Circuits & Related Topics

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1 How do Capacitors work?

1.1 What is Capacitance?

Capacitance measures the ability of a body to store electrical charge. It is defined as:

$$C = \frac{Q}{V}$$

This equation, with Q being the magnitude of the charge stored on each plate. The SI unit of capacitance is the Farad (F). A 1 Farad Capacitor would, when charged with 1 Coulomb, generate a potential difference of 1 Volt.

1.2 Parallel Plate Capacitors

The most common type of capacitor consists of two parallel plates. In fact, the cylindrical capacitors most commonly used in industrial circuits are composed of parallel plates wound into a cylinder.

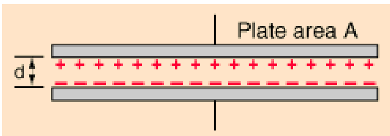

Figure 1: Taken from *hyperphysics.phy-astr.gsu.edu.*

A simple parallel plate capacitor is shown in the diagram above, with a charge of $+Q$ on the upper plate and $-Q$ on the lower plate. To find the capacitance of this capacitor, we must find the voltage difference between the plates, using d as the seperation between the plates and A as the area of one of the identical plates. First electric field caused by each plate is found by using Gauss's Law:

$$\oiint_{S(V)} \mathbf{E} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \frac{Q}{\epsilon_0}$$

Defining the surface to be a rectangluar prism with the dimensions of the plate (enclosing the plate), the field is zero for the faces of the prism perpendicular to to the plate because the dot product evaluates to zero. This leaves the surfaces parallel to the plate, which can be simplified as shown below:

$$\iint_{S(V)} \mathbf{E} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \frac{Q}{\epsilon_0}$$

$$\mathbf{E} \cdot \hat{\mathbf{n}} = |\mathbf{E}||\hat{\mathbf{n}}| \cos\theta = |\mathbf{E}|$$

Since the field is constant at a constant distance from the plate, the integral can be further simplified:

$$|\mathbf{E}| \iint_{S(V)} \mathrm{d}S = \frac{Q}{\epsilon_0}$$

$$|\mathbf{E}| \cdot (A_{top} + A_{bottom}) = \frac{Q}{\epsilon_0}$$

The areas of both the top and bottom are the same, and have the same areas as the plane itself:

$$|\mathbf{E}| = \frac{Q}{2A\epsilon_0}$$

Introducing $\sigma = \frac{Q}{A}$ as the surface charge density, which is constant for each plate:

$$|\mathbf{E}| = \frac{\sigma}{2\epsilon_0}$$

Performing this operation for the other plate would yield the same answer. Now direction of the electric field for each plate must be determined. Positive charge repels the positive plate and is attracted to the negative plate as shown:

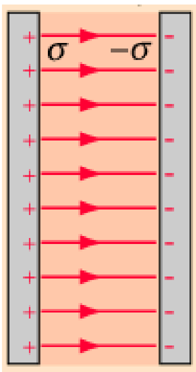

Figure 2: Taken from *hyperphysics.phy-astr.gsu.edu.*

The electric field calculated occurs in the same direction for each plate, and the net electric field can be obtained by adding the electric field of each plate. For the sake of maintaining a vector quantity, the direction of the field is chosen as the positive $\hat{\mathbf{i}}$ direction:

$$\mathbf{E_{net}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{i}} + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{i}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{i}}$$

Next, the potential difference across a parallel plate capacitor can be obtained by integration:

$$V = -\int_{\text{low potential}}^{\text{high potential}} \mathbf{E} \cdot \mathrm{d}l$$

The lowest potential in this system is the negative plate and our path will be from the negative plate to the positive one - a distance d. The field and our path l are in opposite directions, and hence their dot product generates a negative:

$$V = \int_0^d \frac{\sigma}{\epsilon_0}\,\mathrm{d}x = \frac{\sigma d}{\epsilon_0}$$

Now, voltage can be used to find capacitance via the equation in Section 1.1:

$$C = \frac{Q}{V} = \frac{Q}{\frac{\sigma d}{\epsilon_0}} = \frac{Q\epsilon_0}{\sigma d}$$

Plugging back in for sigma:

$$C = \frac{Q\epsilon_0}{\frac{Q}{A}d} = \frac{A\epsilon_0}{d}$$

1.3 Potential Energy of Parallel Plate Capacitors

Since voltage has units of energy per charge, the work needed to move a differential charge $\mathrm{d}q$ is voltage times charge $V\,\mathrm{d}q$. Using the equation from Section 1.1, potential energy can be evaluated as follows:

$$\mathrm{d}U = V\,\mathrm{d}q = \frac{q}{C}\,\mathrm{d}q$$

$$\int_0^{U*} \mathrm{d}U = \int_0^Q \frac{q}{C}\,\mathrm{d}q$$

$$U = \frac{1}{2}\frac{Q^2}{C}$$

This formula can be further variated by switching variables using the formula in Section 1.1:

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

This can again be simplified by plugging in the quantities from Section 1.2 to become a function of volume:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2 = \frac{1}{2}\epsilon_0 E^2 \cdot \text{Volume}$$

By dividing by volume, the volume-energy density η_E can be found:

$$\eta_E = \frac{1}{2}\epsilon_0 E^2$$

1.4 Dielectrics

The capacitance of a parallel plate capacitor can be increased by adding an insulator, called a dielectric. The dielectric is polarized by the electric field storing additional energy within its own polarization.

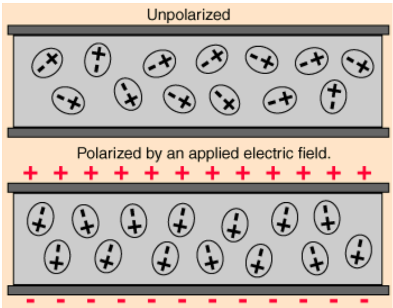

Figure 3: Taken from *hyperphysics.phy-astr.gsu.edu.*

A dielectric constant κ is used in calculations and represents how effective a dielectric is at storing energy. The value of the dielectric constant is reliant on the identity of the dielectric. Without a dielectric, in calculating Field, Voltage, and Capacitance values, ϵ_0 is used as the "permittivity of free space." When a dielectric is introduced, a relative permittivity of $\kappa\epsilon_0$ takes the place of the permittivity of free space in the calculations in Section 1.2. Listed below are the electric field, voltage, and capacitance of a parallel plate capacitor with a dielectric:

$$\mathbf{E} = \frac{\sigma}{\kappa\epsilon_0}\hat{\mathbf{i}} = \frac{\mathbf{E_0}}{\kappa}$$
$$V = \frac{\sigma d}{\kappa\epsilon_0} = \frac{V_0}{\kappa}$$
$$C = \frac{\kappa A\epsilon_0}{d} = \kappa C_0$$

A more complex understanding of the electric field within a capacitor with a dielectric can be found by labeling the charges as follows:

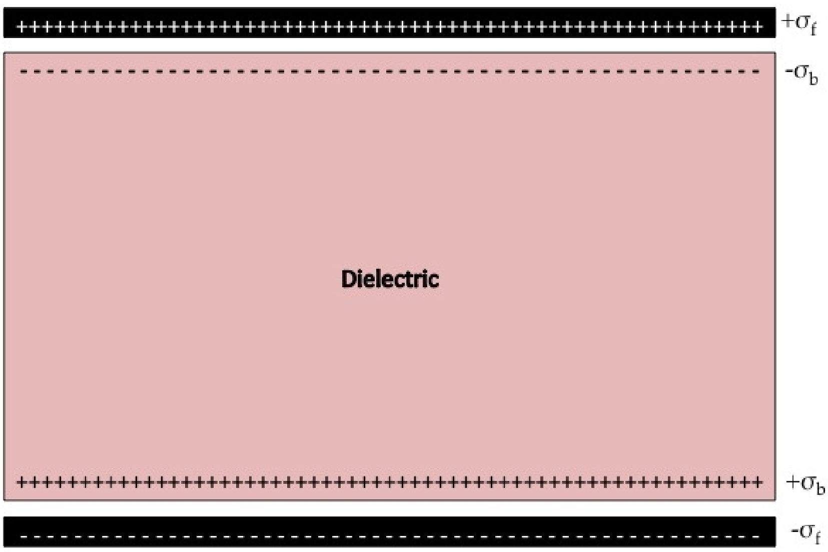

Figure 4: Produced by author.

From this, electric fields can be found for only the charge difference between the plates of the capacitor and only the dielectric.

$$|\mathbf{E_f}| = \frac{\sigma_f}{\epsilon_0} = |\mathbf{E_0}|$$

$$|\mathbf{E_b}| = \frac{\sigma_b}{\epsilon_0}$$

We know the net electric field for a capacitor with a dielectric from above, as $\frac{|\mathbf{E_0}|}{\kappa}$, and we can set the difference of the two fields (because they are in opposite directions) to equal the net electric field, solving for the field purely due to the dielectric and the surface charge density of the dielectric.

$$|\mathbf{E}| = |\mathbf{E_f}| - |\mathbf{E_b}|$$
$$\frac{|\mathbf{E_0}|}{\kappa} = |\mathbf{E_0}| - |\mathbf{E_b}|$$
$$|\mathbf{E_b}| = |\mathbf{E_0}| \left[1 - \frac{1}{\kappa}\right] = \frac{\sigma_f}{\epsilon_0}\left[1 - \frac{1}{\kappa}\right]$$
$$\sigma_b = \sigma_f \left[1 - \frac{1}{\kappa}\right]$$

1.5 Multiple Capacitors in Series

When multiple capacitors are joined in series, charge is conserved (it can go around the circuit but it has to end up on another capacitor eventually).

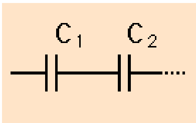

Figure 5: Taken from *hyperphysics.phy-astr.gsu.edu.*

Because the same current must pass through all the capacitors, if all capacitors are charged from zero Coulombs, they must end up with the same charge. Using this, the addition of capacitors in series can be derived:

$$V_1 = \frac{Q}{C_1}, V_2 = \frac{Q}{C_2}, V_3 = \frac{Q}{C_3}, \text{ etc.}$$
$$V_{\text{total}} = V_1 + V_2 + V_3 + ...$$
$$\frac{Q}{C_{\text{eq}}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + ...$$
$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ...$$

1.6 Multiple Capacitors in Parallel

When multiple capacitors are joined in parallel, the voltage drop across each capacitor is the same.

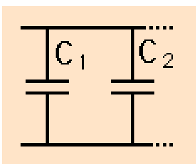

Figure 6: Taken from *hyperphysics.phy-astr.gsu.edu.*

Using this constant voltage, the addition of capacitors in parallel can be derived as follows:

$$Q_1 = C_1 V, Q_2 = C_2 V, Q_3 = C_3 V, \text{ etc.}$$

$$Q_{\text{total}} = Q_1 + Q_2 + Q_3 + ...$$
$$C_{\text{eq}} V = C_1 V + C_2 V + C_3 V + ...$$
$$C_{\text{eq}} = C_1 + C_2 + C_3 + ...$$

2 Multi-path Circuits

2.1 What is Current?

Electric Current is the rate of flow of electric charge through a cross-sectional area.It is defined as:

$$I = \frac{\Delta Q}{\Delta t}$$

The SI unit of current is known as an Ampere (A).

By convention, the direction of current is set as the direction of the flow of positive charge, but in reality it is the electrons that are flowing through a wire.Thus, the direction of current is really the opposite direction of electron flow.

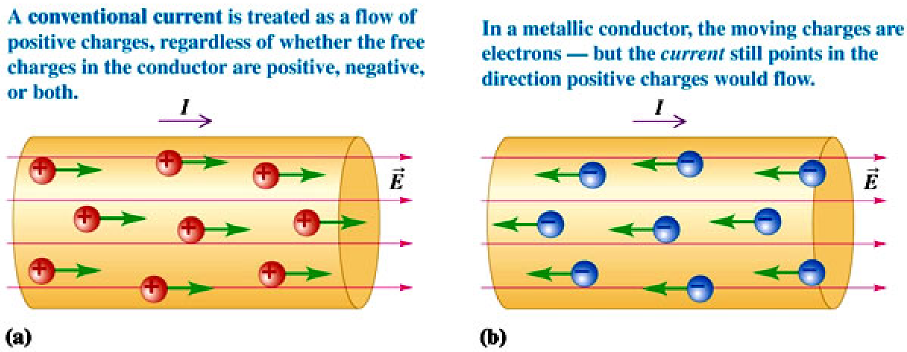

Figure 7: Taken from *sdsu-physics.org*

2.2 Motion of Charges

A common misconception is that electrons zip down electric circuits at high speeds. However, this is not the case as electrons move with a small average known as the drift velocity due to their frequent collisions with latice ions in the wire. Current can then be also found using the following equation

$$\Delta Q = qnAv_d\Delta t$$

Where v_d is the drift velocity, A is the cross area of the wire, and n is the number density of charges. Thus:

$$I = \frac{\Delta Q}{\Delta t} = qnAv_d$$

2.3 Resistance and Ohm's Law

Take a wire segment starting at point a and ending at b, which has length ΔL and cross-sectional area A carryinga current I. The current is made possible by an electric field in the conductor which exerts a force qE on the free charges. Because electric field points in the direction of decreasing potential, the potential at point a is higher than at point b. If we assume the electric field is constant across the points, we can say that the potential difference across the segment is:

$$V = V_a - V_b = E\Delta L$$

The ratio of the potential drop is called the resistance of the segment or:

$$R = \frac{V}{I}$$

From this equation we get Ohm's Law which is simply:

$$V = \frac{I}{R}$$

The SI unit of resistance is called an Ohm (Ω). The resistance of a wire can be found to be:

$$R = \rho\frac{L}{A}$$

where the proportionality constant ρ is called the resistivity of the conducting material.
Resistivity is often used to control current and voltage in a circuit using resistors which have a specified resistance.

2.4 Applications to Circuits

Circuits can consist of a variety of components from transistors to decoders, but the most basic components are resistors, capacitors, and batteries. Circuits are often drawn out using specific representations of wires and components as shown by the figure below:

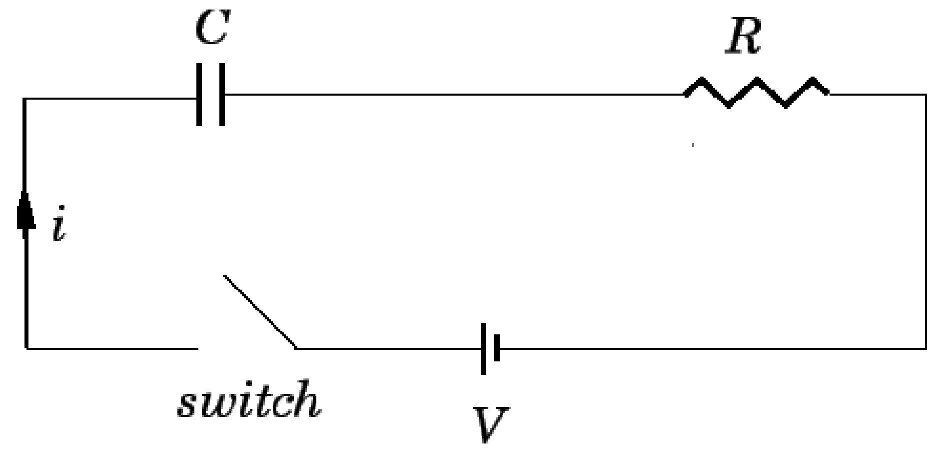

Figure 8: Taken from *farside.ph.utexas.edu*

Often, the analysis of a circuit can be simplified by replacing two or more resistors with a single equivalent resistor that carries the same current with the same potential drop as the original resistors as a group.

Series Resistors When two or more resistors are connected so that they carry the same current, the resistors are said to be in series such as the figure below.

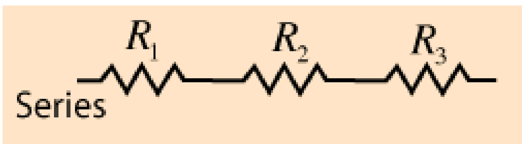

Figure 9: Taken from *hyperphysics.phy-astr.gsu.edu*

The potential drop across R_1 is IR_1 and so on for R_2 and R_3. The total potential drop across all three resistors is:

$$V = IR_1 + IR_2 + IR_3 = I(R_1 + R_2 + R_3)$$

The single equivalent resistance R_{eq} is thus:

$$R_{eq} = R_1 + R_2 + R_3$$

Parallel Resistors Two or more resitors that are connected so they have the same potential difference across them are in parallel such as in the figure below:

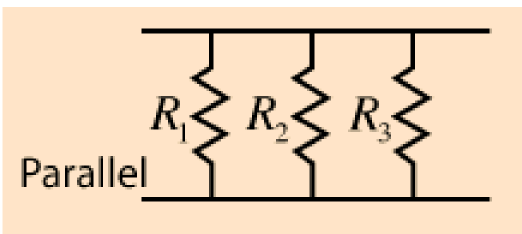

Figure 10: Taken from *hyperphysics.phy-astr.gsu.edu*

The current splits off into three different currents and so the potential difference across any of the three resistors is:

$$V = I_1R_1 = I_2R_2 = I_3R_3$$

The equivalent resistance is:

$$R_{eq} = \frac{V}{I}$$

Thus, solving for I we get:

$$I = \frac{V}{R_{eq}} = I_1 + I_2 + I_3 = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$

The equivalent resistance is then:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

2.5 Kirchhoff's Rules

Some circuits can't be solved using the above steps and thus we need to bring in another set of rules known as Kirchhoff's Rules. These state:

1. When any closed-circuit loop is traversed, the algebraic sum of the changes must equal zero. (loop rule)

2. At any junction point in a circuit where the current caqn divide the sum of the currents into the junction must equal the sum of the currents out of the junction. (junction rule)

2.6 RC Circuits

An RC circuit is simply a circuit containing a resistor and capacitor. The current varies as a function of time because of the capacitor. Consider a series RC circuit with a battery, resistor, and capacitor in series. The capacitor is initially uncharged, but starts to charge when the switch is closed. Initially the potential difference across the resistor is the battery emf, but that steadily drops along with the current as the potential difference across the capacitor increases.