Intro to Quantum Mechanics 2 Solutions

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1 Solutions

1. We will use the rotation operator to argue that $|+z\rangle$ has an eigenvalue of $\frac{\hbar}{2}$.

$$\hat{R}(\theta\mathbf{k})|+z\rangle = e^{\frac{-i\theta\hat{J}_z}{\hbar}}|+z\rangle$$

Using a Taylor series expansion:

$$= \left[1 + \left(\frac{-i\theta\hat{J}_z}{\hbar}\right) + \frac{\left(\frac{-i\theta\hat{J}_z}{\hbar}\right)^2}{2!} + \ldots\right]|+z\rangle$$

$$= \left[|+z\rangle + \left(\frac{-i\theta\hat{J}_z}{\hbar}\right)|+z\rangle + \frac{\left(\frac{-i\theta\hat{J}_z}{\hbar}\right)^2}{2!}|+z\rangle + \ldots\right]$$

Using the fact that $\hat{J}_z$ acting on $|+z\rangle$ yields $\frac{\hbar}{2}|+z\rangle$, we have:

$$= \left[|+z\rangle + \left(\frac{-i\theta}{2}\right)|+z\rangle + \frac{\left(\frac{-i\theta}{2}\right)^2}{2!}|+z\rangle + \ldots\right]$$

$$= \left[1 + \left(\frac{-i\theta}{2}\right) + \frac{\left(\frac{-i\theta}{2}\right)^2}{2!} + \ldots\right]|+z\rangle$$

$$= e^{\frac{-i\theta}{2}}|+z\rangle$$

This is what we hoped it would be since the state has only picked up a phase factor. We leave it to the reader to do basically the exact same thing to repeat with the other eigenvector—there isn't much difference.

One might note that $\hbar$ and $-\hbar$ could therefore have also been the eigenvalues, but based on the results of the Stern-Gerlach experiments as discussed in the previous lecture, we prefer the factor of half. Of course this logic may also seem circular, but we choose these arbitrary definitions so that the math works out nicely.

2. We will use the definition of a basis-representation of an operator. Below is the two dimensional case, since $|+z\rangle$ and $|-z\rangle$ are the basis states:

$$\hat{O} = \begin{pmatrix} \langle +z|\hat{O}|+z\rangle & \langle +z|\hat{O}|-z\rangle \\ \langle -z|\hat{O}|+z\rangle & \langle -z|\hat{O}|-z\rangle \end{pmatrix}$$

In this case, $\hat{O} = \hat{J}_z$. So we'll say:

$$\hat{J}_z = \begin{pmatrix} \langle +z|\hat{J}_z|+z\rangle & \langle +z|\hat{J}_z|-z\rangle \\ \langle -z|\hat{J}_z|+z\rangle & \langle -z|\hat{J}_z|-z\rangle \end{pmatrix}$$

$$\hat{J}_z = \begin{pmatrix} \langle +z|\frac{\hbar}{2}|+z\rangle & \langle +z|\frac{-\hbar}{2}|-z\rangle \\ \langle -z|\frac{\hbar}{2}|+z\rangle & \langle -z|\frac{-\hbar}{2}|-z\rangle \end{pmatrix}$$

$$\hat{J}_z = \begin{pmatrix} \frac{\hbar}{2}\langle +z|+z\rangle & \frac{-\hbar}{2}\langle +z|-z\rangle \\ \frac{\hbar}{2}\langle -z|+z\rangle & \frac{-\hbar}{2}\langle -z|-z\rangle \end{pmatrix}$$

$$\hat{J}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & \frac{-\hbar}{2} \end{pmatrix}$$

This is $\hat{J}_z$ in the z-basis. We can also write this matrix as

$$\hat{J}_z = \frac{\hbar}{2}\sigma_z$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where σ_z is known as the **z-pauli matrix**. This matrix plays a pivotal role in describing angular momentum, and, along with its x and y counterparts and the identity matrix, forms a basis which spans all 2x2 Hermitian matrices. Higher order pauli matrices also exist, but are more complicated.

Let's determine $\hat{J}_z$ in the x-basis now. The calculations are very similar. We have:

$$\hat{J}_z = \begin{pmatrix} \langle +x|\hat{J}_z|+x\rangle & \langle +x|\hat{J}_z|-x\rangle \\ \langle -x|\hat{J}_z|+x\rangle & \langle -x|\hat{J}_z|-x\rangle \end{pmatrix}_x$$

Since the x-states are NOT eigenvectors of the $\hat{J}_z$, we need to express them in terms of the z-states so that we can do a computation. Fortunately, we know from the previous lecture that:

$$|+x\rangle = \frac{\sqrt{2}}{2}\left(|+z\rangle + |-z\rangle\right)$$

$$|-x\rangle = \frac{\sqrt{2}}{2}\left(|+z\rangle - |-z\rangle\right)$$

The bra vectors, of course, are just the complex-conjugate transpose of these two vectors, meaning:

$$\langle +x| = \frac{\sqrt{2}}{2}\left(\langle +z| + \langle -z|\right)$$

$$\langle -x| = \frac{\sqrt{2}}{2}\left(\langle +z| - \langle -z|\right)$$

We can plug these back into the matrix and we can now use the effects of $\hat{J}_z$ acting on the matrix to determine the $\hat{J}_z$ in the x-basis. Note that by simple observation, $\hat{J}_z$ acting on $|+x\rangle$ yields $\frac{\hbar}{2}|-x\rangle$, and it acting on $|-x\rangle$ results in $\frac{\hbar}{2}|+x\rangle$. Using this fact, we are left with:

$$\hat{J}_z = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}_x$$

Once again this can be expressed in terms of a Pauli matrix (the below is in the x-basis, not the z-basis!):

$$\hat{J}_z = \frac{\hbar}{2}\sigma_x$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Similar to before, σ_x is called the **x-pauli matrix**.

3. This problem can be done via the standard approach of writing z in the y-basis, evaluating the effects of $\hat{J}_y$ on the resulting vector and then converting back to the y-basis. There is a much faster way, however. We can rename our axises, provided that they still maintain the order (**i**, **j**, **k**). The permutations that retain this order are: (**k**, **i**, **j**) and (**j**, **k**, **i**). Since we have done a lot of work in the z-basis, we will let **j** map to **k**, meaning the +z-state will go to the +x-state. This makes our new problem a much easier one, in which we try to determine:

$$\hat{R}(\theta\mathbf{k})|+x\rangle$$

$$= \hat{R}(\theta\mathbf{k})\frac{\sqrt{2}}{2}\left(|+z\rangle + |-z\rangle\right)$$

$$= e^{\frac{-i\theta\hat{J}_z}{\hbar}}\frac{\sqrt{2}}{2}\left(|+z\rangle + |-z\rangle\right)$$

Using the results of problem 1, we have:

$$= \frac{\sqrt{2}}{2}\left(e^{\frac{-i\theta}{2}}|+z\rangle + e^{\frac{i\theta}{2}}|-z\rangle\right)$$

$$= e^{\frac{-i\theta}{2}}\frac{\sqrt{2}}{2}\left(|+z\rangle + e^{i\theta}|-z\rangle\right)$$

We can disregard the phase factor out in front since it won't affect the state.

$$= \frac{\sqrt{2}}{2}\left(|+z\rangle + e^{i\theta}|-z\rangle\right)$$

In the beginning we changed our coordinate system, so we need to change it back. Under our adjustment, y would go to z, so when we reverse it, z goes to y. This means we have:

$$\hat{R}(\theta\mathbf{k}) = \frac{\sqrt{2}}{2}\left(|+z\rangle + e^{i\theta}|-z\rangle\right)$$

$$\hat{R}(\theta\mathbf{j}) = \frac{\sqrt{2}}{2}\left(|+y\rangle + e^{i\theta}|-y\rangle\right)$$

In order to express our answer in the z-basis, we will use the facts that:

$$|+y\rangle = \frac{\sqrt{2}}{2}\left(|+z\rangle + i|-z\rangle\right)$$

$$|-y\rangle = \frac{\sqrt{2}}{2}\left(|+z\rangle - i|-z\rangle\right)$$

After plugging in and doing some algebra, we have:

$$\hat{R}(\theta\mathbf{j}) = \frac{1}{2}\left[\left(1+e^{i\theta}\right)|+z\rangle + i\left(1-e^{i\theta}\right)|-z\rangle\right]$$

This answer makes sense when $\theta = \frac{\pi}{2}$, as it simplifies to $e^{i\frac{\pi}{4}}|+x\rangle$, which is just a phase shift of the +x-state, which is what expected from our intuition.

The technique demonstrated in this solution can be extremely useful because we use what we know already instead of re-deriving new quantities. After all, x, y, and z are all just names, and we never used any fact inherent about their specific physical directions to derive any the results.

4. The probability of transmission between adjacent polarizers (given that the photons initially have their transmission axes in the y direction) is

$$|\langle y'|y\rangle|^2 = \cos^2(d\phi)$$

We can write $d\phi = \lim_{N\to\infty}\left(\frac{\phi}{N}\right)$. Since we have N polarizers, the probabilities multiply, and we get

$$P = \lim_{N\to\infty}\left(\cos^2\left(\frac{\phi}{N}\right)\right)^N$$

Now, we must evaluate this limit. This can be done by taking the natural log of both sides, and applying L'Hôpital's rule

$$\ln P = \lim_{N\to\infty} N\cos^2\left(\frac{\phi}{N}\right)$$

$$\ln P = \lim_{N\to\infty}\frac{\frac{\phi}{N^2}\sin\left(2\frac{\phi}{N}\right)}{-\frac{1}{N^2}}$$

$$\ln P = 0 \to P = 1$$

This is an unusual result. It shows that, by staggering the polarizers, you can almost guarantee that a photon will emerge out of the final state. This is unlike just sending a photon through a polarizer tilted at an angle ϕ, whose probability is $\cos^2\phi$.

5. We know, from classical mechanics, that

$$\tau = \frac{dL}{dt} = \frac{d}{dt}\left(\langle L\rangle n\right) = \langle L\rangle N$$

where n measures the number of particles which strike the surface of the disk. The last statement utilizes the fact that $\frac{dn}{dt} = N$, which was a given. To find the expected value of the angular momentum, we have to use the fact that the eigenvalues for the left and right circularly polarized photons are $-\hbar$ and $\hbar$, respectively. Using the formula for expected value:

$$\langle L\rangle = \hbar\left(|a\langle R|x\rangle + b\langle R|y\rangle|^2\right) - \hbar\left(|a\langle L|x\rangle + b\langle L|y\rangle|^2\right)$$

Substituting in results for the inner products

$$\langle L\rangle = \hbar\left(\left|\frac{a}{\sqrt{2}} - \frac{bi}{\sqrt{2}}\right|^2\right) - \hbar\left(\left|\frac{a}{\sqrt{2}} + \frac{bi}{\sqrt{2}}\right|^2\right)$$

Thus

$$\langle L\rangle = \frac{\hbar}{2}\left(|a-bi|^2 - |a+bi|^2\right)$$

After simplification (considering the fact that a and b are complex), we get

$$\langle L\rangle = i\hbar(ab^* - ba^*)$$

This immediately states that a and b must be complex for the expected value of the angular momentum to be non-zero. Thus, the torque exerted on the disk is

$$\tau = i\hbar N(ab^* - ba^*)$$

6. The initial state of a photon is given by

$$|\psi\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle$$

While the light progresses through the crystal, its state is only changed by a phase factor. However, because the crystal is birefringent, the x and y projections are evolved by different phase factors. The phase factor for the x vector is

$$|x\rangle \rightarrow e^{i\phi}|x\rangle$$

This phase factor is $\frac{n_x\omega}{c}d = \frac{2\pi n_x}{\lambda}d$. This is because the velocity of light is reduced to $\frac{c}{n_x}$, and will thus take a time $\frac{dn_x}{c}$ to traverse the crystal. Multiplying this by an angular frequency $\omega = 2\pi f$ gives the appropriate phase factor. Making an additional substitution $\lambda f = c$ (from basic wave mechanics), yields the desired result. Similarly,

$$|y\rangle \rightarrow e^{i\frac{2\pi n_y}{\lambda}d}|y\rangle$$

Thus, after the light travels the full length of the crystal, the new state becomes

$$|\psi\rangle = e^{i\frac{2\pi n_x}{\lambda}d}\cos\phi|x\rangle + e^{i\frac{2\pi n_y}{\lambda}d}\sin\phi|y\rangle$$

We want the probability that the photon will be right-circularly polarized. Thus, we desire

$$\langle R|\psi\rangle = e^{i\frac{2\pi n_x}{\lambda}d}\cos\phi\langle R|x\rangle + e^{i\frac{2\pi n_y}{\lambda}d}\sin\phi\langle R|y\rangle$$

Simplification gives

$$\langle R|\psi\rangle = \frac{1}{\sqrt{2}}e^{i\frac{2\pi n_x}{\lambda}d}\cos\phi - \frac{i}{\sqrt{2}}e^{i\frac{2\pi n_y}{\lambda}d}\sin\phi$$

Taking the absolute value squared of this gives

$$|\langle R|\psi\rangle|^2 = \frac{1}{2} + \sin\phi\cos\phi\sin\left(\frac{2\pi d}{\lambda}(n_y - n_x)\right)$$

$$|\langle R|\psi\rangle|^2 = \frac{1}{2}\left(1 + \sin 2\phi\sin\left(\frac{2\pi d}{\lambda}(n_y - n_x)\right)\right)$$