Basic Postulates of Special Relativity - Solutions

Shankar Balasubramian Ross Dempsey

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1 Lecture Solutions

1. We have the following 4 equations

$$x' = \gamma(x - Vt)$$
$$x = \gamma(x' + Vt')$$
$$x = ct$$
$$x' = ct'$$

Inserting equations 3 and 4 into 2 and 1, we get

$$ct' = \gamma t(c - V)$$
$$ct = \gamma t'(c + V)$$

Plugging the first of the two conditions into the second gives us

$$c\left(\frac{ct'}{\gamma(c - V)}\right) = \gamma t'(c + V)$$
$$\frac{c^2}{(c^2 - V^2)} = \gamma^2 = \frac{1}{1 - \frac{V^2}{c^2}}$$

Precisely agreeing with our Lorentzian transformation. Thus, we can solve for t' and x' easily in terms of x and t, since γ is known.

$$x' = \gamma(x - Vt)$$
$$ct' = \gamma(ct - V\frac{x}{c}) \rightarrow t' = \gamma(t - \frac{Vx}{c^2})$$

2. As mentioned in the lecture, lengths parallel to the direction of motion contract, and lengths perpendicular to the direction of motion remain *constant*. Thus, the y-component of the rod, or $L_0 \sin\theta_0$ will remain constant. The x-component, will change, as dictated by the Lorentzian transformation:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow \Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In our case, Δt is 0, becasue we measure the length of the rod at a particular instant (as supposed to measuring the location of one end of the rod, waiting 2 seconds, measuring the location of the other end of the rod, and taking the difference). Thus

$$x' = \frac{x}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{L_0 \cos\theta_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The angle perceived by the stationary observer is thus

$$\tan\theta = \frac{\Delta y}{\Delta x'} = \frac{L_0 \sin\theta_0}{\gamma L_0 \cos\theta_0} = \frac{1}{\gamma}\tan\theta_0$$

The length of the rod perceived is then

$$L' = \sqrt{x'^2 + y^2} = L_0\sqrt{\gamma^2\cos^2\theta_0 + \sin^2\theta_0}$$

3. The easiest way to approach this scenario is to take a stationary reference frame fixed to Earth, and a moving reference frame fixed to the muon. This gives us the condition $x' = 0$, because the moving frame has its origin fixed to the muon. We start with the Lorentz transformation equation for time:

$$t = \frac{t' - vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

Since $x' = 0$, we can simply write

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

For the conditions described, this gives an observed lifetime of 7.1 µs.

4. To find the velocity in the moving frame, we must be careful and take the derivative with respect to time in the moving frame; i.e., we need to find $\frac{dx'}{dt'}$. To do this, we divide differentials of the Lorentz transformation formulae, noting that the factor of γ will cancel.

$$\frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - v/c^2\,dx}$$

Dividing the numerator and denominator by dt gives the Einstein velocity addition formula:

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}$$

Consider a case in which a particle moves with velocity $-\frac{3}{4}c$ in the x direction, and an observer moving at velocity $\frac{3}{4}c$. We would expect the moving observer to see the particle receding at a velocity of $-\frac{3}{2}c$. However, according to this formula, the actual observed velocity of the particle is

$$u'_x = \frac{-\frac{3}{4}c - \frac{3}{4}c}{1 + \frac{9}{16}} = -\frac{24}{25}c$$

We see in this case that the speed is kept just under c. We can prove that in general, if two speeds are individually less than c, then their combination according to this formula is also less than c. Let $\beta_1 = u_x/c$ and $\beta_2 = -v/c$. Then we can rewrite the velocity addition formula as

$$u'_x = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}c$$

Now, consider the quantity $(\beta_1 - 1)(\beta_2 - 1)$. If β_1 and β_2 are both less than 1 (which is true based on our hypothesis on u_x and v), then the product will be positive. We can use this inequality to show the result we want:

$$\begin{aligned}(\beta_1 - 1)(\beta_2 - 1) &> 0\\ \beta_1\beta_2 - \beta_1 - \beta_2 + 1 &> 0\\ \beta_1 + \beta_2 &< 1 + \beta_1\beta_2\end{aligned}$$

This last inequality demonstrates that the fraction $(\beta_1 + \beta_2)/(1 + \beta_1\beta_2)$ is less than one, so the resultant velocity is less than c.