Calculus

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Contents

1 Introduction

This document is my version of a condensed calculus textbook. It covers both single and multivariable calculus, from the very beginning to the equivalent of Calc 4 in most universities. I will mostly follow the organization of Stewart's *Calculus*, 8th edition.

2 The Inspiration of it All: Limits and Their Properties

2.1 What is a Limit, Anyway?

2.1.1 Introduction to Limits

Suppose you wish to graph the function

$$f(x) = \frac{x^3+1}{x+1}$$

From algebra, you know that this fraction can be simplified into the quadratic $x^2 - x + 1$. However, these two expressions are not quite identical, because $x = -1$ makes the fraction undefined, but not the quadratic.

So the graph of $f(x)$ looks like the quadratic $x^2 - x + 1$, but with a *hole* at the point $(-1, 3)$. While x cannot equal -1, you can have x be arbitrarily close to -1 (say, -0.9999.) And as x moves arbitrarily close to -1, the value of $f(x)$ (i.e. the graph) moves arbitrarily close to the value 3.

So we say that the **limit** of $f(x)$ as x approaches -1 is 3, or in formal notation,

$$\lim_{x \to -1} f(x) = 3$$

Definition. (Informal) If a functional $f(x)$ gets arbitrarily close to a single value L as x approaches c from either side, then the *limit* of $f(x)$ as x approaches c is L. We write

$$\lim_{x \to c} f(x) = L$$

You can estimate a limit graphically or numerically (using a table of values), but those methods are self-explanatory and not very enlightening, so I will not elaborate.

Ex: What is the limit of $f(x)$ as x approaches 1 if f is defined as:

$$\begin{cases} 3, x \neq 1 \\ 5, x = 1 \end{cases}$$

By looking at the graph, it is clear that as x moves arbitrarily close to 1, $f(x)$ moves arbitrarily close to 3. In fact, since $f(x) = 3$ for all x other than 1, the limit must be 3.

The point here is that the *actual* value of $f(1)$ has no influence on what the limit is equal to - in fact, $f(1)$ could take on literally any value (even undefined!), and the limit would be the same.

2.1.2 When a Limit Fails to Exist...

I will discuss 3 common types of limits that fail to exist.

Ex. 1: Behavior that Differs From the Left and Right

Consider $\lim\limits_{x \to 0} \frac{|x|}{x}$.

When x is positive, $f(x) = 1$, and when x is negative, $f(x) = -1$, as shown by the graph. Also note that $f(0)$ is undefined.

According to our informal definition, the limit should be reached as x gets arbitrarily close to 0. But no matter how close to 0 we take x to be, we can pick values of x such that $f(x) = 1$, or $f(x) = -1$.

More formally, suppose that x is within δ of 0, where $\delta > 0$ is just some real number. This means that $-\delta < x < \delta$. Within this range, if you pick x to be in the interval $(-\delta, 0)$, then $f(x) = -1$, and if you pick x to be in the interval $(0, \delta)$, then $f(x) = 1$.

Clearly, this choice is possible regardless of small δ is, which means that the limit does not exist (since it's trying to take on 2 values at the same time.)

This example touches on the concept of δ as well as differentiating between behavior from the left and right, concepts that will show up again later in this chapter.

Ex. 2: Unbounded Functions

Consider $\lim_{x\to 0} \frac{1}{x^2}$.

Clearly, as x approaches 0, $f(x)$ is increasing rapidly without bound. It is not approaching a real number L, but rather shoots off to infinity. Therefore, we say the limit does not exist. (How we deal with infinite limits will be discussed in further detail later.)

Ex. 3: Functions that Oscillate

Consider $\lim_{x\to 0} \cos \frac{1}{x}$.

What is the behavior of $\cos x$? We know that as x goes off to infinity, there are an infinite number of x values that give $\cos x = 1$ and $\cos x = -1$. So for $\cos 1/x$, we get the following table:

x	$1/\pi$	$1/2\pi$	$1/3\pi$	$1/4\pi$	$1/5\pi$	0?
f(x)	-1	1	-1	1	-1	DNE

We can see as x approaches 0, it is always possible to pick a δ such that there are two values of x within the range such that their cosines are equal to 1 and -1. As $f(x)$ is not going towards a singular value as x goes to 0, the limit does not exist.

2.2 The Formalization of a Limit - Epsilon-Delta Proofs

Now that we have explored the basic concept of a limit, and ways in which a limit might fail to exist, we would like to formalize the definition I gave at the beginning of this chapter.

Specifically, we would like to formalize the part where I wrote "$f(x)$ gets arbitrarily close to a single value" and "x approaches c". To do this, we introduce the $\varepsilon - \delta$ **definition of a limit.**

Definition. Let f be a function defined on some interval including c (except possibly being undefined at c itself) and let L be a real number. The statement

$$\lim_{x\to c} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$|x - c| < \delta, \longrightarrow |f(x) - L| < \epsilon.$$

If this is your first foray into calculus, you may be confused by this statement. The bad news is that yes, most math past this is going to have equally jargon-y theorem statements. However, I will now give some examples to make this definition clearer, and also show how to use the definition to prove limits.

Essentially, what the absolute value statement is saying is that as x gets arbitrarily close to c (where the distance is denoted by δ), then the value of $f(x)$ also gets arbitrarily close to the limit, L (where the distance is denoted by ϵ).

In practice, we should be able to write δ as some function of ϵ, which ensures that for each ϵ there exists a δ.

Here is the first example of the formal definition of a limit.

Ex. 1: Linear Function with Specified Value

Given that

$$\lim_{x\to 4} (3x - 5) = 7$$

find δ such that

$$|(3x - 5) - 7| < 0.03 \qquad \text{whenever} \qquad |x - 4| < \delta.$$

Soln. Notice that the first inequality simplifies as

$$|(3x - 5) - 7| < 0.03 \longrightarrow |3x - 12| < 0.03 \longrightarrow 3|x - 4| < 0.03$$

This means that if we pick

$$\delta = \frac{1}{3} \cdot 0.03 = 0.01$$

then we have

$$|x-4| < 0.01 \longrightarrow 3|x-4| < 0.03 \longrightarrow |(3x-5)-7| < 0.03,$$

as desired.

Of course, finding a δ for a specific ϵ is not enough to prove a limit. However, it provides a template for trying to prove that a limit exists.

Now let's actually prove that a limit exists.

Ex. 2: Using the Epsilon-Delta Definition of a Limit

Now, we want to use the definition of a limit to show that

$$\lim_{x\to 4}(3x-5) = 7.$$

Soln. We need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|(3x-5)-7| < \epsilon \,\text{whenever}\, |x-4| < \delta.$$

To find a δ for every ϵ, we need to somehow relate the expressions in the absolute values. But we already did this above, as we saw

$$|(3x-5)-7)| = 3|x-4|$$

Therefore, we should pick $\delta = \epsilon/3$. This works because

$$|x-4| < \delta = \frac{\epsilon}{3}$$

implies

$$|(3x-5)-7| = 3|x-4| < 3\delta = \epsilon$$

and therefore we have found an explicit formula for δ in terms of ϵ, which finishes the proof. ■

Ex. 3: Proving a Limit of a Quadratic Function

We wish to show that

$$\lim_{x\to 10} x^2 - 3 = 97$$

Using the delta-epsilon definition, this means we need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|(x^2-3)-97)| < \epsilon \,\text{whenever}\, |x-10| < \delta.$$

As above, we start by trying to simplify the first inequality:

$$|(x^2-3)-97| < \epsilon \longrightarrow |x^2-100| < \epsilon \longrightarrow |x-10||x+10| < \epsilon \longrightarrow |x-10| < \frac{\epsilon}{|x+10|}$$

However, we arrive at a problem: if we choose, say:

$$\delta = \frac{\epsilon}{|x+10|}$$

this does not give us a formula for δ in terms of ϵ, since there is a dependence on x.

So, what we have to do is artificially bound δ. Usually, we will pick $\delta \le 1$. For large values of ϵ, this is guaranteed to work, and for small values of ϵ, we will show that you can always find a value of δ.

If δ can be at most 1, x has range $[9, 11]$. This implies that

$$19|x-10| \le |x+10||x-10| \le 21|x-10|,$$

and furthermore if you let $21|x-10| < \epsilon$, this immediately implies that $|x+10||x-10| < \epsilon$. So we can take

$$|x-10| < \delta = \frac{\epsilon}{21}$$

and that will satisfy the condition (for small values of ϵ.)

Now we can write the proof.

Soln: I claim that a suitable function for δ is

$$\delta = \min\left(1, \frac{\epsilon}{21}\right)$$

Divide the proof up into two cases.

Case 1: $\epsilon \geq 21$

In this case, we have $\delta = 1$. Then, we can write

$$\begin{gathered} |x-10| < 1 \\ \longrightarrow 21|x-10| < 21 < \epsilon \\ |x+10| < 21 \\ \longrightarrow |x+10||x-10| < 21|x-10| < \epsilon, \end{gathered}$$

as desired.

Case 2: $\epsilon < 21$

In this case, we have $\delta = \epsilon/21$. Note that this still satisfies the condition $\delta \leq 1$. Then, we can write

$$\begin{gathered} |x-10| < \frac{\epsilon}{21} \\ \longrightarrow 21|x-10| < \epsilon \\ |x+10| < 21 \\ \longrightarrow |x+10||x-10| < 21|x-10| < \epsilon, \end{gathered}$$

as desired. This covers all possible values of ϵ, and therefore the proof is finished. ■

In general, this technique of imposing an artificial upper bound on δ and then using a minimum function is how one proves epsilon-delta limits for nonlinear functions.

Ex. 4: Proving a Limit of a Rational Function

We wish to show that

$$\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$$

In other words, we want $\delta > 0$ such that

$$|x-3| < \delta \longrightarrow \left|\frac{1}{x} - \frac{1}{3}\right| < \epsilon$$

We have

$$\left|\frac{1}{x} - \frac{1}{3}\right| < \epsilon \longrightarrow \left|\frac{3-x}{3x}\right| < \epsilon \longrightarrow |x-3| < |3x|\epsilon$$

Again, let's say we must have $\delta < 1$. Then we have

$$|x-3| < 1 \longrightarrow -1 < x-3 < 1 \longrightarrow 2 < x < 4 \longrightarrow 6 < 3x < 12 \longrightarrow |3x| > 6$$

I claim that a suitable function for δ is therefore $\delta = \min(1, 6\epsilon)$. For $\epsilon < 1/6$ this is true because

$$|x-3| < \delta \longrightarrow |x-3| < 6\epsilon \longrightarrow |x-3| < |3x|\epsilon$$

as desired, where we used the fact that $|3x| > 6$. (I omit the second part of the proof, but it is very similar to the previous example.) ■

2.3 An Introduction to Evaluating Limits

Now, I will introduce the most basic properties of evaluating limits.

Theorem 2.1. *Let c and d be real numbers, and let n be a positive integer. Then*

1. $\lim_{x \to c} d = d$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Theorem 2.2. *Let c and d be real numbers, let n be a positive integer, and also say we have functions $f(x)$ and $g(x)$ such that*

$$\lim_{x\to c} f(x) = K, \lim_{x\to c} g(x) = L$$

Then the following are true:

1. $\lim_{x\to c}[bf(x)] = bK$
2. $\lim_{x\to c}[f(x) \pm g(x)] = K \pm L$
3. $\lim_{x\to c}[f(x)g(x)] = KL$
4. $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{K}{L}$
5. $\lim_{x\to c}[f(x)]^n = K^n$

Ex. 1: Find the limit

$$\lim_{x\to -3}(x^2 + 3x)$$

using the above properties.

Soln:

$$\begin{aligned}\lim_{x\to -3}(x^2 + 3x) &= \lim_{x\to -3} x^2 + \lim_{x\to -3} 3x && (2.2.2)\\ &= (-3)^2 + 3\lim_{x\to -3} x && (2.1.3, 2.2.1)\\ &= 9 + 3(-3) = \boxed{0} && (2.1.2)\end{aligned}$$

2.3.1 Direct Substitution

Notice that in the above example, if you just plug -3 into the function, you get the exact same answer. This property is known as *direct substitution*, and it applies to both polynomial and rational functions.

Theorem 2.3. *Let p and q be polynomial functions, and let c be a real number such that $q(c) \neq 0$. Then*

$$\lim_{x\to c} p(x) = p(c)$$

and

$$\lim_{x\to c}\frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

Finally, we can extend our ability to evaluate limits via direct substitution using the following two theorems:

Theorem 2.4. *Let n be a positive integer. Let c be all integers if n is odd, and let c otherwise be non-negative integers if n is even. Then*

$$\lim_{x\to c}\sqrt[n]{x} = \sqrt[n]{c}$$

Theorem 2.5. *Limit of Composition of Functions*

Let f and g be functions such that

$$\lim_{x\to c} g(x) = L, \lim_{x\to L} f(x) = f(L)$$

Then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L)$$

Essentially, this theorem is stating that limits of compositions of polynomial, rational, and/or radical functions can *all* be evaluated via direct substitution. I now give some examples.

Ex. 1: a. Evaluate

$$\lim_{x\to 5}\sqrt[3]{x^2 + 2}$$

Soln: Use the previous two theorems. Because

$$\lim_{x\to 5} x^2 + 2 = 27 \qquad \text{and} \qquad \lim_{x\to 27}\sqrt[3]{x} = 3$$

It follows that

$$\lim_{x\to 5} \sqrt[3]{x^2+2} = 3$$

b. Evaluate

$$\lim_{x\to 7} \frac{(2x-1)^2}{\sqrt{x^3-19}}$$

Soln: This is a composition of polynomial, rational and radical functions, none of which are 0 when $x = 7$. Therefore, we may simply use direct substitution to evaluate

$$\lim_{x\to 7} \frac{(2x-1)^2}{\sqrt{x^3-19}} = \frac{(2(7)-1)^2}{\sqrt{7^3-19}} = \frac{169}{18}$$

(See if you can evaluate this limit step by step by using all the theorems stated.)

Limits of the six basic trigonometric functions can also be evaluated via direct substitution. This can be considered a theorem as well.

Ex. 2: Evaluate

$$\lim_{x\to 0} x^2 \cos x$$

Soln:

$$\lim_{x\to 0} x^2 \cos x = \left(\lim_{x\to 0} x^2\right)\left(\lim_{x\to 0} \cos x\right) = 0^2 \cos 0 = 0$$

2.3.2 More Techniques for Evaluating Limits

In the very first example in this document, we discussed

$$\lim_{x\to -1} \frac{x^3+1}{x+1}$$

The function is undefined at $x = -1$, but we noticed that if we just factor the numerator, the fraction is equivalent to $x^2 + x + 1$, and we were able to determine the limit graphically. In other words, we found that because

$$\frac{x^3+1}{x+1} = x^2 + x + 1$$

Then

$$\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} x^2 + x + 1 = 1$$

Let us formalize this idea with a theorem.

Theorem 2.6. *Functions that Agree at All But One Point*

Let c be a real number, and let f and g be functions such that $f(x) = g(x)$ for all $x \neq c$ in some interval including c. If the limit of $g(x)$ as x approaches c exists, then so does the limit of $f(x)$, and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

This theorem allows us to find limits of some functions that are undefined at the value of x specified.

Ex. 1: Evaluate

$$\lim_{x\to 3} \frac{x^2-5x+6}{x^3-27}$$

Attempting direct substitution leads you to the fraction 0/0. This is called an *indeterminate form*, since the limit cannot be determined. However, we can factor the top and bottom to simplify this fraction:

$$\frac{x^2-5x+6}{x^3-27} = \frac{(x-2)(x-3)}{(x-3)(x^2+3x+9)} = \frac{x-2}{x^2+3x+9}$$

The last fraction is not undefined at $x = 3$, so we can use direct substitution to evaluate the limit. So in total, we have

$$\lim_{x\to 3} \frac{x^2-5x+6}{x^3-27} = \lim_{x\to 3} \frac{x-2}{x^2+3x+9} = \frac{1}{27}$$

Ex. 2: Evaluate

$$\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4}$$

Again, direct substitution yields the indeterminate form 0/0. Here, we can "rationalize the numerator" to rewrite the fraction.

$$\frac{\sqrt{x+5}-3}{x-4} = \frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} = \frac{1}{\sqrt{x+5}+3}$$
$$\longrightarrow \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{6}$$

Theorem 2.7. *Squeeze Theorem*

Given functions f,g,h such that $h(x) \le f(x) \le g(x)$ for all x in an interval including c, except possibly at c itself. Then if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then

$$\lim_{x\to c} f(x) = L$$

as well.

The squeeze theorem is a very powerful tool to use when finding limits. If you can find two functions that "squeeze" $f(x)$ (i.e. they act as upper and lower bounds), and also have the same limit value at $x = c$, this forces $f(x)$ to also take on that limit value.

The squeeze theorem can be used in the proofs of the following 2 notable limits.

Theorem 2.8. *Two Notable Trig Limits*

$$1. \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad 2. \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Here is a proof of 1.

In the diagram above, the circular sector with angle measure x (in radians) is sandwiched between 2 triangles. We can write

$$\frac{\sin x}{2} \le \frac{x}{2} \le \frac{\tan x}{2}$$

which can be rearranged to

$$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

Taking the reciprocal (and reversing the inequality signs) gives

$$1 \ge \frac{\sin x}{x} \ge \cos x$$

The diagram only shows $0 \le x \le \pi/2$, but it actually holds in the entire open interval $-\pi/2 \le x \le \pi/2$, because $\cos(-x) = \cos x$ and $\sin x/x = \sin(-x)/(-x)$. Furthermore,

$$\lim_{x\to 0} \cos x = 1, \lim_{x\to 0} 1 = 1$$

Therefore, we can apply the squeeze theorem to $f(x) = \sin x/x$ to say that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Ex. 3: a. Evaluate

$$\lim_{x\to 0} \frac{\sin x^2}{x}$$

Soln:

$$\lim_{x\to 0} \frac{\sin x^2}{x} = \left(\lim_{x\to 0} \frac{\sin x^2}{x^2}\right)\left(\lim_{x\to 0} x\right) = 1 \cdot 0 = 0$$

(If this doesn't make sense, substitute $y = x^2$ and see how the expression still satisfies Theorem 2.8.)

b. Evaluate

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$$

Soln: We would like to use Theorem 2.8, but we are missing the corresponding x terms. So we try writing

$$\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right)\left(\lim_{x\to 0}\frac{2x}{\sin 2x}\right)$$

but we notice that

$$\frac{\sin 3x}{3x}\frac{2x}{\sin 2x}=\frac{2}{3}\frac{\sin 3x}{\sin 2x}$$

Constants can be factored out of limits (Theorem 2.2), so we can write

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}=\frac{3}{2}\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right)\left(\lim_{x\to 0}\frac{2x}{\sin 2x}\right)=\frac{3}{2}\cdot 1\cdot 1=\frac{3}{2}$$

(Why is $\lim_{x\to 0} x/\sin x = 1$?)

2.4 Existence: Continuity, One-Sided Limits

2.4.1 What is a One-Sided Limit?

We have already seen examples of functions that fail to have limits at certain points. Specifically, we have discussed functions whose behavior differs from the left and right. Let's revisit one such function, $f(x) = |x|/x$.

For negative values of $x, f(x) = -1$. As x approaches 0 from the negative side, $f(x)$ continues to be -1. We say that the limit of $f(x)$ as x approaches 0 *from the left* is -1. This limit is written as

$$\lim_{x\to 0^-} f(x) = -1$$

Similarly, for positive values of $x, f(x) = 1$. So we can say that the limit of $f(x)$ as x approaches 0 *from the right* is 1, or

$$\lim_{x\to 0^+} f(x) = 1$$

Now I give some examples of when one-sided limits appear.

Ex. 1: One-sided limits appear when taking the nth root, for n even. For example,

$$\lim_{x\to -3^+} \sqrt{9-x^2} = 0$$

Notice that

$$\lim_{x\to -3^-} \sqrt{9-x^2}$$

is undefined.

Ex. 2: One-sided limits also appear when working with the floor or ceiling functions. For example,

$$\lim_{x\to 1^-} \lceil x\rceil = 1$$

while

$$\lim_{x\to 1^+} \lceil x\rceil = 2$$

Similar behavior exists for all step functions, due to the inherent discontinuities.

2.4.2 Formalization, Conditions for the Existence of a Limit

Now let's formalize some of the ideas and concepts I have introduced in this section.

Definition. The one-sided limit

$$\lim_{x\to c^+} f(x) = L$$

means that

$$\forall\epsilon > 0,\, \exists\delta > 0 \qquad s.t. \qquad x-c<\delta \longrightarrow |f(x)-L|<\epsilon$$

Similarly, the one-sided limit

$$\lim_{x\to c^-} f(x) = L$$

means that

$$\forall\epsilon > 0,\, \exists\delta > 0 \qquad s.t. \qquad c-x<\delta \longrightarrow |f(x)-L|<\epsilon$$

Recall that in the epsilon-delta formulation of a normal limit, the requirement is that $|x - c| < \delta$. So each one-sided limit takes one side of this absolute value inequality: for limits from the right side, we have $x > c$, so $|x - c| = x - c$, and for limits from the left side, $x < c \longrightarrow |x - c| = c - x$.

Theorem 2.9. *Existence of a Limit*

Let c and L be real numbers, and let f be a function defined in some open neighborhood of c (except possibly at c itself). Then the limit of $f(x)$ as x approaches c is L if and only if the limits from the left and right both exist, and are both equal to L.

In other words,

$$\lim_{x\to c} f(x) = L \longleftrightarrow \lim_{x\to c^-} f(x) = L \qquad \textit{and} \qquad \lim_{x\to c^+} f(x) = L$$

We can use the definitions of one-sided limits to prove the $\longleftarrow$ side, as the other direction follows almost immediately by definition.

Proof sketch: The one-sided limit from the left implies that for all $\epsilon > 0$, there is some $\delta_1 > 0$ such that

$$x - c < \delta_1 \longrightarrow |f(x) - L| < \epsilon$$

and the one-sided limit from the right implies that there is some $\delta_2 > 0$ such that

$$c - x < \delta_2 \longrightarrow |f(x) - L| < \epsilon$$

I want to show that there exists a single $\delta > 0$ such that

$$|x - c| < \delta \longrightarrow |f(x) - L| < \epsilon$$

(Think about this for a second. What δ should I pick?)

I claim that I should consider $\delta = \min(\delta_1, \delta_2)$. (Convince yourself that this guarantees the existence of the two-sided limit.)

2.4.3 Intro to Continuity, Relationship with Limits

We begin with an informal definition of continuity - if the graph of a function is "smooth" on some interval - i.e. there are no breaks or jumps, we say it is continuous. Specifically, to say that some function f is *continuous* at some point $x = c$ implies that there is no interruption in the graph of f at c.

We have already seen that when there is a jump in a function at some point $x = c$, then the one-sided limits at c do not agree. As it turns out, even if the one-sided limits agree at $x = c$, the function still may not be continuous at that point.

As the diagram suggests, there are 3 ways in which a function can NOT be continuous at a point. This leads us to the following definition of continuity at a point:

Definition. A function f is said to be *continuous at c* if the following are all true:

1. $f(c)$ is defined, i.e. c lies in the domain of f.
2. $\lim_{x\to c} f(x)$ is defined.
3. $\lim_{x\to c} f(x) = f(c)$.

If f is defined on an open interval containing c (except possibly at c), and f is not continuous at c, then we say f has a *discontinuity* at c. (It is not entirely accurate to say that f is "discontinuous", since this implies that it is an innate property of the function.) In fact, discontinuities fall into 2 categories: *removable* or *nonremovable*.

add examples, diagram

The key point to take away from this discussion is the relationship between continuity and the (two-sided) limit:

$$f(x) \text{ is continuous at a point } c \longleftrightarrow \text{ 2-sided limit } \lim_{x\to c} f(x) \text{ exists and is equal to } f(c)$$

2.4.4 Properties of Continuity

In this section we will examine properties of continuity and be able to analyze the continuity of specific functions. First, we extend the definition of continuity from points to intervals on the number line.

Continuity on an Open Interval: A function is continuous on an *open interval* (a, b) if it is continuous at each point in the interval. If a function is continuous on the entire real number line $(-\infty, \infty)$, then it is said to be *everywhere continuous*.

Continuity on a Closed Interval: This case is a little different. A function is continuous on a *closed interval* $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a), \lim_{x \to b^-} f(x) = f(b)$$

You can mix and match these definitions to incorporate half-open intervals; for example, the function $f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$.

In section 2.2 we discussed some theorems involving evaluating limits. We've already seen that the existence of a limit implies continuity (and vice versa), so it should come as no surprise that there are similar theorems involving the continuity of functions.

There are certain types of functions that are continuous everywhere in their domains (they are not *everywhere continuous*, since some of these functions are not defined on the entire number line). They are exactly the same kinds of functions that one can evaluate limits for via direct substitution (see Theorems 2.3, 2.4, 2.5):

1. Polynomials: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
2. Rational functions: $f(x) = p(x)/q(x), \qquad q(x) \neq 0$
3. Radicals: $f(x) = \sqrt[n]{x}$
4. Trig functions $f(x) = \sin x$, etc.

Theorem 2.10. *If $f(x)$ and $g(x)$ are functions that are continuous at $x = c$, and b is a real number, then the following functions are also continuous at c:*

1. *bf, scalar multiple*
2. *$f \pm g$, sum or difference*
3. *fg, product*
4. *f/g, quotient $(g(c) \neq 0)$*

Theorem 2.11. *Continuity of a Composition of Functions*

If $g(x)$ is continuous at c and $f(x)$ is continuous at $g(c)$, then the composition of the two functions $(f \circ g)(x) = f(g(x))$ is continuous at c.

This theorem directly implies Theorem 2.5.

With these theorems, we can analyze the continuity of all sorts of functions and composites of functions.

add examples

2.4.5 One Consequence of Continuity: The Intermediate Value Theorem

Theorem 2.12. *Intermediate Value Theorem*

If f is continuous on the closed interval $[a, b]$, then for all L such that $f(a) \leq L \leq f(b)$, there exists at least one $c \in [a, b]$ such that $f(c) = L$.

I will give a simple example of IVT: pumping a bike tire. Suppose you have a tire that is a little flat and has a pressure of 60 psi, and you pump the tire until the pressure is 110 psi. Then for any $60 \leq p \leq 110$, there was a moment in time during the pumping process where the pressure was exactly equal to p psi.

IVT is an *existence theorem*, meaning that the theorem guarantees the existence of such a c, but does not tell you how to actually find the value of c. Note that f being continuous on the interval is critical to the theorem; a function that is not continuous does not necessarily exhibit this property. Now I give two examples of when IVT is useful.

Ex. 1: Existence of a Root in a Specified Range

Show that the function $f(x) = x^3 - x^2 - 5$ has a root in the interval $[2, 3]$.

Proof: This is a polynomial function, so it is everywhere continuous, but specifically it is continuous on the closed interval $[2, 3]$. Furthermore, $f(2) = -1 < 0, f(3) = 13 > 0$. Therefore, we can apply the Intermediate Value Theorem to conclude that there exists some $c \in [2, 3]$ such that $f(c) = 0$, i.e. there exists a root of the function between 2 and 3.

In general, given a function $f(x)$ such that $f(a)$ and $f(b)$ have differing signs, and f is continuous on $[a, b]$, there is guaranteed at least one root c in the interval $[a, b]$.

Ex. 2: Applying IVT to an Arbitrary Function

A hiker starts at 7 am at the base of a mountain and takes 3 hours to reach the top. He camps there overnight, and the next day at 7 am he starts descending along the same path, reaching the bottom in only 2 hours. Show that there is a time t such that the hiker was at the same elevation at t both during the ascent and the descent.

Proof: Let the height of the mountain be h. Consider an elevation function for the upwards hike $u(t), 0 \leq t \leq 3$, so $u(0) = 0$ and $u(3) = h$. Also define a similar function $d(t)$ for the downwards hike, so $d(0) = h, d(2) = d(3) = 0$.

Now consider the difference of the two functions $f(t) = u(t) - d(t)$. As both u and d are continuous functions, their difference must be as well. Furthermore, we have

$$f(0) = u(0) - d(0) = -h, f(3) = u(3) - d(3) = h$$

Now, because f is continuous and $f(0)$ and $f(3)$ have opposite signs, by the Intermediate Value Theorem there is at least one $0 \leq t \leq 3$ such that $f(t) = 0$. This means that for this $t, u(t) - d(t) = 0 \longrightarrow u(t) = d(t)$. $\blacksquare$

2.5 Infinite Limits, Limits at Infinity

In this section we will discuss how limits behave when either x or $f(x)$ goes to infinity, and how to evaluate such limits.
A side note on infinity: I want to be very clear about this one fact: infinity is *not a number*. Rather, infinity is a concept that expresses the *unboundedness* of the real numbers.

2.5.1 Infinite Limits, Vertical Asymptotes

An infinite limit is any limit in which $f(x)$ increases or decreases without bound as x goes to c. We can formalize this in a similar fashion to a regular epsilon-delta statement of a limit:

Definition. Let f be a function defined everywhere in an open interval containing c (except possibly at c). The statement

$$\lim_{x \to c} f(x) = \infty$$

means that for each $M > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \longrightarrow f(x) > M$$

The definition for limits that become unboundedly negative is similar, except we have $M < 0$ and

$$0 < |x - c| < \delta \longrightarrow f(x) < M$$

One-sided limits also exist for infinite limits, in a similar fashion to regular limits.

In section 2.1.2 I said that limits that exhibit unbounded behavior fail to exist. This is still true: the statement $\lim_{x \to c} f(x) = \infty$ does not mean that the limit exists; remember, infinity is not a number. Rather, this statement tells you that the limit fails to exist because it is unbounded as x goes to c.

Definition. If $f(x)$ approaches $\pm\infty$ as x approaches c from either direction, then the line $x = c$ is called a *vertical asymptote* of f.

You have probably already seen and worked with vertical asymptotes of functions. Specifically, vertical asymptotes appear wherever plugging $x = c$ into a function causes a denominator to equal 0 (and the numerator is not 0). The graph of the function appears to "get arbitrarily close" to this vertical line as x gets arbitrarily close to c. The next theorem formalizes this fact:

Theorem 2.13. *Vertical Asymptotes*

Let f and g be continuous on some open interval containing c. If $g(c) = 0$ and $f(c) \neq 0$, and there is some open interval containing c where $g(x) \neq 0$ if $x \neq c$, then the graph of

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

The stipulation that $g(x)$ is not equal to 0 in a neighborhood of c removes the possibility that $g(x)$ is exactly 0, as then $h(x)$ would not be defined to begin with.

Theorem 2.14. *Some Properties of Infinite Limits*

Let f and g be functions that satisfy

$$\lim_{x \to c} f(x) = \infty, \lim_{x \to c} g(x) = L$$

where c and L are real numbers. Then the following limits hold:

1. *Sum and difference:* $\lim_{x \to c}[f(x) \pm g(x)] = \infty$
2. *Product:* $\lim_{x \to c}[f(x)g(x)] = \infty$ *if* $L > 0$, $-\infty$ *if* $L < 0$
3. *Quotient:* $\lim_{x \to c} g(x)/f(x) = 0, \lim_{x \to c} f(x)/g(x) = \infty$ *if* $L > 0$, $-\infty$ *if* $L < 0$.

There are similar theorems for one-sided limits, or when the limit of f(x) is equal to $-\infty$.

Evaluating infinite limits is a little different from evaluating the finite limits we have seen previously in this chapter. As the following example shows, the behavior of infinite limits can differ depending on the nature of the function in question, as well as what side we consider the limit from.

Ex. 1: Some Infinite Limits

(a) Consider the limit

$$\lim_{x \to 5} \frac{1}{x - 5}$$

By Theorem 2.13 we know that this function has a vertical asymptote at $x = 5$. However, the existence of a vertical asymptote says nothing about the behavior of the function on both sides of the asymptote - in other words, we don't know what the one-sided limits are, or even if they are equal.

When evaluating infinite limits of rational functions, be sure to check if the one-sided limits are equal. In this case a table or a sketch of the function is very useful:

The one-sided limits for this function do not agree, so the limit **does not exist**. The one-sided limits are

$$\lim_{x \to 5^-} \frac{1}{x - 5} = -\infty, \qquad \lim_{x \to 5^+} \frac{1}{x - 5} = \infty$$

(b) Now consider

$$\lim_{x \to -2} \frac{2x + 5}{(x + 2)^2}$$

Let's start by looking at the graph:

Notice how unlike the function in part (a), this function actually has one-sided limits that agree with each other, so we can write

$$\lim_{x \to -2} \frac{2x + 5}{(x + 2)^2} = \infty$$

(c) Here is an infinite limit not involving a rational function. Consider

$$\lim_{x \to \pi/2} \sin x \tan^2 x$$

This time, let us try to evaluate this without consulting a graph.

We can use **Theorem 2.14.2:** Let $f(x) = \tan^2 x, g(x) = \sin x$. Then

$$\lim_{x \to \pi/2} f(x) = \infty, \lim_{x \to \pi/2} g(x) = 1$$

and then it follows that

$$\lim_{x \to \pi/2} \sin x \tan^2 x = \infty$$

How do we know that the limit of $\tan^2 x$ at $x = \pi/2$ is ∞? Consider the behavior of $\tan x$ near $\pi/2$. As x approaches $\pi/2$ from the left, $\tan x$ becomes a larger and larger positive number, so $\tan^2 x$ does as well. From the right, $\tan x$ becomes a larger and larger *negative* number (visualize the unit circle), but the behavior of $\tan^2 x$ is the same: it increases without bound. Therefore the one-sided limits agree and the two-sided limit exists.

Analyzing an infinite limit by looking at the behavior of the function from both sides can determine whether or not a two-sided infinite limit exists. For rational functions specifically, can you find a pattern for when a two-sided infinite limit exists and when it does not?

2.5.2 Finite Limits at Infinity, Horizontal Asymptotes

In this section we consider: instead of if the function $f(x)$ increases or decreases without bound, is it possible to characterize and evaluate the limit of a function as x increases or decreases with bound?

Definition. Let L be a real number. The statement

$$\lim_{x\to\infty} f(x) = L$$

means that for each $\epsilon > 0$ there is an $M > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x > M$$

A similar definition exists for $x \to -\infty$.

To say this in simpler terms: no matter how small ϵ is, there exists some finite M such that if $x > M$, $f(x)$ is within the interval $(L - \epsilon, L + \epsilon)$.

The definition of a horizontal asymptote immediately follows from above:

Definition. If

$$\lim_{x\to-\infty} f(x) = L \text{ or } \lim_{x\to\infty} f(x) = L$$

then the line $y = L$ is called a *horizontal asymptote* of f.

It immediately follows that any function can have at most 2 horizontal asymptotes.

2.5.3 Infinite Limits at Infinity

Many functions, especially polynomial functions, do not approach a finite limit as x increases or decreases without bound. We can formalize this idea as follows:

Definition. Let f be a function defined on some open interval including ∞. (Think of any interval (c, ∞)). The statement

$$\lim_{x\to\infty} f(x) = \infty$$

means that for each $M > 0$ there exists an $N > 0$ such that

$$f(x) > M \text{ whenever } x > N$$

Similarly, one can write the equivalent formalization of

$$\lim_{x\to\infty} f(x) = -\infty$$

by making M a negative number and reversing the sign of the corresponding inequality, $f(x) < M$.

There are similar definitions for statements that involves limits that equal $-\infty$.

2.5.4 Evaluating Limits at Infinity

The following theorem will be extremely useful when evaluating limits at infinity.

Theorem 2.15. *Let c be any real number and r be a positive rational number. Then*

$$\lim_{x\to\infty} \frac{c}{x^r} = 0$$

If x^r is defined for negative values of x, then it is also true that

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0$$

Intuitively, this makes sense, as x^r for positive values of r increases monotonically as x gets more and more positive. So as x goes to infinity, so should x^r.

Limits at infinity also have the exact same properties as limits at any real number, as described in Theorem 2.2. With those and the above theorem, we can evaluate limits at infinity for many functions.

Ex. 1: Evaluating Infinite Limits at Infinity

(a) Consider the limit

$$\lim_{x \to -\infty} x^4$$

Intuitively, as x becomes a smaller and smaller number, x^4 will become a larger and larger number. So we can write

$$\lim_{x \to -\infty} x^4 = \infty$$

In similar fashion, we can write

$$\lim_{x \to \infty} -x^3 = -\infty$$

Note the difference in behavior between even and odd powers of x. (This is closely related to the last sentence of Section 2.4.1, as well.)

(b) Consider the limit

$$\lim_{x \to \infty} \left(x^3 - 5x^2 + 4x - 7\right)$$

I could break this up into the limits of each of the individual terms, but if I tried substituting directly I would end up with something like

$$\lim_{x \to \infty} \left(x^3 - 5x^2 + 4x - 7\right) = \infty - \infty + \infty - 7$$

This expression is misleading since I have previously said that ∞ is a not a number. So, we actually don't know what the value of this expression is. We will need to find a different method of evaluating this limit.

Let's think about this intuitively. As x increases, the polynomial also increases. This is because for large values of x, the x^3 term dominates the other terms. So it seems like the answer should be ∞. How can we show this using the theorems we know?

The key is to factor the highest power of x out of every term:

$$\lim_{x \to \infty} \left(x^3 - 5x^2 + 4x - 7\right) = \lim_{x \to \infty} \left[x^3 \left(1 - \frac{5}{x} + \frac{4}{x^2} - \frac{7}{x^3}\right)\right]$$

Now, we can treat the limit as the product of two separate limits. First we have

$$\lim_{x \to \infty} x^3 = \infty$$

Also, by direct substitution and Theorem 2.15 above, we have

$$\lim_{x \to \infty} \left(1 - \frac{5}{x} + \frac{4}{x^2} - \frac{7}{x^3}\right) = 1 - 0 + 0 - 0 = 1$$

Finally, by Theorem 2.14 we can multiply these two limits together to get

$$\lim_{x \to \infty} \left(x^3 - 5x^2 + 4x - 7\right) = \infty \cdot 1 = \infty$$

The major takeaway from this example is that **the behavior of polynomial functions at infinity only depends on the behavior of its leading term**, the highest power of x. In other words,

$$\lim_{x \to \infty} \left(x^3 - 5x^2 + 4x - 7\right) = \lim_{x \to \infty} x^3$$

Ex. 2: Evaluating Finite Limits at Infinity

(a) Evaluate

$$\lim_{x \to -\infty} \left(2 + \frac{195}{x^3}\right)$$

Soln. Using the theorems we know, we can write

$$\begin{aligned} \lim_{x \to -\infty} \left(2 + \frac{195}{x^3}\right) &= \lim_{x \to -\infty} 2 + \lim_{x \to -\infty} \frac{195}{x^3} \\ &= 2 + 0 \\ &= 2 \end{aligned}$$

(b) Evaluate

$$\lim_{x\to-\infty} \frac{5x-7}{2x^2+1}$$

Soln. First, notice that if we try to use direct substitution, we arrive at the form $\frac{-\infty}{\infty}$. This is an example of an *indeterminate form*; more on that shortly.

If direct substitution doesn't work, we can try the same trick of factoring the highest power of x out of the polynomial. In this case, we can factor an x^2 out of the denominator:

$$\lim_{x\to-\infty} \frac{5x-7}{x^2(2+1/x^2)}$$

This is still not very useful, however. In the denominator, the expression in parentheses is not infinite, but the x^2 term is. How can we get rid of the x^2?

Well, we can factor x^2 out of the numerator, as well. Then the x^2 will cancel, leaving us with

$$\lim_{x\to-\infty} \frac{x^2(5/x-7/x^2)}{x^2(2+1/x^2)} = \lim_{x\to\infty} \frac{5/x-7/x^2}{2+1/x^2}$$

Now we can evaluate using direct substitution!

$$\lim_{x\to-\infty} \frac{5/x-7/x^2}{2+1/x^2} = \frac{0-0}{2+0} = 0$$

(c) Evaluate

$$\lim_{x\to\infty} \frac{3x^3+2}{2x^3-7x+9}$$

Soln. Again direct substitution yields the indeterminate form $\frac{\infty}{\infty}$. By factoring out x^3 from both the numerator and denominator, we obtain

$$\lim_{x\to\infty} \frac{3x^3+2}{2x^3-7x+9} = \lim_{x\to\infty} \frac{3+2/x^3}{2-7/x^2+9/x^3} = \frac{3+0}{2-0+0} = \frac{3}{2}$$

(d) Evaluate

$$\lim_{x\to-\infty} \frac{x^4}{50x^2-800}$$

Soln. Another indeterminate form. Factor out an x^2:

$$\lim_{x\to-\infty} \frac{x^4}{50x^2-800} = \lim_{x\to-\infty} \frac{x^2}{50-800/x^2} = \frac{\infty}{50} = \infty$$

After this example, limits to infinity of rational functions should be easy to evaluate:

1. If the degree of the numerator is **greater than** the degree of the denominator, the limit at infinity *does not exist.* (The actual limit may be negative or positive infinity.)
2. If the degree of the numerator is **equal to** the degree of the denominator, the limit at infinity is equal to *the ratio of the leading coefficients.*
3. If the degree of the numerator is **less than** the degree of the denominator, the limit at infinity is exactly 0.

These rules formalize what our intuition tells us: for polynomials and rational functions, the highest powers of x "dictate" the behavior of the limit at infinity.

So far we have only considered polynomial functions. One thing to notice from the previous example is that taking the limit as $x\to\infty$ versus $x\to-\infty$ doesn't change any of the values of the limits themselves. In other words, these rational functions have the same horizontal asymptote on the left and on the right (if the asymptotes exist).

Ex. 3: Non-Polynomial Infinite Limits

(a) Evaluate

$$\lim_{x\to\infty} \frac{5x+6}{\sqrt{x^2+3x-1}} \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{5x+6}{\sqrt{x^2+3x-1}}$$

Notice that the degrees of the top and bottom are technically the same (this is a little unrigorous, essentially both the top and bottom grow at a roughly linear pace).

So we would like to divide both the numerator and denominator by x. However, we have to be careful in the denominator because of the square root.

Soln. For the first limit, as $x \to \infty$, since x is positive we can write $x = \sqrt{x^2}$. Therefore we can proceed as

$$\lim_{x\to\infty} \frac{5x+6}{\sqrt{x^2+3x-1}} = \lim_{x\to\infty} \frac{x(5+6/x)}{\sqrt{x^2}\left(\sqrt{1+3/x-1/x^2}\right)}$$

Then we can evaluate as usual:

$$\lim_{x\to\infty} \frac{x(5+6/x)}{\sqrt{x^2}\left(\sqrt{1+3/x-1/x^2}\right)} = \lim_{x\to\infty} \frac{5+6/x}{\sqrt{1+3/x-1/x^2}} = \frac{5}{\sqrt{1}} = 5$$

For the other limit, as $x \to -\infty$, since x is negative we can write $x = -\sqrt{x^2}$. The rest of the solution is exactly the same, with the addition of the negative sign, so the answer is simply -5.

This is an example of a function with two different horizontal asymptotes.

(b) Evaluate

$$\lim_{x\to-\infty} e^{x^3-5x+1}$$

Here is an example of a limit involving an exponential function. To evaluate this limit, we will need to know the following two limits:

$$\lim_{x\to\infty} e^x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} e^x = 0$$

To evaluate this limit we just need to examine the behavior of the argument of the exponential. We can see that

$$\lim_{x\to-\infty} x^3 - 5x + 1 = -\infty$$

which means that the argument of the exponential goes to $-\infty$, and therefore the exponential itself goes to 0. This is an application of Theorem 2.5, regarding the composition of functions:

$$\lim_{x\to-\infty} x^3 - 5x + 1 = -\infty \qquad \text{and} \qquad \lim_{x\to-\infty} e^x = 0$$

implies that

$$\lim_{x\to-\infty} e^{x^3-5x+1} = 0$$

(c) Evaluate

$$\lim_{x\to\infty} \ln\left(\frac{5x^3}{x^2-6x+1}\right)$$

Again this is a composition of functions. ln, the natural logarithm function, is the inverse of the exponential function, with the limits

$$\lim_{x\to0^+} \ln x = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \ln x = \infty$$

(We will discuss these two functions in greater detail later in this text.)

For this specific limit, because

$$\lim_{x\to\infty} \frac{5x^3}{x^2-6x+1} = \infty \qquad \text{and} \qquad \lim_{x\to\infty} \ln x = \infty$$

we have

$$\lim_{x\to\infty} \ln\left(\frac{5x^3}{x^2-6x+1}\right) = \infty$$

(d) Here's an example of an infinite trigonometric limit. Evaluate

$$\lim_{x\to-\infty} \tan^{-1}\left(e^{-x} - 2e^{-2x}\right)$$

First, we evaluate the exponential limit. Direct substitution yields $\infty - \infty$, an indeterminate form. We can get around this issue using a similar method as the one used on polynomials:

$$\lim_{x\to-\infty} \left(e^{-x} - 2e^{-2x}\right) = \lim_{x\to-\infty} \left(e^{-2x}\left(e^x - 2\right)\right) = \infty \cdot (-2) = -\infty$$

Now we need to know the end behavior of $\tan^{-1}(x)$. Recall that inverse tangent has a range of $(-\pi/2, \pi/2)$, with the limits being

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\pi/2 \qquad \text{and} \qquad \lim_{x\to\infty} \tan^{-1}(x) = \pi/2$$

Therefore, because

$$\lim_{x\to-\infty} \left(e^{-x} - 2e^{-2x}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to-\infty} \tan^{-1}(x) = -\pi/2$$

we have

$$\lim_{x\to-\infty} \tan^{-1}\left(e^{-x} - 2e^{-2x}\right) = -\pi/2$$

Notice that there aren't similar infinite limits for inverse sine and cosine, because the domain of those functions are $[-1, 1]$.

2.5.5 A Quick Note on Indeterminate Forms

We have already seen a number of limits which cannot be evaluated using direct substitution. Oftentimes, direct substitution results in an indeterminate form. There are 7 such forms commonly seen:

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, \infty \cdot 0, 1^\infty, 0^0, \infty^0$$

We have seen the first 3 indeterminate forms, and our tactic for evaluating those limits was to algebraically manipulate the expression into something else that could be evaluated using direct substitution. However, that is not always possible. How we deal with more difficult functions will be discussed in a later chapter.

3 From Limits to Differentiation

3.1 Tangent Lines: A Very Specific Limit

3.1.1 Introduction of the Problem

Now let's step away from limits for a brief moment and discuss a certain problem: finding the equation of a tangent line to a curve.

To start, let's first discuss what a tangent line is. From geometry, one could give the definition that a tangent line to a curve at a point P is a line that "just touches" the curve at P but doesn't "cross" the curve. Or, I could be a bit more general and say that a tangent line to a curve at a point P only intersects the curve at P and nowhere else.

It seems like none of these definitions covers all the possible scenarios, or is wholly satisfactory.

Let's take a step back from the semantics argument and just consider *how* to find the equation of a tangent line to a curve $f(x)$ at a given point $P = (a, f(a))$. We know from analytic geometry that one way to write the equation of a line is in *point-slope form*, $(y - y_1) = m(x - x_1)$ where (x_1, y_1) is a point on the line and m is the slope. We are already given the point, so all we have to do is find the slope.

How are we going to find the slope? Well, we are going to write it as the *limit* of some function, as the title of this section suggests. Usually, to find the slope of a line we need 2 points on the line. The tangent line only intersects the graph at one point, so where to find a second point isn't readily apparent. However, what if our second point was a point on the function *really* close to the original point $(a, f(a))$? Then the line through these two points approximates the tangent line very well.

More formally, we want to find the slope at a point $(a, f(a))$. Consider the line through this point and a second point $(a + \Delta x, f(a + \Delta x))$. This line is called a *secant line* through the two points. The slope of this secant line is simply

$$m = \frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{a + \Delta x - a} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

This expression for the slope is called a *difference quotient.* It is simply the change in y divided by the change in x. From the argument above, as I pick the second point to be closer and closer to the first point, the secant line will approach the tangent line. In other words, I am taking the *limit* as Δx approaches 0.

Definition. Let $f(x)$ be a function defined on some open interval including c. If the limit

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x} = m$$

exists, then the line with slope m passing through $(a, f(a))$ is the *tangent line* to the graph of f at $x = a$.

Let's take a step back and consider what this expression represents. First, the slope of the secant line between two points on a curve is the *average rate of change* of the curve over that interval. That is to say, $m = \dfrac{\Delta y}{\Delta x}$.

As we move the two points closer together on the curve, i.e. we let $\Delta x \to 0, m$ denotes the average rate of change over a smaller and smaller interval, until that interval becomes infinitesimally small. So in the limit as $\Delta x \to 0, m$ denotes the *instantaneous rate of change* of the function at $x = a$.

A classic conceptual example of this would be to take the units of x as time, and the units of $f(x)$ as distance. In this case, the slope of the secant line between points denotes the *average velocity* in that x interval. (Remember, $d = rt$, where d is distance, r is the rate, and t is the time.) As $\Delta x \to 0$, the secant line converges to the tangent line, and the slope of the tangent line is the *instantaneous velocity* at that specific point.

3.1.2 The Solution: Differentiation and the Derivative

Now we have seen a number of different ways to interpret the above limit. In fact, that limit is the definition of one of the two major operations of calculus: *differentiation.*

Definition. The *derivative* of a function f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

if it exists. For all x for which $f'(x)$ exists, f' is also a function of x.

Differentiation is the operation by which one obtains the derivative of a function. The derivative of a function of x is *also* a function of x.

There are many different notations for expressing the derivative of a function. If we are given a function $y = f(x)$, the following are all equivalent notations:

$$f'(x) \qquad y' \qquad \frac{dy}{dx} \qquad \frac{d}{dx}[f(x)] \qquad (Df)(x) \qquad D_x[y] \qquad \dot{y}$$

In this textbook, we will almost exclusively use the first four notations. $f'(x)$ is read as "f prime of x", and $\frac{dy}{dx}$ is read "dy over dx" or just "dy - dx". Note that we have not yet discussed what dy and dx represent, if anything - this will come in a later section.

Ex. 1: Finding Derivatives Using the Definition

(a) Find the derivative of the function $f(x) = 5x - 2$.
Soln. This is simply an exercise in algebra, using the definition:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h \to 0} \frac{(5(x+h)-2)-(5x-2)}{h} \\ &= \lim_{h \to 0} \frac{5h}{h} = \lim_{h \to 0} 5 = 5 \end{aligned}$$

As always, we should ask ourselves if this result matches up with intuition. Recall what the derivative is measuring - the slope of the tangent line to the graph of the function $f(x)$. What is the slope of $f(x) = 5x - 2$? Well, this is a linear function, so it has a constant slope for all values of x, 5. (Recall that this is simply the slope-intercept form of a line, $y = mx + b$.)

(b) Find the derivative of the function $f(x) = x^3 - x^2$, then find the equation of the tangent line to $f(x)$ at $x = 5$.

Soln.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left[(x+h)^3-(x+h)^2\right]-\left[x^3-x^2\right]}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-2xh-h^2}{h} \\
&= \lim_{h\to 0} \left(3x^2+3xh+h^2-2x-h\right) \\
&= 3x^2-2x
\end{aligned}$$

There are multiple ways to determine the equation of the line; in most of these types of problems point-slope form will be the easiest form to use. When $x=5, f(x)=5^3-5^2=100$, so the point on the function through which the tangent line passes is $(5,100)$. The slope of $f(x)$ at $x=5$ is just

$$f'(5) = 3\cdot 5^2 - 2\cdot 5 = 65$$

Therefore, the equation of the line is

$$y-100 = 65(x-5) \longleftrightarrow y = 65x-225$$

I will give 2 examples of non-polynomial derivatives, but the calculations, although a little involved, are still quite straightforward. Feel free to skip them if you feel confident working with the definition of a derivative. In the next section, we will discuss basic rules of differentiation that will make derivatives much easier and faster to calculate.

(c) Find the derivative with respect to t of the function $y(t) = \dfrac{2t}{t+3}$.

Soln.

$$\begin{aligned}
\frac{dy}{dt} &= \lim_{\Delta t\to 0} \frac{y(t+\Delta t)-y(t)}{\Delta t} \\
&= \lim_{\Delta t\to 0} \frac{\dfrac{2(t+\Delta t)}{(t+\Delta t)+3}-\dfrac{2t}{t+3}}{\Delta t} \\
&= \lim_{\Delta t\to 0} \frac{\dfrac{2(t+\Delta t)(t+3)-2t(t+\Delta t+3)}{(t+\Delta t+3)(t+3)}}{\Delta t} \\
&= \lim_{\Delta t\to 0} \frac{2t^2+6t+2t\Delta t+6\Delta t-2t^2-2t\Delta t-6t}{\Delta t(t+\Delta t+3)(t+3)} \\
&= \lim_{\Delta t\to 0} \frac{6\Delta t}{\Delta t(t+\Delta t+3)(t+3)} \\
&= \lim_{\Delta t\to 0} \frac{6}{(t+\Delta t+3)(t+3)} = \frac{6}{(t+3)^2}
\end{aligned}$$

(d) Find the the derivative of the function $f(x)=\sqrt{2x}$.

Soln. At first glance, plugging into the definition doesn't seem to help much, since there isn't anything to be simplified. In this case we will "rationalize the numerator":

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)}-\sqrt{2x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\cdot\frac{\sqrt{2(x+h)}+\sqrt{2x}}{\sqrt{2(x+h)}+\sqrt{2x}}\right) \\
&= \lim_{h\to 0} \frac{2(x+h)-2x}{h\left(\sqrt{2(x+h)}+\sqrt{2x}\right)} \\
&= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2(x+h)}+\sqrt{2x}\right)} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)}+\sqrt{2x}} = \frac{2}{\sqrt{2x}+\sqrt{2x}} = \frac{1}{\sqrt{2x}}
\end{aligned}$$

3.1.3 When a Derivative Fails to Exist...

At this point let's take a step back and consider the question of when derivatives exist. A derivative is really just a specific limit, and we have already seen examples of when limits fail to exist.

In section 2.4 we discussed the concepts of continuity and one-sided limits, and how they tie in to the existence of limits. We can define similar concepts for the derivative, starting with an alternate definition:

Definition. Another way to write the derivative of f at $x = c$ is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

if this limit exists.

Note that if I let $c = x + \Delta x$, we recover the original definition of the derivative ($x \to c$ is the same as $x \to x + \Delta x$, or $\Delta x \to 0$).

Definition. A function is *differentiable at a point* $x = c$ if the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \qquad \text{and} \qquad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal to $f'(c)$. These one-sided limits are called the *derivatives from the left and right*, respectively.

This is the exact same condition described in Theorem 2.9; the derivative is just a special limit and therefore differentiability is a special case of the existence of a limit. Here are some examples of when a derivative of a function *at a specific point* does not exist.

Ex. 1: A Graph with a Sharp Turn

Consider the piecewise function

$$f(x) = \begin{cases} -2x + 3, x \leq 1 \\ x^2, x > 1 \end{cases}$$

What is the value of the derivative at $x = 1$?

First, notice that $f(x)$ is continuous at $x = 1$. This is because the two pieces of the function agree, or using the definition of continuity that we know (in section 2.4.3),

$$f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

But does the derivative exist? The derivative from the right is

$$f'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{-2x + 3 - 1}{x - 1} = -2$$

(As a sanity check, note that for $x \leq 1$ the function is linear and therefore the derivative is just the slope, or -2.) Meanwhile, the derivative from the left is

$$f'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} x + 1 = 0$$

So, the derivative of f at $x = 1$ does not exist, even though the function is continuous at $x = 1$. The graph of the function is "smooth" - i.e, you can draw the graph without lifting your pen from off the paper. However, at $x = 1$, the "direction", or slope, of the graph changes suddenly, creating a sharp turn.

Ex. 2: A Limit that DNE

Consider the function $f(x) = x^{2/3}$. What is $f'(0)$?

Again, first notice that $f(x)$ is everywhere continuous, including at $x = 0$. But the value of the derivative is

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{2/3}}{x} = \lim_{x \to 0} \frac{1}{x^{1/3}} = \infty$$

So, f is not differentiable at $x = 0$, because the limit does not exist. However, the tangent line still exists - it just has an infinite slope, or in other words, it is a *vertical tangent line.* When the derivative at a point is equal to $\pm\infty$, it indicates that the tangent line to that point is vertical.

Now we have seen two examples of functions that were *continuous but not differentiable* at specific points. So, we can say that continuity does **not** imply differentiability. But is the converse true?

Theorem 3.1. *Differentiability Implies Continuity*

If a function f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof: To prove that $f(x)$ is continuous at $x = c$, we need to show that $f(x) \to f(c)$ as $x \to c$. Since the function is differentiable at $x = c$, this means that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Here, notice that $f(x) \to f(c)$ is the same condition as $f(x) - f(c) \to 0$, and we have $f(x) - f(c)$ in this expression. We can isolate it by multiplying by $x - c$, as follows:

$$\begin{aligned}
\lim_{x \to c} [f(x) - f(c)] &= \lim_{x \to c} \left[(x - c) \left(\frac{f(x) - f(c)}{x - c} \right) \right] \\
&= \lim_{x \to c} (x - c) \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\
&= 0 \cdot f'(c) = 0
\end{aligned}$$

Because $f(x) - f(c)$ approaches 0 as $x \to c$, $f(x) \to f(c)$ as $x \to c$, so $f(x)$ is continuous at c. ■

3.2 The Basics of Finding Derivatives

Trying to find the derivative of any remotely complex function with the definition quickly turns into an algebraic nightmare. In this section, we present some of the most basic rules of differentiation. We will work mostly with polynomials in this section, leaving trigonometric functions and logarithms for later.

3.2.1 Basic Properties of Differentiation

Differentiation can be thought of as a function that acts upon functions; the input is $f(x)$ and the output is $f'(x)$. As a function, differentiation obeys two fundamental properties that we will implicitly use in the vast majority of calculations.

Theorem 3.2. *The Constant Multiple Rule*

Let $f(x)$ be a differentiable function and let c be a real number. Then the function $cf(x)$ is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Proof: Use the definition of the derivative.

$$\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= \lim_{h \to 0} c \left[\frac{f(x+h) - f(x)}{h} \right] \\
&= c \lim_{x \to 0} \frac{f(x+h) - f(x)}{h} \\
= cf'(x)
\end{aligned}$$

Recall from Theorem 2.2 that we can factor constants out of limits. So this result should not surprise you either, as again, a derivative is just a special limit.

Theorem 3.3. *Sum and Difference Rules*

The sum (or difference) of two differentiable functions $f(x)$ and $g(x)$ is itself differentiable, and the derivative of the sum (or difference) is the sum (or difference) of the two derivatives of f and g:

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$
$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Proof: Similar to above, use the definition. Remember, the $\pm$ sign can be read two ways: either pick every sign on top or on the bottom. (This is a straightforward calculation so $\mp$ isn't needed.)

$$\begin{aligned}
\frac{d}{dx}[f(x) \pm g(x)] &= \lim_{h\to 0} \frac{[f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)] \pm [g(x+h) - g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \pm \frac{g(x+h) - g(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) \pm g'(x)
\end{aligned}$$

The takeaway from these two theorems is that differentiation preserves addition/subtraction and scalar multiplication. In other words, the order of those operations doesn't matter: adding two functions and then differentiating is the same as differentiating individually and then adding. These two properties together are known as *linearity*:

$$f(\lambda x) = \lambda f(x) \qquad \text{and} \qquad f(x+y) = f(x) + f(y)$$

In practice, these two rules mean that finding derivatives of expressions with sums/differences is as simple as finding the derivatives of each individual term, and then adding/subtracting.

Note that there are no "product" or "quotient" rules. In fact, it is generally **not** true that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x) \qquad \text{or} \qquad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}$$

We will derive rules for products and quotients in the next section.

3.2.2 Basic Formulae: Constant Rule, Power Rule

Now we can begin deriving rules for evaluating derivatives.

Theorem 3.4. *Constant Rule*

If c is a real number, then its derivative is 0. In other words,

$$\frac{d}{dx}[c] = 0$$

Intuitively, the graph of a constant function on the coordinate plane is a horizontal line, which has a slope of 0. (Try to use the limit definition of a derivative to prove this theorem.)

The next theorem forms the basis for a large portion of the calculus that we will do in this entire textbook.

Theorem 3.5. *Power Rule*

Let n be a rational number. Then x^n is differentiable and

$$\frac{d}{dx}x^n = nx^{n-1}$$

Proof: For now, we will only show the proof of this rule for positive integers. To do this, we will need to use the binomial theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

Recall that $\binom{n}{0} = 1, \binom{n}{1} = n$. Now we can use the definition of a derivative:

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n}h^n}{h} \\
&= \lim_{h\to 0} \left[nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n}h^{n-1}\right] \\
&= nx^{n-1} + 0 + \cdots + 0 = nx^{n-1}
\end{aligned}$$

In the numerator of the fraction, after canceling the x^n term, every remaining term except for the first $(nx^{n-1}h)$ is divisible by h^2. Therefore, when dividing by h and then taking the limit as $h \to 0$, every term except for the first one becomes 0.

Now I will give a number of examples to illustrate how to use these rules.

Ex. 1: Some Simple Derivatives

(a) Let's compute the derivative of $f(x) = mx + b$, the slope-intercept equation of a line. We know from our calculation using the limit definition that $f'(x) = m$. Let's use the rules we've derived to calculate the derivative:

$$\begin{aligned}\frac{d}{dx}(mx+b) &= \frac{d}{dx}mx + \frac{d}{dx}b && \text{(Sum Rule)}\\ &= m\left[\frac{d}{dx}x\right] + 0 && \text{(Constant Rule, Constant Multiple Rule)}\\ &= m(1x^0) + 0 = m && \text{(Power Rule)}\end{aligned}$$

This agrees with our previous result. As a sanity check, note that the Power Rule applied to x gives 1, as expected.

(b) The rules we've introduced so far allow us to easily calculate the derivative of any polynomial. For example,

$$\frac{d}{dx}(x^6 - 3x^4 + x^2 - 1) = \frac{d}{dx}x^6 - \frac{d}{dx}3x^4 + \frac{d}{dx}x^2 - \frac{d}{dx}1 = 6x^5 - 12x^3 + 2x^2$$

(c) The Power Rule works for any rational exponent. In certain cases, it may help to rewrite the exponent, as follows:

$$\begin{aligned}\frac{d}{dx}\left[\frac{5}{\sqrt[3]{x^4}}\right] &= \frac{d}{dx}\left[5x^{-4/3}\right] = 5\left(-\frac{4}{3}\right)x^{-7/3} = -\frac{20}{3}x^{-7/3}\\ \frac{d}{dx}\left[\frac{1}{7x^7}\right] &= \frac{d}{dx}\left[\frac{1}{7}x^{-7}\right] = \frac{1}{7}(-7)x^{-8} = -\frac{1}{x^8}\end{aligned}$$

3.2.3 The Product and Quotient Rules

The proofs of these two rules are not as straightforward as the previous rules, and involve the clever addition and subtraction of a quantity.

Theorem 3.6. *Product Rule*

If f and g are differentiable functions then their product is also differentiable, satisfying

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Proof: Start with the definition of the derivative, but we need to add and subtract an additional term (in bold). Recognize the fractions in large parentheses as the definition of the derivative:

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h) \boldsymbol{+ f(x+h)g(x) - f(x+h)g(x)} - f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h}\right]\\ &= \lim_{h\to 0}\left[f(x+h)\left(\frac{g(x+h)-g(x)}{h}\right)\right] + \lim_{h\to 0}\left[g(x)\left(\frac{f(x+h)-f(x)}{h}\right)\right]\\ &= f(x)g'(x) + g(x)f'(x)\end{aligned}$$

Ex. 1:

(a) Find the derivative of $y = (2x^3 - 3x)(x^2 - 5)$.
Soln. Use the Product Rule. In this case $f(x) = 2x^3 - 3x, g(x) = x^2 - 5$.

$$\begin{aligned}y' &= (2x^3 - 3x)\frac{d}{dx}(x^2 - 5) + (x^2 - 5)\frac{d}{dx}(2x^3 - 3x)\\ &= (2x^3 - 3x)(2x) + (x^2 - 5)(6x^2 - 3)\\ &= 4x^4 - 6x^2 + 6x^4 - 3x^2 - 30x^2 + 15\\ &= 10x^4 - 39x^2 + 15\end{aligned}$$

To verify the Product Rule, we could expand the expression for y and then take the derivative:

$$y = (2x^3 - 3x)(x^2 - 5) = 2x^5 - 10x^3 - 3x^3 + 15x = 2x^5 - 13x^3 + 15x$$
$$\longrightarrow y' = 10x^4 - 39x^2 + 15$$

These agree, as expected.

The Product Rule can be expanded to more than 2 functions; for example,

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

(b) Find the derivative of $f(x) = 5x(x-9)(x^2+1)$.
Soln.

$$\begin{aligned} f'(x) &= 5(x-9)(x^2+1) + 5x(1)(x^2+1) + 5x(x-9)(2x) \\ &= (5x^3 + 5x - 45x^2 - 45) + (5x^3 + 5x) + (10x^3 - 90x^2) \\ &= 20x^3 - 135x^2 + 10x - 45 \end{aligned}$$

Theorem 3.7. *Quotient Rule*
If f and g are differentiable functions, then their quotient f/g is differentiable wherever $g(x) \neq 0$. This derivative satisfies

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)^2]}$$

Proof: Again, the same quantity in bold is added and subtracted.

$$\begin{aligned} \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ &= \lim_{h\to 0} \frac{g(x)f(x+h) - f(x)g(x+h)}{hg(x)g(x+h)} \\ &= \lim_{h\to 0} \frac{g(x)f(x+h) \boldsymbol{- f(x)g(x) + f(x)g(x)} - f(x)g(x+h)}{hg(x)g(x+h)} \\ &= \lim_{h\to 0} \left[\frac{\frac{g(x)f(x+h) - g(x)f(x)}{h} - \frac{f(x)g(x+h) - f(x)g(x)}{h}}{g(x)g(x+h)}\right] \\ &= \frac{\lim_{h\to 0}\left[g(x)\frac{f(x+h)-f(x)}{h}\right] - \lim_{h\to 0}\left[f(x)\frac{g(x+h)-g(x)}{h}\right]}{\lim_{h\to 0}[g(x)g(x+h)]} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \end{aligned}$$

The way I like to remember this is "bottom d top, minus top d bottom, over bottom squared".

Ex. 1: Some Simple Derivatives

1. We can use the Quotient Rule to verify our calculation of the derivative of $y(t) = \frac{2t}{t+3}$. Using the rule gives

$$y'(t) = \frac{(t+3)(2) - 2t(1)}{(t+3)^2} = \frac{6}{(t+3)^2}$$

2. Rewriting can also be useful in certain situations. For example, let's take the derivative of $f(x) = \frac{x+6}{x^2}$:

$$\frac{d}{dx}\left[\frac{x+6}{x^2}\right] = \frac{d}{dx}\left[\frac{x}{x^2} + \frac{6}{x^2}\right] = \frac{d}{dx}[x^{-1}] + 6\frac{d}{dx}[x^{-2}] = -\frac{1}{x^2} - \frac{12}{x^3}$$

Here, you could use the Quotient Rule, but recognizing that the fraction can be simplified leads to an easier calculation using the Power Rule.

3. One more example: Find the derivative of $(2x-3)(x+1)^{-2}$.

$$\begin{aligned}
\frac{d}{dx}[(2x-3)(x+1)^{-2})] &= \frac{d}{dx}\left[\frac{2x-3}{(x+1)^2}\right] \\
&= \frac{(x+1)^2(2)-(2x-3)(2x+2)}{(x+1)^4} \\
&= \frac{-2x^2+6x+8}{(x+1)^4} \\
&= -\frac{2(x-4)(x+1)}{(x+1)^4} \\
&= -\frac{2(x-4)}{(x+1)^3}
\end{aligned}$$

Ex. 1: Proof of the Power Rule for Negative Integers

Now we can prove the Power Rule for n negative. (Rational numbers, the last case, will be handled in a later section.) Recall the Power Rule:

$$\frac{d}{dx}x^n = nx^{n-1}$$

We can assume that this is true for n positive. If n is negative, then there exists a positive integer m such that $m=-n$. Then we can write

$$\begin{aligned}
\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} &= \frac{d}{dx}\left[\frac{1}{x^m}\right] \\
&= \frac{x^m(0)-1(mx^{m-1})}{x^{2m}} && \text{(Quotient Rule)} \\
&= -\frac{mx^{m-1}}{x^{2m}} = -mx^{-m-1} \\
&= nx^{n-1} \qquad (m=-n)
\end{aligned}$$

3.2.4 Derivatives of Trig Functions

In this section we derive some of the derivatives of the six basic trig functions (and present the rest without proof). Before we do so, it is relevant to review Theorem 2.8:

$$\lim_{x\to 0}\frac{\sin x}{x} = 0, \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

Theorem 3.8. *Derivatives of Sine and Cosine*

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x$$

Proof: Here is the proof for sine.

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\sin h+\sin x\cos h-\sin x}{h} \\
&= \lim_{h\to 0}\left[\cos x\left(\frac{\sin h}{h}\right)-\sin x\left(\frac{1-\cos h}{h}\right)\right] \\
&= \cos x\left[\lim_{h\to 0}\frac{\sin h}{h}\right]-\sin x\left[\lim_{h\to 0}\frac{1-\cos h}{h}\right] \\
&= \cos x\cdot 1-\sin x\cdot 0 = \cos x
\end{aligned}$$

The proof for cosine is similar and can be left as an exercise.

Theorem 3.9. *Derivatives of Other Trig Functions*

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\csc x = -\csc x\cot x, \qquad \frac{d}{dx}\sec x = \sec x\tan x$$

Proof: Here is the proof for secant. Use the Quotient Rule:

$$\begin{aligned}\frac{d}{dx}[\sec x] &= \frac{d}{dx}\left[\frac{1}{\cos x}\right] \\ &= \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x} \\ &= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \sec x \tan x\end{aligned}$$

3.2.5 Higher Order Derivatives

To illustrate the idea of higher order derivatives, I will expand upon the last paragraph in section 3.1.1, an example from physics. (We will discuss applications of derivatives in much greater detail in the next chapter.)

Suppose we have a particle that can move, but only in one dimension. There is some "position function" $r(t)$ that describes the particle's motion. We've seen now that the derivative of this function with respect to time, $r'(t)$, is a function that describes the *velocity* of the function. Given a time $t, v(t) := r'(t)$ is the velocity of the particle at that time.

What if we take the derivative of the velocity function? The resultant function would be measuring the *rate of change of the velocity.* In physics, there is a term for $\Delta v/\Delta t$: **acceleration**. So the acceleration is the derivative of the derivative of position; this is called the *second derivative* of position. It is denoted

$$a(t) = v'(t) = r''(t)$$

Acceleration is an example of a **higher-order derivative**. It is possible to take the derivative of any function a positive integer number of times. Specifically in physics, there are (somewhat facetious) terms for up to the 6th derivative of position:

$$\begin{aligned}\text{position}: &\quad r(t) \\ \text{velocity}: &\quad v(t) = r'(t) \\ \text{acceleration}: &\quad a(t) = v'(t) = r''(t) \\ \text{jerk}: &\quad j(t) = a'(t) = v''(t) = r'''(t) \\ \text{snap}: &\quad s(t) = j'(t) = a''(t) = v'''(t) = r^{(4)}(t) \\ \text{crackle}: &\quad c(t) = \cdots = r^{(5)}(t) \\ \text{pop}: &\quad p(t) = \cdots = r^{(6)}(t)\end{aligned}$$

Taking a function to the nth derivative can be denoted as

$$f^{(n)}(x) \qquad \frac{d^n y}{dx^n} \qquad \frac{d^n}{dx^n}[f(x)]$$

One thing to be careful about is the notation of $f^{(n)}(x)$. This represents the nth derivative, and is **not equal to** $[f(x)]^n$.

Ex. 1: Some Examples of Higher-Order Derivatives

(a) From the Power Rule, the derivative of a polynomial with degree n is a polynomial with degree $n-1$. Therefore, if we take the derivative of a degree n polynomial enough times, we should eventually arrive at a constant. In fact, if $f(x)$ is a polynomial with degree n, then for any $k \geq n+1$,

$$f^{(k)}(x) = 0$$

(This can be proved using induction.)

(b) Let's find the first four derivatives of $f(x) = \sin x$. We have

$$\begin{aligned}f(x) &= \sin x, \\ f'(x) &= \frac{d}{dx}[\sin x] = \cos x, \\ f''(x) &= \frac{d}{dx}[\cos x] = -\sin x, \\ f'''(x) &= \frac{d}{dx}[-\sin x] = -\cos x, \\ f^{(4)}(x) &= \frac{d}{dx}[-\cos x] = \sin x\end{aligned}$$

So the higher-order derivatives of $f(x) = \sin x$ actually repeat in cycles of 4: $f^{(4)}(x) = f(x)$. This makes it easy to find any higher-order derivative: for example,

$$\frac{d^{105}}{dx^{105}}[\sin x] = \frac{d}{dx}\sin x = \cos x$$

(c) Given a function $f(x)$ such that $f''(x) = \dfrac{x+1}{x-1}$, we want to find $f^{(4)}(x)$. We can obtain the fourth derivative from the second derivative by *taking the derivative twice*:

$$\begin{aligned}
f''(x) &= \frac{x+1}{x-1} \\
f'''(x) &= \frac{d}{dx}\left[\frac{x+1}{x-1}\right] = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} = -\frac{2}{(x-1)^2} \\
f^{(4)}(x) &= \frac{d}{dx}\left[-\frac{2}{(x-1)^2}\right] = -\frac{(x-1)^2(0)-2(2x-2)}{(x-1)^4} = \frac{4}{(x-1)^3}
\end{aligned}$$

(d) Let's use the Product Rule to evaluate higher-order derivatives of a product of functions and see if we can find a pattern. Specifically, if given differentiable functions f and g, what are the first few high-order derivatives of fg? We have

$$\begin{aligned}
(fg)' &= f'g + fg' \\
(fg)'' &= (f'g + fg)' = (f'g)' + (fg')' \\
&= (f''g + f'g') + (f'g' + f'g'') \\
&= f''g + 2f'g' + f'g'' \\
(fg)''' &= (f''g + 2f'g' + fg'')' \\
&= (f'''g + f''g') + (2f''g' + 2f'g'') + (f'g'' + fg''') \\
&= f'''g + 3f''g' + 3f'g'' + fg''' \\
(fg)^{(4)} &= \cdots = f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}
\end{aligned}$$

etc. Can you recognize the pattern? (Hint: look at the coefficients!)

3.2.6 Examples

Now, I will give examples that utilize everything we have learned about derivatives so far.

1. Some straightforward calculation examples.

   (a) Find the derivative of $f(x) = x^2 \cos x$.

   Soln: Use the Product Rule. (Let $h(x) = x^2$ and $g(x) = \cos x$).

$$\begin{aligned}
f(x) &= x^2 \cos x \\
f'(x) &= x^2 \frac{d}{dx}[\cos x] + \cos x \frac{d}{dx}[x^2] \\
&= -x^2 \sin x + 2x \cos x
\end{aligned}$$

   (b) Find the second derivative of $f(x) = \sin x / x$.

   Soln: Use the Quotient Rule twice:

$$f'(x) = \frac{x(\cos x) - \sin x(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

   Before taking the derivative again, notice that taking the derivative of the numerator will involve the Product Rule.

$$\begin{aligned}
f''(x) &= \frac{x^2(\cos x - x\sin x - \cos x) - (x\cos x - \sin x)(2x)}{x^4} \\
&= \frac{-x^3\sin x - 2x^2\cos x + 2x\sin x}{x^4} \\
&= -\frac{2x^2\cos x + (x^2-2)\sin x}{x^3}
\end{aligned}$$

(c) Find the derivative of

$$f(x) = \frac{x \sin x}{\sqrt{x}+1}$$

Soln: Just a little involved algebra-wise, but same rules:

$$\begin{aligned}
f(x) &= \frac{x \sin x}{\sqrt{x}+1} \\
f'(x) &= \frac{(\sqrt{x}+1)(\sin x - x\cos x) - (x\sin x)\dfrac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2} \\
&= \frac{2x\sin x - 2x^2\cos x + 2\sqrt{x}\sin x - 2x\sqrt{x}\cos x - x\sin x}{2\sqrt{x}(\sqrt{x}+1)^2} \\
&= \frac{\sqrt{x}\sin x(\sqrt{x}+2) - 2x^{3/2}\cos x(\sqrt{x}+1)}{2\sqrt{x}(\sqrt{x}+1)^2} \\
&= \frac{\sin x(\sqrt{x}+2) - 2x\cos x(\sqrt{x}+1)}{2(\sqrt{x}+1)^2}
\end{aligned}$$

(d) Find the derivative of

$$y(t) = t^3\left(\frac{t}{t-2} + 2t^{-1}\right)$$

Soln: For this expression it is easier to expand the product first and then take the derivative of the separate parts.

$$\begin{aligned}
y(t) = t^2\left(\frac{t}{t-2} + 2t^{-1}\right) &= \frac{t^3}{t-2} + 2t \\
\frac{d}{dt}[y(t)] &= \frac{d}{dt}\left[\frac{t^3}{t-2}\right] + \frac{d}{dt}[2t] \\
&= \frac{(t-2)(2t^2) - (t^3)(1)}{(t-2)^2} + 2 \\
&= \frac{t^3 - 4t^2}{(t-2)^2} + 2
\end{aligned}$$

2. Find all x for which the function

$$f(x) = \frac{\sin x}{3 - 2\cos x}$$

has a horizontal tangent line.

Soln: A horizontal tangent line has a slope of 0, so we are just looking for values of x for which $f'(x) = 0$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\frac{\sin x}{3 - 2\cos x}\right] \\
&= \frac{(3 - 2\cos x)(\cos x) - \sin x(2\sin x)}{(3 - 2\cos x)^2} \\
&= \frac{3\cos x - 2}{(3 - 2\cos x)^2}
\end{aligned}$$

When is this equal to 0? Notice that this function is defined for all x since $3 - 2\cos x \neq 0$. So the numerator must be equal to 0:

$$3\cos x - 2 = 0 \longrightarrow \cos x = \frac{2}{3}$$

To find all x, note that one value for x is $\arccos\frac{2}{3}$, which is in the first quadrant. The other angle that has $\cos x = \frac{2}{3}$ is $x = -\arccos\frac{2}{3}$. Furthermore, since $\cos x$ is periodic with a period of 2π, the full solutions for x are:

$$x = \arccos\frac{2}{3} + 2\pi n, -\arccos\frac{2}{3} + 2\pi n, n \in \mathbb{Z}$$

3. Find the equation of the tangent line(s) to the curve

$$f(x) = \frac{x^2+1}{x+1}$$

that pass(es) through the point (5,-1).

Soln: This problem is a little trickier than the previous one because we are given a point not on the curve, and we do not know where the tangent line intersects the graph. To find the equation of the line, we are going to need the point on the curve, as well as the slope. Let's start by taking the derivative:

$$f'(x) = \frac{(x+1)(2x) - (x^2+1)(1)}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}$$

So, if we have a value for x, say $x = c$, we can find the slope of the tangent line that intersects the graph at $(c, f(c))$. But how can we find the specific tangent line that passes through $(5, -1)$?

Well, if the point $(5, -1)$ lies on the tangent line to $f(x)$ at the point $(c, f(c))$, then the slope of the line between $(c, f(c))$ and $(5, -1)$ should be equal to $f'(c)$. In other words, we can calculate slope alternatively as the change in y divided by the change in x:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(c)+1}{c-5} = \frac{\frac{c^2+1}{c+1}+1}{c-5}$$

Set m equal to $f'(c)$, and solve for c:

$$\begin{aligned}
\frac{c^2+2c-1}{(c+1)^2} &= \frac{\frac{c^2+1}{c+1}+1}{c-5} \\
(c^2+2c-1)(c-5) &= (c+1)^2\left(\frac{c^2+1}{c+1}+1\right) \\
c^3+2c^2-c-5c^2-10c+5 &= (c^3+c^2+c+1)+(c^2+2c+1) \\
5c^2+14c-3 &= 0 \\
(5c-1)(c+3) &= 0 \\
c &= \frac{1}{5}, -3
\end{aligned}$$

So there are actually 2 values of x that satisfy the given equation, implying that there are 2 tangent lines that pass through $(5, -1)$. Plugging in these values of x to $f(x)$ gives the two points:

$$(x_1, y_1) = \left(\frac{1}{5}, \frac{13}{15}\right), (x_2, y_2) = (-3, -5)$$

To find the equations of the lines, we just need the slope $f'(c)$.

$$\begin{array}{rl}
\text{Point: } (x_1, y_1) = \left(\frac{1}{5}, \frac{13}{15}\right) & (x_2, y_2) = (-3, -5) \\
\text{Slope: } m_1 = f'\left(\frac{1}{5}\right) = -\frac{7}{18} & m_2 = f'(-3) = \frac{1}{2} \\
\text{Line: } y - \frac{13}{15} = -\frac{7}{18}\left(x - \frac{1}{5}\right) & y + 5 = \frac{1}{2}(x+3) \\
\ell_1 : y = -\frac{7}{18}x + \frac{17}{18} & \ell_2 : y = \frac{1}{2}x - \frac{7}{2}
\end{array}$$

3.3 The Chain Rule

3.3.1 The Problem: Composite Functions

So far, we have discussed basic rules for differentiating simple functions, as well as rules for differentiating binary operations of functions (sum/difference, product and quotient). We are currently able to take the derivative of a wide range of trigonometric, polynomial, rational, and combinations of these types of functions. However, there is an even wider range of functions that up to this point we do not know how to derive.

Consider the following two columns. On the left, we have functions we currently know how to take the derivative of. On the right, we have similar functions that we cannot take the derivative of with our current knowledge.

$$\begin{aligned} f(x) &= \sqrt[3]{x} & \qquad f(x) &= \sqrt[3]{x+1} \\ f(x) &= (2x+1)^2 & \qquad f(x) &= (2x+1)^{50} \\ y &= \sin x & \qquad y &= \sin(\tan(x)) \\ y &= \sin x \cos x & \qquad y &= \sin(x^3)\cos^2 x \\ y &= (\sin x \tan x)^{-1} & \qquad y &= (\sin(x \tan x))^{-1} \end{aligned}$$

One thing in common to most of the right column is that those functions are *composite functions*, meaning they can be written in the form $f(g(x))$ for some other, simpler functions f and g. For example, if

$$g(x) = x+1, f(x) = \sqrt[3]{x} \longrightarrow f(g(x)) = \sqrt[3]{x+1}$$

It is possible for a function to be the composite of more than 2 functions. For example, in the last row on the right column:

$$h(x) = x \tan x, g(x) = \sin x, f(x) = x^{-1} \longrightarrow f(g(h(x))) = (\sin(x \tan x))^{-1}$$

In these two examples, notice how individually, we know how to take the derivative of each of the functions f, g, h by themselves, but do not know how to take the derivative of the composite of these functions.

Ex. 1: An Easy Rule?

Let's look at the function

$$y = (2x+1)^3$$

This is a composite function if we define

$$g(x) = 2x+1, f(x) = x^3 \longrightarrow y = f(g(x)) = (2x+1)^3$$

Another way of looking at this, which will come in handy shortly, is to define $u = 2x+1$, so we have

$$y = (2x+1)^3 = u^3 = f(u), \qquad u = 2x+1 = g(x)$$

In other words, I am defining a new function u in terms of x, and then expressing y in terms of u.

Now, I want to determine the derivative $y' = dy/dx$, which measures the rate of change of y with respect to x. However, I'm only given the rate of change of y *with respect to* u (dy/du), and the rate of change of u *with respect to* x (du/dx). How are these three quantities related?

Another way of looking at the question is: can we find a simple relation between

$$\frac{d}{dx}[f(g(x))] \qquad \text{and} \qquad f(x), g(x) = u, f'(x), g'(x)$$

In this case I give a polynomial example because we actually can evaluate this derivative by just expanding the binomial:

$$f(x) = x^3 \qquad g(x) = 2x+1 \qquad f'(x) = 3x^2 \qquad g'(x) = 2$$
$$y = (2x+1)^3 = 8x^3 + 12x^2 + 6x + 1$$
$$y' = 24x^2 + 24x + 6$$

Try and see if you can find an simple expression for y' in terms of f, g, f', g'. If you find one, try to verify it using another simple binomial raised to some power.

...

Spoiler alert: no simple combination of those functions will give the correct derivative.

Let's go back to this function u which acts as an intermediate function between x and the composite function y.

3.3.2 The Solution: The Chain Rule

Once again, we wish to determine dy/dx given dy/du and du/dx.

Theorem 3.10. *The Chain Rule*

Let $y = f(u)$ be a differentiable function of u and let $u = g(x)$ be a differentiable function of x. Then $y = f(g(x))$ is a differentiable function of x and we can write

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

To apply the Chain Rule to a composite function $y = f(g(x))$, where we once again have the intermediate function $u = g(x)$, consider $f(u)$ the *outer function* and $u = g(x)$ the *inner function.* What the Chain Rule says now is that

$$y' = f'(u) \cdot u'$$

the derivative of $f(x)$ *at* $x = u$ multiplied by the derivative of u with respect to x.

Proof: This proof will be a little more technical than any proof we have seen previously. We can start by writing down what we wish to prove using the alternative definition of a derivative (Section 3.1.3), at a specific point $x = c$:

$$[f(g(c))]' = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} = f'(g(c)) \cdot g'(c)$$

Similar to the proofs of the Product and Quotient Rules, we will need to introduce a new term in order to simplify the limit into its desired form. Specifically, we will multiply and divide by the same term:

$$\begin{aligned}
[f(g(c))]' &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{x - c} \cdot \frac{g(x) - g(c)}{g(x) - g(c)} \right] \\
&= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \right] \\
&= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right] \cdot \lim_{x \to c} \left[\frac{g(x) - g(c)}{x - c} \right]
\end{aligned}$$

However, there is a slight problem. It might be the case that as x approaches c, there are values of x for which $g(x) = g(c)$. Specifically, if the function g oscillates infinitely (think $\sin 1/x$), then no matter how close x gets to c, there will be a value of x closer to c with $g(x) = g(c)$. In those cases, the whole limit will be undefined.

To work around this issue, we will define a new function, say $I(y)$, as follows:

$$I(y) = \begin{cases} \dfrac{f(y) - f(g(c))}{y - g(c)}, & y \neq g(c) \\ f'(g(c)), & y = g(c) \end{cases}$$

Don't be confused by all the notation: y is a dummy variable, I could've used any letter I wanted. We are only going to be looking at the specific value $y = g(x)$.

I am going to show that the difference quotient

$$\frac{f(g(x)) - f(g(c))}{x - c}$$

is always equal to

$$I(g(x)) \cdot \frac{g(x) - g(c)}{x - c}$$

by considering both cases. When $g(x) \neq g(c)$, we have the following:

$$\begin{aligned}
I(g(x)) \cdot \frac{g(x) - g(c)}{x - c} &= \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \\
&= \frac{f(g(x)) - f(g(c))}{x - c}
\end{aligned}$$

When $g(x) = g(c)$, then the difference quotient is

$$\frac{f(g(x)) - f(g(c))}{x - c} = \frac{f(g(c)) - f(g(c))}{x - c} = 0$$

and the product in question is

$$I(g(c)) \cdot \frac{g(x) - g(c)}{x - c} = f'(g(c)) \cdot \frac{g(c) - g(c)}{x - c} = 0$$

Now that we have established that the difference quotient is equal to this other quantity, the limit becomes the following:

$$\begin{aligned}
[f(g(c))]' &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} \left[I(g(x)) \cdot \frac{g(x) - g(c)}{x - c} \right] \\
&= \left[\lim_{x \to c} I(g(x)) \right] \cdot \left[\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right]
\end{aligned}$$

First, notice that the second limit is the alternative definition of the derivative. g is given to be a differentiable function of x, so its derivative at $x = c$ exists and is equal to $g'(c)$.

For the function $I(g(x))$, first notice that by definition, I is only defined whenever f is defined. f is assumed to be differentiable at $g(c)$, so because differentiability implies continuity (Theorem 3.1), I is continuous at $g(c)$. Since g is also continuous at c, this implies that $I \circ g$, the composition, is also continuous at c.

Therefore, by the properties of continuity (Section 2.4.3), the limit of $I(g(x))$ as x approaches c exists and is equal to $I(g(c))$. But by definition of the function I, this is exactly equal to $f'(g(c))$. So the overall limit becomes

$$[f(g(c))]' = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} = \left[\lim_{x \to c} I(g(x)) \right] \cdot \left[\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right] = f'(g(c)) \cdot g'(c)$$

and we have proved the Chain Rule. $\blacksquare$

Ex. 1: The Previous Example

We can use the Chain Rule to take the derivative of

$$y = (2x + 1)^3$$

We want to identify the inner and outer functions in order to apply the Chain Rule. In this case, we have

$$g(x) = 2x + 1, f(x) = x^3 \longrightarrow f(g(x)) = (2x + 1)^3$$

Therefore, the derivative is

$$\begin{aligned}
[f(g(x))]' &= f'(g(x)) \cdot g'(x) \\
&= 3(2x + 1)^2 \cdot 2 \\
&= 6(2x + 1)^2 \\
&= 24x^2 + 24x + 6
\end{aligned}$$

which matches with the derivative we found previously.

Just to be absolutely clear, the term $f'(g(x))$ is the derivative of $f(x)$ **evaluated at the point g(x)**. In this example, we have

$$f(x) = x^3 \longrightarrow f'(x) = 3x^2 \longrightarrow f'(g(x)) = f'(2x + 1) = 3(2x + 1)^2$$

I can also write the derivative using the intermediate function $u = g(x), y = f(u)$, and use the alternative form of the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

In this problem again we have

$$y = f(u) = u^3, u = g(x) = 2x + 1$$

So applying the Chain Rule gives

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= 3u^2 \cdot 2 \\
&= 3(2x + 1)^2 \cdot 2
\end{aligned}$$

which agrees, as expected.

3.3.3 Some Examples

When it comes to evaluating derivatives using the Chain Rule, the majority of the work involved will come from simplifying expressions, specifically expanding out complicated products or quotients or collecting like terms.

1. (a) Find the derivative of $f(x) = (2x+1)^{50}$.

   Soln: Similar to above, identify the inner and outer functions, then apply the Chain Rule. Here we have

$$\begin{aligned} u = g(x) &= 2x+1, f(u) = u^{50} \\ f'(x) &= f'(u) \cdot u' \\ &= 50u^{49} \cdot 2 = 100(2x+1)^{49} \end{aligned}$$

   Here's an example of a polynomial that one could not (within reasonable time) differentiate by expanding first. Notice how, using the Chain Rule, the derivative has a nice, compact form. Do not forget to replace u with $g(x)$ in the final few steps when simplifying.

   (b) Find y' for

$$y = \sin(x^3)\cos^2 x$$

   Soln: This problem involves both the Product Rule and the Chain Rule. Let $f(x) = \sin(x^3)$ and $g(x) = \cos^2 x$; we want to calculate their individual derivatives first because we will need them in the Product Rule. We have, by the Chain Rule,

$$\begin{aligned} f(x) = \sin(x^3) &\longrightarrow f'(x) = 3x^2\cos(x^3) \\ g(x) = \cos^2 x = (\cos x)^2 &\longrightarrow g'(x) = -2\cos x \sin x = -2\sin 2x \end{aligned}$$

   Therefore, the derivative of y is

$$\begin{aligned} y &= f(x)g(x) \\ y' &= f'(x)g(x) + f(x)g'(x) \\ &= 3x^2\cos(x^3)\cos^2 x + \sin(x^2)(-2\sin 2x) \\ &= 3x^2\cos(x^3)\cos^2 x - 2\sin(x^2)\sin 2x \end{aligned}$$

   (c) Find the derivative of

$$f(x) = \left(\frac{x-2}{2x^2+1}\right)^3$$

   Soln:

$$\begin{aligned} f'(x) &= 3\left(\frac{x-2}{2x^2+1}\right)^2 \cdot \left[\frac{x-2}{2x^2+1}\right]' \\ &= 3\left(\frac{x-2}{2x^2+1}\right)^2 \cdot \frac{(2x^2+1)(1)-(x-2)(4x)}{(2x^2+1)^2} \\ &= 3\left(\frac{x-2}{2x^2+1}\right)^2 \cdot \frac{-2x^2+8x+1}{(2x^2+1)^2} \\ &= \frac{3(x-2)^2(-2x^2+8x+1)}{(2x^2+1)^4} \end{aligned}$$

   (d) Find the derivative of

$$f(x) = x^2\sqrt[3]{4-x^2}$$

   Soln: This solution will skip a few steps, but should still be easy to follow. Rewrite the cube root as a fractional exponent to apply the Power Rule:

$$\begin{aligned} f'(x) &= 2x\sqrt[3]{4-x^2} + x^2 \cdot \frac{1}{3}(4-x^2)^{-2/3} \cdot (-2x) \\ &= 2x(4-x^2)^{1/3} - \frac{2}{3}x^3(4-x^2)^{-2/3} \\ &= \frac{6x(4-x^2)-2x^3}{3(4-x^2)^{2/3}} \\ &= \frac{-8x^3+24x}{3\sqrt[3]{(4-x^2)^2}} \end{aligned}$$

(e) Find t' for

$$t(x) = \left[\sin\left(x^2 \tan x\right)\right]^{-1}$$

Soln: This function actually has 3 layers instead of 2, as we saw before. The Chain Rule effectively removes one layer, so we will need to apply it twice. Here we show all the steps, including identifying the "layers" of the composite functions:

$$\begin{aligned} t(x) &= f(g(h(x))) = \left[\sin\left(x^2 \tan x\right)\right]^{-1} \\ \longrightarrow f(x) &= x^{-1}, g(x) = \sin x, h(x) = x^2 \tan x \end{aligned}$$

$$\begin{aligned} t'(x) &= [f(g(h(x)))]' \\ &= f'(g(h(x))) \cdot [g(h(x))]' \\ &= -\left[\sin\left(x^2 \tan x\right)\right]^{-2} \cdot \left[\sin\left(x^2 \tan x\right)\right]' \\ &= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \\ &= -\left[\sin\left(x^2 \tan x\right)\right]^{-2} \cdot \cos\left(x^2 \tan x\right) \cdot \left(x^2 \tan x\right)' \\ &= -\left[\sin\left(x^2 \tan x\right)\right]^{-2} \cdot \cos\left(x^2 \tan x\right) \cdot (2x \tan x + x^2 \sec^2 x) \\ &= -\frac{\cos\left(x^2 \tan x\right) \cdot \left(2x \tan x + x^2 \sec^2 x\right)}{\left[\sin\left(x^2 \tan x\right)\right]^2} \end{aligned}$$

2. Find all x for which the tangent lines to the functions

$$f(x) = \cos 2x \qquad \text{and} \qquad g(x) = 2\sin x$$

have the same slope at x.
Soln: A simple word problem that involves setting two derivatives equal. We have:

$$f'(x) = -2\sin 2x \qquad \text{and} \qquad g'(x) = 2\cos x$$

To solve, remember the double angle identity for sine:

$$\sin 2x = 2\sin x \cos x$$

So the solutions for x are:

$$\begin{aligned} -2\sin 2x &= 2\cos x \\ -4\sin x \cos x &= 2\cos x \\ -2\sin x &= 1 \\ \sin x &= -1/2 \end{aligned}$$

For $0 \le x \le 2\pi$, the solutions are in the third and fourth quadrant, at $x = 210° = 7\pi/6$ and $x = 330° = 11\pi/6$. So all x that satisfy the equation and solve the problem are:

$$x = \frac{7\pi}{6} + 2\pi n, \qquad x = \frac{11\pi}{6} + 2\pi n, \qquad n \in \mathbb{Z}$$

3. We can use the Chain Rule to find the derivative of the *absolute value function*, $f(x) = |x|$. To do this, recall that one way to define the absolute value function is:

$$|x| = \sqrt{x^2} = (x^2)^{1/2}$$

We can apply the Chain Rule to the last expression:

$$\begin{aligned} f(x) &= |x| = (x^2)^{1/2} \\ f'(x) &= \frac{1}{2}(x^2)^{-1/2} \cdot 2x \\ &= \frac{x}{(x^2)^{1/2}} = \frac{x}{|x|} \end{aligned}$$

For positive x, the value of the derivative is 1, and for negative x it is -1, as expected. Also note that $f'(0)$ is undefined, since this corresponds to a sharp turn on the graph of $f(x)$ at $x = 0$.

3.4 Implicit Differentiation

We continue to build new differentiation techniques on top of the ones we have learned so far. In this section, we consider how to find the derivative of functions which are not as... straightforward as all the ones we have previously seen.

3.4.1 Implicit vs. Explicit Functions

So far in this text, every function that we have learned to differentiate is written *explicitly*. This is to say, every function has been written in the form

$$y = (\text{some function of only } x)$$

so if we know the value of x, we can plug it directly into the function to obtain y.

However, not every function can be written explicitly. To illustrate what I mean, let's consider the following example.

Ex. 1: Rewriting a Function

Find dy/dx for the function $x^2y = 1$.

Soln: First, notice that this function is not written explicitly. However, you can rewrite the function by solving for y:

$$x^2y = 1 \longrightarrow y = \frac{1}{x^2} \longrightarrow y' = -\frac{2}{x^3}$$

This strategy will work whenever it is possible to solve for y in terms of x. But what about when that is not possible? For example, try solving for y in the following equation:

$$y^2(4 - x^2) = (x^2 + 4y - 4)^2$$

Even though one of the preceding two functions is much more complicated than the other, both have the same form:

$$(\text{function that may involve both } x, y) = (\text{other function that may involve both } x, y)$$

When the relation between x and y is not *explicit*, but rather *implied*, we say the equation is in *implicit form*. How do we find dy/dx for a function in implicit form?

3.4.2 The Chain Rule, Disguised

Let us return to the base assumptions. We have y as an *implicit function of* x. We are trying to differentiate the entire function *with respect to* x. This means that wherever there is a y, we have to think of it as $y(x)$, a function of x. As we have seen already, it is possible for implicit functions to contain terms that involve both y and x. But we know how to differentiate such terms: by using the Chain Rule!

Ex. 1: Differentiating with Respect to x

(a) Differentiate $3x^2$ with respect to x.
Soln: This is not a trick question; the variable being differentiated and the variable in the function agree, so we can use the Power Rule:

$$\frac{d}{dx}(3x^2) = 6x$$

(b) Differentiate $3y^2$ with respect to x.
Soln: In this case, the variables do *not* agree, so we have to assume y is a function of x and use the Chain Rule. To make this more clear, you can think of $3y^2$ as a composite function (of x) as follows:

$$f(x) = 3x^2, y = y(x) \longrightarrow f(y(x)) = 3[y(x)]^2 = 3y^2$$

The outer part is $f(y) = 3y^2$, and the inner part is $y(x)$. So applying the Chain Rule, the derivative is

$$\frac{d}{dx}(3y^2) = 6y \cdot \frac{dy}{dx}$$

If you are still skeptical, try plugging in a test function, for example $y = x^2$.]

(c) Differentiate x^2y^3 with respect to y.
Soln. We have to use both the Product Rule and the Chain Rule:

$$\begin{aligned}\frac{d}{dy}(x^2y^3) &= x^2\frac{d}{dy}(y^3) + \frac{d}{dy}(x^2) \cdot y^3 \\ &= 3x^2y^2 + 2xy^3\frac{dx}{dy}\end{aligned}$$

Now that we understand how to use the Chain Rule, we can implicitly differentiate functions.

3.4.3 Implicit Differentiation

In implicit differentiation, we want to differentiate each term with respect to x. Some terms will end up with dy/dx terms, and some will not. You can then solve for dy/dx by collecting terms.

Let us demonstrate that this technique works by using it on a graph that we know very well by now: the circle.

1. Implicitly differentiate the equation $x^2 + y^2 = 25$, then find the equation of the tangent line to the circle at the point (3,4).
   Soln: Remembering the example we just did, we can proceed as follows:
$$\begin{aligned} x^2 + y^2 &= 25 \\ \frac{d}{dx}\left[x^2 + y^2\right] &= \frac{d}{dx}[25] \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y} \end{aligned}$$
   This is the first derivative we have seen that has a dependence on y in it. So for the point (3,4), the derivative says that the slope of the tangent line is
$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{3}{4}$$
   Thus the equation of the tangent line is
$$y - 4 = -\frac{3}{4}(x-3) \longrightarrow 3x + 4y = 25$$

2. Now, let's rewrite the implicit function explicitly and then differentiate, just to show that implicit differentiation gives the same result. We solve for y in terms of x:
$$\begin{aligned} x^2 + y^2 &= 25 \\ y &= \pm\sqrt{25 - x^2} \end{aligned}$$
   So the explicit equation of a circle consists of two equations for y, with the positive equation corresponding to the top half. Now we can differentiate directly, using the Chain Rule:
$$\begin{aligned} y &= \sqrt{25 - x^2} = \left(25 - x^2\right)^{1/2} \\ y' &= \frac{1}{2}\left(25 - x^2\right)^{-1/2} \cdot (-2x) \\ &= -\frac{x}{\sqrt{25 - x^2}} = -\frac{x}{y} \end{aligned}$$
   This agrees with the previous example.

3. Implicitly differentiate the function
$$y^2(4 - x^2) = (x^2 + 4y - 4)^2$$
   Soln: This example is more complicated than the previous one, as it will involve the Chain Rule multiple times. As always, don't panic! Use the rules you know to peel off one layer of the expression at a time.
$$\begin{aligned} \frac{d}{dx}\left[y^2(4 - x^2)\right] &= \frac{d}{dx}\left[(x^2 + 4y - 4)^2\right] \\ \underbrace{\frac{d}{dx}\left[y^2\right](4 - x^2) + y^2\frac{d}{dx}[4 - x^2]}_{\text{Product Rule}} &= \underbrace{2(x^2 + 4y - 4)\frac{d}{dx}[x^2 + 4y - 4]}_{\text{Chain Rule}} \\ 2y\frac{dy}{dx}(4 - x^2) + y^2(-2x) &= 2(x^2 + 4y - 4)\left(2x + 4\frac{dy}{dx}\right) \\ (8y - 2x^2y)\frac{dy}{dx} - 2xy^2 &= 4x^3 + 16xy - 16x + (8x^2 + 32y - 32)\frac{dy}{dx} \\ (-8x^2 - 24y - 2x^2y + 32)\frac{dy}{dx} &= 4x^3 + 16xy - 16x + 2xy^2 \\ \frac{dy}{dx} &= \frac{2x^3 + 8xy + xy^2 - 8x}{-4x^2 - 12y - x^2y + 16} \end{aligned}$$
   This curve is called a *bicorn*.

4. We can use implicit differentiation to prove the Power Rule for rational exponents. Recall we have already proved the rule for integers (excluding $x = 0$).

   Specifically, let p, q be integers. We want to find the derivative of $x^{p/q}$ and show that it follows the Power Rule. We can use the Power Rule for integers, as we have already proved that.

   Proof: To do this, let $y = x^{p/q}$, get rid of the fractional exponent, and then implicitly differentiate as follows:

$$y = x^{p/q} \longrightarrow y^q = x^p$$

$$\begin{aligned}
\frac{d}{dx}(y^q) &= \frac{d}{dx}(x^p) \\
qy^{q-1}\frac{dy}{dx} &= px^{p-1} \\
\frac{dy}{dx} &= \frac{p}{q}x^{p-1}y^{-(q-1)} \\
&= \frac{p}{q}x^{p-1}x^{-(q-1)\cdot p/q} \qquad \text{(substitute } y = x^{p/q}\text{)} \\
&= \frac{p}{q}x^{p-1}x^{-p+p/q} = \frac{p}{q}x^{p/q-1}
\end{aligned}$$

   In conclusion, we have shown that for $y = x^{p/q}$, the derivative is

$$y' = \frac{dy}{dx} = \frac{p}{q}x^{p/q-1}$$

   which proves the Power Rule for rational exponents. ■