Notes on linear algebra

imyp

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Definition of a linear space

> The concept of a linear space generalizes that of the set of all vectors.

A set $\mathbf{K}$ is called a *linear* (or *affine*) *space over a field* K if

1. Given two elements $x, y \in \mathbf{K}$, there is a rule (the addition rule) leading to a (unique) element $x + y \in \mathbf{K}$, called the *sum* of x and y.

2. Given any element $x \in \mathbf{K}$ and any number $\lambda \in K$, there is a rule (the rule for multiplication by a number) leading to a (unique) element $\lambda x \in \mathbf{K}$, called the *product* of the element x and the number λ.

3. These two rules obey 8 specific axioms.

Properties of the addition rule

1. $x + y = y + x$ for every $x, y \in \mathbf{K}$ (commutative);

2. $(x + y) + z = x + (y + z)$ for every $x, y, z \in \mathbf{K}$ (associative);

3. There exists an element $0 \in \mathbf{K}$ (the *zero vector*) such that $x + 0 = x$ for every $x \in \mathbf{K}$;

4. For every $x \in \mathbf{K}$ there exists an element $y \in \mathbf{K}$ (the *negative element*) such that $x + y = 0$.

Properties of the addition rule

Commutative:

$$x + y = y + x \text{ for every } x, y \in \mathbf{K}$$

Associative:

$$(x + y) + z = x + (y + z) \text{ for every } x, y, z \in \mathbf{K}$$

Identity element:

$$\text{An element } \mathbf{0} \in \mathbf{K} \text{ exists, such that } x + \mathbf{0} = x \text{ for every } x \in K$$

Inverse element:

$$\text{For every } x \in \mathbf{K} \text{ there exist an element } y \in \mathbf{K} \text{ such that } x + y = \mathbf{0}$$

Properties of the rule for multiplication by a number

1. $1 \cdot x = x$ for every $x \in \mathbf{K}$;
2. $\alpha(\beta x) = (\alpha\beta)x$ for every $x \in \mathbf{K}$ and every $\alpha, \beta \in K$;
3. $(\alpha + \beta)x = \alpha x + \beta x$ for every $x \in \mathbf{K}$ and every $\alpha, \beta \in K$;
4. $\alpha(x + y) = \alpha x + \alpha y$ for every $x, y \in \mathbf{K}$ and every $\alpha \in K$.

Properties of multiplication

Identity element:

$$1 \cdot x = x \text{ for every } x \in \mathbf{K}$$

Associative:

$$\alpha(\beta x) = (\alpha\beta)x \text{ for every } x \in \mathbf{K} \text{ and every } \alpha, \beta \in K$$

Distributive with respect to $\mathbf{K}$:

$$(\alpha + \beta)x = \alpha x + \beta x \text{ for every } x \in \mathbf{K} \text{ and every } \alpha, \beta \in K$$

Distributive with respect to K:

$$\alpha(x + y) = \alpha x + \alpha y \text{ for every } x, y \in \mathbf{K} \text{ and every } \alpha \in K$$

Linear dependence

Let $x_1, x_2, \ldots, x_k$ be vectors of the linear space $\mathbf{K}$ over a field K. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be numbers from K. Then the vector $\boxed{y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k}$ is called a *linear combination* of the vectors $x_1, x_2, \ldots, x_k$. The numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ are called the *coefficients* of the linear combination.