# Finite-Volume Spectral Domination Target (Structural)

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## Purpose

We record a single finite-volume coercivity/positivity inequality which, if proven uniformly in volume and along a continuum scaling window, closes the coercivity wall needed for a mass-gap route via transfer-matrix domination.

## Setup (Finite Lattice)

Let $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^4$ be the periodic 4D lattice. Let $\mathcal{H}_L = L^2(\mathcal{A}_L/\mathcal{G}_L)$ be the gauge-invariant Hilbert space of wavefunctions on link variables modulo gauge transformations.

Let $H_L$ be the Kogut–Susskind Hamiltonian (Euclidean time transfer-matrix generator) written schematically as

$$H_L = H_L^{(E)} + H_L^{(B)},$$

where $H_L^{(E)}$ is the electric (Lie-algebra Laplacian) part and $H_L^{(B)}$ is the magnetic (plaquette potential) part.

Denote by $\Omega_L$ the (normalized) ground state vector and by $P_L = |\Omega_L\rangle\langle\Omega_L|$ the ground-state projection.

## Target Inequality (Finite-Volume Domination)

**Definition 1** (Finite-Volume Spectral Domination Constant)**.** *For each $L$, define $\lambda_L$ as the best constant such that*

$$\langle \psi, H_L \psi\rangle \;\geq\; \lambda_L \,\|\psi\|^2 \quad \textit{for all } \psi \in \operatorname{Ran}(1-P_L).$$

*Equivalently,*

$$\lambda_L = \inf\big(\sigma(H_L)\setminus\{0\}\big).$$

**Theorem 1** (Finite-Volume Uniform Domination Target)**.** *There exist constants $\lambda_\star > 0$ and $L_0 \in \mathbb{N}$ such that for all $L \geq L_0$,*

$$\lambda_L \;\geq\; \lambda_\star.$$

## Strengthened Localized Form (Preferred)

Fix a reference block scale $R \geq 1$ and define a local region $B_R \subset \Lambda_L$ (e.g. a cube of side $R$). Let $P_{B_R}$ be the projection onto functions constant along gauge orbits inside $B_R$ (a local gauge-orbit averaging projector), and set $Q_{B_R} = 1 - P_{B_R}$.

**Theorem 2** (Localized Domination Target)**.** *There exist $R \geq 1$, $c_\star > 0$, and $L_0$ such that for all $L \geq L_0$ and all $\psi \in \mathcal{H}_L$,*

$$\langle \psi, H_L \psi \rangle \;\geq\; c_\star \, \|Q_{B_R} \psi\|^2.$$

## Bridge to Mass Gap (Structural)

If either target theorem holds uniformly along a scaling window and survives the continuum limit, then the induced continuum Hamiltonian has a strictly positive spectral gap:

$$\inf\big(\sigma(H) \setminus \{0\}\big) \;>\; 0.$$

This document records only the inequality target and does not claim a proof.