Thermodynamics Lecture Solutions

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1 Solutions

1. This is an interesting problem because the spheres undergo identical expansions. The difference, however, is key: while the suspended sphere's center of mass lowers, the supported sphere's center of mass rises. This gives rise to a different conservation of energy statement for both masses, thus accounting for the variation in their final temperature. For the suspended ball, the conservation of energy statement is

$$\Delta Q = mc\Delta T_1 - mg\Delta r_1$$

Similarily, for the supported ball

$$\Delta Q = mc\Delta T_2 + mg\Delta r_2$$

We want to find $\Delta T_1 - \Delta T_2$. However, we are not given Δr. TO find it, we must use the coefficient of linear expansion. Since the radius is a linear dimension, we can say

$$\Delta r_1 = \alpha R\Delta T_1$$

$$\Delta r_2 = \alpha R\Delta T_2$$

Plugging this back in to our previous equations and solving for ΔT_1 and ΔT_2, we get

$$\Delta T_1 = \frac{\Delta Q}{mc - mg\alpha R}$$

$$\Delta T_2 = \frac{\Delta Q}{mc + mg\alpha R}$$

Subtracting the two

$$\Delta T_1 - \Delta T_2 = \frac{2Rg\alpha\Delta Q}{m\left(c^2 - g^2\alpha^2R^2\right)}$$

2. This is a question of radiation, and thus we must employ the Stephan-Boltzmann equation. Because the object is a black-body, its emissivity is $\epsilon = 1$. Because the substance is enclosed in a spherical container, we can also assume that its surface area is $4\pi R^2$. Thus

$$\frac{dE}{dt} = -\epsilon\sigma AT^4 = -4\pi R^2\sigma T^4$$

To find the rate of change of energy, we must use the first law of thermodynamics

$$U = q - W$$

$$\frac{dU}{dt} = \frac{dE}{dt} = \frac{dq}{dt} - \frac{dW}{dt} = \frac{dq}{dt}$$

because the work done is zero (as we do not assume a volume expansion). Moreover, $\frac{dq}{dt} = Mc\frac{dT}{dt}$,

$$\frac{dE}{dt} = Mc\frac{dT}{dt} = -4\pi R^2\sigma T^4 = -4\pi R^2\sigma T^4$$

This differential equation can be solved easily through integration. The result is

$$-\frac{1}{3}\left(\frac{1}{T^3} - \frac{1}{T_0^3}\right) = -4\pi R^2\sigma t$$

$$T(t) = T_0\left(\frac{1}{12\pi R^2 T_0^3\sigma t + 1}\right)^{1/3}$$

3. In order to find the sphere's temperature as a function of time, we must find its heat flow rate. This is given by

$$\frac{dq}{dt} = \kappa A\frac{\Delta T}{\Delta x}$$

If we build up the sphere through concentric spherical shells of thickness dr, we know that the surface area that the heat will flow through is $4\pi r^2$. This gives us

$$\frac{dq}{dt} = R = \kappa 4\pi r^2\frac{dT}{dr}$$

$$R\frac{dr}{4\pi r^2} = \kappa dT$$

Integrating from $r = a$ to $r = b$ and from $T = T$ to $T = T_0$ gives

$$R = \frac{4\pi\kappa ab(T - T_0)}{b - a}$$

Because we know that $R = -\frac{dq}{dt} = -mc\frac{dT}{dt}$, we get

$$-mc\frac{dT}{dt} = \frac{4\pi\kappa ab(T - T_0)}{b - a}$$

Solving this differential equation,

$$T(t) = T_0 + (T - T_0)e^{-\beta t}$$

$$\beta = \frac{4\pi\kappa ab}{b - a}$$

4. Because we assume that all collisions are elastic, kinetic energy is conserved. The initial energy of the di-atomic molecules is $\frac{5}{2}n_2k_bT_2$ and the initial energy of the mono-atomic molecules is $\frac{3}{2}n_1k_bT_1$. The equilibrium temperature is the same for all constituents. This gives

$$\frac{5}{2}n_2k_bT_2 + \frac{3}{2}n_1k_bT_1 = T\left(\frac{5}{2}n_2k_b + \frac{3}{2}n_1k_b\right)$$

$$T = \frac{5n_2T_2 + 3n_1T_1}{5n_2 + 3n_1}$$

To quantify the situation a bit more, we could also determine the rate of change of energy transfer between the two gases. As you may know, the **mean free path** is a statistically obtained result that describes he average distance a molecule has to travel before it encounters a collision. Because, for our purposes, this length is very small, we can approximate the time between collisions as

$$dt \approx \frac{\lambda_{mean}}{v_{rms}}$$

Through an analysis of the Maxwell-Boltzmann distribution, we can show that

$$\lambda_{mean} = \frac{1}{\sqrt{2}\pi d^2 N_A\left(\frac{n}{V}\right)}$$

where N_A is Avogadro's number, and we define $\frac{n}{V} \equiv \rho$. Let us assume that we want to determine the energy transfer between the mono-atomic and di-atomic gases. In this case, the average distance the monoatomic gas travels before encountering a collision is

$$dt \approx \frac{\lambda_{mean}}{v_{rms}} = \frac{\sqrt{\frac{M}{3RT}}}{\sqrt{2}\pi d^2 N_A \rho}$$

where M is the gas's molar mass. The energy transferred during one collision by all molecules is

$$dP = \frac{3}{2}nc_v\frac{dT}{dt} = \frac{3}{2}nc_v\frac{dT}{\frac{\sqrt{\frac{M}{3RT}}}{\sqrt{2}\pi d^2 N_A \rho}}$$

Upon simplification,

$$dP = \frac{3nc_v\sqrt{6}\pi d^2 N_A R\rho}{2\sqrt{M}}\sqrt{T}dT$$

After we integrate

$$P(T) = \frac{nc_v\sqrt{6}\pi d^2 N_A R\rho}{\sqrt{M}}\left(T^{3/2} - T_0^{3/2}\right)$$

Interestingly, the power is proportional to temperature to the 3/2s power, proportional to the cubed of the root mean velocity of the particles.

6. The efficiency of a cycle is defined as

$$\eta = 1 - \frac{Q_c}{Q_h}$$

The first isochoric process is the heating phase; we can thus find Q_h from this

$$Q_h = nc_v(T_B - T_A)$$

We also know that the ratio of the pressures and temperatures is constant in an isochoric process

$$\frac{\alpha P_0}{P_0} = \alpha = \frac{T_B}{T_A}$$

Substitution of this fact into the previous formula yields

$$Q_h = nc_v T_A(\alpha - 1)$$

Q_c is a bit trickier to calculate. Following from the isochoric process, the adiabatic process requires or expels no heat. Therefore, the only other process is the isobaric. The equation for a heat change in an isobaric process is:

$$Q_c = nc_p(T_A - T_C)$$

In this case, we know everything but T_c . In order to determine its value, we have to work our way around the cycle, including the adiabatic process. Fortunately, we have the adiabatic relationship with us:

$$P^{1-\gamma}T^{\gamma} = constant$$

And remembering the relationship between the temperature at B and A

$$T_B = \alpha T_A$$

We set up the condition to find the temperature at C:

$$(\alpha P_0)^{1-\gamma}(\alpha T_A)^{\gamma} = P_0^{1-\gamma}T_C^{\gamma}$$

$$T_C = T_A\alpha^{\frac{1}{\gamma}}$$

Plugging this equation back in to the result for Q_c:

$$Q_c = nc_pT_A(1-\alpha^{\frac{1}{\gamma}})$$

Since, the efficiency equation prompts for an absolute value, we notice that Q_c is negative, and thus we negate our result to get:

$$Q_c = nc_pT_A(\alpha^{\frac{1}{\gamma}}-1)$$

so, the efficiency is

$$\eta = 1 - \frac{nc_pT_A(\alpha^{\frac{1}{\gamma}}-1)}{nc_vT_A(\alpha-1)}$$

$$\eta = 1 - \gamma\frac{\alpha^{\frac{1}{\gamma}}-1}{\alpha-1}$$

7. For an isothermal expansion, the temperature is constant. This implies that Boyle's law is held:

$$PV = P_0V_0 = nRT_0$$

If the piston is displaced a small value δ, the new pressure is

$$P = \frac{nRT_0}{A(h+\delta)}$$

The restoring force is just the pressure times the surface area of the cylinder

$$PA = \frac{nRT_0}{(h+\delta)}$$

According to newton's second law

$$PA - mg = m\ddot{\delta}$$

$$\frac{nRT_0}{(h+\delta)} - mg = m\ddot{\delta}$$

If we expand this in a taylor series for small displacement values, we get

$$\frac{nRT_0}{h}\left(1-\frac{\delta}{h}\right) - mg = m\ddot{\delta}$$

The first term, when distributed out is an equilibrium term and cancels with the force due to gravity. This leaves

$$-\frac{nRT_0}{h^2}\delta = m\ddot{\delta}$$

This is simple harmonic motion with

$$T = 2\pi\sqrt{\frac{mh^2}{nRT_0}}$$

A similar method could be done with an adiabatic expansion. The result is

$$T = 2\pi\sqrt{\frac{mh^{\gamma+1}}{\gamma P_0 V_0^{\gamma}}}$$