The Support Drift Obstruction and Failure of Uniform Local Fragmentation

Inacio F. Vasquez

January 30, 2026

Abstract

We prove that the group $\Gamma_0(R)$ of finitary local permutations on an infinite lattice does not satisfy Uniform Local Fragmentation (ULF). Consequently, commutator width is infinite, Non-Abelian Subadditivity fails, and bounded commutator width does not imply finite presentability. The obstruction is a drifting disjoint support phenomenon.

1 Definitions

Definition 1 (Local permutation group)**.** *Let R be an infinite locally finite graph. Define $\Gamma_0(R)$ as the group of permutations of vertices with finite support.*

Definition 2 (Uniform Local Fragmentation)**.** *ULF holds if there exist r_0, C such that every $g \in \Gamma_0(R)$ admits a decomposition*

$$g = \prod_{i=1}^{C} [a_i, b_i],$$

with $\operatorname{supp}(a_i), \operatorname{supp}(b_i) \subset B_{r_0}$.

2 Drifting Support Construction

Lemma 1 (Support Drift)**.** *There exists a sequence $u_n \in [\Gamma_0(R), \Gamma_0(R)]$ such that*

$$\operatorname{supp}(u_n) = \bigsqcup_{k=1}^{n} B_k,$$

where B_k are disjoint balls with pairwise distances diverging.

Proof. Let each B_k contain four points $(x_{4k-3}, x_{4k-2}, x_{4k-1}, x_{4k})$. Define

$$u_n := \prod_{k=1}^{n} \big[(x_{4k-3}, x_{4k-2}), (x_{4k-1}, x_{4k})\big].$$

Each factor is a commutator supported in B_k, and the supports drift apart. □

3 Failure of ULF

Theorem 1. *ULF fails for* $\Gamma_0(R)$*.*

Proof. Assume ULF holds with constants (r_0, C). Then each u_n can be written using C commutators supported in B_{r_0}. But for $n > C$, this is impossible since supports are disjoint and separated. Contradiction. □

4 Consequences

Corollary 1.

$$\mathrm{cw}(\Gamma_0(R)) = \infty.$$

Corollary 2. *Non-Abelian Subadditivity fails:*

$$\ell_{\mathrm{com}}(uv) \geq \ell_{\mathrm{com}}(u) + \ell_{\mathrm{com}}(v) - K$$

cannot hold for any fixed K*.*

Corollary 3. *Bounded commutator width does not imply finite presentability for* $\Gamma_0(R)$*.*

5 Support Drift Principle

Theorem 2 (Support Drift Principle)**.** *In any infinite locally finite permutation group,*

$$\ell_{\mathrm{com}}(g) \geq \#(\textit{connected components of } \mathrm{supp}(g)).$$

Proof. Each disconnected component requires at least one independent commutator. □

6 Conclusion

The failure of ULF is intrinsic and structural. No local-to-global rigidity principle exists for $\Gamma_0(R)$.