Centrality in Directed Networks

Gordana Marmulla[0000−0003−3577−2230] and Ulrik Brandes[0000−0002−1520−0430]

Social Networks Lab, ETH Zurich, Zürich, Switzerland
{gmarmulla,ubrandes}@ethz.ch

Abstract. The identification of important nodes in a network is a pervasive task in a variety of disciplines from sociology and bibliometry to geography and chemistry, and an ever growing number of centrality indices is proposed for this purpose. While such indices are often ad-hoc, preservation of the vicinal preorder has been identified as the core axiom shared by centrality rankings on undirected graphs. We extend this idea to directed graphs by defining vertex preorders based on directed neighborhood-inclusion criteria. While, for the undirected case, the vicinal preorder is total on threshold graphs and preserves all standard centrality indices, we show that our generalized preorders are total on certain subclasses of threshold digraphs. We thus provide a consistent formalization of the hitherto rather conceptual notions of radial, medial, and hierarchical centralities.

Keywords: centrality · network analysis · threshold digraph · neighborhood inclusion.

1 Introduction

Network science is the study of empirical phenomena involving data on overlapping dyads [14]. Such data can often be represented as graphs and a particularly common task is to identify nodes that are of structural importance, because they are centrally involved in relationships with others [13,21]. For this purpose, centrality indices are used, which assign values to the vertices of a graph. There is a vast literature on the question of what constitutes a centrality [9, 10, 21]. Formal attempts to settle it are largely focused on the axiomatization of indices [1,3,7,26,29], which serves to characterize indices by virtue of graph modification effects on the values assigned to vertices. The definition of every centrality index, however, incorporates a wealth of assumptions. For instance, they are generally defined to be automorphism invariant, which is therefore also the most common axiom. This makes it difficult, however, to adapt an index to situations in which there are additional attributes or multiple relations.

In a positional approach to network science [12], centrality is therefore not defined by an index, which establishes a complete ranking of all vertices, but in successively extended partial rankings. This allows for the gradual introduction of assumptions until a sufficiently complete ranking is established, or can be ruled out. Attention then shifts to the minimum criteria sufficient for a vertex to be considered more central than another. For undirected graphs, the vicinal preorder [20] has been identified as the common basis of standard variants of centrality [30, 31], leading to the following proposition [31, Proposition 4].

Proposition 1. *A vertex index is a centrality, if and only if it respects the vicinal preorder.*

As threshold graphs [24] are characterized by a complete vicinal preorder, centrality rankings necessarily agree on these graphs [32]. The relevance of the bottom-up approach has been limited by its restriction to undirected graphs, and while ideas for an extension to directed graphs have been floated [13], their implications have not been studied.

We propose three preorders based on directed neighborhood inclusion. They are consistent with the approach for undirected graphs and formalize the conceptual distinction of radial and medial centralities proposed by Borgatti and Everett [10], adding a subclass of radial centralities that we call hierarchical. Radial centralities measure the centrality of a node based on its accessibility, while medial centralities do so based on its bridging function. The claim that they represent "distinct intuitive conceptions" [21, p. 215] of centrality was already corroborated by low correlations found for centrality indices informally categorized as radial and medial [8, 33].

We also identify classes of digraphs on which the preorders yield complete rankings and which can therefore be seen as idealized representations for different notions of centrality. These are shown to be subclasses of threshold digraphs [15], and hence establish a correspondence between certain threshold digraphs and types of neighborhood-inclusions akin to the situation for undirected threshold graphs.

The remainder is organized as follows. After some basic definitions in Section 2, we introduce a family of preorders based on directed neighborhood inclusion in Section 3. Subclasses of threshold digraphs on which these yield complete rankings are determined in Section 4 and their relation to each other and their common superclass are discussed in Section 5. We conclude with suggestions for future work in Section 6.

2 Notation

We consider simple directed graphs (digraphs) $D = (V, E)$ consisting of a finite set V of vertices and a set $E \subseteq (V \times V) \setminus \{(i,i) : i \in V\}$ of directed edges. For a vertex $i \in V$, its in- and out-neighborhood are denoted by $N^-(i) := \{j : (j,i) \in E\}$ and $N^+(i) := \{j : (i,j) \in E\}$. The corresponding closed neighborhoods are defined by $N^-[i] := N^-(i) \cup \{i\}$ and $N^+[i] := N^+(i) \cup \{i\}$. A binary relation $\leqslant$ on V is called a preorder (or partial ranking) if it is reflexive and transitive. A preorder is total (or complete), if $i \leqslant j$ or $j \leqslant i$ for any $i, j \in V$. A total preorder is also called (complete) ranking.

3 Directed Neighborhood-Inclusion Criteria

We propose three extensions of undirected neighborhood inclusion to directed graphs. The first two appeared already in an earlier proposal [13], but without further study. The third, medial, relation defined below is new, because the version proposed previously applies only to pairs of non-adjacent vertices, so that no complete ranking can be obtained unless the graph is empty.

The relations defined in the following are supposed to capture the intuition that having more and better relationships can not make a vertex less central. In the radial case, this means having access to (being accessed by) others, and in the hierarchical case this condition is tightened to also having fewer reverse relationships. This is different in the medial case, where it is advantageous to have both incoming and outgoing edges, and thus be located between others. In the medial case, for comparing an adjacent pair, additional requirements have to be met. The additional requirements ensure that the advantage of one node, which is created through the adjacency, can be compensated by the respective other node. For example, if $(i,j) \in E$, the betweenness of i is enhanced through every walk passing i via a node of $N^-(i)$ and then ends after (i,j) in j. With $N^-(i) \subseteq N^+[j]$, for each of these walks w.r.t. i, there is another walk starting at i and passing j, thus, enhancing the betweenness of j. Visualizations of the neighborhood-inclusion relations are displayed in Figure 1.

Definition 1. *Let $D = (V, E)$ be a digraph.*

(i) The radial outwards and inwards neighborhood-inclusion relations $\leqslant^+, \leqslant^- \subseteq V \times V$ are defined by

$$i \leqslant^+ j \;:\Longleftrightarrow\; N^+(i) \subseteq N^+(j), \quad \textit{and} \quad i \leqslant^- j \;:\Longleftrightarrow\; N^-(i) \subseteq N^-(j)\ .$$

(ii) The hierarchical downwards and upwards neighborhood-inclusion relations $\leqslant^\downarrow, \leqslant^\uparrow \subseteq V \times V$ are defined by

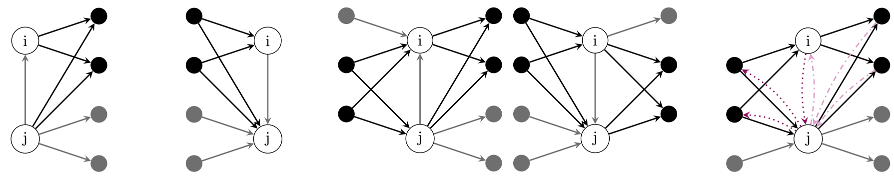
$$\begin{aligned} i \leqslant^\downarrow j \;&:\Longleftrightarrow\; N^+(i) \subseteq N^+(j) \;\wedge\; N^-(i) \supseteq N^-(j) \\ i \leqslant^\uparrow j \;&:\Longleftrightarrow\; N^-(i) \subseteq N^-(j) \;\wedge\; N^+(i) \supseteq N^+(j)\ . \end{aligned}$$

(iii) The medial neighborhood-inclusion relation $\leqslant^{\bowtie} \subseteq V \times V$ is defined by

$$i \leqslant^{\bowtie} j \;:\Longleftrightarrow\; \begin{cases} \quad N^+(i) \subseteq N^+[j] \\ \wedge\ N^-(i) \subseteq N^-[j] \\ \wedge\ N^-(i) \subseteq N^+[j] & \textit{if } (i,j) \in E \\ \wedge\ N^+(i) \subseteq N^-[j] & \textit{if } (j,i) \in E\ . \end{cases}$$

Lemma 1. *The binary relations of Definition 1 are preorders.*

Proof. For the radial (outwards as well as inwards) and hierarchical (downwards as well as upwards) relations, it is straightforward to see that all neighborhood-inclusion relations are reflexive and transitive and, therefore, preorders. For the



(a) $i \leqslant^{+} j$ (b) $i \leqslant^{-} j$ (c) $i \leqslant^{\downarrow} j$ (d) $i \leqslant^{\uparrow} j$ (e) $i \leqslant^{\bowtie} j$

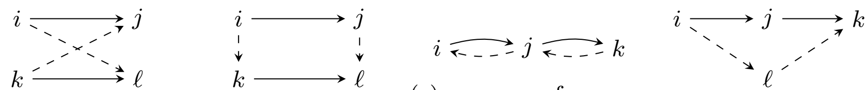
Fig. 1: Visualization of the proposed neighborhood-inclusion relations for directed networks. Gray edges may or may not be present. Dotted and dashdotted edges represent inclusion requirements given an edge between i and j.

medial neighborhood-inclusion relation, reflexivity is also straightforward. It remains to show that the medial relation is transitive. Let $i, j, k \in V$ be distinct vertices with $i \leqslant^{\bowtie} j$ and $j \leqslant^{\bowtie} k$. We consider cases based on the connectivity between i and j; solid edges are present, dashed edges are absent.

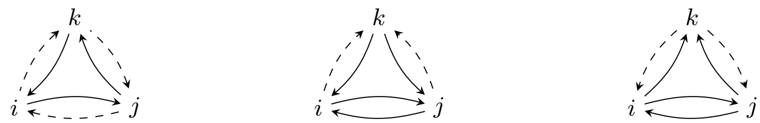
$i \mathrel{\substack{\dashrightarrow \\ \dashleftarrow}} j$: Since $i \leqslant^{\bowtie} j$ and because i and j are disconnected, the neighborhood-inclusions hold in particular for the open neighborhoods, such that $N^{+}(i) \subseteq N^{+}(j) \subseteq N^{+}[k]$ and $N^{-}(i) \subseteq N^{-}(j) \subseteq N^{-}[k]$. What remains to show is that the potential additional conditions hold, dependent on the connection between i and k. Nothing is left to be shown if there is no edge between i and k. If $(i, k) \in E$, then it is $(j, k) \in E$ due to $N^{+}(i) \subseteq N^{+}(j)$. Then, because $j \leqslant^{\bowtie} k$, the additional condition $N^{-}(j) \subseteq N^{+}[k]$ holds, i.e., $N^{-}(i) \subseteq N^{-}(j) \subseteq N^{+}[k]$. If $(k, i) \in E$, then it is $(k, j) \in E$ due to $N^{-}(i) \subseteq N^{-}(j)$. Because $j \leqslant^{\bowtie} k$, the additional condition $N^{+}(j) \subseteq N^{-}[k]$ holds, i.e., $N^{+}(i) \subseteq N^{+}(j) \subseteq N^{-}[k]$. In total, $i \leqslant^{\bowtie} k$.

$i \mathrel{\substack{\rightarrow \\ \dashleftarrow}} j$: Because $j \notin N^{-}(i)$, it holds $N^{-}(i) \subseteq N^{-}(j)$. From $j \leqslant^{\bowtie} k$, it follows then $N^{-}(i) \subseteq N^{-}(j) \subseteq N^{-}[k]$. Since $i \in N^{-}(j) \subseteq N^{-}[k]$ it is $i \in N^{-}[k]$, i.e., $k \in N^{+}(i) \subseteq N^{+}[j]$. Then, for $j \leqslant^{\bowtie} k$ to hold, the additional condition $N^{-}(j) \subseteq N^{+}[k]$ must be fulfilled. This condition entails $i \in N^{-}(j) \subseteq N^{+}[k]$ and thus $k \in N^{-}(i) \subseteq N^{-}[j]$, or equivalently $j \in N^{+}(k)$. With the latter, we have $N^{+}(i) \subseteq N^{+}[j] = N^{+}(j) \cup \{j\} \subseteq N^{+}[k] \cup \{j\} = N^{+}[k]$. Since both $(k, i), (i, k) \in E$, it remains to prove the additional conditions for $i \leqslant^{\bowtie} k$ to hold. Because $(j, k) \in E$ and $j \leqslant^{\bowtie} k$, it holds $N^{-}(j) \subseteq N^{+}[k]$, which entails $N^{-}(i) \subseteq N^{-}(j) \subseteq N^{+}[k]$. Because $(k, j) \in E$ and $j \leqslant^{\bowtie} k$ it holds $N^{+}(j) \subseteq N^{-}[k]$, which entails $N^{+}(i) \subseteq N^{+}[j] = N^{+}(j) \cup \{j\} \subseteq N^{-}[k] \cup \{j\} = N^{-}[k]$. In total, $i \leqslant^{\bowtie} k$.

$i \mathrel{\substack{\dashrightarrow \\ \leftarrow}} j$: Because $j \notin N^{+}(i)$, it holds $N^{+}(i) \subseteq N^{+}(j)$. From $j \leqslant^{\bowtie} k$, it follows then $N^{+}(i) \subseteq N^{+}(j) \subseteq N^{+}[k]$. Since $i \in N^{+}(j) \subseteq N^{+}[k]$, it holds in particular $i \in N^{+}[k]$ and, therefore, equivalently $k \in N^{-}(i)$. The latter entails $k \in N^{-}(j)$ as well, i.e., $(k, j) \in E$. Then, for $j \leqslant^{\bowtie} k$ to hold, the additional condition $N^{+}(j) \subseteq N^{-}[k]$ must be fulfilled. Since $i \in N^{+}(j) \subseteq N^{-}[k]$, this corresponds to the presence of the edge $(i, k) \in E$, as well as $(j, k) \in E$ due to $N^{+}(i) \subseteq N^{+}[j]$.



(a) 2-switch (b) open square (c) one-way of length 2 (d) isolated path of length 2



(e) induced directed 3-cycle (f) diverging triangle (g) converging triangle

Fig. 2: Forbidden configurations. Solid edges are required, dashed edges are forbidden, and no conditions are placed on absent edges.

With the latter edge, it follows $N^-(i) \subseteq N^-[j] = N^-(j) \cup \{j\} \subseteq N^-[k] \cup \{j\} = N^-[k]$. Since both $(i,k),(k,i) \in E$, it remains to prove the additional conditions for $i \leqslant^{\bowtie} k$ to hold. Because $(j,k),(k,j) \in E$ and $j \leqslant^{\bowtie} k$, the inclusions $N^-(j) \subseteq N^+[k]$ and $N^+(j) \subseteq N^-[k]$ hold. Since j and k are reciprocally connected, it follows $N^-(i) \subseteq N^-[j] = N^-(j) \cup \{j\} \subseteq N^+[k] \cup \{j\} \subseteq N^+[k]$ and $N^+(i) \subseteq N^+[j] = N^+(j) \cup \{j\} \subseteq N^-[k] \cup \{j\} = N^-[k]$.

$i \rightleftarrows j$: Because $i \in N^-(j) \subseteq N^-[k]$ and $i \in N^+(j) \subseteq N^+[k]$, vertices i and k are reciprocally connected as well. In particular, $k \in N^-(i) \subseteq N^-[j]$ and $k \in N^+(i) \subseteq N^+[j]$ meaning that j and k must be also reciprocally connected. Therefore, both $N^-(i) \subseteq N^-[j] = N^-(j) \cup \{j\} \subseteq N^-[k] \cup \{j\} = N^-[k]$ and $N^+(i) \subseteq N^+[j] = N^+(j) \cup \{j\} \subseteq N^+[k] \cup \{j\} = N^+[k]$ hold true. The required additional conditions for the pair i,k follow with those from pair j,k and their mutual connection: $N^-(i) \subseteq N^-[j] = N^-(j) \cup \{j\} \subseteq N^+[k] \cup \{j\} = N^+[k]$ and $N^+(i) \subseteq N^+[j] = N^+(j) \cup \{j\} \subseteq N^-[k] \cup \{j\} = N^-[k]$.

4 Uniquely Ranked Directed Graphs

We next determine the classes of digraphs on which the above preorders yield complete rankings. These are important because all centralities preserving the respective preorder will agree. All of them can be defined via forbidden subgraphs depicted in Figure 2.

Radial. Recall that an interval order [19, 34] is an irreflexive binary relation $(V, \prec)$ satisfying the condition $i \prec j \wedge k \prec \ell \implies i \prec \ell \vee k \prec j$. From the graph perspective, interval orders are transitive and acyclic digraphs without a 2-switch [24]. The graphs of interval orders can also be characterized as follows: for all $i,j \in V$ it holds that (i) $N^+(i) \subseteq N^+(j)$ or $N^+(j) \subseteq N^+(i)$, and (ii) $N^-(i) \subseteq N^-(j)$ or $N^-(j) \subseteq N^-(i)$ [5, 17, 18]. In other words, both the out- and in-neighborhoods are linearly ordered by nesting, and therefore interval orders yield exactly those digraphs on which the radial neighborhood-inclusion relations $\leqslant^+$ and $\leqslant^-$ are total.

Interval orders represent extreme cases of acyclicity, essentially reducing centrality to a matter of being positioned at the beginning or the end of a relationship. Due to their transitivity, there is no notion of passing through other vertices via shortest paths, resulting in each vertex conceptually occupying the same medial position. Another implication of transitivity is that the issue of reachability becomes synonymous with determining which vertices belong to which (i.e., out- or in-) neighborhood, rendering distance insignificant while emphasizing the critical role of the chosen direction.

Hierarchical. In general, $\leqslant^+$ need not be the reversed (partial) ranking of $\leqslant^-$, unlike for $\leqslant^\downarrow$ and $\leqslant^\uparrow$ [17]. Digraphs in which $\leqslant^\downarrow$ ($\leqslant^\uparrow$) yields a ranking are the semiorders [23], also called unit interval orders [6]. These are special interval orders satisfying the additional condition $i \prec j \wedge j \prec k \implies i \prec \ell \vee \ell \prec k$ for any other ℓ. From the graph perspective, this means that semiorders do not contain an isolated path of length 2.

Although the choice of direction of edges may influence a node's position in the ranking for interval orders, the hierarchical ranking remains unique up to reversal. Consequently, it only needs to be determined which end of the ranking is deemed more central.

Medial. The medial preorder is total on a class of graphs that, to the best of our knowledge, has not been discussed previously.

Lemma 2. *The medial preorder is total if and only if the digraph contains neither a 2-switch, an open square, a one-way of length* 2*, or a diverging or converging triangle.*

Proof. We prove both directions by contraposition.

$\implies$: Assume D contains one of the forbidden configurations. We refer to the vertex labels as shown in Figure 2. In a 2-switch, vertices i and k are not comparable with respect to $\leqslant^{\bowtie}$ because $j \in N^+(i) \backslash N^+(k)$ while $\ell \in N^+(k) \backslash N^+(i)$. In an open square, on the one hand $N^-(j) \not\subseteq N^-[k]$ because $i \in N^-(j) \backslash N^-(k)$. On the other hand, $N^+(k) \not\subseteq N^+[j]$ because $\ell \in N^+(k) \backslash N^+(j)$. Thus the only possibility is $N^+(j) \subseteq N^+[k]$ and $N^-(k) \subseteq N^-[j]$ implying that j and k are not comparable with respect to $\leqslant^{\bowtie}$. The same line of arguments apply to the pair i and k in a one-way of length 2. In a diverging triangle, the pair i and j requires additional constraints due to their mutual connection. However, neither $N^-(i) \subseteq N^+[j]$ nor $N^-(i) \subseteq N^+[i]$ because $k \in N^-(i), N^-(j)$ but $k \notin N^+[j], N^+[i]$. The analogous argumentation applies to the converging triangle.

$\impliedby$: Assume that the medial neighborhood-inclusion relation is not total. Then, there exist a pair of vertices $i, j \in V$ with $i \not\leqslant^{\bowtie} j$ and $j \not\leqslant^{\bowtie} i$. There can be five distinct reasons why a pair is not comparable. In fact, there are more non-distinct cases, but they can be matched to one of the five following unique cases by changing the roles of i and j accordingly.

$N^-(i) \not\subseteq N^-[j]$ and $N^-(j) \not\subseteq N^-[i]$: Then, there exists a $k \in N^-(i)\backslash N^-[j]$ for which it follows $k \notin \{i,j\}$. At the same time, there is an $\ell \in N^-(j)\backslash N^-[i]$ for which it holds $\ell \notin \{i,j,k\}$. This situation corresponds to a 2-switch.
$N^+(i) \not\subseteq N^+[j]$ and $N^+(j) \not\subseteq N^+[i]$: With the same line of arguments as in the previous case it follows that D contains a 2-switch.
$N^-(i) \not\subseteq N^-[j]$ and $N^+(j) \not\subseteq N^+[i]$: The former implies the existence of $k \notin \{i,j\}$ with $k \in N^-(i)\backslash N^-[j]$. The latter implies the existence of $\ell \notin \{i,j\}$ with $\ell \in N^+(j)\backslash N^+[i]$. If $\ell = k$, this configurations forms an one-way of length 2. If $\ell \neq k$, this configurations forms an open square.
$N^-(i) \subseteq N^-[j]$ and $N^+(i) \subseteq N^+[j]$, but $N^-(i) \not\subseteq N^+[j]$ when $(i,j) \in E$: Then, there is a $k \in N^-(i)\backslash N^+[j]$. In particular, $k \notin \{i,j\}$ and $k \in N^-(i) \subseteq N^-[j]$. Since $k \notin N^+[j]$ and $N^+(i) \subseteq N^+[j]$, it holds $k \notin N^+(i)$. If $(j,i) \in E$, the configuration corresponds to a converging triangle. If $(j,i) \notin E$, the configuration corresponds to an one-way of length 2.
$N^-(i) \subseteq N^-[j]$ and $N^+(i) \subseteq N^+[j]$, but $N^+(i) \not\subseteq N^-[j]$ when $(j,i) \in E$: With the same line of arguments as in the third case it follows that D contains either a diverging triangle or an one-way of length 2.

Medial relations correspond to a notion of centrality, in which central nodes form cores connecting peripheral ones. The medial centrality criterion is related to the level of reciprocity, i.e., the proportion of dyads with edges in both directions. Reciprocity as a network property was found to be significantly associated with the correlations between all different centrality measures (between symmetric, between asymmetric, and between the combination of both), i.e., the more bi-directional connections, the less distinct measures become [33]. This association supports the idea that medial and radial notions of centrality are indiscriminate on undirected graphs. In contrast, directed graphs allow for formal criteria to define these concepts.

5 Threshold Digraphs

All above classes of digraphs on which the our preorders yield rankings are subclasses of threshold digraphs.

Definition 2 (Threshold digraph [15]). *A digraph is called threshold, if it contains neither a 2-switch nor an induced directed 3-cycle.*

Interval orders, being acyclic and lacking a 2-switch, are clearly threshold, and so is their subclass of semiorders. Lemma 2 asserts that digraphs on which $\leqslant^{\bowtie}$ is total contain neither a 2-switch nor a one-way path of length 2. This condition is stronger than the absence of induced directed 3-cycle, and assures that members of our new class of digraphs are threshold, too.

As the name suggests, threshold digraphs were introduced as a generalization of threshold graphs. There have been various other approaches for extending threshold graphs for the directed case: oriented threshold graphs [4], which are generalized by directed threshold graphs [22], which in turn are a subclass of threshold digraphs [15].

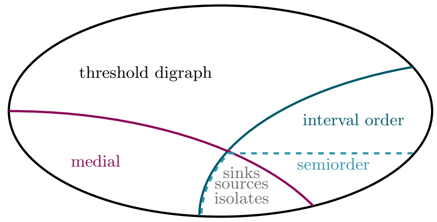


Fig. 3: Unique ranked graphs as subclasses of threshold digraphs.

For interval orders and semiorders, recognizing whether a digraph belongs to one of these types was resolved decades ago and can both be done in linear time $\mathcal{O}(|V| + |E|)$ [25,27]. Whether this can be done similarly efficiently for the class of digraphs on which $\leqslant^{\bowtie}$ yields a ranking remains to be investigated.

For undirected graphs, threshold graphs rank vertices uniquely with respect to the vicinal preorder and thus the distinctness of a given graph to a threshold graph is of interest [32]. Analogously, the determined subsets of threshold digraphs, on which the preorders are complete, might help to estimate how consistent centrality measures will perform on the given digraph. The work of Cloteaux et al. [15], linking degree sequences and threshold digraphs, could serve as a starting point. For semiorders and interval orders, in particular, already established results such as [2,11,16] may be relevant.

There are more equivalent characterizations for threshold digraphs [28] which reveal that these digraphs are Ferrers digraphs [16] with loops removed. A characterization for Ferrers digraphs is that both their out- and in-neighborhoods (but now potentially with loops) are linearly nested. An alternative opportunity to examine the extent and kind of centrality of a threshold digraph could be then the (minimum) number of loops required to transform it into a Ferrers digraph and the type(s) of neighborhood-inclusion that are possible through transformations.

6 Conclusion

In order to establish a common basis for centrality in directed graphs, we proposed three neighborhood-inclusion relations on directed graphs. They provide formalizations of different notions of centrality, namely radial, hierarchical, and medial. As the relations are reflexive and transitive, the resulting rankings are partial in general, and the classes of digraphs on which the rankings are total are non-trivial. These distinct extreme directed graphs suggest that notions of centrality can be discriminated despite their convergence on undirected graphs.

We hope that the findings presented here will have an impact on the definition and application of centrality. The formal criteria can be used to examine which of the neighborhood-inclusion relations a centrality index does or does not preserve. There are also, however, purely theoretical collateral insights. All uniquely ranked directed graphs turned out to be threshold digraphs. Since there are threshold digraphs not covered by any of the neighborhood-inclusion criteria (cf. Figure 3), there may be additional notions of centrality or some other type of ranking hiding. While digraphs with complete rankings obtained from radial and hierarchical preorders are the well-studied interval orders and semiorders, the medial preorders warrant further exploration.

Among the many interesting directions for future work we mention the completeness of our partial rankings on specific graphs, graph-modification distances to uniquely ranked digraphs, and the degree to which select graph characteristics separate different centrality indices.

References

1. Altman, A., Tennenholtz, M.: Axiomatic Foundations for Ranking Systems. Journal of Artificial Intelligence Research **31**, 473–495 (2008). https://doi.org/10.1613/jair.2306
2. Avery, P.: An algorithmic proof that semiorders are representable. Journal of Algorithms **13**(1), 144–147 (1992). https://doi.org/10.1016/0196-6774(92)90010-A
3. Bloch, F., Jackson, M.O., Tebaldi, P.: Centrality measures in networks. Social Choice and Welfare **61**(2), 413–453 (2023). https://doi.org/10.1007/s00355-023-01456-4
4. Boeckner, D.: Oriented threshold graphs. Australasian Journal of Combinatorics **71**(1), 43–53 (2018)
5. Bogart, K.P.: An obvious proof of Fishburn's interval order theorem. Discrete Mathematics **118**(1-3), 239–242 (1993). https://doi.org/10.1016/0012-365X(93)90065-2
6. Bogart, K.P., West, D.B.: A short proof that âĂŸproper = unitâĂŹ. Discrete Mathematics **201**(1-3), 21–23 (1999). https://doi.org/10.1016/S0012-365X(98)00310-0
7. Boldi, P., Vigna, S.: Axioms for Centrality. Internet Mathematics **10**(3-4), 222–262 (2014). https://doi.org/10.1080/15427951.2013.865686
8. Bolland, J.M.: Sorting out centrality: An analysis of the performance of four centrality models in real and simulated networks. Social Networks **10**(3), 233–253 (1988). https://doi.org/10.1016/0378-8733(88)90014-7
9. Borgatti, S.P.: Centrality and network flow. Social Networks **27**(1), 55–71 (2005). https://doi.org/10.1016/j.socnet.2004.11.008
10. Borgatti, S.P., Everett, M.G.: A Graph-theoretic perspective on centrality. Social Networks **28**(4), 466–484 (2006). https://doi.org/10.1016/j.socnet.2005.11.005
11. Bousquet-MÃľlou, M., Claesson, A., Dukes, M., Kitaev, S.: ($2+2$) -free posets, ascent sequences and pattern avoiding permutations. Journal of Combinatorial Theory, Series A **117**(7), 884–909 (2010). https://doi.org/10.1016/j.jcta.2009.12.007, https://linkinghub.elsevier.com/retrieve/pii/S0097316509001885
12. Brandes, U.: Network positions. Methodological Innovations **9**, 1–19 (2016). https://doi.org/10.1177/2059799116630650
13. Brandes, U.: Central Positions in Social Networks. In: Fernau, H. (ed.) Proceedings of the 15th International Computer Science Symposium in Russia (CSR 2020). vol. 12159, pp. 30–45 (2020). https://doi.org/10.1007/978-3-030-50026-9_3
14. Brandes, U., Robins, G., McCranie, A., Wasserman, S.: What is network science? Network Science **1**(1), 1–15 (2013). https://doi.org/10.1017/nws.2013.2

15. Cloteaux, B., LaMar, M.D., Moseman, E., Shook, J.: Threshold Digraphs. Journal of Research of the National Institute of Standards and Technology **119**, 227–234 (2014). https://doi.org/10.6028/jres.119.007
16. Cogis, O.: Ferrers digraphs and threshold graphs. Discrete Mathematics **38**(1), 33–46 (1982). https://doi.org/10.1016/0012-365X(82)90166-2
17. Felsner, S.: Interval Orders: Combinatorial Structure and Algorithms. Dissertation, Technische UniversitÃďt Berlin (1992)
18. Fishburn, P.C.: Intransitive indifference with unequal indifference intervals. Journal of Mathematical Psychology **7**(1), 144–149 (1970). https://doi.org/10.1016/0022-2496(70)90062-3
19. Fishburn, P.C.: Interval orders and interval graphs: a study of partially ordered sets. Wiley, New York (1985)
20. Foldes, S., Hammer, P.L.: The Dilworth Number of a Graph. In: Annals of Discrete Mathematics, vol. 2, pp. 211–219. Elsevier (1978). https://doi.org/10.1016/S0167-5060(08)70334-0
21. Freeman, L.C.: Centrality in social networks conceptual clarification. Social Networks **1**(3), 215–239 (1978). https://doi.org/10.1016/0378-8733(78)90021-7
22. Gurski, F., Rehs, C.: Comparing Linear Width Parameters for Directed Graphs. Theory of Computing Systems **63**(6), 1358–1387 (2019). https://doi.org/10.1007/s00224-019-09919-x
23. Luce, R.D.: Semiorders and a Theory of Utility Discrimination. Econometrica **24**(2), 178–191 (1956). https://doi.org/10.2307/1905751, https://www.jstor.org/stable/1905751?origin=crossref
24. Mahadev, N.V.R., Peled, U.N.: Threshold graphs and related topics. No. 56 in Annals of discrete mathematics, Elsevier, Amsterdam ; New York (1995)
25. Mitas, J.: Minimal representation of semiorders with intervals of same length. In: BouchittÃľ, V., Morvan, M. (eds.) Orders, Algorithms, and Applications, vol. 831, pp. 162–175. Springer-Verlag, Berlin/Heidelberg (1994). https://doi.org/10.1007/BFb0019433
26. Nieminen, U.: On the centrality in a directed graph. Social Science Research **2**(4), 371–378 (1973). https://doi.org/10.1016/0049-089X(73)90010-0
27. Papadimitriou, C.H., Yannakakis, M.: Scheduling Interval-Ordered Tasks. SIAM Journal on Computing **8**(3), 405–409 (1979). https://doi.org/10.1137/0208031
28. Reilly, E., Scheinerman, E., Zhang, Y.: Random Threshold Digraphs. The Electronic Journal of Combinatorics **21**(2), P2.48 (2014). https://doi.org/10.37236/4050
29. Sabidussi, G.: The centrality index of a graph. Psychometrika **31**(4), 581–603 (1966). https://doi.org/10.1007/BF02289527
30. Sadler, E.: Ordinal Centrality. Journal of Political Economy **130**(4), 926–955 (2022). https://doi.org/10.1086/718191
31. Schoch, D., Brandes, U.: Re-conceptualizing centrality in social networks. European Journal of Applied Mathematics **27**(6), 971–985 (2016). https://doi.org/10.1017/S0956792516000401
32. Schoch, D., Valente, T.W., Brandes, U.: Correlations among centrality indices and a class of uniquely ranked graphs. Social Networks **50**, 46–54 (2017). https://doi.org/10.1016/j.socnet.2017.03.010
33. Valente, T.W., Coronges, K., Lakon, C., Costenbader, E.: How Correlated Are Network Centrality Measures? Connections **28**(1), 16–26 (2008)
34. Wiener, N.: A contribution to the theory of relative position. In: Proceedings of the Cambridge Philosophical Society. vol. 17, pp. 441–449 (1914)