Periodic Physics Problems 1 – October

Bryan Lu, Rubaiya Emran, Rohan Venkat, Sohom Paul

October 2019

This is the year's first set of Periodic Physics Problems! Take your time to think about these problems carefully and express your reasoning neatly. You can answer as much or as little as you want in your solutions – we will give feedback on whatever you submit. Note that we definitely don't expect anyone to be able to answer every single question we put on here, so don't be afraid to submit incomplete work or ask questions you still have. Please give us feedback also on these problems – include some sort of difficulty rating so we can adjust accordingly in the future, or ask if we've got something wrong. We want to be able to help you out if you get stuck or need something clarified, so please don't hesitate to ask questions and reach out to us via e-mail or on Messenger!

Submissions that will be considered for our leaderboard will be due by November 2nd, 2019, 11:59 pm, to our email `tjhsstphysicsteam@gmail.com`. Solutions can be LaTeXed, or handwritten (legibly, please!). We will consider the team you attend most frequently as the set of problems we grade for our leaderboard. We will post solutions to these problems at the end of the month, so you can look over these difficult problems. :)

C Team – Calculus

Recall the *derivative* of a function $f(x)$ (or the slope of its tangent line at x is defined by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},\text{ provided the limit exists.}$$

Problem 1. Given $f(x) = 3x + 4$, a) find $f(x + h)$ in terms of x and h. b) Find the derivative of $f(x)$ at $x = 3$. c) Is the derivative different for different x values? Sketch the function. What is it about this function that makes it have the same or different derivative everywhere?

Problem 2. Given $f(x) = x^2$, a) find $f(x + h)$ in terms of x and h. b) Find the derivative of $f(x)$ at $x = 3$. c) Does every x share the same derivative? Sketch the function. What is it about this function that makes it have the same or different derivative everywhere?

Problem 3. What is the difference between displacement and distance? What is the difference between velocity and speed?

Problem 4. Rohan's car is travelling at a velocity of 5 m/s. a) Draw a graph of the velocity versus time from time $t = 0$ to $t = 5$. b) Explain what the area under the curve of represents. (Hint: not distance!)

Problem 5. Suddenly, the velocity of Rohan's car is now a function of time: $v(t) = 5t + 4$. a) Draw a graph of the velocity from time $t = 0$ to $t = 5$. b) The quantity "acceleration" is defined to be the derivative of velocity with respect to time, $\frac{dv}{dt}$. Find the acceleration of the car.

Problem 6 (Challenge)**.** For a function in the form $f(x) = ax + b$, prove the derivative $\frac{df}{dx} = a$.

Problem 7 (Challenge)**.** What does the area under an acceleration vs. time graph represent?

B Team – Harmonic Oscillators and Damping

This month, we'll investigate the harmonic oscillator, and a complication of the standard analysis with a phenomenon that we call *damping*.

Our standard setup will have a spring with spring constant k hanging from a ceiling. The spring will have some equilibrium length as it hangs from the ceiling. We'll also have a mass m, which, when attached to our spring, will stretch the spring vertically, which will lower the equilibrium point of the system. Assume that the mass is constrained to move vertically in one dimension, and that (for now) the system doesn't experience any resistive forces. Take down as the positive direction when dealing with directed quantities.

Problem 1. Set up a free-body diagram with the two relevant forces on the system. If the mass is at rest, how much does the mass stretch the spring from its equilibrium length? (Express your answer in terms of m, k, and any other physical constants.)

Throughout the remainder of this scenario, let $x(t)$ be the displacement of the mass from the unstretched point in the spring, and let $\omega = \sqrt{\frac{k}{m}}$.

Problem 2. Just for this problem, near the equilibrium point of the mass-spring system, we'll use the reference frame of the mass itself to figure out how our mass moves. Let $\tilde{x}(t)$ be the displacement from this new equilibrium point. Find $\tilde{x}(t)$ in general as a function of time by setting up Newton's Second Law, and recognizing the solution to the differential equation.

Problem 3. How do we get $x(t)$ in terms of $\tilde{x}(t)$? Write down $x(t)$ as a function of time t, k, m, ω, or any other physical constants (you don't have to use all of these).

Now, we introduce the damping. Normally, a drag force acting on the object would be proportional to the velocity of the object squared, but we'll simplify this and say that the magnitude of our drag force is directly proportional to the speed of our particle, with constant of proportionality $m\gamma$. Note $\gamma \geq 0$.

Problem 4. What are the standard SI units of γ?

Problem 5. Carefully set up Newton's Second Law to get a differential equation in $x(t)$, and divide out any overall factors. Your answer should be of the form $\frac{d^2x}{dt^2} + A\frac{dx}{dt} + Bx = C$, where A, B, and C are appropriate constants.

This is not an easy differential equation to solve, unfortunately. I'll present a standard way to solve this kind of equation with numbers, and I'll leave it to you to apply the method to your answer to Problem 5.

Consider the differential equation $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 5$. One standard way to solve this equation is to instead consider the equation $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0$ first – the reason for this is because any solution to this modified equation, when added to a solution of the first, will produce another solution the first equation (verify this for yourself!). In this modified equation, if we guess $x(t) = e^{rt}$, look at what we get:

$$r^2e^{rt} + 4re^{rt} + 3e^{rt} = (r^2 + 4r + 3)e^{rt} = 0$$

This is an equation we can solve! The quadratic in r has roots at -3 and -1, so any multiple of e^{-t} and any multiple of e^{-3t} will work for the modified equation.

Now, we just need to get some function that satisfies $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 5$. It's clear that $x(t) = \frac{5}{3}$ works, so our general solution is $x(t) = c_1e^{-3t} + c_2e^{-t} + \frac{5}{3}$. c_1 and c_2 will be determined based on the initial conditions of our system.

Problem 6 (Challenge)**.** Your turn! Use this method to solve the differential equation in Problem 5. There are some difficult caveats you'll have to work around – what happens if the roots of the quadratic are complex? If they're the same? What do your solutions look like as functions of time?

A Team – Thermodynamics and Heat Cycles

This month, we'll look at other heat cycles not discussed in the lectures that see practical use. Carnot gets all of the credit for one of the simplest heat cycles to analyze, but we'll see other cycles that enjoy practical use and are ideally reversible. In all of the below scenarios, assume we are dealing with one mole of a monatomic ideal gas, and that all processes are reversible unless otherwise stated. Draw all cycles as doing positive work on diagrams, and **express all answers in terms of R, not k_B, if needed.**

In case this isn't emphasized enough – recall that *isothermal* processes occur at constant temperature, *isobaric* processes occur at constant pressure, *isochoric* processes occur at constant volume, and *isentropic* processes occur at constant heat.

Problem 1. First, a remark about why the Carnot cycle gets all this hype–consider a TS diagram, which has temperature T on the vertical axis and entropy S on the horizontal axis. Draw a Carnot cycle on a TS diagram – do you see now why it sets a nice standard for all the other cycles?

For a full score, choose three of the following five cycles. Draw a PV diagram, a TS diagram, and compute any other quantities listed. Express all answers in terms of given quantities of the cycles and other physical constants. If you do more than three, mark the three you want to be graded.

The following two cycles that include a *regenerator* heat up and cool down the gas, but this heat is not counted as a part of the input heat to the system.

Problem 2. The *Ericsson cycle* is an ideal cycle that undergoes the following four thermodynamic processes in order, starting at temperature T_1 and pressure P_1:

1. Isothermal compression. The pressure increases to a higher value P_2.
2. Isobaric heat addition. The gas passes through a regenerator during this phase, which heats up the gas to temperature T_2.
3. Isothermal expansion. The pressure decreases back to pressure P_1.
4. Isobaric heat removal. The gas passes back through the regenerator, and returns to temperature T_1.

Compute the work done by one loop of the cycle, and the heat transferred by the regenerator to and from the gas.

Problem 3. The *Stirling cycle* is an ideal cycle that undergoes the following four thermodynamic processes in order, starting at temperature T_1 and volume V_1:

1. Isochoric heat addition. The gas passes through a regenerator, heating the gas to a temperature T_2.
2. Isothermal expansion. The volume increases to V_2.
3. Isochoric heat removal. The gas returns heat to the regenerator, cooling the gas back to T_1.
4. Isothermal compression. The volume decreases to V_1.

Compute the work done by one loop of the cycle, and the heat transferred by the regenerator to and from the gas.

Problem 4. The *Brayton cycle* is intimately related to the Ericsson cycle – it has the same structure, but instead of isothermal processes, it uses isentropic processes. It the cycle model used in gas turbines. Let the gas start at pressure P_1, volume V_1, and temperature T_1. The four steps in the cycle are:

1. Adiabatic (isentropic) compression. The pressure increases to a higher value P_2, decreases to volume V_2, and reaches temperature T_2.
2. Isobaric heat addition. The gas passes through a combustion chamber which uses fuel to heat up the gas to a higher temperature T_3, expanding it to volume V_3.
3. Adiabatic (isentropic) expansion. The pressure decreases back to pressure P_1 as it passes through a turbine,

cooling to temperature T_4.

4. Isobaric heat removal. The gas returns to temperature T_1.

Let $\frac{P_2}{P_1} = \alpha > 1$, and $\frac{V_3}{V_2} = \beta > 1$. Compute, in terms of T_1, α, β, and other physical constants, the work done by the engine and the heat the combustion chamber gives to the gas.

Problem 5. The *Otto cycle* is a model of an internal combustion engine. This is commonly found in car engines, where a spark plug ignites a fuel (gas/air mix) using four strokes. Let the gas start at pressure P_1, volume V_1, and temperature T_1. The stages in this cycle are:

1. Adiabatic (isentropic) compression of the fuel to pressure P_2, volume V_2, and temperature T_2.
2. Isochoric heat addition to a higher temperature T_3. This stage represents an ignition of the fuel in the chamber by the spark plug.
3. Adiabatic (isentropic) expansion of the fuel back to volume V_1, allowing the fuel to reach temperature T_4.
4. Isochoric heat removal, as the spent fuel is exhausted and the gas returns to temperature T_1.

Let $\frac{P_3}{P_2} = \alpha > 1$, and $\frac{V_1}{V_2} = \beta > 1$. Compute, in terms of T_1, α, β, and other physical constants, the work done by the engine and the heat the spark plug gives to the gas.

Problem 6 (Challenge)**.** The *Diesel cycle* is a model of an internal combustion engine, also used in transportation. The fuel is ignited by injecting it into a hot air mixture. Let the gas start at pressure P_1, volume V_1, and temperature T_1. The four stages in this cycle are:

1. Adiabatic (isentropic) compression of the fuel to pressure P_2, volume V_2, and temperature T_2.
2. Isobaric heating of the fuel to volume V_3.
3. Adiabatic (isentropic) expansion of the fuel to pressure P_3, volume V_1, and temperature T_4.
4. Isochoric cooling of the fuel to pressure P_1.

Let $\frac{V_3}{V_2} = \alpha > 1$, and $\frac{V_1}{V_2} = \beta > 1$. Compute, in terms of T_1, α, β, and other physical constants, the work done by the engine and the heat given to the gas.

Problem 7. With all of this analysis, consider another restatement of the Second Law of Thermodynamics:

> "Any heat engine operating between minimum and maximum temperatures T_1 and T_2 (with $T_1 < T_2$) has maximum efficiency of $\eta = 1 - \frac{T_1}{T_2}$, given by the Carnot cycle."

Compute the efficiency η for every non-Carnot cycle you chose in terms of the temperatures of two of the states in each cycle. Is this consistent? Given what you know about entropy and the Carnot cycle, why is this equivalent to the Kelvin-Planck Postulate and the Clausius Postulate?

Hopefully, by the end of this, you've seen why Carnot wears the crown as the king of all heat engines!