Lagrangian Mechanics 8th Period Problem Solutions

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1 Rotating Monomial

So we want to decompose the velocity into two parts, one along the curve, and the other the angular velocity. So the velocity along the curve squared is:

$$\frac{ds}{dt} = \frac{dx}{dt}\sqrt{1 + y'(x)^2}.$$

We can now easily write the Lagrangian as:

$$\mathcal{L} = \frac{1}{2}m\left(\omega^2 x^2 + \dot{x}^2(1 + y'(x)^2)\right) - mgy(x),$$

and by varying x we obtain:

$$\ddot{x}(1 + y'^2)^2 + \dot{x}^2 y' y'' = \omega^2 x - gy'.$$

So now we want the equilibrium value which will occur at $\dot{x} = \ddot{x} = 0$. We get for the equilibrium value x_0:

$$x_0 = \frac{gy'(x_0)}{\omega^2} = a\left(\frac{a^2\omega^2}{ngb}\right)^{1/n-2}$$

To find the frequency of small oscillations about this equilibrium value we set $x(t) = x_0 + \delta(t)$. We will expand the equation of motion in linear terms of *delta*:

$$\ddot{\delta}(1 + y'(x_0)^2) - (\omega^2 - gy''(x_0))\delta = 0$$

$$\Omega^2 = \frac{\omega^2 - gy''(x_0)}{1 + y'(x_0)^2}$$

2 Coupled Pendulums

First we should find how long the spring is for a given θ_1 and θ_2, which will help us to get the potential energy of the spring. Point A is the location where the spring is attached to the pendulum with angle θ_1, and Point B is the same for the other pendulum.

$$\begin{aligned}
(x, y)_A &= (\frac{l}{2}\sin\theta_1, -\frac{l}{2}\cos\theta_1) \\
(x, y)_B &= (\frac{l}{2}\sin\theta_2, -\frac{l}{2}\cos\theta_2) \\
D &= \sqrt{(y_A - y_B)^2 + (x_A - x_B)^2} = \sqrt{\left(\frac{l}{2}(\sin\theta_2 - \sin\theta_1)\right)^2 + \left(\frac{l}{2}(\cos\theta_2 - \cos\theta_1)\right)^2} \\
&= \frac{l}{\sqrt{2}}\sqrt{1 - \cos(\theta_2 - \theta_1)}
\end{aligned}$$

The kinetic and potential energies are then fairly simple:

$$
\begin{aligned}
T &= \frac{1}{2}m(l^2\dot{\theta_1}^2 + l^2\dot{\theta_2}^2) \\
V &= -mgl(\cos\theta_1 + \cos\theta_2) + \frac{1}{2}k\frac{l^2}{2}(1 - \cos(\theta_2 - \theta_1)) \\
\mathcal{L} &= \frac{1}{2}m(l^2\dot{\theta_1}^2 + l^2\dot{\theta_2}^2) + mgl(\cos\theta_1 + \cos\theta_2) - \frac{1}{2}k\frac{l^2}{2}(1 - \cos(\theta_2 - \theta_1))
\end{aligned}
$$

Using the Euler-Lagrange Equation on θ_1 and θ_2:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}}\right) = \frac{\partial \mathcal{L}}{\partial \theta_1} \quad &\Rightarrow \quad ml^2\ddot{\theta_1} = -mgl\sin\theta_1 + \frac{kl^2}{4}\sin(\theta_2 - \theta_1) \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}}\right) = \frac{\partial \mathcal{L}}{\partial \theta_2} \quad &\Rightarrow \quad ml^2\ddot{\theta_2} = -mgl\sin\theta_2 - \frac{kl^2}{4}\sin(\theta_2 - \theta_1)
\end{aligned}
$$

Now the symmetry mentioned in the problem becomes clear. The right hand sides of the differential equations are differing by a minus sign, which is what you would expect for a coupled system like this. Using a small angle approximation for θ_1 and θ_2 we obtain equations for the motion:

$$
\begin{aligned}
\ddot{\theta_1} + \frac{g}{l}\theta_1 &= \frac{kl^2}{4}(\theta_2 - \theta_1) \\
\ddot{\theta_2} + \frac{g}{l}\theta_2 &= -\frac{kl^2}{4}(\theta_2 - \theta_1)
\end{aligned}
$$