Electrostatics PS Solutions

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1 Solutions

1. Clearly the net force on the charge is purely perpendicular to the plane of the charges, due to the symmetry of the situation. For each of the 4 charges Q, the point charge experiences a force as shown in Figure 1.

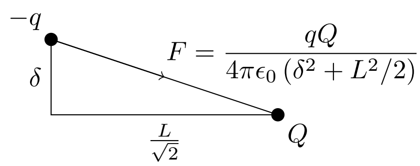

Figure 1: The force due to one of the charges in problem 1.

Since the components in the plane of the charges all cancel, we scale this force by $\delta/\sqrt{\delta^2 + L^2/2}$ to get the perpendicular component. Mulitplying this by 4, we have

$$|\vec{F}_{\text{net}}| = \frac{qQ}{\pi\epsilon_0 \left(\delta^2 + L^2/2\right)^{3/2}}\delta$$

For small δ, this is approximately equal to

$$|\vec{F}_{\text{net}}| = \frac{2qQ}{\pi\epsilon_0 L^3}\delta$$

We see that the force is proportional to the displacement δ; this is indicative of simple harmonic motion, with $k = 2qQ/\pi\epsilon_0 L^3$. Thus, the period of motion is

$$T = 2\pi\sqrt{\frac{m}{k}} = \pi L\sqrt{\frac{2\pi\epsilon_0 mL}{qQ}}$$

2. To solve this problem, we must break the two rods up infinitesimally into pieces of charge dq and determine the force due to all other charges in sight. Thus

$$d|\vec{F}| = \frac{k dq_1 dq_2}{r_{12}^2} \tag{1}$$

where the subscripts represent infinitesimal bits of charge on different rods. r_{12} is the distance between these charged bits, which is given by $x_2 - x_1$. Thus, we get

$$d|\vec{F}| = \frac{kdq_1dq_2}{r_{12}^2} = k\lambda^2 \frac{dx_1dx_2}{(x_2-x_1)^2} \tag{2}$$

where we used the fact that $dq = \lambda dx$. Thus

$$|\vec{F}| = \int\int \frac{kdq_1dq_2}{r_{12}^2} = k\lambda^2 \int_{d+L}^{d+2L} \int_0^L \frac{dx_1dx_2}{(x_2-x_1)^2} \tag{3}$$

Simplifying this integral, we get

$$|\vec{F}| = k\lambda^2 \int_{d+L}^{d+2L} dx_2 \left(\frac{1}{x_2-L} - \frac{1}{x_2}\right) = k\lambda^2 \ln\left(\frac{x_2-L}{x_2}\right)\Big|_{d+L}^{d+2L} = 2k\lambda^2 \ln\left(\frac{d+L}{d(d+2L)}\right) \tag{4}$$

For the case of the two disks, we must first notice that the net force will point in purely in the horizontal direction, so we only need to take account of the horizontal component of the force only. Using a polar coordinate system, we have

$$d|\vec{F}| = \frac{kdq_1dq_2}{r_{12}^2}\cos\theta_1\cos\theta_2 = \frac{kdq_1dq_2}{r_{12}^2}\frac{d^2}{r_{12}^2} \tag{5}$$

Using the fact that $dq = \sigma r dr d\theta$, we obtain

$$d|\vec{F}| = \frac{kdq_1dq_2}{r_{12}^2}\cos\theta_1\cos\theta_2 = \frac{k\sigma^2 r_1r_2dr_1d\theta_1dr_2d\theta_2}{(r_1-r_2)^2+d^2}\frac{d^2}{(r_1-r_2)^2+d^2} = k\sigma^2d^2\frac{r_1r_2dr_1d\theta_1dr_2d\theta_2}{((r_1-r_2)^2+d^2)^2} \tag{6}$$

Thus

$$|\vec{F}| = k\sigma^2d^2 \int_0^{2\pi}\int_0^R\int_0^{2\pi}\int_0^R \frac{r_1r_2dr_1d\theta_1dr_2d\theta_2}{((r_1-r_2)^2+d^2)^2} \tag{7}$$

This gives us the following

$$|\vec{F}| = \frac{k\pi^2\sigma^2}{3d}\left(2d^3\ln\left(\frac{d^2+R^2}{d^2}\right) + 2R^3\left(\tan^{-1}\frac{R}{d} - \tan^{-1}\frac{d}{R}\right) + R^2(\pi R - 2d)\right) \tag{8}$$

This general methodology can be extended to different configurations/shapes, but (as you can see), it quickly gets more complicated.

3. We start by writing the y component of the electric field at the origin emitted by a point particle positioned at (r,θ):

$$E_y = -\frac{q}{4\pi\epsilon_0 r^2}\sin\theta$$

The first factor is the total electric field, and the second term gives the component in the y direction. The negative sign gives the correct direction of the field; points above the x axis give a negative y component, and vice versa. We now need to find a locus of points where this value is constant. We can let $E_y = c_1$ and manipulate the equation:

$$-\frac{q}{4\pi\epsilon_0 r^2}\sin\theta = c_1$$
$$c_2 r^2 = \sin\theta$$

where c_2 absorbs all the constants. The general shape of this curve is shown in Figure 2.

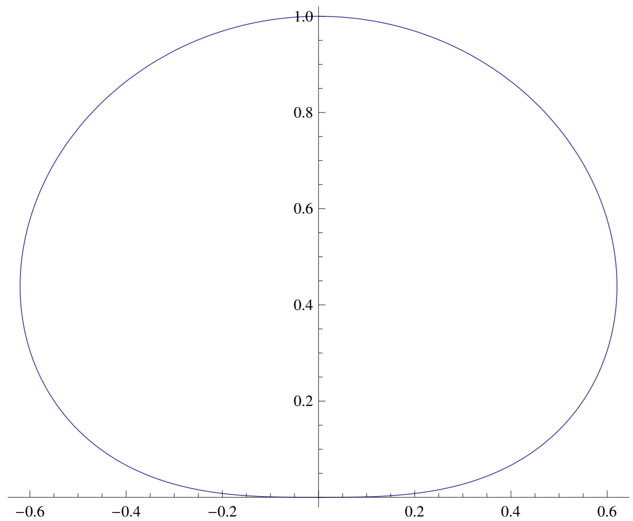

Figure 2: The shape of the solution to problem 3.

4. Using the figure below, we see that the net force points directly downwards for a particular chose point, because the horizontal components cancel. The distance between two charged particles can be determined via the law of cosines. This gives us $d_n^2 = R^2 + R^2 - 2R^2\cos\frac{2\pi n}{N} = 4R^2\sin^2\frac{\pi n}{N}$, where we indicate n to mean the index of the point we choose. The vertical component is given by the force multiplied by $\sin\frac{\pi n}{N}$, using the geometry of the diagram yet again. Thus, the net electric field due to an arbitrary point is

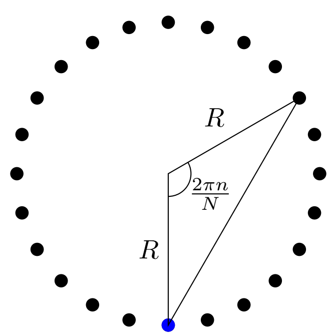

Figure 3: The geometry of problem 4.

$$|\vec{E}_v| = k\frac{Q}{N}\frac{1}{4R^2\sin^2\frac{\pi n}{N}}\sin\frac{\pi n}{N} \tag{9}$$

Thus, the total force due to all the points is

$$|\vec{E}_{net}| = \sum_{n=1}^{N-1} k\frac{Q}{N}\frac{1}{4R^2\sin\frac{\pi n}{N}} = \frac{kQ}{4NR^2}\sum_{n=1}^{N-1}\frac{1}{\sin\frac{\pi n}{N}} \tag{10}$$

If we take the limit as the number of particles approaches infinity, we get

$$|\vec{E}_{net}| = \frac{kQ}{4R^2}\lim_{N\to\infty}\left[\sum_{n=1}^{N-1}\frac{1}{\sin\frac{\pi n}{N}}\frac{1}{N}\right] \tag{11}$$

The quantity within the limit is effectively a Riemann sum, and can be rewritten as an integral.

$$|\vec{E}_{net}| = \frac{kQ}{4R^2}\left[\frac{1}{\pi}\int_0^{\pi}\frac{dx}{\sin x}\right] \tag{12}$$

The integral diverges, so the net electric field is infinitely large. Since the collection of charges becomes a homogeneous ring of charge Q (in the limit as $N\to\infty$), the electric field at a point due to all other charges is infinitely large. However, we are more interested in the electric force, as this behavior allows us to determine the motion of the ring. To find the net electric force at the point, we multiply the electric field by $\frac{Q}{N}$. Using the substitution $dx = \lim_{x\to 0} x = \frac{\pi}{N}$ as previously done, and simplifying the integral, we get

$$|\vec{F}_{net}| = \frac{kQ^2}{4\pi^2R^2}\lim_{x\to 0}\left[x\ln\frac{\tan\left(\frac{\pi}{2}-x\right)}{\tan x}\right] \tag{13}$$

By L'Hopital's rule, this equals 0. Thus, the net force on a charge on the ring is zero. This maintains that the ring is in static equilibrium: if the net force wasn't zero, the ring would have to rotate in order to supply a centrifugal force to counteract the non-zero Coulomb contribution.

5. An alternative expression for the electric potential energy is

$$U = \int \sigma V\, dA$$

Note that we now integrate only over the area of the disk, because the charge density is zero elsewhere. At every point, we must integrate the contributions to the electric potential from every other point. Therefore, we take the quadruple integral

$$U = \frac{\sigma^2}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{R}\int_0^{2\pi}\int_0^{R}\frac{r_1r_2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\left(\theta_1-\theta_2\right)}}\,dr_1\,d\theta_1\,dr_2\,d\theta_2$$

The denominator is the distance between (r_1,θ_1) and (r_2,θ_2) as given by the law of cosines. This quadruple integral does not possess a closed form, so this is the only viable answer we can give. However, one can determine a functional form for how the potential energy of the disk varies with increasing radius.