1 The Local–Global Information Barrier (FO^k Formulation)

1.1 FO^k-Definable Refinement Sequences

[FO^k-definable refinement sequence] Fix integers $k \geq 1$ and $r \geq 0$. An *FO^k-definable refinement sequence* on a graph $G = (V, E)$ is a sequence of colorings

$$\chi_0, \chi_1, \dots, \chi_T : V \to C$$

for some finite color set C, such that for each step $t < T$ there exists a first-order formula

$$\varphi_t(x; \bar{y}) \in \mathrm{FO}^k$$

and a fixed encoding of the current colored radius-r neighborhood of x into parameters $\bar{y}$ so that the update rule is *uniform and isomorphism-invariant*:

$$\chi_{t+1}(v) = F_t\big(\mathrm{type}_r^{\chi_t}(G, v)\big),$$

where $\mathrm{type}_r^{\chi_t}(G, v)$ denotes the rooted isomorphism type of the radius-r neighborhood of v in G with vertices labeled by χ_t, and F_t is induced by φ_t (independent of G and v).

1.2 Global Invariants Beyond FO^k

[Global invariant (relative to FO^k)] A *global invariant* (relative to FO^k) is an isomorphism-invariant graph property

$$P : \mathcal{G} \to \{0, 1\}$$

such that P is *not* definable by any FO^k sentence, i.e.

$$P \notin \mathrm{FO}^k.$$

Equivalently, there exist graphs G, H with $G \equiv_{\mathrm{FO}^k} H$ but $P(G) \neq P(H)$.

1.3 Neighborhood Distributions

[r-neighborhood distribution] For a graph $G = (V, E)$ and radius $r \geq 0$, let $\mathcal{N}_r(G, v)$ denote the rooted radius-r neighborhood of v. The *r-neighborhood distribution* of G is the multiset (or empirical measure)

$$\mu_r(G) := \big\{\!\!\big\{\, [\mathcal{N}_r(G, v)] \;:\; v \in V \,\big\}\!\!\big\},$$

where $[\mathcal{N}_r(G, v)]$ is the rooted isomorphism type.

1.4 CFI Local Homogeneity and FO^k Indistinguishability

[CFI barrier: bounded-degree, connected, FO^k-indistinguishable pairs] For every fixed $k \geq 1$, there exists a family of pairs of graphs

$$(G_k, H_k)$$

such that:

1. G_k and H_k are finite, connected, and have maximum degree bounded by an absolute constant;
2. $G_k \not\cong H_k$;
3. $G_k \equiv_{\mathrm{FO}^k} H_k$ (equivalently, k-WL does not distinguish them);
4. for every fixed radius $r \geq 0$,

$$\mu_r(G_k) = \mu_r(H_k).$$

1.5 Interpretation Closure

[FO^k-interpretation] An *FO^k-interpretation* $\mathcal{I}$ is a uniform construction that, from an input structure G, defines (by FO^k formulas) a domain, relations, and equality on the output structure $\mathcal{I}(G)$.

[No gain under composition of FO^k interpretations] If $G \equiv_{\mathrm{FO}^k} H$ and $\mathcal{I}$ is any finite composition of FO^k-interpretations, then

$$\mathcal{I}(G) \equiv_{\mathrm{FO}^k} \mathcal{I}(H).$$

1.6 The Local–Global Information Barrier

[Local–Global Information Barrier (FO^k form)] Fix $k \geq 1$. There exist finite connected bounded-degree graphs G_k, H_k such that:

1. no FO^k-definable refinement sequence distinguishes G_k from H_k (at any finite depth);
2. yet $G_k \ncong H_k$;
3. hence there exists a global invariant P (relative to FO^k) with

$$P(G_k) \neq P(H_k).$$

Proof. Let (G_k, H_k) be as in the CFI barrier theorem. Since $G_k \equiv_{\mathrm{FO}^k} H_k$, any FO^k-definable procedure (in particular any FO^k-definable refinement sequence, and any finite composition of FO^k-interpretations) produces indistinguishable outcomes on G_k and H_k. Therefore no FO^k-definable refinement can separate them. However $G_k \ncong H_k$, so some isomorphism-invariant property separates them; by definition such a property is not FO^k-definable and is a global invariant relative to FO^k. $\square$

[Separation of knowledge modes] For fixed k,

$$\mathrm{FO}^k\text{-extractable knowledge} \subsetneq \text{isomorphism-invariant knowledge.}$$

In particular, there exist isomorphism-invariant properties not recoverable by any FO^k-definable refinement process, even on bounded-degree connected graphs.