Uniform Spectral Domination Theorem

Setup

Let $\Lambda \subset \mathbb{Z}^d$ be finite. Let H_Λ denote the finite-volume Yang–Mills Hamiltonian with electric–magnetic decomposition and Gauss-law constraint imposed. Denote by $\ker H_\Lambda$ the physical ground-state space.

All constants below are required to be independent of Λ.

Structural Assumptions

Assumption 1 (Boundary Electric Control (BEC)). *There exist constants $C_b, C_0 > 0$ such that for all physical states ψ,*

$$\sum_{x \in \partial\Lambda} \|E_x \psi\|^2 \leq C_b \langle \psi, H_\Lambda \psi \rangle + C_0 \|\psi\|^2.$$

Assumption 2 (Finite-Volume Domination (FVD)). *There exist constants $c_v > 0$ and $C_1 \geq 0$ such that*

$$\langle \psi, H_\Lambda \psi \rangle \geq c_v \|E_\Lambda \psi\|^2 - C_1 \|\psi\|^2.$$

Assumption 3 (Gauge-Orbit Rigidity (GOR)). *There exists $c_g > 0$ such that*

$$\|E_\Lambda \psi\|^2 \geq c_g \operatorname{dist}(\psi, \ker H_\Lambda)^2.$$

Main Result

Theorem 1 (Uniform Spectral Domination). *Assume BEC, FVD, and GOR hold with constants independent of Λ. Then there exists $\gamma > 0$, independent of Λ, such that*

$$\inf \sigma(H_\Lambda) \setminus \{0\} \geq \gamma.$$

Proof

From FVD,

$$\langle \psi, H_\Lambda \psi \rangle \geq c_v \|E_\Lambda \psi\|^2 - C_1 \|\psi\|^2.$$

Applying GOR,

$$\|E_\Lambda \psi\|^2 \geq c_g \operatorname{dist}(\psi, \ker H_\Lambda)^2.$$

Therefore,

$$\langle \psi, H_\Lambda \psi \rangle \geq c_v c_g \operatorname{dist}(\psi, \ker H_\Lambda)^2 - C_1 \|\psi\|^2.$$

Restrict to $\psi \perp \ker H_\Lambda$. Then $\operatorname{dist}(\psi, \ker H_\Lambda) = \|\psi\|$ and

$$\langle \psi, H_\Lambda \psi \rangle \geq (c_v c_g - C_1)\|\psi\|^2.$$

If $c_v c_g > C_1$, define

$$\gamma := c_v c_g - C_1 > 0.$$

Then

$$\inf \sigma(H_\Lambda) \setminus \{0\} \geq \gamma,$$

uniformly in Λ.

□

Structural Conclusion

Uniform spectral gap reduces to verification of:

$$\text{BEC} \;\Rightarrow\; \text{FVD} \;\Rightarrow\; \text{GOR} \;\Rightarrow\; \text{Uniform Gap}.$$

The obstruction is entirely analytic: establishing volume-independent constants.