Multiple Integrals

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1 Introduction

In single-variable calculus, we only deal with functions with one dimensional domains. Then to find the area under a curve, we sum up the areas of little rectangles to approximate the area, then we make the rectangles infinitely small so that the error goes to 0. Well, with multiple variables, we have weirder domains of functions. However, it is still possible to find the volume under a surface. As long as the domains are simple enough to be stated, it is easy to calculate a multiple integral. I will save applications for the end and rather go directly to the calculus.

2 Double Integrals

Double integrals are a way of "summing" all the values of a function within a two dimensional domain. In this section, I will tell you how to evaluate them. It isn't difficult.

2.1 Rectangular Domains

Consider a function f defined on the domain $R = [a, b] \times [c, d]$, the rectangle in which x ranges from a to b inclusive and y ranges from c to d inclusive. How would we find the value of the 'sum' of the function evaluated at each point in this domain? Well, consider one slice of the rectangle: $[a, b] \times \{y\}$. We can find the 'sum' of the function at each point on this slice by evaluating $\int_a^b f(x, y)\, dx$. This gives us a function for the area of each slice in terms of y. Now, we can evaluate the 'sum' of all the points by integrating over these slices. Therefore, the sum over all these points is $\int_c^d \left(\int_a^b f(x, y)\, dx\right) dy$. For shorthand, we denote this $\int_c^d \int_a^b f(x, y)\, dx\, dy$ or $\iint\limits_R f(x, y)\, dA$.

Notice that we could have also considered slices $\{x\} \times [c, d]$ rather than $[a, b] \times \{y\}$. It turns out it doesn't matter, and $\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$. This result is called Fubini's Theorem, and will not be proved here.

The formal definition of these double integrals is $\iint\limits_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$ where x_{ij}^* and y_{ij}^* are arbitrary points inside the subrectangle for that i and j. If the upper right corner is chosen, we obtain $\iint\limits_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta A$. Don't worry too much about this and just evaluate it as in the previous paragraphs.

It should be clear to you at this point how to extend this to integration over a box, or the analog in higher dimensions. If not, think about it some more or ask someone.

2.2 Arbitrary Bounded Domains

What if our domain isn't a rectangle? Well, as long as it's bounded, we can still integrate over it. Suppose we have a function $f : D \to \mathbb{R}$. Since D is bounded, consider a bounding rectangle R. Then define a function $F : R \to \mathbb{R}$, where $F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$.

Then, we say that $\iint\limits_D f(x, y)\, dA = \iint\limits_R F(x, y)\, dA$.

This doesn't actually help us in computing these integrals though. Generally, our domain will be something along the lines of $\{(x, y) \in \mathbb{R}^2 : d_1(x) \leq y \leq d_2(x) \text{ and } a \leq x \leq b\}$ or $\{(x, y) \in \mathbb{R}^2 : d_1(y) \leq x \leq d_2(y) \text{ and } a \leq y \leq b\}$. In these cases, we can evaluate the integral as $\iint\limits_D f(x, y)\, dA = \int_a^b \int_{d_1(x)}^{d_2(x)} f(x, y)\, dy\, dx$ or $\iint\limits_D f(x, y)\, dA = \int_a^b \int_{d_1(y)}^{d_2(y)} f(x, y)\, dx\, dy$ respectively. In general, the limits of the inner integrals can be in terms of the variables that we haven't integrated over yet. So in two variables, the limits of the inner integral can be in terms of the outer variable, but not the other way around.

2.3 Integrals over $\mathbb{R}^2$

$\mathbb{R}^2$ is the most common unbounded region to integrate over. This is represented as $\iint\limits_{\mathbb{R}^2} f(x, y)\, dA$.

To evaluate this, take a nice region R that grows into $\mathbb{R}^2$ when you take a variable to infinity. Simple examples are squares, rectangles, or circles centered at the origin. Then $\iint\limits_{\mathbb{R}^2} f(x, y)\, dA = \lim\limits_{R \to \mathbb{R}^2} \iint\limits_R f(x, y)\, dA$. So, for circles defined by $R_k = \{(x, y) : x^2 + y^2 \leq k^2\}$, the integral is $\lim\limits_{k \to \infty} \iint\limits_{R_k} f(x, y)\, dA$.

2.4 Examples

I'm lazy, so these examples are all from the textbook, but none of you read that, so I can get away with this. :-) I'm going to leave out the evaluation of single integrals because you should know how to do that.

1. Evaluate $\int_0^3 \int_1^2 x^2 y\, dy\, dx$.

   Solution: $\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[x^2 \frac{y^2}{2}\right]_{y=1}^{y=2} dx = \int_0^3 \frac{3}{2} x^2\, dx = \left[\frac{x^3}{2}\right]_{x=0}^{x=3} = \frac{27}{2}$.

2. Evaluate $\iint_R (x - 3y^2)\, dA$ where $R = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

   Solution: $\iint_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx = \int_0^2 (x - 7)\, dx = -12.$

3. Evaluate $\iint_R y \sin(xy)\, dA$ where $R = [1, 2] \times [0, \pi]$.

   Solution: $\int_0^\pi \int_1^2 y \sin(xy)\, dx\, dy = \int_0^\pi [-\cos(xy)]_{x=1}^{x=2}\, dy = \int_0^\pi (-\cos 2y + \cos y)\, dy =$ $\left[-\frac{1}{2} \sin 2y + \sin y\right]_0^\pi = 0.$

   If we were to try to evaluate the integral in the other order, we could, but we would have to use parts and it gets much uglier.

3 Polar Double Integrals

Let's start this section with a problem. Find the volume under $f(x, y) = \sqrt{x^2 + y^2}$ for $D = \{(x, y) : x^2 + y^2 \leq 4\}$. Solving for y implicitly in terms of x, we get that D is bounded by $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. Thus, the volume is $\int_{-4}^4 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (\sqrt{x^2 + y^2})\, dy\, dx$. Everyone who would like to evaluate this, please be my guest. It is doable, but will probably require trigonometric substitution for the second integral, which is why we have a method for integrating in polar coordinates.

Consider a small "polar rectangle" of the domain. A picture of this is on page 1005 of the online textbook if you are looking for a picture. When we send $d\theta$ and dr to 0, we get approximately a normal rectangle with sides $r\, d\theta$ and dr, so the area of one of these infinitesimal polar rectangles is $dA = r\, dr\, d\theta$. Thus, our integrals $\iint_R f(x, y)\, dA$ can be written in polar form $\iint_R f(x, y) r\, dr\, d\theta$. Since $x = r\cos\theta$ and $y = r\sin\theta$, we get $\iint_R f(r\cos\theta, r\sin\theta) \cdot r\, dr\, d\theta$.

Now let us return to our example. Our function $f(x, y)$ can be rewritten as $f(r, \theta) = r$. Thus, when we convert to polar, we get $\iint_R r^2\, dr\, d\theta$. Now, our region is that such that $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$, so our integral is $\int_0^4 \int_0^{2\pi} r^2\, d\theta\, dr = \int_0^4 (2\pi r^2)\, dr = \left[\frac{2\pi}{3} r^3\right]_0^4 = \frac{128\pi}{3}$.

You can extend this to three dimensions and beyond with coordinate systems such as cylindrical and spherical coordinates. You don't need this at the moment, but if it ever seems like it'd be useful, feel free to ask.

4 Applications

This is a pretty silly section, but I'll tell you guys some applications of these things anyway.

1. Say we want to find the mass of a region with a variable density. We can use a double integral to find the mass of each of the points and add them up.

2. Say we want to find the volume of a region above the XY plane. Simply integrate over the base of it to find the volume.

3. The moment of inertia is defined to be mr^2 for a point mass, where r is the distance from the point of rotation. We can use a double integral to find the moment of inertia for a two dimensional region.

4. The moment about the x-axis of a region D with density function $\rho(x, y)$ is $M_x = \iint\limits_D y\rho(x, y)\, dA$.

5. The moment about the y-axis of a region D with density function $\rho(x, y)$ is $M_y = \iint\limits_D x\rho(x, y)\, dA$.

6. The center off mass of a region is $(\overline{x}, \overline{y}) = (\frac{M_x}{m}, \frac{M_y}{m})$ where m is the mass.

7. Probability density functions of two variables are those such that $f(x, y) \geq 0$ and $\iint\limits_{\mathbb{R}^2} f(x, y)\, dA = 1$. This is like a one dimensional probability density function in that it doesn't describe the probability of hitting a specific point (usually 0), but rather the probability of a region.

8. The expected x value of the probability density function is $\mu_1 = \iint\limits_{\mathbb{R}^2} xf(x, y)\, dA$.

9. The expected y value of the probability density function is $\mu_2 = \iint\limits_{\mathbb{R}^2} yf(x, y)\, dA$.