Vectors

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1 Introduction

Scalar quantities: magnitude only

*Ex. A box is pushed for **2 meters***

Vector quantities: both magnitude and direction.

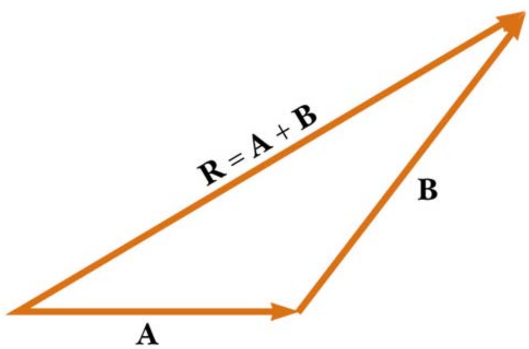
*Ex. A car travels **60 mph North***

A vector is geometrically represented by an arrow.

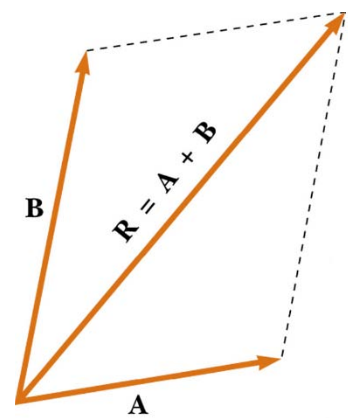
2 Vector Addition

There are two ways to graphically add vectors.

1) Head-to-tail Method



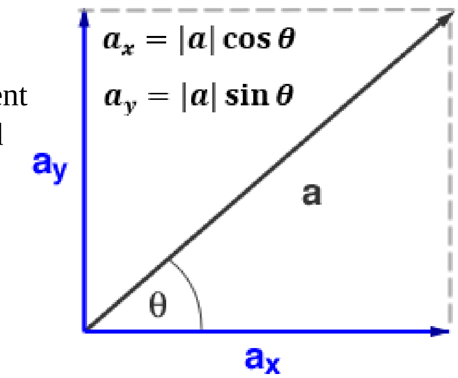
2) Parallelogram Method (tail-to-tail)



3 Components

In a rectangular coordinate system, a vector has an x-component and a y-component. In component form, vector **a** be expressed as <a_1, a_2> in 2D, or <a_1, a_2, a_3> in 3D.

Using components, we can calculate magnitude and direction.



Magnitude: $|a| = \sqrt{a_x^2 + a_y^2}$

Direction: $\tan\theta = \frac{a_y}{a_x}$

*Note: The SAT Subject Test does not allow the use of a calculator, so it is important to know properties of 30-60-90 triangles and 45-45-90 triangles.

Checkpoint Problem: Find the component form of a vector with magnitude 10 and direction angle 120 degrees.

4 Unit Vectors

A unit vector is simply a vector with a magnitude of 1. A vector **v** can be scaled to a unit vector by $\frac{v}{|v|}$

Checkpoint Problem: Find the unit vector of <1,2,2>.

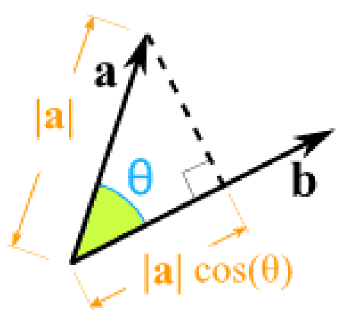
Three important unit vectors (called standard unit vectors**)**:

$$\hat{\imath} =< 1,0,0 > \qquad \hat{\jmath} =< 0,1,0 > \qquad \hat{k} =< 0,0,1 >$$

So **a** = $<a_1, a_2, a_3>$ = $a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

5 Dot Products

Two vectors can be “multiplied” using the Dot Product. The Dot Product of vectors **a** and **b** is the magnitude of **b** multiplied by the projection of **a** onto b (or vice versa), resulting in a scalar quantity. The Dot Product can be calculated in two ways:



$\boldsymbol{a} \cdot \boldsymbol{b} = |a||b| \cos\theta$

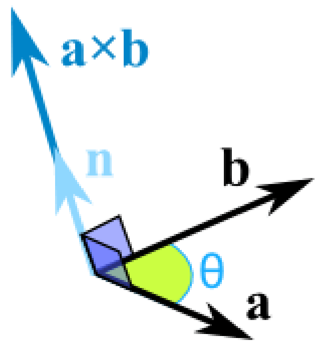
Or, if $\boldsymbol{a} =< a_1, a_2 >$ and $\boldsymbol{b} =< b_1, b_2 >$,

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$

Checkpoint Problem: Find the angle between the vectors $\boldsymbol{a}$=<*2, 5, -1*> and $\boldsymbol{b}$=<*-3, 2, 6*>.

6 Cross Products

Two vectors can also be "multiplied" using the Cross Product. The Cross Product of vectors **a** and **b** is another vector perpendicular to both, as shown in the diagram to the right. The Cross Product can calculated in two ways:



$$\boldsymbol{a} \times \boldsymbol{b} = |a||b|\sin(\theta)\,\boldsymbol{n}$$

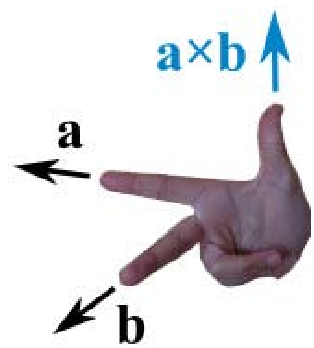
n is the unit vector perpendicular to both **a** and **b**

Or, if $\boldsymbol{a} =< a_x, a_y, a_z >$ and $\boldsymbol{b} =< b_x, b_y, b_z >$, and **a** and **b** are centered at the origin,

$$\boldsymbol{a} \times \boldsymbol{b} =< a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x >$$

Checkpoint Problem: Given that **a** = <-2, 1, 1>, **b** = <2, 1, 1> and **c** = **a** × **b**, find the magnitude of **c**?

Since the Cross Product is a vector perpendicular to two other vectors, it could point either up or down.



Right Hand Rule: Point your index finger along the first vector, and point you middle finger along the second vector. The cross product is in the direction your thumb is pointing.

7 Summary of Vector Operations (3D)

If $\boldsymbol{a} =< a_1, a_2, a_3 >$ and $\boldsymbol{b} =< b_1, b_2, b_3 >$, and k is a scalar:

$$\boldsymbol{a} + \boldsymbol{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$$

$$\boldsymbol{a} - \boldsymbol{b} =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >$$

$$k\boldsymbol{a} =< ka_1, ka_2, ka_3 >$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\boldsymbol{a} \times \boldsymbol{b} =< a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 >$$

8 Problems

1. Which of the following statements does NOT describe a vector quantity?
   (A) An object that has a speed of 20 m/s in the positive x-direction.
   (B) A 30-newton force acts at angle of 30° above the horizontal.
   (C) A car travels a distance of 2.0 kilometers.
   (D) The acceleration of gravity, g, is directed downward.
   (E) A mass is displaced 5.0 meters horizontally.

2. Two beetles run across flat sand, starting at the same point. Beetle 1 runs 0.50 m due east, then 0.80 m at 30° north of due east. Beetle 2 also makes two runs; the first is 1.6 m at 40° east of due north. What must be (a) the magnitude and (b) the direction of its second run if it is to end up at the new location of beetle 1?

3. Three vectors are given by $\vec{a} = 3.0\hat{\imath} + 3.0\hat{\jmath} - 2.0\widehat{k}$, $\vec{b} = -1.0\hat{\imath} - 4.0\hat{\jmath} + 2.0\widehat{k}$, and $\vec{c} = 2.0\hat{\imath} + 2.0\hat{\jmath} + 1.0\widehat{k}$. Find a) $\vec{a} \cdot (\vec{b} \times \vec{c})$, b) $\vec{a} \cdot (\vec{b} + \vec{c})$.

4. Use the definition of scalar product, $\vec{a} \cdot \vec{b} = ab\cos\theta$, and the fact that $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$ to calculate the angle between the two vectors given by $\vec{a} = 3.0\hat{\imath} + 3.0\hat{\jmath} + 3.0\widehat{k}$, and $\vec{b} = 2.0\hat{\imath} + 1.0\hat{\jmath} + 3.0\widehat{k}$.

5. Vector $\vec{A}$ has a magnitude of 6.00 units, vector $\vec{B}$ has a magnitude of 7.00 units, and $\vec{A} \cdot \vec{B}$ has a value of 14.0. What is the angle between the directions of $\vec{A}$ and $\vec{B}$?