Introduction and Problems in Forces - Solutions

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1 Kinematics Problems

1. Consider a projectile launched at an angle θ from a horizontal ground with velocity v_0. Find the angle θ that maximizes

   (a) The horizontal distance traveled (the range)
   Solution: The height of the particle is given as a function of time by $y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$. Letting $y_0 = 0$ and substituting $v_{0y} = v_0 \sin\theta$, we have

   $$y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2$$

   Solving for $y(t) = 0$, we obtain $t = 0$ and $t = \frac{2v_0 \sin\theta}{g}$. We know that the projectile moves at a constant speed in the x direction (because the acceleration is 0), so we have at the end of the trajectory

   $$x(t) = v_0 \cos\theta t = \frac{2v_0 \sin\theta \cos\theta}{g} = \frac{v_0 \sin 2\theta}{g}$$

   This is maximized when $\sin 2\theta = 1$ or $\theta = \pi/4$.

   (b) The time of flight
   Solution: In part (a) we obtained the total time of filght as $\frac{2v_0 \sin\theta}{g}$, which is maximized when $\sin\theta = 1$ or $\theta = \pi/2$.

   (c) the area enclosed by the ballistic trajectory
   Solution: Instead of eliminating t and finding an expression for the height as a function of distance, we find the area by integrating the parametric expressions directly:

   $$\begin{aligned}
   A &= \int_0^{\frac{2v_0 \sin\theta}{g}} y\frac{dx}{dt}\,dt \\
   &= \int_0^{\frac{2v_0 \sin\theta}{g}} \left(v_0 \sin\theta t - \frac{1}{2}gt^2\right) v_0 \cos\theta\,dt \\
   &= \int_0^{\frac{2v_0 \sin\theta}{g}} \frac{t}{2}\sin 2\theta - \frac{t^2}{2}v_0 g\cos\theta\,dt \\
   &= \left[\frac{t^2}{2}\sin\theta\cos\theta - \frac{t^3}{6}v_0 g\cos\theta\right]_0^{\frac{2v_0 \sin\theta}{g}} \\
   &= \frac{2v_0^2\left(3 - 2v_0^2\right)}{3g^2}\sin^3\theta\cos\theta
   \end{aligned}$$

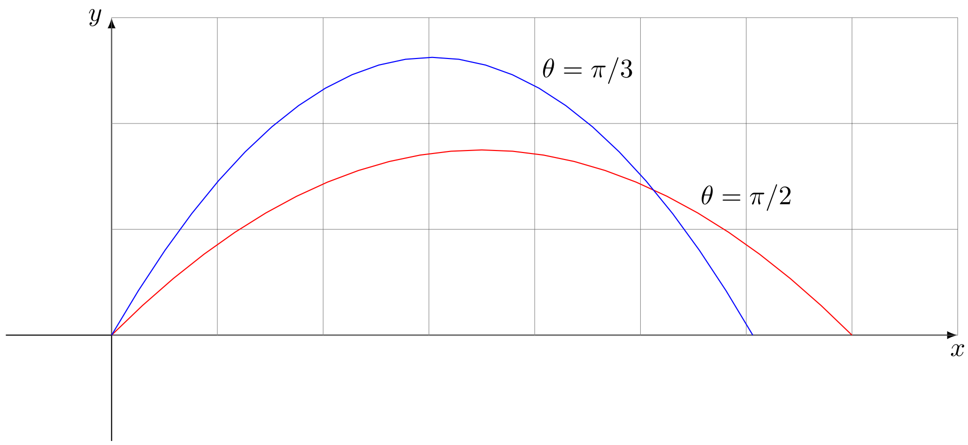

Figure 1: The area under the trajectory is maximized for $\theta = \pi/3$. The trajectories for $\theta = \pi/3$ and $\theta = \pi/4$ are shown in blue and red, respectively.

To find a stationary value of the area, we must solve $\frac{dA}{d\theta} = 0$.

$$0 = \frac{dA}{d\theta} = \frac{2v_0^2\left(3 - 2v_0^2\right)}{3g^2}\sin^2\theta(2\cos 2\theta + 1)$$
$$\sin^2\theta(2\cos 2\theta + 1) = 0$$

The first term is 0 for $\theta = 0$, which is clearly a minimum. This leaves the second term, which vanishes whenever $\cos 2\theta = -\frac{1}{2}$. This yields $\theta = \frac{\pi}{3}$. See Figure 1 for an illustration of this.

(d) The total distance traveled (the arc length)
Solution:[1] We can express the arc length as an integral using the parametric form of the equations of motion:

$$\begin{aligned}
L &= 2\int_0^{v_0\sin\theta/g}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= 2\int_0^{v_0\sin\theta/g}\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}\,dt \\
&= 2v_0\cos\theta\int_0^{v_0\sin\theta/g}\sqrt{1 + \left(\tan\theta - \frac{gt}{v_0\cos\theta}\right)^2}\,dt
\end{aligned}$$

We now let $u = \tan\theta - \frac{gt}{v_0\cos\theta}$ (with $du = -\frac{g}{v_0\cos\theta}dt$) and rewrite the integral as

$$L = -\frac{2v_0^2\cos^2\theta}{g}\int_{\tan\theta}^{0}\sqrt{1+u^2}\,du$$

This integral can be evaluated by letting $u = \sinh\alpha$, or by consulting a table of integrals. The

[1]This solution is adapted from *Introduction to Classical Mechanics* by David Morin.

result is

$$
\begin{aligned}
L &= -\frac{2v_0^2\cos^2\theta}{g}\cdot\frac{1}{2}\left|\left(x\sqrt{x^2+1}+\sinh^{-1}x\right)\right|_{\tan\theta}^{0}\\
&= \frac{v^2\cos^2\theta}{g}\left(\tan\theta\sec\theta+\ln\left(\tan\theta+\sec\theta\right)\right)\\
&= \frac{v^2}{g}\left(\sin\theta+\cos^2\theta\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)\right)
\end{aligned}
$$

We now differentiate with respect to θ:

$$
\frac{dL}{d\theta}=\frac{v^2}{g}\left(\cos\theta-2\cos\theta\sin\theta\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)+\cos^2\theta\left(\frac{\cos\theta}{\sin\theta+1}\right)\frac{\cos^2\theta+(\sin\theta+1)\sin\theta}{\cos^2\theta}\right)
$$

We then set this equal to zero and solve:

$$
\begin{aligned}
0 &= \frac{v^2}{g}\left(\cos\theta-2\cos\theta\sin\theta\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)+\cos^2\theta\left(\frac{\cos\theta}{\sin\theta+1}\right)\frac{\cos^2\theta+(\sin\theta+1)\sin\theta}{\cos^2\theta}\right)\\
0 &= \cos\theta-2\cos\theta\sin\theta\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)+\cos^2\theta\left(\frac{\cos\theta}{\sin\theta+1}\right)\frac{\sin\theta+1}{\cos^2\theta}\\
&= 2\cos\theta\left(1-\sin\theta\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)\right)
\end{aligned}
$$

Clearly there is a stationary value for $\theta=\pi/2$, which gives $L=\frac{v^2}{g}$. The second term can be numerically solved to find $\theta\approx 56.5^\circ$, which one can verify gives the absolute maximum value for L.

2. Repeat problem 1 for the case of a ground inclined at an angle Φ downwards with the horizontal.

   (a) The horizontal distance traveled (the range)
   Solution: We can choose to optimize the horizontal distance or the distance along the ground; both will give the same result. For simplicity, we choose the horizontal distance. The trajectory ends when it intersects the ground, and we can use the parametric equations of motion to determine this point:

$$
\begin{aligned}
y &= -x\tan\Phi\\
v_0\sin\theta\, t-\frac{1}{2}gt^2 &= -\tan\Phi v_0\cos\theta\, t\\
v_0\left(\sin\theta+\cos\theta\tan\Phi\right) &= \frac{1}{2}gt\\
t &= 2\frac{v_0}{g}\left(\sin\theta+\cos\theta\tan\Phi\right)
\end{aligned}
$$

   We now must maximize $x=v_0\cos\theta\, t$.

$$
\begin{aligned}
x &= v_0\cos\theta\left(2\frac{v_0}{g}\left(\sin\theta+\cos\theta\tan\Phi\right)\right)\\
&= \frac{v_0^2}{g}\left(\sin 2\theta+2\cos^2\theta\tan\Phi\right)\\
\frac{dx}{d\theta} &= \frac{v_0^2}{g}\left(2\cos 2\theta-2\tan\Phi\sin 2\theta\right)
\end{aligned}
$$

Setting this derivative equal to zero, we obtain

$$\begin{aligned}
2\cos 2\theta &= 2\tan\Phi\sin 2\theta \\
\cot 2\theta &= \tan\Phi \\
\theta &= \frac{\pi}{4} - \frac{\Phi}{2}
\end{aligned}$$

Note that this solution agrees with problem 1 in the limiting case of $\Phi = 0$.

(b) The time of flight
Solution: We determined the time of flight in part (a). Differentiating with respect to θ,

$$\begin{aligned}
\frac{dt}{d\theta} = 2\frac{v_0}{g}\left(\cos\theta - \sin\theta\tan\Phi\right) &= 0 \\
\cos\theta &= \sin\theta\tan\Phi \\
\theta &= \frac{\pi}{2} - \Phi
\end{aligned}$$

Note that this is simply twice the angle maximizing distance.

(c) The area enclosed by the ballistic trajectory
Solution: We follow essentially the same procedure as in problem 1, except with different bounds on the integral.

$$\begin{aligned}
A &= \int_0^{2v_0(\sin\theta+\cos\theta\tan\Phi)/g} (y + x\tan\Phi)\frac{dx}{dt}\,dt \\
&= \int_0^{2v_0(\sin\theta+\cos\theta\tan\Phi)/g} \left(v_0\sin\theta t - \frac{1}{2}gt^2 + v_0\cos\theta t\tan\Phi\right) v_0\cos\theta\,dt \\
&= \int_0^{2v_0(\sin\theta+\cos\theta\tan\Phi)/g} \frac{t}{2}\sin 2\theta - \frac{t^2}{2}v_0 g\cos\theta + v_0^2\cos^2\theta\tan\Phi t\,dt \\
&= \left[\frac{t^2}{2}\sin\theta\cos\theta - \frac{t^3}{6}v_0 g\cos\theta + \frac{t^2}{2}v_0^2\cos^2\theta\tan\Phi\right]_0^{2v_0(\sin\theta+\cos\theta\tan\Phi)/g} \\
&= \frac{1}{3g^2}v_0^2\sec\Phi\sin(\theta+\Phi)\left(3\sec\Phi\sin 2\theta\sin(\theta+\Phi) - 4v_0^2\cos\theta\sec^2\Phi\sin^2(\theta+\Phi) + 6gv\cos^2\theta\tan\Phi\right)
\end{aligned}$$

To find the angle that maximizes A, we find $\frac{dA}{d\theta}$ and set it to 0. This is best done numerically.

3. A hot air balloon begins accelerating upwards from the ground with acceleration a. At the same time, a cannonball is projected from a cannon a distance L away at an inclination θ. Find the initial velocity v_0 required to impact the balloon.
Solution: We can easily determine the time when the projectile crosses the line of flight of the balloon (assuming its trajectory reaches that far):

$$t = \frac{L}{v_0\cos\theta}$$

We can then use this to determine the height of the projectile and the height of the balloon at this point.

$$\begin{aligned}
y_{\text{proj}} &= v_0 t\sin\theta - \frac{1}{2}gt^2 = L\tan\theta - \frac{gL^2}{2v_0^2\cos^2\theta} \\
y_{\text{balloon}} &= \frac{1}{2}at^2 = \frac{aL^2}{2v_0^2\cos^2\theta}
\end{aligned}$$

Setting these two quantities equal, we obtain

$$L \tan \theta = \frac{(g+a)L^2}{2v_0^2 \cos^2 \theta}$$

$$v_0 = \sqrt{\frac{(g+a)L}{\sin 2\theta}}$$

In addition to this solution, we must add the condition that we assumed from the beginning, that the range of the projectile is at least L:

$$R = \frac{v_0^2}{g} \sin 2\theta = \frac{g+a}{g} L \geq L$$

We see the condition is automatically fulfilled, because $a \geq 0$.