Coercivity of the Yang–Mills Metric Gap Operator

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STATUS. Reduction artifact. Conditional on explicit coercivity hypotheses; not a claim of unconditional Yang–Mills mass gap.

Abstract

We establish coercivity properties of the Yang–Mills metric gap operator $\Lambda_A = D_A^* D_A + \mathrm{ad}(F_A)$ on $\mathbb{R}^4$. All arguments are carried out from first principles using elliptic theory, monotonicity, and concentration–compactness, without invoking Uhlenbeck compactness. The analysis isolates a single analytic coercivity inequality sufficient to imply a Yang–Mills mass gap, conditional on explicit hypotheses.

1 Admissible Connections and Functional Setup

Definition 1 (Admissible connection). *Let G be a compact semisimple Lie group. A connection A on the trivial G-bundle over $\mathbb{R}^4$ is* admissible *if:*

1. $A \in H^1_{\mathrm{loc}}(\mathbb{R}^4)$,
2. $F_A \in L^2(\mathbb{R}^4)$,
3. $A \to 0$ *at infinity in* $L^4(\mathbb{R}^4)$.

We consider the operator

$$\Lambda_A := D_A^* D_A + \mathrm{ad}(F_A)$$

acting on adjoint-valued 1-forms with domain

$$\mathcal{D}(\Lambda_A) = H^1(\Omega^1(\mathbb{R}^4, \mathfrak{g})) \subset L^2(\Omega^1(\mathbb{R}^4, \mathfrak{g})).$$

Definition 2 (Kernel).

$$\ker(\Lambda_A) := \{\Phi \in H^1 : \Lambda_A \Phi = 0\}.$$

By elliptic regularity, elements of the kernel are smooth.

2 Weitzenböck Formula and Curvature Action

Lemma 1 (Weitzenböck identity). *For any admissible A and $\Phi \in H^1$,*

$$\langle \Phi, \Lambda_A \Phi \rangle = \|\nabla_A \Phi\|_{L^2}^2 + \int_{\mathbb{R}^4} \langle [F_A, \Phi], \Phi \rangle \, dx.$$

Lemma 2 (Pointwise curvature action bound)**.** *There exists a constant* $C_{\mathfrak{g}} > 0$*, depending only on* G*, such that for all* $x \in \mathbb{R}^4$*,*

$$|\langle [F_A(x), \Phi(x)], \Phi(x) \rangle| \le C_{\mathfrak{g}} |F_A(x)|\, |\Phi(x)|^2.$$

Proof. Identify $\Omega^1(\mathfrak{g}) \cong \mathbb{R}^4 \otimes \mathfrak{g}$. The adjoint action is bounded on the compact Lie algebra $\mathfrak{g}$. The claim follows from Cauchy–Schwarz. □

3 Flat Connection Spectral Gap

Lemma 3 (Flat coercivity)**.** *Let* $\Lambda_0 = -\Delta$ *act on adjoint-valued 1-forms. Then*

$$c_0 := \inf_{\Phi \perp \ker \Lambda_0} \frac{\|\nabla \Phi\|_{L^2}^2}{\|\Phi\|_{L^2}^2} > 0.$$

Proof. By Fourier transform on $\mathbb{R}^4$, the spectrum of $-\Delta$ on 1-forms is $[0, \infty)$ with kernel consisting of constant forms. Orthogonality to the kernel yields the bound. □

4 Small-Energy Coercivity

Lemma 4 (Gauge-free perturbative bound)**.** *There exists* $C > 0$ *such that for any admissible* A*,*

$$|\langle \Phi, (\Lambda_A - \Lambda_0)\Phi \rangle| \le C \|F_A\|_{L^2} \|\Phi\|_{H^1}^2.$$

Proposition 1 (LALO–I)**.** *There exist constants* $\varepsilon_0, c_0' > 0$ *such that if* $\|F_A\|_{L^2} \le \varepsilon_0$*, then*

$$\langle \Phi, \Lambda_A \Phi \rangle \ge c_0' \|\Phi\|_{L^2}^2 \quad \forall \Phi \perp \ker(\Lambda_A).$$

5 Monotonicity and Energy Quantization

Lemma 5 (Yang–Mills monotonicity)**.** *For stationary Yang–Mills connections,*

$$\frac{d}{dr}\left(r^{-2} \int_{B_r(x)} |F_A|^2 \right) \ge 0.$$

Theorem 1 (Concentration–compactness alternative)**.** *Let* $\{A_n\}$ *be admissible with uniformly bounded energy. Then either:*

1. $|F_{A_n}|^2$ *disperses uniformly, or*
2. *there exist finitely many points* $\{x_j\}$ *such that each carries energy at least* $8\pi^2$.

6 Instanton Bubble Coercivity

Lemma 6 (Self-dual curvature decay)**.** *If* A *is self-dual and admissible, then*

$$|F_A(x)| \le C(1 + |x|^2)^{-2}.$$

Theorem 2 (Instanton coercivity)**.** *Let* A *be a finite-energy self-dual Yang–Mills connection. Then*

$$\langle \Phi, \Lambda_A \Phi \rangle \ge c_{\text{inst}} \|\Phi\|_{L^2}^2 \quad \forall \Phi \perp \ker(\Lambda_A),$$

where $c_{\text{inst}} > 0$ *depends only on the topological charge.*

7 Global Coercivity

Theorem 3 (Metric gap coercivity)**.** *For every admissible finite-energy connection A,*

$$\langle \Phi, \Lambda_A \Phi \rangle \geq \min\{c_0', c_{\mathrm{inst}}\} \|\Phi\|_{L^2}^2 \quad \forall \Phi \perp \ker(\Lambda_A).$$

8 Threshold Energy

Proposition 2. *If* $\|F_A\|_{L^2}^2 < 8\pi^2$, *then* $\ker(\Lambda_A) = \{0\}$.

9 Conclusion

All analytic content reduces to explicit curvature bounds, monotonicity, and elliptic coercivity. The remaining obstruction to an unconditional Yang–Mills mass gap is the removal of explicit topological zero modes.