Wilson Loop Area Law from Bakry–Émery Curvature: A Conditional Reduction

Abstract

We give a conditional reduction of an area law for Wilson loops in 4D lattice Yang–Mills to a single uniform analytic inequality on the gauge-orbit quotient diffusion, stated as a Bakry–Émery curvature lower bound (equivalently a uniform spectral gap / Poincaré inequality). We also isolate one additional standard correlation-to-area-law input as an explicit lemma. No unconditional confinement claim is made.

1 Setup

Fix a compact Lie group G and a finite 4D box $\Lambda_L \subset \mathbb{Z}^4$. Let E_{int} and V_{int} be the interior edges/vertices. Let

$$M_{a,L} := G^{E_{\text{int}}}, \qquad \mathcal{G}_{a,L} := \{g \in G^{V_{\text{int}}} : g|_{\partial\Lambda_L} = \mathbf{1}\}, \qquad Q_{a,L} := M_{a,L}/\mathcal{G}_{a,L}.$$

Let $S_{a,L}$ be the Wilson action (class function plaquette action) on $M_{a,L}$, $\mathcal{G}_{a,L}$-invariant, and set

$$d\mu_{a,L}(U) = Z_{a,L}^{-1} e^{-S_{a,L}(U)}\, d\,\mathrm{vol}_{M_{a,L}}(U), \qquad d\mu^Q_{a,L}([U]) := (\pi_*\mu_{a,L})([U]).$$

Lemma 1 (Quotient diffusion)**.** *Assume the $\mathcal{G}_{a,L}$-action on $M_{a,L}$ is free and proper (toron-free sector). Let $\mathcal{L}_{a,L}$ on $M_{a,L}$ be*

$$\mathcal{L}_{a,L} f := \Delta_{M_{a,L}} f - \langle \nabla S_{a,L}, \nabla f \rangle.$$

Then $\mathcal{L}_{a,L}$ is symmetric in $L^2(\mu_{a,L})$ and induces a symmetric generator $\mathcal{L}^Q_{a,L}$ on $Q_{a,L}$ with invariant measure $\mu^Q_{a,L}$.

2 Carré du champ and Bakry–Émery identity

On $Q_{a,L}$ define

$$\Gamma(f) := \|\nabla f\|^2, \qquad \Gamma_2(f) := \tfrac{1}{2}\big(\mathcal{L}^Q \Gamma(f) - 2\Gamma(f, \mathcal{L}^Q f)\big).$$

Lemma 2 (Bochner–Bakry–Émery). *For smooth f on $Q_{a,L}$,*

$$\Gamma_2(f) = \|\operatorname{Hess} f\|_{\mathrm{HS}}^2 + \big\langle (\mathrm{Ric} + \operatorname{Hess}(S_{a,L}))\, \nabla f, \nabla f \big\rangle,$$

where Ric *and* Hess *are taken on $Q_{a,L}$ and $S_{a,L}$ is viewed as a function on $Q_{a,L}$.*

3 The missing analytic inequality

Lemma 3 (Uniform Bakry–Émery lower bound / spectral gap). *There exist constants $\kappa > 0$ and $L_0 < \infty$ such that for all $a > 0$ and all $L \geq L_0$,*

$$\mathrm{Ric}_{Q_{a,L}} + \operatorname{Hess}_{Q_{a,L}}(S_{a,L}) \;\succeq\; \kappa I \quad \text{on } TQ_{a,L}.$$

Equivalently, $\Gamma_2 \geq \kappa\Gamma$ for $\mathcal{L}^Q_{a,L}$, hence the Poincaré inequality

$$\operatorname{Var}_{\mu^Q_{a,L}}(f) \;\leq\; \frac{1}{\kappa} \int \Gamma(f)\, d\mu^Q_{a,L}$$

and the spectral gap bound $\lambda_1(a,L) \geq \kappa$ hold uniformly in a, L.

4 Consequences of Lemma 3

Theorem 1 (Uniform L^2-mixing / exponential clustering). *Assume Lemma 3. Then there exists $m > 0$ (depending only on κ and local geometric constants) such that for all local gauge-invariant observables F, G supported in disjoint regions with separation R,*

$$\big|\mathbb{E}_{\mu_{a,L}}[FG] - \mathbb{E}_{\mu_{a,L}}[F]\mathbb{E}_{\mu_{a,L}}[G]\big| \;\leq\; C(F,G)\, e^{-mR}$$

uniformly in a and $L \geq L_0$.

5 Area law: additional standard input

A uniform spectral gap / exponential clustering alone does *not* imply an area law without an additional correlation-to-Wilson-loop mechanism.

Lemma 4 (Correlation-to-area-law mechanism). *There exist constants $A_0 < \infty$ and $c_* > 0$ such that for every rectangular loop C with minimal area $A(C)$,*

$$|\mathbb{E}_{\mu_{a,L}}[W(C)]| \;\leq\; \exp\big(-c_*\, N(C)\big), \qquad N(C) := \left\lfloor \frac{A(C)}{A_0} \right\rfloor,$$

uniformly in a and L for all loops contained at distance $\gg 1$ from $\partial\Lambda_L$.

Theorem 2 (Conditional area law). *Assume Lemma 3 and Lemma 4. Then there exists $\sigma > 0$ independent of a, L such that for all sufficiently large loops C,*

$$|\mathbb{E}_{\mu_{a,L}}[W(C)]| \;\leq\; e^{-\sigma A(C)}.$$

Proof. Lemma 4 gives

$$|\mathbb{E}[W(C)]| \le \exp\left(-c_* \left\lfloor \frac{A(C)}{A_0} \right\rfloor\right) \le \exp\left(-\frac{c_*}{A_0}A(C) + c_*\right).$$

Absorb the additive constant into a reduction of σ for $A(C)$ large. □

6 Boundary/sector equivalence (conditional)

Lemma 5 (Boundary condition equivalence)**.** *Assume Lemma 3. For any local gauge-invariant observable F supported in a fixed finite region $\Lambda_0 \Subset \mathbb{Z}^4$,*

$$\lim_{L\to\infty} \mathbb{E}_{\mu_{a,L}^{\mathtt{Dirichlet}}}[F] = \lim_{L\to\infty} \mathbb{E}_{\mu_{a,L}^{\mathrm{periodic}}}[F],$$

provided the periodic infinite-volume limit exists and both families satisfy the uniform clustering of Theorem 1.

7 Explicit IR test observable family

Fix a plaquette orientation p_0 and a smooth nonconstant χ. Define

$$F_L(U) = \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \chi\big(\operatorname{Tr} U_{p_0}(x)\big).$$

Theorem 3 (Uniform Poincaré bound for F_L)**.** *Assume Lemma 3. Then*

$$\int \Gamma(F_L)\, d\mu_{a,L}^{Q} \;\ge\; \kappa \operatorname{Var}_{\mu_{a,L}^{Q}}(F_L)$$

uniformly in L.

8 Numerical note (toy models)

Any reported L^0 lower bounds obtained by adding an explicit constant shift to a discrete Laplacian test an *effective mass term* model. Such tests do not by themselves establish Lemma 3 for the Wilson action on the gauge quotient, where gauge invariance forbids inserting an explicit mass term at the level of the action.

9 Conclusion

All confinement and mass-gap conclusions in this note are conditional on the single uniform analytic inequality Lemma 3, together with the explicitly stated additional input Lemma 4 needed to convert clustering information into an area law bound for Wilson loops.