# Work and Energy

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## 1 Introduction to Energy

In layman's terms, energy is the ability to do something. As we will see, this something is called work. For example, I need energy to push a box. Where does this energy come from? It comes from the food I eat. The food stores energy in its chemical bonds. Most of that energy comes from the sun. The sun releases energy through a series of nuclear reactions. The sun at one point received that energy from another source. We quickly see that energy is only being transfered, never created or destroyed.

This is known as the **law of conservation of energy**. Formally, it states that the net energy remains constant in a closed system. A closed system is a system in which the net external force is zero and the net energy transfer is zero. Energy can enter and leave, but the system does not gain or lose energy. Basically, energy cannot come from nothingness and cannot vanish. It will always be present if it exists, and it must be accounted for. For an open system, the law of conservation of energy can be expressed mathematically as follows:

$$\Delta E_{sys} + \Delta E_{ext} = 0$$

We can add energy to represent energy gain and subtract energy to represent energy loss. If a block is moving and I push it harder, then the energy I transfer is just added to the energy the block had to determine the new total energy. This means:

$$E_T = \sum_g E_g - \sum_l E_l$$

## 2 Types of Energy

Energy can exist in two different forms: kinetic, and potential. Kinetic energy ($K_E$) is the energy associated with an object in motion. Potential energy ($P_E$) refers to energy an object stores in a particular state. Because of the law of conservation of energy, we can state about a closed system:

$$K_E(t) + P_E(t) = T_E$$

$$K_E(t_0) + P_E(t_0) = T_E = K_E(t_f) + P_E(t_f)$$
$$\Delta K_E = -\Delta P_E$$

The constant $T_E$ is the total energy and does not change, even if $K_E$ and $P_E$ vary with time. In an open system, the following holds:

$$K_E(t_0) + P_E(t_0) + E_{in} = K_E(t_f) + P_E(t_f) + E_{out}$$

There are different types of kinetic and potential energy. Kinetic energy consists of energy associated with mechanical motion, electromagnetic radiation, waves, thermal states, and the motion of charges (electricity). Potential energy is associated with energy stored in chemical bonds, nuclear processes, and fields. Field energy is the energy stored in a field, like a gravitational field or an electric field. In mechanics, the two fields we are generally interested in are the gravitational field and the spring force field.

All of these forms of energy are generally measured in joules, the SI unit for energy.

# 3 Work

Work is the transfer of energy. If I take a hot object and touch it to a colder object, thermal energy is transfered. The hot object does positive work on the colder object, and the colder object does negative work on the hotter object.

We introduce the concept of work because sometimes we are not interested in the magnitudes of energy, but rather the difference in energy. For example, if I wanted to make an object move faster, I wouldn't be interested in how much energy it has, but how much energy I would need to add to make it move faster by the amount I want it to.

In mechanics, work is fundamentally a force multiplied by the distance for which it is applied. In the case of a constant force applied to an object moving in one direction, the work is given by:

$$W = |\vec{F}| d \cos\theta$$

The angle $\theta$ is the angle between the force and the direction in which the object moves. When you see this, you should think of a dot product, because that's where this formula comes from mathematically. Physically, this is because Newton's second law tells us that an object will accelerate in the same direction as the net force applied on it. The key word here is "net." If I were to apply a force on a bead on a straight rod so that the bead moves, then the component of my force parallel to the rod causes the bead to move and so does work. The other component of my force is cancelled out by the normal force exerted by the rod. The big takeaway is that only the component of the force parallel to motion counts towards doing work.

Work done now is independent of work done in the future, so this means we can sum up work done by different constant forces over different linear paths to determine the total work.

$$W_T = \sum_i W_i = \sum_i |\vec{F_i}| d_i \cos\theta_i$$

However, not everything in life is constant or moving in one direction. When we have a changing force and/or a complex path, we will need to use calculus. Try to rationalize how we moved from the sum above to the integral below:

$$W = \int_C \vec{F} \cdot d\vec{r}$$

This type of integral is known as a **line integral** or **path integral**. It is different from a regular integral in that we are integrating over the path the object, and not over the x-axis. The vector $\vec{r}$ is the same position vector from kinematics. Each infinitesimal $dW$ is given by the dot product between the force applied at a certain position along the curve and the infinitesimal change $d\vec{r}$ in the position of the object as a result of that force. It is taught in multivariable calculus how to evaluate these integrals for general paths and forces, but for most physics problems, only an understanding of the formula and how to manipulate it for basic forces is required. For example, due to the nature of vectors, the work integral can be decomposed into each direction.

The following manipulation is perhaps the most important. First, we use Newton's second law.

$$\vec{F}_{net} = m\vec{a}$$

Then we substitute and rearrange:

$$\begin{aligned} W &= \int_C \vec{F_{net}} \cdot d\vec{r} = \int_C m\vec{a} \cdot d\vec{r} \\ &= \int_C m\frac{d\vec{v}}{dt} \cdot d\vec{r} = \int_C m d\vec{v} \cdot \frac{d\vec{r}}{dt} \\ &= \int_C m d\vec{v} \cdot \vec{v} = \int_C m\vec{v} \cdot d\vec{v} \\ &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = K_f - K_i = \Delta K_E \end{aligned}$$

This is the work-kinetic energy theorem for mechanics. We are allowed to express the term $\frac{1}{2}mv^2$ as kinetic energy because this is the energy associated with motion (the net force is causing this motion). Please note the "$\frac{1}{2}mv^2$" version is only valid in mechanics - it is better to state that $W = \Delta K_E$. Remember how the hot object being touched to the cold object mentioned earlier still does work? That's part of the reason why. The velocity doesn't really change, but the kinetic energy does.

We also note that the work needed should depend on the path taken. This makes sense intuitively. If I were to push a box across the entire world back to where I started and then push it ten feet more, more energy would be required than if I were to just push the box ten feet. The starting and ending points are the same, but the path is different, and so the work is different.

However, forces that are independent of path when determining work do exist. These forces are known as conservative forces and are important enough that we will consider them below.

# 4 Conservative Forces

A force is conservative if it meets any one of the following conditions. If a force meets any one of the conditions below then it satisfies all of them. This won't be proven here, but a course in the calculus of multivariables will show why.

$$\vec{\nabla} \times \vec{F} = 0$$

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

$$F = -\nabla \Phi$$

The first equation states that the curl of the vector field is zero. The delta is the vector $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ and is more of an operator than a vector. The second statement says applying the force over a path and ending where you started restuls in no net work. The integral sign with a loop in it is just a line integral in which the path ends where it started, and is known as a **closed loop integral**. The third statement states that the field is a gradient of a potential function. This basically means that each component of the field is derivative with respect to that direction of a multivariable function. We can therefore use the fundamental theorem of calculus to state that regardless of path, if a conservative force acts on an object from point $a$ to point $b$, the change in potential energy is:

$$\Delta P_E = \Phi(b) - \Phi(a)$$

This is why we call $\Phi$ a potential function, meaning that the work done on the object is:

$$\Delta K_E = \Phi(a) - \Phi(b)$$

The most common conservative forces that we deal with are gravity $(-mg\hat{j})$ and the spring force $(-kr\hat{r})$. You can use the formulas above to show this. We can calculate the work done by each of these. First, let's consider the force applied by gravity.

$$W = \int_{y_0}^{y_f} -mg\vec{j} \cdot d\vec{r}$$

$$= \int_{y_0}^{y_f} -mg\vec{j} \cdot dy\vec{j}$$

$$= mg(y_0 - y_f) = -mg\Delta y$$

This means that when $y_f > y_0$, the object moves upward, and negative work is done on the object. We increase the gravitational potential energy. The opposite is true when $y_f < y_0$. Also note that the gravitational potential energy is relative to some point, and increases linearly with height. Generally, the ground is set to have zero gravitational potential energy. You can set whatever point you want to have zero gravitational potential energy and accomodate accordingly. It's basically shifting your reference frame.

For the spring force, we do something similar. If we let $r_0$ be the equilibrium length, we have:

$$W = \int_{0}^{r_f - r_0} -kr\hat{r} \cdot d\vec{r}$$

$$= \int_{0}^{r_f - r_0} -k\vec{r} \cdot d\vec{r}$$

$$= -\frac{1}{2}k(r_f - r_0)^2 = -\frac{1}{2}k(\Delta r)^2$$

Observe that this means that the spring does negative work on an object, and it stores the energy through either compression or extension. The energy stored is positive, and the work done is negative (potential vs. kinetic).

On a concluding note, keep in mind that friction is not a conservative force. Friction always opposes motion, and its action generates heat which is released to the environment. You will have to account for the work done by friction when solving problems.

# 5 Summary

Here are some forumlas and explanations that will help with the problems.

Energy is conserved in a closed system. For any system, use the facts $E_{in} + K_i + P_i = K_f + P_f + E_{out}$ to solve problems.

The work-kinetic energy theorem states that $W = \int_C \vec{F} \cdot d\vec{r} = \Delta K$. If an object has not changed in kinetic energy, no net work has been done on it.

Moving an object up by $\Delta y$ changes its gravitational potential energy by $mg\Delta y$. The work that has been done it is simply $-mg\Delta y$, since in a closed system, $-\Delta K_E = \Delta P_E$. We typically set the ground to have zero gravitational potential energy, but this choice is arbitrary.

The spring also does negative work on an object. If a spring is compressed a length $\Delta x$, the potential energy it stores is $\frac{1}{2}k(\Delta x)^2$.

Remember to account for friction. Since friction opposes motion, the work it does is generally $-F_f d$, where $d$ is the distance it acts over.

# 6 Problems

1. I push on a box over a certain distance and friction acts against the motion of the box. I stop pushing, and now the box is at rest. How much work did I do? How much work was done on the box?

2. A block of mass $m$ is pushed off a cliff of height $h$. At what velocity does it hit the ground?

3. A block of mass $m$ resting on a frictionless table is pulled by a string with tension $T$ at an angle $\theta$ with respect to the table. The block is pulled a distance $d$ and then let go. At what velocity does the block move?

4. A block of mass $m$ is on the surface of a planet of mass $M$ and radius $R$. The gravitational force exerted on the block by the planet is $\vec{F}(r) = \frac{-GMm}{r^2}\hat{r}$, where $r$ is the distance from the center of the planet, and $G$ is a constant. What is the minimum energy required to send the block out of the gravitational pull of the planet? What is the escape velocity of the block?

Note: This is the general form of gravitation. When $M >> m$ and $r$ is approximately $R$, this formula simplifies to the well-known formula $F = -mg$.

5. A spring of spring constant $k$ is compressed a distance $x_0$ by a block of mass $m$. Write a differential equation that models the motion by using the law of conservation of energy.