The Basic Postulates of Special Relativity and the Lorentz Transformation

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1 Introduction

If we were to construct a device for measuring the speed of sound and use it outdoors, we would expect results approximately equal to $343\,\mathrm{m\,s^{-1}}$, the speed of sound in air at room temperature. However, particularly on windy days, the measurements would deviate from this value depending on our relative speed with respect to the air. In general, the velocity of a material wave is only a "constant" in a frame of reference fixed to the medium in which it propagates; if we let the measured velocity be $\boldsymbol{v_s'}$, the velocity of the wave with respect to the medium be $\boldsymbol{v_s}$, and the velocity of the observer with respect to the medium be $\boldsymbol{V}$, then we have in general

$$\boldsymbol{v_s'} = \boldsymbol{v_s} - \boldsymbol{V}$$

At the end of the nineteenth century, physicists still believed that light worked in exactly the same manner: a "luminiferous aether" permeating through space defined an absolute frame of reference in which the speed of light was always fixed. An important problem in experimental physics became determining the velocity of earth with respect to this aether. The speed of earth relative to the sun has a ratio on the order of 10^{-4} with the speed of light, so physicists expected the presence of an "aether wind" of this magnitude.

2 Michelson-Morley Experiment

Michelson and Morley devised an experiment in 1887 to detect the presence of an aether wind.[1] The device is shown in an idealized form in Figure 1. Incoming light in the positive x direction meets a half-silvered mirror A inclined at 45°; half continues in the positive x direction towards a mirror B and reflects back, and the other half in the positive y direction towards a mirror D. The returning beams combine at mirror A, and produce beams in the negative x and y directions.

To calculate the time the light beam takes to traverse each path, we imagine ourselves moving with the aether, so that the speed of light is fixed at c. For the horizontal beam, we see mirror B coming towards us with a speed V, so

$$t_{A\to B} = \frac{L_B - V t_{A\to B}}{c}$$

Solving, we obtain $t_{A\to B} = L_B/(c+V)$. Similarly for the returning beam, $t_{B\to A} = L_B/(c-V)$. Therefore, the total time elapsed between the time the beam first arriving at mirror A and the time it return is

$$t_{A\to B\to A} = \frac{L_B}{c+V} + \frac{L_B}{c-V} = \frac{2cL_B}{c^2 - V^2} = \frac{2L_B}{c(1 - V^2/c^2)}$$

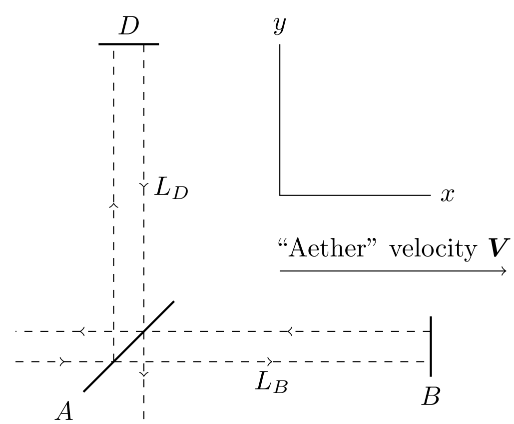

Figure 1: An idealized schematic of the device used by Michelson and Morley in their experiments.

The path for the vertical beam is slightly more complicated relative to the aether. The mirror D appears to travel a distance $Vt_{A\to D}$ to the left as the beam moves towards it, so we have

$$t_{A\to D} = \frac{\sqrt{L_D^2 + V^2 t_{A\to D}^2}}{c}$$

Solving this equation and doubling to obtain the total time (since the situation is obviously symmetric for the returning beam) gives

[1] Here we will actually discuss an apparatus designed by Michelson alone in 1881; it was shown that this device required some optical refinements (realized in the 1887 experiment) before it could accurately confirm or deny the presence of an aether wind.

$$t_{A\to D\to A} = \frac{2L_D}{c\sqrt{1-V^2/c^2}}$$

We see that the two times are not identical; rather, the presence of an aether wind would introduce an anisotropy which offsets the two returning light beams. They will only constructively interfere if $t_{A\to B\to A} = t_{A\to D\to A} \pm n\lambda/c$. We could, for example, set $L_D = L_B/\sqrt{1-V^2/c^2}$, and obtain constructive interference with $n=0$. If we then turn the apparatus so that the aether moves along L_D, then there would no longer be constructive interference. In their 1887 experiment, Michelson and Morley adjusted the parameters of their device so that for an aether wind of $10^{-4}c$ and a wavelength of approximately 600 nm (corresponding to the yellow sodium light they used), rotation through 90° would turn the constructive interference into perfectly destructive interference.

However, when the experiment was performed, rotating the apparatus showed almost no change in the interference. This result was demonstrated again at different times of the year, when the velocity of the earth had changed directions, so it was indicative that the aether happens to travel at the same velocity of earth. The only remaining explanation[2] was that the aether *does not exist*: light does not have a medium through which it propagates. Instead, based on this experiment (and many others which obtained more precise results, including one performed by Georg Joos), special relativity postulates that the speed of light in free space is the same in all inertial frames of reference. In addition, it postulates that the laws of physics are the same in all inertial frames (inclding, especially, Maxwell's laws).

3 The Galilean and the Lorentz Transformations

The kinematics and dynamics that we are accustomed to are considered as "Galilean Transformations". The approach is particularly simple. If we consider an **event** that occurs at a particular distance[3] x from the origin of a fixed coordinate system S. Suppose now, that we introduce a coordinate system that moves with a particular velocity V relative to the fixed coordinate system, which we denote as S'. As seen by an observer moving with S', the distance that he measures from the event will appear to be $x' = x - Vt$, from basic kinematics (see Figure 2). We also can say that $x = x' + Vt$. Moreover, an observer in the stationary reference frame and the moving reference frame must both synchronize their clocks, to assert that their measurements are not staggered. We also assume the clocks are both ticking at the same rate in both reference frames; that is, when two seconds in the stationary observer's clock pass, two seconds from the moving observer's clock will pass. Again, this seems all too obvious. But Einstein... well, he had a different idea.

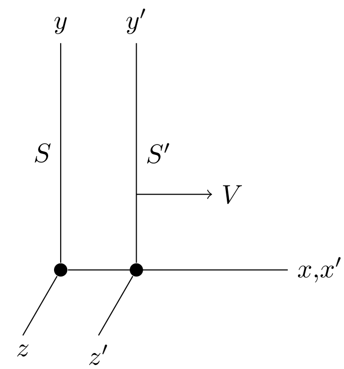

Figure 2: The coordinate system S' moves at a velocity V in the x direction relative to S.

Einstein proposed that there must be two factors that render this statement false. One, lengths, and two, time. He proposed that objects moving with a non-zero velocity will appear to shrink. That is to say, the inertial observer will claim that the moving observer's ruler stick is smaller than it is supposed to be, by a factor γ. Similarly, to the moving observer, the stationary observer appears to be moving *backwards* with a velocity V, and thus can make the exact same claim. The idea that both observers have a symmetrical argument seems almost paradoxical, a major reason why special relativity is hard to grasp. Thus, based on Einstein's implications

$$x' = \gamma(x - Vt)$$
$$x = \gamma(x' + Vt)$$

The second constraint that Einstein adds in is the fact that observers in a moving reference frame have different "clock rates" than those in a stationary reference frame. The first of the two equations assumes that the moving observer is measuring time, while the second equation implies that the stationary observer is measuring time. In both cases, these times scales are different. We will denote this as

$$x' = \gamma(x - Vt)$$
$$x = \gamma(x' + Vt')$$

Our only lurking problem is to solve for γ. For this, we will use the fact that the speed of light is constant in every reference frame. Suppose our event is a light beam turning on. Thus, the light beam will travel a distance x' in t' according

[2]Actually, George Stokes proposed that the earth could "drag" the aether along with it in its orbit. This, of course, soon became an irrelevant theory.

[3]We will stick to 1 dimension for now, as generalizing the situation is straightforward.

to the moving observer, and x in t according to the stationary observer. Since the value of c is constant in both reference frames, we obtain the following 4 equations

$$x' = \gamma(x - Vt)$$
$$x = \gamma(x' + Vt')$$
$$x = ct$$
$$x' = ct'$$

Our task is not to solve for this system of 4 equations. The solutions (work it out in problem 1!) are

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$
$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$
$$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

These are the Lorentzian transformations. Not so simple... However, in the limit that the velocity of the moving frame is much less than the speed of light, we can verify that this breaks down into the familiar Galilean Transformation.

4 Problems

1. Solve the four equations above to derive the Lorentz transformation formulae.
2. A rod of length L_0 at an angle θ_0 with the x axis moves at a velocity v along the x axis. Find the length and the angle measured by a stationary observer.
3. The mean lifetime of a muon is 2.2 µs. If a muon has a speed of $0.95c$, what is its mean lifetime as measured by a stationary observer?[4]
4. Differentiate the Lorentz transformation equations to find the following velocity transformation equation:

$$u_x = \frac{u'_x + v}{1 + vu_x/c^2}$$

   Show from this that the sum of two velocities less than c cannot exceed c.

[4]This change in lifetime is observed in muons incoming on the Earth's surface, and is one piece of evidence that strongly supports the predictions of special relativity.