Rotation

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1 Rotational Variables

Basic translational variables that have rotational analogues include position, velocity, and acceleration.

Translational	Rotational	Relation
$\vec{s}$	$\vec{\theta}$	$\vec{s} = \vec{\theta} \times \vec{r}$
$\vec{v} = \frac{d\vec{r}}{dt}$	$\vec{\omega} = \frac{d\vec{\theta}}{dt}$	$\vec{v} = \vec{\omega} \times \vec{r}$
$\vec{a} = \frac{d\vec{v}}{dt}$	$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$	$\vec{a} = \vec{\alpha} \times \vec{r}$

2 Rotational Kinematics

Kinematic equations apply to rotation as well. Below is a list of the most common rotational kinematic equations given that α is constant.

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \langle\omega\rangle t$$

$$\omega^2 = {\omega_0}^2 + 2\alpha\theta$$

For uniform circular motion, it is important to know that the centripetal (center-seeking) acceleration is given by

$$a_c = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r$$

3 Moment of Inertia

The moment of inertia can be thought of as the rotational analogue of mass. It is defined for a system of discrete objects as

$$I = \sum mr^2$$

Certain continuous objects have specific formulas for moment of inertia that can be derived using calculus. For instance, the moment of inertia of a disk rotating about its center axis is $I_{disk} = \frac{1}{2}mr^2$.

Another equation that can be helpful is the Parallel Axis Theorem, which states that for rotation about a point a distance h from the center of mass:

$$I = I_{com} + mh^2$$

4 Rotational Dynamics

Going back to the idea of the moment of inertia as a rotational analogue of mass, we know from Newton's Second Law of Motion that $\sum \vec{F} = m\vec{a}$. The rotational version of Newton's Second Law replaces force with torque, which as we know is $\vec{\tau} = \vec{r} \times \vec{F}$:

$$\sum \vec{\tau} = I\vec{\alpha}$$

This equation can be used in conjunction with the definition of torque to solve many problems in rotation. Other formulas can also be translated into rotational variables.

Quantity	Translational	Rotational
W	$\vec{F} \cdot \vec{d}$	$\vec{\tau} \cdot \vec{\theta}$
K	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
$\vec{p}$	$m\vec{v}$	$I\vec{\omega}$
P	$\vec{F} \cdot \vec{v}$	$\vec{\tau} \cdot \vec{\omega}$

5 Energy and Angular Momentum

In translational motion we learned that momentum is conserved in the absence of external forces and mechanical energy is conserved when purely conservative forces act on a body. The same principle applies for angular momentum and rotational kinetic energy. That is, $\vec{L_0} = \vec{L_f}$ and $E_0 = E_f$.

In addition, angular momentum is not only defined by $\vec{L} = I\vec{\omega}$, but also as $\vec{p} \times \vec{r}$ or $m\vec{v} \times \vec{r}$ and it exhibits the relationship $\sum \vec{\tau} = \frac{d\vec{L}}{dt}$.

6 Rolling w/o Slipping

When dealing with translational motion, we often work with point particles or blocks. However, we often need to analyze objects that are able to roll, such as balls or rods. When friction is present, it is not enough to consider only translation or rotation; both must be accounted for. When an object rolls without slipping, its angular and translational motion are related with the equations given below.

$$v = \omega r \text{ and/or } a = \alpha r$$

7 Problems

1. TORQUE: Two objects are attached to ropes that are attached to wheels on a common axle. The two wheels are glued together so that they form a single object. Mass m_1 hangs from the wheel with radius $R_1 = 1.0$ m and mass m_2 hangs from the wheel with radius $R_2 = 0.50$ m. The total moment of inertia of the two wheels is 43 $kg \cdot m^2$.

   (a) If $m_1 = 24$ kg, find m_2 such that there is no angular acceleration of the wheels.

   (b) If 12 kg is gently added to the top of m_1, find the angular acceleration of the wheels.

   (c) Find the tension in the rope holding m_1.

   (d) Find the tension in the rope holding m_2.

2. ENERGY: A marble ($I = \frac{2}{5}mr^2$) of mass M and radius R rolls without slipping down the track on the left from a height h_1. The marble then goes up the frictionless track on the right to a height h_2. Find h_2. (Use the following as necessary: M, R, and h_1.)

3. ROLLING: A basketball rolls without slipping down an incline of angle θ. The coefficient of static friction is μ_s. Model the ball as a thin spherical shell ($I = \frac{2}{3}mr^2$). (Use any variable or symbol stated above along with the following as necessary: m for the mass of the ball and g.)

   (a) Find the acceleration of the center of mass of the ball.

   (b) Find the frictional force acting on the ball.

   (c) Find the maximum angle of the incline for which the ball will roll without slipping.

4. ANGULAR MOMENTUM: A small blob of putty of mass m falls from the ceiling and lands on the outer rim of a turntable of radius R and moment of inertia I_0 that is rotating freely with angular speed ω_0 about its vertical fixed-symmetry axis. (Use any variable or symbol stated above as necessary.)

   (a) What is the postcollision angular speed of the turntable-putty system?

   (b) After several turns, the blob flies off the edge of the turntable. What is the angular speed of the turntable after the blob's departure?

5. ANGULAR MOMENTUM: A particle of mass m moves along a straight line at constant speed v. What is its angular momentum about a fixed point a distance d away as a function of time t?

8 Answers to Problems

1. (a) $m_2 = 48$ kg
   (b) $\alpha = 1.29\ ^{rads}/_{s^2}$
   (c) $T_1 = 0.307$ kN
   (d) $T_2 = 0.502$ kN
2. $h_2 = \frac{1}{7}(5h_1 + 2R)$
3. (a) $a = \frac{3}{5} g \sin\theta$
   (b) $f_s = \frac{2}{5} mg \sin\theta$
   (c) $\theta_{max} = \arctan \frac{5}{2}\mu_s$
4. (a) $\omega_f = \frac{I_0\omega_0}{I_0 + mR^2}$
   (b) $\omega = \frac{I_0\omega_0}{I_0 + mR^2}$
5. $L = dmv$