AP Physics - Vector Algrebra Tutorial

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AP Physics Team

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Contents

1 Scalars and Vectors

In our study of Physics this year it will be important to always keep track of the types of quantities we are discussing. Aside from units or intrinsic error, understanding if a quantity is a scalar or a vector fundamental. Both scalars and vectors can be constants, variable, or even functions, but the key difference is that vectors also denote a direction. The table below shows examples of both vector and scalar quantities.

	Data Types	**Examples**	**Physical Quantities**
Scalar	constants variables functions	$\sqrt{2}, 3, \pi$ x, y, t, T e^{-x^2}	time, temperature
Vector	constants variables functions	$\sqrt{2}\hat{i} + 3\hat{j}$ $x\hat{i} + y\hat{j}$ $\cos(\omega t)\hat{i} + \sin(\omega t)\hat{j}$	velocity, force

2 Rectangular and Polar Form

A two dimensional vector can be represented by either a rectangular form (x, y), or a polar form (r, θ).

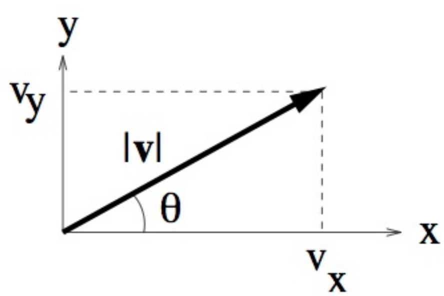


Figure 1: A 2D vector can be represent in rectangular or polar form

In general, a 2D vector can be represented by a linear combination of two unit vectors $\hat{i}$ and $\hat{j}$ that are orthogonal to each other, have length of one, and point in the x and y directions respectively.

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}^1 \tag{1}$$

It is very important to note that in AP Physics C, and generally in all Physics courses, vectors are always written as the linear sum of unit vectors $\hat{i}, \hat{j}$ rather than

[1] Bold face letters are vectors and normal typeface are scalars

as an ordered pair (*n-tuple*) as in (2) below which is taught in some mathematics courses at TJ.

$$\mathbf{v} =< v_x, v_y > \tag{2}$$

Writing vectors using the ordered pair (or triplet) notation will not be considered correct and will result in a loss of points on that question in this course.

We can use figure 1 and basic right triangle trigonometry to form relationships between the magnitude of a vector and the angle the vector makes to the x-axis to the horizontal and vertical components of the vector.

$$v_x = |\mathbf{v}| \cos\theta \tag{3}$$

$$v_y = |\mathbf{v}| \sin(\theta) \tag{4}$$

Equations (3) and (4) now provide us with the polar to rectangular conversions. Conversely, this equation can be inverted to obtain the corresponding rectangular to polar conversation.

$$\theta = \arctan(\frac{v_y}{v_x}) \tag{5}$$

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} \tag{6}$$

We can then generalize this concept for vectors in 3D.

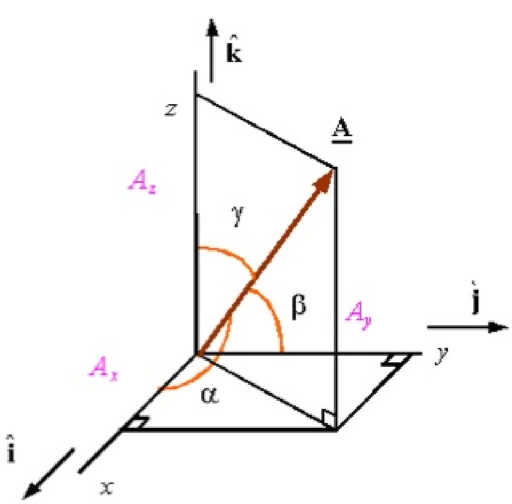


Figure 2: A vector in 3D

From figure 2 is can be seen that α is the angle form the x axis to the vector, β is the angle from the y axis to the vector, and γ is the angle from the z axis to the vector. Consequently we can decompose the vector **A** into:

$$\mathbf{A} = |\mathbf{A}|[\cos(\alpha)\hat{\mathbf{i}} + \cos(\beta)\hat{\mathbf{j}} + \cos(\gamma)\hat{\mathbf{k}}] \tag{7}$$

It is also worth noting that the Pythagorean relationship that defines equation (6) also holds true in higher dimensions. Therefore we can also write down the magnitude (or length) of any dimension vector as:

$$|\mathbf{A}| = \sqrt{\sum_i^n A_i^2} \tag{8}$$

Where n is the dimension of the vector and A_i are the lengths of the projections of the vector **A** onto the basis (or unit) vectors.

3 Vector Operations

In general, we should consider 3D vectors, but it is important to recognize that all but one of these definitions apply to 2D vectors as well.

To begin, let us define two generalized, three dimensional vectors that we will use for all of the following definitions.

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{9}$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} \tag{10}$$

3.1 Vector Addition

When adding vectors we must take care to only add components of vectors that are in the same(or exactly opposite) direction. Therefore we must define the addition of two vectors as:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}} \tag{11}$$

If we have the subtraction of two vectors we can apply equation (11) as long as we recognize that subtraction is simply the addition of a negative.

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = (A_x - B_x)\hat{\mathbf{i}} + (A_y - B_y)\hat{\mathbf{j}} + (A_z - B_z)\hat{\mathbf{k}} \tag{12}$$

To help you visualize the process of vector addition or subtraction we suggest you play around with the vector addition applet:

`http://phet.colorado.edu/sims/vector-addition/vector-addition_en.html`

It is best viewed with "style 3" selected.

3.2 Multiplication of a Scalar and a Vector

The product of a vector and a scalar maintains the direction of the vector, but the has the magnitude scaled(hence the name scalar) by the scalar value. This provides us with the definition:

$$s\mathbf{A} = (sA_x)\hat{\mathbf{i}} + (sA_y)\hat{\mathbf{j}} + (sA_z)\hat{\mathbf{k}} \tag{13}$$

where s can be any real number.

3.3 Multiplication of two Vectors

Any vector operation must account for both the magnitude and the direction of the vectors being operated on. This is no where more apparent than with the multiplication of two vectors with each other. We must define two different ways of multiplying vectors in order to fully define both the scalar (i.e. length) and the directional nature of vector products.

Following from basic algebra, the multiplication of two trinomials by each other is given by:

$$\begin{aligned}(\mathbf{A})(\mathbf{B}) &= (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}})(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) \\ &= A_xB_x\hat{\mathbf{i}}(\hat{\mathbf{i}}) + A_xB_y\hat{\mathbf{i}}(\hat{\mathbf{j}}) + A_xB_z\hat{\mathbf{i}}(\hat{\mathbf{k}}) \\ &\quad + A_yB_x\hat{\mathbf{j}}(\hat{\mathbf{i}}) + A_yB_y\hat{\mathbf{j}}(\hat{\mathbf{j}}) + A_yB_z\hat{\mathbf{j}}(\hat{\mathbf{k}}) \\ &\quad + A_zB_x\hat{\mathbf{k}}(\hat{\mathbf{i}}) + A_zB_y\hat{\mathbf{k}}(\hat{\mathbf{j}}) + A_zB_z\hat{\mathbf{k}}(\hat{\mathbf{k}})\end{aligned} \tag{14}$$

Equation (14) provides us with a structure in which to explore vector multiplication, however we need to further define the product of unit vectors with other unit vectors. There are two ways to interpret this multiplication - one that results in a scalar, the other a vector.

3.3.1 Dot Product

The Dot Product ($\mathbf{A} \cdot \mathbf{B}$) of two vectors results in a scalar, and thus is called the "Scalar Product". This type of vector multiplication is governed by the Kronecker delta function given by:

$$\hat{\mathbf{u}}_{\mathbf{a}} \cdot \hat{\mathbf{u}}_{\mathbf{b}} = \delta_{ab} \begin{cases} 1 & : a = b \\ 0 & : a \neq b \end{cases}$$

Where $\hat{\mathbf{u}}_{\mathbf{a}}$ and $\hat{\mathbf{u}}_{\mathbf{b}}$ are any unit vectors. If we apply the Kronecker delta function to equation (14) we can define the dot product.

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{15}$$

What is the physical interpretation of the dot product? If we consider two nonzero, non-parallel vectors as in the figure below,

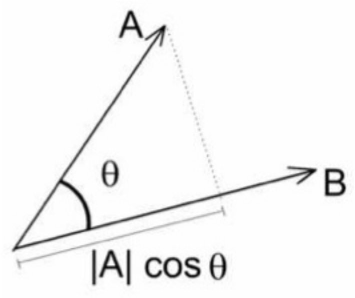


Figure 3:

the magnitude of the dot product can be interpreted by noting that the projection of **A** onto **B** is

$$|\mathbf{A}|\cos(\theta_{AB}) = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|} \tag{16}$$

Which provides us with a second definition for the dot product.

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos(\theta_{AB}) \tag{17}$$

To help you visualize the dot product and the projection of one vector onto another, we suggest you explore this process further using the following website:

`http://www.falstad.com/dotproduct/`

3.3.2 Cross Product

The cross product of two vectors results in a vector, and thus is called the "Vector Product". The rule that governs the vector product is the right hand rule. Looking at figure 4 below, and starting with your right hand with your palm flat, point your fingers towards the x-axis. The curl your fingers (except your thumb) around toward the y-axis making a closed fist. Your thumb should be pointing in the z-direction.

This operation can be expressed as:

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}} \tag{18}$$

Using this procedure, you should also be able to confirm (<u>THIS MEANS DO THIS NOW!</u>):

$$\begin{aligned}
\hat{\mathbf{i}}\times\hat{\mathbf{k}} &= -\hat{\mathbf{j}} & (19)\\
\hat{\mathbf{j}}\times\hat{\mathbf{i}} &= -\hat{\mathbf{k}} & (20)\\
\hat{\mathbf{j}}\times\hat{\mathbf{k}} &= \hat{\mathbf{i}} & (21)\\
\hat{\mathbf{k}}\times\hat{\mathbf{i}} &= \hat{\mathbf{j}} & (22)\\
\hat{\mathbf{k}}\times\hat{\mathbf{j}} &= -\hat{\mathbf{i}} & (23)
\end{aligned}$$

It is also conventional to define

$$\hat{\mathbf{u}}_{\mathbf{a}}\times\hat{\mathbf{u}}_{\mathbf{a}} = 0 \tag{24}$$

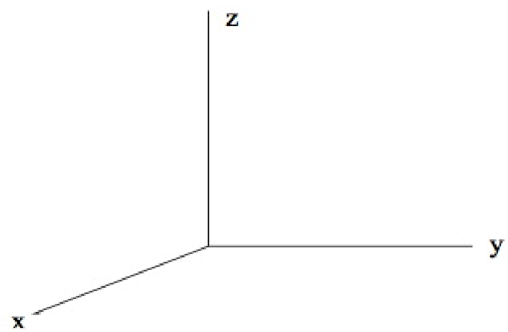


Figure 4: A right handed coordinate system

for any unit vector $\hat{\mathbf{u}}_{\mathbf{a}}$ which then directly leads to

$$\mathbf{A} \times \mathbf{A} = 0 \tag{25}$$

for any vector $\mathbf{A}$.

If we apply the cross product rules to equation (14) we come upon the definition of the cross product.

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\hat{\mathbf{i}} - (A_x B_z - A_z B_x)\hat{\mathbf{j}} + (A_x B_y - A_y B_z)\hat{\mathbf{i}} \tag{26}$$

One way to remember this definition is to consider it to be the determinate of the matrix shown below. While this is a useful and easy to remember memory device, it has no physical meaning or interpretation.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Besides the right hand rule definition, and the determinate based memory aid, there is another memory device for the cross product. Looking at figure 5 below, if you start at $\mathbf{u_x}$ and move around the circle in a clockwise direction (+) towards $\mathbf{u_y}$ you next come to $\mathbf{u_z}$. This represents equation . If you start at $\mathbf{u_y}$ and move clockwise through $\mathbf{u_z}$ you get $\mathbf{u_x}$. This represents equation (21). Reversing your direction on the circle by moving counter-clockwise inserts a negative into your answer.

Like the Dot Product, we should explore the physical interpretation of the cross product. The magnitude of the cross product $|\mathbf{A} \times \mathbf{B}|$ can be shown (i.e. you should do this now) to be equal to the area of the parallelogram formed by the two vectors as in figure 6 below.

This provides us with a final definition of the cross product.

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin(\theta_{AB})\hat{\mathbf{u}}_{\mathbf{AB}} \tag{27}$$

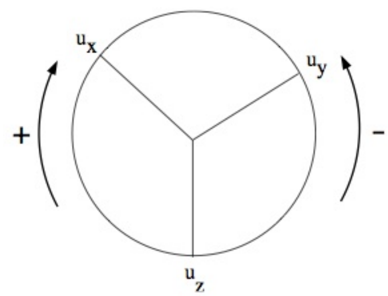


Figure 5: Go around the circle to find the cross product

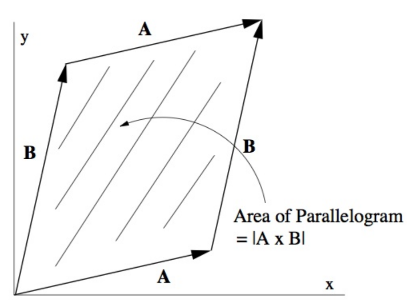


Figure 6:

Where θ_{AB} is the angle between the two vectors and $\hat{\mathbf{u}}_{\mathbf{AB}}$ is a unit vector that is perpendicular to both original vectors in accordance with the right hand rule. If we are only concerned with the magnitude of the cross product, then we can drop the unit vector, resulting in:

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin(\theta_{AB}) \tag{28}$$

To further explore the nature of the cross product we suggest that you investigate the website:

`http://www.surendranath.org/Applets/Math/VectorProduct/VP.html`

4 Problems and Exercises

Using the definitions of vector operations in three dimensions, *PROVE* the following vector identities.

1. Scalar multiplication factors over vector addition
   $s(\mathbf{A} + \mathbf{B}) = (s\mathbf{A}) + (s\mathbf{B})$

2. Association of scalar multiplication in the dot product
   $s(\mathbf{A} \cdot \mathbf{B}) = (s\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (s\mathbf{B})$

3. Associtivity of scalar multiplication in the cross product
   $s(\mathbf{A} \times \mathbf{B}) = (s\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (s\mathbf{B})$

4. Dot product distributes over vector addition
   $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$

5. Community of vector addition
   $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

6. Community of the dot product
   $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

7. Anti-commutivity of cross product
   $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

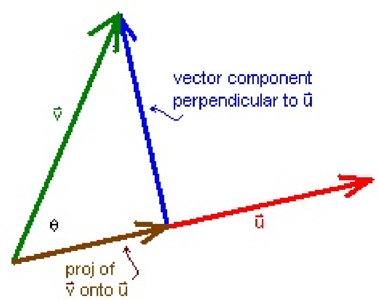
8. Perpendicular nature of cross product
   $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{B}) = 0$

9. Cyclic Identity
   $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$

10. "BAC - CAB" Identity
    $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

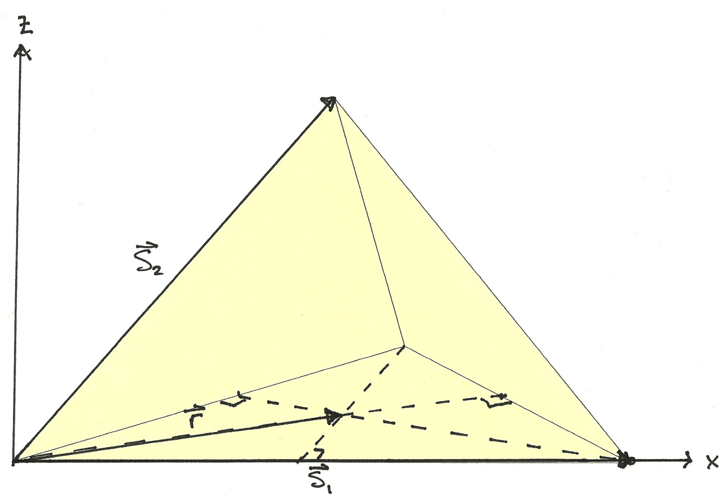
11. Magnitude of a vector in terms of dot product
    $|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$

12. Orthogonal vectors using dot product
    Given two non-zero vectors **u** and **v** which are not proportional by some scalar (i.e. don't point in the same or opposite direction)



    (a) Find an expression for the projection of **v** onto **u** in terms of ONLY the vector dot product and a multiplication of a vector by a scalar.
    (b) Find an expression for the component of **v** that is perpendicular to **u** in terms of ONLY the vector dot product, multiplication of a vector by a scalar, and vector addition.
    (c) Find an expression for the component of **v** that is perpendicular to **u** using only the vector cross product, multiplication by a scalar, and vector dot products

13. Tetrahedron Vectors



A tetrahedron is constructed from 4 equilateral triangles arranged in a pyramid. One vertex of the tetrahedron is located at the origin of a right-handed coordinate system.

- $\vec{r}$ is defined from the vertex A to the intersection of the three medians of the base triangle and is the horizontal projection of $\vec{S_2}$ onto the x-y plane.
- $\vec{S_1}$ points along the x-axis and has magnitude $\left\|\vec{S_1}\right\| = a$
- $\left\|\vec{S_1}\right\| = \left\|\vec{S_2}\right\|$

(a) Write a vector expression for $\vec{r}$ using unit vectors.

(b) Write a vector expression for $\vec{S_2}$ using unit vectors.

(c) Prove that the angle between $\vec{S_1}$ and $\vec{S_2}$ is 60 degrees.

(d) Determine the angle between $\vec{r}$ and $\vec{S_2}$.