Classical Electromagnetic Waves

Allen Cheng

May 23, 2014

This lecture assumes familiarity with most of classical electricity and magnetism. Subscript E will be used to denote electric properties, and B for magnetic properties.

1 Maxwell's Equations

Maxwell's equations sum up the relationships between electricity and magnetism, hence the term "electromagnetism". These four equations (in a vacuum) are:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \tag{Gauss's law}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{(Gauss's law for magnetism)}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (i_C + \epsilon_0 \frac{d\Phi_E}{dt})_{\text{encl}} \tag{Generalized Ampere's law}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{Faraday's Law}$$

Note the generalization of Ampere's law to include the "displacement current" $i_D = \epsilon_0 \frac{d\Phi_E}{dt}$. This generalization makes Ampere's law apply to situations such as capacitors, where there is no actual current between the two plates but the electric flux is still changing.

Maxwell's equations imply the existence of self-propagating electromagnetic waves, i.e. requiring no medium, consisting of electric and magnetic fields that carry each other across space.

2 Plane Electromagnetic Waves

2.1 A Basic Plane Wave

To begin this discussion, consider a so-called plane wave, traveling in a vacuum in the $+x$ direction. The planar front of this wave represents the farthest the wave has traveled, and behind it, let the $\vec{E}$ field be uniform and point in the $+y$ direction, and the $\vec{B}$ field also be uniform but point in the $+z$ direction. Neither field can contain any x-component because otherwise, Gauss's law would be broken. We now show that in order to satisfy Maxwell's equations, the wave must have certain characteristics.

First, consider Gauss's law for both electricity and magnetism. Because the wave is traveling in a vacuum, there is no charge contained within any surface, and the closed-surface Φ_E and Φ_B must both be 0. This implies that neither field has any x component (making the wave transverse), and that in each plane parallel to the wave front, both the electric and magnetic fields are uniform.

Next, consider Faraday's law using a rectangular loop parallel to the xy-plane, with one side of the loop outside the wave front where there is no field. Let the sides parallel to the y-axis have length a. Integrating $\oint \vec{E} \cdot d\vec{l}$ counterclockwise around the rectangle, the only nonzero component is the side parallel to the y-axis, which has value $-Ea$. Thus, Faraday's law gives us

$$-Ea = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -Bac$$

where c is the speed of light, which is the speed of the wave, or

$$E = cB \qquad \text{(Eq. 2.1.1)}$$

Note that $\vec{B}$ cannot point in the $-z$ direction when the wave is traveling in the $+x$ direction, as otherwise, we would have $E = -cB$, which is not possible because all variables must be positive.

Finally, we apply Ampere's law using a similar rectangular loop, this time parallel to the xz-plane. Likewise, let one of the sides (parallel to the z-axis) lie in front of the wave front, and again have length a. Again integrating counterclockwise, $\oint \vec{B} \cdot d\vec{l}$ only has a single nonzero part, Ba. Since the conduction current $i_C = 0$, we have

$$Ba = \oint \vec{B} \cdot d\vec{l} = \mu_0(i_C + \epsilon_0 \frac{d\Phi_E}{dt})_{\text{encl}} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \epsilon_0 \mu_0 Eac$$

or

$$B = \epsilon_0 \mu_0 cE \qquad \text{(Eq. 2.1.2)}$$

Both these equations show that only fields perpendicular to each other propagate. Further, putting together equations 2.1.1 and 2.1.2, we find that

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \qquad \text{(Eq. 2.1.3, speed of light in a vacuum)}$$

This equation agrees with the accepted values of these fundamental constants, thereby showing that this theorized wave satisfies Maxwell's equations.

2.2 The Wave Equation

In mechanical waves, the wave function function expresses the displacement y of any point x at any time t. However, in electromagnetic waves, there are no points physically moving, but instead an oscillation of the electric and magnetic fields. For mechanical waves, the *wave function* $y(x, t)$ satisfies the wave equation, $\dfrac{\partial^2 y(x,t)}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y(x,t)}{\partial t^2}$. In this section, we derive an analogous equation for electromagnetic waves, which we will again assume to be traveling in the $+x$ direction with $\vec{E}$ and $\vec{B}$ parallel to the y- and z- axes, respectively.

First, define two functions as functions of x and t: $E_y(x,t)$ and $B_z(x,t)$, which will be oscillating instead of a physical displacment y. We begin by applying Faraday's law on a loop parallel to the xy-plane, but this time completely contained within the wave front. Again, it has y-length a, and let the other side have length Δx. Then, integrating counterclockwise again, both sides have nonzero components. Thus, the left side still resembles that of equation 2.1.1. However, the right side is different, because velocity is no longer given, instead, $\frac{d\Phi_B}{dt} = \frac{\partial B_z(x,t)}{\partial t} a\Delta x$, as B_z is a function of both x and t. So,

$$a(E_y(x+\Delta x,t) - E_y(x,t)) = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\frac{\partial B_z(x,t)}{\partial t} a\Delta x$$

Letting Δx approach 0 and recognizing the definition of the derivative, we see

$$\frac{\partial E_y(x,t)}{\partial x} = -\frac{\partial B_z(x,t)}{\partial t} \qquad \text{(Eq. 2.2.1)}$$

Now, we apply Ampere's law to a similar rectangle parallel to the xz-plane contained within the wave, again with side lengths Δx and a. The line integral $\oint \vec{B} \cdot d\vec{l}$ is analagous to that over $\vec{E}$ in the previous derivation. As in subsection 1, $i_C = 0$, so only the change of Φ_E need be calcaulated. Approximating a narrow rectangle (small Δx), $\frac{d\Phi_E}{dt} = \frac{\partial E_y(x,t)}{\partial t} a\Delta x$. Substituting into Ampere's law:

$$a(-B_z(x+\Delta x,t) + B_z(x,t)) = \oint \vec{B} \cdot d\vec{l} = \mu_0(i_C + \epsilon_0 \frac{d\Phi_E}{dt})_{\text{encl}} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt} = \epsilon_0\mu_0 \frac{\partial E_y(x,t)}{\partial t} a\Delta x$$

As in the derivation of equation 2.2.1,

$$-\frac{\partial B_z(x,t)}{\partial x} = \epsilon_0\mu_0 \frac{\partial E_y(x,t)}{\partial t} \qquad \text{(Eq. 2.2.2)}$$

Taking a partial with respect to x of equation 2.2.1 and respect to t of 2.2.2,

$$\begin{aligned} \frac{\partial^2 E_y(x,t)}{\partial x^2} &= -\frac{\partial^2 B_z(x,t)}{\partial x \partial t} \\ -\frac{\partial^2 B_z(x,t)}{\partial x \partial t} &= \epsilon_0\mu_0 \frac{\partial^2 E_y(x,t)}{\partial t^2} \end{aligned}$$

Combining, we find the wave equation

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = \epsilon_0\mu_0 \frac{\partial^2 E_y(x,t)}{\partial t^2} \qquad \text{(Eq. 2.2.3, EM wave equation in vacuum for } E\text{)}$$

Comparing with the wave equation for mechanical waves, we see that $\frac{1}{c^2} = \epsilon_0\mu_0$, which agrees with equation 2.1.3.

3 Energy

Electromagnetic waves carry energy. We begin describing this movement using energy densities equations associated with $\vec{E}$ and $\vec{B}$ fields:

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

Substituting equations 2.1.1 and 2.1.3 to express B in terms of E, ϵ_0, and μ_0, we obtain

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}(\sqrt{\epsilon_0\mu_0}E)^2 = \epsilon_0 E^2 \qquad \text{(Eq. 3.1)}$$

Note that energy densities contained within electric and magnetic fields are equal; this equation also shows that energy density is dependent on position and time. The wave propagates energy by extending the region containing this energy density. Thus, letting dV represent the increase in volume over a certain area A, we find $dU = u\ dV = (\epsilon_0 E^2)(Ac\ dt)$. Defining S to be the energy flux (energy per unit time per unit area) through A and substituting equations 2.1.1 and 2.1.3 again,

$$S = \frac{1}{A}\frac{dU}{dt} = \epsilon_0 c E^2 = \frac{EB}{\mu_0} \qquad \text{(Eq. 3.2)}$$

For convenience, we now define a vector $\vec{S}$, the *Poynting vector*, that describes both magnitude and direction of energy flow rate:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \qquad \text{(Eq. 3.3, Poynting vector in a vacuum)}$$

This definition is consistent with both the magnitude found in equation 3.2 and the direction of the wave. Intensity I is defined as the average magnitude of S; however, these are not the same: S is instantaneous while I is average.

4 Momentum

Although electromagnetic waves do not really carry mass, we still consider them as having a sort of momentum. Relativity offers an easy way out in terms of computing this momentum; however, Einstein does not fit into this classical discussion. Instead, consider a point charge q that the wave acts on. This charge oscillates parallel to the y-axis because of the electric field, and this y-velocity $\vec{v_y}$, which happens to always point in the same direction as $\vec{E}$, translates in turn to a magnetic force that always points forward. Then the differential of the magnitude of the momentum, $dp = F_B\ dt = \left|q\vec{v_y} \times \vec{B}\right| dt$. Because $\vec{v_y}$ and $\vec{B}$ are perpendicular,

$$dp = qv_y B\ dt = qv_y \frac{E}{c}\ dt \qquad \text{(Eq. 4.1)}$$

Now, we consider the work that the wave does on the particle. Because magnetic fields do no work, the differential work done is

$$dW = \vec{F_E} \cdot d\vec{x} = qEv_y \; dt \qquad \text{(Eq. 4.2)}$$

It is evident from equations 4.1 and 4.2 (and directly from the relativistic equation $E^2 = (pc)^2 + (mc^2)^2$) that

$$\frac{1}{A}\frac{dp}{dt} = \frac{1}{c}\frac{1}{A}\frac{dW}{dt}$$

However, from 3.3, part of the right side is the magnitude of the Poynting vector S, and so

$$\frac{1}{A}\frac{dp}{dt} = \frac{S}{c} = \frac{EB}{\mu_0 c} \qquad \text{(Eq. 4.3, flow rate of momentum)}$$

This equation implies that electromagnetic waves actually carry momentum, which does cause recoil on objects that light hits. This effect is minute in everyday situations, but significant in some parts of the universe.