(C Team) Gravitation

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1 Introduction

All physical bodies attract each other with a force called gravitational force, which is one of the fundamental forces in nature. This force not only holds you on Earth, but also reaches out across intergalactic space and acts between galaxies. Gravitational force exerts a force mg on everything on earth. Now let's find out if the same gravitational force provides the centripetal force that keeps the moon in its circular orbit surrounding the earth. If we measure its distance from earth's center r and its velocity v, we find its centripetal force $m\frac{v^2}{r}$ is about 3600 times smaller than mg. But we know the moon is about 60 times further from the center of earth than things on earth. So there is a rule of an inverse square distance in play by gravitational force.

2 Newton's Universal Law of Gravitation

Every particle attracts any other particle with a gravitational force. This force has (1) a magnitude that is directly proportional to the product of the masses of the two particles and inversely proportional to the square of the distance between them; and (2) a direction that points along a line connecting the centers of the interacting particles.

$$F(r) = G\frac{M_A M_B}{r^2}, \tag{1}$$

where G is known as the gravitational constant,

$$G = 6.67 \times 10^{-11}\mathrm{N{\cdot}m^2/kg^2}. \tag{2}$$

Now, we can perfectly explain moon's orbital motion around the earth:

$$F_{\text{centripetal}} = G\frac{M_{\text{moon}} M_{\text{earth}}}{r^2} = M_{\text{moon}}\frac{v^2}{r}. \tag{3}$$

We can also show that at the surface of earth:

$$m \cdot (G\frac{M}{R^2}) = mg, \tag{4}$$

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3 Kepler's Laws of Planetary Motion

Newton's Law of Gravitation can provide a complete explanation of Kepler's laws.

Law 1: All planets move in elliptical orbits with the Sun at one focus.

Law 2: A line joining any planet to the Sun sweeps out equal areas in equal time intervals.

Law 3: The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit:

$$T^2 = CR^3, \tag{5}$$

where $C = 4\pi^2/GM$.

4 Gravitational Potential and Energy Conservation

We learned that we can set potential energy to zero at any altitude near the surface of earth because g is constant. In physics, when two objects interact with each other with a force field, it is common to set potential energy to zero as they are infinitely far apart. The gravitational potential energy is:

$$U(r) = -G\frac{M_A M_B}{r}. \tag{6}$$

At the surface of Earth, if we raise a object of mass m from r_1 to r_2, where $r_2 - r_1 = h$, the expression mgh for the potential energy of the object on Earth can be derived:

$$\Delta U = -G\frac{mM_e}{r_2} - (-G\frac{mM_e}{r_1}) = G\frac{mM_e(r_2 - r_1)}{r_1 r_2} = m \cdot G\frac{M_e}{R^2} \cdot h = mgh. \tag{7}$$

If there is no external force, energy is conserved:

$$E_{\text{total}} = K + U. \tag{8}$$

When $E_{\text{total}} < 0$, the orbit is bound. When $E_{\text{total}} > 0$, the orbit is unbound, and they can escape from each other. For a satellite to orbit the Earth, we have:

$$G\frac{m_{\text{s}} M_{\text{e}}}{r^2} = m_{\text{s}}\frac{v^2}{r}, \tag{9}$$

$$E_{\text{total}} = \frac{1}{2}m_{\text{s}}v^2 - G\frac{m_s M_e}{r} = G\frac{m_s M_e}{2r} - G\frac{m_s M_e}{r} = -G\frac{m_s M_e}{2r}. \tag{10}$$

The satellite is bound to Earth. To launch a spaceship to escape Earth's gravity, we have

$$E_{\text{total}} = \frac{1}{2}m_{\text{s}}v^2 - G\frac{m_s M_e}{r} = 0. \tag{11}$$

$$v_{\text{escape}} = \sqrt{\frac{2GM_e}{R}}. \tag{12}$$

5 Problems

1. A planet with twice the mass of Earth and a radius three times that of Earth would have a surface gravity of
(A) g (B) $\frac{2}{9}g$ (C) $\frac{2}{3}g$ (D) $\frac{3}{2}g$ (E) $\frac{9}{2}g$

2. A satellite orbits Earth at height with a period of T. If the satellite were to increase its orbital height to $2h$, what would be its resulting period?
(A) $\frac{1}{2}T$ (B) $\sqrt{2}T$ (C) $2T$ (D) $\sqrt{8}T$ (E) $8T$

3. A satellite of mass m orbits Earth at a height of h and a speed of v. If the satellite were to drop down to an orbit of $\frac{1}{2}h$, what would be its resulting speed?
(A) $\frac{1}{2}v$ (B) $\frac{1}{\sqrt{2}}v$ (C) $\sqrt{2}v$ (D) $2v$ (E) $4v$

4. The gravitational force of attraction between two masses is 16 Newtons. If the distance between the masses is quadrupled, what is the resulting force of attraction?
(A) 1N (B) 2N (C) 4N (D) 8N (E) 16N

5. Which of the planets below would have the same free-fall acceleration on their surfaces?

	Mass	Radius
I	M	$2R$
II	$8M$	$2R$
III	$2M$	R
IV	M	R

(A) I and II (B) II and III (C) III and IV (D) II and IV (E) I and III