V div\, \vec{F}\, d\, V$$

Prob.4 *Evaluate the following double integral.*

$$\int_0^1 \int_0^x e^{y/x} dy dx$$

Sol. $\int_0^1 \int_0^x e^{y/x} dydx$

Since the limits of integrals show that the region is bounded by $y = 0, y = x$ and the lines $x = 0$ and $x = 1$. So we first integrate w.r. to y and then with respect to x

$$\int_0^1 \left[\int_0^x e^{y/x} dy \right] dx = \int_0^1 \left[xe^{y/x} \right]_0^x dx = \int_0^1 (xe - x) dx$$

$$= (e-1)\int_0^1 x\,dx = (e-1)\left[\frac{x^2}{2}\right]_0^1 = \frac{e-1}{2}$$ **Ans.**

Prob.5 *Evaluate the following double integral.*

$$\int_0^1 \int_0^{\sqrt{1-y^2}} y\,dx\,dy$$

Sol. $\int_0^1 \int_0^{\sqrt{1-y^2}} y\,dx\,dy$

Here the region is bounded between two lines $y = 0$, $y = 1$ and the curves $x = 0$ and $x = \sqrt{1-y^2}$ therefore first we integrate w.r. to x and then w.r. to y.

$$= \int_0^1 \left[\int_0^{\sqrt{1-y^2}} dx\right] y\,dy$$

$$= \int_0^1 [x]_0^{\sqrt{1-y^2}} .y\,dy = \int_0^1 y\sqrt{1-y^2}\,dy$$

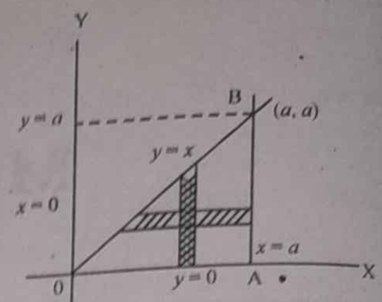
Let $1-y^2 = t^2 \Rightarrow -2ydy = 2tdt$
If $y = 0$; $t = 1$
and if $y = 1$; $t = 0$

then $\int_1^0 -t\,t\,dt = \int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$ **Ans.**

Prob.6 *Change the order of the following integral.*

$$\int_0^a \int_0^x f(x,y)\,dy\,dx$$

Sol. Tracing $x = 0$ and $x = a$, $y = 0$ and $y = x$ gives the region shown in the figure. Both the lines $y = x$, $x = a$ intersect each other at (a, a).



Rotating the paper by 90°, convert the vertical strip bounded between $y = 0$ and $y = x$ into horizontal strip bounded between $x = y$ and $x = a$. Also the shaded region lies between $y = 0$ and $y = a$.

$$\int_0^a \int_0^x f(x,y)\,dy\,dx = \int_0^a \int_y^a f(x,y)\,dx\,dy$$

Prob.7 *Change the order of the following integrals.*

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

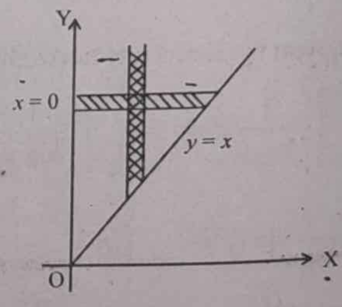
Sol. The given region of integration is shown in the fig. Rotating 90°, *i.e.* when we convert the vertical strip horizontal strip the limits are changed as

$y = 0$ to ∞

$x = 0$ to $x = y$

$$= \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^\infty \left[\int_0^y dx\right] \frac{e^{-y}}{y}\,dy = \int_0^\infty [x]_0^y \frac{e^{-y}}{y}\,dy$$



$$= \int_0^\infty \frac{ye^{-y}}{y}\,dy = \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty = 1$$

II. Long Answer Type Questions

Prob.1 *Transform the integral* $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\,dy\,dx$ *by changing to polar coordinates and hence evaluate*

[R.T.U. 2017, R.T.U. Dec. ...

Sol. The region of integration is

$$0 \le y \le \sqrt{a^2 - x^2}, 0 \le x \le a$$

i.e. the area bounded by the circle $x^2 + y^2 = a^2$, lying in the I quadrant.

Put $x = r\cos\theta$, $y = r.\sin\theta$

$\therefore$ Equation of circle: $(r\cos\theta)^2 + (r\sin\theta)^2 = a^2$

$\Rightarrow r = a$

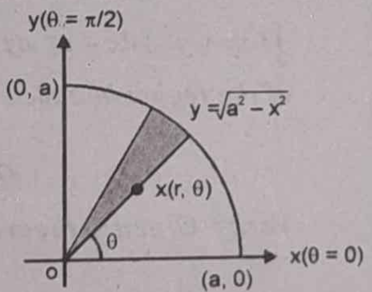
and for I quadrant, θ varies from 0 to $\frac{\pi}{2}$. Thus, in polar coordinates limits of integration are

$$0 \le r \le a,\ 0 \le \theta \le \frac{\pi}{2}$$

$$\therefore I = \int_0^{\pi/2}\int_0^a r^2\sin^2\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\ r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^a r^4\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^5}{5}\right)_0^a \sin^2\theta\,d\theta$$



$$= \frac{a^5}{5}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2}\frac{(1-\cos 2\theta)}{2}d\theta$$

$$= \frac{a^5}{5}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{\pi a^5}{20}$$

Prob.2 ***Evaluate*** $\iiint_V x^2\,dx\,dy\,dz$ ***over the region V enclosed by the planes*** $x = 0, y = 0, z = 0$ ***and*** $x + y + z = a$. *[R.T.U. 2017]*

Sol. Here it is clear from the limits of integration that region of integral is tetrahedron OABC.

Vertical columns are enclosed by planes $z = 0$ and $z = a - x - y$.

Plane $z = a - x - y$ intersect xy-plane at $a - x - y = 0$

So region s of xy-plane on which tetrahedron is situated, is enclosed by lines

$x = 0, x = 1, y = 0$ and $y = a - x$

$$\therefore \iiint_V x^2\,dx\,dy\,dz = \int_0^a\int_0^{a-x}\int_0^{a-x-y} x^2\,dx\,dy\,dz$$

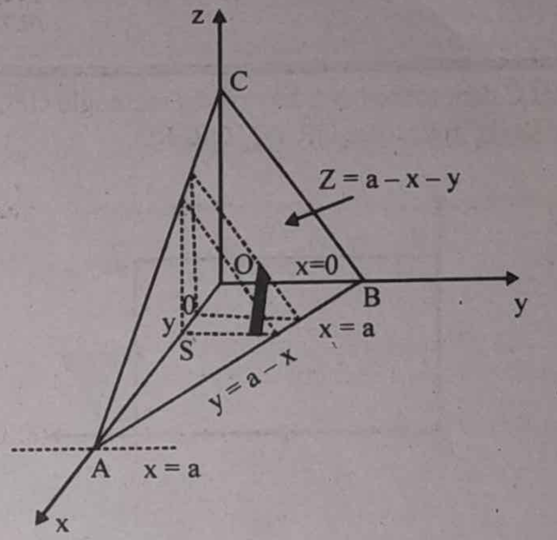


Fig.

$$= \int_0^a\int_0^{a-x}\left\{x^2 z\right\}_0^{a-x-y} dx\,dy$$

$$= \int_0^a\left\{\int_0^{a-x} x^2(a-x-y)dy\right\}dx$$

$$= \int_0^a\left[x^2(a-x)y - \frac{x^2y^2}{2}\right]_0^{a-x} dx$$

$$= \int_0^a\left[x^2(a-x)^2 - \frac{1}{2}x^2(a-x)^2\right]dx$$

$$= \frac{1}{2}\int_0^a x^2(a-x)^2 dx$$

$$= \frac{1}{2}\int_0^a (x^2a^2 - 2ax^3 + x^4)dx$$

$$= \frac{1}{2}\left[\frac{a^2x^3}{3} - \frac{2ax^4}{4} + \frac{x^5}{5}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{1}{3}a^5 - \frac{1}{2}a^5 + \frac{1}{5}a^5\right]_0^a$$

$$= \frac{1}{60}a^5$$

Prob.3 ***Verify stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)i + 2xy\ j$, integrated around the rectangle $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$ and $y = b$.***

[R.T.U. 2017]

Sol. Let C denote the boundary of the rectangle OPQR, then C consists of four lines OP, PQ, QR, RO.

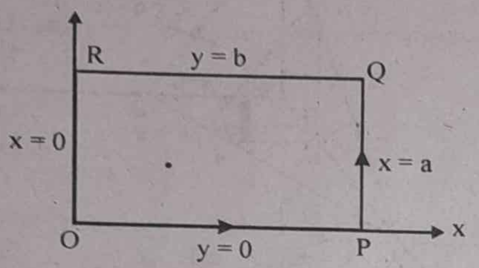


Fig.

$$\bar{r} = x\hat{i} + y\hat{j}$$

$$d\bar{r} = dx\hat{i} + dy\hat{j}$$

$$\therefore \quad \bar{F}.d\bar{r} = (x^2 - y^2)dx + 2xy\,dy$$

Along OP : $y = 0, dy = 0, x : 0 \to a$

$$\int_{OP} \bar{F}.d\bar{r} = \int_0^a x^2 dx = \frac{a^3}{3}$$

Along PQ : $x = a, dx = 0, y : 0 \to b$

$$\int_{PQ} \bar{F}.d\bar{r} = \int_0^b 2aydy = ab^2$$

Along QR : $y = b, dy = 0; x : a \to 0$

$$\int_{QR} \bar{F}.d\bar{r} = \int_a^0 (x^2 - b^2)dx$$

$$= \left[\frac{x^2}{3} - b^2 x\right]_a^0 = ab^2 - \frac{a^3}{3}$$

Along RO : $x = 0, dx = 0, y : b \to 0$

$$\int_{RO} \bar{F}.d\bar{r} = 0$$

$$\therefore \text{L.H.S.} = \int_C \bar{F}.d\bar{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} + 0 = 2ab^2$$

$$\text{Now Curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\hat{k}$$

For the surface, $S.\bar{x} = \hat{k}$

$\therefore$ Curl $\bar{F}.\bar{x} = 4y$

$$\text{R.H.S.} = \iint_S \text{Curl } \bar{F}.\hat{x}\, ds = \int_0^a \int_0^b 4y\, dydx = 2ab^2$$

Hence the stoke's theorem is verified.

Prob.4 ***State stoke's theorem. Verify Green's theor... the plane for $\oint_C [(xy + y^2)dx + x^2 dy]$, where C... closed curve of the region bounde... $y = x^2$ and $y = x$.***

[R.T.U...

OR

Verify Green's theorem in the plan... $\int (xy + y^2)\,dx + x^2\,dy$, where C is the closed... of the region bounded by $y = x$ and $y = x^2$.

[R.T.U...

OR

Verify Green's theorem for $\int_C [(xy + y^2)dx + ...$ where C is bounded by $y = x$ and $y = x^2$.

[R.T.U. 2014, June/July...

Sol. Stoke's Theorem : Let S be an oriented piece... smooth surface that is bounded by a simple close... piecewise smooth boundary curve C with positive orient... Also let $\bar{F}$ be a vector field whose components have conti... partial derivatives on an open region in R^3 that contain...

Then, $\oint_C \bar{F}.d\bar{r} = \iint_S curl\, \bar{F} \cdot \hat{n} dS = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, dS$

Verify Green's theorem : The shaded region shown... represents the positive direction traversed by the closed... C made up of a parabola and a straight line.

Given, $\psi = (xy + y^2)$ and $\phi = x^2$

Evaluating the integral along $y = x^2$, we have

$$= \oint_C P\,dx + Q dy = \int [(x \cdot x^2 + x^4)dx + y\,dy]$$

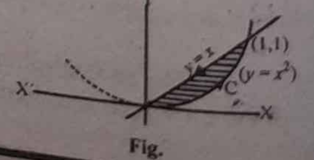


Fig.

$$= \int_{x=0}^{1} (x^3 + x^4)\,dx + \int_{x=0}^{1} y\,dy$$

$$= \left[\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 = \frac{19}{20} \quad \ldots(1)$$

Evaluating the line integral along $y = x$, we have

$$\oint_C P\,dx + Q\,dy = \int[(x^2 + x^2)\,dx + y^2\,dy]$$

$$= \int_{x=1}^{0} 2x^2\,dx + \int_{y=1}^{0} y^2\,dy = -1 \quad \ldots(2)$$

Using eq. (1) and (2),

$$\oint_C P\,dx + Q\,dy = \frac{19}{20} - 1 = -\frac{1}{20} \quad \ldots(3)$$

Also $\dfrac{\partial Q}{\partial x} = 2x$ and $\dfrac{\partial P}{\partial y} = x + 2y$

Thus, $= \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dx\,dy$

$$= \iint_R (2x - x - 2y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} [xy - y^2]_{y=x^2}^{x}\,dx = \int_{x=0}^{1} [x^2 - x^2 - x^3 + x^4]\,dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx = -\frac{1}{20} \quad \ldots(4)$$

It is evident from eq. (3) and eq. (4) that,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -\frac{1}{20}$$

Thus Green's theorem is verified.

Prob.5 ***Find the surface area generated by revolving the cardioid r = a (1 + cos θ) about the initial line.***

[R.T.U. Dec. 2016]

OR

Find the surface of the solid formed by revolving the cardiod r = a (1 + cos θ) about the initial line.

[R.T.U. 2007, Raj.Univ. 2006]

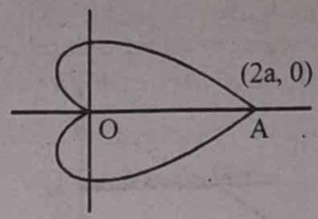
Sol. The equation of the curve is $r = a\,(1 + \cos\theta)$.

Since the given curve revolves about initial line *i.e.* $\theta = 0$

$$\frac{dr}{d\theta} = -a\sin\theta \quad \ldots(i)$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$
$$= a^2\,(1 + \cos\theta)^2 + (-a\sin\theta)^2$$
$$= a^2\,[2 + 2\cos\theta] = 2a^2\,(1 + \cos\theta)$$
$$= 4a^2\cos^2\theta/2$$

$$\therefore \quad \frac{ds}{d\theta} = 2a\cos\frac{\theta}{2} \quad \ldots(ii)$$



The surface area generated by the curve when revolves about its initial line is

$$= 2\pi\int_0^{\pi} x\frac{ds}{d\theta}\,d\theta = 2\pi\int_0^{\pi} (r\cos\theta)\,2a\cos\frac{\theta}{2}\,d\theta,$$

$(\because r = a\,(1 + \cos\theta))$

$$= 4\pi a^2\int_0^{\pi} (1 + \cos\theta)\cos\frac{\theta}{2}\,d\theta$$

$$= 4\pi a^2\int_0^{\pi} (2 - 2\sin^2\frac{\theta}{2})(1 - 2\sin^2\frac{\theta}{2})\cos\frac{\theta}{2}\,d\theta$$

$(\because \cos\theta = 1 - 2\sin^2\frac{\theta}{2})$

$$= 4\pi a^2\int_0^{1} (2 - 2t^2)(1 - 2t^2)(2\,dt)$$

put $t = \sin\dfrac{\theta}{2}$ $= 16\pi a^2\displaystyle\int_0^{1} (1 - t^2)(1 - 2t^2)\,dt$

$$= 16\pi a^2\left[t - t^3 + \frac{2}{5}t^5\right]_0^1$$

$$= 16\pi a^2\left[1 - 1 + \frac{2}{5}\right] = \frac{32\pi a^2}{5} \text{ square units. } \textbf{Ans.}$$

Prob.6 ***Find the volume of the solid generated by revolving the curve (a – x) y² = a²x, about its asymptote.***

[R.T.U. Jan. 2016]

Sol. The given curve is $(a - x)y^2 = a^2x$.

The curve is symmetric about x-axis. The asymptote to the curve is

$$a - x = 0$$

which is parallel to y-axis. Now, the curve revolves round the axis $x = a$. Radius of any element cylindrical disc passing through $P(x, y)$ is $(a - x)$

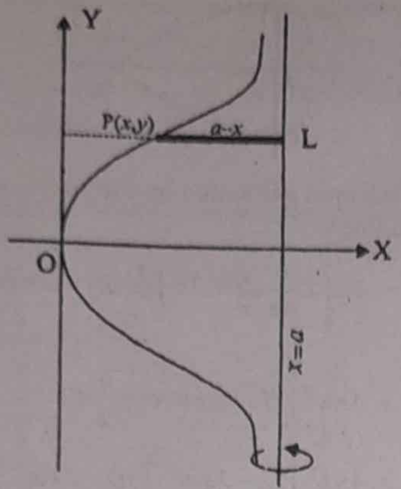
Thus, volume of soild generated

$$V = 2\int_{y=0}^{y=\infty} \pi(a - x)^2\,dy$$

$$= 2\pi\int_0^{\infty}\left[a - \frac{ay^2}{a^2 + y^2}\right]^2 dy \quad \left[\because x = \frac{ay^2}{a^2 + y^2}\right]$$

$$= 2\pi a^6\int_0^{\infty} \frac{1}{(a^2 + y^2)^2}\,dy$$

Put $y = a\tan\theta,\ dy = a\sec^2\theta\, d\theta$



$$V = 2\pi a^6 \int_0^{\pi/2} \frac{a\sec^2\theta d\theta}{a^4\sec^4\theta}$$

$$= 2\pi a^3 \int_0^{\pi/2} \cos^2\theta d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$$

$$= \pi a^3 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi^2 a^3}{2}$$

Prob.7 (a) ***Change the order of integration in the integral*** $\int_0^a \int_{\{a-\sqrt{a^2-y^2}\}}^{\{a+\sqrt{a^2-y^2}\}} xy\, dx\, dy$ ***and then evaluate it.***

(b) ***Show that*** $\int_0^\infty \frac{x^2 dx}{(1+x^4)^3} = \frac{5\pi\sqrt{2}}{128}$.

[R.T.U. Jan. 2016]

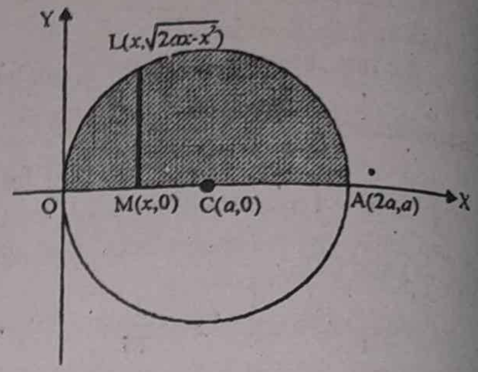
Sol.(a) The region of integration is bounded by the line $x = 0$, $y = a$ and the curve $x = a \pm \sqrt{a^2 - y^2}$ i.e. $(x-a)^2 + y^2 = a^2$, a circle.

After changing the order of integration, the integral I becomes,

$$I = \iint_R xy\, dy\, dx$$

$$= 2\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy\, dy\, dx$$

$$= \int_0^{2a} x\left(\frac{y^2}{2}\right)_0^{\sqrt{2ax-x^2}} dx$$



$$= \frac{1}{2}\int_0^{2a} (2ax - x^2)x\, dx$$

$$= \frac{1}{2}\int_0^{2a} (2ax^2 - x^3)\, dx$$

$$= \frac{1}{2}\left(\frac{2ax^3}{3} - \frac{x^4}{4}\right)_0^{2a} = \frac{2}{3}a^4$$

Sol.(b) Put $x = \sqrt{\tan\theta} \Rightarrow dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta d\theta$

$$\therefore \int_0^\infty \frac{x^2 dx}{(1+x^4)^3} = \int_0^{\pi/2} \frac{\tan\theta . \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta}{(1+\tan^2\theta)^3}$$

$$= \frac{1}{2}\int_0^{\pi/2} (\tan\theta)^{1/2} \sec^{-4}\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^{7/2}\theta d\theta$$

$$= \frac{1}{2}.\frac{1}{2}.\beta\left(\frac{1+\frac{1}{2}}{2}, \frac{1+\frac{7}{2}}{2}\right)$$

$$= \frac{1}{4}\beta\left(\frac{3}{4}, \frac{9}{4}\right)$$

$$= \frac{1}{4}\frac{\Gamma(3/4)\Gamma(9/4)}{\Gamma(3)} = \frac{5\pi\sqrt{2}}{128}$$

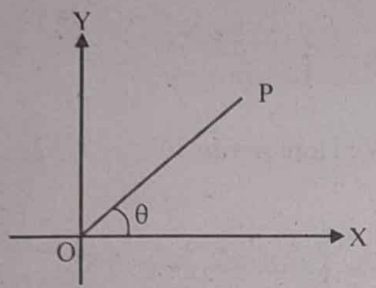
Prob.8 ***Evaluate the integral*** $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ ***by changing to polar coordinates.***

[R.T.U. 2013, Raj. Univ. 2004, 03, 02, 01, 1998]

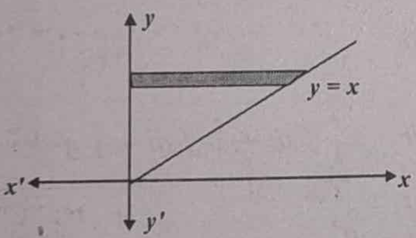
Sol. The region of integration is obtained by the limits $x = 0$ to $x = \infty$ and $y = 0$ to $y = \infty$. So the region is the entire XOY plane which lies in the first quadrant. Changing to polar coordinates by putting $x = r\cos q$, $y = r\sin q$.

So that $x^2 + y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$

limits are $r = 0$ to $r = \infty$ and $\theta = 0$ to $\theta = \frac{\pi}{2}$



$\therefore$ The double integral is transform to



$$\ell = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta \text{, put } r^2 = t \quad \text{so } 2r\,dr = dt.$$

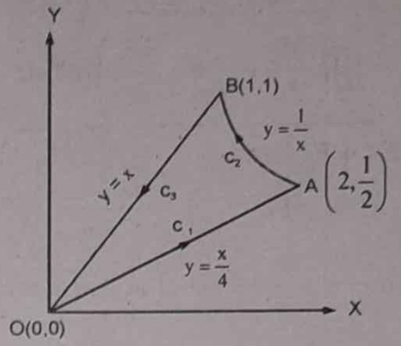
So, $$\ell = \frac{1}{2}\int_0^{\pi/2}\int_0^\infty e^{-t}dt\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(-e^{-t}\right)_0^\infty d\theta$$

$$= \frac{1}{2}\cdot\left[\int_0^{\pi/2} d\theta\right] = \frac{\pi}{4}$$ **Ans.**

Prob.9 ***Using Green's theorem, find the area of the region in the first quadrant bounded by the curves*** $y = x$, $y = \dfrac{1}{x}$ ***and*** $y = \dfrac{x}{4}$. **[R.T.U. 2015]**

Sol. By Green's theorem area A of the region bounded by a closed curve c is given by

$$A = \frac{1}{2}\oint_c x\,dy - y\,dx$$



Here c consists of the curves $c_1 : y = \frac{x}{4}$, $c_2 : y = \frac{1}{x}$, and $c_3 : y = x$.

So $$A = \frac{1}{2}\oint_c = \frac{1}{2}\left[\int_{c_1} + \int_{c_2} + \int_{c_3}\right] = \frac{1}{2}[I_1 + I_2 + I_3]$$

Along $c_1 : y = \frac{x}{4}$, $dy = \frac{1}{4}dx$, $x : 0$ to 2

$$I_1 = \int_{c_1} x\,dy - y\,dx = \int_{c_1} x\frac{1}{4}dx - \frac{x}{4}dx = 0$$

Along $c_2 : y = \frac{1}{x}$, $dy = -\frac{1}{x^2}dx$, $x : 2$ to 1

$$I_2 = \int_{c_2} x\,dy - y\,dx = \int_2^1 x.\left(-\frac{1}{x^2}\right)dx - \frac{1}{x}dx$$

$$= -2\ln\,x\Big|_2^1 = 2\ln 2$$

Along $c_3 : y = x$, $dy = dx$; $x : 1$ to 0.

$$I_3 = \int_{c_3} x\,dy - y\,dx = \int x\,dy - x\,dx = 0$$

$$A = \frac{1}{2}(I_1 + I_2 + I_3)$$

$$A = \frac{1}{2}(0 + 2\ln 2 + 0) = \ln 2$$

Prob.10 ***Verify Gauss' divergence theorem for the function*** $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ ***over the cylindrical region bounded by*** $x^2 + y^2 = 9, z = 0$ ***and*** $z = 2$.

[R.T.U. 2014, 09]

Sol. Statement : The surface integral of normal component of a vector function taken around a closed surface S is equal to the integral of the divergence of $\vec{F}$ taken over the volume V enclosed by the surface.

Mathematically $\iint_S \vec{F}.\hat{n}\,ds = \iiint_V div\,\vec{F}\,dv = \iiint_V (\vec{\nabla}.\vec{F})dV$

In cartesian form $= \iint_S (F_1 dydz + F_2 dzdx + F_3\,dxdy)$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz$$

where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$

$$\hat{\nabla}.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(y\hat{i} + x\hat{j} + z^2\hat{k}\right) = 2z$$

Now $\iiint_V (\vec{\nabla}.\vec{F})dxdydz = 4\int_{z=0}^{2}\int_{0}^{3}\int_{x=0}^{\sqrt{9-y^2}} 2z\,dxdy\,dz$

$$=4\int_{z=0}^{2}\int_{y=0}^{3} 2z\,dy\,dz\,(x)_0^{\sqrt{9-y^2}}$$

$$=4\int_{z=0}^{2}\left[\int_{y=0}^{3} 2z\sqrt{9-y^2}\,dy\right]z\,dz$$

$$=8\int_{z=0}^{2}\left[\int_{0}^{3}\sqrt{9-y^2}\,dy\right]zdz$$

$$=8\int_{z=0}^{2}\left(\sqrt{9-y^2} + \frac{9}{2}\sin^{-1}\frac{1}{3}\right)_0^3 zdz$$

$$=8\int_{z=0}^{2}\left(\frac{9}{2}\cdot\frac{\pi}{2}\cdot zdz\right)z\,dz$$

$$=8.\frac{9\pi}{4}\left(\frac{z^2}{2}\right)_0^2 = 8.\frac{9\pi}{4}.\frac{4}{2} = 36\pi$$

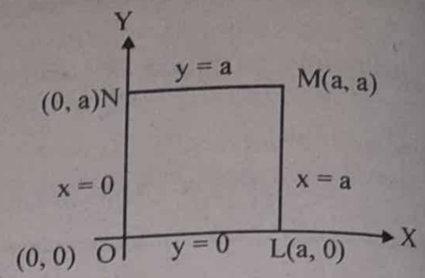
Prob.11 ***Verify Stokes theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square in the plane $z = 0$, whose sides are along the lines $x = y = 0$ and $x = y = a$.*** **[R.T.U. 2012]**

Sol. Here, $f = x^2\hat{i} + xy\,\hat{j}$

$\therefore f.dr = (x^2\hat{i} + xy\,\hat{j})(dxi + dyj)$

$= x^2dx + xydy$

$\therefore \oint_c f.dr = \oint_c (x^2dx + xydy)$



Along OL : here $y = 0$

$\therefore dy = 0$

$$\therefore \int_{OL} f.dr = \int_0^a x^2dx = \frac{a^3}{3}$$

Along LM : Here $x = a$

$\therefore dx = 0$

$$\therefore \int_{LM} f.dr = \int_0^a ay.dy = \frac{a^3}{2}$$

Along MN : Here $y = a$

$\therefore dy = 0$

$$\therefore \int_{MN} f.r = \int_a^0 x^2dx = -\frac{a^3}{3}$$

Along No : Here $x = 0$

$\therefore dx = 0$

$$\therefore \int_{NO} f.dr = 0$$

$$\therefore \oint_c f.dr = \int_{OL} f.dr + \int_{LM} f.dr + \int_{MN} f.dr + \int_{NO} f.dr$$

$$= \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$$

$$= \frac{a^3}{2} \quad ...(i)$$

Now, curl $f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = yk$

Since the surface lies in the xy-plane, $n = k$,

$\therefore$ curl $f.\ n = yk.k = y$

$$\therefore \iint_s curl\,f.xds = \int_{x=0}^{a}\int_{y=0}^{a} y.dxdy = \int_{x=0}^{a}\left[\frac{y^2}{2}\right]_0^a dx$$

$$= \int_0^a \frac{a^2}{2}dx = \frac{a^2}{2}[x]_0^a = \frac{a^3}{2} \quad ...(ii)$$

Equality of Eq. (i) and (ii) verify Stoke's theorem.

Prob.12 ***Find the center of gravity of a plane in the form of a quadrant of an ellipse, the thickness of plate varying as the product of the distance from the bounding radius.***

Sol. The given equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$y=\frac{b}{a}\sqrt{a^2-x^2} \qquad \text{...(i)}$$

and element area can be given as $\delta x\delta y$, thickness at any point of this elementary area = kxy

Let $(\bar{x},\bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int xdm}{\int dm}=\frac{\iint x\rho dxdy}{\iint \rho dxdy}$$

$$=\frac{\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} x.kxydydx}{\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} kxydydx}, \text{ since } p=xyk$$

$$=\frac{\int_{x=0}^{a} x^2\left(\frac{1}{2}y^2\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx}{\int_{x=0}^{a} x\left(\frac{1}{2}y^2\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx}$$

$$=\frac{\int_{x=0}^{a} x^2\left(a^2-x^2\right)dx}{\int_{x=0}^{a} x\left(a^2-x^2\right)dx}$$

$$=\frac{\left[\frac{1}{3}a^2x^3-\frac{1}{5}x^5\right]_0^a}{\left[\frac{1}{2}a^2x^2-\frac{1}{4}x^4\right]_0^a}$$

$$=\frac{\frac{2}{15}a^5}{\frac{1}{4}a^4}=\frac{8}{15}a$$

$$\bar{y}=\frac{\int ydm}{\int dm}=\frac{\iint \rho ydxdy}{\iint \rho dxdy}$$

$$=\frac{\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} y.kxydydx}{\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} kxydydx}$$

$$=\frac{\int_{x=0}^{a} x^2\left(\frac{1}{3}y^3\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx}{\int_{x=0}^{a} x^1\left(\frac{1}{2}y^2\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx}$$

$$=\frac{2b}{3a}\frac{\int_{x=0}^{a} x^2\left(a^2-x^2\right)^{3/2}dx}{\int_{x=0}^{a} x\left(a^2-x^2\right)dx}$$

$$=\frac{2b\left[-\frac{1}{5}\left(a^2-x^2\right)^{5/2}\right]_0^a}{3a\left[-\frac{1}{4}\left(a^2-x^2\right)^2\right]_0^a}$$

$$=\frac{8b}{15a}.\frac{a^5}{a^4}=\frac{8b}{15}$$

Hence the center of gravity of the given curve is $\left(\frac{8a}{15},\frac{8b}{15}\right)$

Prob.13 ***Find the center of gravity of the area enclosed by the curves $y^2 = ax$, and $x^2 = ay$.***

Sol. Assuming density in the given problem is k(constant).

$$\bar{x}=\frac{\int xdm}{\int dm}$$

$$=\frac{\iint x\rho dxdy}{\iint \rho dxdy}$$

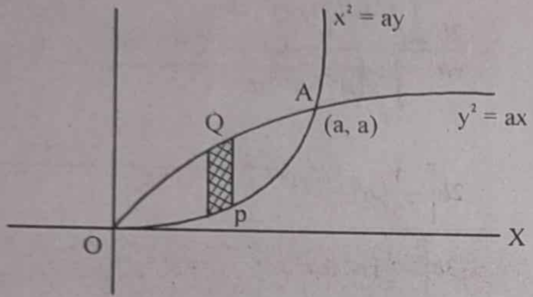
$$=\frac{\int_{x=0}^{a}\int_{x^2/a}^{\sqrt{ax}} kxdydx}{\int_{x=0}^{a}\int_{x^2/a}^{\sqrt{ax}} kdydx}$$

$$=\frac{\int_0^a x[y]_{x^2/a}^{\sqrt{ax}}dx}{\int_0^a [y]_{x^2/a}^{\sqrt{ax}}dx}$$

$$= \frac{\int_0^a x\left[\sqrt{ax}-\frac{x^2}{a}\right]dx}{\int_0^a\left[\sqrt{ax}-\frac{x^2}{a}\right]dx}$$

$$= \frac{\left[\sqrt{a}.\frac{2}{5}x^{5/2}-\frac{x^4}{4a}\right]_0^a}{\left[\sqrt{a}\frac{2}{3}x^{3/2}-\frac{x^3}{3a}\right]_0^a}$$

$$= \frac{a^3\left(\frac{2}{5}-\frac{1}{4}\right)}{a^2\left(\frac{2}{3}-\frac{1}{3}\right)} = \frac{3/20a}{1/3}=\frac{9a}{20}$$



$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\iint \rho y\,dxdy}{\iint \rho\,dxdy}$$

$$= \frac{\int_{x=0}^{a}\int_{x^2/a}^{\sqrt{ax}} ky\,dydx}{\int_{x=0}^{a}\int_{x^2/a}^{\sqrt{ax}} k\,dydx} = \frac{\int_0^a \frac{1}{2}[y]_{x^2/a}^{\sqrt{ax}}\,dx}{\int_0^a [y]_{x^2/a}^{\sqrt{ax}}\,dx}$$

$$= \frac{\frac{1}{2}\int_0^a x\left[ax-\frac{x^4}{a^2}\right]dx}{\int_0^a\left[\sqrt{ax}-\frac{x^2}{a}\right]dx}$$

$$= \frac{\frac{1}{2}\left[a.\frac{x^2}{2}-\frac{x^5}{5a^2}\right]_0^a}{\left[\sqrt{a}\frac{2}{3}x^{3/2}-\frac{x^3}{3a}\right]_0^a}$$

$$= \frac{a(3/10)}{2.\frac{1}{3}} = \frac{9a}{20}$$

Hence C.G. $(\bar{x},\bar{y}) = \left(\frac{9a}{20},\frac{9a}{20}\right)$

Prob.14 ***If $\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$, evaluate $\oint_C \vec{F}$ around a triangle ABC in the xy- plane wit 0), B(2, 0), C(2, 1) in the countercloc direction. Also find the value in the opp direction.***

Sol. The curve C consist three straight lines AB, BC an

$$\therefore \quad \oint_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r}$$

$$= I_1 + I_2 + I_3 \qquad \text{(say)}$$

Along C_1 : The straight line from $A(0, 0)$ to $B(2$ $y = 0 \Rightarrow dy = 0$ and x varies from 0 to 2.

Thus $\vec{r} = x\hat{i} \Rightarrow d\vec{r} = dx\,\hat{i}$

Then $I_1 = \int_{C_1}\vec{F}\cdot d\vec{r}$

$$= \int_{C_1}\{(2x+y^2)\hat{i} + (3y-4x)\hat{j}\}.\{\hat{i}\,dx\}$$

$\because \quad y = 0$

$$\therefore \quad I_1 = \int_{C_1}\vec{F}\cdot d\vec{r} = \int_0^2 \{2x\hat{i} + (-4x)\hat{j}\}.\{\hat{i}\,dx\}$$

$\Rightarrow \quad dx = 0$ and

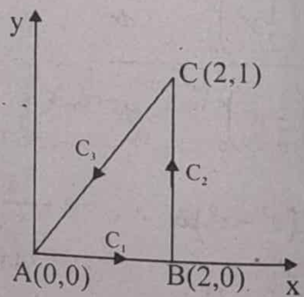


Fig.

$$= \int_0^2 2x\,dx = (x^2)_0^2 = 4$$

Along C_2 : The straight line from $B(2, 0)$ to $C(2, 1)$, $x =$ $dx = 0$ and y varies from 0 to 1.

Thus $\vec{r} = 2\hat{i} + y\hat{j} \Rightarrow d\vec{r} = \hat{j}\,dy$

Then $I_2 = \int_{C_2}\vec{F}\cdot d\vec{r}$

$$= \int_0^1\{(4+y^2)\hat{i} + (3y-8)\hat{j}\}.\{\hat{j}\,dy\}$$

$$= \int_0^1 (3y-8)\,dy = \left(\frac{3y^2}{2} - 8y\right)_0^1 = -\frac{13}{2}$$

Along C_3 : The straight line from $C(2, 1)$ to $A(0, 0)$ is given by $y=\frac{1}{2}\Rightarrow dy=\frac{1}{2}dx$ and x varies from 2 to 0.

Thus $\vec{r}=x\hat{i}+\frac{1}{2}x\hat{j} \quad \Rightarrow d\vec{r}=\left(\hat{i}+\frac{1}{2}\hat{j}\right)dx$

Then $I_3=\int_{C_3}\vec{F}\cdot d\vec{r}$

$$=\int_{C_3}\{(2x+y^2)\hat{i}+(3y-4x)\hat{j}\}.\left[\left\{\hat{i}+\frac{1}{2}\hat{j}\right\}dx\right]$$

$\because \quad y=\frac{1}{2}x$

$\therefore \quad I_3=\int_{C_3}\vec{F}\cdot d\vec{r}$

$$=\int_2^0\left\{\left(2x+\frac{x^2}{4}\right)\hat{i}+\left(\frac{3x}{2}-4x\right)\hat{j}\right\}.\left[\left\{\hat{i}+\frac{\hat{j}}{2}\right\}dx\right]$$

$$=\int_2^0\left(2x+\frac{x^2}{4}+\frac{3x}{4}-\frac{4x}{2}\right)dx$$

$$=\int_2^0\left(\frac{x^2}{4}+\frac{3x}{3}\right)dx=\left(\frac{x^3}{12}+\frac{3x^2}{8}\right)_2^0$$

$$I_3=-\frac{13}{6} \qquad \text{...(iv)}$$

Using eq. (ii), (iii) and (iv) in eq. (i), then the required line integral in the counter clockwise direction will be

$$\oint_C\vec{F}\cdot d\vec{r}=4-\frac{13}{2}-\frac{13}{6}=-\frac{14}{3}$$

The value of line integral in the opposite direction is $\frac{14}{3}$.

Prob.15 *Evaluate $\iint_S\vec{F}.\hat{n}ds$; where $\vec{F}=x^2\hat{i}+y^2\hat{j}+z^2\hat{k}$ and S is the part of plane $x+y+z=1$; which is located in the first octant.* **[R.T. U. 2015]**

Sol. Let S is the surface given by : $f(x, y, z)=x+y+z-1$

$\because$ ∇f is a vector perpendicular to the surface $f(x, y, z)$.

Therefore vector normal to the surface

$$\nabla f=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z-1)$$

$$\nabla f=\hat{i}+\hat{j}+\hat{k}$$

Hence unit normal vector $\hat{n}$ to the surface S is

$$\hat{n}=\frac{\nabla f}{|\nabla f|}=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)$$

The region of integration R is the projection of the surface S (plane ABC) on the xy-plane *i.e.* bounded by x-axis, y-axis and the line $x+y=1$, (say triangle OAB).

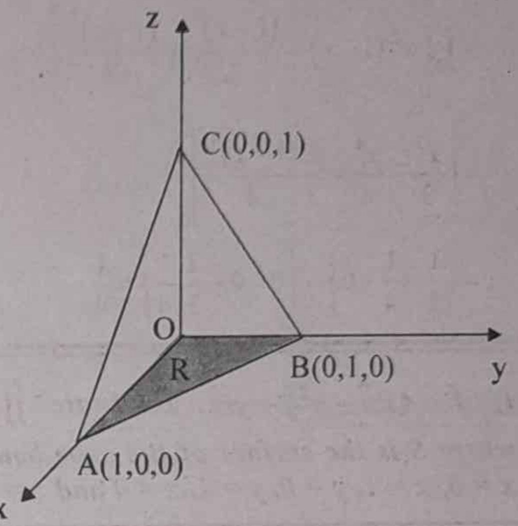


Fig.

Then $\iint_S\vec{F}.\hat{n}\,ds=\iint_R\vec{F}.\hat{n}\frac{dx\,dy}{|\hat{n}.\hat{k}|}$ ... (i)

Now $\vec{F}.\hat{n}=\left(x^2\hat{i}+y^2\hat{j}+z^2\hat{k}\right).\frac{\left(\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{3}}$

$$=\frac{x^2+y^2+z^2}{\sqrt{3}}$$

$$\Rightarrow \quad \vec{F}.\hat{n}=\frac{x^2+y^2+(1-x-y)^2}{\sqrt{3}} \qquad \text{... (ii)}$$

$[\because x+y+z=1]$

and $\hat{n}.\hat{k}=\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}=\frac{1}{\sqrt{3}}$ ... (iii)

Using eq. (ii) and eq. (iii) in eq. (i), we have

$$\iint_S\vec{F}.\hat{n}\,ds=\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{x^2+y^2+(1-x-y)^2}{\sqrt{3}}\right]\frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$=\int_{x=0}^{1}\left\{\int_{y=0}^{1-x}[x^2+y^2+(1-x-y^2)]dy\right\}dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} - \frac{(1-x-y)^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[\left\{ x^2(1-x) + \frac{(1-x)^3}{3} - \frac{(1-x-1+x)^3}{3} \right\} - \left\{ 0 + 0 - \frac{(1-x)^3}{3} \right\} \right] dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} + \frac{(1-x)^3}{3} \right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{2}{3}\frac{(1-x)^4}{4} \right]_0^1$$

$$= \left\{ \frac{1}{3} - \frac{1}{4} - 0 \right\} - \left\{ 0 - 0 - \frac{2}{3}.\frac{1}{4} \right\} = \frac{1}{4}$$

Prob.16 ***If*** $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$. ***Evaluate*** $\iint_S \vec{F}.\hat{n}\,ds$, ***where S is the surface of the cube bounded by*** $x = 0, x = 1, y = 0, y = 1, z = 0$ ***and*** $z = 1$.

Sol. Here the surface consists of 6 sub-surfaces.

S_1 : AFGO, S_2 : OGDC
S_3 : ABCO, S_4 : DEFG
S_5 : ABEF, S_6 : BCDE

Then $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds + \iint_{S_3} \vec{F}\cdot\hat{n}\,ds + \iint_{S_4} \vec{F}\cdot\hat{n}\,ds + \iint_{S_5} \vec{F}\cdot\hat{n}\,ds + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds$

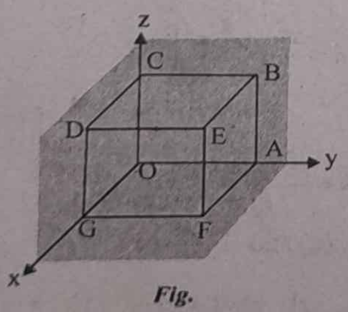


Fig.

$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6$ (say)... (i)

S_1 (Face AFGO) : Surface AFGO in *xy*-plane *i.e.* $z = 0$, unit outward normal to AFGO is $\hat{n} = -\hat{k}$

Then $I_1 = \iint_{AFGO} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right).\left(-\hat{k}\right)\frac{dx\,dy}{|-\hat{k}.\hat{k}|}$

$$= \int_0^1\int_0^1 -yz\,dx\,dy$$
$$= 0$$

S_2 (Face OGDC) : Surface OGDC in xz –plane *i.e.* y unit outward normal to S_2 is $\hat{n} = -\hat{j}$

Then $I_2 = \iint_{OGDC} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right).\left(-\hat{j}\right)\frac{dx\,dz}{|-\hat{j}.\hat{j}|}$

$$= \int_0^1\int_0^1 \left(4xz\hat{i}\right).\left(-\hat{j}\right) dx\,dz \quad (\because y$$
$$= 0$$

S_3 (Face ABCO) : Surface S_3 in *yz*-plane *i.e.* $x = 0$, outward normal to S_3 is $\hat{n} = -\hat{i}$.

Then $I_3 = \iint_{ABCO} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right).\left(-\hat{i}\right)\frac{dy\,dz}{|-\hat{i}.\hat{i}|}$

$$= \int_0^1\int_0^1 \left(-y^2\hat{j} + yz\hat{k}\right)\left(-\hat{i}\right) dy\,dz$$
$$= 0$$

S_4 (Face DEFG) : Surface S_4 is parallel to *yz*-plane $x = 1$, unit outward normal to S_4 is $\hat{n} = \hat{i}$.

Then $I_4 = \iint_{DEFG} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right).(\hat{i})\frac{dy\,dz}{|\hat{i}.\hat{i}|}$

$$= \int_0^1\int_0^1 \left(4z\hat{i} - y^2\hat{j} + yz\hat{k}\right).\left(\hat{i}\right) dy\,dz$$

$$= \int_0^1\int_0^1 4z\,dy\,dz = \int_0^1 4z\left\{\int_0^1 dy\right\} dz$$

$$= \int_0^1 4z.(1)dz = 2$$

S_5 (Face ABEF) : Surface S_5 is parallel to *xz*-plane and 1, unit outward normal to S_5 is $\hat{n} = \hat{j}$

$$I_5 = \iint_{ABEF} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right).\hat{j}\frac{dx\,dz}{|\hat{i}.\hat{j}|}$$

$$= \int_0^1\int_0^1 \left(4xz\hat{i} - \hat{j} + z\hat{k}\right).\hat{j}\;dx\,dz$$

$$= \int_0^1\int_0^1 -dx\,dz = \int_0^1\left\{\int_0^1 -dx\right\}dz = \int_0^1 -1\,dz$$

$$I_5 = -1$$

S_6 (Face BCDE) : Surface S_6 is parallel to *xy*- plane and 1, unit outward normal to S_6 is $\hat{n} = \hat{k}$

Then $I_6 = \iint_{BCDE} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right).\hat{k}\frac{dx\,dy}{|\hat{k}.\hat{k}|}$

$$I_6 = \int_0^1\int_0^1\left(4x\hat{i} - y^2\hat{j} + yz\hat{k}\right).\hat{k}\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 y\left\{\int_0^1 dx\right\}dy \qquad (\because z = 1)$$

$$I_6 = \frac{1}{2} \qquad \text{...(vii)}$$

Adding eq. (ii) to eq. (vii), and using in eq. (i), we have

$$\iint_S \vec{F}.\hat{n}ds = 0+0+0+2-1+\frac{1}{2} = \frac{3}{2}$$

Prob.17 *Evaluate* $\iint_S \vec{F}.\hat{n}ds$,

where $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$

and S is the closed surface of the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ ***and the planes*** $x = 0$, $x = 2$, $y = 0$ ***and*** $z = 0$.

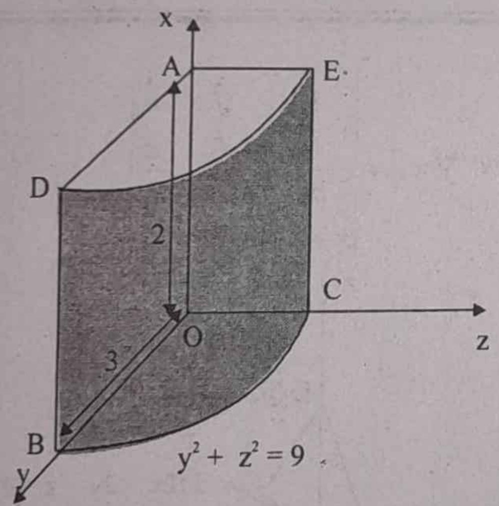


Fig.

Sol. The surface consists of 5 sub surfaces S_1 : rectangular face OADB, S_2 : rectangular face OAEC, S_3 : circular quadrant OBC in yz-plane S_4 : circular quadrant ADE and S_5 : curved surface DBCE.

Thus $$\iint_S \vec{F}.\hat{n}\,ds = \iint_{S_1}\vec{F}.\hat{n}\,ds + \iint_{S_2}\vec{F}.\hat{n}\,ds$$

$$+\iint_{S_3}\vec{F}.\hat{n}\,ds + \iint_{S_4}\vec{F}.\hat{n}ds + \iint_{S_5}\vec{F}.\hat{n}\,ds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 \qquad \text{...(i)}$$

S_1 (OADB) : The surface S_1 in xy-plane *i.e.* $z = 0$, unit outward normal to S_1 is $\hat{n} = -\hat{k}$

Then $I_1 = \iint_{S_1}\vec{F}.\hat{n}\,ds$

$$= \iint_{OADB}\left(2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}\right).(-\hat{k})\frac{dx\,dy}{\left|-\hat{k}\cdot\hat{k}\right|}$$

$$= \iint_{S_1} -4xz\,dz \qquad (\because z = 0)$$

$$\therefore \quad I_1 = 0 \qquad \text{...(ii)}$$

S_2 (OAEC) : The surface S_2 in xz-plane *i.e.* $y = 0$, unit outward normal to S_1 is $\hat{n} = -\hat{j}$.

Then $I_2 = \iint_{S_2}\vec{F}.\hat{n}\,ds$

$$= \iint_{OAEC}\left(2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}\right).\left(-\hat{j}\right)\cdot\frac{dx\,dz}{\left|-\hat{j}.\hat{j}\right|}$$

$$= \iint_{OAEC} y^2\,dx\,dz \qquad (\because y = 0)$$

$$\therefore \quad I_2 = 0 \qquad \text{...(iii)}$$

S_3 (OBC) : The section OBC in yz-plane *i.e.* $x = 0$, the unit outward normal to S_3 is $\hat{n} = -\hat{i}$

$$\therefore \quad I_3 = \iint_{S_3}\vec{F}.\hat{n}\,ds$$

$$= \iint_{OBC}\left(2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}\right).\left(-\hat{i}\right)\frac{dy\,dz}{\left|-\hat{i}.\hat{i}\right|}$$

$$= \iint_{OBC} -2x^2y\,dy\,dz = 0 \quad (\because x = 0) \quad \text{...(iv)}$$

S_4 (ADE) : The section ADE is parallel to yz-plane and $x = 2$, the unit outward normal to S_4 is $\hat{n} = \hat{i}$

Then $I_4 = \iint_{S_4}\vec{F}.\hat{n}\,ds$

$$= \iint_{ADE}\left(2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}\right).(i)\frac{dy\,dz}{\left|\hat{i}.\hat{i}\right|}$$

$$= \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}} 2x^2y\,dy\,dz \qquad (\because x = 2)$$

$$= \int_{z=0}^{3}\left\{\int_{y=0}^{\sqrt{9-z^2}} 8y\,dy\right\}dz$$

$$= \int_{z=0}^{3} 4\left(\sqrt{9-z^2}\right)^2 dz = \int_{z=0}^{3} 4\left(9-z^2\right)dz$$

$$= \int_{z=0}^{3}\left(36-4z^2\right)dz = \left(36z-\frac{4z^3}{3}\right)_0^3 = 72$$

...(v)

S_5 (DBCE) The curve surface S_5 is given by $f = y^2 + z^2 - 9$, the unit outward normal is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2y\hat{j}+2z\hat{k}}{\sqrt{4y^2+4z^2}} = \frac{y\hat{j}+z\hat{k}}{3}$$

and $\quad ds = \dfrac{dx\,dy}{|\hat{n}.\hat{k}|} = \dfrac{dx\,dy}{\frac{z}{3}}$

(Region of integration is the projection of the surface S_5 on xy-plane)

$$\therefore \quad I_5 = \iint_{S_5}\vec{F}.\hat{n}\,ds = \iint_{OADB}\vec{F}.\hat{n}.\frac{dx\,dy}{\frac{z}{3}}$$

$$= \iint_{OABD}\left(2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}\right).\frac{\left(y\hat{j}+z\hat{k}\right)}{3}\frac{dx\,dy}{\frac{z}{3}}$$

$$= \int_{x=0}^{2}\int_{y=0}^{3}\left(-y^3+4xz^3\right)\frac{dx\,dy}{z}$$

Putting $y = 3\sin\theta$, $z = 3\cos\theta$, we have
($\because y^2 + z^2 = 9$, so $y = 3\sin\theta$, $z = 3\cos\theta$)
parametric eq.

$$= \int_{x=0}^{2}\left\{\int_{\theta=0}^{\pi/2}\left[\frac{-27\sin^3\theta}{3\cos\theta} + 4x\left(9\cos^2\theta\right)\right]3\cos\theta\,d\theta\right\}dx$$

$$= \int_{x=0}^{2}\left\{\int_{\theta=0}^{\pi/2} -2\pi\sin^3\theta\,d\theta + x\int_{\theta=0}^{\pi/2}108\cos^3\theta\,d\theta\right\}dx$$

$$\boxed{\int_0^{\pi/2}\cos^m\theta\sin^n\theta\,d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}}$$

$$\therefore \quad I_5 = \int_{x=0}^{2}\left\{\frac{-27\,\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{3+0+2}{2}\right)} + 180x\frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{3+0+2}{2}\right)}\right\}dx$$

$$= \int_0^2\left\{-27\frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{5}{2}\right)} + 180x\frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{5}{2}\right)}\right\}dx$$

$$= \int_0^2\left\{-27\frac{1\cdot\Gamma(1)\,\Gamma\left(\frac{1}{2}\right)}{2\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} + 180x\frac{1\cdot\Gamma(1)\,\Gamma\left(\frac{1}{2}\right)}{2\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\right\}dy$$

$$= \int_0^2\left\{\frac{-27\times 2}{3} + \frac{180x\times 2}{3}\right\}dx$$

$$= 108$$

Using eq. (ii), (iii), (iv), (v) and eq. (vi) in eq. (i),

Hence $\iint_S \vec{F}.\hat{n}\,ds = 0+0+0+72+108 = 180$

Prob.18 ***Evaluate $\iiint_V \nabla\times\vec{A}\,dV$,***

where $\vec{A} = (x+2y)\hat{i} - 3z\hat{j} + x\hat{k}$ and V is closed region in the first octant bounded by plane $2x + 2y + z = 4$.

Sol.

$$\text{Now} \quad \nabla\times\vec{A} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+2y & -3z & x\end{vmatrix}$$

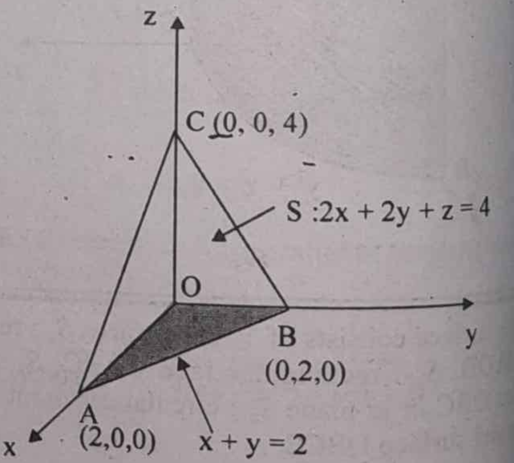


Fig.

$$= \hat{i}\left[\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(-3z)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x+2y) - \frac{\partial}{\partial x}(x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-3z) - \frac{\partial}{\partial y}(x+2y)\right]$$

$$= 3\hat{i} - \hat{j} - 2\hat{k}$$

$$\therefore \quad \iiint_V \nabla\times\vec{A}\,dV = \iiint_V\left(3\hat{i} - \hat{j} + 2\hat{k}\right)dx\,dy\,dz$$

Here z varies from 0 to $4-2x-2y$
y varies from 0 to $2-x$
x varies from 0 to 2.

$$\iiint_V \nabla\times\vec{A}dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}\left(3\hat{i}-\hat{j}+2\hat{k}\right)dx\,dy\,dz$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}\left\{\int_{z=0}^{4-2x-2y}\left(3\hat{i}-\hat{j}+2\hat{k}\right)dz\right\}dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}\left(3\hat{i}-\hat{j}+2\hat{k}\right).(z)_0^{4-2x-2y}\,dxdy$$

$$=\left(3\hat{i}-\hat{j}+2\hat{k}\right)\int_{x=0}^{2}\left\{\int_{y=0}^{2-x}(4-2x-2y)dy\right\}dx$$

$$=2\left(3\hat{i}-\hat{j}+2\hat{k}\right)\int_{x=0}^{2}\left\{\int_{y=0}^{2-x}(2-x-y)dy\right\}dx$$

$$=2\left(3\hat{i}-\hat{j}+2\hat{k}\right)\int_{x=0}^{2}\left[(2-x)y-\frac{y^2}{2}\right]_0^{2-x}dx$$

$$=\left(3\hat{i}-\hat{j}+2\hat{k}\right)\int_{x=0}^{2}\left[2(2-x)^2-(2-x)^2\right]dx$$

$$=\left(3\hat{i}-\hat{j}+2\hat{k}\right)\int_{x=0}^{2}(2-x)^2dx$$

$$=\left(3\hat{i}-\hat{j}+2\hat{k}\right)\left(\frac{(2-x)^3}{-3}\right)_0^2$$

$$=\frac{8}{3}\left(3\hat{i}-\hat{j}+2\hat{k}\right)$$

Prob.19 ***Find the volume enclosed between the two surfaces $S_1 : z = 8 - x^2 - y^2$ and $S_2 : z = x^2 + 3y^2$***

Sol. Eliminating z from the given two surfaces S_1 and S_2, then

$8-x^2-y^2 = z = x^2+3y^2$ *i.e.*
$x^2+2y^2 = 4.$

Hence the given two surface S_1 and S_2 intersect on $x^2 + 2y^2 = 4$

So the solid region between S_1 and S_2 is covered when z varies from x^2+3y^2 to $8-x^2-y^2$,

y varies from $-\sqrt{\dfrac{4-x^2}{2}}$ *to* $\sqrt{\dfrac{4-x^2}{2}}$

and x varies from -2 to 2

Hence the required volume V enclosed between the two surface S_1 and S_2 is given by

$$V=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2}dx\,dy\,dz$$

$$=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\left\{\int_{x^2+3y^2}^{8-x^2-y^2}dz\right\}dx\,dy$$

$$=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\left\{8-x^2-y^2-x^2-3y^2\right\}dx\,dy$$

$$=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\left(8-2x^2-4y^2\right)dy\,dx$$

$$V=\int_{-2}^{2}\left[\left(8-2x^2\right)y-\frac{4y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}dx$$

$$V=\int_{-2}^{2}\left[2\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}}\right]dx$$

$$=\int_{-2}^{2}\left[\frac{4\left(4-x^2\right)^{\frac{3}{2}}}{\sqrt{2}}-\frac{4}{3}\frac{\left(4-x^2\right)^{\frac{3}{2}}}{\sqrt{2}}\right]dx$$

$$=\frac{8}{3\sqrt{2}}\int_0^2\left(4-x^2\right)^{\frac{3}{2}}dx$$

Put $x^2 = 4t2x\ dx = 4\ dt$

Then $$V=\frac{8}{3\sqrt{2}}\int_0^1(4-4t)^{\frac{3}{2}}\frac{2dt}{2\sqrt{t}}$$

$$=\frac{8}{3\sqrt{2}}\int_0^1 8(1-t)^{\frac{3}{2}}\frac{dt}{\sqrt{t}}$$

$$=8\pi\sqrt{2}$$

Prob.20 ***Evaluate $\iint_S \vec{A}.\hat{n}\,ds$, where $\vec{A}=z\hat{i}+x\hat{j}-3y^2z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z = 0$ and $z = 5$.*** ***[R.T. U. 2008]***

Sol. Let S be the surface given by $f(x, y, z) = x^2 + y^2 - 16$, in the first octant between $z = 0$ and $z = 5$

$\because$ ∇f is a vector perpendicular to the surface $f(x, y, z)$

$\therefore$ Vector normal to the surface is

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 - 16\right)$$

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$

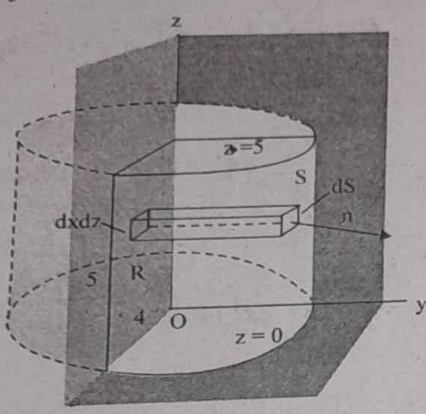


Fig.

The unit normal vector to S is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(2x^2) + (2y)^2}} = \frac{x\hat{i} + y\hat{j}}{4}$$

The region of integration is the projection of S on xz-plane *i.e.* bounded by x-axis, z-axis, and the line $z = 5$ and $x =$ 4. Thus the surface integral

$$\iint_S \vec{A}.\hat{n}\, ds = \iint_R \vec{A}.\hat{n}\frac{dx\,dz}{|\hat{n}.\hat{j}|}$$

Now $\vec{A}.\hat{n} = \left(z\hat{i} + x\hat{j} - 3y^2z\hat{k}\right)\left(\frac{x\hat{i} + y\hat{j}}{4}\right) = \frac{1}{4}(xz + xy)$

and $\hat{n}.\hat{j} = \frac{x\hat{i} + y\hat{j}}{4}.\hat{j} = \frac{y}{4}$

$$\therefore \iint_S \vec{A}.\hat{n}\, ds = \int_{x=0}^{4}\int_{z=0}^{5} \frac{1}{4}(xz + xy)\frac{dxdz}{\frac{y}{4}}.$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16 - x^2}} + x\right)dz\,dx$$

$$(\because y^2 = 16 - x^2)$$

$$= \int_{z=0}^{5}\left[\left\{\int_{x=0}^{4}\frac{xz}{\sqrt{16 - x^2}} + x\right\}dx\right]dz$$

$$= \int_{z=0}^{5}\left\{-z\sqrt{16 - x^2} + \frac{x^2}{2}\right\}_0^4 dz$$

$$= \int_{z=0}^{5}(4z + 8)\,dz$$

$$= \left[\frac{4z^2}{2} + 8z\right]_0^5 = \left[\frac{4(5)^2}{2} + 8(5)\right] = 90$$

□□□