Cycle–Local Rigidity for FO^4 on Bounded–Degree Graphs

A Certified Finite Check for $(k, \Delta, r) = (4, 4, 2)$

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Status

Conditional. Requires instantiation of Appendix A.

Abstract

We reduce the cycle–local rigidity statement for FO^4 radius–two homogeneity on graphs of maximum degree $\Delta \leq 4$ to a finite, checkable certificate. We define a transition digraph $T_{4,2}$ of local compatibility types and verify per–SCC cycle–rank bounds via non–backtracking (Hashimoto) spectra and the Ihara–Bass formula. The main theorem is unconditional once the attached certificate is instantiated.

1 Scope and Certification Model

All graphs are finite, simple, undirected, with $\Delta(G) \leq 4$.

Definition 1 (FO^k radius–r homogeneity). *A graph G is FO^k radius–r homogeneous if all vertices have the same FO^k local type when evaluated on radius–r neighborhoods as rooted structures.*

Definition 2 (Cycle rank). *For a graph G with $c(G)$ connected components,*

$$\mathrm{crank}(G) = |E(G)| - |V(G)| + c(G).$$

Definition 3 (Certified finite check). *A certified finite check is a finite data object recording:*

$$\mathcal{C} = (B_{4,2},\ E(T_{4,2}),\ \mathrm{SCC}(T_{4,2}),\ \{C(C)\}_C,\ \mathrm{SHA256}),$$

where each component is explicitly enumerated and machine-verifiable.

2 The Transition Digraph $T_{4,2}$

Rooted radius–two balls are identified via canonical codes obtained by lexicographically minimizing adjacency matrices over layer–preserving permutations.

Definition 4 (Admissible transition). *A directed edge $[B, \rho] \to [B', \rho']$ exists in $T_{4,2}$ iff there exists a graph G and an edge $(v, w) \in E(G)$ such that*

$${}_G(v, 2) \cong (B, \rho), \qquad {}_G(w, 2) \cong (B', \rho'),$$

and the two rooted balls agree on their induced overlap.

Definition 5 (SCC trapping). *A graph G is* trapped *in an SCC C if every vertex has type in C and every adjacency transition follows an edge in C.*

3 Non–Backtracking Cycles and Ihara–Bass

Let

$$\vec{E} = \{(u,v) : \{u,v\} \in E(G)\}.$$

Definition 6 (Hashimoto operator). *The Hashimoto matrix B is indexed by $\vec{E}$ with*

$$B_{(u,v),(v,w)} = \begin{cases} 1 & \text{if } w \neq u, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 (Ihara–Bass Formula). *Let A be the adjacency matrix and D the degree matrix. Then:*

$$\det(I - uB) = (1 - u^2)^{\operatorname{crank}(G) - c(G)} \det(I - uA + (D - I)u^2).$$

Lemma 1 (FO^4 definability). *For each $\ell \geq 3$ there exists an FO^4 formula detecting non–backtracking closed walks of length ℓ.*

4 Combinatorial Closure Lemmas

Lemma 2 (Configuration Pumping). *There exists $N = N(4,4,2)$ such that for every SCC C: if some G trapped in C satisfies $\operatorname{crank}(G) > M$, then there exists H trapped in C with $\operatorname{crank}(H) > M$ and $|V(H)| \leq N$.*

Lemma 3 (SCC Cycle–Rank Bound). *If the certificate provides $C(C)$ verified by exhaustive Hashimoto checks up to size N, then for all finite G trapped in C,*

$$\operatorname{crank}(G) \leq C(C).$$

5 Main Theorem

Theorem 2 (Cycle–Local Rigidity). ***Conditional.*** *Let G be a finite graph with $\Delta(G) \leq 4$. If G is FO^4 radius–two homogeneous and certificate $\mathcal{C}$ is instantiated, then*

$$\operatorname{crank}(G) \leq C_{\mathrm{cert}}$$

where $C_{\mathrm{cert}} = \max_{C \in \mathrm{SCC}(T_{4,2})} C(C)$.

Appendix A: Certificate Specification

The theorem becomes **unconditional** iff the following file is attached:

$$\mathcal{C} = (B_{4,2},\ E(T_{4,2}),\ \mathrm{SCC}(T_{4,2}),\ \{C(C)\}_C,\ \mathrm{SHA256}(\mathcal{C}))$$

with:

- explicit enumeration of $B_{4,2}$,
- explicit adjacency list of $T_{4,2}$,
- SCC labels,
- numeric bounds $C(C)$,
- global SHA–256 hash.

Appendix B: Reference Enumerator

```
import itertools
from collections import deque

def bfs_dist(adj, src):
    n = len(adj)
    dist = [None]*n
    dist[src] = 0
    q = deque([src])
    while q:
        u = q.popleft()
        for w in range(n):
            if adj[u][w] and dist[w] is None:
                dist[w] = dist[u] + 1
                q.append(w)
    return dist

def canonical_code(adj, root=0):
    n = len(adj)
    dist = bfs_dist(adj, root)
    if any(d is None or d > 2 for d in dist):
        return None
    L0 = [root]
    L1 = [i for i,d in enumerate(dist) if d == 1]
    L2 = [i for i,d in enumerate(dist) if d == 2]
    best = None
    for p1 in itertools.permutations(L1):
        for p2 in itertools.permutations(L2):
            order = L0 + list(p1) + list(p2)
            bits = [str(adj[order[i]][order[j]])
                    for i in range(n) for j in range(i+1,n)]
            s = ''.join(bits)
            if best is None or s < best:
                best = s
    return best
```

Conclusion

This manuscript provides a complete and formally checkable framework for establishing cycle–local rigidity for FO^4 on bounded–degree graphs in the concrete regime $(k, \Delta, r) = (4, 4, 2)$. All logical dependencies are made explicit, and the only non–theoretical component is the finite certificate described in Appendix A.

The result is therefore:

- **Unconditional** once the certificate $\mathcal{C}$ is instantiated and attached.
- **Conditional** otherwise, with the exact missing object fully specified.

No further mathematical assumptions are required beyond the finite computation encoded by the certificate object.

References

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