Introduction to Wave Motion

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1 Introduction

When we discussed the simple harmonic oscillator, you may have wondered if the idea of oscillatory motion can be generalized. Of course, we see this in strings, vibrating tuning forks, and even water waves. In this lecture, we hope to elucidate the wave equation: a fundamental descriptor of many natural phenomena. These phenomena transcend into the realm of electromagnetism, and even quantum mechanics. As does all of physics, the wave equation is an innocuous expression which somehow acts as an overarching motif for everything.

2 Derivation of the Wave Equation

We will derive the wave equation from a purely physical standpoint first. Consider a string, which is oscillating back and forth. We can describe the string's motion by 2 variables: time and position. To make this clearer, let's first start by following a single point on the string. As the string undulates, the point moves up and down, demonstrating how the string's state is temporally dependent. Now, let us take a snapshot of the whole string at a particular moment. As we move across the string, we see that it changes spatially too. These two principles allow us to depict the state of the string as a multivariable function.

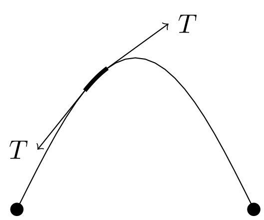

Figure 1: The force on a differential element of the string is due to variation in the angle of the tension.

In order to extract the most from this model, let us start by considering a small piece of the string. As the string is curved, there is naturally an imbalance in the net force acting on it. The two forces in effect are the tension forces pulling the string and straightening it (see Figure 1). The forces in the horizontal direction should cancel out with each other, as the string should not move horizontally. And, indeed, they do, only if the tension in the string remains relatively constant. In the vertical direction, however, the string is forced to move in a particular manner. This is given by

$$T\sin(\theta+\Delta\theta) - T\sin\theta = a_y dm = \mu ds \frac{\partial^2 y(x,t)}{\partial t^2} \tag{1}$$

Now, we will make another approximation, that vertical deviations are negligible. Thus, a string which is strongly stretched from its equilibrium position will immediately be eliminated as a candidate for the wave equation. In this case, θ is small, and $\sin\theta \approx \theta \approx \tan\theta$. Small vertical deviations also imply that $ds \approx dx$. This gives

$$T\left(\frac{\tan(\theta+\Delta\theta)-\tan\theta}{dx}\right)=\mu\frac{\partial^2 y(x,t)}{\partial t^2} \tag{2}$$

Notice, from the definition of a derivative, that $\tan(\theta+\Delta\theta)=y'(x+\Delta x)$ and $\tan(\theta+\Delta\theta)=y'(x)$. This gives us

$$T\frac{\partial^2 y(x,t)}{\partial x^2}=\mu\frac{\partial^2 y(x,t)}{\partial t^2} \tag{3}$$

Which can be rewritten as

$$\frac{\partial^2 y(x,t)}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} \tag{4}$$

Where $v=\sqrt{\frac{T}{\mu}}$ and is defined as the speed of the string's propagation. This is known as the wave equation. If the rope were subject to an external force per unit length $F(x,t)$ (like a sinusoidally driving force, or even gravity), the new equation of motion (in its most general form) is given by

$$T(t)\frac{\partial^2 y(x,t)}{\partial x^2}=\mu(x)\frac{\partial^2 y(x,t)}{\partial t^2}+F(x,t) \tag{5}$$

2.1 Acoustic Wave Equation

Consider air in a tube. Playing an instrument is a classic example of a perturbation of this particular medium: a force induces a particular pressure difference, driving the system to behave like a longitudinal wave. This longitudinal wave satisfies the wave equation

$$\frac{\partial^2 \xi(x,t)}{\partial x^2}=\frac{\rho}{\kappa}\frac{\partial^2 \xi(x,t)}{\partial t^2} \tag{6}$$

where ρ is the density of the air, and κ is a certain bulk elasticity constant, which measures the ability of air to expand by a particular volume at a given pressure. However, it's important to notice that ξ represents over and under pressure: that is deviation from atmospheric pressure.

3 Solutions of the Wave Equation

The simplest solutions to the wave equation, both mathematically and physically, are sinusoidal waves. It is easy to verify that both sines and cosines with appropriate frequencies and wavelengths form solutions to the wave equation:

$$y=A\cos(kx-\omega t+\phi) \tag{7}$$
$$y=A\sin(kx-\omega t+\phi) \tag{8}$$

We can simplify things even further by using a complex exponential instead of trigonometric functions.

$$y=Ae^{i(kx-\omega t+\phi)}=A\left(\cos(kx-\omega t+\phi)+i\sin(kx-\omega t+\phi)\right) \tag{9}$$

Since the wave equation is linear, this linear combination of solutions is also a solution. In fact, the most amazing take-away point of the wave equation is that it admits an infinite number of solutions. The solutions need only be *purely functions of (x-vt)* in order to satisfy the wave equation.

We can determine a simple constraint on the parameters ω and k by substituting any of the solutions into the wave equation.

$$\frac{\partial^2 y}{\partial t^2}=v^2\frac{\partial^2 y}{\partial x^2}$$
$$\omega^2 y=v^2k^2y$$
$$\frac{\omega}{k}=v$$

For a fixed wave speed, we can choose one of ω or k, and the other is determined by this relationship. These two parameters are closely related to frequency and wavelength, which is easily verified by examining the form of the sinusoid:

$$\omega = 2\pi f$$
$$k = \frac{2\pi}{\lambda}$$

We usually describe waves by ω and k rather than the more familiar f and λ because it simplifies notation and is easily extended to higher dimensions. We call ω the angular frequency and k the angular wavenumber (or sometimes just the wavenumber).

3.1 Dispersion Relation

A *dispersion relation* is a general equation relating the angular frequency of a wave to its wavenumber. The waves we have just described have a simple linear dispersion relation:

$$\omega = vk \tag{10}$$

The dispersion relation allows us to define two important quantities, the phase velocity and the group velocity.

$$\text{Phase velocity} = v_p = \frac{\omega}{k} \tag{11}$$
$$\text{Group velocity} = v_g = \frac{d\omega}{dk} \tag{12}$$

Clearly, these two quantities are the same for the waves we have dealt with so far. This leads to common confusion between the meanings of the phase and group velocities, because differentiating between them requires examining waves more complicated than these simple examples. The v in the wave equation is properly associated with the phase velocity. The group velocity is the speed of propagation of the wave envelope of a wave packet, like the one shown in Figure 2.

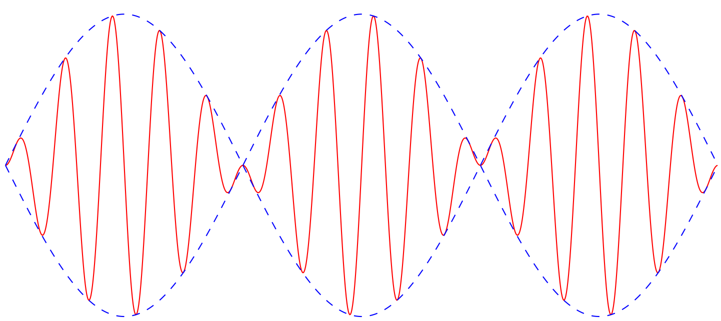

Figure 2: A sinusoid modulated by another sinusoid of lower frequency, forming a wave packet (in blue) surrounding the wave itself (in red).

In some situations, the speed of the wave itself differs from the speed of this surrounding envelope. Most often, the group velocity can be interpreted as the signal velocity. AM radio, for example, transmits information in the height of the wave packet (hence why it is called Amplitude Modulation).

4 Standing Waves

A standing wave is one in which every point in space has an oscillation with amplitude constant in time, and so does not appear to be traveling. They can be thought of as two waves of equal frequency traveling in opposite directions.

$$y = \frac{A}{2}\left(\sin(kx - \omega t) + \sin(kx + \omega t)\right) \tag{13}$$

We can simplify this using the sine addition identity, $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.

$$y = A\cos(\omega t)\sin(kx)$$

We see that the time dependence is separated from the space dependence. This means that the wave will not appear to move through space as time passes: at position x, the standing wave leads to a vibration with an amplitude $2A\sin(kx)$ and angular frequency ω. We define the positions were this amplitude is 0 to be the nodes of the wave. The nodes occur at $x = 0, \lambda/2, \lambda, \ldots$.

Oftentimes, we have to find solutions to the wave equation with particular boundary conditions that fix the endpoints:

$$y(0,t) = 0 \tag{14}$$

$$y(L,t) = 0 \tag{15}$$

These conditions must be met for all times t, so the wave cannot travel. Additionally, only standing waves with particular wavelengths can satisfy these boundary conditions. In order to have nodes at both 0 and L, L must be a multiple of $\lambda/2$. We can use this to find the allowed values of k:

$$L = n\frac{\lambda}{2}$$
$$L = n\frac{\pi}{k}$$
$$k = \frac{n\pi}{L}$$

We can then write the solution to the wave equation with these boundary conditions as a series of standing waves indexed by $n = 1, 2, 3, \ldots$.

$$y = A\cos(\omega t)\sin\left(\frac{n\pi}{L}x\right) \tag{16}$$

5 The Wave Equation in Many Dimensions

So far, our discussion has been confined to describing wave motion in one dimension. In general, however, we have to keep in mind that surfaces are waves as well. The ripples in a drum also satisfy the wave equation... but we have more coordinates to deal with. Without proof, the wave equation in n dimensions is given by

$$\nabla^2\psi(x_i,t) = \frac{1}{v^2}\frac{\partial^2\psi(x_i,t)}{\partial t^2} \tag{17}$$

Where i runs through all the position coordinates of the system. Keep in mind that we are using the variable ψ. This choice of variable suggests that wave motion is not restricted to position. Rather, ψ can represent many different things, such as electromagnetic fields, and pressure.

To shed more light on the importance of this general result, let's consider a circular vat on which a sheet of plastic is firmly attached on. Because we are dealing with two coordinates, out equation of motion is given by

$$v^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^2 z}{\partial t^2} \tag{18}$$

Now, if the radius of the vat is given by a, we need to enforce the *boundary condition* that $z(a) = 0$, because we assume that the vat is holding the plastic sheet in place. In this case, we see that Cartesian coordinates are ineffective. Instead, we can convert to polar coordinates, giving us the result

$$v^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right) = \frac{\partial^2 z}{\partial t^2} \tag{19}$$

Here, we used the Laplacian of a function in terms of polar coordinates. We will now guess the ansatz $z = T(t)\Theta(\theta)R(r)$, and see where it will take us. Plugging this trial function into the differential equation, we get

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} = \frac{1}{v^2}\frac{T''(t)}{T(t)} \tag{20}$$

Now, we should notice that our ansatz gives us a particularly unique property. The left hand side is purely a function of r and θ, while the right hand side is purely a function of t. The only way to reconcile this apparent contradiction would be if both the LHS and the RHS were constants, which we will call $-\omega^2$. The $T(t)$ equation thus gives us

$$T''(t) = -(\omega v)^2 T(t) \tag{21}$$

whose solution is $T(t) = A\cos(\omega v t) + B\sin(\omega v t)$. The LHS equation gives us $\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} + \omega^2 = 0$. We can physically understand that the vibrations of a drum are periodic in an angular sense: in order to meet the boundary conditions, the drums vibrational patterns must repeat after every rotation. This rotational invariance implies that $\Theta(\theta) = C\cos(m\theta) + D\sin(m\theta)$. Furthermore, this implies that

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -m^2 \tag{22}$$

Putting this all together, we have

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \omega^2 - \frac{m^2}{r^2} = 0 \tag{23}$$

$$\omega^2 r^2 + r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = m^2 \tag{24}$$

The solution to this equation is given by Bessel Functions $R(r) = J_m\left(\frac{\alpha_n}{a}r\right)$, where α_n denotes the *nth* positive root of the Bessel function that is the solution for a particular value m. Thus, the presence of 2 free parameters allows a plethora of different vibrational patterns to form. The final equation of motion is given by

$$z = \left(A\cos(\omega v t) + B\sin(\omega v t)\right)\left(C\cos(m\theta) + D\sin(m\theta)\right) J_m\left(\frac{\alpha_n}{a}r\right) \tag{25}$$

6 Problems

1. Consider a three-dimensional wave inside a cube with side length L centered at the origin. Find a set of solutions to the wave equation that is zero on the boundaries on the box. Where are the nodes for these waves?

2. Consider a rope composed of two pieces, one with density μ_1 and the other with density μ_2. They are joined at their ends, and then a pulse is sent through the section with density μ_1. Find the amplitudes of the reflected and transmitted waves.

3. Verify that the following function is a solution to the three-dimensional wave equation. *Hint:* $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$.

$$f(r) = \frac{1}{r}e^{i(kr-\omega t)}$$

4. In light, the phase velocity $v_p = c/n$, where the index of refraction of n can vary with frequency. Show that the group velocity in a medium with $n = A/\lambda_0$, where λ_0 is the vacuum wavelength and A is an arbitrary constant, is half the phase velocity. *Hint:* try finding the inverse of the group velocity.