# Non Uniform Circular Motion Solution Set

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## 1 Introductory Problems

1. The force required to induce a constant acceleration to a particle in uniform circular motion is given by

$$\vec{F} = \hat{r}(m\ddot{r} - m\omega^2 r) + \hat{\theta}(2m\dot{r}\omega + mr\alpha) \tag{1}$$

Since the object is in uniform circular motion, the derivatives of $r$ are dropped. We want a constant acceleration, so $\omega = \alpha t$. Thus, our force is

$$\vec{F} = -m\alpha^2 t^2 R\hat{r} + mR\alpha\hat{\theta} \tag{2}$$

As one can see, the radial term starts to dominate the tangential term for large $t$. This makes sense, as the centripetal acceleration would become indefinitely large as $\omega$ grows without bound.

## 2 Intermediate and Advanced Problems

1. The forces acting on the mass only arise from the tension in the string, as we assume that the normal force provided by the table neutralizes the effect of gravity. Since we know that the tension is directed radially, we can say

$$\vec{F}_{net} = T\hat{r} = \hat{r}(m\ddot{r} - m\omega^2 r) + \hat{\theta}(2m\dot{r}\omega + mr\alpha) \tag{3}$$

Using the fact that the tangential force is zero, we can immediately write down

$$2m\dot{r}\omega = -mr\alpha \tag{4}$$

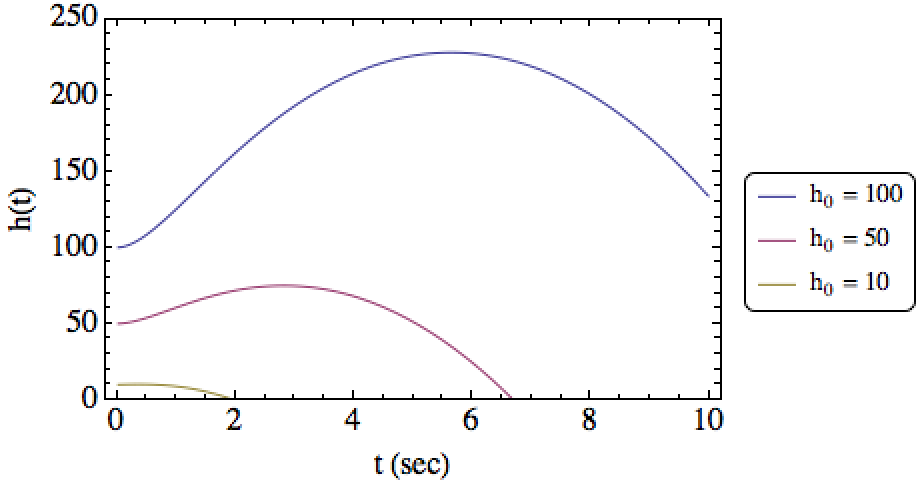
Using the fact that $\dot{r} = -V$ and $r(t) = r_0 - Vt$

$$2\frac{-V dt}{r_0 - Vt} = -\frac{d\omega}{\omega} \tag{5}$$

$$\rightarrow \omega(t) = \omega_0 \left(\frac{r_0}{r_0 - Vt}\right)^2 \tag{6}$$

The general solution given any $r(t)$ is

$$\omega(t) = \omega_0 \left(\frac{r_0}{r(t)}\right)^2 \tag{7}$$




2. Decompose the system into cylindrical coordinates. Along the plane of the revolving mass, we must apply the equations associated with non-uniform circular motion

$$T \sin\theta = m\dot{\phi}^2 R \tag{8}$$
$$-b\dot{\phi}R = m\ddot{\phi}R \tag{9}$$

From the second equation, we get

$$\dot{\phi}(t) = \omega_0 e^{\frac{-bt}{m}} \tag{10}$$

Thus

$$T \sin\theta = m\omega_0^2 R e^{\frac{-2bt}{m}} \tag{11}$$

Forces in the z-direction give us the following equation as well

$$T\cos\theta - mg = m\ddot{h} \tag{12}$$

Dividing the previous two equations

$$\cot\theta = \frac{h}{R} = \frac{m\ddot{h} + mg}{m\omega_0^2 R e^{\frac{-2bt}{m}}} \tag{13}$$

$$h = \frac{\ddot{h} + g}{\omega_0^2 e^{\frac{-2bt}{m}}} \tag{14}$$

The general solution to this is

$$h(t) = C_1 e^{\omega_0 t e^{-\frac{\mathrm{bt}}{m}}} + C_2 e^{\omega_0 t\left(-e^{-\frac{\mathrm{bt}}{m}}\right)} + \frac{g e^{\frac{2\mathrm{bt}}{m}}}{\omega_0^2} \tag{15}$$

where $C_1$ and $C_2$ are constants chosen from initial conditions. The graph on the previous page displays the solution $h(t)$ with all relevant parameters set to 1 (except $g$). These shows three different graphs for different values of the initial height (called $h_0$).

3. In the case of the frictionless bead/rod system, the only force acting is the normal force that the rod exerts on the bead (in the tangential direction). However, with the presence of a frictional force, the bead will exhibit a damped motion in the radial direction. Because the frictional force is given by the normal force times the coefficient of friction,

$$\vec{F}_{net} = -\mu N\hat{r} + N\hat{\theta} = \hat{r}(m\ddot{r} - m\omega^2 r) + \hat{\theta}(2m\dot{r}\omega + mr\alpha) \tag{16}$$

Now, we are left with the daunting task for solving for the two unknowns, $N$ and $r(t)$. First, we can eliminate the second derivative term in $\theta$ since we know that the rod moves with a constant angular velocity $\omega$. The normal force can easily be eliminated from the equation, from inspection of the tangential component

$$\begin{align} N &= 2m\dot{r}\omega \tag{17} \\ \rightarrow -2\mu m\dot{r}\omega &= m\ddot{r} - m\omega^2 r \tag{18} \\ \rightarrow \ddot{r} + 2\mu\dot{r}\omega - \omega^2 r &= 0 \tag{19} \end{align}$$

This differential equation can be solved using ODE techniques. The solution that one obtains from this is

$$r(t) = r_0 e^{-\mu\omega t}\left(\frac{\mu\sinh\left(\sqrt{\mu^2+1}\omega t\right)}{\sqrt{\mu^2+1}} + \cosh\left(\sqrt{\mu^2+1}\omega t\right)\right) \tag{20}$$

we can verify this result by substituting $\mu = 0$ to determine whether we obtain the answer in the frictionless case ($r(t) = r_0\cosh(\omega t)$). It is quite comforting to see that we do...

4. In order to solve this problem, we must first consider the mass oscillating about some equilibrium position. By virtue of a simple pendulum, the forces that act on the mass are tension and gravity. The force of gravity can be componentized into the radial and tangential directions, thus giving us the following equations

$$\vec{F}_{net} = (mg\cos\theta - T)\hat{r} + mg\sin\theta\hat{\theta} = \hat{r}(m\ddot{r} - m\omega^2 r) + \hat{\theta}(2m\dot{r}\omega + mr\alpha) \tag{21}$$

Furthermore, we know that the rope is reeled inwards with a velocity $V$. This simplifies our equations to

$$(mg\cos\theta - T)\hat{r} + mg\sin\theta\hat{\theta} = \hat{r}(-m\omega^2 L) + \hat{\theta}(2mV\omega + mL\alpha) \tag{22}$$

We can easily see that $T = mg\cos\theta + m\omega^2 L2$. We would like to perturb the system at its maximum tension in order to induce a higher probability that it will swing in a horizontal circle. The tension, in this case, is maximized when the angular velocity is at its maximum. (Note that this particular case does not correlate to $\theta = 0$, as the reeling velocity offsets the system's motion by inducing a Coriolis term to appear). Moreover, the angular acceleration at this point is also zero. Using this, we obtain

$$mg\sin\theta = 2mV\omega \tag{23}$$

$$T - mg\cos\theta = m\omega^2 L \tag{24}$$

Substitution and elimination yields

$$T - mg\cos\theta = m\frac{g^2 L\sin^2\theta}{4V^2} \tag{25}$$

The tension that is necessary for a conical pendulum is given by $T = \frac{mg}{\cos\theta}$. Substituting in this result, we obtain

$$\frac{mg}{\cos\theta} - mg\cos\theta = m\frac{g^2 L\sin^2\theta}{4V^2} \tag{26}$$

$$\rightarrow \frac{4V^2}{g} = L\cos\theta = H \tag{27}$$

This implies that we can determine the height at which a pendulum will transform into a conical one via a perturbation (near the equilibrium position). As one may imagine, for low reeling velocities, the height at which the transition will occur is small. This result could be generalized if we choose to perturb the system at an arbitrary angular acceleration $\alpha$.

# 3 Constraint Equations

1. We start by writing Newton's second law for each mass:

$$T - m_1 g = m_1 a_1$$
$$T - m_2 g = m_2 a_2$$

By using the length constraint, we obtained the relation $2A = a_1 + a_2$. We can thus write both equations in terms of $a_1$:

$$T - m_1 g = m_1 a_1$$
$$T - m_2 g = m_2(2A - a_1)$$

We can then subtract the equations and solve for $a_1$:

$$(m_2 - m_1)g = (m_1 + m_2)a_1 + 2m_2 A$$
$$a_1 = \frac{(m_2 - m_1)g + 2m_2 A}{m_1 + m_2}$$
$$a_2 = \frac{(m_2 - m_1)g + 2m_1 A}{m_1 + m_2}$$

where we have obtained $a_2$ using the constraint equation. Either of these expressions can be substituted back into the original equations to obtain the tension:

$$T = \frac{2m_1 m_2}{m_1 + m_2}(g + A)$$

Note that this is exactly the expression we would have obtained for a stationary Atwood's mahcine, except that $g$ is replaced with $g + A$.

2. We start by writing Newton's second law for every mass.

$$
\begin{aligned}
T - mg &= ma_1 \\
2T - 2mg &= 2ma_2 \\
T - 3mg &= 3ma_3
\end{aligned}
$$

We have 3 equations in 4 variables, so we need to add the constraint equation. While we could derive it formally from the conservation of the string length, it is evident that $a_1 + a_3 = -2a_2$. This gives us a system of 4 linear equations in 4 variables, which can be readily solved to give:

$$
\begin{aligned}
T &= \frac{6mg}{5} \\
a_1 &= \frac{g}{5} \\
a_2 &= \frac{g}{5} \\
a_3 &= -\frac{3g}{5}
\end{aligned}
$$

3. We start, as always, by writing Newton's second law for every mass:

$$
\begin{aligned}
2T - m_1 g &= m_1 a_1 \\
2T - m_2 g &= m_2 a_2 \\
&\vdots \\
2T - m_n g &= m_n a_n
\end{aligned}
$$

We have $n$ equations in $n+1$ variables; the last equation is the constraint equation. If there are two masses, then it is easy to see that the constraint is $a_1 + a_2 = 0$. We can extend this reasoning to $N$ masses, and introduce the final equation $\sum_{i=1}^{N} a_i = 0$. This gives a sufficient number of equations, so we can proceed with algebra. Manipulating any one of the above $N$ equations yields

$$a_i = \frac{2T}{m_i} - g$$

We can substitute these accelerations into the constraint equation, and arrive at

$$\sum_{i=1}^{N} \left( \frac{2T}{m_i} - g \right) = 2T \sum_{i=1}^{N} \frac{1}{m_i} - Ng = 0$$

Upon slight rearrangement, we then have

$$T = \frac{Ng}{2 \sum \frac{1}{m_i}}$$

Substituting this expression for the tension back into one of the $N$ statements of Newton's second law gives

$$\frac{Ng}{\sum \frac{1}{m_i}} = m_i g + m_i a_i$$

Finally, rearrangment gives

$$a_i = \frac{N}{\sum \frac{1}{m_i}} \frac{g}{m_i} - g$$

Here we have split the first term to show that it contains the harmonic mean of the masses. Upon inspection, we see that if an individual mass is greater than the harmonic mean, then it will sink; if it is less than the harmonic mean, it will rise.

4. In a slightly different approach to other pulley problems, we will write Newton's second law for the pulleys themselves instead of the masses. Let the tension in the top string be $T_1$ and all successive tensions indexed accordingly. Similarly, the acceleration of the first mass will be $a_1$, and the rest follow. Then Newton's second law gives a simple equation for every pulley:

$$\begin{aligned} T_1 &= 2T_2 \\ T_2 &= 2T_3 \\ &\vdots \end{aligned}$$

We find immediately that the tensions decline in a geometric progression as we move deeper into the pulley system. Now, in a manner similar to many problems involving infinite sequences, consider all the pulleys except the first one. These pulleys form a system identical to the full system, except that it is accelerating downwards at a rate $a_1$. Thus, the effective gravitational acceleration is $g - a_1$ (this decrease in the gravitational acceleration is the same decrease you experience when an elevator begins accelerating downwards). Therefore, the tension $T_2$ in the top string must be $\frac{g-a_1}{g}T_1$, because changing the effective gravity changes the tension in the same proportion (see problem 1 for an example of this). However, we have already determined that $T_2 = \frac{1}{2}T_1$. For both these relations to hold, we must have $a_1 = \frac{g}{2}$.