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Nontransitive Preferences in Decision Theory

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Abstract

Intransitive preferences have been a topic of curiosity, study, and debate over the past 40 years. Many economists and decision theorists insist on transitivity as the cornerstone of rational choice, and even in behavioral decision theory intransitivities are often attributed to faulty experiments, random or sloppy choices, poor judgment, or unexamined biases. But others see intransitive preferences as potential truths of reasoned comparisons and propose representations of preferences that accommodate intransitivities.

This article offers a partial survey of models for intransitive preferences in a variety of decisional contexts. These include economic consumer theory, multiattribute utility theory, game theory, preference between time streams, and decision making under risk and uncertainty. The survey is preceded by a discussion of issues that bear on the relevance and reasonableness of intransitivity.

The later sections of this article discuss quantitative representations of preferences that accommodate preference cycles and milder forms of intransitivity. Many also feature additivity, as in nontransitive additive conjoint measurement, additive differences, skew-symmetric bilinear utility, and skew-symmetric additive utility for decision under uncertainty. The effects on the representations of adding transitivity or weak order to their axioms will be noted.

Prior to that discussion, the next two sections offer a point-and-counterpoint medley on the theme of transitivity. Section 1 uses the simple preference cycle $x > y > z > x$ to argue for transitivity by showing what is wrong or unreasonable about this cycle. Section 2 counters by exposing weak points or fallacies of those arguments.

The rest of this introduction defines terms and mentions three topics not covered later: partially ordered preferences, choice probabilities, and the distinction between normative and descriptive decision theory.

Our starting point is a binary relation of preference on a set X. It is customary to adopt one relation as primitive and then to define other relations from it. Some writers begin with an *is preferred to* relation $>$ and define *is indifferent to* ($\sim$) and *is preferred or indifferent to* ($\gtrsim$) by

$$x \sim y \text{ if neither } x > y \text{ nor } y > x,$$
$$x \gtrsim y \text{ if } x > y \text{ or } x \sim y.$$

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Others begin with $\gtrsim$ and define $>$, $\sim$, and $\|$ (incomparability) by

$$x > y \text{ if } x \gtrsim y \text{ and not}(y \gtrsim x),$$
$$x \sim y \text{ if } x \gtrsim y \text{ and } y \gtrsim x,$$
$$x \| y \text{ if neither } x \gtrsim y \text{ nor } y \gtrsim x.$$

This article uses the former approach unless noted otherwise. In a few cases we will include the *cannot be compared to* relation $\|$ explicitly, and in those cases $\gtrsim$ will be the primitive. The latter approach reduces to the former if the primitive $\gtrsim$ is complete (i.e., $x \gtrsim y$ or $y \gtrsim x$ always obtains).

We will assume that $>$ is asymmetric, so $x > y \Rightarrow \text{not}(y > x)$. If $>$ is transitive also (i.e., $x > y$ and $y > z \Rightarrow x > z$), then $>$ is a *partial order*. If both $>$ and $\sim$ are transitive, then $>$ is a *weak order*, and in this case $\gtrsim$ is transitive and complete. Weak orders are widely assumed in preference theory and apply whenever there is a utility function $u: X \rightarrow \mathfrak{R}$ such that, for all $x, y \in X$,

$$x > y \Leftrightarrow u(x) > u(y).$$

Examples include basic ordinal utility theory, additive conjoint measurement in which $u(x_1, \ldots, x_n) = u_1(x_1) + \ldots + u_n(x_n)$, as axiomatized by Debreu (1960) and Krantz et al. (1971), the expected utility theories of von Neumann and Morgenstern (1944) and Savage (1954), and the nonlinear theories of Allais (1979), Machina (1982), and Schmeidler (1989).

Partial orders admit the representation $x > y \Rightarrow u(x) > u(y)$, but not $x \sim y \Rightarrow u(x) = u(y)$ when indifference is intransitive. Wiener (1914), Luce (1956), and Aumann (1962) illustrate seminal contributions, and further information is available in Fishburn (1970a, 1970b, 1985) and Fishburn and Monjardet (1991). Although intransitive indifference fits the theme of the article, the focus here shall be on the more interesting and more controversial case of intransitive strict preference.

Another topic not detailed later is notions of stochastic transitivity in probabilistic choice theory. Attention here focuses on the probability $p(a, A)$ that alternative a will be chosen from set A when A is the available set of alternatives, and on the binary specialization $p(ab) = p(a, \{a, b\})$. The p function is usually the primitive, and $>$ and $\gtrsim$, when considered at all, are defined by $a > b$ if $p(ab) > 1/2$, and $a \gtrsim b$ if $p(ab) \geq 1/2$, or in some related way. An example of stochastic transitivity is $\{p(ab) > 1/2, p(bc) > 1/2\} \Rightarrow p(ac) > 1/2$. Luce (1959), Luce and Suppes (1965), Fishburn (1973a), and Suppes et al. (1989) provide broad coverage of the p approach.

Decision theorists often take care to distinguish between normative theory and descriptive theory when discussing axioms, examples, and empirical findings: see, for example, Savage (1954), Tversky and Kahneman (1986), Fishburn (1988), Brown (1989), and Rapoport (1989). Normative theory deals with ideals and principles of "good" or "rational" decision making. It is heavily influenced by Platonism and Scottish common-sense philosophy, and asserts that knowledge of the correct principles of decision making can be arrived at by intuition and introspection. One might go further and suggest that all reasonable and clear-thinking people will agree on the principles or axioms of rational decision making, since these will be self-evident once they are carefully specified. An analogous viewpoint is expressed by

> We hold these truths to be self-evident, that . . . (The unanimous Declaration of the thirteen united States of America, In Congress, July 4, 1776).

Normative theory used as the basis for decision making in practice, and with the cognizance of human limitations, gives rise to prescriptive decision theory (Brown, 1989) and decision analysis (Raiffa, 1968).

On the other hand, descriptive or behavioral decision theory is concerned with people's actual judgments and choices. The situations it studies may involve relatively unstructured settings or tightly controlled laboratory environments. It is Aristotelian and guided by scientific methodology.

Most decision theorists insist on transitivity or weak order as a foundational principle of normative theory. Some acknowledge the reasonableness of intransitivities within the realm of descriptive theory. For example, the largely empirical work reported in Flood (1951-1952), May (1954), Tversky (1969), and MacCrimmon and Larsson (1979) suggests that intransitive preferences are most likely to be observed in comparisons between multiattribute alternatives where differential attribute weighting, within-factor discrimination, and perhaps interdependencies among attributes or factors can lead to preference cycles. Whether people who exhibit such preferences would want to amend them to adhere to transitivity after instruction in normative theory is a moot question.

There is no question that the nontransitive preference representations in later sections are candidate models for descriptive decision theory, since their descriptive accuracy in various situations is a matter for empirical research. But some of them might be considered as candidate models for normative theory. Since this is an unorthodox position, it deserves elaboration.

As described above, a particular normative theory is a creed, a set of principles adhered to and proclaimed by its followers. The same is true, of course, for descriptive theory and its creed of scientific method. And creeds, whether concerned with physical reality, religion, ethics, or the principles of rational action, tend to be culturally conditioned. In normative theories of decision under uncertainty, this is evident from differences between the Anglo-American school of Ramsey (1931) and Savage (1954) and the Franco-European school of Allais (1953a, 1953b). These differences, brought into focus at a colloquium in Paris in 1952, persist (Fishburn, 1987). Interestingly enough, transitivity or weak order is one of the few tenets of both schools (see item 4, section 3, for others), so the debates of principles have revolved around other matters.

Transitivity is obviously a great practical convenience and a nice thing to have for mathematical purposes, but long ago this author ceased to understand why it should be a cornerstone of normative decision theory. In the realm of decision under uncertainty, the normative position adopted here is roughly approximated by the theory in chapter 9 of Fishburn (1988) mentioned in section 8 of this article. This theory is very much like that of Savage (1954) without transitivity and the things that it implies, and says something about the culturally induced beliefs of this article's author. The task of the next two sections is to explain why some people no longer regard transitivity as a viable normative principle.

1. Examples

Like a religious faith, a normative decision theory is bound to the believer's psyche and cannot be proved right or wrong by logic or scientific inquiry. But both have their advocates and defenders who argue their cause, and it is at this level that we consider transitivity.

For discussion purposes, we suppose throughout this section and the next that a person, P, has preferences $\{x > y, y > z, z > x\}$, or $x > y > z > x$ for short. It will be helpful in what follows to think of preference in the binary choice mode, so that $x > y$ indicates that P would choose x from $\{x, y\}$. But it should be kept in mind that this moves $>$ away from primitive status, and that characterization of preference in terms of choice raises difficult issues (see item 1 in section 2).

We proceed to consider why $x > y > z > x$ might be problematic, foolish, or irrational.

Item 1 (maximization). P's cyclic pattern $x > y > z > x$ gives no help in making a choice from $\{x, y, z\}$, since there is no most-preferred alternative. This cannot occur under transitivity, since then every finite set has at least one maximally preferred alternative.

Item 2 (money pump). Suppose P has title to x. Since $z > x$, P would surely pay something to exchange x for z; given z, P would surely pay to exchange z for the preferred y; and given y, P would pay again to exchange y for the preferred x. Thus P begins and ends at x but is poorer in the process.

Item 3 (decision under risk; simplified preference reversal). Let x be a lottery that wins \$100 with probability 0.3, nothing otherwise; let y be \$25 as a sure thing; and let z be a lottery that wins \$30 with probability 0.9, nothing otherwise. Then our cyclic pattern says that P prefers z to x but values z at less than \$25 and x at more than \$25, and that seems rather strange.

Item 4 (decision under uncertainty; dominance). Suppose f^* and g^* are two acts in a three-state context with equally likely states and consequence matrix

	1	2	3
f^*	x	y	z
g^*	y	z	x

Since $x > y$ in column 1, $y > z$ in column 2, and $z > x$ in column 3, an incontestable state-by-state dominance principle says that P must prefer f^* to g^*: P is better off with f^* no matter which state obtains. But this preference is inane, since f^* and g^* reduce to precisely the same thing, i.e., a lottery on $\{x, y, z\}$ with probability 1/3 for each consequence.

Item 5 (decision under uncertainty; reduction). Suppose x, y, and z are three acts in a situation with three equally likely states and prize matrix

	1	2	3
x	\$5000	\$4000	\$3000
y	\$4000	\$3000	\$5000
z	\$3000	\$5000	\$4000

Here P prefers x to y because x has a larger prize than y in two of the three states. Similarly, $y > z$ and $z > x$. But that is silly since, whichever act P chooses, it amounts to an even chance lottery on the three prizes. The only reasonable course is for P to be indifferent among the acts.

Item 6 (sequential choice). P must make choices at time 1 and at a later time 2. At time 1, P chooses between x and z; at time 2, P chooses between y and the choice at time 1. Assume that P chooses the preferred alternative at time 2. Then, if the preferred z is chosen at time 1, P ends up with y; if the less preferred x is chosen at time 1, P ends up with x. Hence P ends up with x or y. Since $x > y$, P will end up with the preferred x only if the less preferred alternative is chosen at time 1. This strange result will not happen if preferences are transitive.

2. Discussion

The preceding items suggest two things: the presence of intransitive preferences complicates matters, and these preferences are irrational. The first point seems incontestable; however, it is not cause enough to reject intransitivity. An analogous rejection of non-Euclidean geometry in physics would have kept the familiar and simpler Newtonian mechanics in place, but that was not to be. Indeed, intransitivity challenges us to consider more flexible models that retain as much simplicity and elegance as circumstances allow. It challenges old ways of analyzing decisions and suggests new possibilities. It may not be unlike cyclical majorities in multicandidate elections (Condorcet, 1785) that stimulated Arrow (1951) and others to develop the rich and expanding theory of social choice (Fishburn, 1973b; Peleg, 1984; Merrill, 1988).

So intransitive preferences can make life more difficult but perhaps also more interesting. Are they irrational? Let us reconsider the items of the preceding section.

Item 1. This item begs the following question: When a choice is required from a set of three or more alternatives, what justifies basing it on *binary* preferences, even when they are transitive? This raises issues of the relationship between preference and choice, and that relationship is far from obvious according to writers like Churchman (1961) and Cowan and Fishburn (1988). At the very least, the tenet that good choices should be based on preferences presumes a principle of extendibility from the binary realm to ternary, quaternary, and higher contexts. At the worst, one might conclude that there is no necessary relationship between the two. In any event, research on relationships between preference and choice, and among choices from subsets of alternatives of various sizes, deserves emphasis in the years ahead.

Suppose we adopt the *extendibility principle* and agree that maximally preferred alternatives in subsets are good choices from those subsets. Given $x > y > z > x$, this principle appears to leave the choice from $\{x, y, z\}$ entirely open. But more can be said. Social choice theorists have wrestled for over 200 years with the problem of choosing from $\{x, y, z\}$ when majority comparisons cycle, and have proposed numerous ways to resolve it. One of these that involves lotteries (Kreweras, 1965; Fishburn, 1984a) has a counterpart for individual decisions. In particular, if P's preferences on the set L of all probability distributions on $\{x, y, z\}$ satisfy axioms described in section 9, then there is a *unique* $p^* \in L$ that is preferred or indifferent to everything else in L (Kreweras, 1961; Fishburn, 1984b). The extendibility principle then advises P to use p^* to make the choice of x, y, or z. This is not unlike the use of mixed strategies in game theory, although it arises from different considerations. Moreover, it is "justified" by the same extendibility principle that applies to subset choice when preferences are transitive.

Item 2. The first thing to say about the money pump process is that it transports us into a dynamic realm with possibilities of strategy and deception that transcend the basic problem of choosing something from $\{x, y, z\}$. It is a clever device, but one that applies transitive thinking to an intransitive world. Moreover, it involves a non sequitur that calls for the additional hypothesis that P would willingly participate in a process that leaves P worse off. It is indeed foolish to be a money pump, but it seems unlikely that an intelligent person would suffer it unless he or she were conned into it. On the other hand, the mere possibility of a money pump could encourage people to reexamine expressed preferences. This would likely lead to transitive revisions in some cases.

Item 3. The earlier example is a simplified version of the widely studied *preference reversal phenomenon*, the most common manifestation of which occurs when a high-probability-for-a-modest-prize lottery z is preferred to a low-probability-for-a-bigger-prize lottery x, but x is "valued" higher than z. The earliest study is by Slovic and Lichtenstein (1968); the latest include those of Goldstein and Einhorn (1987), Bostic, Herrnstein, and Luce (1990), Slovic, Griffin, and Tversky (1990), Tversky, Slovic, and Kahneman (1990), Schkade and Johnson (1989), and Casey (1990). Insofar as the $>$'s in $x > y > z$ are granted to be true preferences, such reversals, while strange, are accounted for by intransitivity. However, the elicitation of values for lotteries x and z has not usually been done by direct preference comparison. The early studies typically asked for the lowest price $\$_x$ for x or $\$_z$ for z at which the subject would sell title to x or z and, when $z > x$, would say that a reversal occurred if $\$_x > \$_z$. Recent research has shown that different ways of eliciting values for lotteries give different answers, in violation of what Tversky and others refer to as *procedure invariance*, so that the phenomenon may be due more to such violations than to intransitivities. For example, a novel experimental design in Tversky et al. (1989) yielded data in which 90% of the reversals were attributable to violations of procedure invariance and only 10% to intransitivities.

These results say two things for our present study. First, intransitivity may arise even for the simplest lottery comparisons. Second, elicitation procedures and biases rather than true intransitivities might lie behind the expressed intransitivities.

Item 4. Two other principles for decision under uncertainty are involved here. Let S be the set of states, assumed finite for now, and let $f, g, \ldots$ (acts) be functions from S into an outcome set C. Our first principle is a strong form of Savage's (1954) sure-thing principle:

strong dominance: $f > g$ if $f(s) > g(s)$ for all $s \in S$.

Given π as the person's subjective probability measure on S, let $\pi_f(c) = \pi\{s \in S : f(s) = c\}$ for all $c \in C$, so π_f is the measure on C induced by act f. Our second principle is *reduction:* $f > g \Leftrightarrow f' > g'$ whenever $\pi_f = \pi_{f'}$ and $\pi_g = \pi_{g'}$.

Allais and Savage subscribe to both principles along with weak order. Asymmetry and reduction imply the *identity reduction* principle, which says that $\pi_f = \pi_g \Rightarrow f \sim g$.

In the example for item 4 of the preceding section, strong dominance requires $f^* > g^*$, and reduction gives $f^* \sim g^*$, so the two are incompatible when $x > y > z > x$. Since strong dominance seems more compelling than reduction, $f^* > g^*$ is here considered to be reasonable in that example. The problem with reduction is that it divorces the comparison from its context by obliterating the state-by-state alignments of outcomes for the two acts. Whether one attributes the importance of alignments to the likelihood of being better off in the end or to ex post remembering effects of what might have been, such as regret and rejoicing (Loomes and Sugden, 1982, 1987), introspection suggests that they may have an important bearing on preference comparisons.

An argument *for* reduction says that after an act has been chosen, say f from $\{f, g\}$, only its distribution π_f on outcomes should matter; therefore only π_f and π_g should matter when f and g are compared. This viewpoint has two weaknesses. First, its premise supposes that remembering has no place in rational choice, which may be fine for machines but seems doubtful for people. Second, its conclusion does not follow from the premise. No rule of logic is violated if alignments matter for preference comparisons and, once the decision is made, only the probability distribution on outcomes for the chosen act matters.

Although it has little bearing on transitivity, we note that strong dominance is *not* viable in some contexts. An example in public risks evaluation (Fishburn and Straffin, 1989) that stems from Diamond (1967) involves holistic equity considerations that cannot be accounted for by intrastate comparisons. Suppose $S = \{1, 2\}$ and the state that obtains will be determined by a toss of a fair coin. There is a prize of \$10,000, two worthy individuals a and b, and two acts, f and g. If state 1 obtains, f uses a second coin toss to determine whether a or b gets the prize, but g awards the prize to a outright. If state 2 obtains, f uses a second coin toss to determine whether a or b gets the prize, but g awards the prize to b outright. It is common to regard f as preferred to g within each state on the basis of fairness. Strong dominance then requires $f > g$. But standard theory would assign f and g the same expected utility.

Item 5. The earlier example with acts x, y, and z and a transitive outcome matrix gives another look at reduction, which fails if $x > y > z > x$. Strong dominance is not a factor here, but we can get close to it by expanding to one million states in all but one of which x gives a larger prize than y. A similar expansion in the number of acts then produces a preference cycle if, as modified, x is preferred to y, or y is preferred to x. The latter preference would be expected for some people if x is one cent better than y for each of the 999,999 states at which x is preferred, and y is \$9,999.99 better than x on the other state.

Item 6. The given example is adapted from Fishburn and LaValle (1988), who discuss sequential choice and related topics under convexification as described for L in item 1 of this section. The result of the example may seem odd, but the whole process, including its look-ahead feature, is quite rational.

Intransitivities in the sequential choice domain can have other odd effects. Suppose the choice space is partitioned into subsets $S_1, S_2, \ldots, S_n$. The problem is to choose an S_i, perform and analyze an experiment tied into S_i, and then choose something from S_i as a terminal act. One way to proceed is to choose an overall act that is holistically maximal for $>$, do its experiment, and then implement the terminal act assigned to the result of the experiment. The problem with this holistic approach is that, once an S_i is chosen, there might be a terminal strategy that is clearly superior to the terminal strategy dictated by the original choice. It is therefore tempting to switch to a different terminal strategy in midstream, once S_i has been chosen. However, such a switch negates the rationale for overall choice used initially. A different procedure says to first determine the maximal elements within each S_i and then make the initial choice assuming that one of these maximals will be used later, but this raises other problems that are described in section 5 of Fishburn and LaValle (1988).

These and related anomalies vanish if weak order is assumed. A sort of converse to this also exists: if, given other plausible assumptions in a dynamic decision setting, the decision maker is dynamically consistent in certain specified ways, then preferences are transitive. The flavor of this has been suggested by the initial example, but a detailed discussion is not possible here. Hammond (1988) an LaValle (1990) are suggested for further reading.

The purpose of the preceding discussion has been to show that, whereas intransitivity can have unfamiliar implications and effects, its status as a principle of normative decision theory is tenuous. Because our items avoided explicit multiattribute examples in which intransitivities seem most likely to arise, this section concludes with two such examples.

Professor P is about to change jobs. She knows that if two offers are far apart on salary, then salary will be the determining factor in her choice. Otherwise, factors such as the prestige of the university will come into play. She eventually receives three offers, described in part as follows:

	Salary	Prestige
x	\$65,000	Low
y	\$50,000	High
z	\$58,000	Medium

On reflection, P concludes that $x > y, y > z$, and $z > x$.

Professor P is about to change jobs. His four most important factors are salary, university prestige, department reputation, and location. They are of roughly equal importance and substantially outweigh other factors. He eventually receives four offers and ranks these under each factor on a scale of 1 (minimally acceptable) to 4 (couldn't be better) as follows:

	Salary	Prestige	Reputation	Location
a	4	3	1	2
b	3	2	4	1
c	2	1	3	4
d	1	4	2	3

He finds that if one offer is better than another on at least three of the four factors, he prefers the former. Hence $a > b > c > d > a$. Further reflection on the other two comparisons leads to $a > c$ and $b \sim d$.

3. Nontransitive representations

The rest of this article outlines axioms for preferences and associated quantitative representations that accommodate intransitivities. The basic form for most of the representations is either

$$x > y \Leftrightarrow \phi(x,y) > 0,$$

or

$$x \gtrsim y \Leftrightarrow \phi(x,y) \geq 0,$$

in which ϕ maps $X \times X$ into $\mathfrak{R}$. The cycle $x > y > z > x$ is accounted for by $\phi(x,y) > 0$, $\phi(y,z) > 0$, and $\phi(z,x) > 0$. In most cases ϕ has additional properties, such as additivity over factors or bilinearity when X is a convex set, that are axiomatized by appropriate independence or cancellation conditions. The effect of weak order will often be noted. It sometimes implies that ϕ can be decomposed as $\phi(x,y) = u(x) - u(y)$, and in such cases we obtain the familiar representation $x > y \Leftrightarrow u(x) > u(y)$.

Many of our representations have nice uniqueness properties. We say that a functional ϕ that satisfies $\mathfrak{S}$ is *unique up to proportionality transformations*, or unique up to multiplication by a positive constant, if the set of all functionals that satisfy $\mathfrak{S}$ is $\{\alpha\phi : \alpha > 0\}$. Functionals $\phi_1, \phi_2, \ldots, \phi_n$ that satisfy $\mathfrak{S}$ are *unique up to similar proportionality transformations* if the set of all such $(\phi_1, \ldots, \phi_n)$ is $\{(\alpha\phi_1, \ldots, \alpha\phi_n) : \alpha > 0\}$. A function u that satisfies $\mathfrak{S}$ is *unique up to positive affine transformations* if the set of all functionals that satisfy $\mathfrak{S}$ is $\{\alpha u + \beta : \alpha > 0, \beta \in \mathfrak{R}\}$.

The representations of ensuing sections are all relevant to all traditional decision theory contexts such as economic consumer theory, multiattribute preferences, time-stream preferences, n-person games, decision under risk, and decision under uncertainty. When $X = X_1 \times X_2 \times \cdots \times X_n$ with elements $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), \ldots, x_k$ could denote the quantity of consumer good k purchased, a value or level of attribute or criterion k, whatever happens in time period k, a mixed strategy for player k, or the outcome or lottery that obtains when state k is the true state. In many cases the representations have implications for applied topics that also follow from less flexible weak-order representations such as the existence of economic equilibria, Nash equilibria in n-person noncooperative games, and maximally preferred lotteries or acts in decision under risk or uncertainty. Some of these are detailed in chapter 6 in Fishburn (1988), and they are not discussed here.

We mention two representational approaches for product sets $X = X_1 \times X_2 \times \cdots \times X_n$ that are not covered later. The first is a correspondence approach based on preferred-to sets $\{y \in X : y > x\}$ that is used in equilibrium theory by Sonnenschein (1971), Mas-Colell (1974), and others. It does not generally presume that preferences are transitive or complete, and usually assumes that the preferred-to sets are convex and may have smooth boundaries. The sets $\{y \in X : y > x\}$ might be represented by $\{y \in X : \phi(y,x) > 0\}$ with ϕ continuous in an appropriate sense in its first argument.

The second approach combines a lexicographic importance ordering on the factors (Fishburn, 1974) with a local interfactor tradeoff structure or intrafactor ordering that applies when a threshold for the lexicographic aspect is not exceeded. We illustrate this with three models for $n = 2$. In each, X_1 is the dominant factor up to a threshold governed by a semiorder (Luce, 1956; Fishburn, 1985), and u_k is a real valued function on X_k. For the semiorder aspect, ϵ is a nonnegative constant, and $\delta : X_1 \to \mathfrak{R}^+$.

Tversky's (1969) lexicographic semiorder model has

$$x > y \text{ if } u_1(x_1) > u_1(y_1) + \epsilon \text{ or } \{|u_1(x_1) - u_1(y_1)| \leq \epsilon, u_2(x_2) > u_2(y_2)\}.$$

The penultimate example in the preceding section is based on this idea. Luce (1978) uses within-threshold additivity for the representation

$$\begin{aligned} x > y \Leftrightarrow u_1(x_1) > u_1(y_1) + \delta(y_1) \text{ or } \{&u_1(x_1) \leq u_1(y_1) + \delta(y_1), \\ &u_1(y_1) \leq u_1(x_1) + \delta(x_1), u_1(x_1) + u_2(x_2) > u_1(y_1) + u_2(y_2)\}. \end{aligned}$$

Fishburn (1980) uses within-threshold additive differences and Tversky's simpler threshold model for

$$\begin{aligned} x > y \Leftrightarrow u_1(x_1) > u_1(y_1) + \epsilon \text{ or } \{&|u_1(x_1) - u_1(y_1)| \leq \epsilon, \\ &F_1[u_1(x_1) - u_1(y_1)] + F_2[u_2(x_2) - u_2(y_2)] > 0\}, \end{aligned}$$

where F_k is a strictly increasing, continuous, and odd $[F_k(\alpha) + F_k(-\alpha) = 0]$ functional on a real interval symmetric around 0. We say more about additive differences in section 7. With suitably rich structure, the imposition of weak order reduces the Luce and Fishburn models to either the fully lexicographic representation $[x > y \Leftrightarrow u_1(x_1) > u_1(y_1)$ or $\{u_1(x_1) = u_1(y_1), u_2(x_2) > u_2(y_2)\}]$ or the additive conjoint representation.

4. Additivity without completeness

Let $X = X_1 \times X_2 \times \cdots \times X_n$. This section considers the *nontransitive additive* (NA) *representation*

$$x \gtrsim y \Leftrightarrow \sum_{k=1}^{n} f_k(x_k, y_k) \geq 0$$

with $\gtrsim$ as primitive, and then looks at

$$x > y \Leftrightarrow \sum_{k=1}^{n} f_k(x_k, y_k) > 0$$

with $>$ as primitive. Both cases assume that $f_k : X_k \times X_k \to \Re$ with $f_k(x_k, x_k) = 0$ for all $x_k \in X_k$ and all k. The NA representation is intermediate between the weak-order additive representation

$$x \gtrsim y \Leftrightarrow \sum_{k=1}^{n} u_k(x_k) \geq \sum_{k=1}^{n} u_k(y_k),$$

referred to henceforth as the *A representation*, and the correspondence approach of the preceding section. It accommodates incomparability, since $x \parallel y \Leftrightarrow \Sigma f_k(x_k, y_k) < 0$ and $\Sigma f_k(y_k, x_k) < 0$. In addition, $>$ is asymmetric, $\sim$ is reflexive ($x \sim x$) and symmetric ($x \sim y \Rightarrow y \sim x$), and $\parallel$ is irreflexive and symmetric.

The NA representation with primitive $\gtrsim$ implies the single-factor independence property

$$(x_k, x_{(k)}) \gtrsim (y_k, x_{(k)}) \Leftrightarrow (x_k, y_{(k)}) \gtrsim (y_k, y_{(k)}),$$

where $(r_k, s_{(k)}) = (s_1, \ldots, s_{k-1}, r_k, s_{k+1}, \ldots, s_n)$. It follows that $\gtrsim_k$ on X_k, defined by $x_k \gtrsim_k y_k$ if $(x_k, x_{(k)}) \gtrsim (y_k, x_{(k)})$ for all $x_{(k)}$ is reflexive: it may or may not be complete or transitive when the NA representation holds. Even when each $\gtrsim_k$ is transitive and complete, $\gtrsim$ need not be transitive.

More generally, if $\{J, K\}$ is a nontrivial two-part partition of $\{1, \ldots, n\}$ and coordinates are grouped to write x as (x_J, x_K), the NA representation for $\gtrsim$ implies

$$(x_J, x_K) \gtrsim (y_J, x_K) \Leftrightarrow (x_J, y_K) \gtrsim (y_J, y_K).$$

When $\gtrsim_J$ on $X_J = \times_{j \in J} X_j$ is defined in the usual way, $\gtrsim_J$ is reflexive but need not be complete.

The NA representation in either form is fairly new (Fishburn, 1984c; Bouyssou, 1986; Vind, 1991) although specializations, including the additive difference model, are older. In what may be its first axiomatization (based on $\gtrsim$) that accommodates incompleteness, Bouyssou (1986) presents necessary and sufficient axioms when $n = 2$ and X is countable. His version of the representation requires $f_k(a, b) f_k(b, a) \leq 0$ for all $a, b \in X_k$, $k = 1, 2$. His three axioms say that $\gtrsim_1$ and $\gtrsim_2$ are complete, that $\{x_1 \gtrsim_1 y_1, x_2 \gtrsim_2 y_2\} \Rightarrow x \gtrsim y$ (with $x > y$ when one $\gtrsim_k$ is strict), and that

$$\{x_1 x_2 \gtrsim y_1 y_2, a_1 a_2 \gtrsim b_1 b_2\} \Rightarrow x_1 a_2 \gtrsim y_1 b_2 \text{ or } a_1 x_2 \gtrsim b_1 y_2,$$

where parentheses are omitted for convenience. He refers to this last condition as *weak cancellation*. Either weak cancellation or a closely related condition is used in other axiomatizations of the representation.

Vind (1991) adopts a topological approach to axiomatize the NA representation with $>$ primitive and $n \geq 4$. Only one of many interesting results in his paper is outlined here. To do this, we reformulate the basic structure as follows. Assume $|X_k| \geq 2$ for all k to rule out trivial factors. Let

$$T_k = X_k \times X_k, T_K = \times_{k \in K} T_k, T_{(k)} = \times_{j \neq k} T_j, T = T_1 \times \cdots \times T_n.$$

Also let D_k denote the *diagonal* $\{(x_k, x_k) : x_k \in X_k\}$ of T_k, and define D_K, $D_{(k)}$, and D by analogy. This reformulation maps $(x,y) \in X \times X$ into $t = (t_1, \ldots, t_n) = ((x_1,y_1), \ldots, (x_n, y_n))$ in T. We write $t \in P$ to denote $x > y$ so that P is the strict preference subset of T. The NA representation for $>$ in this format is

$$t \in P \Leftrightarrow \sum_{k=1}^{n} f_k(t_k) > 0$$

with $f_k(d_k) = 0$ for all $d_k \in D_k$ and all k.

Vind assumes for every $t_k \in T_k$ and all k that there exist $t_{(k)}, s_{(k)} \in T_{(k)}$ for which $(t_k, t_{(k)}) \in P$ and $(t_k, t_{(k)}) \notin P$. His main independence condition (cf. weak cancellation) is that for every partition $\{J, K\}$ of $\{1, \ldots, n\}$, all $t_J, s_J \in T_J$ and all $t_K, s_K \in T_K$,

$$\{t_J t_K \in P, s_J s_K \in P\} \Rightarrow t_J s_K \in P \text{ or } s_J t_K \in P.$$

This induces $\gtrsim_k$ on T_k in the natural way, and the $\gtrsim_k$ are used to endow the T_k with order topologies. T receives the product topology of the T_k. The other axioms assert that P is open in T; D is a subset of the closure of P, minus P; every T_k is connected in its order topology; and $n \geq 4$. It follows that there exist continuous $f_k : T_k \rightarrow \mathfrak{R}$ that satisfy the NA representation, and that $f_1, \ldots, f_n$ are unique up to similar proportionality transformations.

If weak order is added to Vind's axioms, the NA representation reduces to the A representation with $u_1, \ldots, u_n$ unique up to similar (same $\alpha > 0$) positive affine transformations. As shown at the end of the next section, weak order need not have the same effect on Bouyssou's result.

5. Skew symmetry

A functional g on $Y \times Y$ is *skew symmetric* if $g(a, b) + g(b, a) = 0$ for all $a, b \in Y$. Skew symmetry for the f_k in the NA representation is suggested by the decomposition $f_k(a, b) = u_k(a) - u_k(b)$ that yields the weak-order additive representation, and it has some intuitive appeal otherwise. When every f_k is skew symmetric in the NA representation based on $\gtrsim$, $\|$ is empty, $\gtrsim$ is complete, $x > y \Leftrightarrow \text{not}(y \gtrsim x)$, and $x \sim y \Leftrightarrow \Sigma f_k(x_k, y_k) = 0$. We refer to the NA representation in which every f_k is skew symmetric as the *NASS representation*.

Bouyssou (1986) and Fishburn (1990a) give necessary and sufficient conditions for the NASS representation when $n = 2$. Bouyssou uses completeness and the cancellation condition.

$$\{x_1x_2 \gtrsim y_1y_2, a_1y_2 \gtrsim b_1x_2, y_1a_2 \gtrsim x_1b_2\} \Rightarrow a_1a_2 \gtrsim b_1b_2,$$

with X countable. Fishburn uses a more general cancellation axiom and a denseness condition for X of arbitrary cardinality.

Other axiomatizations use the T reformulation of the preceding section so that the NASS model is $t \in P \Leftrightarrow \Sigma f_k(t_k) > 0$, with $f_k(t_k) + f_k(t_k^{-1}) = 0$ for all $t_k \in T_k$ and all k. Here t_k^{-1} denotes the *inverse* of t_k: when $t_k = (x_k, y_k)$, $t_k^{-1} = (y_k, x_k)$.

For all $t \in T$, let $t^{-1} = (t_1^{-1}, \ldots, t_n^{-1})$. Also let $P^{-1} = \{t^{-1} : t \in P\}$ and $I = T \setminus (P \cup P^{-1})$ so that $t \in I$ corresponds to $x \sim y$ when $t = ((x_1, y_1), \ldots, (x_n, y_n))$. Our basic independence or cancellation axiom in this format uses the subset E_4 of $T \times T \times T \times T$ defined by

$$(t^1, t^2, t^3, t^4) \in E_4 \text{ if for all } k \in \{1, \ldots, n\} \text{ and all } a \in T_k,\ |\{j : t_k^j = a\}| = |\{j : t_k^j = a^{-1}\}|.$$

This holds precisely when the kth components of the four t^j are evenly balanced by inverse pairs for $k = 1, \ldots, n$. When the NASS representation holds, it implies that $\Sigma_k(f_k(t_k^1) + f_k(t_k^2) + f_k(t_k^3) + f_k(t_k^4)) = 0$. This shows that the independence condition

$$\{(t^1, t^2, t^3, t^4) \in E_4; t^1, t^2, t^3 \in P \cup I\} \Rightarrow t^4 \notin P$$

is necessary for the representation.

Fishburn (1990a) proves that, when every X_k is finite, the straightforward extension of the preceding independence condition (to E_m, $m \geq 4$) is necessary and sufficient for the NASS representation. Fishburn (1989, 1990a) also gives two sufficient-condition axiomatizations of the model for $n \geq 3$. Both assume that for every $t_k \in T_k$ and all k there is a $t_{(k)} \in T_{(k)}$ such that $(t_k, t_{(k)}) \notin P$, and both use the preceding independence axiom for E_4. The other axioms are largely structural. Fishburn (1989) assumes that X_k is a connected and separable topological space, that for each k there is some $(t_k, d_{(k)}) \in P$ with $d_{(k)} \in D_{(k)}$, and that $\{t_k \in T_k : (t_k, w) \in P\}$ is open in $X_k \times X_k$ whenever $k \in \{1, \ldots, n\}$ and $w \in T_{(k)}$. The NASS representation follows with continuous f_k that are unique up to similar proportionality transformations.

Fishburn (1990a) uses a more flexible algebraic structure for $n \geq 3$ that replaces the preceding topological axioms by a solvability condition and an Archimedean axiom. The f_k in this case are also unique up to similar proportionality transformations.

When weak order is added to the two preceding axiomatizations for $n \geq 3$, the NASS representation reduces to the A representation. This is not true, however, in the general case. For example, suppose the NASS representation holds for $n = 2$ with $X_1 = \{a_1, b_1, c_1\}$, $X_2 = \{a_2, b_2, c_2\}$, and $(f_1(a_1b_1), f_1(b_1c_1), f_1(a_1c_1)) = (1, 3, 5)$, $(f_2(a_2b_2), f_2(b_2c_2), f_2(a_2c_2)) = (2, 4, 4)$. Then $>$ on X is a strict order with

$$a_1a_2 > b_1a_2 > a_1b_2 > b_1b_2 > a_1c_2 > c_1a_2 > c_1b_2 > b_1c_2 > c_1c_2.$$

But the A representation cannot hold because it would contradict $b_1a_2 > a_1b_2$, $a_1c_2 > c_1a_2$, and $c_1b_2 > b_1c_2$. This example is easily extended to $n \geq 3$.

6. Additive differences

When $v : Y \rightarrow \mathfrak{R}$, let $\Delta(v) = \{v(a) - v(b) : a, b \in Y\}$, the set of all v differences. Given $X = X_1 \times X_2 \times \cdots \times X_n$, the *additive difference* (AD) *representation* is

$$x > y \Leftrightarrow \sum_{k=1}^{n} F_k[u_k(x_k) - u_k(y_k)] > 0$$

where $u_k : X_k \to \Re$ and $F_k : \Delta(u_k) \to \Re$ with F_k strictly increasing and odd. This is the specialization of the NASS representation in which $f_k(a, b) = F_k[u_k(a) - u_k(b)]$. Skew symmetry for f_k corresponds to oddness for F_k. The AD representation is due to Morrison (1962) and Tversky (1969).

The related representation

$$p(xy) \geq p(zv) \Leftrightarrow \sum_{k=1}^{n} F_k[u_k(x_k) - u_k(y_k)] \geq \sum_{k=1}^{n} F_k[u_k(z_k) - u_k(v_k)],$$

in which p is a binary choice probability function, is axiomatized for $n \geq 2$ in chapter 17 of Suppes et al. (1989). As noted there, $p(xy) \geq p(zv)$ can be replaced by the quarternary expression $xy \succsim_s zv$, where $\succsim_s$ applies to comparisons of preference differences. Standard axioms can then be used to obtain the ordinal model $xy \succsim_s zv \Leftrightarrow u(xy) \geq u(zv)$, and the p axioms can be applied to u, or to $\succsim_s$ under suitable emendations, to obtain $xy \succsim_s zv \Leftrightarrow \Sigma F_k[u_k(x_k) - u_k(y_k)] \geq \Sigma F_k[u_k(z_k) - u_k(v_k)]$. Their u_k are unique up to positive affine transformations and, given the u_k, the F_k are unique up to similar proportionality transformations.

Axiomatizations of the AD representation appear in Bouyssou (1986), Fishburn (1980, 1990c), and Croon (1984). The first two of these assume $n = 2$, and Bouyssou's representation allows the F_k to be merely increasing. Fishburn (1980) uses topological assumptions similar to those for the NASS representation for $n \geq 3$ in Fishburn (1989) in conjunction with a cancellation condition that involves sextuples of elements in X. Croon (1984) axiomatizes the $n = 2$ case with an algebraic structure, and outlines his approach for $n = 3$.

Fishburn (1990c) extends the topological and algebraic axioms for the NASS representation of the preceding section for $n \geq 3$ to obtain axioms for the AD representation. The key addition is a straightforward intrafactor monotonicity axiom that is similar to the sextuple condition in Block and Marschak (1960). Using the T format, it says that for all k, for all $a_k, \ldots, z_k \in X_k$, and for all $w, w', w^* \in T_{(k)}$, if

$$((a_k, b_k), w), ((b_k, c_k), w'), ((a_k, c_k), w^*) \in I;$$
$$((x_k, y_k), w), ((y_k, z_k), w') \in P \cup I,$$

then $((x_k, z_k), w^*) \in P \cup I$. This is all that is needed in the topological approach to obtain the AD representation from the NASS representation. In this case the u_k are continuous and unique up to positive affine transformations and, given the u_k, each $\Delta(u_k)$ is a nondegenerate real interval and the F_k are continuous and unique up to similar proportionality transformations. The algebraic approach adds an intrafactor solvability condition to the preceding monotonicity axiom. Its AD representation has the same properties as the topological approach for the u_k and F_k, minus continuity.

Weak order reduces the AD representation to the A representation under the axioms in Fishburn (1980, 1990c). Slight changes in the example at the end of the preceding section shows that the A representation is not generally implied by the AD representation plus weak order.

7. Special homogeneous cases

We assume throughout this section that $X_1 = \cdots = X_n = C$ with $X = C^n$, and consider specializations of the NASS representation for this homogeneous format. Applicable contexts include time streams with the same consumption or event possibilities in each period as well as decision under uncertainty with the same outcome set for each state.

Two representations that make use of homogeneous structure are the *homogeneous NASS representation*

$$x > y \Leftrightarrow \sum_{k=1}^{n} \pi_k f(x_k, y_k) > 0,$$

in which the π_k are positive factor weights and f from $C \times C$ into $\Re$ is skew symmetric, and the *homogeneous AD representation*

$$x > y \Leftrightarrow \sum_{k=1}^{n} F_k[u(x_k) - u(y_k)] > 0,$$

with $u : C \to \Re$, $F_k : \Delta(u) \to \Re$, and F_k strictly increasing and odd for each k. Neither of these is implied by the other. The first says that the f_k of the NASS representation are scale multiples of each other, or that the f_k are related by proportionality transformations. The second says that the u_k in the AD representation are related by positive affine transformations and hence may be transformed into the same u before the F_k are applied. Unlike the first, the second requires each intrafactor $>_k$ to be a weak order, the same for all k, and it allows the f_k, defined by $f_k(a, b) = F_k[u(a) - u(b)]$, to be very different from one another.

Fishburn (1990c) notes that the homogeneous AD representation follows from the $n \geq 3$ axioms for the AD representation in the penultimate paragraph of the preceding section, augmented by an *interfactor uniformity* condition. This new condition is expressed in the T format by

$$\{(r \text{ in position } j, w) \in I, (r \text{ in position } k, w') \in I,$$
$$(s \text{ in position } j, w) \in P\} \Rightarrow (s \text{ in position } k, w') \in P$$

whenever $r, s \in C \times C$, $w \in T_{(j)}$ and $w' \in T_{(k)}$. The usual properties then hold for u and the F_k.

The homogeneous NASS representation has appeared in several places in the decision under uncertainty literature, with the π_k interpreted as subjective probabilities for the n states under the $\Sigma\pi_k = 1$ normalization. Loomes and Sugden (1982, 1987) and Bell (1982) describe how $f(a, b)$ can accommodate notions of regret and rejoicing within scaling procedures based on comparisons of preference differences. Fishburn (1988) axiomatizes the homogeneous NASS representation under the heading of SSA (skew symmetric additive) utility through a simple change in Savage's (1954) axioms for $>$ on his set of acts: see also Fishburn (1987). This approach presumes that the set S of states is infinite, and it yields the homogeneous NASS representation written above for every finite partition of S. The only substantial change in Savage's approach is to weaken his weak order axiom. The probability measure π on S is identical to Savage's, and f on $C \times$

C is unique up to proportionality transformations. If weak order is restored, Savage's representation holds. A somewhat different approach to the homogeneous NASS representation that is based on lotteries over C within each factor (time period, state) is described in Fishburn and LaValle (1987).

An approach to the homogeneous NASS representation based solely on $>$ on $X = C^n$ is presented in Fishburn (1990b). It uses the topological structure of Fishburn (1989) for the $n \geq 3$ axiomatization of the NASS representation recalled in section 6, but drops the assumption that for every $t_k \in T_k$ and all k there is a $t_{(k)} \in T_{(k)}$ such that $(t_k, t_{(k)}) \notin P$, which is no longer needed. The addition of interfactor uniformity and an additivity condition based on three-part partitions of $\{1, \ldots, n\}$ completes the axioms. The π_k are unique, given $\Sigma\pi_k = 1$, and f is unique up to proportionality transformations.

The further specialization to the finite-states subjective expected utility representation

$$x > y \Leftrightarrow \sum_{k=1}^{n} \pi_k(u(x_k) - u(y_k)) > 0,$$

is elegantly axiomatized by Wakker (1989) in the topological mode. A representation that is intermediate between this one and the homogeneous NASS representation is

$$x > y \Leftrightarrow \sum_{k=1}^{n} \pi_k F[u(x_k) - u(y_k)] > 0$$

where $F : \Delta(u) \rightarrow \mathfrak{R}$ with the usual properties. This follows from the axioms of the preceding paragraph plus intrafactor monotonicity. It is very close to the subjective expected utility model but allows a warping of utility differences within factors by means of F that accommodates intransitivity.

8. SSB utility

Our final main nontransitive representation is the SSB (skew symmetric bilinear) generalization of the von Neumann and Morgenstern (1944) expected utility model introduced by Kreweras (1961, 1965) and axiomatized by Fishburn (1982, 1988). We take $>$ as primitive on a nonempty convex set P_0 of probability measures defined on a Boolean algebra of subsets of a set X. Convexity means that if $p, q \in P_0$ and $0 \leq \lambda \leq 1$ then $\lambda p + (1 - \lambda)q \in P_0$.

The three basic axioms for SSB theory apply to all $p, q, r \in P_0$ and all $0 < \lambda < 1$:

continuity: $p > q > r \Rightarrow q \sim \alpha p + (1 - \alpha)r$ for some $0 < \alpha < 1$;

convexity: $\{p > q, p \gtrsim r\} \Rightarrow p > \lambda q + (1 - \lambda)r$; $\{p \sim q, p \sim r\} \Rightarrow p \sim \lambda q + (1 - \lambda)r$; $\{q > p, r \gtrsim p\} \Rightarrow \lambda q + (1 - \lambda)r > p$;

symmetry: $\{p > q > r, p > r, q \sim \frac{1}{2}p + \frac{1}{2}r\} \Rightarrow \{\lambda p + (1 - \lambda)r \sim \frac{1}{2}p + \frac{1}{2}q \Leftrightarrow \lambda r + (1 - \lambda)p \sim \frac{1}{2}r + \frac{1}{2}q\}$.

Convexity implies that the sets of measures preferred to p, indifferent to p, and less preferred than p are convex. The symmetry condition is a balance axiom for $>$ and its inverse that leads to skew symmetry in the representation. The axioms are necessary and sufficient for the existence of a skew symmetric and bilinear $\phi : P_0 \times P_0 \rightarrow \mathfrak{R}$ such that, for all $p, q \in P_0$,

$$p > q \Leftrightarrow \phi(p, q) > 0.$$

Bilinearity means that ϕ is linear separately in each argument: for the first argument, $\phi(\lambda p + (1 - \lambda)q, r) = \lambda\phi(p, r) + (1 - \lambda)\phi(q, r)$, and similarly for the second argument. We refer to this as the *SSB representation*. When it holds, ϕ is unique up to proportionality transformations.

One generalization of the SSB representation, referred to as the nontransitive convex representation, is discussed in Fishburn (1988). It retains the continuity and convexity axioms but drops symmetry. Its ϕ is linear only in the first argument and has $\phi(p, q) > 0 \Leftrightarrow \phi(q, p) < 0$ in place of skew symmetry. A different generalization in Nakamura (1990) introduces a variable threshold of discrimination and represents $>$ by

$$p > q \Leftrightarrow \phi(p, q) > w(p, q)$$

in which ϕ is SSB, w is symmetric $[w(q, p) = w(p, q)]$ and bilinear on $P_0 \times P_0$, $w(p, q) > -|\phi(p, q)|$ when $\phi \neq 0$, and $w(p, q) \geq 0$ when $\phi = 0$. The constant threshold SSB representation

$$p > q \Leftrightarrow \phi(p, q) > 1,$$

is axiomatized in Fishburn and Nakamura (1991).

Suppose the SSB representation holds and P_0 contains every one-point measure. Define ϕ on $X \times X$ by $\phi(x, y) = \phi(p, q)$ when $p(x) = q(y) = 1$. Then bilinearity implies

$$\phi(p, q) = \sum_{x \in X} \sum_{y \in X} p(x)q(y)\phi(x, y)$$

for all $p, q \in P_0$ that have finite supports. Expectation with respect to the product measure $p \times q$ is extended to $\phi(p, q) = \int \int \phi(x, y)dp(x)dq(y)$ for arbitrary measures in Fishburn (1984b, 1988). Additional dominance and monotonicity axioms are needed for the extension.

The axioms of the von Neumann and Morgenstern theory that are weakened by the SSB theory are weak order and the independence axiom $\{p > q, 0 < \lambda < 1\} \Rightarrow \lambda p + (1 - \lambda)r > \lambda q + (1 - \lambda)r$. If independence is added to the SSB axioms, we ge the von Neumann–Morgenstern representation $p > q \Leftrightarrow u(p) > u(q)$, where u is a linear functional on P_0 that is unique up to positive affine transformations. If weak order but not independence is added, the SSB representation reduces to the *weighted linear representation*

$$p > q \Leftrightarrow u(p)w(q) > u(q)w(p)$$

in which u and w are linear functionals on P_0 that do not vanish simultaneously, and $w \geq 0$. This was introduced by Chew and MacCrimmon (1979). It and transitive generalizations are discussed in Chew (1982, 1983, 1985), Fishburn (1983, 1988), Nakamura (1985), and Dekel (1986).

The multiattribute theme for the SSB and weighted linear representations is examined in Fishburn (1984c). Suppose $X = X_1 \times X_2 \times \cdots \times X_n$ and the SSB representation holds with P_0 as the set of all finite-support measures on X. For nonempty $J \subset \{1, \ldots, n\}$, let P_J bet the set of marginal distributions on $X_J = \times_{j \in J} X_j$. Also let $P_k = P_{\{k\}}$ with p_k the marginal of p on X_k.

One multiattribute independence assumption in this setting says that $p \sim q$ whenever $p_k = q_k$ for $k = 1, \ldots, n$. This implies that ϕ can be written as

$$\phi(p, q) = \sum_{k=1}^{n} \phi_k(p_k, q_k) + \sum_{j<k} [\phi_{jk}(p_j, q_k) - \phi_{jk}(q_j, p_k)],$$

where each ϕ_k on $P_k \times P_k$ is SSB and each ϕ_{jk} on $P_j \times P_k$ is bilinear. Another independence assumption says that for every nontrivial two-part partition $\{J, K\}$ of $\{1, \ldots, n\}$, and for all $p, q, r, s \in P_0$, if (grouping coordinates) $p_J = q_J, r_J = s_J, p_K = r_K$, and $q_K = s_K$, where p_J is the marginal of p in P_J, and so forth, then

$$p > r \Leftrightarrow q > s.$$

When this holds and $\sim$ is not transitive, ϕ can be decomposed as

$$\phi(p, q) = \sum_{k=1}^{n} \phi_k(p_k, q_k),$$

where ϕ_k on $P_k \times P_k$ is SSB. When combined with the basic SSB representation, this gives a version of the NASS representation in which each ϕ_k, or f_k, is bilinear as well as skew symmetric.

9. Conclusions

Transitivity has been the cornerstone of traditional notions about order and rationality in decision theory. Three lines of research during the past few decades have tended to challenge its status. First, a variety of experiments and examples that are most often based on binary comparisons between multiple-factor alternatives suggest that reasonable people sometimes violate transitivity, and may have good reasons for doing this. Second, theoretical results show that transitivity is not essential to the existence of maximally preferred alternatives in many situations. Third, fairly elegant new models that do not presume transivity have been developed, and sometimes axiomatized, as alternatives to the less flexible traditional models.

The purpose of this article has been to outline the cultural context in which the special position afforded to transitivity arose, to present a series of arguments for and against transitivity as a necessary tenet of rational decision making, and to illustrate numerical representations of preference that accommodate intransitivities. If the variety of representations is more confusing than illuminating, one would hope that further research during the next few decades will help to identify the most viable models on the basis of philosophical arguments, empirical robustness, and applications potential. General but elegant models that are capable of representing what most researchers agree are reasonable patterns of preference will likely prevail. Some of these surely await discovery.

References

Allais, Maurice. (1953a). "Fondements d'une théorie positive des choix comportant un risque et critique des postulats et axiomes de l'école americaine." Memoir III, *Économétrie*, CNRS, Vol. XL, Paris, 257–332.

Allais, Maurice. (1953b). "Le comportement de l'homme rationnel devant le risque: critique des postulats ex axiomes de l'école americaine," *Econometrica* 21, 503–546.

Allais, Maurice. (1979). "The So-called Allais Paradox and Rational Decisions under Uncertainty." In M. Allais and O. Hagen (eds.), *Expected Utility Hypotheses and the Allais Paradox*. Dordrecht, Holland: Reidel, pp. 437–681.

Arrow, Kenneth J. (1951). *Social Choice and Individual Values*. New York: Wiley. Second edition, 1963.

Aumann, Robert J. (1962). "Utility Theory without the Completeness Axiom," *Econometrica* 30, 445–462.

Aumann, Robert J. (1964). "Utility Theory without the Completeness Axiom: A Correction," *Econometrica* 32, 210–212.

Bell, David. (1982). "Regret in Decision Making under Uncertainty," *Operations Research* 30, 961–981.

Block, H. D. and Jacob Marschak. (1960). "Random Orderings and Stochastic Theories of Responses." In I. Olkin et al. (eds.), *Contributions to Probability and Statistics*. Stanford, CA: Stanford University Press, pp. 97–132.

Bostic, Raphael, R. J. Herrnstein, and R. Duncan Luce. (1988). "The Effect on the Preference-Reversal Phenomenon of Using Choice Indifferences," *Journal of Economic Behavior and Organization* 13, 193–212.

Bouyssou, Denis. (1986). "Some Remarks on the Notion of Compensation in MCDM," *European Journal of Operational Research* 26, 150–160.

Brown, Rex V. (1989). "Toward a Prescriptive Science and Technology of Decision Aiding," *Annals of Operations Research* 19, 467–483.

Casey, Jeff T. (1990). "Reversal of the Preference Reversal Phenomenon." *Organizational Behavior and Human Decision Making*

Chew Soo Hong. (1982). "A Mixture Set Axiomization of Weighted Utility Theory," Discussion Paper 82-4, College of Business and Public Administration, University of Arizona, Tucson.

Chew Soo Hong. (1983). "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox," *Econometrica* 51, 1065–1092.

Chew Soo Hong. (1985). "From Strong Substitution to Very Weak Substitution: Mixture-monotone Utility Theory and Semi-weighted Utility Theory," preprint, Department of Political Economy, Johns Hopkins University, Baltimore.

Chew Soo Hong and Kenneth R. MacCrimmon. (1979). "Alpha-nu Choice Theory: A Generalization of Expected Utility Theory," Working Paper 669, Faculty of Commerce and Business Administration, University of British Columbia, Vancouver.

Churchman, C. West (1961). *Prediction and Optimal Decision*. Englewood Cliffs, NJ: Prentice-Hall.

Condorcet, Marquis de. (1785). Essai sur l'application de l'analyse á la probabilité des décisions rendues á la pluralité des voix. Paris.

Cowan, Thomas and Peter C. Fishburn. (1988). "Foundations of Preference." In G. Eberlein and H. Berghel (eds.), *Essays in Honor of Werner Leinfellner. Theory and Decision Series*. Dordrecht, Holland: Reidel, pp. 261–271.

Croon, M. A. (1984). "The Axiomization of Additive Difference Models for Preference Judgments." In E. Degreef and G. Van Buggenhaut (eds.), *Trends in Mathematical Psychology*. Amsterdam: North-Holland, pp. 193–27.

Debreu, Gerard. (1960). "Topological Methods in Cardinal Utility Theory." In K. J. Arrow, S. Karlin, and P. Suppes (eds.), *Mathematical Methods in the Social Sciences, 1959*. Stanford, CA: Stanford University Press, pp. 16–26.

Dekel, Eddie. (1986). "An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom," *Journal of Economic Theory* 40, 304–318.

Diamond, Peter A. (1967). "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparison of Utility: Comment," *Journal of Political Economy* 75, 765–766.

Fishburn, Peter C. (1970a). *Utility Theory for Decision Making*. New York: Wiley.

Fishburn, Peter C. (1970b). "Intransitive Indifference in Preference Theory: A Survey," *Operations Research* 18, 207–228.

Fishburn, Peter C. (1973a). "Binary Choice Probabilities: On the Varieties of Stochastic Transitivity," *Journal of Mathematical Psychology* 10, 327-352.

Fishburn, Peter C. (1973b). *The Theory of Social Choice*. Princeton, NJ: Princeton University Press.

Fishburn, Peter C. (1974). "Lexicographic Orders, Utilities and Decision Rules: A Survey," *Management Science* 20, 1442–1471.

Fishburn, Peter C. (1980). "Lexicographic Additive Differences," *Journal of Mathematical Psychology* 21, 191–218.

Fishburn, Peter C. (1982). "Nontransitive Measurable Utility," *Journal of Mathematical Psychology* 26, 31–67.

Fishburn, Peter C. (1983). "Transitive Measurable Utility," *Journal of Economic Theory* 31, 293–317.

Fishburn, Peter C. (1984a). "Probabilistic Social Choice Based on Simple Voting Comparisons," *Review of Economic Studies* 51, 683–692.

Fishburn, Peter C. (1984b). "Dominance in SSB Utility Theory," *Journal of Economic Theory* 34, 130–148.

Fishburn, Peter C. (1984c). "Multiattribute Nonlinear Utility Theory," *Management Science* 30, 1301–1310.

Fishburn, Peter C. (1985). *Interval Orders and Interval Graphs*. New York: Wiley.

Fishburn, Peter C. (1987). "Reconsiderations in the Foundations of Decision under Uncertainty," *Economic Journal* 97, 825–841.

Fishburn, Peter C. (1988). *Nonlinear Preference and Utility Theory*. Baltimore: Johns Hopkins University Press.

Fishburn, Peter C. (1989). "Continuous Nontransitive Additive Conjoint Measurement," forthcoming in *Mathematical Social Sciences*.

Fishburn, Peter C. (1990a). "Nontransitive Additive Conjoint Measurement," *Journal of Mathematical Psychology*.

Fishburn, Peter C. (1990b). "Skew Symmetric Additive Utility with Finite States," *Mathematical Social Sciences* 19, 103–115.

Fishburn, Peter C. (1990c). "Additive Differences and Simple Preference Comparisons," *Journal of Mathematical Psychology*.

Fishburn, Peter C. and Irving H. LaValle. (1987). "A Nonlinear, Nontransitive and Additive-probability Model for Decisions under Uncertainty," *Annals of Statistics* 15, 830–844.

Fishburn, Peter C. and Irving H. LaValle. (1988). "Context-dependent Choice with Nonlinear and Nontransitive Preferences," *Econometrica* 56, 1221–1239.

Fishburn, Peter C. and Bernard Monjardet. (1991). "Norbert Wiener on the Theory of Measurement (1914, 1915, 1921)," *Journal of Mathematical Psychology*.

Fishburn, Peter C. and Yutaka Nakamura. (1991). "Nontransitive Measurable Utility with Constant Threshold," *Journal of Mathematical Psychology*.

Fishburn, Peter C. and Philip D. Straffin. (1989). "Equity Considerations in Public Risks Evaluation," *Operations Research* 37, 229-239.

Flood, Merrill M. (1951-1952). "A Preference Experiment," The Rand Corporation Papers P-256, P-258, and P-263, Santa Monica, CA.

Goldstein, William and Hillel J. Einhorn (1987). "Expression Theory and the Preference Reversal Phenomenon," *Psychological Review* 94, 236–254.

Hammond, Peter J. (1988). "Consequentialist Foundations for Expected Utility," *Theory and Decision* 25, 25–78.

Krantz, David H., R. Duncan Luce, Patrick Suppes, and Amos Tversky. (1971). *Foundations of Measurement*, Volume I. New York: Academic Press.

Kreweras, Germain. (1961). "Sur une possibilité de rationaliser les intransitivités," *La Décision*, CNRS, Paris, 27–32.

Kreweras, Germain. (1965). "Aggregation of Preference Orderings." In Saul Sternberg et al. (comp.), *Mathematics and Social Sciences I*. Paris: Mouton, pp. 73–79.

LaValle, Irving H. (1990). "Small Worlds and Sure Things: Consequentialism by the Back Door." In W. Edwards (ed.), *Utility: Theories, Measurements, and Applications*.
Loomes, Graham and Robert Sugden. (1982). "Regret Theory: An Alternative Theory of Rational Choice under Uncertainty," *Economic Journal* 92, 805–824.
Loomes, Graham and Robert Sugden. (1987). "Some Implications of a More General Form of Regret Theory," *Journal of Economic Theory* 41, 270–287.
Luce, R. Duncan. (1956). "Semiorders and a Theory of Utility Discrimination," *Econometrica* 24, 178–191.
Luce, R. Duncan. (1959). *Individual Choice Behavior: A Theoretical Analysis*. New York: Wiley.
Luce, R. Duncan. (1978). "Lexicographic Tradeoff Structures," *Theory and Decision* 9, 187–193.
Luce, R. Duncan and Howard Raiffa. (1957). *Games and Decisions*. New York: Wiley.
Luce, R. Duncan and Patrick Suppes. (1965). "Preference, Utility, and Subjective Probability." In R. D. Luce, R. R. Bush, and E. Galanter (eds.), *Handbook of Mathematical Psychology, III*. New York: Wiley, pp. 249-410.
MacCrimmon, Kenneth R. and Stig Larsson. (1979). "Utility Theory: Axioms Versus 'Paradoxes'." In M. Allais and O. Hagen (eds.), *Expected Utility Hypotheses and the Allais Paradox*. Dordrecht, Holland: Reidel, pp. 333-409.
Machina, Mark J. (1982). " 'Expected Utility' Analysis without the Independence Axiom," *Econometrica* 50, 277-323.
Mas-Colell, Andreu. (1974). "An Equilibrium Existence Theorem without Complete or Transitive Preferences," *Journal of Mathematical Economics* 1, 237–246.
May, Kenneth O. (1954). "Intransitivity, Utility, and the Aggregation of Preference Patterns," *Econometrica* 22, 1–13.
Merrill, Samuel. (1988). *Making Multicandidate Elections More Democratic*. Princeton, NJ: Princeton University Press.
Morrison, H. W. (1962). "Intransitivity of Paired Comparison Choices," Ph.D. dissertation, University of Michigan, Ann Arbor.
Nakamura, Yutaka. (1985). "Weighted Linear Utility," preprint, Department of Precision Engineering, Osaka University, Osaka, Japan.
Nakamura, Yutaka. (1990). "Bilinear Utility and a Threshold Structure for Nontransitive Preferences," *Mathematical Social Sciences* 19, 1–21.
Peleg, Bezalel. (1984). *Game Theoretic Analysis of Voting in Committees*. Cambridge, England: Cambridge University Press.
Raiffa, Howard. (1968). *Decision Analysis: Introductory Lectures on Choice under Uncertainty*. Reading, MA: Addison-Wesley.
Ramsey, Frank P. (1931). "Truth and Probability." In *The Foundations of Mathematics and Other Logical Essays*. London: Routledge and Kegan Paul, pp. 156–198. Reprinted in H. E. Kyburg and H. E. Smokler (eds.), *Studies in Subjective Probability*. New York: Wiley, 1964, pp. 61–92.
Rapoport, Anatol. (1989). *Decision Theory and Decision Behaviour*. Dordrecht, Holland: Kluwer Academic.
Savage, Leonard J. (1954). *The Foundations of Statistics*. New York: Wiley.
Schkade, David A. and Eric J. Johnson. (1989). "Cognitive Processes in Preference Reversals," *Organizational Behavior and Human Decision Processes* 44, 203–231.
Schmeidler, David.(1989). "Subjective Probability and Expected Utility without Additivity," *Econometrica* 57, 571-587.
Slovic, Paul, Dale Griffin, and Amos Tversky. (1990). "Compatibility Effects in Judgment and Choice." In R. M. Hogarth (ed.), *Insights in Decision Making: Theory and Applications*. Chicago: University of Chicago Press, pp. 5-27.
Slovic, Paul and Sarah Lichtenstein (1968). "The Relative Importance of Probabilities and Payoffs in Risk-taking," *Journal of Experimental Psychology Monograph Supplement* 78, part 2, 1–18.
Sonnenschein, Hugo F. (1971). "Demand Theory without Transitive Preferences, with Applications to the Theory of Competitive Equilibrium." In J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein (eds.), *Preferences, Utility, and Demand*. New York: Harcourt Brace Jovanovich, pp. 215–223.
Suppes, Patrick, David H. Krantz, R. Duncan Luce, and Amos Tversky. (1989). *Foundations of Measurement*, Volume II. New York: Academic Press.
Tversky, Amos. (1969). "Intransitivity of Preferences," *Psychological Review* 76, 31–48.

Tversky, Amos and Daniel Kahneman. (1986). "Rational Choice and the Framing of Decisions." In R. M. Hogarth and M. W. Reder (eds.), *Rational Choice.* Chicago: University of Chicago Press, pp. 67–94.

Tversky, Amos, Paul Slovic, and Daniel Kahneman. (1990). "The Causes of Preference Reversal," *American Economic Review* 80, 204–217.

Vind, Karl. (1991). "Independent Preferences," *Journal of Mathematical Economics* 20, 119–135.

von Neumann, John and Oskar Morganstern. (1944). *Theory of Games and Economic Behavior.* Princeton, NJ: Princeton University Press. Third edition, 1953.

Wakker, Peter P. (1989). *Additive Representations of Preferences.* Dordrecht, Holland: Kluwer Academic.

Wiener, Norbert. (1914). "A Contribution to the Theory of Relative Position," *Proceedings of the Cambridge Philosophical Society* 17, 441–449.