Explicit SU(2) Plaquette Barrier Lemma

Lemma (Explicit SU(2) Barrier). Let $U \in \mathrm{SU}(2)$ and define the Wilson plaquette observable

$$W(U) := 1 - \frac{1}{2}\mathrm{ReTr}(U).$$

If

$$\|U - I\| \geq \delta,$$

then

$$W(U) \geq \frac{\delta^2}{2}.$$

Proof. Every $U \in \mathrm{SU}(2)$ admits the parametrization

$$U = \cos\theta\, I + i \sin\theta\, \vec{n} \cdot \vec{\sigma},$$

where $\theta \in [0, \pi]$, $\vec{n} \in S^2$, and $\vec{\sigma}$ are the Pauli matrices.

The trace satisfies

$$\mathrm{Tr}(U) = 2\cos\theta.$$

Thus

$$W(U) = 1 - \cos\theta.$$

Using the identity

$$1 - \cos\theta = 2\sin^2(\theta/2),$$

we obtain

$$W(U) = 2\sin^2(\theta/2).$$

The operator norm distance satisfies

$$\|U - I\| = 2|\sin(\theta/2)|.$$

Therefore,

$$\|U - I\| \geq \delta \quad \Longrightarrow \quad 2|\sin(\theta/2)| \geq \delta,$$

which implies

$$|\sin(\theta/2)| \geq \frac{\delta}{2}.$$

Substituting,

$$W(U) = 2\sin^2(\theta/2) \geq 2\left(\frac{\delta}{2}\right)^2 = \frac{\delta^2}{2}.$$

□