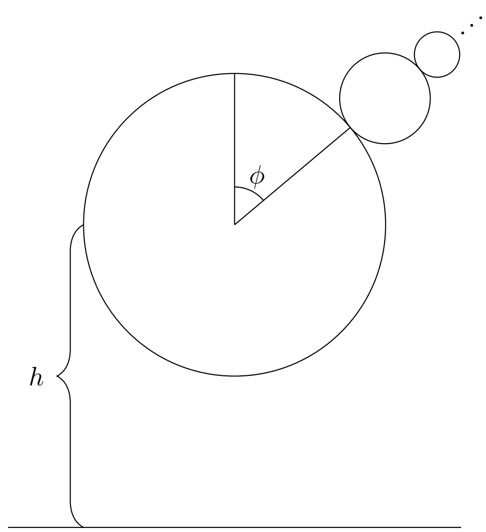
To Infinity and Beyond

Shankar Balasubramanian *

November 2014

1 Problem Statement

Consider problem **8** on the conservation of energy/momentum problem set 2. Instead of just two balls, consider N balls on top of each other, with each ball placed at an angle ϕ from the ball below it. Each ball has a mass much smaller than the ball below it. Determine the range of the trajectory of the top-most ball (assuming that the total radii of the balls is much smaller than the height at which they were dropped).



2 Solution

We first will consider the $(i+1)$st mass on top of the ith mass. Let us assume that the ith mass is moving with a velocity $\langle v_{i,x}, v_{i,y}\rangle$ during the collision with the falling mass, moving at a velocity $\langle 0, -V\rangle$. (We will call $\sqrt{2gh} \equiv V$ from now on). The relative velocity is thus $\vec{v}_{rel} = \langle v_{i,x}, v_{i,y} + V\rangle$. This effectively converts the situation to the stationary reference frame of the much larger ball. In this center of mass reference frame, the small ball's scattering angle also changes, and it will non-trivially deflect off of the larger ball (as opposed to the 2-ball case, where both balls were initially moving only in the vertical direction). Call this scattering angle Φ.

The smaller ball will deflect by an angle $2\left(\phi - \left(\frac{\pi}{2} - \Phi\right)\right)$. Thus, relative to the horizontal, the ball will appear at an angle $\frac{\pi}{2} - \left(\phi + \phi - \left(\frac{\pi}{2} - \Phi\right)\right) = \pi - (2\phi + \Phi)$. See the diagram above for more details. The speed of the ball is unchanged because the collision is elastic, and it remains at $|\vec{v}_{rel}|$. Thus, the ball emerges with a velocity

*Diagram by Ross Dempsey

$$\begin{aligned}\vec{v}'_{rel} &= \langle |\vec{v}_{rel}|\cos(\pi-(2\phi+\Phi)), |\vec{v}_{rel}|\sin(\pi-(2\phi+\Phi))\rangle \\ &= \langle -|\vec{v}_{rel}|\cos(2\phi+\Phi), |\vec{v}_{rel}|\sin(2\phi+\Phi)\rangle\end{aligned}$$

Expanding the trig functions using identities, and using the fact that $|\vec{v}_{rel}|\cos\Phi = v_{i,x}$ and $|\vec{v}_{rel}|\sin\Phi = v_{i,y}+V$, we get

$$\vec{v}'_{rel} = \langle \left[-v_{i,x}\cos 2\phi + (v_{i,y}+V)\sin 2\phi\right], \left[v_{i,x}\sin 2\phi + (v_{i,y}+V)\cos 2\phi\right]\rangle$$

Thus, converting back to the stationary frame

$$\vec{v}' = \langle \left[-v_{i,x}\cos 2\phi + (v_{i,y}+V)\sin 2\phi + v_{i,x}\right], \left[v_{i,x}\sin 2\phi + (v_{i,y}+V)\cos 2\phi + v_{i,y}\right]\rangle = \langle v_{i+1,x}, v_{i+1,y}\rangle$$

This gives us a coupled set of recursive relationships, which can be casted into the following matrix form

$$\begin{pmatrix} 1-\cos 2\phi & \sin 2\phi & V\sin 2\phi \\ \sin 2\phi & 1+\cos 2\phi & V\cos 2\phi \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_{i,x} \\ v_{i,y} \\ 1\end{pmatrix} = \begin{pmatrix} v_{i+1,x} \\ v_{i+1,y} \\ 1\end{pmatrix}$$

The eigenvalues of the transformation matrix are $\{2,1,0\}$, and the eigenvectors corresponding to these eigenvalues are $\{\langle\tan\phi,1,0\rangle, \langle 0,V,-1\rangle, \langle -\cot\phi,1,0\rangle\}$. The initial vector is given by $\langle 0,V,1\rangle$, which can be written as a linear combination of eigenvectors $-e_2 + 2V(\sin^2\phi e_1 + \cos^2\phi e_3)$. Thus, the matrix equation looks like

$$\hat{M}^i(-e_2 + 2V(\cos^2\phi e_1 + \sin^2\phi e_3)) = \begin{pmatrix} v_{i+1,x} \\ v_{i+1,y} \\ 1\end{pmatrix}$$

$$(2^{i+1}V\cos^2\phi)e_1 - (1^i)e_2 + (0^i 2V\sin^2\phi)e_3 = \begin{pmatrix} v_{i+1,x} \\ v_{i+1,y} \\ 1\end{pmatrix}$$

This simplifies to

$$\begin{pmatrix} 2^i V\sin 2\phi \\ V(2^{i+1}\cos^2\phi - 1) \\ 1\end{pmatrix} = \begin{pmatrix} v_{i+1,x} \\ v_{i+1,y} \\ 1\end{pmatrix}$$

It is reassuring that the $\phi = 0$ case gives us the answer to 7(b) on the Problem Set, and $N = 1$ (we will call $i \equiv N$) gives us the answer to 8 on the problem set. There is a particularly interesting value of N for which the ball is launched horizontally. Nevertheless, we are interested in finding the range of the smallest ball's motion. Since the range is given by $R = \frac{2v_x v_y}{g}$,

$$R = \frac{2^{N+1}V^2\sin 2\phi(2^{N+1}\cos^2\phi - 1)}{g}$$

This is maximized when the derivative is 0. The general value of ϕ for arbitrary n is difficult, because the equation becomes quartic. When setting the derivative to 0, we get as far as saying

$$2\cos 2\phi\left(2^{N+1}\cos^2\phi - 1\right) - 2^{N+1}\sin^2 2\phi = 0$$

Now, in the limit that $N \to \infty$, the constant term in the parenthesis disappears, and we get

$$2\cos 2\phi \left(2^{N+1}\cos^2\phi\right) - 2^{N+1}\sin^2 2\phi = 0$$

$$\sin\phi = \frac{1}{2} \rightarrow \phi = \frac{\pi}{6}$$

This is a very interesting result. In the limit that the number of balls goes to infinity, the angle they are placed at should be this particular value for the range to be maximized. Of course, the range will be infinitely large either way, but some infinities are larger than others.

Another interesting result is the resultant angle the "infinith" ball makes with the horizontal. For the $(N+1)$st ball, the angle is given by

$$\tan\Psi = \frac{v_y}{v_x} = \frac{V(2^{i+1}\cos^2\phi - 1)}{2^i V \sin 2\phi}$$

In the limit that N goes to infinity, this reduces to

$$\tan\Psi = \frac{2\cos^2\phi}{\sin 2\phi} = \cot\phi$$

Thus, $\Psi = \frac{\pi}{2} - \phi$. This is another unusual result, a convergent value which is independent of V and g.