Thermodynamics Problem Set Answers

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March 2015

These are just answers. If you want to see a solution to a specific question, feel free to email either of us.

1 Thermal Properties of Matter

1. $\Delta T_1 - \Delta T_2 = \frac{2Rg\alpha\Delta Q}{m(c^2-g^2\alpha^2R^2)}$
2. $t = \frac{\rho L d^2}{2\kappa T}$

2 Statistical Mechanics

1. a. $\frac{dm}{dt} = -\rho A\sqrt{\frac{2}{m}\left(C - \frac{m}{\rho}P_0\right)}$,
   where C is a constant equal to $\frac{3}{2}n_0RT_0 + P_0V_0$.
   b. $t = \frac{\sqrt{2}}{3}\frac{M}{\rho A}\sqrt{\frac{\mathcal{M}}{RT_0}}$
   c. $v(t) = \frac{\rho A3RT_0M}{\mathcal{M}}\int_0^t \frac{dt'}{m(t')(m+m(t'))}$,
   where t is determined from part b, and $m(t)$ is the general expression that must be obtained to solve part b. The integrals in part a and c are convergent, in fact, and can be symbolically obtained. However, the answers are not as clean as that given in part b.
2. a. $P = \frac{nRT_0}{V}\left(\frac{V_0}{V}\right)^2$
   b. $\frac{T}{T_0} = \left(\frac{\ell}{\ell - Vt}\right)^2 = \left(\frac{\ell}{x}\right)^2$

3 First Law of Thermodynamics

1. $\rho = \frac{29}{7}\frac{P_0}{gL}$, which makes $\frac{y_2}{y_0} = \frac{7}{36}$
2. $T = \frac{2\pi}{P_0A}\sqrt{MnRT_0}$. There are many ways in which the final answer can be written.

4 Second Law of Thermodynamics

1. $P(t) = P_h\left(1 - \frac{T_0}{T_h}e^{\frac{P_h(t-T_0)}{mcT_h}}\right)$ where $t_0 = \frac{mlT_h}{P_hT_0}$
2. a. $W = c_p(T_1 + T_2 - 2T_f)$
   b. $T_f \geq \sqrt{T_1T_2}$ from considering that the total entropy change is at least 0.
   c. $W = c_p(\sqrt{T_1} - \sqrt{T_2})^2$

5 Miscellaneous

1. a. $\Delta T = 0$. Can you see why?

   b. 60 centimeters from the left

   c. $\Delta S = 3.24$ J/K

2. $\epsilon = 1 - \left(\frac{1}{\alpha}\right)^{\gamma-1}$

3. 2 atmospheres. Hint: there are two partial pressure gradients, but one is restricted due to the impermeability of the glass bulb.

4. Solution is due to Dr. Jonathan Osborne and is on the **next two pages**. Notice that the solution AAPT gives is incorrect. The actual solution is much more complicated (but surprisingly still neat).

On the 'Potato Gun'
Dr. Osborne

Consider the problem of a 'potato gun', a tube containing an ideal gas with ℓ degrees of freedom initially trapped at a closed end of the tube by a mass *m* with the same cross-sectional area *A* of the tube held in place by some mechanism. The other end of the tube is considered open to the atmosphere. The trapped gas is initially at the same temperature as the surrounding atmosphere, mass, and tube, occupies a length L_0 of the tube, and initially has pressure $a_0 p_a$, where p_a is the atmospheric pressure and $a_0 > 1$. The mass is released at time $t = 0$ and allowed to slide freely down the tube. The question is, "what is the optimal length of the tube so that the mass will be moving the fastest on exit, and what is this maximum speed?". We assume that the process takes place quickly enough that there is no heat exchanged between the surroundings and the trapped gas during its expansion.

Since there is no heat exchanged, the change in internal energy of the trapped gas is associated only with the kinetic energy gained by the mass. Assuming no friction (we will be able to relax this assumption later on, provided a friction force independent of speed), we have

$$\frac{1}{2}mv^2 = \frac{\ell}{2}\left(a_0 p_a A L_0 - a p_a A L\right).$$

Here, *v*, *a*, and *L* are all functions of time. Differentiating gives

$$mvdv = -\frac{\ell}{2}\left(\dot{a}L + a\dot{L}\right)p_a A dt = -\frac{\ell}{2}\left(\dot{a}L + av\right)p_a A dt .$$

The net force on the mass is given by the difference in pressure between the atmosphere and the trapped gas, so we have

$$m\frac{dv}{dt} = \left(a-1\right)p_a A .$$

Multiplying by *dt* and comparing our two expressions gives

$$\left(a-1\right)vp_a A dt = mvdv = -\frac{\ell}{2}\left(\dot{a}L + av\right)p_a A dt .$$

Re-arranging gives the linear first order differential equation

$$\dot{a} + \left(1 + \frac{2}{\ell}\right)\frac{v}{L}a = \frac{2v}{\ell L}$$

for *a*. Happily, the ratio v/L is easily integrated even though we do not have an explicit form for either function. The integrating factor is L^γ, where I have taken $\gamma \equiv 1 + 2/\ell$. Integrating the equation gives

$$L^\gamma a - L_0^\gamma a_0 = \frac{2}{\gamma\ell}\left(L^\gamma - L_0^\gamma\right),$$

or

$$a = \frac{2}{\gamma\ell} + \left(\frac{L_0}{L}\right)^\gamma\left(a_0 - \frac{2}{\gamma\ell}\right) = \frac{2}{\ell+2} + \left(\frac{L_0}{L}\right)^\gamma\left(a_0 - \frac{2}{\ell+2}\right).$$

It is obvious from this equation that the quantity aL^γ is *not* constant, as it would be in an adiabatic expansion. Note that we are not 'out of the woods' yet, as we still do not know *L* as a function of time. This function is related to the function *a* by the above equation as well as Newton's second law.

The latter can be used in concert with the former to determine a second order *nonlinear* differential equation in L:

$$\frac{d^2L}{dt^2} = \left[\frac{1}{\gamma} + \left(\frac{L_0}{L}\right)^{\gamma}\left(a_0 - \frac{2}{\ell+2}\right)\right]\frac{p_a A}{m}.$$

This, along with the initial conditions $L(0) = L_0$; $\dot{L}(0) = 0$, would allow us to completely solve the problem if we could only solve the differential equation. We may be able to linearize this equation using a substitution of the form $u = \left(L_0/L\right)^{\gamma}$ or something similar, but, happily, this is *not* necessary in order to solve the problem given.

It is often the case in physics problems that finding positions and speeds as a function of *time* is more difficult than finding out the speed at a given position or finding out where the speed will take a given value. This is the foundation of the ease with which conservation of energy can be used to solve many problems so easily, while related questions associated with *time* have no such simple solution. The current problem is no different, as we are essentially using a conservation of energy argument. We already have an expression for a in terms of L, and these two functions are the only ones we care about as far as the questions asked are concerned. It is clear from Newton's second law that the mass will be speeding up whenever $a > 1$ and slowing down whenever $a < 1$. Therefore, the maximum exit speed will be attained when $a = 1$ at the end of the tube. Setting $a = 1$ and solving for L, we find

$$L_f = \left(\gamma a_0 - 2/\ell\right)^{1/\gamma} L_0$$

is the optimal length and the maximum kinetic energy is

$$\frac{1}{2}mv_f^2 = \frac{\ell}{2}\left[a_0 - \left(\gamma a_0 - 2/\ell\right)^{1/\gamma}\right]p_a A L_0.$$

The maximum speed of the mass is therefore

$$v_f = \sqrt{\ell\left[a_0 - \left(\gamma a_0 - 2/\ell\right)^{1/\gamma}\right]\frac{p_a A L_0}{m}}.$$

For a more-or-less realistic mass of 0.2 kg, initial length of 1 cm, cross-sectional area of $1\,\text{cm}^2$, and initial pressure of 5 atmospheres, a diatomic gas gives about 1.7 m/s for the maximum speed associated with a tube of length 3.85 cm. Note that the length of the mass is not taken into account here. The length of the tube is taken to be the final length associated with the trapped gas itself. If the mass's length is larger than 2.85 cm, it will be sticking out of the tube at the beginning of the experiment.

We may worry that the quantity taken to the power of $1/\gamma$ in these expressions could be negative or larger than a_0. This would lead to situations that make no sense. Re-arranging this quantity, we have

$$\gamma a_0 - 2/\ell = \frac{\left(\ell+2\right)a_0 - 2}{\ell} = a_0 + \frac{2}{\ell}\left(a_0 - 1\right).$$

The speed will therefore be zero if $a_0 = 1$, as expected. I leave it to you to show that $\left(\gamma a_0 - 2/\ell\right)^{1/\gamma} < a_0$, provided $a_0 > 1$, because $\ell > 0$. Also, you should see if you can re-do the analysis assuming a constant kinetic friction force. The final value of a must be larger than 1 in this case… .