20 Maximum Matching in Bipartite Graphs

Bipartite Graph: Connected undirected graph such that

1. The vertices of G are partitioned into two sets X and Y.

2. Every edge of G has one end point in X and the other in Y.

Matching M in G is a set of edges that have no end points in common

Maximum Bipartite Matching Problem: Find a matching with the greatest number of edges (over all matchings).

An Application: X represents a set of college courses, Y represents a set of classrooms. Let there be an edge joining x in X and y in Y if course x can be taught in classroom y (based on audio-visual needs, enrollment, etc.).

20.1 Reduction to the Maximum Flow Poblem

Set up the flow network

- Let G be the bipartite graph in which we want to find a max matching (whose vertices are partitioned into sets X and Y.)
- Create a flow network H as follows:
- H contains all the vertices of G plus a new source vertex s and a new sink vertex t.
- Add every edge of G to H (but direct it so that it "goes" from the vertex in X to the vertex in Y).
- Add a directed edge from s to each vertex in X.
- Add a directed edge from each vertex in Y to t.
- Assign a capacity of 1 to each edge.

Solve the maximum matching problem.

- Now, run the max-flow algorithm on H.

- The flow in each edge is either 0 or 1 (why?).
- For each vertex in X, there is at most one outgoing edge with a flow of 1 (why?).
- For each vertex in Y, there is at most one incoming edge with a flow of 1 (why?).
- So, we get a matching if we pair vertices $x \in X$ and $y \in Y$ such that there is a flow of 1 from x to y. (So, any flow gives a matching).
- Want to make sure that every matching M has a corresponding flow (otherwise we might miss out on a matching).
- Construct a flow by assigning a flow of 1 to each edge in M.
- Make sure that flow conservation is satisfied at each vertex.
- The value of the flow is equal to the size of the matching.
- So a max flow corresponds to a maximum matching.

Complexity Analysis

- Construct H from G: $O(V + E)$.
- Run FF algorithm: $(|f| \times E)$.
- But $|f| = |M| \leq V/2$.
- So $O(VE)$.