Advanced Topics and Errata in Waves

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1 Introduction

When we discussed wave motion, we entered a new realm of physics: we started to describe a complex system in terms of a particular state variable. As abstract as this seems, such a notation is very useful in analyzing properties of solid materials. We will continue to use this nascent notation when discussing particular properties of waves. This notation will come back to haunt us (hopefully not) in our studies of quantum mechanics, so it won't be the last time we'll come face-to-face with it.

2 Energy and Waves

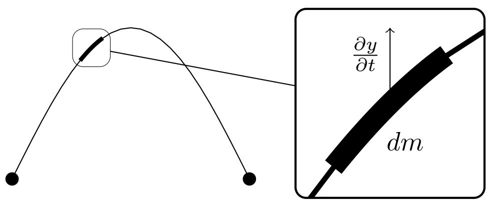

Figure 1: A small section of the string has mass dm and velocity $\frac{\partial y}{\partial t}$.

By definition, a wave is a physical "packet of energy". It is solely responsible for transporting energy form one place to another. Notice that we never described a wave as such until now. This is because we are now ready to talk about the energetics of wave motion. Consider your average wave (travelling or standing), and analyze a small portion ds of it. If this portion is taken to be infinitesimally small, then it appears to be a straight line (see Figure 1). The Kinetic Energy of this small piece is given by

$$dK = \frac{1}{2}\, dm \left(\frac{\partial y}{\partial t}\right)^2 \tag{1}$$

where we use the rate of change of the y position, because this is the source of the *only* motion of the wave. Our general solution to the wave equation is $f(x - vt)$, so using the chain rule gives us

$$dK = \frac{1}{2} v^2 f'^2(z)\, dm \tag{2}$$

where $z \equiv x - vt$. It is convenient to derive an equation for the kinetic density of the wave, which is given by

$$\frac{dK}{dx} = \frac{1}{2}\mu v^2 f'^2(z) \tag{3}$$

The wave also contains a particular potential energy. This energy is derived from the work done to create the initial "pluck" that propagates the wave. The work done by the tension forces is negative (since the string moves opposite to the restoring tension), so the potential energy of the string is positive. It is given by the work done from stretching the string from an original length dx to a new length ds. Since the tension acts along the direction of motion at all times, the dot product vanishes, giving us

$$dU = T(\,ds - \,dx) = T(\sqrt{(\,dy)^2 + (\,dx)^2} - \,dx) \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \,dx \tag{4}$$

where the last result uses the binomial approximation. Writing our answer in terms of a potential energy density, we have

$$\frac{dU}{dx} = \frac{1}{2}Tf'^2(z) \tag{5}$$

Noting that $\mu v^2 = T$, we see that $K = U$ and the total energy density is given by $E = Tf'^2(z)$. As a corollary, one can determine the total energy stored in one period of a particular waveform.

3 Wave Propagation: Attenuation and Impedance

Our current description of wave motion was developed based on the assumption of an undamped and uniform string. It is useful to look at some models of waves that start with more general conditions and see what sorts of behavior they lead to.

3.1 Attenuation

Consider a vibrating string immersed in some fluid. We will assume that in addition to the tension of the string, the fluid exerts a drag force proportional to the transverse velocity: $F_d = -\beta v_y$. Then we can write Newton's second law for the transverse motion of the string:

$$\mu\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} - \beta\frac{\partial y}{\partial t} \tag{6}$$

The first term on the right is based on a small-angle approximation; see the lecture on Wave Motion for the complete derivation. To solve this equation, we can guess a solution of the same form as the solutions to the undamped wave equation, $y = Ae^{i(\omega t - kx)}$. Plugging this in and rearranging terms, we obtain

$$\omega^2 - i\frac{\beta}{\mu}\omega = k^2\frac{T}{\mu} \tag{7}$$

This gives a relationship between ω and k, which we recognize from the last lecture as the dispersion relation. It depends only on β/μ and T/μ; we will call the former Γ, and the latter we recognize from the last lecture as v^2. The equation then becomes

$$\omega^2 - i\Gamma\omega = k^2 v^2 \tag{8}$$

Solving for k, we obtain

$$k = \frac{1}{v}\sqrt{\omega^2 - i\Gamma\omega} = K - i\kappa \tag{9}$$

where we have labeled the real and imaginary parts as K and κ. Inserting this form back into the original equation, we have

$$y = Ae^{-\kappa x}e^{i(\omega t - Kx)}. \tag{10}$$

The behavior of this function depends crucially on the sign of κ. Returning to its definition, we see that κ is proportional to the imaginary part of the square root of a complex number with a negative imaginary part. Such a square root also has a negative imaginary part, and so κ must be positive. This means that the function represents an oscillation with an exponentially decaying amplitude. Figure 2 shows the real part of the solution, with the exponential envelope highlighted.

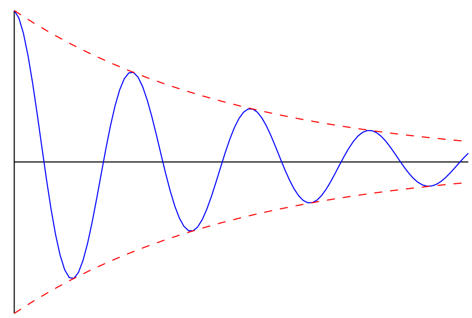

Figure 2: The amplitude of the wave falls off exponentially with x.

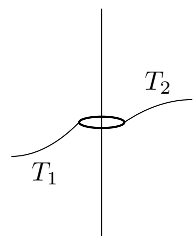

Figure 3: Two strings of differing tension both joined to a ring on a pole.

3.2 Impedance

When the properties of a string (i.e., its tension or its density) are changing, then the simple model we used previously also fails to capture all of the behavior. We can investigate the effects of changing string properties by considering two strings joined at a ring on a pole, as in Figure 3.

Now, consider an incident wave $y_i(x,t)$ on the right hand side. When it reaches the pole, it will generally produce a reflected wave and a transmited wave, which we call $y_r(x,t)$ and $y_t(x,t)$. Since these all satisfy the wave equation, they can be written as $f(x-vt)$ or equivalently as $f(t-x/v)$. With this representation, the wave on the left string is $f_i(t-x/v)+f_r(t-x/v)$ and the wave on the right string is $f_t(t-x/v)$. Since both strings are attached to the string,

$$f_i(t)+f_r(t)=f_t(t) \tag{11}$$

Additionally, the vertical forces must cancel at $x=0$, because otherwise the massless string would experience an infinite vertical acceleration.

$$-\frac{T_1}{v_1}f_i'(t)+\frac{T_1}{v_1}f_r'(t)=-\frac{T_2}{v_2}f_t'(t) \tag{12}$$

Note that the reflected wave has a negative velocity. Integrating and simplifying gives

$$T_1v_2f_i(t)-T_1v_2f_r(t)=T_2v_1f_t(t) \tag{13}$$

We can solve these for the reflected and transmitted waves:

$$f_r(t)=Rf_i(t)=\frac{T_2/v_2-T_1/v_1}{T_1/v_1+T_2/v_2} \tag{14}$$

$$f_t(t)=Tf_i(t)=\frac{2T_2/v_2}{T_1/v_1+T_2/v_2} \tag{15}$$

Clearly, the quantity $Z=T/v=\sqrt{T\mu}$ is important here. We call this quantity the impedance. In terms of Z, the reflection and transmission coefficients become

$$R=\frac{Z_2-Z_1}{Z_1+Z_2}$$
$$T=\frac{2Z_2}{Z_1+Z_2}.$$

We can see the importance of the impedance when the coefficients are written in this form. If the impedances are the same, then there is no reflection ($Z_2 - Z_1 = 0$), and all the energy is transmitted. This is obviously the case for a uniform string where tension and density are constant, but less obviously the case when $T_2 = 5T_1$ and $\mu_2 = \mu_1/5$.

4 Representation of Different Waveforms: An Introduction to the Fourier Series

In the section on attenuation, we saw one example of an important wave that is not a simple sine or cosine. On one hand, we want to investigate these more complicated functions and understand their behavior; on the other hand, sines and cosines are much easier to work with. As it turns out, we can have our cake and eat it too, using the Fourier series.

Consider, for a moment, a vector space such as $\mathbb{R}^3$. In principle, we could choose any three non-coplanar vectors and write arbitrary vectors as linear combinations of them. For example, any vector could be written as a linear combination of $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$ and $\langle 1,1,1\rangle$. However, it is much more convenient to use the basis of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. This is because they are orthogonal, so we can extract components of an arbitrary vector like so.

$$\vec{r} \cdot \hat{\imath} = x \tag{16}$$

$$\vec{r} \cdot \hat{\jmath} = y \tag{17}$$

$$\vec{r} \cdot \hat{k} = z \tag{18}$$

Since the basis vectors are orthogonal, dotting a vector with one of the basis vectors extracts the corresponding component.

Now, consider a function $f(x)$. The function is a correspondence between every real number x and some other real number, just as the vector is a correspondence between some integers and real numbers. If we step back, we can see some similarities between the two cases. In fact, functions do occupy a vector space, and some of the properties of the vectors – such as decompositions into basis vectors – carry over into the realm of functions. If we set out to find the component of a function "along" a particular basis function, then we must answer two questions: what are the basis functions, and how do we take dot products?

The second question is the easier one to answer. For a more general vector space, dot products are called inner products and are denoted $\langle f, g\rangle$ instead of $f \cdot g$, but the concept is the same: we have to add products of corresponding components. For functions, adding components is done through integration:

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx \tag{19}$$

We have to decide on the bounds a and b. Sometimes they are $\pm\infty$, and other times they specify a finite range; for the Fourier series, we will consider $a = -\pi$ and $b = \pi$. Now, to answer the first question, consider the infinite set of functions $\{1 = \cos 0x, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$. The inner product of a function in the set with itself can be easily determined:

$$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \frac{1-\cos nx}{2}\,dx = \pi \tag{20}$$

$$\int_{-\pi}^{\pi} \cos^2 nx\,dx = \int_{-\pi}^{\pi} \frac{1+\cos nx}{2}\,dx = \pi \tag{21}$$

with the exception that for $n = 0$, both integrals give 2π. Inner products between distinct members of the set can be determined using other trigonometric identities:

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x - \cos(n+m)x \, dx = 0 \tag{22}$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x + \cos(n-m)x \, dx = 0 \tag{23}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x + \sin(n-m)x \, dx = 0 \tag{24}$$

Remarkably, the inner products between different functions vanish: they form an orthogonal set. We might envision writing a function as a linear combination of members of the set:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots \tag{25}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{26}$$

Now, following the same process as with the finite vectors, we take the inner product of both sides with one of the basis elements, say $\cos 2x$:

$$\langle f(x), \cos 2x \rangle = a_2 \langle \cos 2x, \cos 2x \rangle \tag{27}$$

$$a_2 = \frac{1}{\pi} \langle f(x), \cos 2x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 2x \, dx \tag{28}$$

In general, we can write the following expressions for the a_n and b_n.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \tag{29}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \tag{30}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \tag{31}$$

This defines the Fourier series of a function f on the interval $[-\pi, pi]$. For a function such as $\sin x$, it is evident that these formulas will give $b_1 = 1$ and all other coefficients zero. The more interesting case is for a function such as $f(x) = x$, which does not have any representation as a finite sum of trigonometric functions. We can try computing the coefficients of the Fourier series:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0 \tag{32}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx = \frac{1}{\pi} \left(\left[\frac{x}{n} \sin nx \right]_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} \sin nx \, dx \right) = 0 \tag{33}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{1}{\pi} \left(\left[-\frac{x}{n} \cos nx \right]_{-\pi}^{\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos nx \, dx \right) = (-1)^{n+1} \frac{2}{n} \tag{34}$$

So, supposedly, $x = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x - \frac{1}{2} \sin 4x + \dots$ on the interval $[-\pi, \pi]$. A plot of the sum of the first 20 terms shows that the series does indeed converge towards x. Additionally, it shows that outside of the interval, the series gives a periodic extension of x.

5 The Doppler Effect

The idea of the Doppler Effect is a ubiquitous one: one can experience it by standing before and ambulance passing by. When it passes by, you can hear a distinct drop in its pitch (check out the wonderful example here:

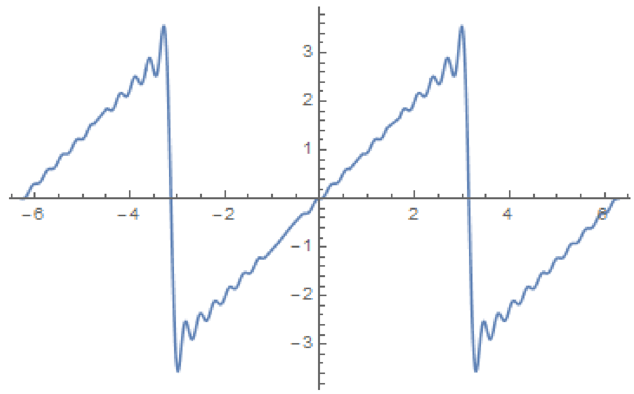

Figure 4: The first 20 terms in the Fourier series of x.

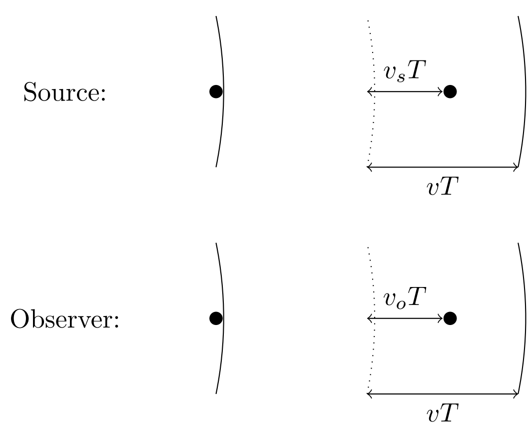

Figure 5: Moving observers and moving sources both see distorted wavelengths.

https://www.youtube.com/watch?v=imoxDcn2Sgo). This may seem like an act of magic, but it actually occurs because the waveforms emanating from the siren are getting crunched by the motion of the ambulance.

The proof of the Doppler effect is relatively simple once this key observation has been made. Consider the crests of a moving object coming towards a source (which we will assume in generality to be moving with a particular velocity). The velocity of the source is v_s, the velocity of the observer is v_o, and the velocity of the propagating waveforms (in a stationary reference frame) is v. See Figure **??** for details. Relative to the observer, after a period T, the waves seem to move a distance $\lambda_o = (v - v_s)T$. Relative to the source, $\lambda_s = (v - v_o)T$. Thus, $\frac{\lambda_o}{\lambda_s} = \frac{v-v_s}{v-v_o}$. Since the apparent frequency and wavelength of the waves might change from different points of view, but the speed of the wave remains the same, this implies that

$$\frac{f_o}{f_s} = \frac{1}{\lambda_o/\lambda_s} = \frac{v - v_o}{v - v_s} \tag{35}$$

This is known as the Doppler equation. For a stationary observer, and a source which is moving towards the observer ($v_o = 0$ and v_s is positive), the frequency seems to increase, which is an expected result from observation. For a stationary observer, but with relativistic effects not neglected, there is a modified form for the doppler effect, creatively dubbed the *relativistic doppler effect*

$$\frac{f_o}{f_s} = \sqrt{\frac{1-\beta}{1+\beta}} \tag{36}$$

where $\beta = v/c$, assuming that the wave emitted is from the electromagnetic spectrum. For a proof, see the problem set.

6 Problems

1. Consider a drum of radius a, which produces rippling waves with angular symmetry. The wave function is given by a function of radius and angle $\psi(r, \theta)$. Assuming a surface tension γ in units of energy per unit area, determine the potential energy of the configuration for the given ψ in terms of an integral expression.

2. Determine the *total* energy of a Gaussian wave form $y = Ae^{-\frac{(x-vt)^2}{2\sigma}}$. Assume that the string which contains the pulse is arbitrarily long.

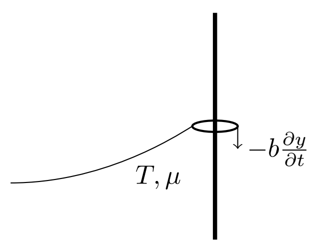

Figure 6: Two strings of differing tension both joined to a ring on a pole.

3. A very long string of mass density μ and tension T is attached to a small hoop with *negligible* mass. The hoop slides on a greased vertical rod and experiences a vertical force $F_y = b\frac{\partial y}{\partial t}$ when it moves (see Figure 6).
(a) Apply Newton's law to the hoop to find the boundary condition at the end of the string. Express your result in terms of the partial derivatives of $y(x, t)$ at the location of the rod.
(b) Show that the boundary condition is satisfied by an incident pulse $f(x - vt)$ and a reflected pulse $g(x + vt)$. Find g in terms of f.
(c) Show that your result has the correct behavior in the limits $b \to 0$ (the string is free to slip) and $b \to \infty$ (the string is firmly clamped).

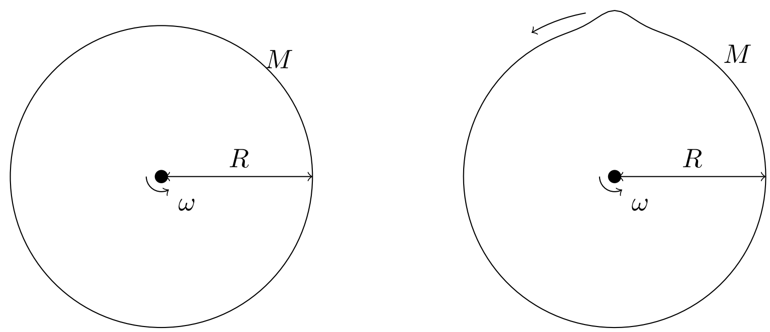

Figure 7: A hoop of rope rotates with angular velocity ω.

4. A hoop of rope with a total mass M and a radius R rotates with a constant angular velocity ω (see Figure 7). Determine the tension in the rope. A kink is then introduced in the hoop. Determine the possible motion(s) of the kink as it propagates around the hoop.

5. Prove the relativistic Doppler formula. Hint: in the frame of the observer, time dilation is in effect.

6. Consider the regular Doppler effect, except that the observer and the source have velocity vectors that differ by an angle θ. How would the derivation of the equations change?

7. Consider the rectangular function

$$f(x) = \begin{cases} 0 & \text{if } |x| > \frac{1}{2} \\ \frac{1}{2} & \text{if } |x| = \frac{1}{2} \\ 1 & \text{if } |x| > \frac{1}{2} \end{cases}$$

and determine a Fourier series representation for it. Describe the behavior outside of the interval you choose to expand in (a good choice would be $[-1, 1]$) and the behavior at the discontinuous point(s).