S.Y.J.C (2019-20)	**MATHS PAPER - I**	**12/12/19** **Thursday**	**Time:- 1 ½ hrs.** **Marks: 40**
Unit Test 1	***Topic: Derivative; Integration & Definite Integration & Continuity***		**08.00 am to 09.30 am**

Q.1. Attempt All Questions:- **(8)**

(i) Find the value of 'k' if the function

$$f(x)=\frac{\tan 7x}{2x}, \quad \text{for } x\neq 0$$
$$= k, \quad \text{for } x=0$$

is continuous at $x=0$

(ii) Evaluate : $\int x\cos x\,dx$.

(iii) Evaluate : $\int \frac{1}{x(3+\log x)}dx$

(iv) Find $\frac{dy}{dx}$ if $y=\cos^{-1}\left(\sqrt{x}\right)$

Q.2. Attempt All Question:- **(12)**

Q-2.

(i) Find $\frac{dy}{dx}$ if $y=\tan^{-1}\left(\frac{6x}{1-5x^2}\right)$

(ii) Find $\frac{dy}{dx}$ if $x=e^{2t}$, $y=e^{\sqrt{t}}$

(iii) Examine the continuity of the following function :

$$\left.\begin{aligned} f(x) &= x^2-x+9, && \text{for } x\leq 3\\ &= 4x+3, && \text{for } x>3 \end{aligned}\right\} \text{at } x=3$$

(iv) If 'f' is continuous at $x=0$, then find $f(0)$.

$$f(x)=\frac{15^x-3^x-5^x+1}{x\tan x}, \quad x\neq 0$$

Q.3. Attempt All Question:- **(20)**

Q-3.

(i) Evaluate : $\int \frac{(1+\log x)}{x(2+\log x)(3+\log x)} dx$

(ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot x}$

(iii) Discuss the continuity of the function f at $x = 0$,

where $f(x) = \frac{5^x + 5^{-x} - 2}{\cos 2x - \cos 6x}$, for $x \neq 0$

$= \frac{1}{8}(\log 5)^2$, for $x = 0$

(iv) If $x^y = e^{x-y}$ Show $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

(v) $\int_0^3 \frac{dx}{x+\sqrt{9-x^2}}$

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Unit Test 1	*Topic: Derivative; Integration & Definite Integration & Continuity*		**Solution**

(1)

1) (1) $f(x) = k$ (for $x=0$)

$f(0) = k$ —— (I)

$f(x) = \dfrac{\tan 7x}{2x}$ $(x \neq 0)$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\tan 7x}{2x}$

$= \dfrac{1}{2} \lim\limits_{x\to 0} \dfrac{\tan 7x}{7x} \cdot 7$

$= \dfrac{7}{2} \lim\limits_{x\to 0} \dfrac{\tan 7x}{7x}$

$= \dfrac{7}{2}(1)$

$= \dfrac{7}{2}$ —— (II) —— 1 Mark

$\because f$ is continuous at $x=0$

By defn

$\lim\limits_{x\to 0} f(x) = f(0)$

$\dfrac{7}{2} = k$ —— 1 Mark

ii) $I = \int x \cos x \, dx$

By parts

$x\int \cos x \, dx - \int\left(\dfrac{d}{dx} x \int(\cos x \, dx)\right)dx$ —— (1) Mark

$= x \sin x - \int(1 \times \sin x)dx$

$= x \sin x - \int \sin x \, dx$

$= x \sin x - (-\cos x) + c$

$= x \sin x + \cos x + c$ —— (1) Mark

(2)

(3) $\int \frac{1}{x(3+\log x)} dx$

put $3+\log x = t$

diff w.r.t x

$0 + \frac{1}{x} = \frac{dt}{dx}$

$\frac{dx}{x} = dt$ 1 Mark

$I = \int \frac{1}{t} dt$

$= \log|t| + C$

$= \log|3+\log x| + C$ 1 Mark

(iv) $y = \cos^{-1}\sqrt{x}$

diff w.r.t x

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(\sqrt{x})^2}} \frac{d}{dx}\sqrt{x}$ (1)

$= \frac{-1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$

$= \frac{-1}{2\sqrt{x}\sqrt{1-x}}$ Ans (1)

Q 2) (i) $y = \tan^{-1}\left(\frac{6x}{1-5x^2}\right)$

$= \tan^{-1}\left(\frac{5x+x}{1-5x\cdot x}\right)$

(3)

Using $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}x + \tan^{-1}y$

(1) mark

$y = \tan^{-1}(5x) + \tan^{-1}x$

diff w.r.t x

$$\frac{dy}{dx} = \frac{1}{1+(5x)^2}\frac{d}{dx}(5x) + \frac{1}{1+x^2}$$

$$= \frac{1}{1+25x^2}\cdot 5 + \frac{1}{1+x^2}$$

$$= \frac{5}{1+25x^2} + \frac{1}{1+x^2} \quad \text{Ans}$$ 1 mark

ii) $x = e^{2t}$

diff w.r.t t

$$\frac{dx}{dt} = e^{2t}\frac{d}{dt}2t$$

$$= e^{2t}\cdot 2$$

$$= 2e^{2t}$$ — (I)

(1) mark

$y = e^{\sqrt{t}}$

diff w.r.t t

$$\frac{dy}{dt} = e^{\sqrt{t}}\frac{d}{dt}\sqrt{t}$$

$$= e^{\sqrt{t}}\frac{1}{2\sqrt{t}}$$

(1) mark

$$= \frac{e^{\sqrt{t}}}{2\sqrt{t}}$$ — (II)

Using parametric rule

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{e^{\sqrt{t}}}{2\sqrt{t}}}{2e^{2t}} = \frac{e^{\sqrt{t}}}{2\sqrt{t}} \times \frac{1}{2e^{2t}} = \frac{e^{\sqrt{t}}}{4\sqrt{t}e^{2t}}$$

(1) Ans

(4)

(iii) LHS Limit

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} x^2 - x + 9$$

$$= 3^2 - 3 + 9$$

$$= 15 - I \quad \text{1 mark}$$

RHS limit

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3} 4x + 3 = 4(3)+3$$

$$= 15 - II$$

1 mark

from I & II

$$\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$$

∴ Limit does not exist & By defn

given function is not continuous at $x = 3$

1 mark

(iv) $f(x) = \dfrac{15^x - 3^x - 5^x + 1}{x \tan x}$ $(x \neq 0)$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{15^x - 3^x - 5^x + 1}{x \tan x}$$

$$\lim_{x \to 0} \frac{(3 \cdot 5)^x - 3^x - 5^x + 1}{D}$$

$$\lim_{x \to 0} \frac{3^x \cdot 5^x - 3^x - 5^x + 1}{D}$$

I

$$= \lim_{x \to 0} \frac{3^x(5^x - 1) - 1(5^x - 1)}{D}$$

$$= \lim_{x \to 0} \frac{(5^x - 1)(3^x - 1)}{x \tan x}$$ 1 mark

Divide N & D by x^2

$x \to 0$, $x \neq 0$, $x^2 \neq 0$

$$\lim_{x \to 0} \frac{\frac{(5^x - 1)(3^x - 1)}{x^2}}{\frac{x \tan x}{x^2}} \quad (x \to 0,\ x \neq 0)$$

$$\frac{\lim_{x \to 0} \frac{(5^x - 1)}{x} \lim_{x \to 0} \left(\frac{3^x - 1}{x}\right)}{\lim_{x \to 0} \frac{\tan x}{x}}$$

$$= \frac{\log 5 \cdot \log 3}{1}$$ 1 mark

$\lim_{x \to 0} f(x) = \log 5 \cdot \log 3$ — I

Given f is continuous at $x = 0$

$\therefore$ By defn

$\lim_{x \to 0} f(x) = f(0)$

$\log 5 \cdot \log 3 = f(0)$ Ans 1 mark

6

Q3) (ii) $\int \frac{1+\log x}{x(2+\log x)(3+\log x)}\,dx$

Put $\log x = t$

diff w.r.t. x

$\frac{1}{x} = \frac{dt}{dx}$

$\frac{dx}{x} = dt$ — 1 mark

$I = \int \frac{1+t}{(2+t)(3+t)}\,dt$

We use partial fraction

$\frac{A}{2+t} + \frac{B}{3+t} = \frac{1+t}{(2+t)(3+t)}$

$A(3+t) + B(2+t) = 1+t$

Put $3+t=0$

$t = -3$

$0 + B(2-3) = 1-3$

$-B = -2$

$B = 2$ 1 mark

Put $2+t=0$

$\therefore t = -2$

$A(3-2) + 0 = 1-2$ 1 mark

$A = -1$

(7)

$$= \frac{-1}{2+t} + \frac{2}{3+t} = \frac{1+t}{(2+t)(3+t)}$$

$$\therefore I = \int \left(\frac{-1}{2+t} + \frac{2}{3+t} \right) dt$$

$$= -1 \int \frac{1}{2+t}\, dt + 2 \int \frac{1}{3+t}\, dt$$

$$= -1 \ln|2+t| + 2 \ln|3+t| + C$$

$$= -\ln|2+\tan x| + 2 \ln|3+\tan x| + C$$

$$2 \ln|3+\tan x| - \ln|2+\tan x| + C$$ (1) mark

ii) $$I = \int_0^{\pi/2} \frac{dx}{1+\cot x}\, dx$$

$$= \int_0^{\pi/2} \frac{1}{1+\frac{\cos x}{\sin x}}\, dx$$

$$\int_0^{\pi/2} \frac{1}{\frac{\sin x+\cos x}{\sin x}}\, dx$$

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx \quad \text{— (1)}$$ 1 mark

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

replace x by $\pi/2 - x$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\, dx$$

(8)

$$I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx - \text{I}$$

Add I & II — 1 mark

$$I + I = \int_0^{\pi/2} \frac{}{}\,dx + \int^{\pi/2} \frac{}{}\,dx$$

$$2I = \int_0^{\pi/2} \left(\frac{}{} + \frac{}{} \right) dx$$

$$= \int_0^{\pi/2} \frac{\cancel{\sin x + \cos x}}{\cancel{\sin x + \cos x}}\,dx$$

$$= \int_0^{\pi/2} 1\,dx \quad \text{1 mark}$$

$$2I = \left[x \right]_0^{\pi/2}$$

$$2I = \left[\frac{\pi}{2} - 0 \right)$$

$$\boxed{I = \frac{\pi}{4}} \text{ Ans} \quad \text{1 mark}$$

(9) $f(x) = \frac{1}{8}(\log 5)^2 \quad (x = 0)$

$f(0) = \frac{1}{8}(\log 5)^2$ — (1) 1 mark

$$f(x) = \frac{5^x + 5^{-x} - 2}{\cos 2x - \cos 6x} \quad (x \neq 0)$$

(a)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{5^{x} + 5^{-x} - 2}{\underbrace{\cos 2x - \cos 6x}_{D}}$$

$$= \lim_{x\to 0} \frac{5^{x} + \frac{1}{5^{x}} - 2}{2\sin\left(\frac{2x+6x}{2}\right)\sin\left(\frac{6x-2x}{2}\right)}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{\frac{(5^{x})^2 + 1 - 2\cdot 5^{x}}{5^{x}}}{\sin 4x \, \sin 2x}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{5^{2x} - 2\cdot 5^{x} + 1}{\sin 4x \, \sin 2x \, 5^{x}}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{(5^{x}-1)^2}{D}$$ — 1 mark

Divide N & D by x^2

As $x \to 0$, $x \neq 0$, $x^2 \neq 0$

$$\frac{1}{2}\lim_{x\to 0}\left(\frac{\frac{(5^{x}-1)^2}{x^2}}{\frac{\sin 4x \, \sin 2x \, 5^{x}}{x^2}}\right)$$

$$\frac{\frac{1}{2}\lim_{x\to 0}\left(\frac{5^{x}-1}{x}\right)^2}{\lim_{x\to 0}\left(\frac{\sin 4x}{x}\cdot\frac{\sin 2x}{x}\cdot 5^{x}\right)}$$ 1 mark

(1c)

$\frac{1}{2} \lim_{x \to 0} \left(\frac{5^x - 1}{x}\right)^2$

$\lim_{x \to 0} \frac{\sin 4x}{4x} \cdot 4 \quad \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 \quad \lim_{x \to 0} 5^x$

$\frac{1}{2} \quad \frac{(\ln 5)^2}{1 \times 4 \times 1 \times 2 \times 5^0}$

$\lim_{x \to 0} f(x) = \frac{1}{16}(\ln 5)^2$ — II

for I & II

$\lim_{x \to 0} f(x) \neq f(0)$

By defn

f is not continuous at $x = 0$ 1 Mark

(1d) $x^y = e^{x-y}$

take log on both sides

$\ln x^y = \ln e^{x-y}$

$y \ln x = (x-y)\ln e$

$y \ln x = (x-y)1$

$y \ln x = x - y$

$y + y\ln x = x$

$y(1 + \ln x) = x$ 1 Mark

(ii)

$$y = \frac{x}{1+\ln x} \quad \text{--- 2 mark}$$

differentiate w.r.t x

using $\frac{u}{v}$ rule

$$\frac{dy}{dx} = \frac{(1+\ln x)\frac{d}{dx}x - x\frac{d}{dx}(1+\ln x)}{(1+\ln x)^2}$$

$$= \frac{(1+\ln x).1 - \cancel{x}(0+\frac{1}{\cancel{x}})}{D}$$

$$= \frac{1+\ln x - 1}{D}$$

$$= \frac{\ln x}{(1+\ln x)^2} \quad \text{(proved)}$$

2 mark

1 mark

(v) $$I = \int_0^3 \frac{dx}{x+\sqrt{9-x^2}}$$

put $x = 3\sin\theta$

diff w.r.t θ

(4) mark

$$\frac{dx}{d\theta} = 3\cos\theta$$

$$dx = 3\cos\theta\, d\theta$$

$x = 3\sin\theta$	$x = 3\sin\theta$
$x = 0$	$x = 3$
$0 = 3\sin\theta$	$3 = 3\sin\theta$
$0 = \sin\theta$	$1 = \sin\theta \Rightarrow \theta = \pi/2$ mark
$\Rightarrow \theta = 0$	

(12)

$$I = \int_0^{\pi/2} \frac{3\cos\theta \, d\theta}{3\sin\theta + \sqrt{9 - 9\sin^2\theta}}$$

$$\int_0^{\pi/2} \frac{3\cos\theta \, d\theta}{3\sin\theta + \sqrt{9(1-\sin^2\theta)}}$$

$$\int_0^{\pi/2} \frac{3\cos\theta}{3\sin\theta + 3\cos\theta} \, d\theta$$

$$I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta} \, d\theta \quad \text{— (I)} \quad \text{1 Mark}$$

Us $\Rightarrow \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

replace x by $\frac{\pi}{2} - x$

$$I = \int_0^{\pi/2} \frac{\cos(\frac{\pi}{2} - \theta)}{\sin(\frac{\pi}{2} - \theta) + \cos(\frac{\pi}{2} - \theta)} \, d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin x}{\cos\theta + \sin\theta} \, d\theta \quad \text{1 Mark}$$

Add I & II

$$I + I = \int_0^{\pi/2} \text{—} \, d\theta + \int_0^{\pi/2} \text{—} \, d\theta$$

$$= \int_0^{\pi/2} \left(\text{—} + \text{—} \right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\cos\theta + \sin\theta}{\sin\theta + \cos\theta} \right) d\theta$$

$$= \int_0^{\pi/2} 1 \, d\theta \quad \text{1 Mark}$$

$$= (\theta)_0^{\pi/2}$$

$$2I = \frac{\pi}{2} - 0 \quad \therefore \boxed{I = \frac{\pi}{4}} \quad \text{Ans}$$