Lagrangian Mechanics

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1 Some Definitions

$\frac{\partial z}{\partial x}$ - Partial derivative of z with respect to x, calculated by assuming all variables except for x to be constant

$\dot{x}$ - First time derivative of x or $\frac{dx}{dt}$

$\ddot{x}$ - Second time derivative of x or $\frac{d^2x}{dt^2}$

∇U - defined *in cartesian coordiantes* as vector $\frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}}$

2 The Gradient in Polar Coordinates

We define the gradient in cartesian coordinates as

$$\nabla U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}}$$

The gradient also has to satisfy

$$dU = \nabla U \cdot \vec{dr}$$

Making the substitution for dU in Polar Coordinates

$$\frac{\partial U}{\partial r}dr + \frac{\partial U}{\partial \theta}d\theta + \frac{\partial U}{\partial z}dz = \nabla U \cdot \vec{dr}$$

$$\frac{\partial U}{\partial r}dr + \frac{\partial U}{\partial \theta}d\theta + \frac{\partial U}{\partial z}dz = (\nabla U)_r dr + (\nabla U)_\theta r d\theta + (\nabla U)_z dz$$

Now equating each component of dr on each side we get

$$(\nabla U)_r = \frac{\partial U}{\partial r}$$

$$(\nabla U)_\theta = \frac{1}{r}\frac{\partial U}{\partial \theta}$$

$$(\nabla U)_z = \frac{\partial U}{\partial z}$$

Therefore the gradient in Polar Coordinates comes out to be

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}}$$

We will end up using this fact later on when we work in these coordinates.

3 The Basics

The idea behind Lagrangian Mechanics is the ability to use its equations in any situation, no matter what coordinate system we choose to use.

3.1 The Lagrangian

The Lagrangian is an expression that forms the basis of Lagrangian Mechanics. It is defined as

$$\mathcal{L} = T - U$$

where T is the total kinetic energy of the system and U is the total potential energy. Notice that $\mathcal{L}$ is a function of position (due to the potential energy) and velocity (due to the kinetic energy). Also be grateful that we are not dealing with any vectors.

3.2 The Euler-Lagrange Equation

Now we need to use the Lagrangian expression to determine the laws of physics of a system. To accomplish this, we use the Euler-Lagrange Equation. Assume we are using a coordinate system with coordinates x_1, x_2, x_3... and each coordinate has a respective velocity component $\dot{x_1}$, $\dot{x_2}$, $\dot{x_3}$..., so then for the nth coordinate

$$\frac{\partial \mathcal{L}}{\partial x_n} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x_n}}$$

So for a system using m coordinates, there will be m such equations to write. To really understand and get a hang of using these ideas, practicing them in examples will really help.

4 Examples

4.1 Particle moving in 2D conservative force field using Cartesian Coordinates

This is the simplest case to consider. It is also convenient that the results we get should agree with Newtonian Mechanics since it is used in this coordinate system. This can help us verify the method of Lagrangian Mechanics.

The first step is to define the position and velocity in the coordinate system.

Position: $x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$

Velocity: $\dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}$

Now we use these definitions to plug values into the Lagrangian expression.

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)$$

Finally we will apply the Euler-Lagrangian equation to each coordinate.

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = \frac{d}{dt}(m\dot{x})$$

Notice that $-\frac{\partial U}{\partial x} = F_x$

$$F_x = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$
$$-\frac{\partial U}{\partial y} = \frac{d}{dt}(m\dot{y})$$

Again notice that $-\frac{\partial U}{\partial y} = F_y$

$$F_y = m\ddot{y}$$

Let us take a moment and see if we can make some generalizations. The left side of the Euler-Lagrange equation came out to be force components. The expression is therefore often called the *generalized force.* Similarly, the right side produced a rate of change of momentum. Therefore the term $\frac{\partial \mathcal{L}}{\partial \dot{x}}$ is called the *generalized momentum.* So the Euler-Lagrange equation reads: *the generalized force is the rate of change of the generalized momentum.* This is very similar to Newton's Second Law!

4.2 Particle moving in 2D conservative force field using Polar Coordinates

We will use the same scenario as last time, but now we will use polar coordinates to analyze it. Again we will first define the position and velocity vectors.

Position: $r\cos\theta\hat{\mathbf{\imath}} + r\sin\theta\hat{\mathbf{\jmath}}$

Velocity: $(\dot{r}\cos\theta - r\sin\theta\dot{\theta})\hat{\mathbf{\imath}} + (\dot{r}\sin\theta + r\cos\theta\dot{\theta})\hat{\mathbf{\jmath}}$

Again we will find the Lagrangian expression from these definitions.

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r,\theta)$$

Now we apply the Euler-Lagrange equation in both the r and θ coordinates.

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$mr\dot{\theta}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(m\dot{r})$$

Notice that $-\frac{\partial U}{\partial r} = F_r$

$$F_r = m(\ddot{r} - r\dot{\theta}^2) = ma_r$$

This expression directly states the idea of centripetal force, with a_r being centripetal acceleration. Now let's see what happens in the θ coordinate.

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$
$$-\frac{\partial U}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta})$$

From the definition of the gradient in polar coordinates, we see that $-\frac{\partial U}{\partial \theta} = rF_\theta = \tau$. So the left side turns into torque, and by more examination, we see that $mr^2\dot{\theta} = L$ or angular momentum. We then get the common result

$$\tau = \frac{dL}{dt}$$

or the rotational form of Newton's Second Law.

These last two examples served to show us that Lagrangian Mechanics works with scenarios we are familiar with. Now let's try to solve a more complex problem using it.

4.3 Particle confined to cylinder

Problem: Consider a particle of mass m constrained to move on a frictionless cylinder of radius R as shown below. In addition to the force of constraint (the normal force of the cylinder) the only force on the mass is a force $\vec{F} = -k\vec{r}$ directed to the origin. Solve for the motion of the mass.

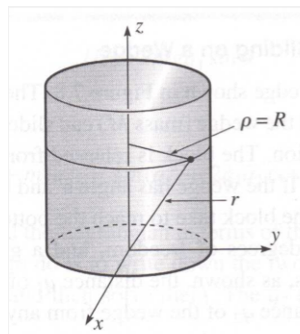


Figure 1: Diagram of Scenario

First we set up our coordinate system as usual.

Position: $R\cos\theta\hat{\mathbf{i}} + R\sin\theta\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Velocity: $(-r\sin\theta\dot{\theta})\hat{\mathbf{i}} + (r\cos\theta\dot{\theta})\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$

This time we know the conservative force acting on the particle $\vec{F} = -k\vec{r}$. So, the potential energy associated with the force is $U = \frac{1}{2}kr^2$ by integration. Notice that $r^2 = R^2 + z^2$ by the geometry in the problem. Now we can write the Lagrangian expression

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2)$$

In this case, our two changing variables are θ and z. Now we use the Euler-Lagrange equation on each of these.

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

$$-kz = m\ddot{z}$$

This tells us that the particle expresses simple harmonic motion (SHM) in the vertical direction with period $T = 2\pi\sqrt{\frac{m}{k}}$.

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$

$$0 = \frac{d}{dt}(mR^2\dot{\theta})$$

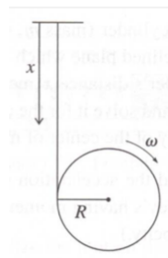
This equation tells us that angular momentum is conserved, as expected, and therefore the particle moves with constant angular velocity.

Something important to note when using Lagrangian Mechanics is that it is important to solve all the Euler-Lagrange equations because, as seen in the examples above, each will have its own unique interpretation and implication.

5 Problems

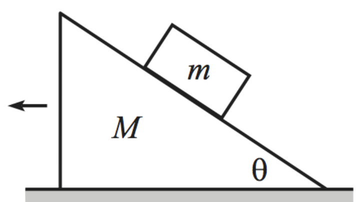
1. Determine the acceleration of each mass, m_1 and m_2, in an Atwood Machine, assuming a massless pulley and massless string.

2. Below is a crude model of a yo-yo. A massless string is suspended vertically from a fixed point and the other end is wrapped several times around a uniform cylinder of mass m and radius R. When the cylinder is released it moves downward, rotating as the string unwinds. Determine the downward acceleration of the cylinder/yo-yo.



3. An object with mass m, radius R, and moment of inertia I rolls without slipping straight down an inclined plane which is at an angle α from the horizontal. Determine the linear acceleration of the object as it rolls.

4. A block of mass m is held motionless on a frictionless incline of mass M with an angle of inclination θ. The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the incline?



5. Two massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at an end. One stands on top of the other, as shown below. The bottom end of the lower stick is hinged at the ground. They are held such that the lower stick is vertical, and the upper one is tilted at a small angle ϵ with respect to the vertical. They are then released. At this instant, what are the angular accelerations of the two sticks? Assume ϵ is very small.