IFT6561 Homework 1

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Exercise 1 a)

What is the number of ways of choosing the first t cards, for t <= 52, taking order into account?

taking order into account (permutations) - we want the total number of t-permutations of n - the state space can be represented by a tree - starting with 52 nodes, then 51 for each of the 52… - e.g. if t = 3 : (3-permutations of 52) = 52 * 51 * 50 - this is usually denoted as a factorial which stops at n - t - 52! / (52 - t)!

without taking order into account (combinations) - we want the total number of t-combinations of n - the intuition is the same as for the t-permutation - there are multiple repetitions in the permutations - 52! / (((52 - t)!)* (t!))

Exercise 1 b)

What is the minimal period length of the generator, and the minimal number of bits needed to represent its state, to make sure that every possibility can happen for the first t cards? Give expressions that are functions of t.

- The simplest way would be to have 52! / (52 - t)! states.
- These states can be encoded in ceiling(log2(number_of_states)) bits.
- The shortest periodicity of such PRNG would have to be of number_of_states - 1
  - otherwise some state would be impossible to reach (violating the condition)

Exercise 1 c)

What is that minimal number of bits for t = 52? (Give a numerical value.)

```
def factorial(n):
  """iterative factorial"""
  ret = 1
  for i in range(1, n+1):
    ret *= i
  return ret

threshold = factorial(52) / factorial(52 - 52)

necessary_bits = 1
while 2**necessary_bits < threshold:
  necessary_bits += 1

print(necessary_bits)

# this yields a value of 226
```

This yield a value of 226 bits to represent 52!. If the order is not important, 0 bits would be needed since there would be only one possible state (all of the cards).

Exercise 1 d)

If the period length is 2ˆ31 -2 (many widely-used classical LCGs have that value), what is the maximal value of t for which we can have all the possibilities?

```
# in the same fashion as (c)

from math import e, pi, sqrt, log2  # get the necessary

def stirling(n):
  """stirling approximation of factorial(n)"""
  assert n>0 and isinstance(n, int)
  return sqrt(2*pi*n) * ((n / e)**n)

def number_of_states(t):
  """calculate the number of states needed for t"""
  assert t>0 and isinstance(t, int)
  return stirling(52) / stirling(52 - t )
```

```
t = 1
while number_of_states(t+1) < ((2**31)-2):
  t += 1

print(t)
```

This yields a value of t = 6. There would be enough states to represent

Exercise 2 a)

Implement the SWB generator of Example 1.16, whose parameters are (b, r, k) = (2ˆ31 , 8, 48), and real-valued output defined by un = x 2n /2 62 + x2n+1 /2 31 , and use it to generate three-dimensional points in [0, 1) 3 , defined by ui = (u25i, u25i+20, u25i+24) for i = 0, …, m-1, for m = 10ˆ4.

Partition the unit cube into k = 10ˆ6 subcubes by partitioning each axis into 100 equal intervals.

Number these subcubes from 0 to k - 1 (in any way), find the number of the subcube in which each point u i has fallen, and count the number C of collisions as in Example 1.6. Repeat this 10 times, to obtain 10 "independent" realizations of C, and compare their distribution with the Poisson approximation given in Example 1.6. You can do the latter comparison informally; there is no need to perform a formal statistical test.

These is the total number of collisions observed in the subcubes, for each run of the experiment (total of 10 realizations) using MathematicaSWB:

[2170, 2137, 2100, 2104, 2127, 2086, 2111, 2114, 2130, 2158]

The expected value calculated with the Poisson approximation is the following:

```
def poisson_estimate_collisions(number_of_points, number_of_states):
  """ calculates the estimated number of collisions according to
  ((numberOfPoints^2) / (2 * numberOfCases))"""
  return (number_of_points**2) / (2 * (number_of_states))

estimate = estimated_collisions(10000, 10**6)

# this is equal to 50.0
```

We can see that there are way more collisions than expected. This is a bad sign for the generator, indicating that values do not seem uniform (that could be exploited by an attacker).

Exercise 2 b)

Redo the same experiment, but this time using a better generator, such as MRG32k3a in SSJ, for example. Discuss your results.

These is the total number of collisions observed in the subcubes, for each run of the experiment (total of 10 realizations) using MRG32k3a:

[41, 66, 53, 50, 54, 55, 53, 44, 53, 59]

Since the expected value calculated by Poisson approximation is of 50, this pseudorandom number generator yields a much more uniform distribution, which is a good thing for a pseudorandom number generator…

However, since many other qualitites are needed to evaluate the performance of a pseudorandom number generator, it cannot be said with just those simple observations that this is a good generator.

Exercise 3 a)

Voir l'exercice 1.18 des notes. Il n'y a pas de simulation a implanter pour cette question. L'idee est de comprendre ce qui se passe si on estime le volume d'une sphere de rayon 1 en s dimensions par la methode Monte Carlo. Il s'agit bien sur d'un exercice purement academique, puisqu'on connait deja le volume de cette sphere, mais il permet de comprendre un type de difficulte qui survient dans de nombreuses applications pratiques. Pour estimer le volume, on tire n points au hasard dans le cube (0, 1) s , on calcule la fraction pn de ces points qui tombent dans la sphere (pour estimer la fraction p du cube occupe par la sphere), et l'estimateur du volume est un = 2 s pn . (a) Prouvez que cet estimateur est sans biais. Donnez aussi (avec preuve) des formules exactes pour la variance et l'erreur relative de cet estimateur, en fonction de s.

Exercise 3 b)

Pour avoir une erreur relative constante en fonction de s, disons inferieure a 0.01 pour tout s, a quelle vitesse (ou de quelle maniere) doit-on augmenter n en fonction de s, lorsque s est grand? Donnez une formule pour n en fonction de s et expliquez ce que cela implique pour les grandes valeurs de s.

Exercise 3 c)

Calculez les valeurs numeriques de p, V s , et de l'erreur relative au carre de u n , RE 2 [˜u n], pour s = 2, 5, 10, 20.

C'est un cas de "hit-or-miss" estimator, 1.3.5 dans le livre.

Exercise 4 a)

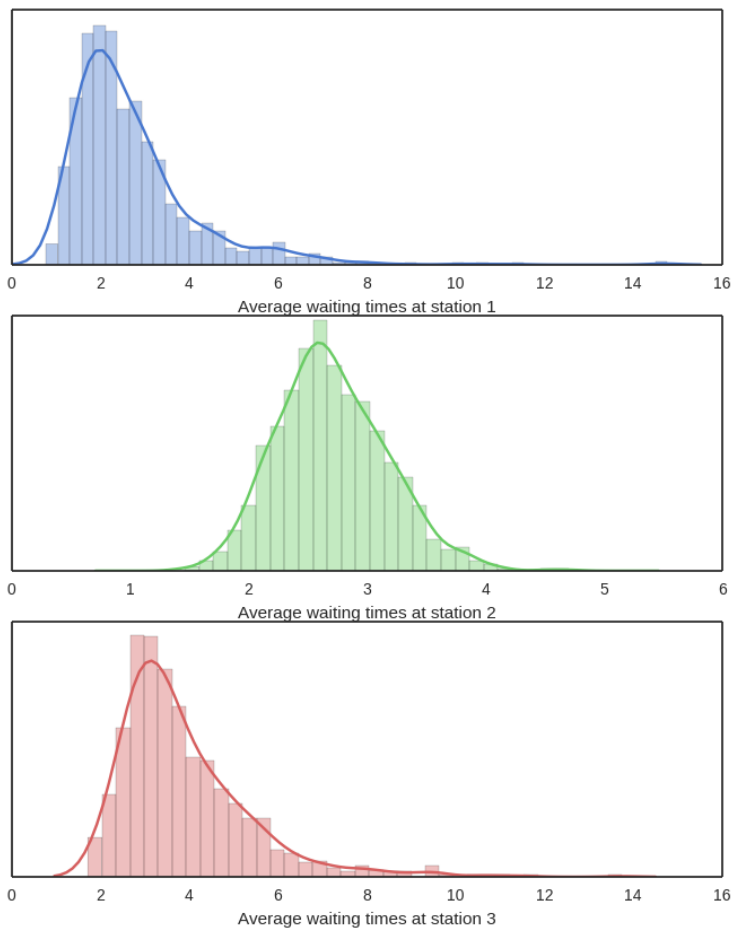
Repetez cette simulation $n = 1000$ fois, puis calculez la valeur estimee et un intervalle de confiance a 95% pour l'esperance de chacune des 5 quantites calcules (temps total d'attente et temps de blocage a chaque station).

```
REPORT on Tally stat. collector ==> Average waiting times for station 1
    num. obs.      min          max        average     standard dev.
       1000        0.773       14.758        2.715         1.446

REPORT on Tally stat. collector ==> Average blocking times for station 1
    num. obs.      min          max        average     standard dev.
       1000        0.011        0.244        0.074         0.032

REPORT on Tally stat. collector ==> Average waiting times for station 2
    num. obs.      min          max        average     standard dev.
       1000        1.462        4.706        2.706         0.462

REPORT on Tally stat. collector ==> Average blocking times for station 2
    num. obs.      min          max        average     standard dev.
       1000     -2.2E-15      2.4E-15     -1.1E-17       7.8E-16

REPORT on Tally stat. collector ==> Average waiting times for station 3
    num. obs.      min          max        average     standard dev.
       1000        1.718       13.739        3.881         1.458
```

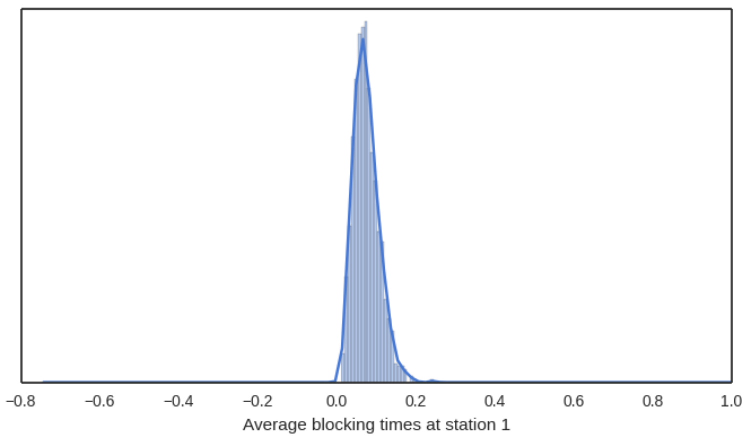
Exercise 4 b)

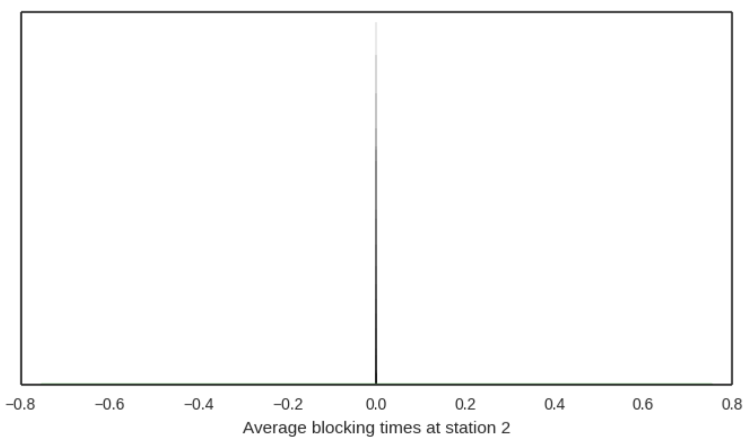
Faites aussi tracer des histogrammes des 1000 valeurs observes pour chacune de ces 5 quantites.

Waiting time histograms



Blocking time histograms





The histograms were made using *Seaborn* library in python3. The script is available with the rest of the code. The number of bins was decided using the Freedman–Diaconis formula.

Exercise 4 c)

Discutez ce que vous observez; par exemple ou observe-t-on davantage d'attente ou de blocage?

We see more