Vector Calculus

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1 Vector Operations and Definitions

We'll be working with vectors in 3D space. We will, in the context of this lecture, define any vector $\mathbf{a} = \langle a_x, a_y, a_z \rangle$ and define the vector's magnitude to be a. Then we define the dot product $\mathbf{a} \bullet \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = ab\cos\theta$ and the cross product $\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$ with a magnitude of $ab\sin\theta$. Note that the dot product is a scalar while the cross product is a vector. From the trignometric definition of the dot product, we can see that it is the product of a and the magnitude of the component of $\mathbf{b}$ in the direction of $\mathbf{a}$, or vica-versa. From the trignometric definition of the magnitude of the cross product, we see that the magnitude of the cross product is the product of a and the magnitude of the component of $\mathbf{b}$ in the direction perpendicular to $\mathbf{a}$, or vica-versa. These are important so don't forget them.

There is one vector that pervades this whole subject. It is $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. This is called the del (the fools couldn't spell Dr. Dell's name right) or, if you're Canadian, the nabla. It has no geometric meaning, but acts as an operator. We can do several things with it. If you have a scalar function $f(x, y, z)$, then $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ is called the gradient of f and this vector points in the direction in which the instantaneous rate of decrease for f is greatest. If you have a vector function $\mathbf{F}$, then $\nabla \bullet \mathbf{F}$ is called the divergence and $\nabla \times \mathbf{F}$ is called the curl of $\mathbf{F}$.

2 Line Integrals and Conservative Vector Fields

Consider a vector function defined in 3-space. That is, a vector-valued function $\mathbf{F}(x, y, z)$ that takes on a different value at each point in space. We define the line integral as $\int \mathbf{F} \bullet \mathbf{dr}$. What this is saying is that, for any given path in space defined by the position function $\mathbf{r}(t)$, for every infinitesimal change in $\mathbf{r}$ ($\mathbf{dr} = \mathbf{r}(t + dt) - \mathbf{r}(t)$), we are taking the dot product of $\mathbf{F}$ and $\mathbf{dr}$ (the component of $\mathbf{F}$ in the direction of motion)and summing all of these dot products. We need a practical way to evaluate this integral. Thus, $\int \mathbf{F} \bullet \mathbf{dr} = \int \mathbf{F} \bullet \frac{\mathbf{dr}}{dt} dt$. This we can evaluate because we have $\mathbf{F}$ and $\frac{\mathbf{dr}}{dt}$.

A conservative vector field $\mathbf{F}$ is one in which the line integral around a closed loop is zero. This seems like an inherent property of vector fields, but its not (convince yourself of this by finding a non-conservative vector field). In any case, let us analyze these conservative fields because, as it turns out, they have some neat properties. Take two points in space, $\mathbf{r}(a)$ and $\mathbf{r}(b)$. Then, by our closed loop requirement, we have that $\int_{\mathbf{r}(a)}^{\mathbf{r}(b)} \mathbf{F} \bullet \mathbf{dr} + \int_{\mathbf{r}(b)}^{\mathbf{r}(a)} \mathbf{F} \bullet \mathbf{dr} = 0$. Note that nowhere have we specified the path taken between $\mathbf{r}(a)$ and $\mathbf{r}(b)$. It is because the statement is true regardless of path. This implies that $\int_{\mathbf{r}(a)}^{\mathbf{r}(b)} \mathbf{F} \bullet \mathbf{dr}$ is path independent, that is, it only depends $\mathbf{r}(a)$ and $\mathbf{r}(b)$. Thus, we can define a function $f = \displaystyle\int_{\mathbf{O}}^{\mathbf{r}} \mathbf{F} \bullet \mathbf{dr}$ and we call this function the potential function. The potential is a function only of position $\mathbf{r}$ and the point $\mathbf{O}$ is just a reference point from which this potential is defined. Note that potential function is adjustable by a constant depending on the choice of $\mathbf{O}$. A neat property of f is that $\nabla f = \mathbf{F}$ and you should derive this result on your own.

Because $\nabla f = \mathbf{F}$, we can find f quite easily from $\mathbf{F}$. If $\mathbf{F} = \langle F_x, F_y, F_z \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$, then we know that $\int F_x dx + e(y,z) = \int F_y dy + g(x,z) = \int F_z dz + h(x,y) = f$ where e, g, h are abritrary functions, the constants of integration. Solving this equation gives you f. If the equation has no solution, then you know that $\mathbf{F}$ is not conservative.