Rotation Problem Set Solutions

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October 2014

1 Solutions

1. The most challenging part of this question is correctly accounting for all the forces at play. The forces are shown in Figure 1, with the walls omitted for clarity.

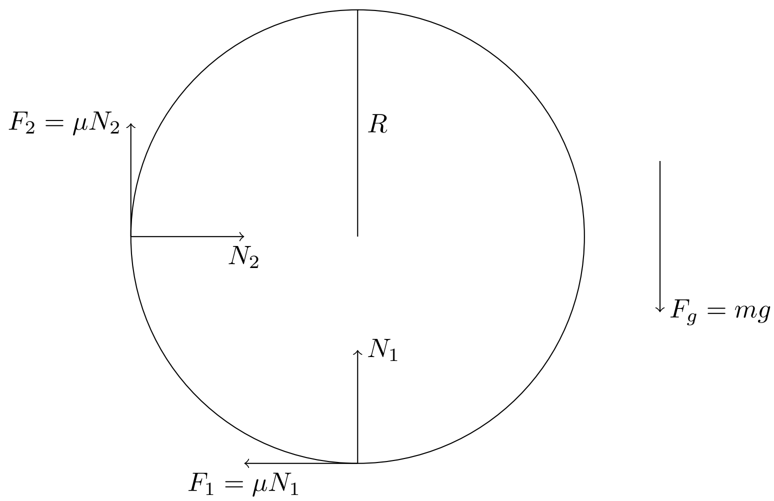

Figure 1: The forces acting in Problem 1.

In the x direction, we note that there is a balance between the normal force from the vertical wall and the frictional force from the horizontal wall. Therefore, $F_1 = \mu N_1 = N_2$. We can use this relationship to rewrite F_2 as $\mu^2 N_1$. Now we can use the equilibrium in the y direction to solve for N_1:

$$\begin{aligned} N_1 + F_2 - mg &= 0 \\ (1+\mu^2)N_1 - mg &= 0 \\ N_1 &= \frac{mg}{1+\mu^2} \end{aligned}$$

With this, we can write the torque all in terms of known parameters:

$$\begin{aligned} \tau &= R(F_1 + F_2) \\ &= R(\mu N_1 + \mu^2 N_1) \\ &= \frac{\mu + \mu^2}{1+\mu^2} Rmg \end{aligned}$$

To minimize the time it takes to stop the wheel, we must maximize this torque. Thus, we take the derivative with respect to μ:

$$\begin{aligned} \frac{d\tau}{d\mu} = \frac{(1+2\mu)(1+\mu^2) - (\mu+\mu^2)(2\mu)}{(1+\mu^2)^2}&(1+\mu^2)^2 = 0 \\ 1 + 2\mu - \mu^2 &= 0 \\ \mu &= 1 + \sqrt{2} \end{aligned}$$

2. This problem is fairly tricky because very few people tread through the road not followed. You may be inclined to use the conservation of angular momentum. But wait, think...

   When the wheels rub against each other, we know that one wheel slows down and the other wheel speeds up because of a frictional force. The frictional force is directed upowards/downwards on both wheels, thus allowing the net force on either wheel to be upwards or downwards. To prevent the configuration from fly apart, one must apply vertical pressure, thus indicating the presence of an *external torque*. Thus, angular momentum is NOT conserved.

   To solve this problem, we must retrace our steps back to the fundamental definitions. The frictional force will point upwards for M_2 and thus downwards for M_1, by Newton's third law. Thus, the kinematic equations follow

$$\omega_1(t) = \omega_0 - \frac{F_f R_1}{I_1} t$$
$$\omega_2(t) = \frac{F_f R_2}{I_2} t$$

   Because the wheels will act like a pair of gears when an "equilibrium" condition is achieved, their tangential velocities will also be equal

$$\omega_1 R_1 = \omega_2 R_2$$

   Solving for t, we get

$$\omega_0 R_1 - \frac{F_f R_1^2}{I_1} t = \omega_1 R_1 = \omega_2 R_2 = \frac{F_f R_2^2}{I_2} t$$
$$\omega_0 R_1 = t F_f \left(\frac{R_1^2}{I_1} + \frac{R_2^2}{I_2} \right) = \frac{F_f t}{\beta} \left(\frac{1}{M_1} + \frac{1}{M_2} \right)$$

   where we defined $I - \beta M R^2$. Thus, we can determine the final angular velocities ω_1 and ω_2.

$$\omega_1 = \omega_0 - \frac{F_f R_1}{I_1} \frac{\omega_0 R_1}{\frac{F_f}{\beta}\left(\frac{1}{M_1} + \frac{1}{M_2}\right)} = \omega_0 \left(1 - \frac{\frac{1}{M_1}}{\frac{1}{M_1} + \frac{1}{M_2}} \right)$$

$$\omega_1 = \omega_0 \frac{M_1}{M_1 + M_2}$$

$$\omega_2 = \omega_0 \frac{M_1}{M_1 + M_2} \frac{R_1}{R_2}$$

3. In order to determine the horizontal motion of the bar, we must determine the frictional forces acting at each wheel. To find these forces, we must solve for the normal forces. We can write one equation relating them very easily, since the bar does not move vertically:

$$N_1 + N_2 = Mg$$

   To find a second equation, we note that since the bar does not tilt off of the wheels (or so we presume), the net torque is also zero. We can choose any point about which to compute the torques, but the easiest is the point on the bar midway between the wheels. If we call the position of the center of mass of the bar with respect to this point x, then

$$xMg + lN_1 = lN_2$$

   With these two equations, we can solve for the normal forces in terms of known parameters:

$$N_1 = \frac{1}{2}\left(1 - \frac{x}{L}\right) Mg$$
$$N_2 = \frac{1}{2}\left(1 + \frac{x}{L}\right) Mg$$

The net horizontal force on the bar is then the sum of the frictional forces:

$$F = \mu(N_1 - N_2) = -\frac{\mu M g}{l} x$$

We now note that this force resembles a spring force, with $k = \frac{\mu M g}{l}$. Thus, the center of mass of the bar undergoes simple harmonic oscillation around its equilibrium position between the wheels:

$$x = A\cos\left(\sqrt{\frac{\mu g}{l}} t - \phi\right)$$

where A and ϕ are parameters chosen to meet initial conditions. If the wheels were rotating in the opposite direction, then all our equations would remain the same except for the force, which would become $\frac{\mu M g}{l} x$. Negating the force changes the solution from oscillation to exponential growth:

$$x = Ae^{\sqrt{\frac{\mu g}{l}} t}$$

4. The torque due to gravity points out/in to the page, so, angular momentum is not conserved. However, the angular momentum in the vertical direction IS conserved (the vertical angular momentum is due to the spins of the cone and the particle), so we can use the conservation of angular momentum ONLY in that direction. The other angular momentum (that is, due to the linear velocity of the particle down the cone) is not conserved. [1]

$$I_0\omega_0 = (I_0 + mR^2)\omega$$
$$\frac{I_0\omega_0}{I_0 + mR^2} = \omega$$

To determine the linear velocity of the particle before it leaves the cone, we must appeal to the conservation of energy

$$\frac{1}{2} I_0\omega_0^2 + mgh = \frac{1}{2}(I_0 + mR^2)\left(\frac{I_0\omega_0}{I_0 + mR^2}\right)^2 + \frac{1}{2}mv^2$$
$$\sqrt{\frac{2}{m}\left(\frac{1}{2} I_0\omega_0^2 + mgh - \frac{1}{2}\left(\frac{I_0^2\omega_0^2}{I_0 + mR^2}\right)\right)} = v$$

where I_0 is the moment of inertia of the cone rotated about the vertical axis, $\frac{3}{10}MR^2$.

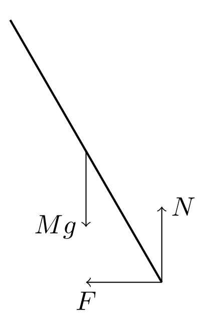

Figure 2: A free body diagram for the situation in problem 5.

5. A free body diagram for this problem is shown in Figure 2. Note that we are looking straight along the plane of the biker, so that he appears as a leaning line. In order to maintain this perspective, we must fix the position our coordinate

[1] We can still use this fact to solve the problem, since $\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$. However, this is a bit trickier method to pursue.

system with the biker, which means that we are working in a non-inertial reference frame. To account for this, we add the inertial force MV^2/R pointing out of the circle.

We can see immediately that $N = Mg$ (for equilibrium in the y direction) and $F = MV^2/R$ (for equilibrium in the y direction). We can determine the angle the bike makes with the vertical by solving for rotational equilibrium. We can sum the torques about the center of mass of the bike and let the sum be the rate of change of the wheel's spin angular momentum:

$$N \cdot 2l \sin\phi - F \cdot 2l \cos\phi = \left|\frac{d\vec{L}}{dt}\right| = L_s \Omega \cos\phi$$

where Ω is the precession velocity. Moreover, we know that $L_s = (ml^2 + ml^2)\omega$, where ω is the spin angular velocity of the wheel(s). Since the bike is rolling without slipping and moving with a velocity V, $L_s = (ml^2 + ml^2)\frac{V}{l}$. Thus

$$N \cdot 2l \sin\phi - F \cdot 2l \cos\theta = 2ml^2 \frac{V}{l} \cos\phi \frac{V}{R}$$

$$Mg \cdot 2l \sin\phi - \frac{MV^2}{R} \cdot 2l \cos\phi = 2m\frac{V^2}{R} l \cos\phi$$

where, the second statement used substitutions from Newton's Laws. Thus we get

$$\tan\phi = \frac{V^2}{gR}\left(1 + \frac{m}{M}\right)$$

6. We will work through the question part by part

   (a) One bit that wasn't stated in the problem statement explicitly: we assume that the ring can only move about the pivot point P and we want to find the angular momentum about P.

   With this information, let us proceed to determining the angular momentum for the first part. Taking note of the diagram below, we notice that the angle between $\vec{p}$ and $\vec{r}$ is $\frac{\phi}{2}$. Thus

$$|\vec{L}_p| = |\vec{r}||\vec{p}| \sin\frac{\phi}{2}$$

   Using the law of cosines, $|\vec{r}|^2 = R^2 + R^2 - 2R\cos\phi$, which gives us $|\vec{r}| = 2R\sin\frac{\phi}{2}$. Moreover, $|\vec{p}| = mV$. This gives

$$|\vec{L}_p| = 2mvR\sin^2\frac{\phi}{2}$$

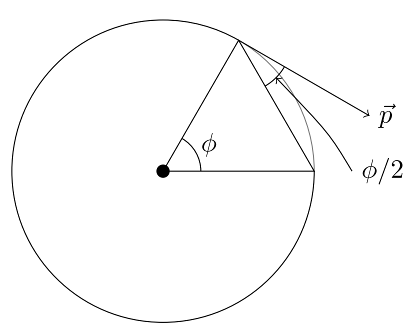

Figure 3: The geometry of the situation in problem 6a.

(b) Since the ring is rotating, the bug moves along a circle of radius $|\vec{r}| = 2R\sin\frac{\phi}{2}$ at an angular velocity Ω. Thus

$$|\vec{L}_p| = I\Omega = m|\vec{r}|^2\Omega = 4m\Omega R^2\sin^2\frac{\phi}{2}$$

(c) The total angular momentum can be obtained due to three factors. The first two types of angular momentum (motion of the bug along the ring, and rotational inertia of the bug) contribute to two of the terms. The third term is the inherent spin angular momentum of the ring itself. Since we take the angular momentum about P, we know that the quantity is conserved. When the bug initially passes through point P, we know that it has no angular momentum relative to the pivot point, and the ring does not rotate (and thus doesn't have any angular momentum either). Thus, the total angular momentum is 0. Using the conservation of angular momentum (and the parallel axis theorem), we get

$$0 = 2MR^2\Omega + 4m\Omega R^2\sin^2\frac{\phi}{2} + 2mvR\sin^2\frac{\phi}{2}$$

$$\Omega = -\frac{mv\sin^2\frac{\phi}{2}}{MR + 2mR\sin^2\frac{\phi}{2}}$$

(d) Using the answer from the previous question, we know that

$$\Omega = \frac{d\theta}{dt} = \frac{d\theta}{d\phi}\frac{d\phi}{dt}$$

where θ is the angle that the ring has rotated by (about P). However, we know that the bug's linear velocity about the ring is constant, so $\frac{d\phi}{dt} = \frac{v}{R}$. Thus

$$\Omega = \frac{d\theta}{d\phi}\frac{v}{R} = -\frac{mv\sin^2\frac{\phi}{2}}{MR + 2mR\sin^2\frac{\phi}{2}}$$

$$\Delta\theta = -\frac{R}{v}\int_{\phi_0}^{\phi}\frac{mv\sin^2\frac{\phi}{2}}{MR + 2mR\sin^2\frac{\phi}{2}}d\phi$$

$$\Delta\theta = -\int_0^{\frac{vt}{R}}\frac{\sin^2\frac{\phi}{2}}{\frac{M}{m} + 2\sin^2\frac{\phi}{2}}d\phi$$

where the last step changed the bounds of the integration (since we know that ϕ increases linearly). This integral is not easily determinable, and we thus will not require you to compute it.