# Introduction to Fluid Mechanics

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## 1 Introduction

In general, our mechanics has been dealing with rigid bodies, where we assume that our bodies are not deformed in any manner. Now, we will enter the realm of fluids, where such is not the case. Fluids behave in a more exotic manner, which will prompt us to introduce new and important properties; however, these conservation laws are very familiar to what we've have seen before. We will end this lecture with the most cumulative conservation law of all: the Navier-Stokes Equation.

## 2 What are Fluids? What is Pressure?

According to Wikipedia, "In physics, a fluid is a substance that continually deforms (flows) under an applied shear stress." In short, a fluid can be deformed, and occupies the volume in which it is contained in. Yet, the ability to be deformed comes with the idea of *pressure*. Pressure is defined as a force per unit area which acts normal to the surface of an object. In this sense, we cannot define pressure as a vector. Instead, we define pressure as a scalar, whose value equals the the magnitude of the force acting on a unit areal element of an object.

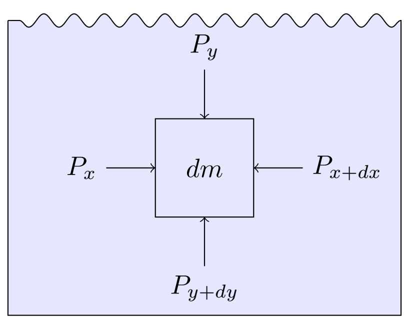


Figure 1: Free body diagram for a differential mass element in a body of fluid.

The simplest application of pressure is determining how it changes as a function of altitude. Consider a static body of water. In this case, we can assert that the fluid is in "hydrostatic equilibrium". If we consider a small chunk of the fluid with a mass $dm$, it is easy to see from Figure 1 that the horizontal components of the pressure (which tend to pull the fluid apart) cancel each other out. Assuming that the fluid is incompressible (which we haven't defined yet...), this approximation is fairly valid. The vertical pressures, however, must supply a force equal and opposite to the weight of the fluid. Thus

$$A\Delta P = g\Delta m \tag{1}$$

where $A$ is the surface area of the piece of the fluid in question. Since $\Delta m = \rho A \Delta h$, we obtain

$$\Delta P = \rho g \Delta h \tag{2}$$

Or $\frac{dP}{dh} = \rho g$. Conventionally, we assume that $\rho$ is constant, enabling us to derive the equation $P(h) = P_0 + \rho g \Delta h$. However, $rho$ is commonly non constant, and is many times a function of $P$ and $h$ (not to mention that $g$ also depends on height too). Again, this only works for a perfect fluid: no viscosity, intermolecular forces, or other factors which complicate the situation.

The most general expression for a fluid's motion (under a particular external force) is given by

$$-\nabla P + \nabla \Phi = \rho \vec{a} \tag{3}$$

where $\vec{a}$ is the acceleration of the object, and $\Phi$ is a potential associated with the *conservative* external field.

# 3 Velocity Fields are the Fluid Gods

Often times, it is practical to represent fluid motion as a vector field. As you may know, a vector field is a mathematical quantity which ascribes to every point in space a vector of a particular magnitude and direction. The most fundamental vector field we will discuss is the *velocity vector field*, given by $\vec{V} = u(x,y,z,t)\hat{\imath} + v(x,y,z,t)\hat{\jmath} + w(x,y,z,t)\hat{k}$. This type of analysis, using vector fields and such, is called "Eulerian Analysis". The other method of analyzing fluid flow, the "Lagrangian Analysis", involves tracking individual flows of particles that the fluid is composed of.

One important property that should arise from vector fields is the idea of steady and unsteady flow. Steady flow is defined to be the critical points of the vector field. As this condition is very restrictive, most fluids experience unsteady flows. One can also determine the *streamlines* of a particular velocity field. For a two dimensional field $u(x,y)\hat{\imath} + v(x,y)\hat{\jmath}$, we have that $\tan\theta = \frac{v}{u}$. However, the tangent of the angle is the positional derivative of the field, and thus $\frac{dy}{dx} = \frac{v}{u}$. Such functions which satisfy this differential equation represent a visual path that a fluid takes, when experienced by a particular velocity field.

## 3.1 Acceleration Fields

As a corollary to the velocity field, we can determine the acceleration field of a particular fluid. The acceleration is given by the derivative of the velocity field, so

$$\vec{a} = \frac{d}{dt}\left(u(x,y,z,t)\hat{\imath} + v(x,y,z,t)\hat{\jmath} + w(x,y,z,t)\hat{k}\right) \tag{4}$$

By the chain rule of multivariable calculus, this becomes

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + \left(\frac{\partial u}{\partial x}v_x + \frac{\partial u}{\partial y}v_y + \frac{\partial u}{\partial z}v_z\right)\hat{\imath} + \left(\frac{\partial v}{\partial x}v_x + \frac{\partial v}{\partial y}v_y + \frac{\partial v}{\partial z}v_z\right)\hat{\jmath} + \left(\frac{\partial w}{\partial x}v_x + \frac{\partial w}{\partial y}v_y + \frac{\partial w}{\partial z}v_z\right)\hat{\jmath} \tag{5}$$

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \tag{6}$$

Where we make use of a new notation, known as the *material derivative*. This notation will be extremely useful and lucrative for us.

# 4 Buoyancy

When we force an object into a fluid, we see that it wants to push back upwards. We call this restoring force a *buoyant force*. The Buoyant Force is defined as the force exerted on the object that restores it to equilibrium, with a magnitude equal to the weight of the fluid displaced by the object. Or, mathematically, $|\vec{F}_B| = \rho_f g V_{sub}$.

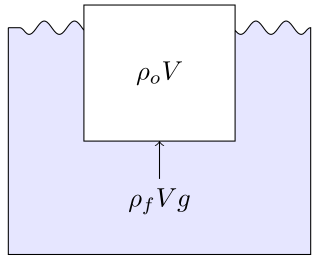


Figure 2: The free body diagram for a body of density $\rho_o$ in a fluid of density $\rho_f$.

We can use the principle of buoyancy to determine the floating/sinking conditions for a particular system. If an object were immersed in a fluid with a density $\rho_f$, and the object's density is $\rho_o$, a free body diagram for the object would look like that in Figure 2. Thus, balancing gravitational and buoyant forces gives

$$\rho_o V g - \rho_f V g = \rho_o V a \tag{7}$$

Dividing, we get

$$\rho_o - \rho_f = \rho_o \frac{a}{g} \tag{8}$$

From here, it's easy to see that if $\rho_o > \rho_f$, the object will have a tendency to float, and vice-versa for sinking. However, the condition where $\rho_o = \rho_f$ warrants some consideration, as this represents the case where the object remains fully submerged in the fluid, but does not sink. Again, all this should agree with our intuition.

# 5 Conservation of Mass: Reynold's Transport Theorem and the Continuity Equation

The idea of conserving mass is quite fundamental: so fundamental that we included it in a lecture on the conservation of momentum. However, mass conservation appears just as often in fluid dynamics, except under a more fancy name: the *Reynold's Transport Theorem*.

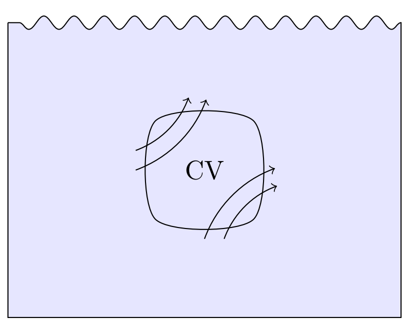


Figure 3: An example of a control volume in a fluid.

Let us first start out by defining what is known as a *control volume* (see Figure 3). In essence, a control volume is an artificial construct (a closed surface) which acts as a placeholder to following the fluid's motion. This control volume is fixed, which means that the fluid in the control volume will likely be displaced after a particular time. By conserving the mass of our system, we see that, after a small amount of time $\Delta t$ from when the control volume was initially "full",

$$M_{sys} = M_{CV} + M_{leak} \tag{9}$$

where $M_{leak}$ represents the mass that leaked out of the control volume. Notice that this works for any extensive property $B$. Now, if we take a derivative of both sides, we get

$$\frac{DM_{sys}}{Dt} = \frac{\partial M_{CV}}{\partial t} + F_{leak} \tag{10}$$

where $F_{leak}$ represents the flux of the velocity field through the control volume. Writing the total mass as an integral of a density function,

$$\boxed{\frac{DM_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, dV + \iint_{\partial CV} \rho\vec{V}\cdot d\vec{S}} \tag{11}$$

where $V_c$ represents the control volume. Notice that we utilized the material derivative notation: this is because there is a particular time dependence of mass in the control volume, as well as positional dependence. This comes straight from the fact that the velocity vector field is a multivariable function.

Since our control volume is defined as a closed surface, the divergence theorem gives us

$$\frac{DM_{sys}}{Dt} = \iiint_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] dV \tag{12}$$

In most cases, our system's mass remains conserved, so we get

$$-\frac{\partial \rho}{\partial t} = \nabla\cdot(\rho\vec{V}) \tag{13}$$

This is a fundamental equation of mass conservation in fluid dynamics. There are some fluids which are dubbed as *incompressible*: that is, the material density of a particular point in the fluid remains constant as it evolves across its trajectory. As a corollary to $\frac{D\rho}{Dt} = 0$, one can show that the divergence of an incompressible fluid's velocity field is 0 (see problem set).

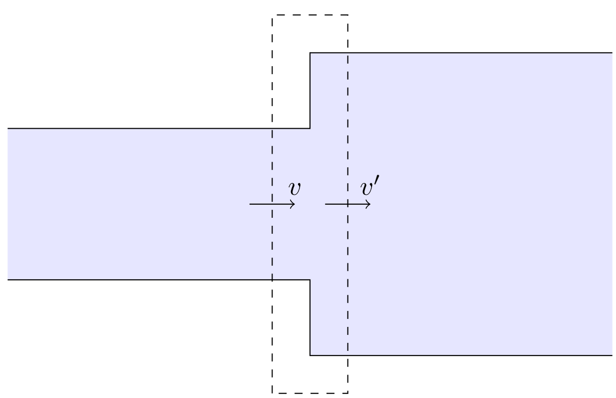


Figure 4: Two pipes of different sizes with a control volume (dashed) surrounding the junction.

Let us consider a simple example of Reynold's Theorem. A pipeline with a cross sectional area $A$ is attached to a larger pipe with an area $2A$. We would like to determine the flow of water through this construct if it was initially sent through the small tube at a velocity $v$. If we consider our control volume to be a box around the junction between the two tubes, we should first take note that the material derivative of the mass of the total system is 0. If we make our box small enough, we can evolve the system such that our control volume won't lose mass, and the first partial derivative is rendered null (see Figure 4). Thus

$$\iint_{\partial CV} \rho\vec{V} \cdot d\vec{S} = 0 \tag{14}$$

This surface integral is simple, because we have chosen our surface such that only two of its faces contribute. Thus, we get

$$\rho(2A)v' - \rho Av = 0 \rightarrow v' = \frac{v}{2} \tag{15}$$

The idea that the surface integral is 0 is obtained by judiciously choosing our control volume. Such a result is known as the *Continuity Equation*.

# 6 Conservation of Momentum

When discussing the conservation of mass, we noted that Reynold's transport theorem can apply to any extensive property of a fluid. If we apply this idea to momentum, we get

$$\left(\frac{D\vec{p}}{Dt}\right)_{CV} = \frac{\partial}{\partial t} \iiint_{CV} \rho\vec{b}\, dV + \iint_{\partial CV} \rho\vec{b} \left(\vec{V} \cdot d\vec{S}\right) \tag{16}$$

where $\vec{b}$ is a particular intensive property of the system associated with the extensive property. In this case, we choose $\vec{b} = \vec{V}$ in order for our statement to be dimensionally correct, thus obtaining

$$\left(\frac{D\vec{p}}{Dt}\right)_{CV} = \frac{\partial}{\partial t} \iiint_{CV} \rho\vec{V}\, dV + \oiint_{\partial CV} \rho\vec{V} \left(\vec{V} \cdot d\vec{S}\right) \tag{17}$$

If we choose our control volume most judiciously, we can determine the rate of change of the system's total momentum. Noting that the rate of change of momentum is the sum of the external forces from Newton's Second law, we arrive at the following equation for momentum conservation

$$\boxed{\left(\sum \vec{F}_{ext}\right)_{CV} = \frac{\partial}{\partial t} \iiint_{CV} \rho\vec{V}\, dV + \oiint_{\partial CV} \rho\vec{V} \left(\vec{V} \cdot d\vec{S}\right)} \tag{18}$$

There are a couple simple applications that can be illustrated by this principle

## 6.1 Forcing the Continuity Equation Back

We return to our discussion of the continuity equation, where two pipes of unequal areas ($A$ and $2A$) are attached together. As one may know, there is an external force that exists which pushes the configuration as the water abruptly shifts between the two pipes. This external force is given by

$$\vec{F}_{ext} = \oiint_{\partial CV} \rho\vec{V} \left(\vec{V} \cdot d\vec{S}\right) \tag{19}$$

because the internal momentum of our control volume (the same as before) does not change with time. The surface integral is trivialized by choice of control volume: the only non-zero contributions come from the cross sections of either pipe. This gives us

$$\vec{F}_{ext} = \rho(2A)\left(v'\right)^2 - \rho Av^2 \tag{20}$$

Since we deduced that $v' = v/2$,

$$\vec{F}_{ext} = \rho(2A)\frac{v^2}{4} - \rho A v^2 = -\frac{\rho A v^2}{2}\hat{\imath} \tag{21}$$

As expected, this is a reactionary force, and pushes the tubes back by Newton's 3rd law. Thus, one must supply such a force in the opposite direction to prevent the setup from moving.

## 6.2 Pipe in a Twisty Situation

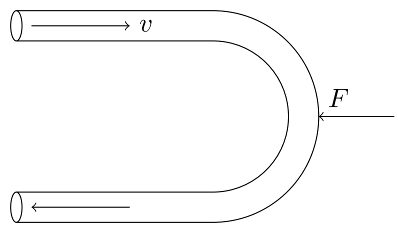


Figure 5: The force $F$ keeps the pipe stable as water moves through with velocity $v$.

Now, we will consider a pipe in a U-shaped bend. Water is sent through the pipe with a velocity $v$, and we are asked to determine the external force acting on the pipe, and the reactionary force needed to keep the pipe stable (see Figure 5). Applying the conservation of momentum, we get

$$\vec{F}_{ext} = \oiint_{\partial CV} \rho\vec{V}\left(\vec{V}\cdot d\vec{S}\right) \tag{22}$$

like before. In this case, the surface integrals are trivial too, if we choose our control volume to consist of the pipe itself. However, signs can be tricky...

$$\vec{F}_{ext} = \oiint_{\partial CV} \rho\vec{V}\left(\vec{V}\cdot d\vec{S}\right) = \rho v(vA) + \rho(-v)(-vA) = 2\rho v^2 A \tag{23}$$

Which is a result that you may have seen before when working with the conservation of momentum with changing mass.

# 7 Conservation of Energy and Bernoulli's Equations

The idea of the Conservation of Energy comes from the 1st law of thermodynamics, which we will talk about in great deal a bit later. Explicitly, the first law of thermodynamics states that

$$\dot{E} = \dot{Q}_{CV} + \dot{W}_{CV} \tag{24}$$

where $Q$ represents the heated added the system, and $W$ represents the work done to the system. According to Reynold's transport theorem for an arbitrary extensive property $\vec{B}$,

$$\left(\frac{DB}{Dt}\right)_{CV} = \frac{\partial}{\partial t}\iiint_{CV} \rho b\, dV + \iint_{\partial CV} \rho b\left(\vec{V}\cdot d\vec{S}\right) \tag{25}$$

Since we will choose $B = E$, we will also choose $b = e$, where $e$ represents the energy per unit mass of the fluid in the chosen control volume. Using the first law of thermodynamics, we have

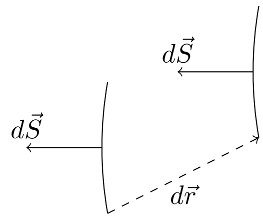


Figure 6: External pressure acting on a differential fluid element does work $\vec{F} \cdot d\vec{r} = P\, d\vec{S} \cdot d\vec{r}$.

$$\dot{Q}_{CV} + \dot{W}_{CV} = \frac{\partial}{\partial t} \iiint_{CV} \rho e \, dV + \iint_{\partial CV} \rho e \left( \vec{V} \cdot d\vec{S} \right) \tag{26}$$

In general, heat is added to the system by an external source. However, the work done to the system comes from two distinct sources: a propeller or pump which converts mechanical energy to the fluid and the external pressure which tries to conform the fluid in a particular manner. The second source of work is particularly easy to mathematically express. Consider a situation like in Figure 6, where a parcel of fluid with a mass $dm$ is being pushed by an external pressure field. The $\hat{n}$ vector points in the direction normal to the surface at the parcel. The work which is done to this parcel is $\delta W = P\, d\vec{S} \cdot d\vec{r}$. The rate of change of this quantity is $\delta \dot{W} = -P\, d\vec{S} \cdot \vec{V}$. The negative sign is introduced because the pressure acts inwardly normal to the surface. Thus, the total work done by external pressures is

$$\dot{W}_{pext} = - \oiint_{\partial CV} P \left( \vec{V} \cdot d\vec{S} \right) \tag{27}$$

We can thus simplify our energy equation and write [1].

$$\boxed{\left( \dot{Q} + \dot{W}_{mech} \right)_{CV} = \frac{\partial}{\partial t} \iiint_{CV} \rho e \, dV + \oiint_{\partial CV} \rho \left( e + \frac{P}{\rho} \right) \vec{V} \cdot d\vec{S}} \tag{28}$$

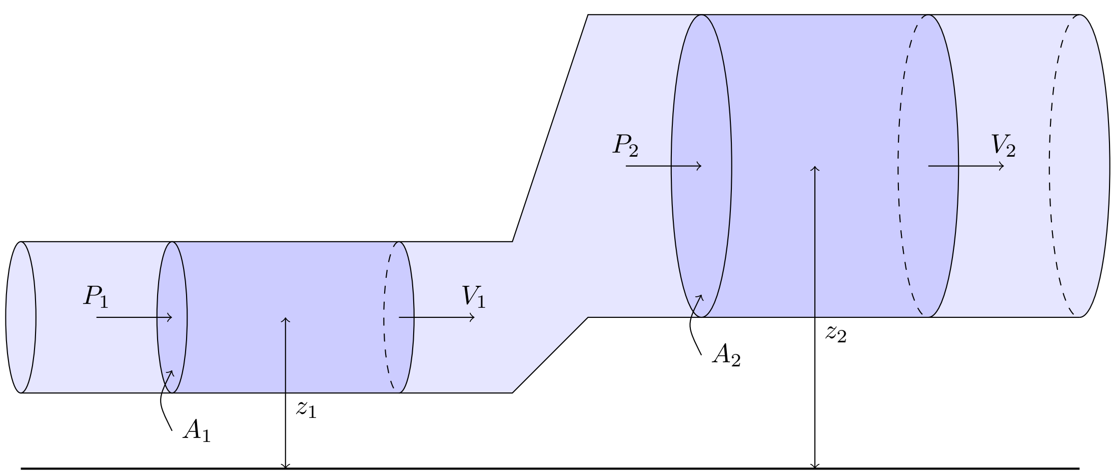


Figure 7: A tube of fluid moving to the right, with pressure, velocity, height, and cross-sectional area indicated.

[1] You may notice that we are abusing notation, interchanging plain surface integrals with closed surface integrals. This abuse is not pernicious, however, because our control volume is always a closed surface

This is the fundamental equation of energy conservation. In general, $e$ refers to the energy per unit mass that is contained in the control volume. If we assume that this includes internal energy (associated with the motion of individual fluid molecules), kinetic, and gravitational potential energy, we can say that $e = u + \frac{V^2}{2} + gz$. If we were to consider the following system in Figure 7, we can apply the conservation of energy. We do not have any mechanical devices or sources of heat transfer, so $\dot{Q} + \dot{W}_{mech} = 0$. Furthermore, the first integral is 0 because the energy content of the system does not change with time. Thus, we conclude that the flux across the control volume (defined to the the surface of the construct) must be 0. By now, these surface integrals are obvious, and we get

$$\left(u + \frac{V_1^2}{2} + gz_1 + \frac{P_1}{\rho}\right)\rho A_1 V_1 = \left(u + \frac{V_2^2}{2} + gz_2 + \frac{P_2}{\rho}\right)\rho A_2 V_2 = constant \tag{29}$$

From the continuity equation, $\rho AV = constant$. Thus, we arrive at the famed Bernoulli's equation, that $\frac{V^2}{2} + gz + \frac{P}{\rho} = constant$.

# 8 Navier-Stokes Equation

When conservation principles are not enough to solve a problem, sometimes we have to resort to the Navier-Stokes equations. This is a nonlinear partial differential equations that gives an expression for the material derivative of the velocity field:

$$\boxed{\rho\frac{D\vec{V}}{Dt} = -\nabla P + \mu\nabla^2\vec{V} + \frac{1}{3}\mu\nabla(\nabla\cdot\vec{V}) + \vec{F}} \tag{30}$$

We have used $\mu$ to represent the viscosity of the fluid, a factor indicating the strength of internal frictional forces, and $\vec{F}$ to represent an external force. For incompressible fluids, the equation reduces somewhat to

$$\rho\frac{D\vec{V}}{Dt} = -\nabla P + \mu\nabla^2\vec{V} + \vec{F} \tag{31}$$

because incompressible fluids have divergence-free velocity fields (see problem 5). While we will not discuss the derivation of these equations here, we can give a qualitative explanation for each term.

On the left hand side, we have $\rho\frac{D\vec{V}}{Dt}$. This is a mass-like factor multiplied with a time derivative of velocity, so we can think of it as the force. The first term on the right hand side, $-\nabla P$, is a pressure term representing resistance to motion due to normal stresses: the fluid pushes on itself to resist compression. The next two terms, $\mu\nabla^2\vec{V} + \frac{1}{3}\mu\nabla(\nabla\cdot\vec{V})$, are collectively called the *stress deviator tensor*. These terms are responsible for the viscous flows and turbulence that give the Navier-Stokes equations their notorious complexity. The final term, $\vec{F}$, is simply an external force acting on the fluid.

The pressure term and the stress deviator tensor are actually a decomposition of the stress tensor. The pressure term is also called the *volumetric stress tensor*, and is associated with resistance to changes in volume. The stress deviator tensor represents all the deviations from this purely volumetric behavior, i.e., forces that distort the fluid. The particular expressions in the equations above are derived for the particular (and common) case of Newtonian fluids.

We can actually solve the Navier-Stokes equations for the very simple case of water sitting in a pool. Since the initial velocity field is everywhere zero, the viscous terms vanish. Then we simply have

$$\rho\frac{D\vec{V}}{Dt} = -\nabla P + \vec{F}. \tag{32}$$

The only external force is gravity, so $\vec{F} = -\rho\vec{g}$. We found in section three that $\nabla P = -\rho\vec{g}$ as well, so the right hand side is zero. Therefore, the velocity field is not changing with time, as we would expect.

As we mentioned earlier, the Navier-Stokes equations are very difficult to work with, both analytically and computationally. As a mathematical problem, it made the Clay Mathematics list of the seven Millennium Prize problems. Specifically, you can win $1,000,000 for proving or disproving the statement: "in three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations." Even when we need only a numerical approximation to a solution, the Navier-Stokes equations are unyielding: methods of approximation can be extremely laborious, often requiring supercomputers for sufficiently large systems. But a million bucks is a million bucks: who knows, maybe YOU will be its recipient...

# 9 Problems

1. Consider our atmosphere to be made up of an ideal gas with uniform temperature. Argue, from the ideal gas law, that the pressure will exponentially die down as a function of height.

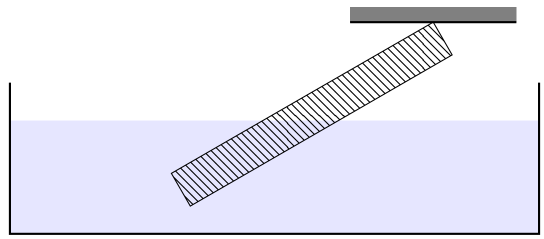

Figure 8: The situtation described in problem 2.

2. (F = ma 2013) A uniform rod is partially in water with one end suspended, as shown in figure 8. The density of the rod is 5/9 that of water. At equilibrium, what portion of the rod is above water?

3. When we discussed the conservations of mass, energy, and momentum, we often implied that our control volume was stationary. If our control volume is moving with a particular velocity, then all we need to do is replace the velocity vector field with a relative velocity vector field in the reference frame of the control volume. Using this fact, derive the rocket equation from a fluid dynamics conservation of momentum perspective.

4. Consider the vector field $v = \langle 3x^2yt^2, 5y^4x^6t^3\rangle$. Determine (a) streamlines and (b) the acceleration field.

5. From the continuity equation, prove that the divergence of the velocity vector field is 0 if the material derivative of the density is 0. Hint: expand out the material derivative in terms of its definition and combine with the continuity equation. Such a fluid is called incompressible.

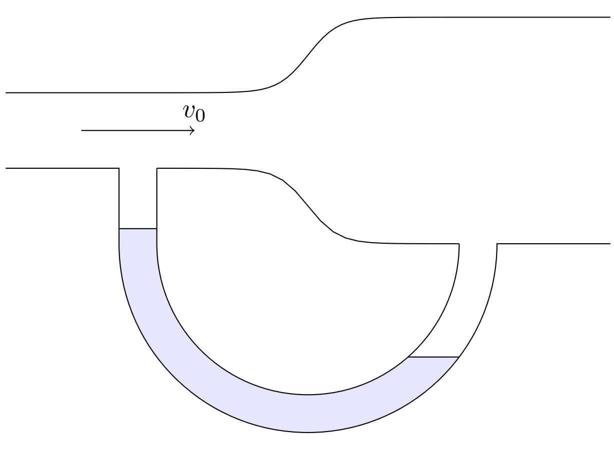


Figure 9: A Venturi meter, in which air flowing through a constriction creates a pressure differential.

6. Consider a Venturi meter in Figure 9. Determine the pressure difference between points $a$ and $b$ assuming that the initial velocity of the fluid is $v_0$. Assume that the cross sectional area goes from $A$ to $2A$.

7. We call a fluid's vorticity $\vec{\omega} = \nabla \times \vec{V}$. Using this definition, prove Kelvin's theorem: the material derivative of the circulation of a fluid $\Gamma = \oint_C \vec{\omega} \cdot d\vec{\ell}$ is zero, assuming that all potentials are derived from conservative forces[2]. Ignore viscosity, and other such effects, and assume that the *density of the fluid is only a function of pressure*. Hint: use the definition of the material derivative, as well as the relationship between pressure and potentials.

8. In the last problem, we proved Kelvin's theorem as viewed in a *stationary* reference frame. Now, consider a rotating reference frame, where $\vec{v}_{rot} = \vec{v}_{stat} + \vec{\Omega} \times \vec{r}$. Derive a modified version of Kelvin's theorem using this standpoint. Consider the following problem: a vat of water is flowing into a water drain. The vat's radius is $r_0$, and the volumetric flux rate of water through the drain is $\gamma$. If the tangential velocity at the rim of the vat is initially $v_0$, determine the velocity vector field of the water as a function of position and time. Use the rotated-reference frame version of Kelvin's theorem since we know that the rotation of earth results in the Coriolis-like effect of this scenario.

[2] Kelvin's theorem is essentially a conservation principle for circulating currents, or vortices. If the air had no viscosity, for example, then smoke rings would continue on indefinitely. This remarkably property led Kelvin to hypothesize that atoms are vortices in the ether. This theory was developed by J.J. Thomson, but is only of historical interest now.