Thomas Jefferson Physics Olympiad

A high school physics contest

This exam covers concepts related to electromagnetism and superconductors and explores current insights into superconductor research and applications.

TJ Physics Team <tjhsstphysicsteam@gmail.com>

Electromagnetism and Superconductors

If you have any questions or concerns, please email TJ Physics Team.

Contents

Rules

Teams will have a total of four days to complete the TJPhO. All teams are required to submit their response with a cover page listing the team name and number, team member names, team email, and the date. Each submitted page should also have the problem number. All other formatting decisions are delegated to the teams themselves, with no one style favored over another. We suggest that teams use LaTeX for typesetting. The easiest way we've found to typeset LaTeX is with Overleaf.

There will be two types of problems on the online exam:

- **Background:** Background problems will test knowledge of classical physics and involve higher-level conceptual thinking and application of fundamental formulas and calculations.
- **Applied:** Applied problems will explore the applications of superconductors, use known concepts to generalize to higher systems, and combine knowledge from a variety of different topics.

Even if you are not able to fully solve a problem, please write up what you can. We want to see how you approached the problem, and partial credit may be awarded.

Collaboration

Students participating in the competition may only correspond with other members of their team. No other correspondence is allowed, including mentors, teachers, professors, and other students. Teams are not allowed to use online/print resources or post content on online forums asking questions related to the exam. We welcome teams to email us if there are any questions or concerns. Teams may use any computational resources they might find helpful, such as Wolfram Alpha/Mathematica, Matlab, Excel, or programming languages (C++, Java, Python, etc).

Submission

Teams must submit their solutions by email to tjhsstphysicsteam@gmail.com by the deadline 11:59 PM EST on April 10th. Late submissions will not be considered. Solutions should be written in English and submitted as a single PDF document with the .pdf extension. The email must contain "Submission" in the subject line. Only one person per team, the person who registered through the Google Form, should send the final submission.

Awards

The top five teams will receive awards.

1st place: $200, 3 electronic copies of *Superconductivity: A Very Short Introduction* by Stephen J. Blundell, 3 one-year subscriptions to WolframAlpha Notebook Edition

2nd place: $150, 3 electronic copies of *Superconductivity: A Very Short Introduction* by Stephen J. Blundell, 3 one-year subscriptions to WolframAlpha Notebook Edition

3rd place: $100, 3 electronic copies of *Superconductivity: A Very Short Introduction* by Stephen J. Blundell, 3 one-year subscriptions to WolframAlpha Notebook Edition

4/5th place: 3 electronic copies of *Superconductivity: A Very Short Introduction* by Stephen J. Blundell, 3 one-year subscriptions to WolframAlpha Notebook Edition

Sponsors

We are incredibly grateful to our sponsors, Dr. Adam Smith and Mr. Robert Culbertson, and Thomas Jefferson High School for Science and Technology for their support. We would also like thank our sponsors WolframAlpha, the Thomas Jefferson Partnership Fund, and MinutePhysics for making this Olympiad possible.

Introduction

In 1911, Dutch physicist Heike Kamerlingh Onnes was experimenting with low-temperature mercury when he noticed a strange property — when he cooled the mercury to the same temperature as liquid helium, the resistance completely disappeared. This new phenomenon called superconductivity emerged as an entire field of research. For the past century, physicists have been on the race to create substances that exhibit superconductivity at lower pressures and higher temperatures. Just this past October, superconductivity was discovered in a compound consisting of carbon, sulfur and hydrogen at a record-high temperature of 15° C. These breakthroughs in superconductivity open the door for physicists and engineers alike to develop advanced and impactful technologies.

The TJ Physics Team is proud to announce the third annual TJ Physics Olympiad (TJPhO)! We hope this contest will introduce you to the fascinating world of superconductivity and provide a good starting point to explore electromagnetism. As per the nature of superconductivity, there will be quite a bit of electromagnetism involved in solving certain problems. To accommodate those that may not be familiar with these subjects, we have included an overview of multivariable calculus and background problems to build an understanding of electromagnetism.

This contest covers a wide range in superconductor research, from the underlying principles to the current applications such as MRIs, quantum computers, and particle accelerators. We hope you have as much fun learning about superconductors as we had writing this contest.

Happy exploring!

TJ Physics Team Officers

Multivariable Calculus

This section is dedicated to the conceptual background of multivariable calculus necessary for the exam. You do not need to mathematically understand the topics in this section, as all problems may be done using single-variable calculus.

For a function $f(x, y)$, the partial derivative of f with respect to x is given by

$$\frac{\partial f}{\partial x} = \lim_{u \to x} \frac{f(u, y) - f(x, y)}{u - x}.$$

The idea is that all other variables except the variable that the derivative is being taken with respect to is held constant.

A vector field $\mathbf{F}(x, y, z)$ is a function that outputs a vector at every point in space. The del operator $\nabla = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is the vector differential operator used to define the divergence and curl. The divergence of a vector field $\nabla \cdot \mathbf{F}$ represents how much a vector field acts like a "source" or a "sink" at a given point. The curl of a vector field $\nabla \times \mathbf{F}$ represents the vector "circulation" at a certain point. One may also think of the divergence and curl as the dot and cross product between the del operator and $\mathbf{F}$ respectively. You do not need to calculate divergence and curl for this exam.

Line integrals are used to integrate a vector field $\mathbf{F}$ over a curve C, parameterized by $\mathbf{r}(t)$. You may have seen this when computing the work done to an object along a path. The line integral essentially takes infinitesimal steps along C, taking the dot product of the displacement vector with the vector field at each step. A line integral starting at point A and ending at point B is denoted by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r}.$$

If C is a closed loop, the line integral is denoted by

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Surface integrals are used to integrate a vector field $\mathbf{F}$ across a surface S by determining the flux of the field through the surface, or how much of $\mathbf{F}$ is going through it. This is done by taking a dot product of $\mathbf{F}$ with a normal vector to S at every point on the surface, and then integrating over both parameters. Essentially, the surface integral adds up all of the contributions of the vector field that go out of the surface, multiplied by the area of each small section of the surface. A surface integral is denoted by

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Note that line and surface integrals are difficult to calculate in general. All integrals of these two types on the exam may be simplified to a single-variable integral.

1 Background Problems

Electric and Magnetic Fields (20 pts.)

1 Charged particles, such as protons or electrons, will produce an electric field $\mathbf{E}$. By Coulomb's Law, the electric field of a particle with charge q at a position $\mathbf{r}$ relative to the particle is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}},$$

where ε_0 is the permittivity of free space, with $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C}^2 \cdot \mathrm{N}^{-1} \cdot \mathrm{m}^{-2}$. Electric current is the net charge flowing through a region in space per unit time. Moving charges will produce a magnetic field $\mathbf{B}$. By Ampere's Law, the line integral of $\mathbf{B}$ around a closed loop C is proportional to the current passing through C.

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I,$$

where μ_0 is the permeability of free space, with $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T} \cdot \mathrm{m} \cdot \mathrm{A}^{-1}$.

The force on a particle with charge q moving with velocity $\mathbf{v}$ through an electric and magnetic field is given by the Lorentz force $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$.

a) Two charges $-q$ and $2q$ are at $x = -a$ and $x = a$ respectively. A test charge is placed at $x = d$. Find d if the force on the test charge is zero.

b) Suppose an infinite straight wire has constant current I. Taking C to be a circular loop with radius r and appealing to symmetry, use Ampere's Law to determine the magnitude of the magnetic field at a distance r from the wire.

c) An electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are directed along the $+z$-axis. A mass m with charge q starts at the origin with velocity v along the $+x$ axis. Find the position of the particle as a function of time and describe the motion.

The Lorentz equation and Maxwell's equations form the basis of classical electrodynamics. According Maxwell's equations, time dependent electric and magnetic fields in a vacuum will interact such that they propagate with speed $c = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}} = 3 \times 10^8\ \mathrm{m} \cdot \mathrm{s}^{-1}$.

Conductors (20 pts.)

2 The electric potential V is defined as potential energy per unit charge. The electric potential difference ΔV is the work per unit charge needed to move a particle from point A to point B along any path.

$$\Delta V = V_B - V_A = -\int_A^B \mathbf{E} \cdot d\boldsymbol{\ell}.$$

A conductor allows charged particles to move freely within it. The ability to conduct electrical current is characterized by the conductivity σ. For a conductor with cross-sectional area A, Ohm's Law states that an electric field $\mathbf{E}$ will produce a current density $\mathbf{J} = \sigma\mathbf{E}$, where $J = \dfrac{I}{A}$. For an resistor with resistance R, an alternate and perhaps more familiar version of Ohm's Law states $I = \dfrac{V}{R}$, where V is the potential difference across the resistor.

a) Find the resistance of a conductor with cross-sectional area A, length L, and conductivity σ in terms of the given variables.

Even though there exists a net current flow for a conductor in an electric field, the electrons are moving randomly, which causes scattering. Electron scattering produces an drag force $F_{drag} = \dfrac{mv}{\tau}$, where m is the mass of the electron and τ is the average time between collisions.

b) After a long time, the electrons in a current will move with constant drift velocity v_d . Find v_d for a conductor placed in a uniform electric field $\mathbf{E}$.

c) Suppose the conductor has electron density n and conductivity σ. Show the average time between electron collisions is $\tau = \dfrac{m\sigma}{e^2 n}$.

Below a critical temperature, the resistance of an ideal superconductor is zero, allowing superconductors to carry current for years without noticeably decaying. This effect is known as supercurrent.

Circuits (30 pts.)

3 Voltage drop is defined as the electric potential difference between two points in a circuit. Kirchhoff's loop rule states that the sum of the electric potential differences in any closed circuit loop is zero.

a) Two resistors with resistances R_1 and R_2 are connected in series across a battery. The circuit may be reduced to a battery and a single resistor with resistance R_{eq}. Using Kirchhoff's loop rule, find R_{eq} in terms of R_1 and R_2. Determine R_{eq} in the limit as $R_2 \to 0$.

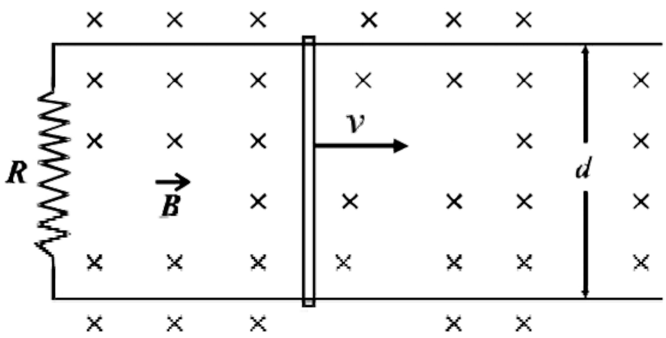
b) Repeat part a) with the two resistors connected in parallel.

Magnetic flux Φ_B is defined as the total magnetic field passing through a surface. For a magnetic field $\mathbf{B}$ and surface S,

$$\Phi_B = \iint_S \mathbf{B} \cdot d\mathbf{S}.$$

The Faraday-Lenz law states that a time dependent magnetic flux through the area bounded by a circuit will induce an opposing electromotive force (EMF) $\mathcal{E} = -\frac{\Phi_B}{dt}$, with positive direction determined by the right hand rule. The EMF may be thought of as an electric potential difference, or voltage from a battery.

A freely moving conducting rod and a resistor with resistance R are placed across two parallel conducting rails separated by a distance d. The rod moves with constant velocity v to the right, perpendicular to a uniform constant magnetic field B directed into the page as shown in the figure below.



c) Find the magnitude and direction of the induced EMF.

d) Find the force from the magnetic field on the moving charges in the circuit.

e) Joule's first law states the resistor will dissipate power $P = IV = I^2R$ in the form of heat. By conversation of energy, and equal power must be delivered to the circuit. Where does this power come from?

Superconductors have gained attention for their ability to effectively eliminate resistive Joule heating, thereby carrying supercurrents and enabling perfect transfer of energy.

Diamagnetism (40 pts.)

4 Let $\hat{\mathbf{n}}$ be the unit vector normal to the plane of a current loop, with positive direction determined by the right hand rule. The magnetic moment $\mathbf{m}$ of a circular loop of radius r carrying current I is $I\pi r^2\hat{\mathbf{n}}$. The orbital motion of electrons will form tiny current loops with a permanent magnetic moment.

An applied magnetic field in a material will cause the permanent magnetic moments to change depending on the magnetization $\mathbf{M}$, which is defined as the net magnetic moment per unit volume. When a magnetic field $\mathbf{B}_0$ is applied to a material with magnetization $\mathbf{M}$ parallel to $\mathbf{B}_0$, the internal magnetic field is

$$\mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M} = \mathbf{B}_0(1 + \chi_m),$$

where χ_m is the magnetic susceptibility. Diamagnetism occurs when the applied magnetic field induces magnetic moment in the opposite direction of $\mathbf{B}_0$.

To understand why diamagnetism occurs, consider N uniformly spaced electrons, each moving with speed v is a circle of radius r. A uniform magnetic field $\mathbf{B}_0$ is then applied. Suppose half the electrons magnetic moments are parallel to $\mathbf{B}_0$ and half are antiparallel. Let m and e be the mass and charge of the electron respectively.

a) Working with the assumption $B_0 \gg \dfrac{mv}{re}$, find the change in speed Δv of the electrons due to the applied magnetic field.

b) Determine the magnetic moment of a single spin up electron. Then summing over all electrons, determine the net magnetic moment.

c) Find χ_m in terms of the electron density ρ, r, and fundamental constants.

Due to electromagnetic induction on the atomic scale, all materials exhibit diamagnetism. However, the effects may be masked by other stronger properties such as ferromagnetism, which is responsible for the magnetic properties of typical magnets (e.g. iron, nickel).

Magnetic Dipoles (40 pts.)

5 In spherical coordinates, every point is described by r, θ, and ϕ, representing the radial distance from the origin, the azimuthal angle from the $+z$-axis, and the polar angle in the xy-plane respectively. To check your understanding, it may help to draw a diagram and show the following conversions,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

A magnetic dipole produces a magnetic field similar to that of a north and south pole pair. The strength of the dipole is the magnetic moment, which points from south to north. For a dipole placed at the origin with dipole moment $\mathbf{m}$ along the $+z$-axis, the magnetic field in spherical coordinates is given by

$$B_r = \frac{\mu_0 m}{2\pi r^3}\cos\theta, \quad B_\theta = \frac{\mu_0 m}{4\pi r^3}\sin\theta, \quad B_\phi = 0.$$

where B_r, B_θ, and B_ϕ are the components of the magnetic field along the directions of increasing r, θ, and ϕ respectively.

a) A magnetic dipole is placed a uniform magnetic field $\mathbf{B}_0$ directed opposite to the dipole moment. Find R for which no magnetic field lines pass through the shell of with radius R centered at the dipole.

The magnetic dipole is replaced with a homogeneous conducting sphere with radius R, surface charge density σ, and identical magnetic moment. The external magnetic field produced is the same as that of the removed dipole. Internally, the magnetic field produced is $\frac{2}{3}\mu_0\mathbf{M}$, where $\mathbf{M}$ is the magnetization of the conductor.

b) Find the B_r, B_θ, and B_ϕ components of the total magnetic field as a function of r and θ.

c) Find the speed of the charge carriers moving on the surface of the conductor.

With the right current distribution on the shell of radius R, it is possible to eliminate magnetic fields in the interior. The ability for superconductors to expel applied magnetic fields is explored in greater depth via the London equations and the Meissner effect.

Meissner Effect (50 pts.)

6 Superconductors exhibit a phenomenon called the Meissner effect, wherein the superconductor repels all external magnetic fields. The effect explains why placing a superconductor on top of a magnet will cause it to levitate.

To treat more complex systems, consider the local differential forms of the Faraday-Lenz law and Ampere's law, which respectively state

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

By the results from problem 2 and Ohm's law, the current density $\mathbf{J} = \dfrac{ne^2\tau}{m}\mathbf{E}$. Taking the time derivative of both sides gives the first London equation,

$$\frac{d\mathbf{J}}{dt} = \frac{d(n\tau)}{dt}\frac{e^2}{m}\mathbf{E} = \frac{n_s e^2}{m}\mathbf{E},$$

where the number density $n_s = \dfrac{d(n\tau)}{dt}$.

a) The Helmholtz equation for the magnetic field states

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda^2}\mathbf{B}.$$

where the Laplacian operator $\nabla^2 = \nabla \cdot \nabla$ is the divergence of the gradient. Use the first London equation to determine the penetration depth λ in terms of m, μ_0, n_s, and e.

Start by taking the curl of Ampere's law. Then, apply Faraday's law and integrate with respect to time. You are given the magnetic field identity

$$\nabla \times \nabla \times \mathbf{B} = \nabla^2 \mathbf{B}.$$

b) Suppose the magnetic field within the superconductor only varies in one dimension, meaning the operator ∇ is $\dfrac{d}{dx}$. For a certain superconductor, the magnetic field 500 nm deep into the superconductor is 5 times weaker than the magnetic field at the surface. Using the Helmholtz equation, find the penetration depth to the nearest nanometer.

Your analysis in part b) should mathematically justify the Meissner effect. The explanation behind how the effect works is given by the second London equation,

$$\nabla \times \mathbf{J} = -\frac{n_s e^2}{m}\mathbf{B},$$

derived by taking the curl of the first London equation and applying Faraday's law. The equation states that an external magnetic field induces a current in the superconductor, which produces an internal magnetic field that cancels the external field, hence the Meissner effect.

c) Two isolated conductors sit far apart at room temperature. Conductor A has a small external magnetic field applied to it, while conductor B does not. They are both cooled down below their superconducting critical temperatures. An identical small external magnetic field is then applied to B. Using conclusions from the London equations, describe what happens to the external and internal magnetic fields for each superconductor.

d) Now the external magnetic fields are turned off for A and B. Describe what happens to the external and internal magnetic fields for each superconductor.

e) Suppose that A is a perfect conductor instead of a superconductor. Repeat part d) for the perfect conductor by describing the new behavior of the external and internal magnetic fields.

The Meissner effect is a special and unexpected property of superconductors, different from perfect conductivity or perfect diamagnetism. The effect also manifests differently in type I and II superconductors, with the latter forming magnetic flux tubes rather than displaying complete Meissner effects.

2 Applied Problems

Cooling Superconductors (50 pts.)

7 A refrigerator may be used to cool a superconductor to the critical temperature T_c by absorbing heat Q_c from a cold reservoir and releasing heat Q_h to a hot reservoir. Because this process reverses heat flow, it requires an external work W. A refrigerator operates optimally when the total entropy, a measure of the disorder in the system, stays constant. For a system at constant temperature T, the change in entropy is $\Delta S = \frac{Q}{T}$, where Q is the heat transferred to the system.

a) The coefficient of performance (COP) of a refrigerator is $\frac{Q_h}{W}$. Find the COP of an optimal refrigerator operating between cold and hot reservoirs at temperatures T_c and T_h. Express your answer in terms of T_c and T_h.

Specific heat c is defined as the heat per unit mass required to increase temperature by one kelvin. By the equipartition theorem, the energy per mole associated with each molecular degree of freedom is $\frac{1}{2}kTN_A$, where $k = 1.38 \times 10^{-23}$ J $\cdot$ K^{-1} is the Boltzmann constant and $N_A = 6.02 \times 10^{23}$ mol^{-1} is the Avogadro constant.

b) Considering translational, rotational, and vibrational degrees of freedom, calculate the specific heat of a solid with molar mass M. Express your answer in terms of T, M, and fundamental constants.

The Debye model corrects the equipartition theorem at low temperatures by treating molecules as quantum harmonic oscillations with discrete energies. With the Debye model, the energy per mole for a solid with Debye temperature T_D is

$$\frac{9kT^4N_A}{T_D^3}\int_0^{T_D/T}\frac{x^3}{e^x-1}\,dx.$$

c) Find the Debye specific heat in the high temperature limit $T \gg T_D$. Explain why this answer is expected.

d) Find the Debye specific heat in the low temperature limit $T \ll T_D$. You are given the following integral.

$$\int_0^{\infty}\frac{x^3}{e^x-1}\,dx = \frac{\pi^4}{15}.$$

Suppose a refrigerator operating optimally between a superconductor of mass m and a hot reservoir with constant temperature T_h is used to cool the superconductor from an initial temperature T_i to T_{cr}.

e) Find the work W required and the change in entropy ΔS in terms of m, T_i, T_c, T_D, and fundamental constants. Comment on the disorder in the superconductor as it is cooled. Assume the low temperature limit for the Debye specific heat holds during the entire process.

f) Using your expressions in part e), calculate W and ΔS if 50 g of tin initially at 100 K is cooled to the critical temperature of 7.2 K. For tin, $T_D = 200$ K.

The cost of cooling superconductors to a few kelvin using liquid helium often offsets the advantages of eliminating Joule heating. As a result, research is centered around high-temperature superconductors, which have critical temperatures above 77 K, the boiling point of liquid nitrogen. In October 2020, scientists at the University of Rochester in New York discovered the first room-temperature superconductor, with a critical temperature of 288 K under 267 gigapascals of pressure.

Magnetic Resonance Imaging (MRI) Machines (40 pts.)

8 An inductor is a cylindrical coil of wire, shaped similarly to a slinky, that produces a constant magnetic field **B** oriented parallel to the length inside the solenoid and no magnetic field outside. The inductance L of an inductor gives the tendency to resist a time-varying current, which produces an opposing electromotive force $\mathcal{E} = -L\frac{dI}{dt}$. For a inductor with cross-sectional area A, length ℓ, and n turns per unit length, $L = \mu_0 n^2 A\ell$.

a) Using Ampere's Law with a rectangular loop, find the magnetic field B inside the inductor in terms of L, A, ℓ, I, and fundamental constants.

An MRI machine uses a large superconducting inductor with inductance L to produce a strong magnetic field. The inductor is connected in parallel with a persistence resistor with resistance R_P and a DC current source until the final magnetic field strength B_f is achieved. The current source is then disconnected and the resistor is cooled until it becomes superconducting. The superconducting loop then carries the current in persistence mode. Suppose due to impurities and incomplete Meissner effects, the inductor and resistor both have a small but non-zero resistance $R \ll R_P$ when below the supercritical temperature.

b) Suppose the current source delivers a constant current I. Find $B(t)$ in terms of L, R_P, $\eta = \frac{R}{R_P}$, and I. Determine B_{max}, the maximum value of $B(t)$. How does B_{max} vary with η?

c) Find $B_P(t)$, where t is the time after transitioning to persistence mode, in terms of B_f, R, and L.

d) Find R if the magnetic field of a 1.5 T MRI machine with diameter 0.8 m, length 2.5 m, and initial current 500 A decays by 1 mT after one year. Express your answer in scientific notation rounded to the nearest hundredth.

e) In persistence mode, quenching may occur if the resistor is suddenly heated and loses superconducting properties. Calculate the heat Q released through quenching using the numeric values given in part d).

Superconducting Qubits (50 pts.)

9 A capacitor is a circuit device composed of two parallel oppositely charged plates that stores energy in an electric field. The voltage drop across the capacitor is $V = \frac{Q}{C}$, where Q is the charge on the capacitor plates and C is the capacitance.

An LC circuit composed of a capacitor and inductor serves as the foundation of the superconducting qubit. Suppose the electric and magnetic field energies are analogous to kinetic and potential energies respectively.

a) Find the total energy in an LC circuit.

b) The single particle quantum harmonic oscillator with potential function $\frac{1}{2}kx^2$ has allowed energy values $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where $\omega = \sqrt{\frac{k}{m}}$ and n is a positive integer. Drawing on some familiarity with the kinetic and potential energy of a particle, find the allowed energy values for a quantum LC circuit.

c) Why are the allowed energies in a quantum LC circuit problematic for the implementation of a two energy level qubit?

A Josephson junction is composed of two superconductors separated by a thin insulating wafer and carries a supercurrent via quantum tunnelling. The current and voltage drop across the junction are given by the Josephson equations,

$$I = I_c \sin(\phi), \quad V = \frac{\hbar}{2e}\frac{d\phi}{dt}.$$

where I_c is the critical current and ϕ is the phase difference across the insulator. Suppose the inductor in an LC circuit is replaced by a Josephson junction.

d) Find the potential energy function of the superconducting qubit. Explain why the addition of the junction solves the problem identified in part c).

e) Calculate the maximum temperature for a useful superconducting qubit operating at 8 GHz to the nearest mK.

Quantum computing initiatives at Google, IBM, Rigetti, etc. use superconducting qubits to create quantum devices. In October 2019, scientists at Google announced their 53-qubit quantum computer surpassed the capabilities of the best classical supercomputers.

Particle Accelerators (60 pts.)

10 The Large Hadron Collider at CERN uses superconducting magnets to accelerate protons to relativistic speeds. The protons have mass $m = 1.67 \times 10^{-27}$ kg and are injected into a circular tube with radius $r = 27$ km. Two equal and opposite currents with magnitude I are sent through two circular superconducting wires with radii $r \pm d$, with $d = 2$ cm. The superconductors have a small resistance R and create a magnetic field perpendicular to the velocity of the protons. Suppose the protons are injected with zero initial speed and the current is increased at a constant rate α, which accelerates the protons to a final speed $v = \beta c$, where $\beta = 1 - 10^{-4}$.

The radius of motion of a particle in a magnetic field $\mathbf{B}$ perpendicular to velocity is given by the gyroradius $r_g = \dfrac{\gamma m v}{|q|B}$, where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$. An accelerating charged particle will emit radiation in the form of electromagnetic waves. By the Larmor formula, the power radiated is

$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3},$$

where the acceleration a is equal to $\dfrac{\gamma}{m}\left|\dfrac{d\mathbf{P}}{dt}\right|$. The relativistic momentum four-vector $\mathbf{P} = (\gamma mc, \gamma m\mathbf{v})$ satisfies

$$|\mathbf{P}|^2 = (\gamma m v)^2 - (\gamma mc)^2 = \gamma^2 m^2 c^2\left(\frac{v^2}{c^2} - 1\right) = -m^2c^2.$$

a) If the protons are kept at the final speed, find the power dissipated by the superconducting wires. Express your answer as R times a constant in scientific notation to the nearest hundredth.

b) Find the total heat dissipated by the superconducting wires as the protons are accelerated to the final speed. Express your answer as $\dfrac{R}{\alpha}$ times a constant in scientific notation to the nearest hundredth.

c) Find the minimum external power needed to accelerate one proton to the final speed. Express your answer symbolically using the given variables.

Superconducting magnets enable particle accelerators to achieve faster relativistic speeds. For comparison, the first particle accelerators used a voltage difference to accelerate protons to 1 MeV, whereas the LHC accelerates protons to 6.5 TeV.

3 BCS Theory

Bardeen-Cooper-Schrieffer (BCS) theory proposed the first microscopic treatment of superconductors. This section is dedicated to the foundations of BCS theory, culminating with the derivation of Cooper pairs.

Quantum Statistics (75 pts.)

11 In quantum mechanics, the wave function $\psi(\mathbf{r})$ is a complex-valued function that describes the state of a system. The magnitude of $\psi(\mathbf{r})$ is the probability amplitude, meaning the particle is in the interval $(\mathbf{r}_0, \mathbf{r}_0 + d\mathbf{r})$ with probability $|\psi(\mathbf{r}_0)|^2 d^3\mathbf{r}$, where $d^3\mathbf{r} = dr_x\, dr_y\, dr_z$.

Two particles of the same species are indistinguishable in quantum mechanics. For a two particle wave function $\psi(\mathbf{r}_1, \mathbf{r}_2)$, switching the position vectors will not change the probability amplitude. According to the spin-statistics theorem,

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_2, \mathbf{r}_1) \text{ for bosons,} \quad \psi(\mathbf{r}_1, \mathbf{r}_2) = -\psi(\mathbf{r}_2, \mathbf{r}_1) \text{ for fermions.}$$

a) Using the spin-statistics theorem, show that bosons have integral spin while fermions have half-odd-integral spin.

b) Justify the Pauli exclusion principle, which states that two fermions can not occupy the same quantum state.

c) At low temperatures around one kelvin, liquid ^{4}He is a superfluid, flowing with zero viscosity and no kinetic energy losses, whereas ^{3}He is a liquid. What does this indicate about the electrons in a superconducting metal?

The Schrödinger equation determines the properties of a wave function $\psi(\mathbf{r})$. In three dimensions, the time independent equation is

$$-\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}),$$

where $\nabla_{\mathbf{r}}^2 = \dfrac{\partial^2}{\partial \mathbf{r}^2}$. The terms $-\dfrac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2$ and $V(\mathbf{r})$ are the kinetic and potential energies of the state respectively.

Suppose the conduction electrons in a metal are modelled as a free Fermi gas, non-interacting fermions with $V(\mathbf{r}) = 0$. Furthermore, impose the condition that the wave function repeats with period L along the x, y, and z directions.

The energy density of states per unit volume $D(E)$ is defined such that the number of states with energy between E and $E + dE$ is $\Omega D(E)dE$, where Ω is the volume of the system.

d) Let the mass of the electron be m. Find all values of $\mathbf{p}$ which satisfy the Schrödinger equation and the imposed periodic boundary conditions.

e) Find $D(E)$ in terms of m, E and fundamental constants. It may help to plot values of $\mathbf{p}$ and partition the space into homogeneous volume elements.

f) Suppose $F(\mathbf{p})$ is a function of the momentum $\mathbf{p}$. Find the constant α for which

$$\sum_{\mathbf{p}} F(\mathbf{p}) = \frac{\Omega}{\alpha}\int F(\mathbf{p})\, d^3\mathbf{p}$$

where $\mathbf{p}$ is summed over values found in part d) and Ω is the volume of the system. Although this result is derived for large cubic volumes, it holds for large systems of arbitrary shape.

The energy density of states per unit volume is useful in dealing with many-particle systems. Let $N \gg 1$ be the total number of particles in a boson gas with chemical potential μ as the thermodynamic conjugate variable. Using the grand canonical ensemble and Bose-Einstein statistics, the average number of particles in state i at temperature T is

$$\langle n_i(\mu, T)\rangle = \frac{1}{e^{(\varepsilon_i - \mu)/kT} - 1},$$

where ε_i is the energy of state i and μ is the chemical potential. Because bosons do not follow the Pauli exclusion principle, below a critical temperature T_c, the bosons will begin to simultaneously occupy the ground state with $\varepsilon_0 = 0$ while the rest occupy excited states with $\mu = 0$.

g) Find all possible values of $\langle n_i(\mu, T)\rangle$ for bosons and fermions. Then express $\langle n_0(T)\rangle$ for bosons in terms of N, T, and T_c. Verify your answer in the limiting cases. Why is your expression for $\langle n_0(T)\rangle$ problematic when comparing superfluids and superconductors?

Working in the grand canonical ensemble, there are certain subtleties that must be taken into account when considering superfluids and superconductors. Most notably, superfluids will exhibit spontaneously broken symmetry, while superconductors have a different property called topological order.

Cooper Pairs (75 pts.)

12 Superconductivity is due to the formation of Cooper pairs, pairs of electrons that form a bound state with an attractive potential. Despite electrons having the same charge, the interaction is attractive due to quantum effects. As a simplified classical explanation, consider a slowly moving electron which pulls positive ions in the surrounding lattice towards it. This creates a small region of positive charge, attracting another electron.

Consider two electrons at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ interacting via a potential $V(\mathbf{r}_1 - \mathbf{r}_2)$. Let $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ be the relative position and $\mathbf{R} = \dfrac{\mathbf{r}_1 + \mathbf{r}_2}{2}$ be the center of mass.

a) Write the Schrödinger equation for the wave function $\psi(\mathbf{r}_1, \mathbf{r}_2)$. Then rewrite $\psi(\mathbf{r}_1, \mathbf{r}_2)$ as the wave function $\psi(\mathbf{r}, \mathbf{R})$ by expressing the Laplacian in terms of $\mathbf{r}$ and $\mathbf{R}$ using the chain rule for multivariate functions,

$$\frac{\partial}{\partial \mathbf{r}_a} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_a}\frac{\partial}{\partial \mathbf{r}} + \frac{\partial \mathbf{R}}{\partial \mathbf{r}_a}\frac{\partial}{\partial \mathbf{R}}.$$

b) Since the potential does not depend on $\mathbf{R}$, the wave function $\psi(\mathbf{r}, \mathbf{R})$ is of the form $\psi(\mathbf{r})e^{-i\mathbf{P}\cdot\mathbf{R}/\hbar}$. Simplify the expression you obtained in part a) and then set $P = 0$.

The wave function $\psi(\mathbf{r})$ is expressed in position space. There exists an wave function $\phi(\mathbf{p})$ in momentum space equivalent to $\psi(\mathbf{r})$. The Fourier transform $\mathcal{F}$ and inverse Fourier transform relates $\psi(\mathbf{r})$ and $\phi(\mathbf{p})$ by the equations

$$\phi(\mathbf{p}) = \mathcal{F}[\psi(\mathbf{r})] = \int_{-\infty}^{\infty} \psi(\mathbf{r}) e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\, d^3\mathbf{r}$$
$$\psi(\mathbf{r}) = \mathcal{F}^{-1}[\phi(\mathbf{p})] = \frac{1}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} \phi(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\, d^3\mathbf{p},$$

c) Apply the Fourier transform to show that the Schrödinger equation for two interacting electrons in momentum space is

$$\frac{\Omega}{(2\pi\hbar)^3} \int V(\mathbf{p}-\mathbf{p}')\phi(\mathbf{p}')\, d^3\mathbf{p}' = (E - 2\varepsilon_{\mathbf{p}})\phi(\mathbf{p}),$$

where $V(\mathbf{p}-\mathbf{p}')$ is the normalized potential function operator in momentum space. Ω is the volume of the system and $\varepsilon_{\mathbf{p}} = \dfrac{p^2}{2m}$.

The electrons in a superconducting material will experience an attractive interaction if their energies are above the Fermi energy threshold $\varepsilon_F \gg \hbar\omega_D$, where ω_D is the Debye frequency. ε_F is the maximum energy of an electron at absolute zero, and ω_D is the frequency of the phonons that mediate the electron-electron interaction.

To simplify the problem, assume the attractive potential is given by

$$V(\mathbf{p}-\mathbf{p}') = \begin{cases} -V_0 & \varepsilon_{\mathbf{p}} - \varepsilon_F, \varepsilon_{\mathbf{p}'} - \varepsilon_F < \hbar\omega_D \\ 0 & \text{otherwise} \end{cases}$$

where V_0 is a constant.

d) Briefly describe the qualitative properties of the attractive potential $V(\mathbf{r})$.

e) Find the function $f(\mathbf{p})$ for which

$$\Omega V_0 \int f(\mathbf{p})\, d^3\mathbf{p} = 1.$$

f) Suppose the interaction is weak with $V_0 \ll \dfrac{\beta}{\Omega\sqrt{\varepsilon_F}}$, where β is a small constant with appropriate units. Write the integral

$$\int f(\mathbf{p})\, d^3\mathbf{p}$$

as an integral over energy

$$\int g(\varepsilon)\, d\varepsilon.$$

g) Find $2\varepsilon_F - E$ in terms of m, V_0, ε_F, ω_D, Ω, and fundamental constants. How does $2\varepsilon_F - E$ behave as $V_0 \to 0$? Why does the relationship show the existence of Cooper pairs?

4 Current Research

Here is an enumeration of articles and web pages on current superconductor research, for those interested. Click on the hyperlinks for more information.

Properties

- Fe-based Superconductors
- Graphene Superconductors
- Low Pressure Superconductors
- Room Temperature Superconductors

Applications

- Chip Interconnects
- Energy Storage Rings
- Generators with Superconducting Wires in Rotors
- IR Sensors
- Josephson Junctions
- Magnetic Bearings
- Magnetic Separation
- Magnets for Fusion Devices
- Magnets for Magnetically Levitated Trains (MAGLEV)
- Magnets for Magnetic Resonance Imaging
- Power Transmission Lines
- Ship Propulsion (Motors)
- SQUIDS (Superconducting Quantum Interference Devices)