Beams and Chains

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1 Problem Formulation

We start by investigating the behavior of a finite collection of hanging beams. In general, we consider N beams, each with possibly varying lengths l_i and each with linear mass density σ. The left end of the first beam is fastened to the point (x_0, y_0) and the right end of the final beam is fastened at (x_n, y_n). There is a uniform gravitational field with acceleration g in the negative y direction.

Our objective is to determine the equilibrium configuration of the beams. This configuration can be expressed as the $n-1$ coordinates where the joints between the beams rest: $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})$. Figure 1 shows the solution to one instance of this problem.

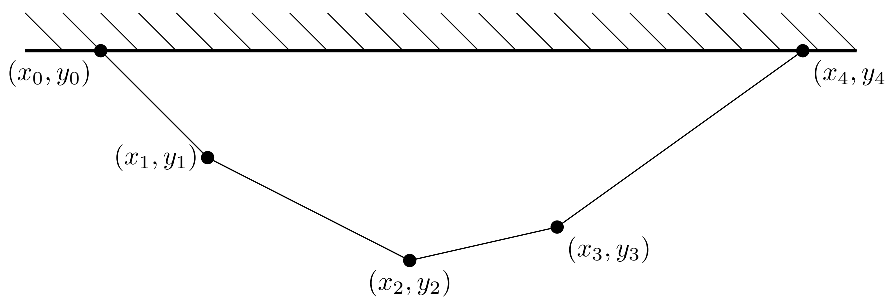

Figure 1: The equilibrium configuration for a set of $N = 4$ beams.

2 Approach

This problem is certainly solvable by introducing vectors representing all the forces involved, and solving Newton's second law for each beam. However, this is tedious, and the use of vectors makes it even more so. Instead of finding the positions of the joints for which the net forces and torques vanish, we will minimize the energy of the configuration.

In simple problems, minimization of potential energy is little more than a substitute for finding the point at which the net force is 0, because the potential energy is defined by

$$\nabla U = -\vec{F}$$

So the absence of force implies a stationary point on the potential energy surface. For example, for a simple harmonic oscillator, we could use either the force $-kx\hat{\imath}$ or the potential energy $\frac{1}{2}kx^2$. Whether we let the force equal zero or solve for a stationary point of the potential energy, we find the same equilibrium solution $x = 0$.

However, the problem of the hanging beams is not so simple. We have $2n-2$ coordinates $x_1, x_2, \ldots, x_{n-1}$ and $y_1, y_2, \ldots, y_{n-1}$. If there were no constraints on these coordinates, we would need only take partial derivatives of the potential energy with respect to each one and set each derivative to zero to solve for all equilibrium points. However, there are a number of auxliary conditions that must hold for a valid solution: the distance between (x_i, y_i) and (x_{i+1}, y_{i+1}) must be equal to the length l_i. If we are to take advantage of the utility of the energy-minimization method, we must find a method for handling these constraints.

3 Lagrange Multipliers

Remarkably, mathematics offers a general tool for dealing with auxiliary conditions in optimization problems. This tool is known as the method of Lagrange multipliers.

To introducde this method, we will consider the simple problem of maximizing the quantity xy on the ellipse $\frac{x^2}{4} + y^2 = 1$. We denote the objective function by f and the constraints (of which there are only one, in this case) by g:

$$f(x, y) = xy$$
$$g(x, y) = \frac{x^2}{4} + y^2$$

If we had no constraints, then we would find a point at which ∇f vanishes, and so there is no direction in which we can move such that f has first-order changes in value. However, due to the presence of the constraint, we need only find a point on the ellipse such that no direction of motion *permitted by the constraint* will result in first-order changes in f. Graphically, as shown in Figure 2, we must find a point on ellipse where ∇f is perpendicular to the tangent to the ellipse.

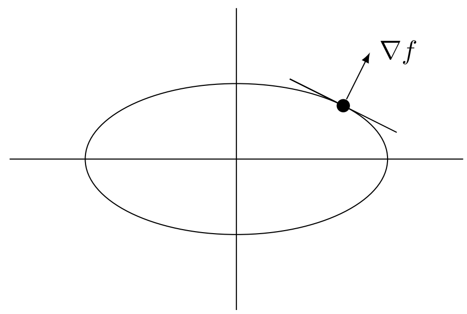

Figure 2: In the constrained optimization problem, we must find a point on the ellipse where ∇f is perpendicular to the tangent to the ellipse.

This condition is easily formalized by noting that the tangent to the ellipse is perpendicular to the gradient of g. In fact, it is generally true that a level set of a function (such as the elliptical level set given by $g(x, y) = 1$) is everywhere perpendicular to the gradient of that function. Thus, we are searching for a point at which ∇f is perpendicular to a tangent which is in turn perpendicular to ∇g. This is equivalent to stating that ∇f is a scalar multiple of ∇g:

$$\nabla f = -\lambda \nabla g$$

For the example above, this vector equation would yield two scalar equations (one for $\frac{\partial}{\partial x}$ and one for $\frac{\partial}{\partial y}$):

$$y + \frac{\lambda}{2}x = 0$$
$$x + 2\lambda y = 0$$

However, we have introduced a third variable, λ. The constraint equation is the third equation that completes the system:

$$\frac{x^2}{4} + y^2 - 1 = 0$$

In principle, we can now solve these three equations in three variables to find the constrained stationary points of f. However, the equations are nonlinear, and so we do not have a general method for solving them. In this particular case, the solutions are easily found by substitutions, and are $(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}})$.

In the case where there are multiple constraints, the process is similar. The gradient ∇f simply has to lie in the space spanned by all the gradients ∇g_i. Therefore, we write

$$\nabla f = -\sum_i \lambda_i \nabla g_i$$

This generates n equations (assuming f and g_i are all functions mapping $\mathbb{R}^n$ into $\mathbb{R}$), which along with the m constraint equations $g_i = 0$ can be solved for all $n + m$ variables. However, this can be stated much more elegantly by defining a new function:

$$L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m) = f(x_1, x_2, \ldots, x_n) + \sum_i \lambda_i g(x_1, x_2, \ldots, x_n)$$

If we set the partial derivatives of L with respect to each of its arguments to 0, we arrive at the $n + m$ equations derived above. We call this new composite function the Lagrange function. To summarize, the following process can be used to find constrained minima (although the exact same process replacing the word "minima" with "maxima"):

1. Restate a problem in the following form:

$$\begin{aligned} \text{minimize } & f(x_1, x_2, \ldots, x_n) \\ \text{subject to } & g_1(x_1, x_2, \ldots, x_n) = 0 \\ & g_2(x_1, x_2, \ldots, x_n) = 0 \\ & \vdots \\ & g_m(x_1, x_2, \ldots, x_n) = 0 \end{aligned}$$

2. Form the composite Lagrange function:

$$L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m) = f(x_1, x_2, \ldots, x_n) + \sum_i \lambda_i g_i(x_1, x_2, \ldots, x_n)$$

3. Find all the stationary points of L by solving the $n+m$ equations contained in $\nabla L = 0$, and determine which ones represent minima of f.

4 Minimizing the Beam Energy

We can now proceed to follow the process above for the problem of the beam. First, we need an expression for the potential energy of the system in terms of the joint coordinates. The center of mass of the ith beam is at a height of $\frac{1}{2}(y_{i-1} + y_i)$. Thus, the potential energy is given by the sum

$$V = \frac{\sigma g}{2} \sum_{i=1}^{N} (y_{i-1} + y_i)\, l_i$$

The length constraints are also simple to express. If we let Δx_i represent $x_i - x_{i-1}$ and similarly $\Delta y_i = y_{i+1} - y_i$, then each constraint equation takes the form

$$\Delta x_i^2 + \Delta y_i^2 = l_i^2$$

Therefore, the Lagrange function, which we will denote here by $\bar{V}$, is

$$\bar{V} = \frac{\sigma g}{2} \sum_{i=1}^{N} (y_{i-1} + y_i)\, l_i + \sum_{i=1}^{N} \lambda_i \left(\Delta x_i^2 + \Delta y_i^2 - l_i^2\right)$$

If we take the partial with respect to x_i, we obtain

$$\frac{\partial \bar{V}}{\partial x_i} = 2\Delta x_i \lambda_i - 2\Delta x_{i+1} \lambda_{i+1} = 0$$

Therefore, $\Delta x_i \lambda_i = c$. The partial with respect to y_i is given by

$$\frac{\partial \bar{V}}{\partial y_i} = \frac{1}{2}\left(l_i + l_{i+1}\right) + 2\Delta y_i \lambda_i - 2\Delta y_{i+1} \lambda_{i+1} = 0$$

If we substitute $\lambda_i = \frac{c}{\Delta x_i}$, then we arrive at the difference equation

$$2c\left(\frac{\Delta y_{i+1}}{\Delta x_{i+1}} - \frac{\Delta y_i}{\Delta x_i}\right) = \frac{1}{2}\left(l_i + l_{i+1}\right)$$

If we specify the orientation of the first beam and c, this equation allows us to recursively compute the orientations of all subsequent beams. We can determine this first orientation and c by specifying the coordinates (x_0, y_0) and (x_n, y_n).

5 Chains

If we let N grow large, the problem of linked beams becomes a problem of determining the equilibrium configuration of a hanging chain. If we let N grow without bound, then the problem becomes a continuous one and we can replace Δ's with d's. Letting $\alpha = \frac{1}{2c}$:

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \alpha$$

We can substitute the definition of $\frac{d}{ds}$ to obtain the differential equation

$$\frac{y''}{\sqrt{1 + y'^2}} = \alpha$$

Or, upon rearrangement,

$$y''^2 - \alpha^2 y'^2 - \alpha^2 = 0$$

The exact form of the solution depends on the initial conditions; for the simple case of a rope hanging at two points from a ceiling, we have the solution

$$y = \frac{1}{2\alpha}\left(e^{\alpha x} + e^{-\alpha x}\right) = \frac{1}{\alpha}\cosh \alpha x$$

This is the familiar equation for a catenary. Figure 3 shows one particular solution.

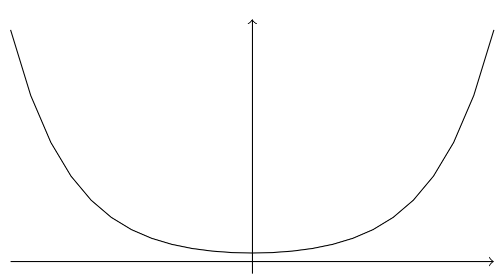

Figure 3: A plot of a catenary, the function describing the shape of a hanging rope.