Matrix Factorizations

$A = LU = \begin{pmatrix}\text{lower triangular } L \\ \text{1's on the diagonal}\end{pmatrix} \begin{pmatrix}\text{upper triangular } U \\ \text{pivots on the diagonal}\end{pmatrix}$

Requirements: No row exchanges as Gaussian elimination reduces square A to U

$A = LDU = \begin{pmatrix}\text{lower triangular } L \\ \text{1's on the diagonal}\end{pmatrix} \begin{pmatrix}\text{pivot matrix } D \\ \text{is diagonal}\end{pmatrix} \begin{pmatrix}\text{upper triangular } U \\ \text{1's on the diagonal}\end{pmatrix}$

Requirements: No row exchanges. The pivots in D are divided out to leaves 1's on the diagonal of U. If A is symmetric then U is L^T and $A = LDL^T$.

$PA = LU =$ (permutation matrix$P$to avoid zeros in the pivot positions.)

Requirements: A is invertible. Then P,L,U are invertible.

$EA = R\,(m$ by m invertible $E)$ (any m by n matrix $A) =$ (reduced row echelon form R)

Requirements: None!

$S = C^T C =$ (lower triangular) (upper triangular) with $\sqrt{D}$ on both diagonals

Requirements: S is symmetric and positive definite.

$A = QR =$ (orthonormal columns in Q) (upper triangular R)

Requirements: A has independent columns. Those are orthogonalized in Q by the Gram-Schmidt process.

$A = X\Lambda X^{-1} =$ (eigenvectors in X) (eigenvalues in Λ) (left eigenvectors in X^{-1})

Requirements: A must have n linearly independent eigenvectors.

$S = Q\Lambda Q^T =$ (orthogonal matrix Q) (real eigenvalue matrix Λ) (Q^T is Q^{-1})

Requirements: S is real and symmetric: $S^T = S$. This is the **Spectral Theorem.**

$A = U\Sigma V^T = \begin{pmatrix}\text{orthogonal} \\ U \text{ is } m \times m\end{pmatrix} \begin{pmatrix}m \times n \text{ singular value matrix} \\ \sigma_1, \sigma_2, \cdots, \sigma_r \text{ on its diagonal}\end{pmatrix} \begin{pmatrix}\text{orthogonal} \\ V \text{ is } n \times n\end{pmatrix}$

Requirements: None.

Eigenvalues and Eigenvectors of Special Matrices

Matrix	**Eigenvalues**	**Eigenvectors**
Symmetric: $S^T = S = Q\Lambda Q^T$	real eigenvalues	orthogonal $x_i^T x_j = 0$
Orthogonal: $Q^T = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\bar{x}_i^T x_j = 0$
Skew-symmetric: $A^T = -A$	imaginary λ's	orthogonal $\bar{x}_i^T x_j = 0$
Positive Definite: $x^T S x > 0$	all $\lambda > 0$	orthogonal $x_i^T x_j = 0$
Similar: $A = BCB^{-1}$	$\lambda(A) = \lambda(C)$	B times eigenvector of C
Projection: $P = P^2 = P^T$	$\lambda = 1;\ 0$	column space; null space
Rank One: uv^T	$\lambda = v^T u;\ 0, \cdots, 0$	u ; whole plane $v^\perp$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Diagonalizable: $A = X\Lambda X^{-1}$	diagonal of Λ	columns of X