Fuzzy and Overlapping Communities Detection: An Improved Approach Using Formal Concept Analysis

Martin WAFFO KEMGNE[1[0000−0002−0668−8825]], Christophe DEMKO[1[0000−0003−0740−8466]], Karell BERTET[1[0000−0002−9741−4570]], and Jean-Loup GUILLAUME[1[0000−0002−4615−1563]]

L3i (Laboratoire Informatique, Image et Interaction),
La Rochelle Université, La Rochelle, France
{martin.waffo_kemgne, christophe.demko, karell.bertet, jean-loup.guillaume}@univ-lr.fr

Abstract. This paper introduces an innovative method for identifying fuzzy and overlapping communities in graphs using Formal Concept Analysis (FCA). While several initial works have explored the application of FCA to non-overlapping community detection, this paper aims to extend one of these approaches to the detection of overlapping communities. We conduct experiments on various benchmark graphs, including both synthetic and real networks. The performance of the proposed method is evaluated using well-known metrics and compared with other algorithms in the literature.

Keywords: Overlapping communities · Fuzzy communities· FCA · graphs.

1 Introduction

In an era marked by an exponential proliferation of data, modeling complex systems through networks or graphs has become fundamental for understanding the underlying structures that govern the dynamics of social, biological, and technological systems. These networks, which model connections between elements or actors, reflect the intrinsic complexity of the systems they represent, ranging from protein interactions to social networks and technological infrastructures.

At the heart of this exploration lies the notion of community structure, a pervasive feature in networks that manifests the tendency of nodes to cluster into groups or communities, where internal links are numerous while inter-community links are scarcer. The study of community structures within complex networks has garnered significant attention in the field of network science, providing deep insights into the organization, dynamics, and function of various real systems [22].

Traditionally, community detection algorithms have sought to partition networks into distinct groups or communities, with the underlying assumption that each node is a member of only one community [12]. Although useful, this approach often fails to capture the nuanced relationships inherent in many complex networks where nodes can belong to multiple communities, reflecting a more realistic scenario of overlapping or fuzzy community memberships [30].

While Formal Concept Analysis (FCA) [14] has been effective in detecting non-overlapping communities [2,11,13,20], its application to fuzzy and overlapping communities remains unexplored. In this paper, we propose an adapted FCA approach for the detection of fuzzy and overlapping communities. Building upon our previous work, we introduce a refinement step to enhance community detection accuracy. We conduct extensive experiments on various benchmark graphs, including synthetic and real networks, comparing the performance of our method with existing approaches.

The structure of this paper is organized as follows: Section 2 provides an overview of previous work on community detection. We discuss existing FCA-based approaches and their limitations in Section 3. Our proposed approach and motivation are detailed in Section 4. To evaluate the consistency and performance of our method, we present general simulations in Section 5. Finally, we conclude in Section 6 by summarizing our contributions and outlining future research directions.

2 Related works

Community structure analysis in networks is an increasingly studied area within network science, focusing on the detection of both overlapping and non-overlapping communities.

2.1 Preliminary Definitions

Let $G = (V, E)$ be a graph with $n = |V|$ and $m = |E|$.

Clique: A clique in a graph is a subset of nodes $M \subseteq V$ where every pair of nodes is interconnected. A clique is maximal (MC) if adding any other node disrupts the clique condition, indicating it is not contained within a larger clique.

Partition: A partition divides the node set V of graph $G = (V, E)$ into disjoint subsets, or communities, where $\bigcup_{i=1}^{k} V_i = V$ and $V_i \cap V_j = \emptyset$ for all distinct i and j. Each node is exclusively in one community.

Fuzzy Partition: Differing from a classical partition, a fuzzy partition permits nodes to belong to multiple communities, represented by a membership degree α_{ic} ranging from 0 to 1. Here, 0 indicates no membership, and 1 denotes full membership. A membership matrix $A = [\alpha_{ic}]$ defines this structure, allowing nodes to exhibit varying affiliations, thus more accurately reflecting complex community dynamics.

2.2 Detection of Non-Overlapping Communities

The detection of non-overlapping communities in networks has been extensively studied, leading to the development of various algorithms. Pioneering methods include Newman and Girvan's algorithm [23] and the widely-used Louvain algorithm by Blondel et al. [4]. Santo Fortunato provided a comprehensive review of over fifty community detection methods [12], highlighting approaches based on modularity optimization, such as Newman and Girvan's [15]. However, optimizing modularity is NP-hard [5], leading to the use of heuristic methods like the Louvain algorithm [4]. Despite their popularity, modularity-based methods suffer from resolution limit issues [12] and other drawbacks [25]. Alternative approaches, such as the Infomap algorithm [27], Label Propagation algorithm [26], and Stochastic Block Models (SBM) [17], offer effective community detection without relying on modularity. For a detailed comparison of methods, Fortunato's review is a valuable resource [12].

Despite all the methods previously mentioned, there exists one that we have not discussed, which is the focus of this paper. These methods rely on the notion of cliques and FCA to detect communities [2,11,13]. These methods, while effective in identifying community structures, overlook the possibility that a node may belong to multiple communities simultaneously. In the following section, we will explore in depth the methods and techniques dedicated to the detection of overlapping communities.

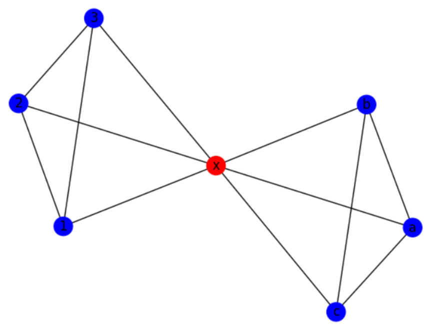


Fig. 1: Graph with Two Fuzzy Communities and a Vertex with Equal Membership Degree to Both Communities

2.3 Detection of Overlapping Communities

While traditional methods have been effective in identifying non-overlapping communities, they often fall short in capturing the intricate interactions found in various domains. For instance, social networks, interdisciplinary research, ecology, and the medical field all exhibit scenarios where entities belong to multiple categories simultaneously. Recognizing these limitations, recent research has shifted towards detecting overlapping communities. Early approaches include those based on the line graph [1,10], as well as methods involving seed extension or contraction [29].

Another method, among the most well-known not involving the *line graph*, is the CPM (*Clique Percolation Method*) algorithm [24]. Several other methods for detecting overlapping communities exist, particularly fuzzy communities [16] where, in addition to being overlapping, a degree of membership to a community is associated with each node. To this end, several modified versions of modularity have been proposed by different authors [21,28] to extend the modularity of Newman and Girvan [15], but the one we will use in this article is that of Shen et al. [28]. In the following section, we will present in detail those methods based on FCA.

3 Detection of Communities Based on FCA

FCA [14] is a method of data analysis and knowledge representation that deals with information in the form of hierarchies of concepts constructed from binary relations. It uses two important formalisms: the notion of a formal context and a formal concept. FCA has been used in several studies for the detection of non-overlapping communities, providing a rigorous framework based on lattice theory to identify groups of elements sharing common attributes [2,11,13]. In FCA, community detection can essentially be summarized as the search for *hidden* attributes shared by certain objects. This section presents the approaches used in [2,11,13]. Historically, Freeman was the pioneer in 1996 in using FCA to analyze social networks and identify communities by focusing on maximal overlapping cliques [13]. However, this approach excluded actors from bridging cliques. Falzon, in 2000, proposed an improvement by including these actors in the analysis without excluding them [11]. Ali in 2014 further developed this idea by incorporating all actors in the network, even those not belonging to the maximal cliques [2]. However, its application to detecting overlapping and fuzzy communities remains underexplored. Our objective is to develop an FCA-based method that overcomes these limitations by revealing hidden overlapping fuzzy communities.

3.1 Preliminary Definitions

Formal Context: A formal context, denoted as $\mathbb{K} = (G, M, I)$, comprises a finite set of objects G, a finite set of attributes M, and an incidence relation $I \subseteq G \times M$ indicating which objects possess which attributes.

Galois Connection Operators: Formal Concept Analysis introduces two operators, $\alpha : 2^G \to 2^M$ and $\beta : 2^M \to 2^G$, establishing a Galois connection between the sets 2^G and 2^M. These operators are defined as follows:

- $M \supseteq \alpha(A) = \{a \in M | \forall o \in A, (o, a) \in I\}$ for $A \subseteq G$,
- $G \supseteq \beta(B) = \{o \in G | \forall a \in B, (o, a) \in I\}$ for $B \subseteq M$.

Formal Concept: A formal concept, denoted as (A, B), consists of subsets $A \subseteq G$ and $B \subseteq M$ satisfying $\alpha(A) = B$ and $\beta(B) = A$. Here, A represents the extension and B the intension of the concept.

Formal concepts, ordered by the relation $(A_1, B_1) \geq (A_2, B_2) \Leftrightarrow A_1 \supseteq A_2 \Leftrightarrow B_2 \supseteq B_1$, form a complete lattice.

3.2 Freeman and Falzon's Methods for Community Detection

As mentioned earlier, Falzon's method [11] is an improvement of Freeman's method [13], which takes into account the actors from bridging cliques in the detection phase.

Definition of the Notion of Overlap We generalize Freeman's [13] definition of overlapping node sets to c-overlapping node sets as follows.

The binary relation *c-overlap*, $o_c \subseteq L_k \times L_k$, is defined as:

(i) $(n_i, n_i) \in o_c$ — each node set overlaps with itself, hence o_c is reflexive.
(ii) $|n_i \cap n_j| > c \Rightarrow (n_i, n_j) \in o_c$ — two node sets having more than c elements in common overlap, hence o_c is symmetric.
(iii) $(n_i, n_j) \in o_c$ and $(n_j, n_k) \in o_c \Rightarrow (n_i, n_k) \in o_c$ — o_c is transitive.

Note that for $c = 0$, the *0-overlap* coincides with *overlap*. It is clear that o_c is an equivalence relation and partitions L_k into subsets (disjoint for $c = 0$) that form the basis of the level k groups, which are the communities.

Principle of Freeman's Method Consider a graph $G = (V, E)$ and let $C = \{C_1, C_2, \ldots, C_n\}$ be the set of maximal cliques in G that contain at least three vertices. The choice of cliques of size at least three is intuitive, as triangles, which are cliques of size three, exemplify the principle that *the friends of my friends are my friends*. The first step involves constructing the context $\mathbb{K} := (V, C, I)$ where I is the binary relation defined by: for every $x \in V$ and $M \in C$, $I(x, C_i) = 1$ if the node x belongs to the maximal clique C_i and $I(x, C_i) = 0$ otherwise, $\forall i \in \{1, 2, ..., n\}$.

The second step is to construct the concept lattice associated with the above context. Then, to detect communities, Freeman [13] relies on the notion of overlapping cliques in the Galois lattice. He identifies, from the Galois lattice of the previously constructed context, the set of nodes of the lattice where at least two paths leading from their current position in the lattice to the infimum are not of the same length, which he calls bridging cliques.He then proceeds to eliminate the edges departing from these nodes to obtain disjoint groups. These groups represent the communities. To obtain them, it is sufficient to apply the overlap relation on the remaining maximal cliques after the elimination of the bridging cliques; these maximal cliques are at the first level of the lattice. The equivalence classes for this relation represent the communities (by uniting the elements or maximal cliques of the same equivalence class).

However, in large networks, there can often be multiple bridging MCs, and often, by eliminating them, some network actors are also eliminated. This is why Falzon [11] proposed another formalism to address this limit and take into account the actors from bridging cliques.

Principle of Falzon's Method T The step of constructing the context is similar to that described in subsection 3.2.

The second step involves constructing the concept lattice associated with the previous context. The construction of the concept lattice, is based on the algorithm that Falzon [11] proposed for building concept lattice.

In the final phase, Falzon [11] introduces a community detection method utilizing the properties of maximal cliques in a concept lattice. Her algorithm identifies community structures by generating groups G_k at each lattice level, which represent equivalence classes formed based on overlaps among lattice elements. This approach effectively detects communities, including nodes in bridging cliques, offering an improvement over Freeman's method [13], which excludes such nodes. The method has been applied to several graphs and has yielded good results; see [11] for more details. Despite the various improvements to this method, its application to fuzzy communities remains unexplored. This is the subject of the following section.

4 Proposed Approach

While traditional methods based on FCA, such as Falzon's method [11], have been effective in identifying non-overlapping communities, they are unable to detect overlapping communities. To illustrate this, consider the graph shown in Figure 1, where it is intuitive that there are at least two communities. Falzon's method, when applied to the same graph, fails to detect these communities, concluding that there is no possible decomposition into communities. Given this observation, our objective is to propose a new method that addresses these shortcomings and is thus capable of detecting overlapping fuzzy communities.

4.1 Principle of the Method

The principle of the method is similar to that of Falzon [11].

Let $G = (V, E)$ be a graph and $C = \{C_1, C_2, ..., C_n\}$ the set of maximal cliques in the graph. The first step involves constructing the context $\mathbb{K} := (V, C, I)$ following the same principle as in subsection 3.2. Thus, the step of constructing the lattice is identical to that described in [11]. The second step is the generation of communities.

First case: If for all $C_i, C_j \in C$ such that $i \neq j$, $C_i \cap C_j = \emptyset$, then the set of communities is $\pi = C = \{C_1, C_2, ..., C_n\}$.

Second case: Conversely, if there exists $C_i, C_j \in C$ such that $i \neq j$ and $C_i \cap C_j \neq \emptyset$. First determine the cardinality of each non-empty element of the lattice. Let c be the smallest cardinality; $c \geq 1$ by definition. Let $\mathcal{B}$ be the concept lattice previously constructed.

Then, apply Falzon's algorithm to our lattice $\mathcal{B}$, where the overlap function has been modified by the *c-overlap*. Thus, the communities to be obtained are overlapping. Therefore, to refine the method introduced in our first work, we incorporate the following two small steps.

Assume that we initially detected m communities noted $\pi_1 = \{G_1, G_2, \ldots, G_m\}$.

First, we identify all nodes that have not been classified, i.e., those that do not belong to any maximal clique of size at least 3. We denote this set of unclassified nodes by $U = V \setminus (\bigcup_{i=1}^{m} G_i)$.

Next, for each node $n \in U$, we generate $G(a)$ as the set of $G_i \in \pi_1$ with which a has a link, meaning that a is adjacent to at least one node of G_i.

After that, we identify nodes a that are linked to only one community; this corresponds to nodes for which $G(a)$ contains a single element. If $G(a)$ contains a single element and a is not adjacent to any other unclassified node $b \in U$ with $b \neq a$, then we add node a to the single community in $G(a)$ with which it is linked. Thus, all unclassified nodes that are linked to a single community and are not linked to any other node outside this community are integrated into this community. This reduces the number of unclassified nodes, and we obtain modified new communities $\pi_2 = \{G'_1, G'_2, \ldots, G'_m\}$, where each G'_i corresponds to the community G_i augmented by one or more new nodes.

Secondly, the new unclassified nodes are now $U' = V \setminus (\bigcup_{i=1}^{m} G'_i)$. To augment meaningful communities, we proceed as follows: We identify the connected components of the subgraph G' of G defined by the graph $G' = (V \setminus U, E)$. Connected components of size at least 3 will be considered as new communities, which we note π_3. The final communities will be $\pi = \pi_2 \cup \pi_3$.

To compute the membership degree α_{xK_i} of element x to community K_i, given a family $\pi = \{K_1, K_2, ..., K_m\}$ of communities, use the formula: $\alpha_{xK_i} = \frac{\sum_{y \in K_i} w(x,y)}{\sum_{z \in V} w(x,z)}$. where $w(x, y)$ represents the edge weight between x and y.

Illustrative Examples: Our method identifies two communities in Figure 1: {{1,2,3,x}, {a,b,c,x}}, unlike Falzon's method, which fails to detect any.

4.2 Advantages of Our Approach

Our approach provides several advantages over existing methods: It detects overlapping communities for deeper insights into network interactions, ensures determinism for consistent and reproducible results, simplifies calculations with a non-parametric framework, and effectively addresses the resolution limit problem by utilizing cliques instead of modularity.

5 General Simulations

In this section, we compare the performance of our *FALO* (Falzon Overlap) method against established techniques for detecting overlapping communities. We selected the following methods due to their recognized effectiveness: *k-clique* [24], an early method in community detection; *SLPA* [31], known for its precision; *DEMON* [7], favored for large networks [8]; *UMSTMO* [3], a recent non-parametric approach; and *Core Expansion (COREE)* [6], which tackles group fragmentation. This selection provides a robust evaluation of *FALO*.

Remark: In our simulations, we choose the cover with the highest extended modularity [28] in our dendrogram as the best option.

5.1 Description of Data

In synthetic network evaluations, we used the LFR benchmark, known for simulating realistic networks [18]. For Real-world Data Sets, We analyzed several diverse real networks to evaluate algorithm performance: **Zachary's Karate Club Network**, **Football Network**, **Dolphin Social Network** and **Jazz Musicians Network**.

5.2 Evaluation Metrics

In our network analysis, we employ three key metrics: extended modularity (EQ) [28], Overlapping Normalized Mutual Information (ONMI) [19], and F1-score[9]. ONMI and F1-score necessitate ground truth data and are thus utilized solely for synthetic networks.

5.3 Performance Evaluation

For these experiments, the algorithms k-clique, DEMON, SLPA, UMSTMO, and COREE were sourced from the Python library cdlib, and the code for FALO can be found on our GitLab repository at `https://gitlab.univ-lr.fr/mwaffo01/falo-algorithm-for-overllaping-community-detection`. We used the default values for all the algorithms that offer adjustable parameters. The average result was computed based on 10 independent runs. All experiments were performed on a desktop computer equipped with a 12th generation Intel(R) Core(TM) i9-12900H processor at 2.50 GHz and 32 GB of RAM.

Evaluations on Synthetic Networks In evaluations on synthetic networks using the LFR benchmark, we set standard parameters following Lancichinetti et al. [18]. These parameters include: $n = 1000$ nodes, average node degree $\hat{k} = 20$, maximum degree $k_{\max} = 50$, minimum community size $c_{\min} = 20$, maximum community size $c_{\max} = 100$, power law exponents $\tau_1 = -2$ and $\tau_2 = -1$, number of overlapping community memberships $Om = 2$, and number of overlapping nodes $On = 200$ with mixing parameter $\mu = 0.1$. To comprehensively assess algorithm behavior, we varied parameters μ, Om, On, and $\hat{k}$. Furthermore, considering that the plots associated with the three metrics exhibit similar patterns for each parameter, we have chosen to include only one metric for each parameter due to space constraints, with the exception of μ, for which images of all associated metrics are presented. Nonetheless, we provide conclusions related to all metrics for each parameter, even if the associated plots are not included.

Effect of μ (mixing parameter): In our study, μ ranged from 0.1 to 0.8, with other parameters fixed as in Section 5.3. As shown in Fig. 2, all detection methods' performance declined with increasing μ, indicating the growing difficulty of community detection with higher mixing. FALO and SLPA exhibited remarkable stability and outperformed other algorithms for all the metrics.

Effect of Om (number of memberships of overlapping nodes): We varied Om from 2 to 8 while keeping other parameters constant. As depicted in Fig. 3a, most algorithms exhibited declining performance with increasing Om, reflecting the increased complexity introduced by multiple memberships. But in summary, SLPA demonstrated robustness across all levels of membership overlap, while FALO showed strong early performance, albeit with diminishing returns as Om increased.

Effect of On (number of overlapping nodes): We studied the impact of On, by varying it from 200 to 800. FALO and SLPA demonstrated notable performance across all metrics, while DEMON and COREE performed comparably, as depicted in Fig. 3b.

Effect of $\hat{k}$ (Average Degree): We explored how varying ($\hat{k}$) from 10 to 25 impacts algorithm performance. Overall, FALO and SLPA emerged as the most effective algorithms, especially in networks with varying densities, while UMSTMO consistently showed lower performance; from the Fig. 3c.

FALO vs SLPA: Previous simulations have demonstrated the effectiveness of both *FALO* and *SLPA*, with *SLPA* showing slight superiority. However, considering that *SLPA* is parametric, depending on two parameters, r and t, and we have used the default parameters for general simulations, it is pertinent to question what would happen if we vary these parameters. To investigate this, we adjusted t from 5 to 25. The analysis from Figures 4 indicates scenarios where *FALO* outperforms *SLPA* under certain parameterizations, specifically when $5 \leq t \leq 15$.

Evaluations on Real Networks We assessed *FALO*'s performance against other algorithms on five real-world networks, using the EQ metric for effectiveness in the absence of ground truth data.

	FALO	Kclique	DEMON	SLPA	UMSTMO	COREE
Karate	**0.306**	0.185	0.145	0.175	0.106	*0.302*
Football	0.420	0.190	0.285	**0.561**	0.074	*0.436*
Dolphins	**0.405**	0.361	0.153	*0.393*	0.039	0.290
Polbooks	*0.441*	0.436	0.113	**0.467**	0.081	0.292
Jazz	*0.282*	0.002	0.060	**0.326**	0.000	0.183

Table 1: Performance of algorithms on real-world data sets. Values in **bold** indicate the best performance, while values in *italic* indicate the second best performance.

The table 1 summarizes the performance of different algorithms on five real-world datasets, measured using the EQ metric. *FALO* and *SLPA* consistently demonstrate strong performance across the datasets, with *SLPA* achieving the highest EQ score in the Football, Polbooks and Jazz datasets. In contrast, *FALO* achieves the highest EQ score in the Karate and Dolphins indicating its effectiveness in identifying overlapping communities. Overall, *FALO* and *SLPA* emerge as top-performing algorithms across the datasets.

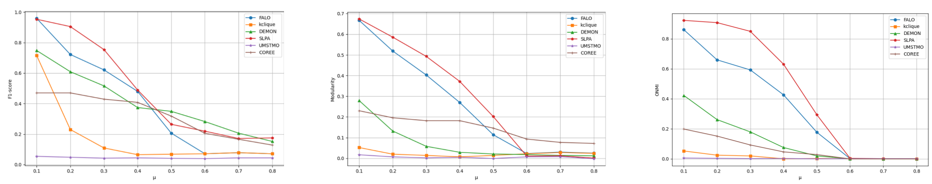
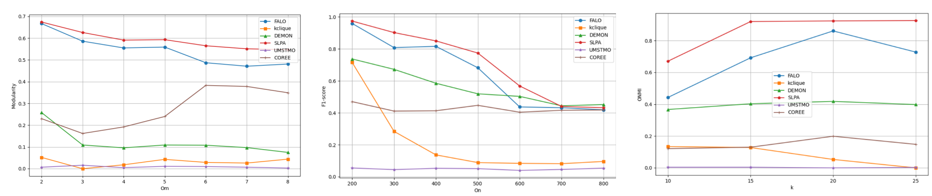


Fig. 2: Comparative Analysis of Algorithm Performance on LFR Networks with Parameter μ Varying from 0.1 to 0.8



(a) Modularity for Om (b) F1-Score for On (c) NMI for $\hat{k}$

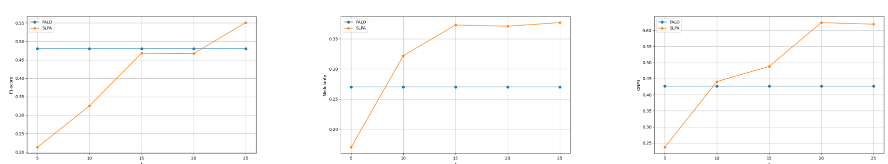


Fig. 4: Comparative Analysis between FALO and SLPA, with Parameter t Varying from 5 to 25

6 Conclusion

This paper introduces a novel method leveraging FCA for detecting fuzzy and overlapping communities in complex networks. The approach, validated through experiments on synthetic and real-world networks, proves its effectiveness. Our method offers enhanced capabilities over traditional approaches by accommodating overlapping communities. This research contributes to a deeper comprehension of community structures, with potential applications in various domains, including artificial intelligence for tasks like clustering.

Future research directions include addressing the challenge of assigning nodes not belonging to any maximal clique to communities systematically. Additionally, further analysis of the algorithmic complexity is warranted for comprehensive insights.

Acknowledgments. This research work is supported by the National Research Agency through the project "SmartFCA - ANR-21-CE23-0023: Formal Concept Analysis : an intelligent tool for the analysis of complex data."

References

1. Ahn, Y.Y., Bagrow, J.P., Lehmann, S.: Link communities reveal multiscale complexity in networks. nature **466**(7307), 761–764 (2010)
2. Ali, S.S., Bentayeb, F., Missaoui, R., Boussaid, O.: An efficient method for community detection based on formal concept analysis. In: Foundations of Intelligent Systems: 21st International Symposium, ISMIS 2014, Roskilde, Denmark, June 25-27, 2014. Proceedings 21. pp. 61–72. Springer (2014)
3. Asmi, K., Lotfi, D., El Marraki, M.: Overlapping community detection based on the union of all maximum spanning trees. Library Hi Tech **38**(2), 276–292 (2020)
4. Blondel, V.D., Guillaume, J.L., Lambiotte, R., Lefebvre, E.: Fast unfolding of communities in large networks. Journal of statistical mechanics: theory and experiment **2008**(10), P10008 (2008)
5. Brandes, U., Delling, D., Gaertler, M., Görke, R., Hoefer, M., Nikoloski, Z., Wagner, D.: On finding graph clusterings with maximum modularity. In: Graph-Theoretic Concepts in Computer Science: 33rd International Workshop, WG 2007, Dornburg, Germany, June 21-23, 2007. Revised Papers 33. pp. 121–132. Springer (2007)
6. Choumane, A., Awada, A., Harkous, A.: Core expansion: a new community detection algorithm based on neighborhood overlap. Social Network Analysis and Mining **10**, 1–11 (2020)
7. Coscia, M., Rossetti, G., Giannotti, F., Pedreschi, D.: Demon: a local-first discovery method for overlapping communities. In: Proceedings of the 18th ACM SIGKDD international conference on Knowledge discovery and data mining. pp. 615–623 (2012)
8. Coscia, M., Rossetti, G., Giannotti, F., Pedreschi, D.: Uncovering hierarchical and overlapping communities with a local-first approach. ACM Transactions on Knowledge Discovery from Data (TKDD) **9**(1), 1–27 (2014)
9. Epasto, A., Lattanzi, S., Paes Leme, R.: Ego-splitting framework: From non-overlapping to overlapping clusters. In: Proceedings of the 23rd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. pp. 145–154 (2017)
10. Evans, T.S., Lambiotte, R.: Line graphs, link partitions, and overlapping communities. Physical review E **80**(1), 016105 (2009)

11. Falzon, L.: Determining groups from the clique structure in large social networks. Social networks **22**(2), 159–172 (2000)
12. Fortunato, S.: Community detection in graphs. Physics reports **486**(3-5), 75–174 (2010)
13. Freeman, L.C.: Cliques, galois lattices, and the structure of human social groups. Social networks **18**(3), 173–187 (1996)
14. Ganter, B., Wille, R.: Formal concept analysis: mathematical foundations. Springer Science & Business Media (2012)
15. Girvan, M., Newman, M.E.: Community structure in social and biological networks. Proceedings of the national academy of sciences **99**(12), 7821–7826 (2002)
16. Gregory, S.: Fuzzy overlapping communities in networks. Journal of Statistical Mechanics: Theory and Experiment **2011**(02), P02017 (2011)
17. Holland, P.W., Laskey, K.B., Leinhardt, S.: Stochastic blockmodels: First steps. Social networks **5**(2), 109–137 (1983)
18. Lancichinetti, A., Fortunato, S., Radicchi, F.: Benchmark graphs for testing community detection algorithms. Physical review E **78**(4), 046110 (2008)
19. McDaid, A.F., Greene, D., Hurley, N.: Normalized mutual information to evaluate overlapping community finding algorithms. arXiv preprint arXiv:1110.2515 (2011)
20. Missaoui, R., Messaoudi, A., Ibrahim, M.H., Abdessalem, T.: Detecting communities in complex networks using formal concept analysis. In: Advances in Knowledge Discovery and Management: Volume 9, pp. 77–105. Springer (2022)
21. Nepusz, T., Petróczi, A., Négyessy, L., Bazsó, F.: Fuzzy communities and the concept of bridgeness in complex networks. Physical Review E **77**(1), 016107 (2008)
22. Newman, M.E.: The structure and function of complex networks. SIAM review **45**(2), 167–256 (2003)
23. Newman, M.E., Girvan, M.: Finding and evaluating community structure in networks. Physical review E **69**(2), 026113 (2004)
24. Palla, G., Derényi, I., Farkas, I., Vicsek, T.: Uncovering the overlapping community structure of complex networks in nature and society. nature **435**(7043), 814–818 (2005)
25. Peixoto, T.P.: Hierarchical block structures and high-resolution model selection in large networks. Physical Review X **4**(1), 011047 (2014)
26. Raghavan, U.N., Albert, R., Kumara, S.: Near linear time algorithm to detect community structures in large-scale networks. Physical review E **76**(3), 036106 (2007)
27. Rosvall, M., Bergstrom, C.T.: Maps of random walks on complex networks reveal community structure. Proceedings of the national academy of sciences **105**(4), 1118–1123 (2008)
28. Shen, H., Cheng, X., Cai, K., Hu, M.B.: Detect overlapping and hierarchical community structure in networks. Physica A: Statistical Mechanics and its Applications **388**(8), 1706–1712 (2009)
29. Wang, X., Jiao, L., Wu, J.: Adjusting from disjoint to overlapping community detection of complex networks. Physica A: Statistical Mechanics and its Applications **388**(24), 5045–5056 (2009)
30. Xie, J., Kelley, S., Szymanski, B.K.: Overlapping community detection in networks: The state-of-the-art and comparative study. Acm computing surveys (csur) **45**(4), 1–35 (2013)
31. Xie, J., Szymanski, B.K., Liu, X.: Slpa: Uncovering overlapping communities in social networks via a speaker-listener interaction dynamic process. In: 2011 ieee 11th international conference on data mining workshops. pp. 344–349. IEEE (2011)