Introduction to Quantum Mechanics 3

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1 Introduction

This lecture is a continuation of the previous two lectures given on quantum mechanics. Our purpose is to introduce quantum mechanics at a basic level, so we will not be introducing advanced topics. However, we hope that these lectures inspire you to examine these more advanced topics in your free time.

2 Commutators and Properties

Matrices, in general, do not commute: try it out with an arbitrary pair of matrices. Thus, we define the commutator of two matrices $\hat{A}$ and $\hat{B}$ as

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

There are several properties of commutators. Here is a list of the most important ones:

$$[\hat{A},\hat{B}] + [\hat{B},\hat{A}] = 0$$
$$[\hat{A},\hat{A}] = 0$$
$$[\hat{A},\hat{B}+\hat{C}] = [\hat{A},\hat{B}] + [\hat{A},\hat{C}]$$
$$[\hat{A}+\hat{B},\hat{C}] = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]$$
$$[\hat{A},\hat{B}+\hat{C}] = [\hat{A},\hat{B}] + [\hat{A},\hat{C}]$$
$$[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$$
$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

Let's also develop a conjecture on what it means for two matrices to commute (commutator is 0). If such matrices commute, we can say that

$$[\hat{A},\hat{B}]|\psi\rangle = 0$$
$$\hat{A}\hat{B}|\psi\rangle = \hat{B}\hat{A}|\psi\rangle$$

if we conveniently assume that $|\psi\rangle$ is an eigen-vector of both $\hat{A}$ and $\hat{B}$ with eigen-values a and b, respectively, we see that

$$ab|\psi\rangle = ba|\psi\rangle$$

an identically true statement. This suggests that operators which commute with each other share at least one eigen-state. Operators that share more than one eigen-state in common are said to be **degenerate**.

*With contributions from Arun Kannan

3 Commutators of Angular Momentum

Recall the rotation matrices in three dimensions

$$\hat{S}_z = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{S}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$\hat{S}_y = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

If we use small angle approximations for sine and cosine, we get

$$\hat{S}_z = \begin{pmatrix} 1-\frac{\Delta\phi^2}{2} & -\Delta\phi & 0 \\ \Delta\phi & 1-\frac{\Delta\phi^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{S}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\Delta\phi^2}{2} & -\Delta\phi \\ 0 & \Delta\phi & 1-\frac{\Delta\phi^2}{2} \end{pmatrix}$$

$$\hat{S}_y = \begin{pmatrix} 1-\frac{\Delta\phi^2}{2} & 0 & \Delta\phi \\ 0 & 1 & 0 \\ -\Delta\phi & 0 & 1-\frac{\Delta\phi^2}{2} \end{pmatrix}$$

Taking the commutator of $\hat{S}_x$ and $\hat{S}_y$, we get:

$$[\hat{S}_x, \hat{S}_y] = \begin{pmatrix} 0 & -\Delta\phi^2 & 0 \\ \Delta\phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{S}_z\left(\Delta\phi^2\right) - \hat{I}$$

Where we retain powers of $\Delta\phi^2$ or lower, and $\hat{S}_z\left(\Delta\phi^2\right)$ is the rotation operator for $\Delta\phi^2$ in the z direction. We know that the rotation matrices, as defined in quantum mechanics, are $\hat{S}_n = e^{\frac{-i\hat{J}_n\Delta\phi}{\hbar}}$. We can thus say that

$$\hat{S}_x = e^{\frac{-i\hat{J}_x\Delta\phi}{\hbar}}$$
$$\hat{S}_y = e^{\frac{-i\hat{J}_y\Delta\phi}{\hbar}}$$
$$\hat{S}_z = e^{\frac{-i\hat{J}_z\Delta\phi}{\hbar}}$$

If we do a Taylor series expansion to the second powers of the angle and use the commutation relationship, we get

$$\left\{1 - \frac{i\hat{J}_x\Delta\phi}{\hbar} - \frac{1}{2}\left(\frac{i\hat{J}_x\Delta\phi}{\hbar}\right)^2\right\}\left\{1 - \frac{i\hat{J}_y\Delta\phi}{\hbar} - \frac{1}{2}\left(\frac{i\hat{J}_y\Delta\phi}{\hbar}\right)^2\right\} -$$
$$\left\{1 - \frac{i\hat{J}_y\Delta\phi}{\hbar} - \frac{1}{2}\left(\frac{i\hat{J}_y\Delta\phi}{\hbar}\right)^2\right\}\left\{1 - \frac{i\hat{J}_x\Delta\phi}{\hbar} - \frac{1}{2}\left(\frac{i\hat{J}_x\Delta\phi}{\hbar}\right)^2\right\} = \left(1 - \frac{i\hat{J}_z\Delta\phi^2}{\hbar}\right) - 1$$

Equating terms of 2nd order, we get

$$\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x = i\hbar\hat{J}_z$$
$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$$

Similarly, through cyclic permutation of indices

$$[\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x$$
$$[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$$

4 Eigenstates and Eigenvalues of Angular Momentum

We will start by defining the following operator

$$\hat{\boldsymbol{J}}^2 = \hat{J_x}^2 + \hat{J_y}^2 + \hat{J_z}^2$$

Using properties of the commutator and the relationships we derived

$$[\hat{J}_z, \hat{\boldsymbol{J}}^2] = [\hat{J}_z, \hat{J_x}^2] + [\hat{J}_z, \hat{J_y}^2] + [\hat{J}_z, \hat{J_z}^2]$$
$$= \hat{J}_x[\hat{J}_z, \hat{J}_x] + [\hat{J}_z, \hat{J}_x]\hat{J}_x + \hat{J}_y[\hat{J}_z, \hat{J}_y] + [\hat{J}_z, \hat{J}_y]\hat{J}_y$$
$$= i\hbar\left([\hat{J}_x, \hat{J}_y] + [\hat{J}_y, \hat{J}_x]\right) = 0$$

Because the commutator is 0, $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$ share a common eigenstate. This means that the states are called $|\lambda, m\rangle$, where the following is true:

$$\hat{\boldsymbol{J}}^2|\lambda, m\rangle = \lambda\hbar^2|\lambda, m\rangle$$

$$\hat{J_z}^2|\lambda, m\rangle = m\hbar|\lambda, m\rangle$$

5 Raising and Lowering Operators

Let us consider the matrix

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

Take a moment to prove the following

$$\hat{J_+}^\dagger = \hat{J}_-$$
$$[\hat{J}_z, \hat{J}_\pm] = \pm\hat{J}_\pm$$

Next, we must set out to determine the action of the operator on an eigenstate of $\hat{J}_z$

$$\hat{J}_z\hat{J}_+|\lambda, m\rangle = \left([\hat{J}_z, \hat{J}_+] + \hat{J}_+\hat{J}_z\right)|\lambda, m\rangle$$

$$\hat{J}_z\hat{J}_+|\lambda, m\rangle = \left(\hbar\hat{J}_+ + \hat{J}_+ m\hbar\right)|\lambda, m\rangle$$

$$\hat{J}_z\hat{J}_+|\lambda, m\rangle = \hbar(m+1)\left\{\hat{J}_+|\lambda, m\rangle\right\}$$

This suggests that $\hat{J}_+|\lambda, m\rangle$ is an eigen state of $\hat{J}_z$ with an eigen value $(m+1)\hbar$. Since the normal eigen value associated with the angular momentum eigen state is incremented by a multiple of Planck's constant, this operator is known as the **Raising Operator**. Similarly

$$\hat{J}_z\hat{J}_-|\lambda, m\rangle = \hbar(m-1)\left\{\hat{J}_-|\lambda, m\rangle\right\}$$

This operator is known as the **Lowering Operator**.

6 Possible Values of Angular Momentum

It turns out that Angular Momentum is quantized, or, in other words, can exist only in discrete values. We will explore this. We know that a particle with a certain angular momentum is bounded by an upper and lower value: that is, spinning as fast as possible in either direction. Because of the argument of symmetry, we can bound all possible values of angular momentum between $[-j, j]$. However, we have yet to find out how sparse the effects of quantization is. We know that, if we are at a state of maximum angular momentum,

$$\hat{J}_+|\lambda, j\rangle = 0$$

because the raising operator cannot raise the state above this threshold. This means that

$$\hat{J}_-\hat{J}_+|\lambda, j\rangle = 0$$
$$\left(\hat{\boldsymbol{J}}^2 - \hat{J_z}^2 + \hbar\hat{J}_z\right)|\lambda, j\rangle = 0$$
$$\left(\lambda - j^2 - j\right)\hbar^2|\lambda, j\rangle = 0$$
$$\lambda = j(j+1)$$

Starting from the lowering operator, we obtain something similar

$$\lambda = j'(j'-1)$$

Setting these values equal to each other

$$j'(j'-1) = j(j+1)$$
$$1) \rightarrow j' = -j$$
$$2) \rightarrow j' = j+1$$

The second case violates the assumption of a bound angular momentum, so we have rigorously proved that the angular momentum ranges from $[-j, j]$. The raising or lowering operator changes $j \rightarrow j+1$ or $j \rightarrow j-1$ for any j not equal to the maximum or minimum values. This asserts that $j - (-j) = 2j$ must be divisible by 1, or, in other words, an integer. This gives us half integer possibilities for the maximum and minimum angular momentum. As a result, we can deduce that

$$\hat{\boldsymbol{J}}^2|\lambda, j\rangle = j(j+1)\hbar^2|\lambda, j\rangle$$
$$\hat{J}_z|\lambda, j\rangle = m\hbar|\lambda, j\rangle$$

For example, if we were to take a spin 3/2 particle, the maximum and minimum angular momentum (in the z direction) are $\frac{3}{2}$ and $\frac{-3}{2}$, respectively. Thus the allowed values are

$$-\frac{3}{2}\hbar, -\frac{1}{2}\hbar, \frac{1}{2}\hbar, \frac{3}{2}\hbar$$

7 Uncertainties in Angular Momentum

So far, we have discussed the various values that the angular momentum of a particle can take on, but we have yet to quantify the uncertainty of a measurement. Indeed, we will set out to prove the more general form of the Heisenburg Uncertainty Principle. Let us first assume **Hermitian** operators which satisfy

$$[\hat{A}, \hat{B}] = i\hat{C}$$

Let us now make use of a simple, yet powerful inequality called the Cauchy-Schwarz inequality. For inner products, this states that

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

In our Cartesian coordinate system, where the inner products are now dot products, we can argue that

$$(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) \geq |(\mathbf{a}\cdot\mathbf{b})|^2$$

$$a^2b^2 \geq a^2b^2\cos^2\theta_{ab}$$

which makes sense. However, this is not the way to prove the statement in a Hilbert Space (see problem 2). Nevertheless, let's choose

$$|\alpha\rangle = (\hat{A} - \langle A\rangle)|\psi\rangle$$
$$|\beta\rangle = (\hat{B} - \langle B\rangle)|\psi\rangle$$

Now, we just plug this into our inequality

$$\langle\alpha|\alpha\rangle = \langle\psi|(\hat{A} - \langle A\rangle)^2|\psi\rangle = (\Delta A)^2$$
$$\langle\beta|\beta\rangle = \langle\psi|(\hat{B} - \langle B\rangle)^2|\psi\rangle = (\Delta B)^2$$
$$\langle\alpha|\beta\rangle = \langle\psi|(\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)|\psi\rangle$$

However, the last statement needs some pruning and simplification. Let us call $(\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle) = \hat{Q}$. Thus, we can say that

$$\hat{Q} - \hat{Q}^\dagger = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A}, \hat{B}] = i\hat{C}$$

Because of the hermicity of $\hat{A}$ and $\hat{B}$. However, $\frac{\hat{Q}-\hat{Q}^\dagger}{2}$ preserves, in a sense, the complex portion of the matrix, as how $\frac{a-a^*}{2} = Im(a)$. This means that $\frac{\hat{C}}{2}$ is the imaginary portion of the matrix. The quantity

$$\langle\alpha|\beta\rangle = \langle\psi|(\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)|\psi\rangle = \langle\psi|\hat{Q}|\psi\rangle$$

Gives the expected value for $\hat{Q}$. Thus, $|\langle\alpha|\beta\rangle|^2$ gives the sum of the squares of the "Real" and "Imaginary" parts of the matrix $\hat{Q}$, that is

$$|\langle\alpha|\beta\rangle|^2 = |\langle R\rangle|^2 + |\langle I\rangle|^2 \geq |\langle I\rangle|^2$$

From the Schwarz inequality (follow along carefully!), we also know that

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = (\Delta A)^2(\Delta B)^2 \geq |\langle\alpha|\beta\rangle|^2 = |\langle R\rangle|^2 + |\langle I\rangle|^2 \geq |\langle I\rangle|^2$$
$$(\Delta A)^2(\Delta B)^2 \geq |\langle I\rangle|^2$$

The imaginary part of the matrix $\hat{Q}$ is $\frac{\hat{C}}{2}$, so we get

$$(\Delta A)^2(\Delta B)^2 \geq \frac{|\langle C\rangle|^2}{4}$$
$$(\Delta A)(\Delta B) \geq \frac{|\langle C\rangle|}{2}$$

which concludes the proof. Since, we know that

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$$
$$[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$$
$$[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$$

We can say that

$$\Delta J_x\Delta J_y \geq \frac{\hbar}{2}|\langle J_z\rangle|$$
$$\Delta J_y\Delta J_z \geq \frac{\hbar}{2}|\langle J_x\rangle|$$
$$\Delta J_z\Delta J_x \geq \frac{\hbar}{2}|\langle J_y\rangle|$$

8 Stern Gerlach Mechanics for Different Spin Particles

This is just an eigen-value problem, where we use matrix representations of the angular momentum operators to determine the eigen states and, thus, their respective probabilities.

9 Problems

1. Introduce $\cos\theta = \frac{J_z}{\mathbf{J}}$. Determine this angle for a spin-1/2 particle, a spin-1 particle, a spin-3/2 particle and a macroscopic top.

2. Derive the Schwarz Inequality. If we get to it, I'll give you the hint.

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

3. Verify the uncertainty principle for the following spin states

$$|\psi\rangle = |+z\rangle \tag{1}$$

$$|\psi\rangle = |+x\rangle \tag{2}$$

$$|\psi\rangle = |+y\rangle \tag{3}$$

$$|\psi\rangle = \frac{i}{2}|+z\rangle + \frac{\sqrt{3}}{2}|-z\rangle \tag{4}$$