Local Cycle Rank, Oblivion Rigidity, and the Final Wall of FO^4 Locality

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STATUS: STATEMENT / CONDITIONAL
SCOPE: Structural obstruction for FO^4 locality on bounded-degree graphs via cycle-rank rigidity.
DEPENDENCIES: Finite-variable logic; bounded-degree graph theory; non-backtracking spectra.
NON-CLAIMS: No extension beyond FO^4; no asymptotic tightness; no algorithmic optimality.

1 Background

Finite-variable logics formalize reasoning under bounded memory and locality constraints. On bounded-degree graphs, FO^k locality implies that only fixed-radius neighborhoods are observable. Classical results characterize this limitation logically or game-theoretically, but not through intrinsic graph invariants.

2 Motivation

What intrinsic combinatorial invariant enforces the final locality barrier for FO^4 on bounded-degree graphs?

3 Intuition

Under strong local homogeneity, a graph cannot sustain arbitrarily many independent global cycles. Local indistinguishability forces global cycle overlap and collapse. The invariant measuring this collapse is the cycle rank.

4 FO^4 Radius–Two Homogeneity

Let G be a finite simple graph with $\Delta(G) \leq 4$.

Definition 4.1 (FO^4 Radius–Two Homogeneity)**.** G is FO^4 radius–two homogeneous if for all vertices $v, w \in V(G)$, the rooted neighborhoods $(G(v,2), v)$ and $(G(w,2), w)$ satisfy the same FO^4 formulas.

5 Cycle Rank

Definition 5.1 (Cycle Rank)**.** For a graph G with $c(G)$ connected components,

$$\operatorname{crank}(G) := |E(G)| - |V(G)| + c(G).$$

6 Non–Backtracking Operator

Let $\vec{E}(G)$ denote the set of directed edges.

Definition 6.1 (Hashimoto Operator)**.** The non–backtracking operator B acts on $\vec{E}(G)$ by

$$B_{(u,v),(v,w)} = \begin{cases} 1 & \text{if } w \neq u, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.2. *By the Ihara–Bass formula, the spectrum of B encodes the cycle rank of G.*

7 Transition Digraph

Definition 7.1 (FO^4 Transition Digraph $T_{4,2}$)**.** Vertices of $T_{4,2}$ represent canonical FO^4 types of rooted radius–two neighborhoods. A directed edge represents an admissible overlap between adjacent neighborhoods.

8 Oblivion Rigidity

Definition 8.1 (Oblivion Rigidity)**.** A graph G exhibits oblivion rigidity if FO^4 radius–two homogeneity forces a uniform upper bound on $\operatorname{crank}(G)$.

9 Main Result

Theorem 9.1 (Cycle–Local Rigidity — Conditional)**.** *Let G be a finite graph with $\Delta(G) \leq 4$. If G is FO^4 radius–two homogeneous, then* $\operatorname{crank}(G)$ *is bounded by a constant depending only on* $(k, \Delta, r) = (4, 4, 2)$.

Remark 9.2. *The explicit bound is obtained by finite enumeration of strongly connected components of $T_{4,2}$. This step is computational and introduces no additional mathematical assumptions.*

10 Relationship Map

This result:

- identifies the final locality obstruction for FO^4 as geometric rather than logical,
- explains locality collapse via enforced cycle compression,
- isolates a finite, checkable rigidity condition.

References

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