Boundary Electric Control Lemma (Structural Target)

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Purpose

We record a local-to-boundary control inequality intended to convert local gauge-invariant fluctuations into boundary electric energy. This is a structural rigidity ingredient toward uniform electric coercivity.

Setup

Let $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^4$ be the periodic lattice.

Let $\mathcal{H}_L = L^2(\mathcal{A}_L/\mathcal{G}_L)$ be the gauge-invariant Hilbert space.

Let E_ℓ^a denote the electric field operator on link ℓ, and define the electric energy

$$H_L^{(E)} = \sum_\ell \sum_{a=1}^{3} (E_\ell^a)^2.$$

Let $B_R(x) \subset \Lambda_L$ be a cubic region of side length R centered at x.

Let $\partial B_R(x)$ denote the set of spatial links crossing the boundary of $B_R(x)$.

Local Fluctuation Functional

For $\psi \in \mathcal{H}_L$ define the local variance in $B_R(x)$ by

$$\mathrm{Var}_{B_R(x)}(\psi) = \|\psi - \mathbb{E}_{B_R(x)}\psi\|^2,$$

where $\mathbb{E}_{B_R(x)}$ denotes averaging over gauge orbits supported inside $B_R(x)$.

Boundary Electric Control Target

Lemma 1 (Boundary Electric Control Target)**.** *There exist constants $R \geq 1$ and $C > 0$ independent of L such that for all L and all gauge-invariant ψ,*

$$\mathrm{Var}_{B_R(x)}(\psi) \;\leq\; C \sum_{\ell \in \partial B_R(x)} \sum_{a=1}^{3} \|E_\ell^a \psi\|^2.$$

Interpretation

If true, this lemma states:

Local gauge-invariant fluctuations inside a region must be detected by nonzero electric derivatives on the boundary of that region.

Summing over x yields:

$$\mathcal{F}_L(\psi) \;\le\; C' \mathcal{E}_L(\psi),$$

which implies uniform spectral coercivity of $H_L^{(E)}$.

Status

This document records the inequality target only. No proof is claimed.