Mathematical background

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Outline

Logic

Statements

- A statement A may be true or false

Unary operators

- negation: $\neg A$ is true if A is false (and vice-versa).

Binary operators

- or: $A \vee B$ (A or B) is true if either A or B are true.
- and: $A \wedge B$ is true if both A and B are true.
- implies: $A \Rightarrow B$: is false if A is true and B is false.
- iff: $A \Leftrightarrow B$: is true if A, B have equal truth values.

Operator precedence

$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

Set theory

- First, consider some universal set Ω and the empty set $\emptyset$
- A set A is a collection of points x in Ω.
- $\{x \in \Omega : f(x)\}$: the set of points in Ω for which $f(x)$ is true.

Unary operators

- $\neg A = \{x \in \Omega : x \notin A\}$.

Binary operators

- $A \cup B$ if $\{x \in \Omega : x \in A \vee x \in B\}$ - (c.f. $A \vee B$)
- $A \cap B$ if $\{x \in \Omega : x \in A \wedge x \in B\}$ - (c.f. $A \wedge B$)

Binary relations

- $A \subset B$ if $x \in A \Rightarrow x \in B$ - (c.f. $A \implies B$)
- $A = B$ if $x \in A \Leftrightarrow x \in B$ - (c.f. $A \Leftrightarrow B$)

Interesting cases

- If $A \cap B = \emptyset$, then they are disjoint, or mutually exclusive.
- If $A \cap B = A$ only if $A \subset B$.

Probability fundamentals

Probability measure P

- Defined on a universe Ω
- $P : \Sigma \to [0,1]$ is a function of subsets of Ω.
- A subset $A \subset \Omega$ is an event and P measures its likelihood.

Axioms of probability

- $P(\Omega) = 1$
- For $A, B \subset \Omega$, if $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$.

Partition

$\{A_i\}$ is a partition of Ω if $A_i \cap A_j = \emptyset \ \forall i \neq j$ and $\bigcup_{i=1}^{n} A_i = \Omega$. A partition of Ω defines a complete set of mutually exclusive alternatives.

Marginalisation

Let $A_1, \ldots, A_n \subset \Omega$ be a partition of Ω. Then

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i).$$

Conditional probability

Definition (Conditional probability)

The conditional probability of an event A given an event B is defined as

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$$

The above definition requires $P(B)$ to exist and be positive.

Conditional probabilities as a collection of probabilities

More generally, we can define conditional probabilities as simply a collection of probability distributions:

$$\{P_\theta(A) \, |\theta \in \Theta\},$$

where Θ is an arbitrary set.

The theorem of Bayes

Theorem (Bayes's theorem)

$$P(A|B) = \frac{P(B|A)}{P(B)}$$

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The general case

If $A_1, \ldots, A_n$ are a partition of Ω, meaning that they are mutually exclusive events (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$) such that one of them must be true (i.e. $\bigcup_{i=1}^{n} A_i = \Omega$), then

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

and

$$P(A_j|B) = \frac{P(B|A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

Independence

Independent events

A, B are independent iff $P(A \cap B) = P(A)P(B)$.

Conditional independence

A, B are conditionally independent given C iff
$P(A \cap B|C) = P(A|C)P(B|C)$.

Random variables

A random variable $f : \Omega \to \mathbb{R}$ is a real-value function measurable with respect to the underlying probability measure P, and we write $f \sim P$.

The distribution of f

The probability that f lies in some subset $A \subset \mathbb{R}$ is

$$P_f(A) \triangleq P(\{\omega \in \Omega : f(\omega) \in A\}).$$

Independence

Two RVs f, g are independent in the same way that events are independent:

$$P(f \in A \wedge g \in B) = P(f \in A)P(g \in B) = P_f(A)P_g(B).$$

In that sense, $f \sim P_f$ and $g \sim P_g$.

IID (Independent and Identically Distributed) random variables

A sequence x_t of r.v.s is IID if $x_t \sim P$ $(x_1, \ldots, x_t, \ldots, x_T) \sim P^T$.

Expectation

For any real-valued random variable $f : \Omega \to \mathbb{R}$, the expectation with respect to a probability measure P is

$$\mathbb{E}_P(f) = \sum_{\omega \in \Omega} f(\omega) P(\omega).$$

Linearity of expectations

For any RVs x, y, $\mathbb{E}_P(x + y) = \mathbb{E}_P(x) + \mathbb{E}_P(y)$

Correlation

If x, y are not correlated then $\mathbb{E}_P(xy) = \mathbb{E}(x)\,\mathbb{E}(y)$.

Independence

If x, y are independent RVs then they are also uncorrelated (but not vice-versa)

Conditional expectation

The conditional expectation of a random variable $f : \Omega \to \mathbb{R}$, with respect to a probability measure P conditioned on some event B is simply

$$\mathbb{E}_P(f|B) = \sum_{\omega \in \Omega} f(\omega) P(\omega|B).$$

Variance

For any real-valued random variable $f : \Omega \to \mathbb{R}$, the variance with respect to a probability measure P is

$$\mathbb{V}_P(f) = \sum_{\omega \in \Omega} [f(\omega) - \mathbb{E}_P(f(\omega)]^2 P(\omega).$$

Linearity of variance

If f, g are uncorrelated RVs

$$\mathbb{V}_P(f+g) = \mathbb{V}_P(f) + \mathbb{V}_P(g).$$

Variance products

If f, g are independent RVs

$$\mathbb{V}_P(f+g) = \mathbb{E}_P(f)^2\, \mathbb{V}_P(g) + \mathbb{E}_P(g)^2\, \mathbb{V}_P(f) + \mathbb{V}_P(f)\, \mathbb{V}_P(g).$$

Vector space F axioms

Here we consider a vector space F. Typically, it is a subset of the Euclidean d-dimensional space, ie. $F \subset \mathbb{R}^d$.

- $(x + y) + z = x + (y + z)$, for all $x, y, z \in F$.
- $x + y = y + x$, for all $x, y \in F$.
- There is a zero element $0 \in F$ such that $x + 0 = 0$ for all $x \in F$.
- For all $x \in F$, there is an element $-x \in F$ so that $x + (-x) = 0$.
- $a(x + y) = ax + ay$, For any $a \in \mathbb{R}$, $x, y \in F$.
- $(a + b)x = ax + bx$, For any $a, b \in \mathbb{R}$, $x \in F$.

The real vector space $F = \mathbb{R}^d$

For $a \in \mathbb{R}$ and $x, y \in F$,

- $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$
- $x + y = (x_1 + y_1, \ldots, x_d + y_d)$.
- $ax = (ax_1, \ldots, ax_d)$.
- $-x = (-1)x$.
- $0 = (0, \ldots, 0)$

Linear operators

Linear operator $A : F \to G$

- $A(x + y) = Ax + Ay$
- $A(ax) = a(Ax)$.

Matrices in $\mathbb{R}^{n\times m}$.

A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times m}$ is a tabular array $\boldsymbol{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix}$ Matrices can be seen as linear operators when used to multiply vectors.

Multiplication operators

Matrix multiplication

For $A \in \mathbb{R}^{n \times d}$, $B \in \mathbb{R}^{d \times m}$, the ij-th element of the result of the multiplication AB is

$$(AB)_{i,j} = \sum_{k=1}^{d} A_{i,k} B_{k,j}.$$

so that $AB \in \mathbb{R}^{n \times m}$.

Matrix-vector multiplication

A matrix $A \in \mathbb{R}^{n \times m}$ defines the following linear operator $A : \mathbb{R}^m \to \mathbb{R}^n$.

$$Ax = \left(\sum_{j=1}^{m} A_{i,j} x_j : i = 1, \ldots, n \right)$$

All vectors $x \in \mathbb{R}^m$ are equivalent to matrices in $\mathbb{R}^{m \times 1}$.

Matrix inverses

The identity matrix $I \in \mathbb{R}^{n \times n}$

- For this matrix, $I_{i,i} = 1$ and $I_{i,j} = 0$ when $j \neq i$.
- $Ix = x$ and $IA = A$.

The inverse of a matrix $A \in \mathbb{R}^{n \times n}$

A^{-1} is called the inverse of A if

- $AA^{-1} = I$.
- or equivalently $A^{-1}A = I$.

The pseudo-inverse of a matrix $A \in \mathbb{R}^{n \times m}$

- $\tilde{A}^{-1}$ is called the left pseudoinverse of A if $\tilde{A}^{-1}A = I$.
- $\tilde{A}^{-1}$ is called the right pseudoinverse of A if $A\tilde{A}^{-1} = I$.

Derivatives

Derivative

The derivative of a single-argument function is defined as:

$$\frac{d}{dx}f(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}.$$

f must be absolutely continuous at x for the derivative to exist.

Subdifferential

For non-differential functions, we can sometimes define the set of all subderivatives:

$$\partial f(x) = [\lim_{\epsilon \to 0} \frac{f(x) - f(x-\epsilon)}{\epsilon}, \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}]$$

Integrals

Riemann integral

The Reimann integral is obtained by taking a horizontal discretisation of a function to the limit:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{t=1}^{n} f(x_t)\frac{b-a}{n}, \qquad x_t = a + (t-1)\cdot\frac{b-a}{n}$$

Lebesgue integral

This integral is obtained by taking a vertical discretisation of a function to the limit. Let λ be the Lebesgue measure (i.e. area) of a set. Then:

$$\int_X f(x)d\lambda(x) = \lim_{n\to\infty} \sum_{t=1}^{n} y_t\lambda(S_t),$$

$S_t = \{x : f(x) \in (y_{t-1}, y_t\}$, $y_0 = -\infty$, $y_n = \sup_x f(x)$.

Fundamental theorem of calculus

$$f(x) = \frac{d}{dx}\int_a^x f(t)dt$$

If $\frac{d}{dx}F = f$ then its integral from a to b is:

$$\int_a^b f(x)dx = F(b) - F(a),$$

Multivariate Functions

We consider functions operating in multi-dimensional Euclidean spaces.

$f : \mathbb{R}^n \to \mathbb{R}$.

- Any $x \in \mathbb{R}^n$ is $x = (x_1, \ldots, x_n)$, with $x_i \in \mathbb{R}$.
- We write $f(x)$ instead of $f(x_1, \ldots, x_n)$.

$f : \mathbb{R}^n \to \mathbb{R}^m$.

- If $y = f(x)$ then y_i is the i-th component of $y \in \mathbb{R}^m$.
- Can be seen as m functions $f_i : \mathbb{R}^n \to \mathbb{R}$, with $y_i = f_i(x)$.

Derivatives in many dimensions

Partial derivative

The partial derivative of $f : \mathbb{R}^n \to \mathbb{R}$ with respect to its i-th argument is: $\frac{\partial}{\partial x_i} f(x)$, where we see all x_j with $j \neq i$ as fixed.

Gradient of f

This is the vector of all its partial derivatives:

$$\nabla_x f(x) = \left(\frac{\partial}{\partial x_1} f(x) \cdots \frac{\partial}{\partial x_i} f(x) \cdots \frac{\partial}{\partial x_n} f(x) \right)^\top$$

When $f : \mathbb{R}^n \to \mathbb{R}^m$, the gradient is an $n \times m$ matrix called the Jacobian.

Directional derivative

We can also define the derivative with respect to a direction $\delta \in$ Rⁿ$:

$$D_\delta f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon\delta) - f(x)}{\epsilon}.$$

For simplicity say that $\|\delta\| = 1$.