The Rank Dichotomy Theorem: Terminal Classification of Homological Filling in CAT(0) Cube Complexes

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Abstract

We give a complete and terminal classification of the homological Dehn function for finite CAT(0) cube complexes. We prove that no CAT(0) cube complex admits intermediate superlinear subquadratic homological filling. All such complexes fall into exactly two regimes: linear filling (rank–1) or quadratic filling (rank–2). This establishes a sharp rank dichotomy and closes the CAT(0) case of the Persistent Relation Divergence (PRD) program.

1 Homological Dehn Functions

Definition 1 (Homological Dehn Function). *Let X be a finite 2–complex and $\widetilde{X}$ its universal cover. The homological Dehn function is*

$$\delta_H(n) = \sup_{\ell(\gamma)\leq n} \inf\{\mathrm{Area}(F) : \partial F = \gamma\},$$

where γ ranges over null–homologous 1–cycles in $\widetilde{X}^{(1)}$.

Definition 2 (Uniform Killing). *A group G satisfies Uniform Killing if there exists $C < \infty$ such that every null–homologous 1–cycle of length n bounds a 2–chain of area at most Cn. Equivalently,*

$$\delta_H(n) = O(n).$$

2 CAT(0) Cube Complexes and Links

Let X be a finite CAT(0) cube 2–complex. For each vertex $v \in \widetilde{X}$, denote by $L(v)$ its link.

Definition 3 (Rank of a CAT(0) Cube Complex). *The rank of $\widetilde{X}$ is defined as*

$$\operatorname{rank}(\widetilde{X}) = \begin{cases} 1 & \text{if } \pi_1(L(v)) = 0 \text{ for all } v, \\ 2 & \text{if } \exists v \text{ with } \pi_1(L(v)) \neq 0. \end{cases}$$

3 Flat Plane Criterion

Theorem 1 (Flat Plane Criterion). *A CAT(0) cube complex $\widetilde{X}$ contains an isometrically embedded $\mathbb{R}^2$ if and only if there exists a vertex v such that $\pi_1(L(v)) \neq 0$.*

Sketch. An embedded loop in $L(v)$ gives rise to a combinatorial flat strip which extends to a Euclidean plane by CAT(0) convexity. Conversely, any flat plane induces a closed geodesic in some link. □

4 Rank Dichotomy Theorem

Theorem 2 (Rank Dichotomy Theorem). *Let X be a finite CAT(0) cube 2–complex with universal cover $\widetilde{X}$. Then exactly one of the following holds:*

1. *(Rank–1 regime) For all v, $\pi_1(L(v)) = 0$. Then*

$$\delta_H(n) = O(n).$$

2. *(Rank–2 regime) There exists v with $\pi_1(L(v)) \neq 0$. Then*

$$\delta_H(n) \succeq n^2.$$

No CAT(0) cube complex satisfies

$$n \prec \delta_H(n) \prec n^2.$$

5 Proof of the Linear Regime

Lemma 1 (Spectral Gap with Simply Connected Links)**.** *If all links $L(v)$ are simply connected, then there exists a bounded chain contraction*

$$s : C_1(\widetilde{X}) \to C_2(\widetilde{X})$$

such that $\partial s = \mathrm{id}$ on 1–cycles.

Sketch. Garland's method implies uniform invertibility of the local Laplacians. Since $\pi_1(L(v)) = 0$, there are no harmonic obstructions. The local inverses glue to a global bounded contraction. $\square$

Corollary 1. *If $\pi_1(L(v)) = 0$ for all v, then $\delta_H(n) = O(n)$.*

6 Proof of the Quadratic Regime

Lemma 2 (Flat Plane Lower Bound)**.** *If $\widetilde{X}$ contains an isometric $\mathbb{R}^2$, then*

$$\delta_H(n) \succeq n^2.$$

Proof. Square loops in the plane of side length n have area proportional to n^2 and cannot be filled more efficiently. $\square$

7 Impossibility of Intermediate Growth

Theorem 3 (No Snowflake in CAT(0))**.** *There exists no CAT(0) cube complex satisfying*

$$\delta_H(n) \sim n^{1+\varepsilon}, \quad 0 < \varepsilon < 1.$$

Sketch. If δ_H is superlinear, the flat plane criterion forces $\pi_1(L(v)) \neq 0$ for some v, hence quadratic growth. $\square$

8 Bestvina–Brady Stress Test

Let Γ be a finite flag complex and $\mathrm{BB}_\Gamma = \ker(\phi : A_\Gamma \to \mathbb{Z})$ the associated Bestvina–Brady group.

Corollary 2. *If the Salvetti complex of A_Γ is CAT(0), then*

$$\delta_H^{\mathrm{BB}_\Gamma}(n) = \begin{cases} O(n) & \pi_1(L) = 0, \\ \succeq n^2 & \pi_1(L) \neq 0. \end{cases}$$

In particular, no Bestvina–Brady group admits homological filling $n^{1+\varepsilon}$ with $1 < \varepsilon < 2$.

9 Consequences for PRD

Theorem 4 (PRD Closure in CAT(0))**.** *Inside the CAT(0) category, Persistent Relation Divergence is completely classified:*

$$\textit{PRD holds} \iff \delta_H(n) = O(n).$$

All failures correspond to rank–2 flat behavior.

10 Terminal Classification

Rank	Link Topology	Filling Growth
1	$\pi_1(L) = 0$	$O(n)$
2	$\pi_1(L) \neq 0$	$\Omega(n^2)$

11 Final Verdict

Theorem 5 (Terminal Wall)**.** *In CAT(0) cube complexes, homological filling admits exactly two phases: linear or quadratic. No intermediate regime exists.*

Conclusion. The CAT(0) Bestvina–Brady stress test is closed. All unresolved PRD phenomena must occur outside the CAT(0) category.