Momentum/Energy Problem Set 2 Solutions

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1 A Quick Comment

This problem set is intended to be extremely difficult. Expect problems of Easy-Medium difficulty to appear on the F=ma exam, and problems of Hard difficulty to appear on the Semifinal exam. The other questions are extremely difficult; I guarantee you that nothing Very Hard or above will appear on USAPhO exams.

2 Solutions

1. Since momentum is conserved and the collision is inelastic, we have the following

$$(120\,\mathrm{kg})\left(2\,\mathrm{m\,s^{-1}}\right) - (40\,\mathrm{kg})\left(5\,\mathrm{m\,s^{-1}}\right) = (160\,\mathrm{kg})v$$

$$0.25\,\mathrm{m\,s^{-1}} = v$$

2. Since the collision is elastic, the path of the ball is essentially reflected when it meets the wall. In fact, we can consider the trajectory of the ball after it bounces a continuation of its original trajectory except in the opposite horizontal direction. Thus, the situation is actually equivalent to one in which Diego stands 30 m from Jordi and there is no intervening wall. Remembering the range of a projectile from kinematics, we obtain:

$$\begin{aligned} \frac{v^2}{g}\sin 2\theta &= 30\,\mathrm{m} \\ v^2 &= 200\sqrt{3}\,\mathrm{m^2/s^2} \\ v &= 18.6\,\mathrm{m\,s^{-1}} \end{aligned}$$

3. Although there exists a coefficient of restitution, there is still no net external force, implying that momentum is conserved. If momentum is conserved, then we know that the velocity of the center of mass is constant. This means that we can condense both objects into a single point that moves with a velocity V_{cm}. This velocity is just

$$V_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{(2kg)\left(5\frac{m}{s}\right)}{2kg + 10kg} = \frac{5}{6}\frac{m}{s}$$

After 60 seconds, the center of mass will travel a distance $\frac{5}{6}\frac{m}{s} * 60s = 50m$. The position of the smaller mass, however, falls within a 5 meter uncertainty (since it can lie in any place in the 5 meter long box.) So, the distance traveled is between 45 and 55 meters, which is closest to choice **B**.

4. In the first case, the sand falls with no horizontal momentum and lands on the freight car, where it begins moving with velocity v. The force is simply the rate of change of momentum:

$$F = \frac{dp}{dt} = v\frac{dm}{dt}$$

The second case is trickier. We might be tempted to think that the situation is reversed, and that an opposing force is required to stop the freight car from gaining speed. However, when the sand leaves the freight car, there is actually no change in momentum: the sand may have left the car, but it continues moving in the air with the same horizontal speed. Only when it reaches the ground does its momentum change, so there is a force on the ground, but none on the freight car.

5. The more straightforward solution is much longer and more tedious. We will illustrate a slick solution which instantly gives the answer. The conservation of momentum statement says that

$$m_1\vec{v_0} = m_1\vec{v_1} + m_2\vec{v_2}$$

Taking the dot product with itself gives

$$\begin{aligned} m_1^2\vec{v_0}\cdot\vec{v_0} &= m_1^2\vec{v_1}\cdot\vec{v_1} + m_2^2\vec{v_2}\cdot\vec{v_2} + 2m_1m_2\vec{v_1}\cdot\vec{v_2} \\ m_1^2v_0^2 &= m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2\vec{v_1}\cdot\vec{v_2} \\ v_0^2 &= v_1^2 + v_2^2 + 2\vec{v_1}\cdot\vec{v_2} \end{aligned}$$

where the last statement assumes that the masses are equal. Using the conservation of energy (and cancelling out mass factors), we obtain

$$v_0^2 = v_1^2 + v_2^2$$

Subtracting the two equations, we get $2\vec{v_1}\cdot\vec{v_2} = 0$, or that the final velocity vectors are perpendicular to each other.

6. a) First, we find the velocity of the center of mass of the system and change to that reference frame:

$$\begin{aligned} v_{CM} &= \frac{m_1v_1 + m_2v_2}{m_1+m_2} \\ v_{1C} &= v_1 - v_{CM} = \frac{m_2(v_1-v_2)}{m_1+m_2} \\ v_{2C} &= v_2 - v_{CM} = \frac{m_1(v_2-v_1)}{m_1+m_2} \end{aligned}$$

We see from this that the momentum in the COM frame is zero. We can then write the conservation of momentum and of energy for the velocities after a collision:

$$\begin{aligned} &m_1v'_{1C} + m_2v'_{2C} = 0 \\ &m_1v'^2_{1C} + m_2v'^2_{2C} = m_1v^2_{1C} + m_2v^2_{2C} = \frac{m_1m_2(v_1-v_2)^2}{m_1+m_2} \end{aligned}$$

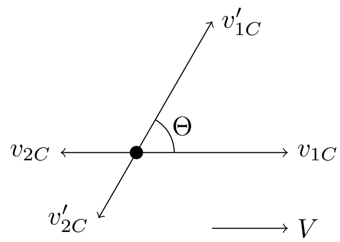
We can use the first equation to substitute for v'_{2C} in the second equation:

$$\begin{aligned} m_1v'^2_{1C} + \frac{m_1^2}{m_2}v'^2_{1C} &= \frac{m_1m_2(v_1-v_2)^2}{m_1+m_2} \\ \left(1+\frac{m_1}{m_2}\right)v'^2_{1C} &= \frac{m_2(v_1-v_2)^2}{m_1+m_2} \\ v'^2_{1C} &= \frac{m_2^2(v_1-v_2)^2}{(m_1+m_2)^2} = v^2_{1C} \end{aligned}$$

We could follow a very similar proof to show that the magnitude of v_{2C} is conserved through the collision. The momentum is zero before and after the collision, so both sets of vectors must be in direct opposition. Thus, the only possibility is that they are both rotated by the same angle Θ.

b) We see in the diagram below that the y-component of the lab velocity is the same as the y-component of the COM velocity, and the x-component in the lab frame is the sum of the velocity of the COM and the x-component of the COM frame velocity. This leads directly to the formula:

$$\tan\theta = \frac{v' \sin\Theta}{V + v' \cos\Theta}$$



c) Without loss of generality, we can move the lab frame so that $v_2 = 0$. Then, moving to the center of mass frame, we have

$$v_{CM} = \frac{m_1 v_1}{m_1 + m_2}$$
$$v_{1C} = \frac{m_2 v_1}{m_1 + m_2}$$

We also know from part a that $|v_{1C}| = |v'_{1C}|$, so $V/v' = V/v = m_1/m_2$.

d) If m_1 is the smaller mass, then we see immediately that the denominator can be made zero and the small mass can be rotated through a full 90°. In the other case, we take the derivative, obtaining:

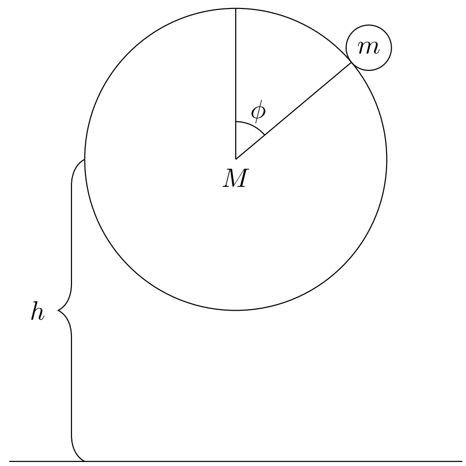
$$\frac{d(\tan\theta)}{d\Theta} = \frac{\cos\Theta(m_1/m_2 + \cos\Theta) + \sin^2\Theta}{(m_1/m_2 + \cos\Theta)^2} = \frac{1 + (m_1/m_2)\cos\Theta}{(m_1/m_2 + \cos\Theta)^2} = 0$$

We see that $\cos\Theta = -m_2/m_1$, and substituting back into the original equation and simplifying considerably, we arrive at $\tan\theta = 1/\sqrt{m_1^2/m_2^2 - 1}$ or $\sin\theta = m_2/m_1$. This gives the upper bound on the scattering angle for the larger mass.

7. (a) We will assume that the height $h \gg d$, which will simplify the problem (the general case is far more tedious, and we can illustrate the general method thought this simplification without toiling through unnecessary work). Since the mass of the first ball is much larger than that of the second ball, we can effectively treat the larger ball like a wall for which the smaller ball elastically bounces off of. Working in the frame of the larger ball, after it bounces off the ground, it immediately meets the falling smaller ball. To the larger ball, the smaller ball is falling towards it at a velocity $2v$ because their velocities are pointing in opposite directions. Thus after the collision, the smaller ball will appear to move *upwards* at a velocity $2v$. Converting back to the stationary frame, the small ball actually moves upwards at a velocity $3v$. Now, since the height an object rises to is proportional to the square of its velocity, the small ball will rise to a height $9h$. Compensating for the extra distance $d \ll h$, the small ball rises to a height $9h + d$.
(b) The same assumptions will be used. When the ith ball collides with the $i+1$st ball, we can treat the collision as a "brick wall" collision and work in the frame of the more massive ball. If the ball moves upwards at a velocity v_i and the smaller ball moves downwards at a velocity v_{i+1}, the small ball appears to be approaching the larger ball at a velocity $v_i + v$. After the collision, the small ball's velocity is reversed and appears to move at, in the stationary frame, $v_i + (v_i + v) = 2v_i + v$. This gives us

$$v_{i+1} = 2v_i + v$$
$$v_1 = v$$

by inspection, our solution is $v_n = (2^n - 1)v$. Since final height is proportional to the square of the velocity, the height the small ball travels is given by $h' = \ell + (2^n - 1)^2 h$.



8. Like in the previous problem, we will work in the reference frame of the much larger ball. Notice, in the diagram, that the smaller ball will appear to come at a velocity $2v$ towards the larger ball, in the reference frame of the larger ball. However, the small ball will deflect elastically about the tangent line to the surface of the larger ball at an angle ϕ from the vertical. Due to the deflection of the ball across the normal line (by a total angle 2ϕ), the ball's final velocity will appear to be at an angle $\frac{\pi}{2} - 2\phi$ from the horizontal (see the above figure). Thus, relative to the larger ball, the small ball will appear to move with a velocity

$$\vec{v}_{rel} = 2v \sin 2\phi \hat{i} + 2v \cos 2\phi \hat{j}$$

Since the massive ball moves upwards with a velocity v, the resulting velocity vector in the inertial frame will be

$$\vec{v} = 2v \sin 2\phi \hat{i} + (2v \cos 2\phi + v)\hat{j}$$

If we remember from kinematics, the range of a projectile's motion is $R = \frac{2v_0^2 \cos\theta \sin\theta}{g} = \frac{2(v_0 \cos\theta)(v_0 \sin\theta)}{g} = \frac{2v_x v_y}{g}$. Thus, the range of the small ball's motion is

$$R = \frac{2(2v \sin 2\phi)(2v \cos 2\phi + v)}{g} = \frac{4v^2 (\sin 2\phi)(2 \cos 2\phi + 1)}{g}$$

Since we want to maximize the range, we take a derivative and set it to zero. This is equivalent to setting the derivative of $(\sin 2\phi)(2 \cos 2\phi + 1) = \sin 4\phi + \sin 2\phi$ to zero. Using the substitution $\tilde{\phi} = 2\phi$, we get a quadratic equation. The positive root is $\cos\tilde{\phi} = \frac{\sqrt{33}-1}{8}$. From this, $\sin\tilde{\phi} = \frac{\sqrt{30+2\sqrt{33}}}{8}$. Thus,

$$R = \frac{4v^2(\frac{\sqrt{30+2\sqrt{33}}}{8})(\frac{\sqrt{33}+3}{4})}{g} = \frac{v^2(\sqrt{30+2\sqrt{33}})(\sqrt{33}+3)}{8g}$$

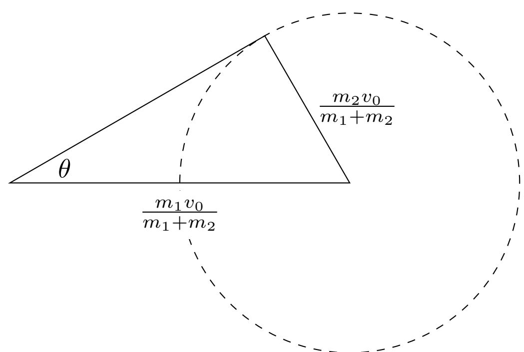
From the conservation of energy, $v^2 = 2gh$. This finally gives us the maximum range,

$$R = \frac{(\sqrt{30+2\sqrt{33}})(3+\sqrt{33})}{4}h \approx 14h$$

9. (a) In problem 6, we found that the maximum scattering angle for a mass m_1 in an elastic collision is given by $\sin\theta = m_2/m_1$. In every collision, we need $\theta = \pi/N$. If we take the small angle approximation $\sin\theta \approx \theta$, then we must have

$$\frac{\pi}{N} \leq \frac{M/N}{m}$$
$$\pi \leq \frac{M}{m}$$

(b) The best way to look at this part is to derive the constraint $\sin\theta \leq m_2/m_1$ in a different way. In the center of mass frame, we know that the collision results simply in a rotation of the velocity vector. Thus, the final velocity in the COM frame is of fixed magnitude but lies anywhere on a circle of radius $m_2v_0/(m_1+m_2)$. We can return to the lab frame by adding back v_{CM}, so that the final velocity vector lies on a circle as shown: The maximum scattering angle is obtained when



the velocity vector is tangent to the circle, so that the triangle in the diagram is right. Therefore, we can easily compute the final velocity from the Pythagorean theorem:

$$v_f = v_0\frac{\sqrt{m^2 - (M/N)^2}}{m + M/N}$$

The mass M/N is small compared to m, so we can approximate to first order in μ/m:

$$v_f \approx v_0\frac{m}{m+M/N} \approx v_0\left(1 - \frac{M/m}{N}\right)$$

Thus, the velocity is multiplied by a factor of $\left(1 - \frac{M/m}{N}\right)$ in each collision. Raising to the Nth power and taking the limit as N goes to infinity, the factor becomes $e^{-M/m}$. In part a, we found that in the minimum case, this becomes $e^{-\pi}$.

10. This problem is extremely difficult because of the number of approximations that are needed (enough to make the computation doable and not too many that the problem becomes trivialized). The first

approximation we will make is that the string winds up so slowly, that the contact point between the rope and the pole does not change appreciably. In this case, the potential energy of the mass can be approximated as just $-mgL\cos\theta$. Since dE is 0 from the conservation of energy, we get

$$d\left(\frac{1}{2}mv^2 - mgL\cos\theta\right) = 0$$
$$\frac{1}{2}md(v^2) + mgL\sin\theta d\theta = 0$$

To find v, we will appeal to Newton's Laws. At a particular time t, when the string has a length L and is inclined by θ, the second law equations read

$$T\cos\theta = mg$$
$$T\sin\theta = \frac{mv^2}{L\sin\theta}$$

Dividing the two equations gives $\tan\theta = \frac{v^2}{gL\sin\theta}$, or $v^2 = gL\sin\theta\tan\theta$. Plugging this into the conservation of energy statement gives us

$$d(L\sin\theta\tan\theta) + 2L\sin\theta d\theta = 0$$
$$\sin\theta\tan\theta dL + Ld\theta\left(\cos\theta\tan\theta + \frac{\sin\theta}{\cos^2\theta}\right) + 2L\sin\theta d\theta = 0$$
$$dL\frac{\sin^2\theta}{\cos\theta} + Ld\theta\left(3\sin\theta + \frac{\sin\theta}{\cos^2\theta}\right) = 0$$

Because this is separable and integrable, we get

$$-\frac{dL}{L} = \left(\frac{1}{\sin\theta\cos\theta} + 3\cot\theta\right)d\theta$$
$$\ln L = -3\ln\sin\theta + \ln\cot\theta + C$$
$$L = \tilde{C}\frac{\cos\theta}{\sin^4\theta}$$

From initial conditions, we get that $\tilde{C} = L_0\frac{\sin^4\theta_0}{\cos\theta_0}$. Notice that $\theta = \frac{\pi}{2}$ when the mass hits the pole. Now, we have the task of finding the height at which the mass hits the pole. We can do this by noting that when a small piece of string dL wraps around the pole, the mass moves up by a distance $dL\cos\theta$. Thus, the total distance the mass moves up is given by

$$\Delta y = \int d\left(\tilde{C}\frac{\cos\theta}{\sin^4\theta}\right)\cos\theta$$
$$\Delta y = L_0\frac{\sin^4\theta_0}{\cos\theta_0}\int d\left(\frac{\cos\theta}{\sin^4\theta}\right)\cos\theta$$

This integral can be simplified by using integration by parts (or by actually expanding out the differential, which is a much more tedious task). The result is

$$\Delta y = L_0 \frac{\sin^4\theta_0}{\cos\theta_0}\left(\frac{\cos^2\theta}{\sin^4\theta} - \frac{1}{2\sin^2\theta}\right)\Bigg|_{\theta_0}^{\frac{\pi}{2}}$$

where the bounds are for θ. This gives the result

$$\Delta y = -L_0\cos\theta_0\left(1 - \frac{\sin^2\theta_0}{2}\right)$$

The ball, thus, rises a distance $\frac{1}{2}L_0\sin^2\theta_0\cos\theta_0$ during the course of its motion. Finally, we can determine the ratio of the ball's final and initial velocity when it hits the pole using the conservation of energy.

$$\begin{aligned}
\frac{1}{2}mv_f^2 + \frac{1}{2}mgL_0\sin^2\theta_0\cos\theta_0 &= \frac{1}{2}m(gL_0\sin\theta_0\tan\theta_0) \\
\frac{1}{2}mv_f^2 &= \frac{1}{2}mgL_0\sin^2\theta_0\left(\frac{1}{\cos\theta_0} - \cos\theta_0\right) \\
v_f^2 &= gL_0\frac{\sin^4\theta_0}{\cos\theta_0} \\
\frac{v_f}{v_i} &= \sqrt{\frac{gL_0\frac{\sin^4\theta_0}{\cos\theta_0}}{gL_0\sin\theta_0\tan\theta_0}} = \sin\theta_0
\end{aligned}$$

a surprisingly simple result for so much work...

11. We start by writing the equations for conservation of momentum and energy, where v is the velocity of the sliding block relative to the hemisphere and V is the velocity of the hemisphere:

$$\begin{gathered}
mv\cos\theta = (m+M)V \\
MV^2 + mv^2\sin^2\theta + m(v\cos\theta - V)^2 = 2mgr(1-\cos\theta)
\end{gathered}$$

We can simplify these by introducing $\mu = M/m$:

$$\begin{gathered}
v\cos\theta = (1+\mu)V \\
\mu V^2 + v^2\sin^2\theta + (v\cos\theta - V)^2 = 2gr(1-\cos\theta)
\end{gathered}$$

Then, substituting for V in the second equation, we obtain

$$\begin{aligned}
\left(\frac{\mu}{(1+\mu)^2}\cos^2\theta + \sin^2\theta\right)v^2 + v^2\cos^2\theta\left(1 - \frac{1}{1+\mu}\right)^2 &= 2gr(1-\cos\theta) \\
v^2 &= \frac{2gr(1-\cos\theta)}{1 - \frac{1}{1+\mu}\cos^2\theta}
\end{aligned}$$

If we neglect the effects of the acceleration of the hemisphere, we simply need to find the angle for which $mv^2/r = mg\cos\theta$ (that is, where the centripetal force required is equal to the radial component of gravity). Thus, we substitute and solve:

$$\begin{gathered}
2mg(1-\cos\theta) = mg\cos\theta\left(1 - \frac{1}{1+\mu}\cos^2\theta\right) \\
\frac{1}{1+\mu}\cos^3\theta - 3\cos\theta + 2 = 0
\end{gathered}$$

12. (a) The time between collisions is the time it takes for both blocks to together traverse a distance of $2L$, or $2L/(V+v)$. Thus, the recurrence relation for the lengths is

$$L_{n+1} = L_n - V_n t = L_n - \frac{2L_n V_n}{V_n + v_n} = \frac{v_n - V_n}{v_n + V_n} L_n$$

Now we can employ the fact that relative speeds are preserved by elastic collisions. Remembering that v is negated after each collision when it bounces off the wall, this gives $v_n + V_n = v_{n+1} - V_{n+1}$. Substituting this in, we obtain a recurrence relation that is much easier to work with:

$$L_{n+1} = L_n - V_n t = L_n - \frac{2L_n V_n}{V_n + v_n} = \frac{v_n - V_n}{v_{n+1} - V_{n+1}} L_n$$

The utility of this new form can be seen by finding the subsequent length:

$$L_{n+2} = \frac{v_{n+1} - V_{n+1}}{v_{n+2} - V_{n+2}} L_{n+1} = \frac{v_n - V_n}{v_{n+2} - V_{n+2}} L_n$$

We see that in general, the product telescopes, and we have

$$L_n = \frac{v_0 - V_0}{v_n - V_n} L_0$$

When the block is at its closest approach, its velocity is approximately zero and conservation of energy gives $v = \sqrt{M/m} V_0$. This gives the final result:

$$L_{\min} = \sqrt{\frac{m}{M}} V_0$$

(b) We can write the collision as a matrix transformation, which can be verified directly from the conservation of momentum:

$$\begin{pmatrix} V_{n+1} \\ v_{n+1} \end{pmatrix} = \frac{1}{M+m} \begin{pmatrix} M-m & -2m \\ 2M & M-m \end{pmatrix} \begin{pmatrix} V_n \\ v_n \end{pmatrix}$$

The eigenvalues and eigenvectors of this transformation (including the factor of $1/(M+m)$ are

$$\lambda_1 = \frac{M - m - 2i\sqrt{Mm}}{M+m}, \quad e_1 = \begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}$$

$$\lambda_2 = \frac{M - m + 2i\sqrt{Mm}}{M+m}, \quad e_2 = \begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix}$$

We can simplify this considerably by noting that the eigenvalues are unimodular and conjugate complex numbers, so we can define $\lambda_1 = e^{-i\theta}$ and $\lambda_2 = e^{i\theta}$, where $\theta = \tan^{-1}(2\sqrt{Mm}/(M-m))$. Initially, $V = V_0$ and $v = 0$, so the initial vector is $\frac{V_0}{2}(e_1 + e_2)$. Using the recurrence relation N times, we obtain

$$\begin{aligned} \begin{pmatrix} V_N \\ v_N \end{pmatrix} &= \frac{V_0}{2} \left(\lambda_1^N e_1 + \lambda_2^N e_2 \right) \\ &= \frac{V_0}{2} \left(e^{-iN\theta} \begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix} + e^{iN\theta} \begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix} \right) \\ &= V_0 \begin{pmatrix} \cos N\theta \\ \sqrt{\frac{M}{m}} \sin N\theta \end{pmatrix} \end{aligned}$$

At its closest approach, the velocity of the larger block is zero, so $N\theta = \pi/2$. Substituting for θ, we have

$$N = \frac{\pi}{2 \tan^{-1} \frac{2\sqrt{Mm}}{M-m}}$$

At this point, we have yet to use the fact that $m \ll M$. We can simplify considerably by taking $\frac{\sqrt{Mm}}{M-m} \approx \sqrt{\frac{m}{M}}$ and then using the approximation that $\tan^{-1}(x) \approx x$:

$$N = \frac{\pi}{4}\sqrt{\frac{M}{m}}$$