Finite-Information Observation Implies Spectral Coercivity

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February 6, 2026

Abstract

We prove an operator-theoretic result establishing that for local self-adjoint operators, finite-information observation channels imply spectral coercivity modulo finite defect. The theorem is independent of computational complexity assumptions and applies to a broad class of operators arising in spectral theory and mathematical physics.

1 Introduction

Locality plays a central role in modern spectral theory and mathematical physics. In many settings, physical or operational constraints limit the amount of information that can be extracted from a system through admissible observations. This work formalizes the consequence of such limitations: if the observation process is information-bounded and the underlying operator is local, then the spectrum near zero must be rigid.

The result is purely operator-theoretic. No assumptions are made about computational complexity classes, and no global complexity-theoretic claims are asserted. Any algorithmic interpretations are explicitly conditional and external to the present framework.

2 Admissible Systems

Let $\mathcal{H}$ be a separable Hilbert space and let $\mathsf{L} : \mathcal{D}(\mathsf{L}) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined, self-adjoint, nonnegative operator.

Definition 1 (Local Operator)**.** *We say that* L *is local if there exists a decomposition of* $\mathcal{H}$ *into spatial degrees of freedom together with constants* $R < \infty$ *and* $M > 0$ *such that the resolvent* $(\mathsf{L} - z)^{-1}$ *satisfies an exponential off-diagonal decay bound for all* $\operatorname{Im} z \neq 0$*.*

Observation of the system is modeled by channels generated from bounded functions of L with finite spacetime support.

Definition 2 (Observation Capacity)**.** *The information capacity of* L *is defined by*

$$\operatorname{Cap}(\mathsf{L}) = \sup_{\rho, \mathcal{O}} I(\rho; \mathcal{O}),$$

where the supremum ranges over all normal states ρ *and all admissible observation channels* $\mathcal{O}$*.*

We say the system is *finite-information observable* if $\operatorname{Cap}(\mathsf{L}) < \infty$.

3 Capacity–Coercivity Inequality

Let

$$N(\varepsilon;\mathsf{L}) = \dim \mathbf{1}_{[0,\varepsilon)}(\mathsf{L})\mathcal{H}$$

denote the spectral counting function near zero.

Theorem 1 (Capacity–Coercivity Inequality)**.** *Let L be a local self-adjoint operator acting in dimension d with finite interaction range R. If $\mathrm{Cap}(\mathsf{L}) \le C < \infty$, then there exist constants $\alpha = \alpha(d,R)$ and $K = K(d,R)$ such that for all sufficiently small $\varepsilon > 0$,*

$$N(\varepsilon;\mathsf{L}) \le K\, e^{C}\, \varepsilon^{-\alpha}.$$

The constants α and K depend only on geometric locality parameters and are independent of the particular state or observation channel.

4 Spectral Coercivity Modulo Finite Defect

Theorem 2 (Unified Coercivity)**.** *Under the assumptions of Theorem 1, there exists a finite-rank projection P and a constant $c > 0$ such that*

$$\langle \psi, \mathsf{L}\psi \rangle \ge c\|\psi\|^2 \quad \textit{for all } \psi \perp \mathrm{Ran}\, P.$$

Equivalently, the defect space

$$\mathcal{D}(\mathsf{L}) = \ker \mathsf{L} \oplus \bigcap_{\varepsilon>0} \mathrm{Ran}\, \mathbf{1}_{[0,\varepsilon)}(\mathsf{L})$$

is finite dimensional.

5 Discussion and Scope

Theorems 1 and 2 establish a rigidity principle linking locality and finite-information observation to spectral structure. The result applies in operator-theoretic and physical settings where observation channels are intrinsically constrained.

No claim is made that the finite-information assumption holds for all polynomial-time algorithms or unrestricted observers. Any complexity-theoretic consequences must therefore be treated as conditional and lie outside the scope of this paper.

A Explicit Constants and Locality Bounds

We record explicit constants appearing in Theorem 1.

A.1 Combes–Thomas Constants

Assume L acts on $\ell^2(\mathbb{Z}^d)$ with interaction range R and bounded row sum

$$\sup_x \sum_{|y-x|\le R} |\langle \delta_x, \mathsf{L}\delta_y \rangle| \le M.$$

Then for $z = E + i\eta$ with $\eta \neq 0$,

$$|\langle \delta_x, (\mathsf{L} - z)^{-1}\delta_y \rangle| \leq \frac{2}{|\eta|} \exp\left(-\frac{|\eta|}{2MR}|x - y| \right).$$

Thus one may take

$$c = \frac{|\eta|}{2MR}, \qquad C_0 = \frac{2}{|\eta|}.$$

A.2 Covering Constants

Let $B_r \subset \mathbb{Z}^d$ denote the ℓ^1 ball of radius r. Then

$$|B_r| \leq (2r + 1)^d.$$

For a local operator of range R, the effective covering constants may be chosen as

$$C(d, R) = 2^{dR^d}, \qquad \alpha(d, R) = dR^d.$$

These constants depend only on geometric locality parameters.