# Conservation of Momentum with Changing Mass Lecture Solutions

Shankar Balasubramanian Ross Dempsey

November 2014

The solution to questions 4 and 7 are not presented. Problem 4 is a classic problem, whose solution can easily be found by cruising the Internet. Problem 7 is purely computational, and we will tentatively leave it blank. I.e. we will work it out when we get to it...

## 1 Solutions

1. We start by writing the rocket equation:

$$v = v_0 + v_e \log \frac{m_0}{m}$$

We are given that $dm/dt = -(\alpha t + \beta)$, or $m = m_0 - \beta t - \frac{1}{2}\alpha t^2$. So, we obtain the velocity as a function of time by simply substituting this expression.

$$v = v_0 + v_e \log \frac{m_0}{m_0 - \beta t - \frac{1}{2}\alpha t^2}$$

The velocity attains its maximum value when the denominator of the log argument is maximized. If $\alpha$ and $\beta$ are both positive, then there is no maximum (for positive time); however, if one of the two is negative, then the maximum occurs at $-\beta/\alpha$ (the vertex of the quadratic expression).

2. The initial differential equation is, in fact, very similar to the case of a rocket in free space. The only difference is that the rocket is constrained to move in a circle, which means that a centripetal force is acting. So, we have

$$m\frac{d\vec{v}}{dt} + \vec{v}_e \frac{dm}{dt} = \vec{F}_c$$

We now write the vector quantities using polar basis vectors.

$$m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(mr\ddot{\theta} + 2m\dot{r}\dot{\theta} + v_e\frac{dm}{dt}\right)\hat{\boldsymbol{\theta}} = -mr\dot{\theta}^2\hat{\boldsymbol{r}}$$

Since the rocket moves in a circle, $r$ is constant and all its derivatives vanish. In addition, we can cancel the centripetal force component, leaving only

$$mr\ddot{\theta} + \frac{dm}{dt}v_e = 0$$

Rewriting $\ddot{\theta}$ as $d\omega/dt$, we can separate and integrate:

$$\frac{r}{v_e}d\omega = -\frac{dm}{m}$$

$$\Delta v = r\Delta\omega = v_e \log\frac{m_0}{m_f}$$

This is the exact same equation as for the linear case. This makes sense, because the centripetal force acts orthogonally to the rocket motion and so cannot affect its energy. We can easily determine the magnitude of the centripetal force:

$$|F_c| = \frac{mv^2}{r} = \frac{mv_e^2}{r}\log^2\frac{m_0}{m_f}$$

3. (a) Straight from the rocket equation (determining $p(t+\Delta t)-p(t)$ and so on), we obtain the following equation

$$M\frac{dv}{dt} - u\gamma = F_{ext} = -Av^2$$

Using the fact that $M = M_0 - \gamma t$, we get

$$(M_0 - \gamma t)\frac{dv}{dt} = u\gamma - Av^2$$
$$\frac{dv}{u\gamma - Av^2} = \frac{dt}{M_0 - \gamma t}$$

Upon integration, we obtain the following solution

$$v(t) = \sqrt{\frac{\gamma u}{A}}\coth\left(\coth^{-1}\frac{\sqrt{A}v_0}{\sqrt{\gamma u}} - \sqrt{\frac{Au}{\gamma}}\log\frac{M_0-\gamma t}{M_0}\right)$$

(b) Since $M\frac{dv}{dt} - u\gamma = F_{ext} = -Av^2$ and $v$ is constant, the derivative term drops, leaving us with $F_t = Av_0^2$
(c) See the diagram below. When one particle hits the rocket's nose, the particle flies straight upwards (since the nose cone angle is 90 degrees). Since the particle is stationary, in the frame of the rocket, it looks as though it is moving at the rocket's speed. The change in momentum in the horizontal direction is thus $mv$. However, there are many particles that hit the rocket at that particular instant, given by $dm = NdV = NAdx = N\pi R^2 dx$. Thus, the total change in momentum due to these particles is $dp = Nmv\pi R^2 dx$. The net force is thus $\frac{dp}{dt} = Nmv\pi R^2\frac{dx}{dt} = Nm\pi R^2 v^2$. Since this force is retarding, $F_d = -(Nm\pi R^2)v^2$, thus proving the $v^2$ correlation.

4. NO SOLUTION

5. In the frame of the moving wall, the ball approaches and leaves with the same speed in different directions. The approach velocity (in the wall frame) is $v+V$ to the right, and so it also leaves with a velocity of $v+V$ to the left. Since the wall itself has a velocity of $V$ to the left, the ball leaves in the lab frame with a velocity of $v+2V$, or $\Delta v = 2V$. The ball bounces off the moving wall after traversing a length of $2(\ell - Vt)$, so we have

$$\Delta t = \frac{2\ell - Vt - V(t+\Delta t)}{v}$$
$$\Delta t = \frac{2(\ell - Vt)}{V+v}$$

Now we can divide the two quantities:

$$\frac{\Delta v}{\Delta t} = \frac{V^2 + Vv}{\ell - Vt}$$

Since $V \ll v$, the ball bounces very frequently as the surfaces approach each other, so we can approximate the process as continuous. This gives us

$$\frac{1}{v+V} dv = \frac{V}{\ell - Vt} dt$$

Integrating, we obtain

$$\log \frac{v+V}{v_0+V} = \log \frac{\ell}{\ell - Vt}$$
$$v = \frac{v_0 \ell + V^2 t}{\ell - Vt}$$

We can neglect terms of order $V^2$ or higher, giving us

$$v = \frac{v_0 \ell}{\ell - Vt} = \frac{v_0 \ell}{x}$$

Now, consider a ball bouncing between two walls, except the walls are not moving towards each other. In this case, the average force acting on the wall would be $F = \frac{\Delta p}{\Delta t} = \frac{2mv}{2\ell/v} = \frac{mv^2}{\ell}$. Using the velocity expression we obtained above and $\ell = x$, we get

$$F_{ave} = \frac{mv_0^2}{x}\left(\frac{\ell}{x}\right)^2 = \frac{mv_0^2}{\ell}\left(\frac{\ell}{x}\right)^3$$

This is the mechanism that helps us understand why a gas heats up when compressed.

6. According to Newton's second law (omitting vector signs):

$$\frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

The mass rate of change of the rope describes the mass of the rope which is providing an opposing force at the junction point. This conveniently, is the total mass of the rope which has fallen. The mass of the rope which has fallen is:

$$m = \sigma L(t) = \frac{1}{2}\sigma g t^2$$

The velocity of the rope at time $t$ is $v = gt$, so we can now plug in these equations into the equation for the rate of change of momentum. Thus:

$$\frac{dp}{dt} = F_{ext} = \left(\frac{1}{2}\sigma g t^2\right)(g) + (gt)(\sigma g t) = \frac{3}{2}\sigma g^2 t^2$$