Introduction to General Relativity — Lesson 1

Stephen Drodge

April 21, 2008

General Relativity is a theory of curved spacetime. The familiar concepts of Euclidean Geometry must therefore be abandoned along with our model of spacetime as a vector space. Without these, we are left with no concept of direction or distance, and no method for conducting calculus. In order to retrieve these notions in the arena of curved spacetime, we must unfortunately build extensive mathematical machinery which will initially be very unfamiliar for most people. For this reason, I urge you to work through all equations and concepts in the familar $\mathbb{R}^2$. 2 dimensional space has the benefit of being simple enough to visualize yet complicated enough to exhibit some complex behavior.

To simplify matters we will start by requiring that all the spaces we work with be Euclidean. We will label our underlying space M (and for now $M = \mathbb{R}^n$). We will also be forcing ourselves to forget any resemblance M might have to a vector space. The only structure we will assume for M is that a coordinate system exists. We will later formalize the exact nature of M and then show that our constructions are actually possible.

Our first construction is called the tangent space. There will be a tangent space associated with every point $p \in M$ which we will denote T_pM. We will define the tangent space in a rigorous fashion once we formalized the notion of curved space. For now we will simply say that each tangent space consists of all the directional derivative operators at point p. In other words, a basis of each tangent space can be defined as:

$$e_a(f) = \frac{\partial f}{\partial x^a}\Big|_p$$

where $f : M \to \mathbb{R}$ is a differentiable function and x^a represents some set of coordinates for M.

example:

let's use the familiar coordinates x, y on $\mathbb{R}^2$. In any tangent space T_pM, the equation $e_a(f) = \frac{\partial f}{\partial x^a}\Big|_p$ becomes the two equations:

$$\hat{x}(f) = e_x(f) = \frac{\partial f}{\partial x}\Big|_p$$

$$\hat{y}(f) = e_y(f) = \frac{\partial f}{\partial y}\Big|_p$$

We can also use the coordinates r, θ to come up with a different basis for each tangent space:

$$\hat{r}(f) = e_r(f) = \frac{\partial f}{\partial r}\Big|_p$$

$$\hat{\theta}(f) = e_\theta(f) = \frac{\partial f}{\partial \theta}\Big|_p$$

Any vector in a T_pM on $\mathbb{R}^2$ can now be expressed in terms of either of these bases:

$$\vec{v} = v^x e_x + v^y e_y = v^x \hat{x} + v^y \hat{y} = v^x \frac{\partial}{\partial x}\Big|_p + v^y \frac{\partial}{\partial y}\Big|_p$$

$$\vec{v} = v^r e_r + v^\theta e_\theta = v^r \hat{r} + v^\theta \hat{\theta} = v^r \frac{\partial}{\partial r}\Big|_p + v^\theta \frac{\partial}{\partial \theta}\Big|_p$$

I'll take a moment now to talk about the notation I am using. When raised indices are used, as in the case of $e_a = \frac{\partial}{\partial x^a}$ they are not meant to represent exponents. The specific meaning of raised and lowered indices will be explained later, but for now it should be understood that each index is meant to span the dimension of M. In the last example, a spanned $1 \cdots 2$ because we were working in $\mathbb{R}^2$. The advantage of this notation is that we can write down equations of an unspecified number of dimensions in an extremely compact manner. Another useful convention which I will make repeated use of is called the Einstein Summation Convention. When the same index appears once lowered and once raised on the same side of an equation, a sum over that index is implied. for example: $\vec{v} = v^a e_a$ is the same as writing $\vec{v} = \sum_a v^a e_a$. However, I will not use the summation convention when coordinates appear in an equation. I will always write coordinates with raised indicies, but I will also always explicity write out a sum when coordinates are involved. for example: $\vec{v} = \sum_a v^a \frac{\partial}{\partial x^a}\Big|_p$

Now that we know what the tangent space is, and how to get a component-basis decomposition. We want to know how our decomposition is effected by a change in the coordinate system on M. We can use the chain rule to immediately see that, $e'_b = \frac{\partial}{\partial x'^b}\Big|_p = \sum_a \frac{\partial}{\partial x^a}\frac{\partial x^a}{\partial x'^b}\Big|_p$ In other words, under a change of coordinates from x^a to x'^a the basis of each tangent space transforms by this rule:

$$e'_b = \sum_a e_a \frac{\partial x^a}{\partial x'^b}\Big|_p$$

This type of transformation is known as a **Covariant Transformation**. You can remember the name by the fact that both the old basis vectors and the old coordinates appear in the numerator. The other type of transformation is called a **Contravariant Transformation** and is given below:

$$v'^b = \sum_a v^a \frac{\partial x'^b}{\partial x^a}\Big|_p$$

This name is easy to remember by noting that the old components are in the numerator while the old coordinates are in the denominator. The fact that I used tangent vector components in defining the contravariant transformation law leads directly to the first exercise:

Exercise 1: Given that the basis vectors of the tangent spaces tranform covariantly with a change of coordinates, prove that tangent vector components transform contravariantly under the same coordinate change. Use abstract index notation.
Hint: $\vec{v} = v^a e_a = v'^a e'_a$

Next we will introduce the cotangent space. There is one cotangent space associated with every point in M in the same way that there is a tangent space at each point. We shall denote it as T^*_pM and as the notation implies, we define it as the dual space to the tangent space. This means that $\omega \in T^*_pM \iff \omega(a\vec{u} + b\vec{v}) = a\omega(\vec{u}) + b\omega(\vec{v})$ where $\omega : T_pM \to \mathbb{R}$. In other words, T^*_pM is the set of all linear functionals that map T_pM to the reals (not $\mathbb{R}^n$).

Exercise 2: Show that the cotangent space is a vector space.
Hint: use the properities of real addition and multiplication in conjunction with the fact that all members of the cotangent space are linear functionals.

We now define the natural dual basis in terms of the tangent basis. $e^a(e_b) = \delta^a_b$ where $a = b \Rightarrow \delta^a_b = 1$ and $a \neq b \Rightarrow \delta^a_b = 0$

Exercise 3: Show that the dual basis forms a basis for the cotangent space
Hint: if a linear functional maps everything to 0, it is the 0 vector

We can now safely define any cotangent vector (also referred to as a one-form) as $\vec{\omega} = \omega_a e^a$ and we can see that $\vec{\omega}(\vec{v}) = \omega_a e^a(v^a e_a) = \omega_a v^a$. We will also sometimes use the notation $e_a = \partial_a$ and $e^a = dx^a$ which will be justified later.

Exercise 4: Show that the natural dual basis for the cotangent space transforms contravariantly with a coordinate change. Also show that one-form components transform covariantly. Use abstract index notation.
Hint: Prove that the components transform covariantly first.

We have now constructed two vector spaces at every point in M. We have also seen that the difference between these two spaces is in how their component-basis decomposition changes when coordinates on M are changed. In the next lesson we will learn how to use these new structures to recover the concept of distance and direction on M. We will also see how T_pM serves as the dual of T_p^*M in the same way that T_p^*M serves as the dual of T_pM. The introduction of notation and fairly rigorous mathematical construction does serve to obscure the true simplicity of the concepts in this lesson, but as you familiarize yourself with these concepts and the abstract index notation, the picture should become far clearer. For now, one should visualize the tangent space as the possible velocities a particle can have as it travels through $p \in M$. The cotangent space is best visualized as possible infinitesimal displacements out from p. The significance of these interpretations will become clearer in later lessons. As for now, if you have completed the exercises and feel comfortable manipulating equations in the new notation, you will have little trouble with future lessons, and you have replicated some of the founding of Riemmannian Geometry, good work.