Exercise 2

a)

In Example 4.2, explain in detail how to generate from the sampling density g that corresponds to the hat function h of Figure 4.2, by inversion.

The rejection method

The goal is to generate (x, y) uniformly in $S(f)$.

The idea

The trick to achieve that is to generate (x, y) in a space that encloses S(f) and check if $(X, Y) \in S(f)$. If it does belong, then we return it as a random point of the density $S(f)$, otherwise we try again.

The problem

So if the enclosing space is huge in relation to the volume of $S(f)$, it would be expected that very few points actually land in $S(f)$ and therefore most of the randomly generate points are lost.

Therefore, depending on the shape of the enclosing space and $S(f)$, this procedure can become extremely wasteful or work pretty well.

The hat function trick (majoring function)

The idea is to fit the enclosing space as closely as possible to $S(f)$ to avoid generating points for nothing.

In this exercise, the hat function is a piecewise constant uniform distribution in $(0, 1)$. This hat function h has breakpoints optimized to diminish the area over the curve.

The inversion works simply by generating $(x, y) \in U(0, 1)^2$. We scale y to the hat function height (in the particular piece) and check if the point is in $S(f)$. If it is, we accept and return x, otherwise we generate the next point.

The expected number of points that will be wasted is the ratio of the area over the curve $S(f)$ to the total area. By selecting the breakpoints optimally, this ratio can be minimized.

b)

The implementation is in *exercise2.java*. In the runtime results, you can see that the runtime of the 3 pieces algorithm is slightly less than the one of the naive algorithm. Very close to $\frac{1.38997}{\frac{16}{9}}$, the expected ratio.

c)

The 4 breakpoints were optimized by supposing that there were two distinct areas on each side of $x = \frac{2}{3}$.

The optimization for the breakpoints was:

$max\{x_1 * (\frac{16}{9} - f(x_1)) + (x_2 - x_1) * (\frac{16}{9} - f(x_2))\}$

$max\{(1 - x_4) * (\frac{16}{9} - f(x_4)) + (x_4 - x_3) * (\frac{16}{9} - f(x_3))\}$

this yields the values (by wolfram alpha):

$$x_1 = 0.202604$$
$$x_2 = 0.381073$$
$$x_3 = 0.847968$$
$$x_4 = 0.932157$$

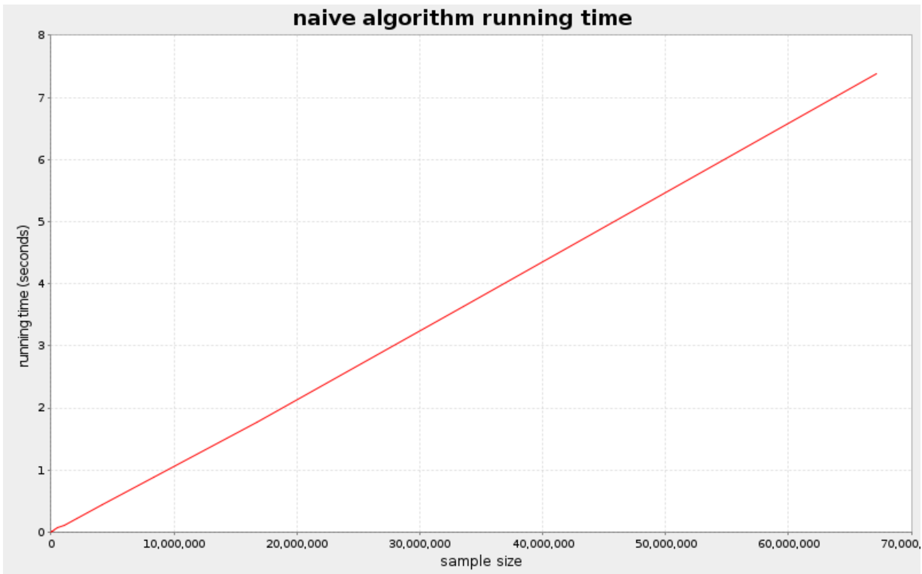
These new breakpoints yield a total area of

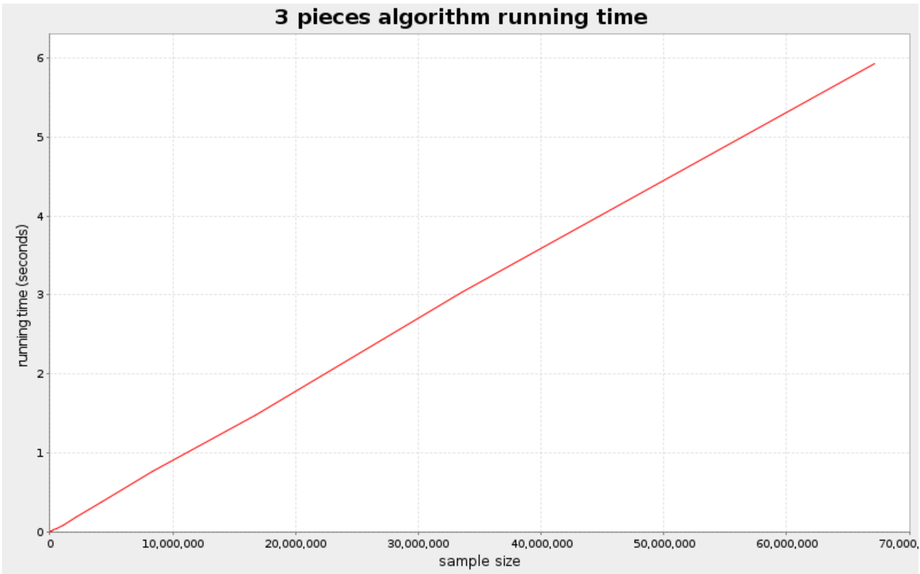
$Area = x_1 f(x_1) + (x_2 - x_1) f(x_2) + (x_3 - x_2)\frac{16}{9} + (x_4 - x_3) f(x_3) + (1 - x_4) f(x_4)$
$Area = 1.26053$

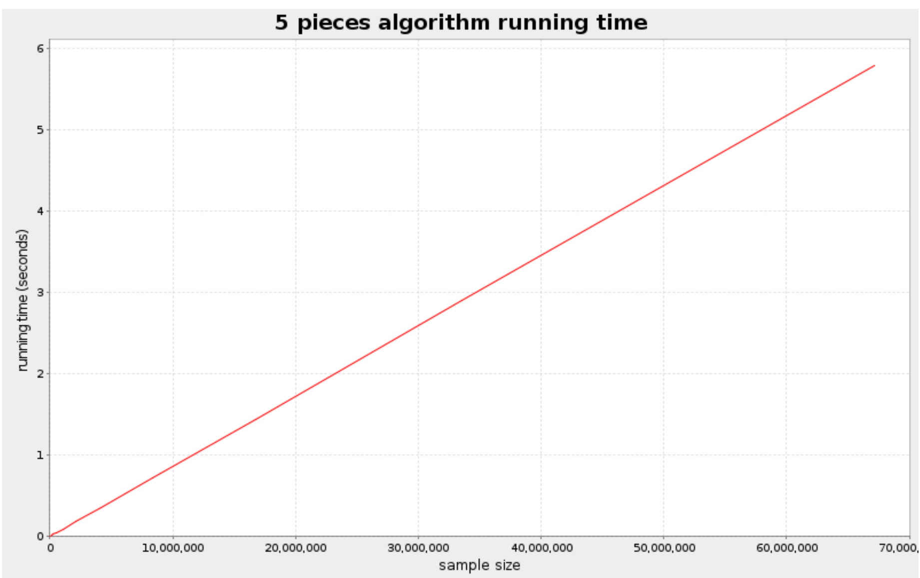
This is slightly better than the 3 pieces version, which has a total area of 1.38997.

Runtime results

Points in the range of 2^0 to 2^{27} were generated by using the three different algorithms.

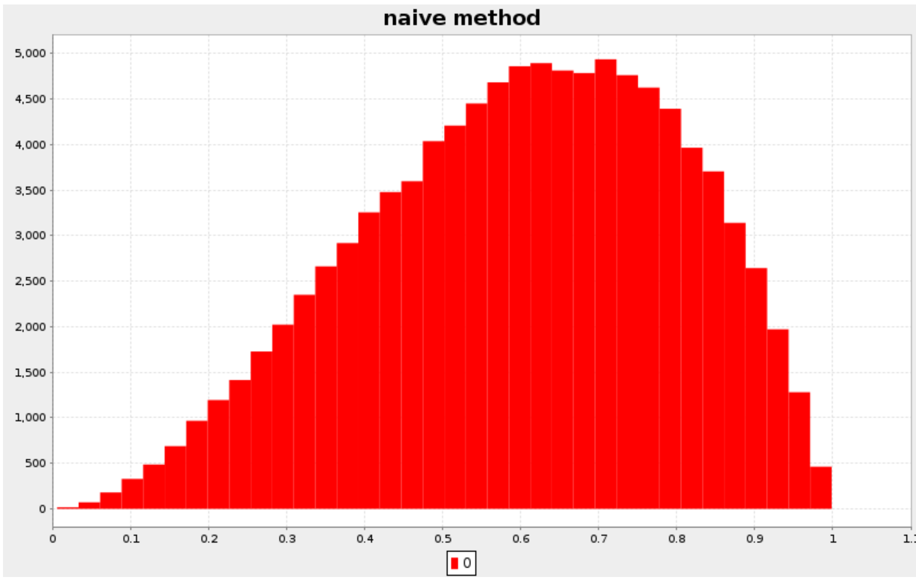


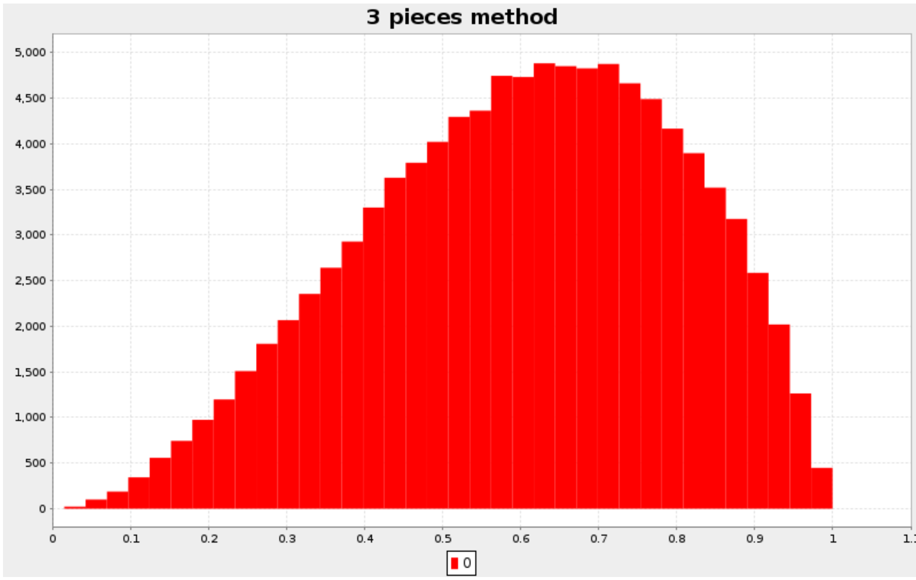

3 pieces algorithm running time
running time (seconds)
0
1
2
3
4
5
6
0
10,000,000
20,000,000
30,000,000
40,000,000
50,000,000
60,000,000
70,000,
sample size

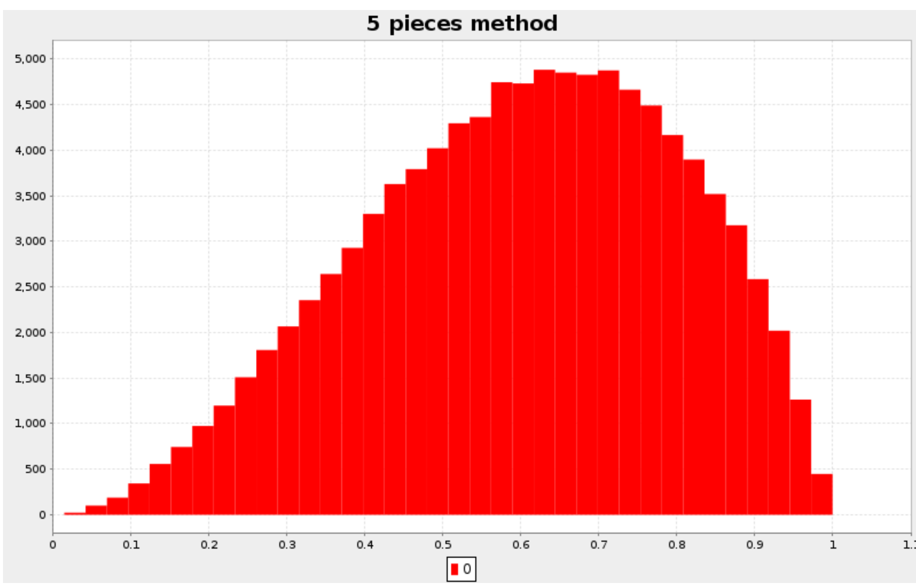

5 pieces algorithm running time
running time (seconds)
0
1
2
3
4
5
6
0
10,000,000
20,000,000
30,000,000
40,000,000
50,000,000
60,000,000
70,000,
sample size

Correctness results

100000 points were generated by each algorithm and the empirical distribution is graphed here. This confirms that the algorithm works fine.




3 pieces method
5,000
4,500
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
0


5 pieces method
5,000
4,500
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
0