AVL Tree Rotation Exercises

Problems 1–8

Problem Setup

In all problems, we use the following conventions:

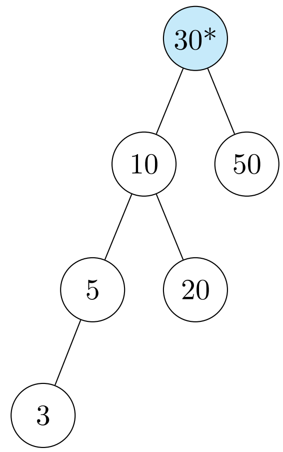
- The height of an empty subtree is $h(\text{NULL}) = -1$
- For non-empty nodes: $h(v) = 1 + \max(h(v_{\text{left}}), h(v_{\text{right}}))$
- Balance factor: $\text{bf}(v) = h(v_{\text{left}}) - h(v_{\text{right}})$
- A node is *balanced* if $|\text{bf}(v)| \leq 1$

Each tree shown represents the state *after exactly one insertion* into a previously valid AVL tree. Exactly one node (marked with $*$) is unbalanced.

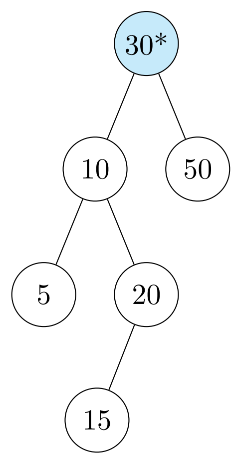
For each problem, you must:

1. Identify the first unbalanced node z on the path from the inserted leaf to the root
2. Compute relevant heights and balance factors
3. Classify the case as LL, LR, RL, or RR
4. Perform the required rotation(s) and draw the final balanced tree

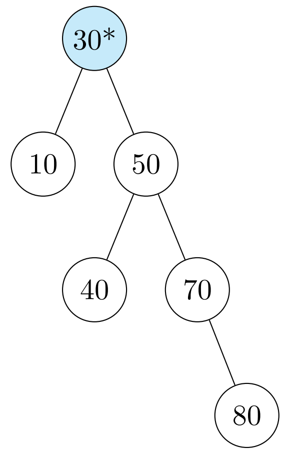
Problem 1 (LL case in a larger tree)**.** This tree results from inserting key `3` into a tree that already contained $\{5, 10, 20, 30, 50\}$.



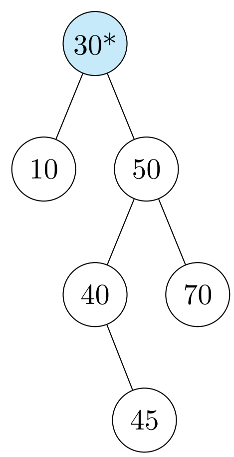
Problem 2 (LR case in a larger tree)**.** This tree results from inserting key `15` into a tree that already contained $\{5, 10, 20, 30, 50\}$.



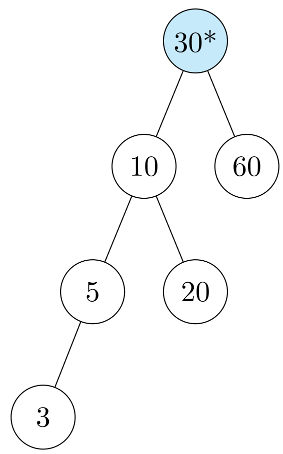
Problem 3 (RR case in a larger tree)**.** This tree results from inserting key `80` into a tree that already contained $\{10, 30, 40, 50, 70\}$.



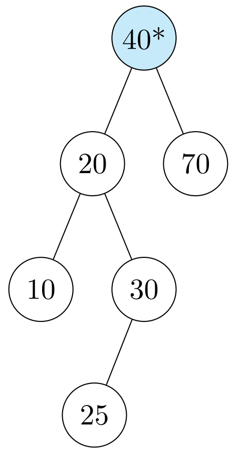
Problem 4 (RL case in a larger tree)**.** This tree results from inserting key `45` into a tree that already contained $\{10, 30, 40, 50, 70\}$.



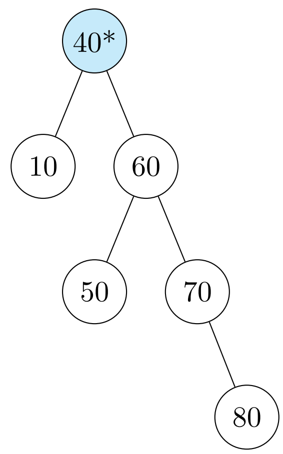
Problem 5. The following tree results from inserting one new key into a previously valid AVL tree. Exactly one node (marked with ∗) is unbalanced. Identify the case and perform the necessary rotation(s).



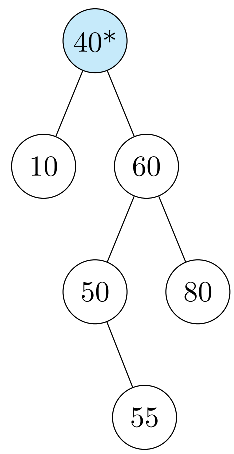
Problem 6. The following tree results from inserting one new key into a previously valid AVL tree. Exactly one node (marked with $*$) is unbalanced. Identify the case and perform the necessary rotation(s).



Problem 7. The following tree results from inserting one new key into a previously valid AVL tree. Exactly one node (marked with $*$) is unbalanced. Identify the case and perform the necessary rotation(s).



Problem 8. The following tree results from inserting one new key into a previously valid AVL tree. Exactly one node (marked with ∗) is unbalanced. Identify the case and perform the necessary rotation(s).



Step-by-Step Solutions

Solution to Problem 1

Solution. Step 1: Compute heights and balance factors

Starting from the leaves and working upward:

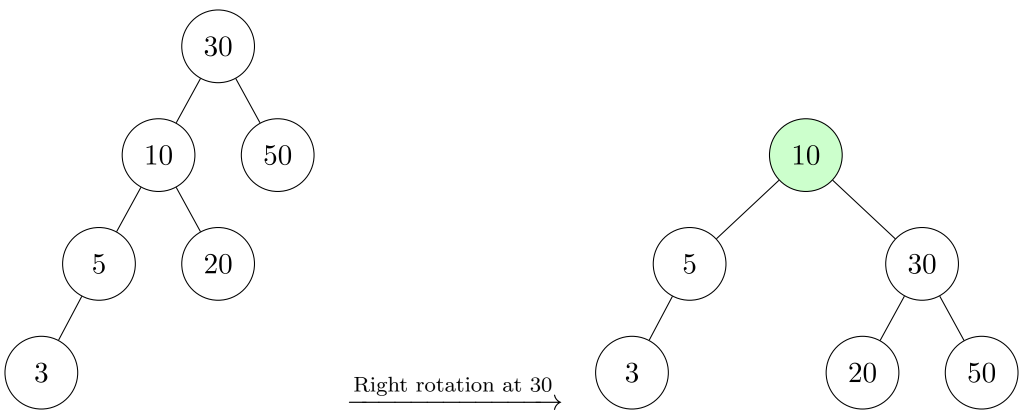
$$\begin{aligned}
h(3) &= 0, \quad \mathrm{bf}(3) = 0 \\
h(5) &= 1 + \max(0, -1) = 1, \quad \mathrm{bf}(5) = 0 - (-1) = 1 \\
h(20) &= 0, \quad \mathrm{bf}(20) = 0 \\
h(10) &= 1 + \max(1, 0) = 2, \quad \mathrm{bf}(10) = 1 - 0 = 1 \\
h(50) &= 0, \quad \mathrm{bf}(50) = 0 \\
h(30) &= 1 + \max(2, 0) = 3, \quad \mathrm{bf}(30) = 2 - 0 = \boxed{+2}
\end{aligned}$$

Node `30` is unbalanced with $\mathrm{bf}(30) = +2$.

Step 2: Identify the case

The path from inserted node `3` to the root is: $3 \to 5 \to 10 \to 30$. The first unbalanced node is $z = 30$. Its heavy child is $y = 10$ (left child), and the heavy direction within y is also to the left ($x = 5$). This is an **LL case** $\Rightarrow$ single right rotation at $z = 30$.

Step 3: Perform rotation



The final tree is balanced and maintains BST ordering.

Solution to Problem 2

Solution. **Step 1: Compute heights and balance factors**

$$
\begin{aligned}
h(5) &= 0, \quad \mathrm{bf}(5) = 0 \\
h(15) &= 0, \quad \mathrm{bf}(15) = 0 \\
h(50) &= 0, \quad \mathrm{bf}(50) = 0 \\
h(20) &= 1 + \max(0, -1) = 1, \quad \mathrm{bf}(20) = 0 - (-1) = 1 \\
h(10) &= 1 + \max(0, 1) = 2, \quad \mathrm{bf}(10) = 0 - 1 = -1 \\
h(30) &= 1 + \max(2, 0) = 3, \quad \mathrm{bf}(30) = 2 - 0 = \boxed{+2}
\end{aligned}
$$

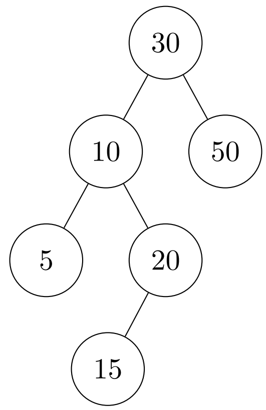
Node `30` is unbalanced with $\mathrm{bf}(30) = +2$.

Step 2: Identify the case

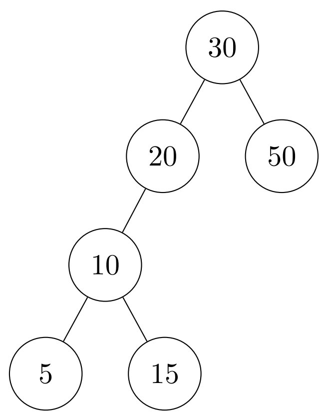
The heavy child $y = 10$ is right-heavy ($\mathrm{bf}(10) = -1$). The inserted node lies in the left subtree of 10's right child. This is an **LR case** $\Rightarrow$ double rotation: left at $y = 10$, then right at $z = 30$.

Step 3: Perform rotations

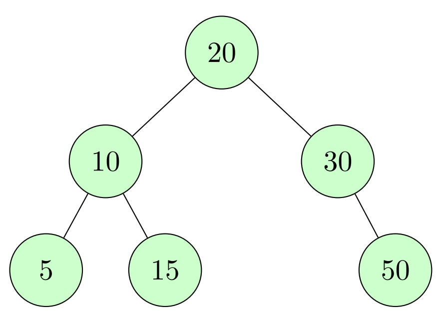
Initial tree:



↓ *Left rotation at 10*



↓ *Right rotation at 30*



The final tree is balanced.

Solution to Problem 3

Solution. Step 1: Compute heights and balance factors

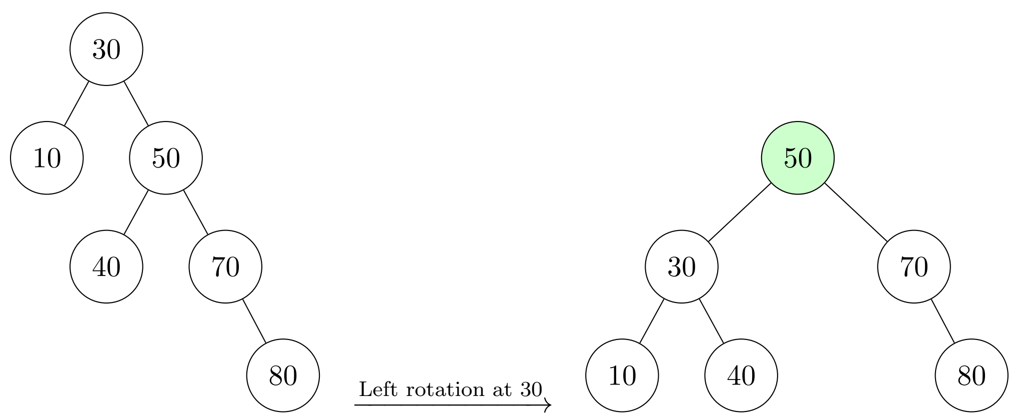
$$
\begin{aligned}
&h(10) = 0, \quad \mathrm{bf}(10) = 0\\
&h(40) = 0, \quad \mathrm{bf}(40) = 0\\
&h(80) = 0, \quad \mathrm{bf}(80) = 0\\
&h(70) = 1 + \max(-1, 0) = 1, \quad \mathrm{bf}(70) = -1 - 0 = -1\\
&h(50) = 1 + \max(0, 1) = 2, \quad \mathrm{bf}(50) = 0 - 1 = -1\\
&h(30) = 1 + \max(0, 2) = 3, \quad \mathrm{bf}(30) = 0 - 2 = \boxed{-2}
\end{aligned}
$$

Node `30` is unbalanced with $\mathrm{bf}(30) = -2$.

Step 2: Identify the case

The heavy child is $y = 50$ (right child), which is also right-heavy. This is an **RR case** $\Rightarrow$ single left rotation at $z = 30$.

Step 3: Perform rotation



The final tree is balanced.

Solution to Problem 4

Solution. **Step 1: Compute heights and balance factors**

$$
\begin{aligned}
&h(10) = 0, \quad \mathrm{bf}(10) = 0\\
&h(45) = 0, \quad \mathrm{bf}(45) = 0\\
&h(70) = 0, \quad \mathrm{bf}(70) = 0\\
&h(40) = 1 + \max(-1, 0) = 1, \quad \mathrm{bf}(40) = -1 - 0 = -1\\
&h(50) = 1 + \max(1, 0) = 2, \quad \mathrm{bf}(50) = 1 - 0 = 1\\
&h(30) = 1 + \max(0, 2) = 3, \quad \mathrm{bf}(30) = 0 - 2 = \boxed{-2}
\end{aligned}
$$

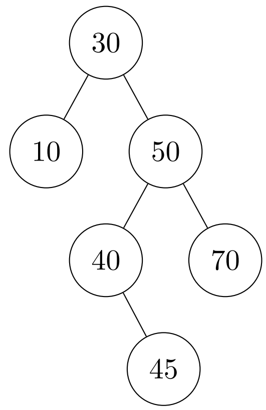
Node `30` is unbalanced with $\mathrm{bf}(30) = -2$.

Step 2: Identify the case

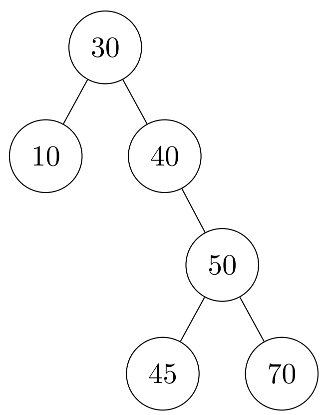
The heavy child is $y = 50$ (right child), but y is left-heavy ($\mathrm{bf}(50) = +1$). This is an **RL case** $\Rightarrow$ double rotation: right at $y = 50$, then left at $z = 30$.

Step 3: Perform rotations

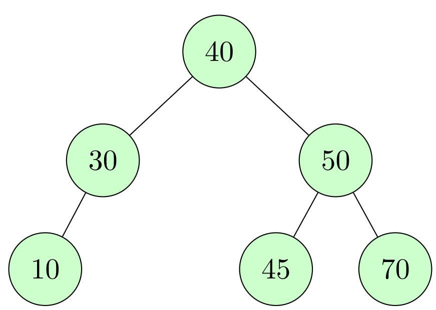
Initial tree:



↓ *Right rotation at 50*



↓ *Left rotation at 30*



The final tree is balanced.

Solution to Problem 5

Solution. **Step 1: Compute heights and balance factors**

Starting from the leaves and working upward:

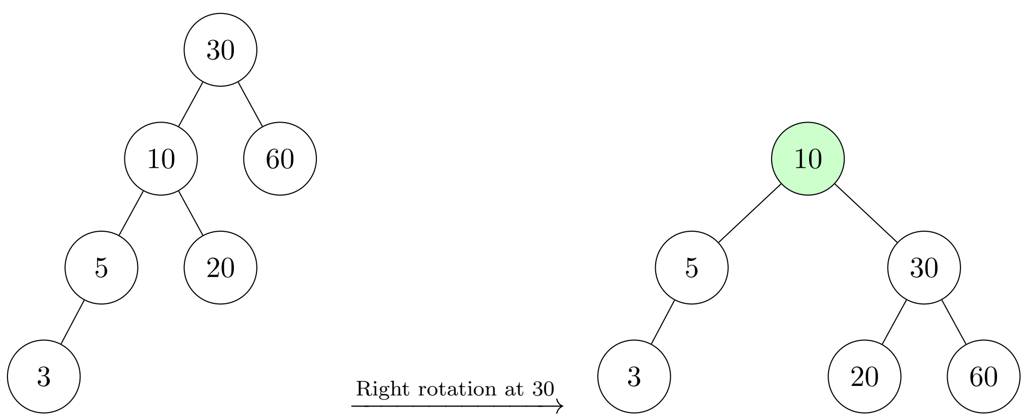
$$\begin{aligned}
h(3) &= 0, \quad \mathrm{bf}(3) = 0 \\
h(5) &= 1 + \max(0, -1) = 1, \quad \mathrm{bf}(5) = 0 - (-1) = 1 \\
h(20) &= 0, \quad \mathrm{bf}(20) = 0 \\
h(10) &= 1 + \max(1, 0) = 2, \quad \mathrm{bf}(10) = 1 - 0 = 1 \\
h(60) &= 0, \quad \mathrm{bf}(60) = 0 \\
h(30) &= 1 + \max(2, 0) = 3, \quad \mathrm{bf}(30) = 2 - 0 = \boxed{+2}
\end{aligned}$$

Node `30` is unbalanced with $\mathrm{bf}(30) = +2$.

Step 2: Identify the case

The path from inserted node `3` to the root is: $3 \to 5 \to 10 \to 30$. The first unbalanced node is $z = 30$. Its heavy child is $y = 10$ (left child), and the heavy direction within y is also to the left ($x = 5$). This is an **LL case** $\Rightarrow$ single right rotation at $z = 30$.

Step 3: Perform rotation



The final tree is balanced and maintains BST ordering.

Solution to Problem 6

Solution. **Step 1: Compute heights and balance factors**

$$
\begin{aligned}
&h(10) = 0, \quad \mathrm{bf}(10) = 0 \\
&h(25) = 0, \quad \mathrm{bf}(25) = 0 \\
&h(70) = 0, \quad \mathrm{bf}(70) = 0 \\
&h(30) = 1 + \max(0, -1) = 1, \quad \mathrm{bf}(30) = 0 - (-1) = 1 \\
&h(20) = 1 + \max(0, 1) = 2, \quad \mathrm{bf}(20) = 0 - 1 = -1 \\
&h(40) = 1 + \max(2, 0) = 3, \quad \mathrm{bf}(40) = 2 - 0 = \boxed{+2}
\end{aligned}
$$

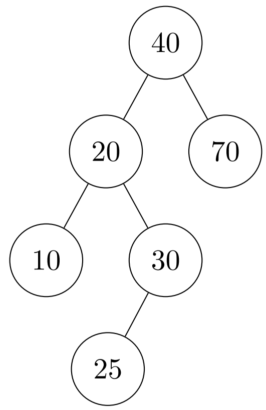
Node `40` is unbalanced with $\mathrm{bf}(40) = +2$.

Step 2: Identify the case

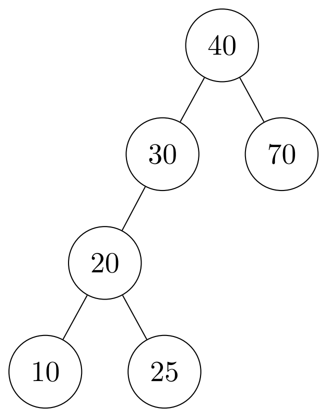
The heavy child $y = 20$ (left child) is right-heavy ($\mathrm{bf}(20) = -1$). The inserted node `25` lies in the right subtree of the left child. This is an **LR case** $\Rightarrow$ double rotation: left at $y = 20$, then right at $z = 40$.

Step 3: Perform rotations

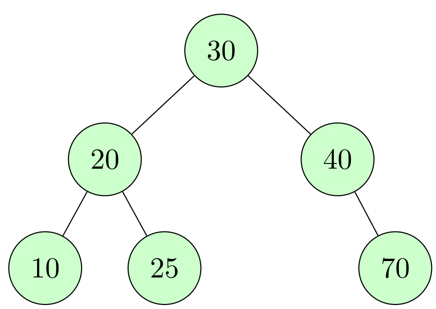
Initial tree:



↓ *Left rotation at 20*



↓ *Right rotation at 40*



The final tree is balanced.

Solution to Problem 7

Solution. Step 1: Compute heights and balance factors

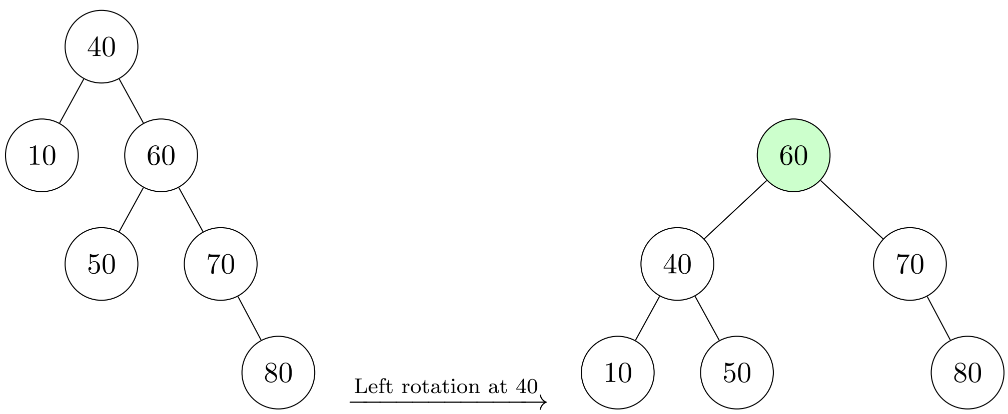
$$\begin{aligned}
&h(10) = 0, \quad \text{bf}(10) = 0 \\
&h(50) = 0, \quad \text{bf}(50) = 0 \\
&h(80) = 0, \quad \text{bf}(80) = 0 \\
&h(70) = 1 + \max(-1, 0) = 1, \quad \text{bf}(70) = -1 - 0 = -1 \\
&h(60) = 1 + \max(0, 1) = 2, \quad \text{bf}(60) = 0 - 1 = -1 \\
&h(40) = 1 + \max(0, 2) = 3, \quad \text{bf}(40) = 0 - 2 = \boxed{-2}
\end{aligned}$$

Node `40` is unbalanced with $\text{bf}(40) = -2$.

Step 2: Identify the case

The heavy child is $y = 60$ (right child), which is also right-heavy ($\text{bf}(60) = -1$). The inserted node `80` lies in the right subtree of the right child. This is an **RR case** $\Rightarrow$ single left rotation at $z = 40$.

Step 3: Perform rotation



The final tree is balanced.

Solution to Problem 8

Solution. **Step 1: Compute heights and balance factors**

$$
\begin{aligned}
&h(10) = 0, \quad \mathrm{bf}(10) = 0\\
&h(55) = 0, \quad \mathrm{bf}(55) = 0\\
&h(80) = 0, \quad \mathrm{bf}(80) = 0\\
&h(50) = 1 + \max(-1, 0) = 1, \quad \mathrm{bf}(50) = -1 - 0 = -1\\
&h(60) = 1 + \max(1, 0) = 2, \quad \mathrm{bf}(60) = 1 - 0 = 1\\
&h(40) = 1 + \max(0, 2) = 3, \quad \mathrm{bf}(40) = 0 - 2 = \boxed{-2}
\end{aligned}
$$

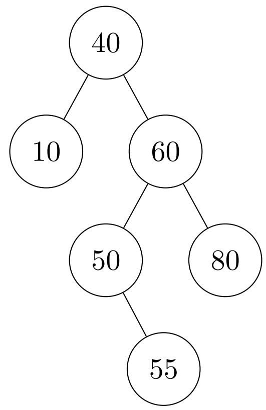
Node `40` is unbalanced with $\mathrm{bf}(40) = -2$.

Step 2: Identify the case

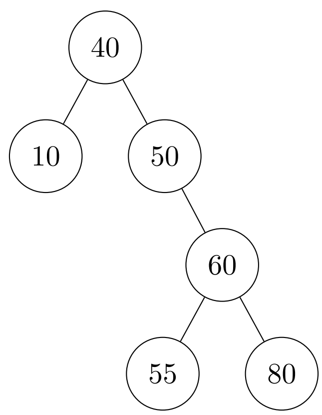
The heavy child is $y = 60$ (right child), but y is left-heavy ($\mathrm{bf}(60) = +1$). The inserted node `55` lies in the left subtree of the right child. This is an **RL case** $\Rightarrow$ double rotation: right at $y = 60$, then left at $z = 40$.

Step 3: Perform rotations

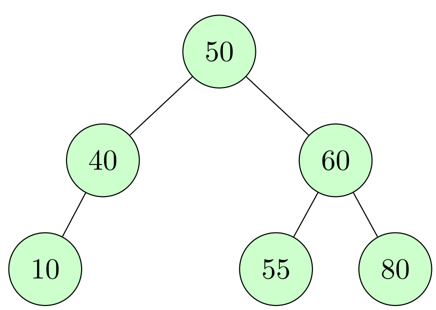
Initial tree:



↓ *Right rotation at 60*



↓ *Left rotation at 40*



The final tree is balanced.