Opinion and self-confidence in influence networks: a coupled dynamics model

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Abstract. In this paper, we develop a novel model of opinion dynamics that takes into account the variability of self-confidence. This agent-based model aims at combining both opinion and self-confidence, two notions often assimilated in many modeling works, although well distinct in practice. We analyze our model under two types of opinion space, namely a standard interval and a circular structure under which the system presents some intriguing specificity. We give some insights concerning the global equilibria of the co-evolving dynamics, and also highlight some salient features of the global system's trajectory. Finally, we conduct several numerical simulations to support the analytical results.

Keywords: opinion dynamics · bounded-confidence· influence networks.

1 introduction

1.1 Research context and motivation

In many opinion dynamics models, the variable of opinion is taken as a simple scalar value. But since in general, opinion is a complex and fuzzy system of beliefs and reasoning, it can hardly be projected into a simple interval. Then, one shall ask whether the scalar value presented as an "opinion" in the literature is finally nothing else but a confidence degree (with respect to an underlying social norm or behaviour not explicitly defined). To this title, some authors [1,2] propose models where agents hold an opinion on each topic, and the different opinions of the same agent can be structured and influence each other. Hence, the two concepts of *opinion* and *confidence* are confusedly assimilated. The current work aims at developing a framework where the distinction is clearly done: each agent is endowed with an opinion value (which can be in the interval or in a cyclic structure), *plus* a scalar value comprised between 0 and 1, measuring how confident the agent is in his own opinion.
To define the opinion dynamics (OD) structure, we rely on the well-established

bounded-confidence (BC) interaction, also known under the name of *Hegselman-Krause model* (HK model for short). This socially-inspired mechanism aims at encapsulating both *homophily* and *selective exposure*, two natural psycho-social trends one may follow to reduce cognitive dissonance. Since the pioneering works of Rainer and Krause [3], the BC model has been the scope of intensive study (see e.g. [4,5] and references therein), even until very recently [6], and by researchers coming from various horizons. The interested reader can easily refer to the recent survey of C. Bernardo and her coauthors [7].
Moreover, we also analyse the coupling of opinion and self-confidence in the case where the opinion space is *toric*. This modelling has the main advantages to preserve a total symmetry between all the opinions, and to prevent the *border-effect*.

1.2 Related literature

Circular opinion has already been investigated in the past. In [8] and [9], the authors opt for a statistical-physics approach and combine attractive and repelling mechanism. In [10], the problem of convergence time of the HK model *on the circle* is tackled via rigorous and involved mathematical methods. But none of these already existing works encompass in their frame a self-confidence dynamics. This concept of self-confidence is indeed very close to *social power* or *influence*, notions which have been already investigated: authors of [11] indeed consider a dynamical social power furnished to each agent, combined with stubbornness. Nevertheless, they rather focus on network evolution and convergence. Authors of [12] pursue in the same vein but here consider a multiplicity of topics. Some authors indeed mention self-confidence and make a clear distinction with the opinion variables, but the self-confidence is taken constant along time [13]. In his notes of 2016 [14], C. Wang considers dynamical self-confidence, but is designed as an autonomous process, which does not depend of the opinion variables. Finally, to the best of our knowledge, the explicit combination of opinion and self-confidence with an emphasis on the interaction of the two variables still has yet to be explored.

1.3 Paper scope and organisation

The goal of this study is to investigate the impact of the self-confidence on the opinion when the former is driven by a dynamics induced by a *network effect*, while the latter is driven by a BC dynamics. In the case we are interested in, we assume that two agents with a low self-confidence are close one to each other, whatever their respective opinion. At the extreme case, two agents with zero self-confidence coincide, whatever their opinion. It is indeed this assumption which acknowledges the very nature of *self-confidence*, and then gives rise to the (semi-)ball structure. If we do not make this assumption, then we simply obtain as phase space a rectangle instead of a semi-ball, and a cylinder's border instead of a ball. In Section 2, we formally introduce the multi-agent model coupling

opinion with self-confidence. Then, the following section 3 is devoted to its analysis, including an additional variant: the opinion space taken as a *toric* structure (namely the circle $\mathbb{C}$). Finally, the last section 4 before concluding consists of a substantial amount of numerical experiments illustrating the analytical investigations and pave the way for future research.
Notations. For any $K \geq 1$, we write $[K]$ to shorten $\{1, 2, ..., K\}$. We write $\mathbb{1}_U$ for the indicator map of event U, and for any positive scalar $h > 0$ we write $\Delta_h x(t) := x(t+h) - x(t)$ the difference operator. We denote the unitary circle as $\mathbb{C} := \{(x, y) \in \mathbb{R}^2 : ||(x, y)||^2 = 1\}$, the ball $\mathbb{B} := \{(x, y) : ||(x, y)||^2 \leq 1\}$, and the semi-ball $\mathbb{B}_+ := \mathbb{B} \cap \big(\mathbb{R} \times \mathbb{R}_+\big)$. Finally, $|P|$ stands for the cardinality of any set P, $\perp\!\!\!\perp$ stands for the symbol of independence between two random variables, and $\lambda(.)$ is the Lebesgue measure.

2 the model

We define a population of agents of size $K \in \mathbb{N}^*$, where each agent is indexed by an integer $k \in [K]$. The time step is fixed and small: $h \ll 1$. The time axis is taken discrete along the set $\mathbb{N}h$, and at each time point $t \in \mathbb{N}h$, for each agent $k \in [K]$ is assigned an opinion value $\theta_k(t)$. In the first part of the analysis 3.1, we take the simple interval $[0, \pi]$. Subsequently, we extend the model to the circle $\mathbb{C} := [0, 2\pi]\big/_{0 \sim 2\pi}$. In this latter case, the curbless structure of $\mathbb{C}$ gives birth to interesting changes in the model's behaviour. In addition to his opinion $\theta_k(t)$, agent k is also endowed with a scalar value, standing for his *self-confidence* degree in his own opinion. We note it $\rho_k(t) \in [0, 1]$. Hence, one can represent the global state of agent k in polar coordinates: $u_k(t) := (\theta_k(t), \rho_k(t)) \in \mathbb{B}_+$ in the case of the semi-ball (resp. $\mathbb{B}$ for the full circle case studied in subsection 3.2). As mentioned in the last subsection of the introduction 1.3, it is precisely the (semi-)ball structure which makes the zero state the only equidistant point of the arc of maximal confidence. Mathematically, one can see the semi-ball (respectively the ball) as the simple rectangle $[0, \pi] \times [0, 1]$ (respectively $\mathbb{C} \times [0, 1]$) where all the points of the horizontal line (respectively circle-shaped section) $\{(\theta, \rho) : \rho = 0\}$ has merged to form one single point.
Since the self-confidence (written as a radius) and the opinion (written as an angle) co-evolve, the confidence profile $(\rho_j(t))_j$ appears in the evolution equation of the opinion $\theta_k(t)$ for agent k and especially his own confidence $\rho_k(t)$. First, define the *radius map* R and the directed *influence network* represented by the matrix-valued map ϕ, two structural elements in the dynamics:

$$R : \rho \in [0, 1] \mapsto R(\rho) := c_0 \pi (1 - \rho),\ c_0 \in [0, 1], \text{ and}$$
$$\phi : (\mathbb{B}_+)^K \mapsto \{0, 1\}^{K^2} \text{ as } \phi_{kj}(u) := \mathbb{1}_{\big\{|\theta_k - \theta_j| < R(\rho_k)\big\}},$$

$\forall u \in (\mathbb{B}_+)^K$. The scalar c_0 is a parameter defined beforehand. Hence, if $\phi_{kj} = 1$, then k is influenced by j. Notice that the map R decays according to the confidence variable, and delimits the zone of influence. Let us also define the *neighbour set*. For a given configuration $u \in (\mathbb{B}_+)^K$, denote by $\mathcal{N}_k(u)$ the set of neighbours of agent $k \in [K]$: $\mathcal{N}_k(u) := \Big\{ j \in [K] : |\theta_j - \theta_k| < R(\rho_k) \Big\}$. When it is clear from the context, we write $\mathcal{N}_k(t)$ instead of $\mathcal{N}_k(u(t))$ for brevity, and respectively $\phi_{kj}(t)$ instead of $\phi_{kj}(u(t))$. Thus, we are now able to state the global (θ, ρ)-dynamics:

$$\Delta_h \theta_k(t) = \frac{h}{K} \sum_j \rho_j (1 - \rho_k) \phi_{kj}(t)(\theta_j - \theta_k), \tag{1}$$

$$\Delta_h \rho_k(t) = h \left(\frac{|\mathcal{N}_k(t)|}{K} - \rho_k(t) \right). \tag{2}$$

The first equation corresponds to a classical BC dynamics with $\rho_j(1 - \rho_k)$ as weighting coefficients. This refinement seems indeed intuitive: the more agent j is self-confident, the more persuasive is she. On the contrary, the more agent k is self-confident, the less steerable is she. This is why this coefficient grows with ρ_j and decays with ρ_k.
The second equation of the dynamics relies on a standard herding behaviour known as *network effect*: each agents gains in self-confidence as long as his neighbour size increases. His ideological proximity with alter egos tends to reinforce him in his own opinion. By contrast, an isolated agent has low self-confidence, and remains able to be influenced. In accordance with the network-effect formalism [15, 16], the self-confidence may be interpreted as the *utility to be connected* to others.

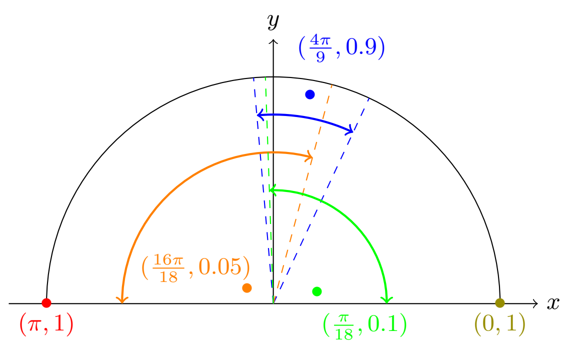


Fig. 1: A simple scheme to explain dynamics (1) with 5 agents and their respective radius of influence.

Figure 1 illustrates a situation with $K = 5$ agents, represented in coloured dots in B_+ whose associated tuples are their respective polar coordinates. The red and brown point respectively represent a left radical and right radical, both with maximal confidence. The blue point correspond to a quasi-centrist agent with high (but not maximal) confidence. On the contrary, the orange and green agents are rather extreme, but with a low self-confidence. This explains why their

respective influence zone is larger than the blue one: it is the zone delimited by two dashed segments of the same colour. We thus can see that the blue agent both influences the green and the orange one, but the reciprocal is false. Finally, the maximally-confident agents (the red and brown) cannot be influenced since $R(1) = 0$.
The choice of the initial agents states impacts the system's dynamics. In the rest of the analysis, we consider an *independent and identically distributed* (i.i.d) sequence of tuples: $(\theta_k(0), \rho_k(0))_{k\in[K]} \sim_{\text{i.i.d}} \theta_{\text{in}} \otimes \rho_{\text{in}}$, where the angular and radius distribution are also independent: $\theta_k(0) \perp\!\!\!\perp \rho_k(0)$ for all $k \in [K]$.

Remark 1. Note that if we simply take $(\theta_k, \rho_k)_k \sim_{\text{i.i.d}} \text{Unif}[0, \pi]\otimes\text{Unif}[0, 1]$, then a concentration on the center occurs, as illustrated figure 2a. In order to construct a sample of K i.i.d agents uniformly distributed on the semi-ball B_+, one shall use the sample $(\theta_k, \sqrt{\rho_k})_k$, where $(\theta_k, \rho_k)_k \sim_{\text{i.i.d}} \text{Unif}^{\otimes 2}[0, \pi] \times [0, 1]$. For a proof, see the appendix.

3 Analysis

3.1 General analysis

First, we characterise the rest points of dynamics (1). Relying on the analysis of the classical BC dynamics, which is very similar to the first line of (1), one can assume that clusters emerge in finite time, i.e. there exists a time $T_{\text{clust}} > 0$ such that for any $t > T_{\text{clust}}$, there exists an integer $P \geq 1$ (the actual number of clusters) and a map (assigning to each agent the cluster she belongs to) $\mathfrak{C} : [K] \mapsto [P]$ such that $\mathfrak{C}(k) = \mathfrak{C}(j) \iff |\theta_k(t) - \theta_j(t)| < \min\big((R(\rho_j(t)), R(\rho_k(t))\big)$. In such a clustered configuration, the influence network map ϕ is symmetric and block-wise diagonal, whose blocks are complete graphs, and the size of the blocks correspond to the size of the clusters. We then have two cases according to the number of clusters.

- *one single cluster:* $P = 1$. In that case, we have $|\mathcal{N}_k(T_{\text{clust}})| = K \ \forall k$, implying that
$$\Delta_h \rho_k(t) = h(1 - \rho_k(t)) \text{ for } t \geq T_{\text{clust}} \implies \rho_k(t) \approx (1 - C_k e^{-t}).$$
However, when $\rho_k(t)$ becomes almost 1, the graph becomes disconnected leading to ρ_k decreasing again. This in turn, makes the graph reconnected and so forth. Thus, we expect a slower convergence speed to exact consensus, i.e., $\lim_{t\to+\infty}(\theta_j - \theta_k)(t) = 0$ for all $k, j \in [K]$ asymptotically.
- *several clusters:* in the case where $P \geq 2$, then we have $\frac{|\mathcal{N}_k(t)|}{K} < 1$, and due to dynamics (1) we have
$$\rho_k(t) \underset{t\longrightarrow+\infty}{\longrightarrow} \rho_k^* := \frac{|\mathcal{N}_k(T_{\text{clust}})|}{K}, \tag{3}$$

where the convergence occurs exponentially fast from the time instant T_{clust}. Note that in the case where the clusters are of equal size, we have $\rho_k^* = \frac{1}{P}$ for all $k \in [K]$. After clustering and quasi convergence of $\rho_k(t)$, we then get

$$\begin{aligned}
\Delta_h \theta_k(t) &\approx \frac{(\rho_j^*)(1-\rho_k^*)}{K} \sum_{j \in \mathfrak{C}(k)} (\theta_j - \theta_k)(t), \\
\implies \forall j \in \mathfrak{C}(k), &\quad (\theta_j - \theta_k)(t) \underset{t \to +\infty}{\longrightarrow} 0.
\end{aligned}$$

The last convergence is due to the fact that the interaction graph of the variables $(\theta_i, \theta_j)_{i,j \in \mathfrak{C}(k)}$ is complete.

3.2 Circle case

We now consider the *full circle* (FC) case, that is $\theta_k(t) \in \mathbb{C}$. Choosing the circle $\mathbb{C}$ presents two main modelling advantages: first, there is neither extreme opinions nor central ones, whose connotation is not neutral. More precisely, extreme opinions can be implicitly considered as *negative*, while on the contrary central or *moderate* opinions can be considered as *positive*. A toric modeling of the opinion space is then an interesting way to evacuate this kind of preconceptions. In the FC case, all opinions play a symmetric role, and there is no "natural convergence point" supposed to be in the center. There are indeed many situations where there is no extreme viewpoints, despite an area for disagreement. Second, in many models where the opinion space is an interval, the consensus is reached in the center, which clearly resembles to a modelling artefact of type *edge effect*. Over the circle, this effect does not occur. This topological modification of the opinion space requires to redefine the notion of subtraction and distance, which is done in the next paragraph.
Giving two angles θ_k and θ_j, there are always two paths to reach θ_j from θ_k: the first path d_0 crosses 0, while the other $d_\emptyset$ does not. The latter is of length $d_\emptyset = |\theta_k - \theta_j|$, while the former is of length $d_0 = 2\pi - |\theta_k - \theta_j|$. We can now define the proper subtraction of angles:

$$(\theta_j -_{\mathbb{C}} \theta_k) := \begin{cases} \text{sign}(\theta_k - \theta_j) d_0 & \text{if } d_0 < \pi, \text{ and} \\ (\theta_j - \theta_k) & \text{otherwise.} \end{cases} \tag{4}$$

More simply, we write $d_{\mathbb{C}}(\theta_k, \theta_j)$ for the distance in the circle: $d_{\mathbb{C}}(\theta_k, \theta_j) := \min(d_0, d_\emptyset)$.

In the FC case, all subtraction of angles are made according to the preceding formula (4). Specifically, we then have $\phi_{kj} = \mathbb{1}_{\{d_{\mathbb{C}}(\theta_k, \theta_j(t)) < R(\rho_k)\}}$ and $\mathcal{N}_k = \{j : d_{\mathbb{C}}(\theta_k, \theta_j) < R(\rho_k)\}$.

Although the full-circle case displays a similar behaviour to the semi-circle case, there is indeed one specificity which deserves to be revealed: a *temporal phase separation* between the two variables θ and ρ. Taking $(\theta_{\text{in}}, \rho_{\text{in}}) = \text{Unif}(\mathbb{B})$, there exists a time instant T_{ps} such that

$$\Delta_h \theta_k(t) = 0, \forall t \leq T_{\text{ps}}, \text{ with } T_{\text{ps}} = \inf\{t \geq 0 : \Delta_h \rho_k(t) = 0\},$$

for all $k \in [K]$. To see this, let us compute the opinion dynamics of a single agent from the initial configuration $(\theta_{\rm in}, \rho_{\rm in}) \in \mathrm{Unif}(\mathbb{C}) \otimes [0,1]$. Then, we have the following convergence as the number of agents tends to be infinite:

$$\lim_{K \to +\infty} \big(\theta_j, \rho_j\big)_{j \in [K]} = \frac{d\theta d\rho}{2\pi} \text{ in law,} \tag{5}$$

by Glivencko-Cantelli theorem. Next, for $k \in [K]$ and $(\theta_k(0), \rho_k(0))$ being fixed, we can write

$$\frac{1}{h}\Delta_h \theta_k(t) = \frac{1}{K}\sum_j \rho_j(1-\rho_k)\mathbb{1}_{\{d_{\mathbb{C}} < R(\rho_k)\}}(\theta_j -_{\mathbb{C}} \theta_k)(t) \approx$$
$$(1-\rho_k)\int_{\rho=0}^{\rho=1} \rho \int_{\theta \in \mathbb{C}} \mathbb{1}_{\{d_{\mathbb{C}}(\theta_k,\theta) < R(\rho_k)\}}(\theta -_{\mathbb{C}} \theta_k)\frac{d\theta d\rho}{2\pi},$$

where the second equation relies on the asymptotics (5) mentioned above, with in addition K to be large. Furthermore, we have for any pair (θ_k, ρ_k)

$$\int_{\theta \in \mathbb{C}} \mathbb{1}_{\{d_{\mathbb{C}}(\theta,\theta_k) < R(\rho_k)\}}(\theta -_{\mathbb{C}} \theta_k) d\theta = \int_{\theta_k - R(\rho_k)}^{\theta_k + R(\rho_k)} (\theta - \theta_k) d\theta = 0.$$

We can then conclude that for any $h > 0$ and any initial state $(\theta_k(0), \rho_k(0))$, we have $\frac{1}{h}\Delta_h \theta_k(t) \equiv 0$, provided the entrance law for θ is uniform over the circle. Nevertheless, one has to keep in mind that the uniform density $\mathrm{Unif}(\mathbb{C})$ is an unstable equilibrium measure, this is why in all numerical experiments presented in the next section, since K is always large but finite, fluctuations arise and the angular symmetry is broken.

As we have shown that the opinion remains constant at least during the first phase of the process, it is then possible to estimate the *pseudo radius equilibrium* (PRE) $\rho^\star$, which is clearly different of the *true* limiting value ρ^* defined in equation (3) above. Concretely, $\rho^\star$ is the limiting value if we consider a *continuum of agents* written as a density map instead of a large but finite number. This value can be obtained by a repeated use of equation (5):

$$\frac{|\mathcal{N}_k(t)|}{K} = \frac{1}{K}\sum_j \phi_{kj} = \int_{\theta \in \mathbb{C}} \mathbb{1}_{\{d_{\mathbb{C}}(\theta,\theta_k) < R(\rho_k(t))\}}\frac{d\theta}{2\pi} = \int_{\theta_k - R(\rho_k)}^{\theta_k + R(\rho_k)} \frac{d\theta}{2\pi} = \frac{R(\rho_k)}{\pi}.$$

Thus, by definition of R, $R(\rho) = c_0\pi(1-\rho)$ for $c_0 > 0$, then the PRE for any agent $k \in [K]$ satisfies the following:

$$\frac{1}{h}\Delta_h \rho_k^\star = 0 \iff \frac{\mathcal{N}_k}{K} - \rho_k^\star = 0 \iff c_0(1-\rho_k^\star) = \rho_k^\star \iff \rho_k^\star = \frac{c_0}{1+c_0}.$$

This convergence result is finely corroborated by the numerical experiments presented in the next section.

To avoid the phase separation described above, one shall consider an alternative initial distribution for the angular variables:

$$\theta_k(0) \sim_{\text{i.i.d}} f_n : x \in [0, 2\pi] \mapsto \frac{1+\cos(nx)}{2\pi}, \tag{6}$$

for some $n \geq 0$. It is easy to show that f_n is a density map supported on $[0, 2\pi]$. Note that for $n = 0$ and $n = +\infty$, we retrieve the uniform law. We have conducted an experiment with such a non uniform law, and the result is given figure 3b.

4 Numerical Simulations

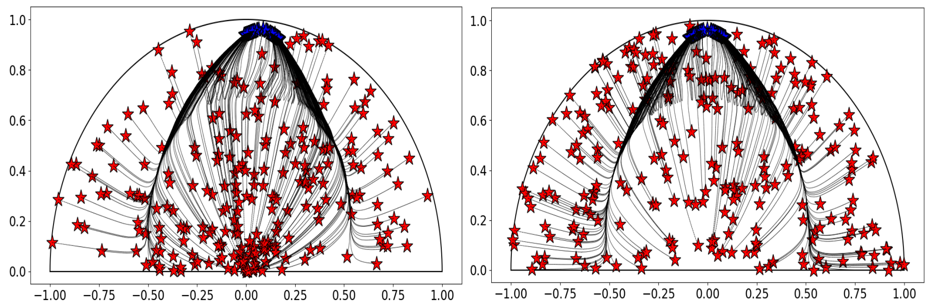
In this section, we present a sample of the results obtained from numerical experiments, conducted with various parameters: the first part 4.1 deals with the semi-ball case, whereas the second part 4.1 shows the system's behaviour when the opinion is circular-valued. We systematically take as population size $K = 250$ and step size $h = 0.01$. On the one hand, we took $c_0 = 1$ in the definition of the influence radius map R for the semi-circle case, while on the other hand, we took $c_0 = \frac{1}{2}$ for the FC case. We represent the initial configurations with red stars, and the final ones with blue stars. In the three first cases, it is interesting to notice that all the agents seem to slide across an oval-shaped curve.

4.1 The semi-circle case

In the next two figures, the opinion variables evolve in the interval: $\theta_k(t) \in [0, \pi]$ for all $k \in [K]$, and at all time instant $t \geq 0$. The initial angular distribution is the same in the two cases: $\theta_{\text{in}} = \text{Unif}[0, \pi]$, but two types of entrance laws for the self-confidence distribution ρ_{in} are tested: in the first case (figure 2a), we took $\rho_k(0) \sim_{\text{i.i.d}} \text{Unif}[0, 1]$. We can therefore observe a concentration on the the center at initial time. In the second case (figure 2b), we use the spatially uniform distribution $(\theta_k, \rho_k) \sim_{\text{i.i.d}} \text{Unif}(\mathbb{B}_+)$. In both cases, a quasi-consensus seems to occur around the value $(\theta^*, \rho^*) \approx (\frac{\pi}{2}, 1)$. At equilibrium, all the agents get a maximum confidence because consensus is reached, as depicted in the preceding section. See also that in this case, the convergence is slow due to the exponentially decaying terms $(1 - \rho_k(t))$ in the angular dynamics. Since the entrance law θ_{in} is uniform, the limiting opinion θ^* is unsurprisingly around the center of the interval $[0, \pi]$, whence the interest for the full-circle case.

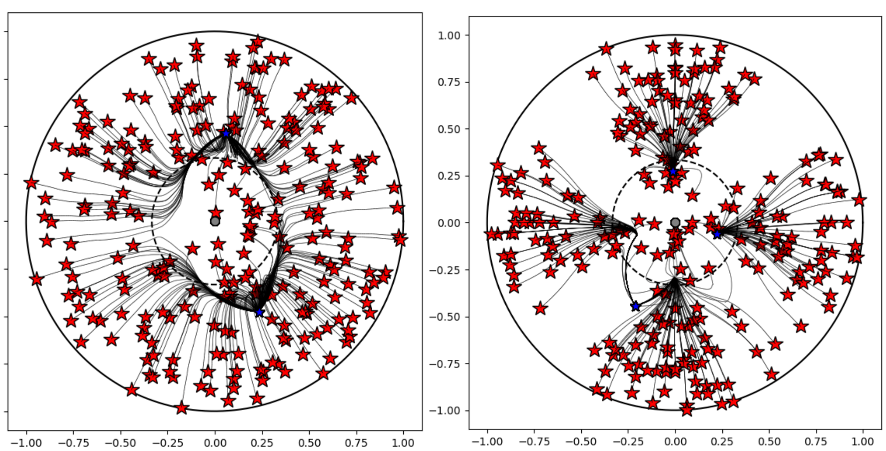
4.2 The full circle case

The last two figures display the coupled dynamics when the opinion space is circular: $\theta_k(t) \in \mathbb{C}$ for all $k \in [K]$. In both cases, we took $\rho_k(0) = \sqrt{r_k}$, where $(r_k)_k$ is an i.i.d sample of random variables uniform over $[0, 1]$. But in the first case (figure 3a), we took $\theta_{\text{in}} = \text{Unif}(\mathbb{C})$. The temporal phase separation is then clearly visible: in the first instants of the process, all agents describe a quasi-exact radial trajectory. Subsequently, as they get closer to the circle of radius $\rho^\star = \frac{1}{3}$, the radial dynamics decays and the 'finite size' effect prevails. Finally, all the agents converge with a self-confidence $\rho^* = \frac{1}{P}$, here $P = 2$.
In the last picture 3b, we have chosen a non-uniform angular entrance law: $\theta_{\text{in}} \sim f_4$ (see equation (6)). The temporal phase separation does not occur, which proves that this phenomenon is very dependent of the initial configuration.



(a) (b)

Fig. 2: Comparison of different entrance laws. On the left (a), we took as *entrance law* uniform in the opinion, that is $\theta_{\text{in}} = \text{Unif}[0, \pi]$, but with concentration on the center by choosing $\rho_{\text{in}} = \text{Unif}[0, 1]$. And on the right (b), we took as *entrance law* the uniform spatial measure: $(\theta_{\text{in}}, \rho_{\text{in}}) = \text{Unif}(\mathbb{B}_+)$.



(a) (b)

Fig. 3: Comparison of different entrance laws over the full circle. On the left (a), the entrance law $(\theta_{\text{in}}, \rho_{\text{in}})$ is uniform over the whole ball. *We clearly observe two phases of the process: during the first phase is only the radii evolve. Once the agents' self-confidence get closer to the PRE value, here* $\rho^\star = \frac{1}{3}$*, the radius dynamics decelerates and the angular dynamics expands, namely due to fluctuations, also called 'finite-type' effect.*
And on the right (b), we took as entrance law $(\theta_{\text{in}}, \rho_{\text{in}}) = f_4 \otimes \text{Unif}[0, 1]$ (see equation (6)). *By contrast with the preceding experiment with uniform angular entrance law, both the angles and the radii evolve from the initial time of the process.*

5 Conclusion

In this paper, we developed and analyzed a tractable multi-agent system intertwining both the opinion and the self-confidence of each agent. We saw that injecting the self-confidence within the classical BC scheme is very intuitive, but the converse is not as straightforward. In the design of the self-confidence dynamics, we opted for a network effect, relating self-confidence and herding behaviour, but this choice deserves to be discussed and refined. More generally, the interaction between the key features of the classical BC model – namely the influence radius map R and the self-confidence dynamics – shall be explored in a more general framework. This constitutes an interesting avenue of research for the future.

Appendix. We show that when we have $(\theta, \rho) \sim \mathrm{Unif}^{\otimes 2}[0, 2\pi] \times [0, 1]$, then the couple $(\theta, \sqrt{\rho})$ is distributed according to a Lebesgue restricted to the ball. Compute the density of the random variable $\sqrt{\rho}$: for any $x \in [0, 1]$,

$$\mathbb{P}(\sqrt{\rho} < x) = \mathbb{P}(\rho < x^2) = x^2 \implies \sqrt{\rho} \sim \big(2x\big)_{x \in [0,1]}.$$

Now, for any two angles $\theta_1 < \theta_2$ and any positive value $\rho_0 \in]0, 1[$, define the parcel $\mathcal{P}$ in polar coordinates as $\mathcal{P} := \{(\theta, \rho) \in \mathbb{B} : \theta \in [\theta_1, \theta_2] \text{ and } \rho > \rho_0\}$. It is sufficient to show that $\mathbb{P}\left((\theta, \sqrt{\rho}) \in \mathcal{P}\right) = \frac{\lambda(\mathcal{P})}{\lambda(\mathbb{B})} = \frac{(\theta_2 - \theta_1)(1 - \rho_0^2)}{2\pi}$:

$$\mathbb{P}\left((\theta, \sqrt{\rho}) \in \mathcal{P}\right) = \int_{u=\theta_1}^{u=\theta_2} \int_{x=\rho_0}^{x=1} 2x dx \frac{du}{2\pi} = \frac{(\theta_2 - \theta_1)}{2\pi}(1 - \rho_0^2),$$

which terminates the proof.

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