A numerical tsunami simulation generated with the Finite Elements method

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[This report had to be done individually since my partner decided to drop the course at the last moment.]

Contents

1 Introduction

The physical world is described by partial differential equations (PDEs). From the explosion of a supernova to the movement of the electrons around the atom, the vast majority of the known physical phenomena can be accurately described by PDEs. Sadly, this type of equations are, in general, impossible to solve analytically. Therefore, we are forced to rely on numerical methods that allow us to approximate the solutions. One of these methods is the Finite Elements method, which will be studied in this report through the simulation of a tsunami.

Tsunamis are water waves usually originated in the deep sea, normally by a an earthquake, that be as high as 30 meters. Being able to model the propagation of a tsunami is very important in order to prevent flood damage and to be prepared in case a natural disaster occurs. Therefore, the goal here is to be able to simulate how the tsunami waves will travel through the sea surface of the Earth. To do this, we need to solve the equations (wave equation) for the water pressure u, with the corresponding BC. A tsunami is a very complex process, but the model presented here will be a simplification where we neglect aspects like the curvature of the Earth, the accurate shape of the continents or the loss of energy due to the friction with the bottom of the sea. A more adequate type of the equation for simulating tsunamis is known as the shallow water wave equation.

2 Mathematical model

Naturally, the mathematical model to simulate a tsunami is based on the wave equation. A tsunami is the propagation of a pressure wave, originated by some type of source, through the sea surface of the Earth. Thus, we are dealing with a non-homogeneous wave equation, described by:

$$-\nabla^2 u - k^2 u = f \qquad in\ \Omega \tag{1}$$

where u represents the pressure, $k = \frac{2\pi}{\lambda}$ is the wavev number and f is the source function. The domain of the PDE is the sea surface of the Earth, which will be called Ω. As a tsunami can only propagate through water, it is natural to ask for the tsunami wave to vanish when it reaches land: this can be expressed as the boundary condition (BC):

$$u = 0 \qquad in\ \Gamma_1 \tag{2}$$

Where Γ_1 represents the coast line of the Earth. In this report we will use a flat Earth model; therefore, some kind of BC must be imposed in the outer border of the Earth, which will be called Γ_2 (this can be seen as the edge of the world). Two types of BC will be explored for this boundary.

First, it can be reasonable to impose an absorbing BC in the edge of the world (Robin type of BC). This means that the waves that encounter Γ_2 will be absorbed and not reflected. An incoming wave that is solution of the homogeneous wave equation can be represented by $u = e^{-i\vec{k}\cdot\vec{x}}$. In addition, this wave is also solution of the equation $\frac{\partial u}{\partial n} + iks = 0$. Note that the wave $e^{+i\vec{k}\vec{x}}$, travelling in the opposite direction, does not fulfill this equation. Therefore the absorbing BC has the form:

$$\frac{\partial u}{\partial n} + iku = 0 \qquad in\ \Gamma_2 \tag{3}$$

The whole PDE+BC problem with absorbing BC can be expressed as:

$$\begin{cases} -\nabla^2 u - k^2 u = f \qquad in\ \Omega \\ u = 0 \qquad in\ \Gamma_1 \\ \frac{\partial u}{\partial n} + iku = 0 \qquad in\ \Gamma_2 \end{cases} \tag{4}$$

The second type of BC that will be studied for Γ_2 are periodic boundary conditions. These will be further explained in section 3.1.

3 Numerical method

In a first approximation we will simulate the tsunami in a flat earth and with a very simple mesh. The domain of the simulation is the sea area in the earth, which is enclosed by the coast line of the continents Γ_1 and, in the case of a flat earth, the border of the world Γ_2.

The first step toward a finite element approach of the problem is to get the weak formulation of the the PDE+BC problem, defined by:

$$-\nabla^2 u - k^2 u = f \qquad in\ \Omega \tag{5}$$

$$u = 0 \qquad in\ \Gamma_1 \tag{6}$$

$$\frac{\partial u}{\partial n} + iku = 0 \qquad in\ \Gamma_2 \tag{7}$$

To get the weak formulation we first multiply the equation above by a test function ϕ, for which we only require to be sufficiently smooth so that all the expressions below make sense. Then we integrate in the domain Ω:

$$\int_\Omega -\phi\nabla^2 u - \phi k^2 u - \phi f = 0 \tag{8}$$

Carrying out integration by parts:

$$\int_{\partial\Omega} -\phi\frac{\partial u}{\partial n} + \int_\Omega \nabla\phi\nabla u - \phi k^2 u - \phi f = 0 \tag{9}$$

where:

$$\int_{\partial\Omega} = \int_{\Gamma_1} + \int_{\Gamma_2} \tag{10}$$

Substituting in 9 we get :

$$\int_{\Gamma_1} -\phi\frac{\partial u}{\partial n} + \int_{\Gamma_2} -\phi\frac{\partial u}{\partial n} + \int_\Omega \nabla\phi\nabla u - \phi k^2 u - \phi f = 0 \tag{11}$$

The term associated with Γ_2 can be easily computed using the natural boundary condition $\frac{\partial u}{\partial n} = -iku\ in\ \Gamma_2$ (it is natural because it is integrated in the weak formulation organically, with no need of imposing it to the test functions). The boundary term associated with Γ_2 can only be dealt with if we enforce the test function ψ to be 0 in $\Gamma 1$. Therefore, the weak formulation of the problem is:

Find $u \in \Sigma = \{v : v\ is\ smooth\ enough\ and\ v = 0\ in\ \Gamma_1\}$ *such that* :

$$\int_{\Gamma_2} \phi(iku) + \int_\Omega \nabla\phi\nabla u - \phi k^2 u - \phi f = 0$$

$$\forall\psi \in \Sigma_0 = \{p : p\ is\ smooth\ enough\ and\ \psi = 0\ in\ \Gamma_1\}$$

So we need to impose the essential boundary condition $\phi = 0$ in Γ_1 to all the basis functions. We soon will see how can we do this.

We now proceed to apply the finite elements method to the weak problem we just stated. For that we will change our space of solutions from Σ to the subspace $\Sigma^n = \{u^n\ such\ that\ u^n = \sum_{i=1}^{n+m_2} c_i\phi_i\} \leq \Sigma$, where $\{1, ..., n\}$ comprises all the vertices in the boundary Ω and $\{n, ..., n + m_2\}$ all the vertices in the boundary Γ_2. Note that we are leaving out the vertices in the border Γ_1 to ensure that the solution verifies the essential boundary condition $u^n = 0$ in Γ_1. This means that our active vertices (vertices with basis functions associated) will only be sea vertices and the coast vertices.

We are restricting the solutions to a finite dimensional space use a discrete function u^n. The choice of the basis function is very important. In this case we'll use the typical linear triangular basis functions.

Plugging the expression $u^n = \sum_{i=1}^{n+m_2} c_i\phi_i$ in the weak formulation and changing $\phi \rightarrow \phi_i$ we get the Garlekin's equations:

$$\sum_{j=1}^{n+m_2} \left(\int_{\Gamma_2} ik\phi_i\phi_j + \int_\Omega \nabla\phi_i\nabla\phi_j - k^2\phi_i\phi_j \right) c_j = \int_\Omega \phi_i f \ , \qquad \forall\phi_i = 1, ..., n \tag{12}$$

$$\rightarrow \sum_{k=1}^{n+m_2} (S_{ij} + B_{ij}) \ c_j = b_i \tag{13}$$

This is a system of $n+m_2$ linear equations which can be expressed as $(S+B)\cdot c = b$, where S and B are $(n+m_2)$x$(n+m_2)$ matrix and c and b are $(n+m_2)$x1 vectors. These objects can be expressed as a sum of the element objects defined locally in each element k:

$$S_{ij} = \sum_{k=1}^{n} S_{ij}^k = \sum_{k=1}^{n} \int_{l_k} ik\phi_i\phi_j \tag{14}$$

$$B_{ij} = \sum_{k=n}^{n+m2} B_{ij}^k = \sum_{k=1}^{n+m_2} \int_{be_k} \nabla\phi_i\nabla\phi_j - k^2\phi_i\phi_j \tag{15}$$

$$b_i = \sum_{k=1}^{n} b_i^k = \sum_{k=1}^{n} \int_{l_k} \phi_i f \tag{16}$$

We need some sort of numerical method to compute these element integrals. We will use a second order Newton-Côtes quadrature approximation, where the basis functions in each element are interpolated through three points in the internal elements (the three vertices in the triangles) or two points in the boundary elements (the two extreme points). The basis functions in an element k can be defined locally in an element k as $\phi_i = \alpha_i + \beta_i x + \gamma_i y$ and $\phi_i(x_p) = \delta_{ip}$, being x_p one of the points used in the Newton-Côtes quadrature. To approximate the element integrals above we first approximate the integrand and the integrate. Using baryocentric coordinates, a function in a internal (boundary) element can be approximated by:

$$\phi_i \approx \sum_{p=1}^{3(2)} \lambda_p(x_p)\phi_i(x_p) = \lambda_i(x_i) \tag{17}$$

where the relation $\phi_i(x_p) = \delta_{ip}$ was used. Applying this approximation and using the Holand Bell's theorem to integrate the baryocentric coordinates we get the following expressions for the element integrals:

$$S_{ij}^k = \int_{l_k} ik\phi_i\phi_j \approx \frac{ik}{6}|be_k|(1+\delta_{ij}) \tag{18}$$

$$B_{ij}^k = \int_{be_k} \nabla\phi_i\nabla\phi_j - k^2\phi_i\phi_j \approx |l_k| \left((\beta_i\beta_j + \gamma_i\gamma_j) + \frac{k^2}{12}(1+\delta_{ij}) \right) \tag{19}$$

$$b_i^k = \int_{l_k} \phi_i f \approx f(\vec{x}_i)\frac{|l_k|}{3} \tag{20}$$

where $|l_k|$ represents the area of the internal triangular element and $|be_k|$ the length of the boundary element. Once these element integrals are computed, S_k, B_k need to be assembled in the general matrix S and b_k in the general vector b. In order to do this, we have to consistently assign a global index (which will represent the position in the general matrix/vector) to each of the local vertices in the elements (the function "localtoglobal" in the code). Then the system of linear equations $(S+B)\cdot c = b$ is solved and we get the coefficient vector c, thus also getting the approximate solution u^n.

3.1 Periodic boundary conditions

Obviously, the behaviour of the tsunami waves when imposing absorbing boundary conditions in the edge of the world is not realistic: the Earth does not have an edge! We have neglected the curvature of the earth. There is two ways around this issue:

- To create a 3D spherical mesh in Gmsh.
- To use a flat Earth model but to impose periodic BC in the edge of the world instead of absorbing BC.

Only the latter solution will be studied in this report. The periodic BC in a flat Earth represented by a square in the xy plane with $x \in (a, b)$, $y \in (c, d)$ are:

$$u(x, c) = u(x, d) \tag{21}$$

$$u(a, y) = u(b, y) \tag{22}$$

These are essential BC that need to be directly imposed to the basis function ϕ_j (also note that since we no longer consider absorbing BCs, we do not have the boundary term arising from those natural BC). The way to do this is to assign the same coefficients to both the vertices from the left and right border that have the same y coordinate (same for the top and bottom border vertices with the same x coordinate). Mathematically, the expression of a Finite Elements solution fulfilling periodic BC would be:

$$u^n = \sum_{j=1}^{n} c_j \phi_j + \sum_{j=n}^{n+m_2} c_j (\phi_j + \bar{\phi}_j) \tag{23}$$

where n are the interior vertices, m_2 are the vertices in the border (edge of the world) and ϕ_j and $\bar{\phi}_j$ are the basis functions associated with vertices that are respectively in the top and bottom (or left and right) border and have the same the x (y) coordinate. This type of periodic boundary conditions still does not accurately described a spherical geometry. The fact that a plane surface and a sphere are not isomorfic makes it impossible to accurately represent a spherical surface in a plane. Nevertheless, these BCs are probably more adequate for a toroidal geometry

In terms of code, we first need to retrieve the coordinates and the global indices of the vertices in the border. Each side of the border is called "top", "bottom", "left" and "right". We then need to make sure that the vertices in the top and bottom side that have the same coordinate are assigned with the same global index, by tweaking the "localtoglobal" function. After this we have to erase these vertices from the global matrix S and then add the corresponding solutions for these vertices manually (see code).

4 Results

The coding for this report was done in Julia and the software used to construct the meshes was Gmsh 4.6.

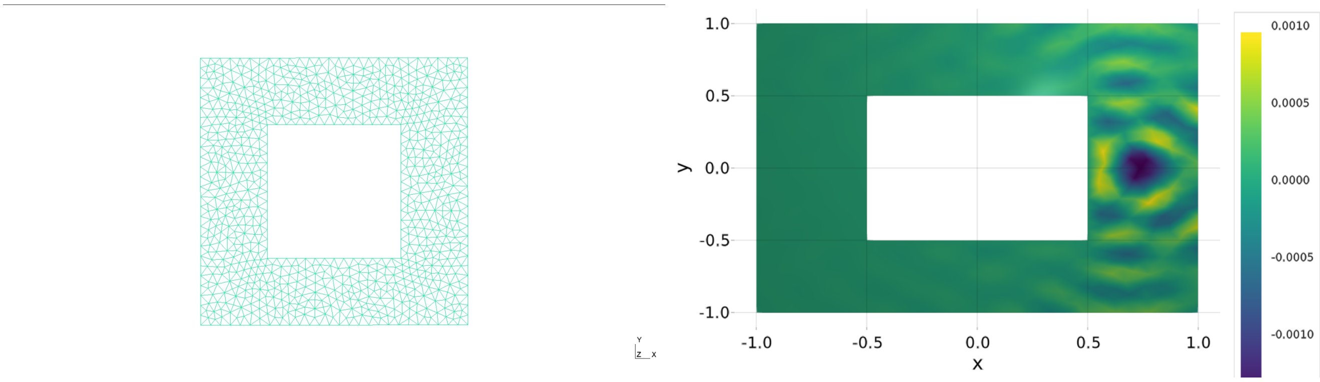
4.1 Absorbing boundary conditions

To plot the simulations we were asked to use a wavelength $\lambda = 4\ 000$ km. Given that the diameter of the Earth is around 13 000 km and that the plotting is represented in a 1x1 square, when normalizing we get a value of $k \approx 6\pi$, corresponding to a wavelength $\lambda = 1/3$. . The source function on the RHS of the PDE 5 is chosen to be a rapidly decaying gaussian function with the form:

$$f = e^{-\alpha\left[\cdot(x-x_0)^2+(y-y_0^2)\right]} \tag{24}$$

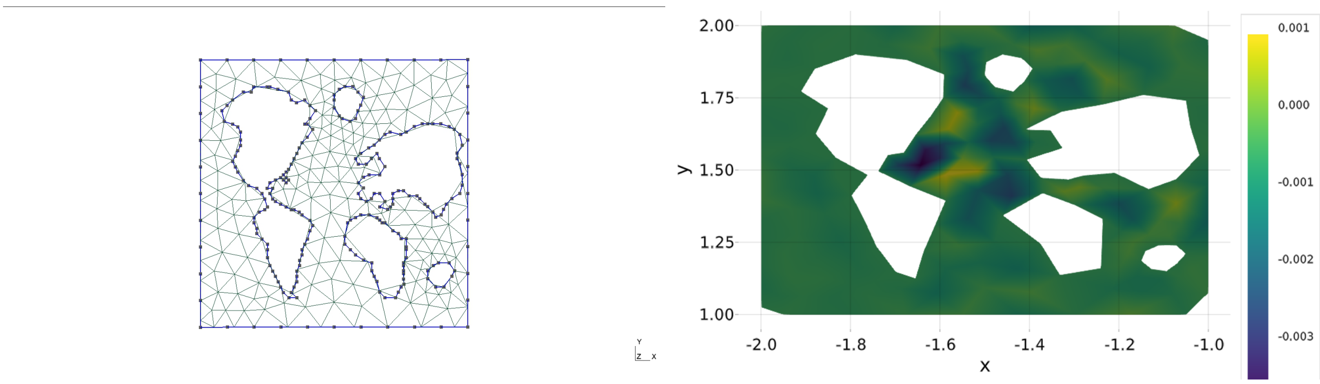
In a first approximation a very simple mesh was used, where the earth is considered flat and the continents and the sea are represented by rectangles. The mesh and the simulation are shown in figure 1. The inner square represents the continents and the sea surface is enclosed between the coast line and the outer border, which can be seen as the edge of the world.

One can easily note that the above representation of the continents is not very realistic. To improve the mesh, a more detailed outline of the continents can be done. Two versions of this more accurate mesh were generated, one with a bigger size of the internal elements (figure 2) and a second one with a finer partitioned mesh (figure 3). One can easily note that the latter predicts a more realistic continuous, realistic wave, as it should be since the grid is smaller.



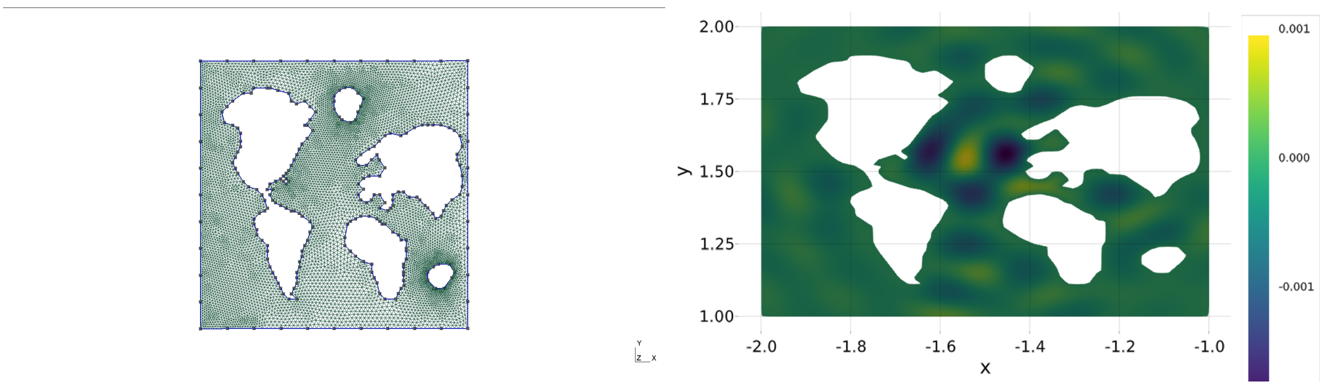
(a) Mesh (b) Tsunami representation

Figure 1: Simplest mesh approximation, where the continents are represented by the inner white square and the sea is the area enclosed in between the coast line and the outer border, which can be seen as the edge of the world. The source of the tsunami is. The parameters of the source function f are: $\alpha = 100$, $x_0 = 0.75$, $y_0 = 0$



(a) Mesh (b) Tsunami representation

Figure 2: Finer representation of the continents with a relatively big basic elements. The parameters of the source function f are: $\alpha = 100$, $x_0 = -1.5$, $y_0 = 1.5$

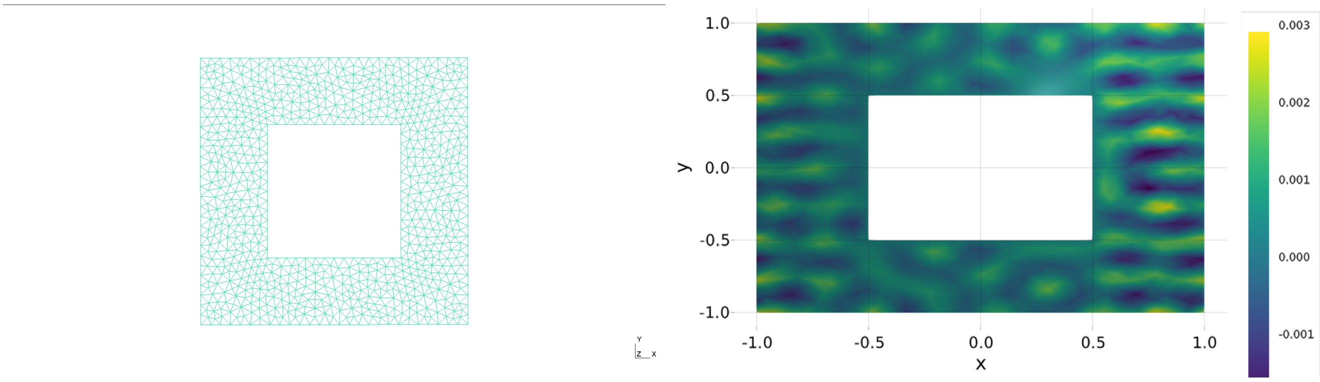


(a) Mesh (b) Tsunami representation

Figure 3: Same continent representation as in figure 2 but with a smaller basic element size.The parameters of the source function f are: $\alpha = 100$, $x_0 = -1.5$, $y = 1.5$

4.2 Periodic boundary conditions

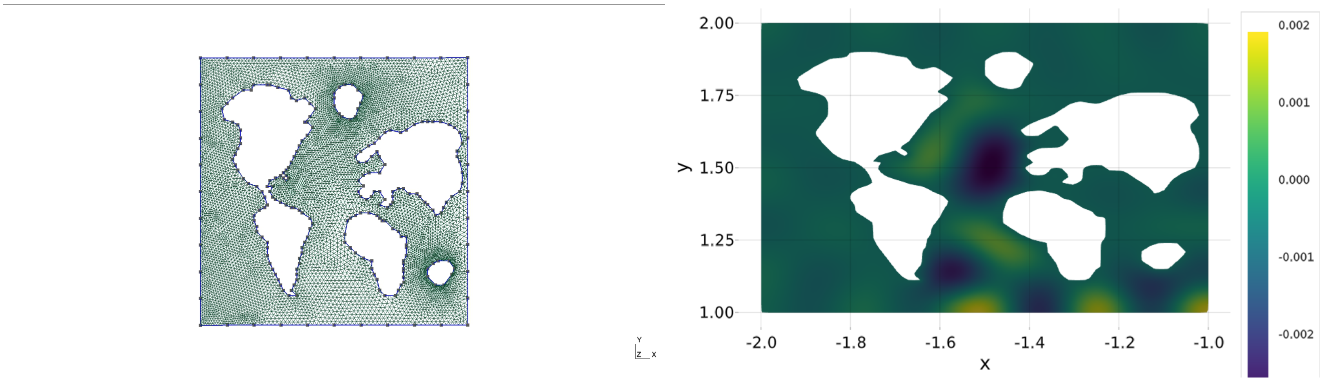
A more realistic simulation is presented now, with periodic BC in the the top/bottom and left/right sides. The simulation was done with two different meshes: a mesh representing the continents as a square 4 and the more realistic continent delimitation presented before 5. In both cases we can see that the wave pattern has been changed. In the case of figure 4, the wave pattern is quite symmetric due to the symmetry of the mesh itself. Also, since there is only one continent in the mesh, the waves can freely propagate and interfere in every point of the surface, hence that we see such a rapidly varying pattern. In the case of figure 5, continents stop the propagation of the waves so they attenuate sooner.



(a) Mesh

(b) Tsunami representation

Figure 4: Periodic BC in a mesh with the continents represented by the white square . The parameters of the source function f are: $\alpha = 100$, $x_0 = 0.75$, $y_0 = 0$



(a) Mesh

(b) Tsunami representation

Figure 5: Tsunami simulation with periodic boundary conditions. The parameters of the source function f are: $\alpha = 100$, $x_0 = -1.5$, $y_0 = 1.5$

5 Conclusions

In this report the simulation of a tsunami has been carried out. The surface of the Earth was represented by a squared plane, and the continents were represented with different levels of fidelity. The impact of the size of the grid elements of the mesh in the result of the simulation was also studied. It was found out that imposing periodic BC on the border of the mesh gives more realistic results than imposing absorbing BC. Still, the results given in this report are limited by the geometry of the used meshes. This suggest that a spherical mesh that takes in account the curvature of the Earth is very likely needed in order to get high fidelity results, specially studying in a very large portion of the surface of the earth as it was done here.