Central Forces Lecture

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1 Introduction

Many problems in physics reduce to the determination of the motion of a particle under the influence of a force directed inwards towards a single point and with a magnitude varying only with radius. Two of the most prominent examples are the Coulomb attraction force and Newtonian gravitation. Both of these forces exhibit properties shared by all central forces.

2 Conservation of Angular Momentum

One of the most important properties of central forces is that they conserve angular momentum about an axis through the central object. Recall that the angular momentum of a particle is given by

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}.$$

The time derivative can be found easily by the product rule:

$$\frac{d\boldsymbol{L}}{dt} = \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\tau}.$$

The $\dot{\boldsymbol{r}} \times \boldsymbol{p}$ term vanishes because the velocity is parallel to momentum. This is the general result; in the case of a central force, $\boldsymbol{F}$ is parallel to $\boldsymbol{r}$ by definition (when $\boldsymbol{r}$ is measured from an axis through the center of the force field). Therefore, there is no torque, and $\boldsymbol{L}$ remains constant. This has multiple important consequences:

- Central forces lead to planar motion. Since the angular momentum is constant, $\boldsymbol{r}$ and $\boldsymbol{p}$ must both lie in a plane perpendicular to it. Therefore, the particle does not move out of this plane.
- Particles under the influence of central forces have a constant areal velocity. In the special case of gravity, this is referred to as Kepler's second law. It follows directly from the interpretation of the angular momentum as a cross product describing the area swept out per time. Consider the vector $\frac{1}{2m}\boldsymbol{L}$:

  $$\frac{1}{2m}\boldsymbol{L} = \frac{1}{2}\boldsymbol{r} \times \boldsymbol{v} = \frac{1}{2}\frac{\boldsymbol{r} \times d\boldsymbol{x}}{dt}$$

  Figure 1 shows that this represents the area swept out per time. Since angular momentum is constant, so is this areal velocity.

3 Types of Central Forces

In this section we will review two of the most common central forces. In fact, *Bertrand's theorem* proves that these are the only two forces whose bounded orbits are also closed. We will not present a proof of this theorem, as it goes far beyond the level of physics presented in this lecture. However, we recommend that the inquisitive reader view the Wikipedia page associated with the theorem.

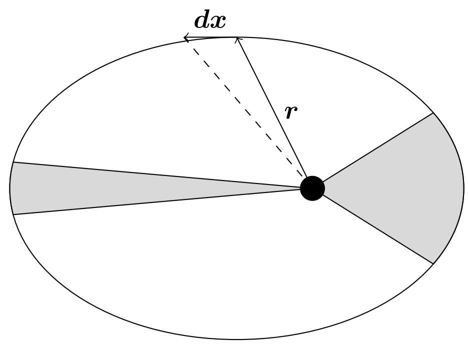


Figure 1: Equal areas are swept out in equal times because $\boldsymbol{L}$ is conserved.

3.1 Linear Force

This force is representative of a spring, and is proportional to the distance from a particular central point. Much work has likely been expended on your part to studying the mechanics of a spring; this central force shouldn't be anything non-trivial.

3.2 Inverse Square Forces

In classical mechanics, inverse square forces are by far the most important. These forces have additional properties that are of interest in problems involving gravity or electromagnetism. We have already proved Kepler's second law in the general case; his first and third laws are unique to inverse square forces.

Kepler's first law states that planets move in elliptical orbits. This is approximately true; in fact, a particle under the influence of an inverse-square force can have an elliptical, parabolic, or hyperbolic orbit. These possibilities can be summarized in a single equation, which we present without its unwieldy derivation:

$$r = \frac{L^2}{m^2\alpha}\frac{1}{1+e\cos\theta}$$

The parameter α describes the strength of the force: $F/m = -\alpha/r^2$. The shape of the path is determined by e, the eccentricity. When $e < 1$ the orbit is elliptical; when $e = 1$, the orbit is parabolic; and when $e > 1$, the orbit is hyperbolic.

Kepler's third law relates the period t of an orbit to its semi-major axis a as $t^2 \propto a^3$. The elliptical case is tricky, but we can derive it for uniform circular motion in a gravitational potential. The gravitational force must be equal to the centripetal force:

$$\begin{aligned}
m\frac{\alpha}{r^2} &= m\frac{v^2}{r} \\
v &= \sqrt{\frac{\alpha}{r}} \\
t = \frac{2\pi r}{v} &= \sqrt{\frac{4\pi^2 r^3}{\alpha}} \\
t^2 &= \frac{4\pi^2}{\alpha}r^3
\end{aligned}$$

4 Effective Potentials

4.1 System with One Mass

Having discussed the most common central forces that occur in nature, let us now take a closer look of studying the motion of an object due to a central force.

We first will write the total energy of an object subject to a central force $F(r)\hat{r}$

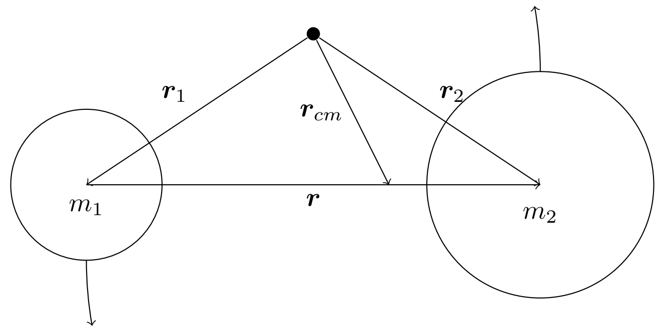


Figure 2: Two Orbiting Masses

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$$

Where the first term denotes the kinetic energy due to the radial motion of the object, and second term is the kinetic energy due to the angular motion of the object, and the third term represents the potential energy as derived from the force using $-\nabla V(r) = F(r)$. The second term can be further simplified, using the fact that the angular momentum of the particle stays constant ($mr^2\dot{\theta} = L$). Using this substitution, we can say that the second term is given by $\frac{L^2}{2mr^2}$, which gives us

$$E = \frac{1}{2}m\dot{r}^2 + \left(\frac{L^2}{2mr^2} + V(r)\right)$$

The term grouped on the right is collectively known as the *effective potential energy* of the particle. Thus, if we were to conserve a particle's energy under the assumption of a central force, we must add this extra term to account for the conservation of angular momentum, and the azimuthal kinetic energy of the object.

As before, energy is and will be conserved (unless some of it is released as sensible heat, possibly due to dissipation). Thus, we can solve for the radial velocity of the object as

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{L^2}{2mr^2} - V(r)}$$

This form, although simple to set-up as an integral, is not necessarily the nicest form. One can play around with the algebra to show that the following, modified equation is equivalent (and perhaps more tractable) than the above result.

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2mV(r)}{L^2}$$

Using $V(r) = -\frac{\alpha}{r}$ gives us the result that we presented earlier, without derivation. For the more fastidious readers, we have decided to make the details of the derivation a problem in this lecture's corresponding problem set.

4.2 System with Many Masses

Consider two masses orbiting each other. This a fairly accurate model for most planetary behavior. Now, how do we model the energy of the system? Let us consider the following energy invariant

$$E = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 + V(|\mathbf{r}_1 - \mathbf{r}_2|)$$

Figure 2 details why we have written down the following function. Now, let us define the center of mass vector of the system as outline in the diagram above. This center of mass position vector is equivalent to

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Thus, we can rewrite in the individual position vectors of the object in terms of the center of mass position vector ($\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1+m_2}\mathbf{r}$ and $\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1+m_2}\mathbf{r}$), where $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$. Reinserting these equations into the conservation of energy statement gives us

$$E = \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{m_1+m_2}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{m_1+m_2}\dot{\mathbf{r}}\right)^2 + V(|\mathbf{r}|)$$

This can be simplified to

$$E = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r)$$

Where $\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}$. This successfully changes the system to a function of two convenient coordinates. One of which, the center of mass kinetic energy, is redundant, since the velocity of the center of mass of the system remains constant. So, we technically have one changeable (and thus relevant) coordinate, the distance between the two objects.

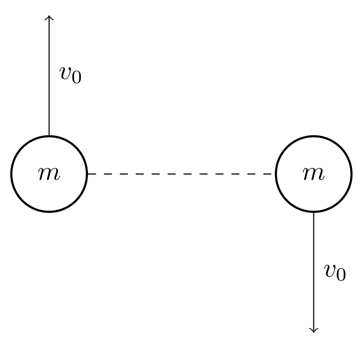
5 Problems

1. Using the inverse square potential, derive the equation presented in section 3.2. Hint: using the polar form will be most ideal.

2. Two masses m_1 and m_2 orbit each other in a circular motion. Determine the period of their orbit.

3. A particle moves in a $V(r) = \beta r^2$ potential. Show that its path is an ellipse.

4. (Semifinal Exam 2012) Two masses m separated by a distance ℓ are given initial velocities v_0 as shown in the diagram. The masses interact only through universal gravitation.
(a) Under what conditions will the masses eventually collide?
(b) Under what conditions will the masses follow circular orbits of diameter ℓ?
(c) Under what conditions will the masses follow closed orbits?
(d) What is the minimum distance achieved between the masses along their path?



5. (F=ma 2008) Two satellites are launched at a distance R from a planet of negligible radius. Both satellites are launched in the tangential direction. The first satellite launches correctly at a speed v_0 and enters a circular orbit. The second satellite, however, is launched at a speed $\frac{1}{2}v_0$. What is the minimum distance between the second satellite and the planet over the course of its orbit?

6. A particle of mass m travels in a hyperbolic orbit past a mass M, whose position is assumed to be fixed. The speed at infinity is v_0 and the impact parameter is b. Determine the angle through which the particle is deflected.