Central Forces Solution Set

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1 Solutions

1. We begin with the polar differential equation for the path of the particle given at the end of section 4.1.

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2m(E-V(r))}{L^2} - \frac{1}{r^2} \tag{1}$$

Now, we can make the very helpful substitution $u = 1/r$, so that $\frac{du}{d\theta} = -u'(\theta)/u^2$.

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2mE}{L^2} + \frac{2m\alpha}{L^2}u - u^2 \tag{2}$$

Now, perhaps contrary to intuition, we can simplify the differential equation by differentiating further and thereupon canceling $\frac{du}{d\theta}$:

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} = \frac{2m\alpha}{L^2}\frac{du}{d\theta} - 2u\frac{du}{d\theta} \tag{3}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{m\alpha}{L^2} \tag{4}$$

This is a non-homogenous second order differential equation. To solve such an equation, we first need a specific solution. In this case, we see that $u = \frac{m\alpha}{L^2}$ is a trivial solution. Then, to generalize, we add to this specific solution all the solutions of the corresponding homogenous equation:

$$\frac{d^2u}{d\theta^2} + u = 0 \tag{5}$$

This gives us, in total,

$$u(\theta) = A\sin\theta + B\cos\theta + \frac{m\alpha}{L^2} \tag{6}$$

This is the general solution for u; we can prescribe some initial conditions to give meaning to the parameters A and B. Consider a particle of mass m at distance a and at $\theta = 0$ launched at speed $v = L/ma$ perpendicular to the radius vector (i.e., upwards). In this case, $u'(0) = 0$, so that A must vanish. Additionally,

$$u(0) = \frac{1}{a} = B + \frac{m\alpha}{L^2}$$

$$B = \frac{1}{a} - \frac{m\alpha}{L^2}$$

So, given these initial conditions, we arrive at

$$u(\theta) = \left(\frac{1}{a} - \frac{m}{\alpha}L^2\right)\cos\theta + \frac{m\alpha}{L^2} \tag{7}$$

$$= \frac{m\alpha}{L^2}\left(\left(\frac{L^2}{m\alpha a} - 1\right)\cos\theta + 1\right) \tag{8}$$

Taking the reciprocal, we have the final result:

$$r(\theta) = \frac{L^2}{m\alpha}\frac{1}{1 + e\cos\theta} \tag{9}$$

where we have defined the eccentricity $e = \frac{L^2}{m\alpha a} - 1$.

2. From the lecture, we made the realization that, for two orbiting bodies,

$$E = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r) \tag{10}$$

taking a time derivative yields

$$\frac{dE}{dt} = 0 = \mu\dot{\mathbf{r}}\ddot{\mathbf{r}} + \dot{\mathbf{r}}\left(\nabla V(r)\right) \tag{11}$$

notice that the first term vanished because we know that the speed of the center of mass of the system remains constant. This gives us

$$\mu\ddot{\mathbf{r}} = -\nabla V(r) = F(r) = \frac{Gm_1m_2}{d^2} \tag{12}$$

where d is the distance between the objects. We notice that the acceleration of the object is centripetal in nature, so

$$\mu\omega^2 d = \frac{\mu 4\pi^2 d}{T^2} = \frac{Gm_1m_2}{d^2} \tag{13}$$

using the fact that $\mu = \frac{m_1m_2}{m_1+m_2}$, we can solve for T

$$T = 2\pi\sqrt{\frac{d^3}{G(m_1+m_2)}} \tag{14}$$

notice how this problem was made easy by the fact that we converted a two-body system into a one body system.

3. Without loss of generality, assume the particle moves in the xy-plane. Then we have

$$U = \beta r^2 = \beta\left(x^2 + y^2\right) \tag{15}$$

This is very easily separable into potentials for each axis, and the problem reduces to simple harmonic motion along each axis:

$$ma_x = -2\beta x \to x = A\cos\left(\sqrt{\frac{2\beta}{m}}t + \delta\right) \tag{16}$$

$$ma_y = -2\beta y \to y = B\cos\left(\sqrt{\frac{2\beta}{m}}t + \phi\right) \tag{17}$$

You can convince yourself that the path of the particle is an ellipse with semimajor axis a, semiminor axis b, and angle (relative to the x axis) of φ, with these parameters given by

$$a = \sqrt{A^2\cos^2\delta + B^2\cos^2\phi} \tag{18}$$

$$b = \sqrt{A^2\sin^2\delta + B^2\sin^2\phi} \tag{19}$$

$$\tan\varphi = \frac{B\cos\phi}{A\cos\delta} \tag{20}$$

4. a). The two masses will orbit around each other (in any fashion), as long as a tangential velocity is supplied to allow centrifugal effects to occur. Thus, without an initial velocity, the object will be linearly forced and will collide with each other. So, they will collide when $v_0 = 0$.

   A more interesting question would be to ask the time it takes for the two masses to collide. This can be found by energy conservation methods

$$-\frac{Gm^2}{\ell} = -\frac{Gm^2}{\ell'} + 2\left(\frac{1}{2}mv^2\right) \tag{21}$$

$$Gm\left(\frac{1}{\ell'} - \frac{1}{\ell}\right) = v^2 \tag{22}$$

   By symmetry, we can conclude that the two velocities of the masses are equal at every instant (thus why we combined the kinetic energy terms). The rate at which the distance between the objects decreases is $2v$, because the motion of both objects captures twice the distance than just one object. Thus, $v = \frac{1}{2}\frac{d\ell'}{dt}$. This gives us the following differential equation

$$2\sqrt{Gm\left(\frac{1}{\ell'} - \frac{1}{\ell}\right)} = \frac{d\ell'}{dt} \tag{23}$$

$$t = \int_{\ell}^{0} \frac{d\ell'}{2\sqrt{Gm\left(\frac{1}{\ell'} - \frac{1}{\ell}\right)}} \tag{24}$$

$$t = \frac{\pi}{4}\frac{\ell^{3/2}}{\sqrt{Gm}} \tag{25}$$

   b). The condition for the circular orbit is straight from Newton's laws

$$\frac{Gm^2}{\ell^2} = \frac{mv_0^2}{\ell/2} \tag{26}$$

   Thus

$$\frac{Gm}{2\ell} = v_0^2 \tag{27}$$

c). If the objects were to escape to infinity, then they would have enough energy to achieve a potential of 0 (this logic mirrors the derivation for an object's escape velocity). Thus, in order for the object to be trapped in a closed orbit, it's total energy cannot exceed zero; that is, $E_{tot} < 0$. The total energy of the system is $E = mv_0^2 - \frac{Gm^2}{\ell}$, so

$$mv_0^2 - \frac{Gm^2}{\ell} < 0 \tag{28}$$

$$v_0^2 < \frac{Gm}{\ell} \tag{29}$$

Another way of seeing this is by noticing that the eccentricity of the objects orbit must be less than 1 (otherwise it will be parabolic/hyperbolic in nature). Thus, since $e = \sqrt{1 + \frac{2EL^2}{\mu\alpha^2}}$, we see immediately that $E < 0$ for the eccentricity to be less than 1.

d). We can solve this question by using conservation of energy and angular momentum (about the center of mass of the system).

$$mv_0^2 - \frac{Gm^2}{\ell} = mv_{min}^2 - \frac{Gm^2}{\ell_{min}} \tag{30}$$

$$mv_0\ell = mv_{min}\ell_{min} \tag{31}$$

Eliminating v_{min} gives us

$$mv_0^2 - \frac{Gm^2}{\ell} = mv_0^2\left(\frac{\ell}{\ell_{min}}\right)^2 - \frac{Gm^2}{\ell_{min}} \tag{32}$$

$$\ell_{min}^2\left(v_0^2 - \frac{Gm}{\ell}\right) + \ell_{min}\left(Gm\right) - v_0^2\ell^2 = 0 \tag{33}$$

Solving the quadratic equation gives us

$$\ell_{min} = \ell, \frac{\ell}{\frac{Gm}{v_0^2\ell} - 1} \tag{34}$$

The first root is smaller than the second root for $\frac{Gm}{v_0^2\ell} < 2$ and the second is smaller than the first for $\frac{Gm}{v_0^2\ell} > 2$.

5. First, we need an expression for v_0, which we can find by equating the centripetal and gravitational forces for the first satellite.

$$\frac{v_0^2}{R} = \frac{GM}{R^2} \tag{35}$$

$$v_0 = \sqrt{\frac{GM}{R}} \tag{36}$$

We can then write the conservations of energy and angular momentum for the second satellite, assuming that the angular momentum is simply mvr (which is true for the points of closest and furthest approach).

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{8}m\frac{GM}{R} - \frac{GMm}{R} = -\frac{7}{8}\frac{GMm}{R} \tag{37}$$

$$mvr = \frac{1}{2}m\sqrt{\frac{GM}{R}}R = \frac{1}{2}m\sqrt{GMR} \tag{38}$$

From the second equation, we have $v = \sqrt{GMR}/2r$. Substituting back into the first equation,

$$\frac{1}{8}m\frac{GMR}{r^2} - \frac{GMm}{r} = -\frac{7}{8}\frac{GMm}{R} \tag{39}$$

$$\frac{7}{R}r^2 - 8r + R = 0 \tag{40}$$

The two solutions to this equation are $r = R$ and $r = R/7$. The second solution is clearly the answer.

6. The angle of deflection is given by

$$\phi = \pi - 2\arctan\left(\frac{b}{a}\right) \tag{41}$$

where b and a are the length of the semi minor and semi major axes of the hyperbola (we are essentially determining the slope of the tangent line, or the asymptotes, to the hyperbola). Additionally, b is what we define as the "impact parameter". Now, we know that, for a parabola

$$\frac{b}{a} = \sqrt{e^2 - 1} \tag{42}$$

Where e is the eccentricity of the hyperbola. Since $e^2 = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}$, and we know that $E = \frac{mv_0^2}{2}$ and $L = mv_0 b$, we get that $\frac{b}{a} = \frac{v_0^2 b}{GM}$. Thus

$$\phi = \pi - 2\arctan\left(\frac{v_0^2 b}{GM}\right) \tag{43}$$