Show that for d, the true-distance[1] to the true-nearest[1] food and n, the number of remaining food,

$$h(d,n) = \begin{cases} d+n-1 & n \geq 1 \\ 0 & n = 0 \end{cases}$$

is a valid heuristic for all $n \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$.

Proof. We will show that:

1. h never overestimates the cost of reaching the goal.
2. The consistency inequality where d' and n' are d and n after taking one action[2]:

$$h(d,n) \leq 1 + h(d',n')$$

is always satisfied.

Lemma 1 shows admissibility and Lemma 2 shows consistency.

Lemma 1: h is admissible. Pac-man must always travel d or more to the true-nearest food dot. After that, it must travel at least $n-1$ farther for each food.
$\therefore$ h never overestimates the cost of reaching the goal.

Lemma 2: h is consistent.
Base Case: h is consistent when $d = 1$ (Pac-man is right next to the food).
If Pac-man moves towards the true-nearest food and $n = 1$, then $n' = 0$ and h returns 0. The consistency inequality is:

$$h(d,n) \leq 1 + h(d',n')$$

$$1+1-1 \leq 1+0$$

$$1 \leq 1$$

If Pac-man moves towards the true-nearest food and $n \geq 2$, then $n' = n-1$ and $d' \in \mathbb{Z}^+$. The consistency inequality is:

$$h(d,n) \leq 1 + h(d',n')$$

$$1+n-1 \leq 1+d'+n'-1$$

$$n \leq d'+n-1$$

$$1 \leq d'$$

If Pac-man moves away from the true-nearest food, then $n' = n$ and $d' \geq d$. The consistency inequality is:

$$h(d,n) \leq 1 + h(d',n')$$

[1]"True-" refers to manhattan distance and accounting for walls.
[2]One move up, down, left, or right

$$1 + n - 1 \leq 1 + d' + n' - 1$$
$$n \leq d' + n$$
$$0 \leq d'$$

The consistency inequality is satisfied.

Inductive Hypothesis: If h is consistent when $d = k, k \in \mathbb{Z}^+$, then h is consistent when $d = k + 1$.

Inductive Step: [I must show that $h(d_{k+1}, n) \leq 1 + h(d'_{k+1}, n)$]
If Pac-man moves towards the true-nearest food, then $d'_{k+1} = d_{k+1} - 1$ and $n' = n$. The consistency inequality is:

$$h(d_{k+1}, n) \leq 1 + h(d'_{k+1}, n')$$
$$d_{k+1} + n - 1 \leq 1 + d_{k+1} - 1 + n - 1$$
$$0 \leq 0$$

If Pac-man moves away from the true-nearest food, then $d'_{k+1} \geq d_{k+1}$ and $n' = n$. The consistency inequality is:

$$h(d_{k+1}, n) \leq 1 + h(d'_{k+1}, n')$$
$$d_{k+1} + n - 1 \leq 1 + d'_{k+1} + n' - 1$$
$$d_{k+1} \leq 1 + d'_{k+1}$$
$$d_{k+1} \leq d'_{k+1} \leq 1 + d'_{k+1}$$

$\therefore$ By the principle of induction, h remains consistent for all $d \in \mathbb{Z}^+$ and all $n \in \mathbb{Z}^+$

$\blacksquare$