Solution to Diffy Eqns Problems

1. We guess $x = e^{\alpha t}$, and then $\frac{dx}{dt} = \alpha e^{\alpha t}$ and $\frac{d^x}{dt^2} = \alpha^2 e^{\alpha t}$. This makes the given differential equation

$$\alpha^2 + 3\alpha + 2 = 0 \tag{1}$$

(canceling the $e^{\alpha t}$). So the solutions for α are -2 and -1, and our general solution to this differential equation reads as

$$x(t) = Ae^{-t} + Be^{-2t} \tag{2}$$

where A and B are constants determined by initial conditions. In this problem, the initial conditions are $x(0) = 5$ and $x'(0) = 0$. The first is equivalent to

$$x(0) = Ae^{-0} + Be^{-2*0} = A + B = 5 \tag{3}$$

The second initial condition is equivalent to

$$x'(0) = -Ae^{-0} - 2Be^{-2*0} = -A - 2B = 0 \tag{4}$$

(Remember that $x'(t) = \frac{dx}{dt}$.) Solving Eqns. 3 and 4 for A and B gives $A = -5$, $B = 10$, and plugging these values into the general formula for $x(t)$ given by Eqn. 2 gives

$$x(t) = -5e^{-t} + 10e^{-2t} \tag{5}$$

Thus,

$$x(7) = -5e^{-7} + 10e^{-14}$$

2. Applying Newton's Second Law to this block gives

$$\sum F_x = -k\Delta x = m\frac{d^2x}{dt^2} \tag{6}$$

where Δx is the distance that the spring is stretched from its initial position. Let's simplify this differential equation by choosing our origin at the natural length of the spring, so that

$$\Delta x = x$$

Some rearranging of Eqn. 6 gives

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{7}$$

This gives a general solution of

$$x(t) = Ae^{i\sqrt{\frac{k}{m}}t} + Be^{-i\sqrt{\frac{k}{m}}t} \tag{8}$$

And yes, those exponents have imaginary numbers. That is not a mistake– you'll see why in a bit. Let's apply the given initial conditions, which are $x(0) = D$ and $x'(0) = 0$. The first condition rewrites as

$$x(0) = A + B = D \tag{9}$$

The second condition rewrites as

$$x'(0) = (i\sqrt{\frac{k}{m}})A + (-i\sqrt{\frac{k}{m}})B = 0 \tag{10}$$

which gives values of $A = \frac{D}{2}$ and $B = \frac{D}{2}$. So let's plug these values back into our general solution given by Eqn. 8 to get

$$x(t) = \frac{D}{2}e^{i\sqrt{\frac{k}{m}}t} + \frac{D}{2}e^{-i\sqrt{\frac{k}{m}}t} \tag{11}$$

But for those of you who are clever and remember Euler's formula ($e^{i\theta} = \cos(\theta) + i\sin(\theta)$), Eqn. 11 simplifies to

$$x(t) = D\cos(\sqrt{\frac{k}{m}}t) \tag{12}$$

Clearly this block oscillates on whatever surface it's on with a frequency of $\sqrt{\frac{k}{m}}$. If you do not agree that this statement is clear based solely on Eqn. 12, then you should learn some more about trigonometric functions. (I recommend Wikipedia or MathWorld for these kind of ideas in general.)

Regarding 2^{nd} Order Differential Equations: A Few Problems

1.
$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$
If $x(0) = 5$ and $x'(0) = 0$, find $x(7)$.

2. A block of mass m is attached to a spring of spring constant k, so that it moves horizontally across a frictionless surface. Show that the frequency of this block's motion is $\sqrt{\frac{k}{m}}$, and find an equation describing the block's motion along the surface, given that it is initially stretched D from its equilibrium position and moves starting from rest.