Introduction to Quantum Mechanics 1

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The Stern-Gerlach Experiments

Before we delve into the algebra of quantum mechanics, let us consider a thought experiment based on the Stern-Gerlach experiments.

The electron has a spin angular momentum vector, similar to how a top has an angular momentum vector. When the electron passes through an inhomogenous magnetic field oriented in a specific direction (let it be the z-direction for simplicity), the measurement of the spin angular momentum will be quantized. This means it can take a specific discrete range of values. For an electron, a spin-1/2 particle, the allowed values are $\hbar/2$ and $-\hbar/2$, and this measurement is called S_z. However, these numbers are so small, that in a large scale scenario, angular momentum is continuous for all intents and purposes.

What is even more strange is that if we make these measurements on a statistical sample of electrons, about half will be spin-up, and about half will yield spin-down.

What happens if we orient the magnetic field in the y-direction? The exact same thing. And the same thing for the x-direction. That yields the question of putting it through the z-directed field, then putting it through another directed field. Then we will make our measurement on the spin angular momentum vector.

In one part of the Stern-Gerlach experiments, the spin-down particles from the z-directed field were blocked, and the spin-up particles were allowed to pass through an x-directed field. Strikingly, half of the particles had $S_x = \hbar/2$, and the other half had $S_x = -\hbar/2$. We expected there to be zero angular momentum in the x-direction because the electron had just come out of a z-oriented field! The more discomforting fact arose when scientists added another z-directed field and blocked the spin-down particles from the x-directed field. We would expect that all the particles were in the up-state, as we initially blocked all the down particles from the original z-machine, but in turned out that about half were up and about half were down. It turns out that the state (which we will define later) is not deterministic and evolves through time, and is a superposition of basis states (think vectors).

The Basics and Bra-Ket Notation

In quantum mechanics, we define quantities describing a system (energy, momentum, angular momentum) to be **observables**. Our system is initially in a **state**, which essentially describes the physical nature of the system at a specific time. The observables that are associated with quantities like angular momentum can be determined by forcing an **operator** to act on the state. Like a matrix transformation, the operator may change the state according to a prescribed algorithm.

States in quantum mechanics are described as vectors. However, to condense notation, we will use the following representation:

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ ... \\ \psi_n \end{pmatrix}$$

This is called a **ket** vector. Note that the vector components can be complex numbers. We also have to deal with the **dual space** to a ket vector. This is known as a bra vector, which is chosen so that when we combine them, we have a bra-ket (bracket). Their combinations will prove useful later in determining probablilites,amplitudes and special matrix operators.

$$\langle\psi| = \begin{pmatrix} \psi_1^* & \psi_2^* ... & \psi_n^* \end{pmatrix}$$

The bra and ket vectors are related by We also must introduce the meaning of the following entity: $\langle\psi|\psi\rangle$. This is known as a *hermitian inner product.* It is just a multiplication of a vector $|\psi\rangle$ with its dual vector $\langle\psi|$, and results a complex scalar. Since $|\psi\rangle$ is generally normalized, the hermitian inner product with its bra-vector should yield a complex number of mangitude one. A fundamental theorem of the inner product states that $\langle\psi|\psi\rangle \geq 0$. The zero equality occurs if and only if the vector is the zero vector.

What is the physical interpretation of a Hermitian product? If a particle starts at a state ψ_1 and is deflected in a manner to change it to a different state ψ_2, then $|\langle\psi|\psi\rangle|^2$ represents the probability that a measurement of a particle's angular momentum in the new state will yield a measurement prescribed by that state.

Simple Two-State Systems

Now that we have the algebra basics down, we can discuss simple two-state systems. In the Stern-Gerlach experiments, we focused on measuring the angular momentum in the z-direction of a spin-1/2 particle, which took either a positive value or negative value. We will call the state which yields a positive value (spin-up) $|+z\rangle$ and the state which yields a negative value (spin-down) $|-z\rangle$, which is why this is a two-state system. Please note that $|+z\rangle$ is not $-|-z\rangle$. In a classical linear algebra sense, we have the following:

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

All these vectors tell us is that about the state is that if we were to make a measurement of the angular momentum in the z-direction, we would expect $\pm\hbar/2$. There are obviously more attributes we can add, like the position, velocity, or momentum of the state, but currently this state does not address those quantities.

The reason that these vectors are two dimensional is because we are dealing with spin-1/2 particles. For example, a spin-1 particle would have three column vectors. We have also choosen these vectors not only because they span the entire C^2-space (in which each element is a complex number), but because they are orthonormal (orthogonal and normalized).

Since these two z-vectors form a basis, we will call it the z-basis, and it is somewhat arbitrary that we have chosen it. We could have chosen the x-basis for example, but historically the z-basis has been used. This means that for an arbitrary state $|\psi\rangle$:

$$|\psi\rangle = c_1|+z\rangle + c_2|-z\rangle$$

There are a few constrains that we would like on this state. First of all, the state needs to normalized. That means we have the following:

$$1 = |\langle\psi|\psi\rangle|^2$$

$$1 = |(c_1^*\langle+z| + c_2^*\langle-z|)(c_1|+z\rangle + c_2|-z\rangle)|^2$$

Because everything here is linear (we are just using a new notation), we can re-express the above as follows:

$$1 = c_1^2 + c_2^2$$

Different Bases

We have modeled several situations by using the z basis as our primary basis. However, we would also like to determine ways of measuring angular momentum in the x or y basis. To do this, we need a relationship between our old basis, and the new basis.

According to the Stern-Gerlach experiments, if our machine sends particles whose angular momentum point entirely in the +x direction, and send the through a deflected magnetic field in the y-direction, we would expect to see exactly half the particles to possess a +y angular momentum, and the other half to have an angular momentum of -y.

We will first write down a general formula for the x and y state. Let us say

$$|+x\rangle = \frac{e^{i\delta_+}}{\sqrt{2}}|+z\rangle + \frac{e^{i\delta_-}}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{e^{i\gamma_+}}{\sqrt{2}}|+z\rangle + \frac{e^{i\gamma_-}}{\sqrt{2}}|-z\rangle$$

This properly normalizes the states. By the result of the Stern- Gerlach experiment, and the definition of the inner product, we can assert that:

$$|\langle +y|+x\rangle|^2 = \frac{1}{2}$$

After much simplification, we get

$$|\langle +y|+x\rangle|^2 = \frac{1}{4}\left[1+e^{i(\delta-\gamma)}\right]\left[1+e^{-i(\delta-\gamma)}\right]$$

$$|\langle +y|+x\rangle|^2 = \frac{1}{2}\cos(\delta-\gamma)$$

where $\delta = \delta_- - \delta_+$ and $\gamma = \gamma_- - \gamma_+$. To assert that this probability is a half $\delta - \gamma = \pm\frac{\pi}{2}$. In this case, we will choose the positive value. We will use the common convention to set $\delta = 0$. We will also ignore the phases δ_+ and γ_+. This gives the following forms for the x and y state

$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle$$

Expected Value and Standard Deviation

When we deal with expected vaue, most of you may be familiar with

$$\langle A\rangle = \sum_i P(A_i)A_i$$

This definition applies exactly to quantum mechanics. For a two state system, there are two outcomes that are possible: a positive or negative $\hbar/2$ for a measurement of a particle's angular momentum.

Standard Deviation is a different story. As with all statistical samples, there follow a distribution of possible expectation values for a sample size. The standard deviation is defined as

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle$$
$$(\Delta A)^2 = \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2$$
$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$$

This equation is a matter of plugging and chugging; however, it is vital to deriving the uncertainty relationships which form the basis for quantum mechanics

An Introduction to Operators

An operator, unlike a vector, is represented as a square matrix. For example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Is known as the identity operator because it does not do anything to a state when acted on it. The operator

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotates the shadow of a state in the R^2 basis by a counterclockwise angle θ.

An important equation that pops up several times in quantum mechanics is the famed matrix equation. If an operator acts on a state, it will transform the state by a series of transformations. All such transformations are linear, because they can be represented as the matrix equation.

$$\mathbf{O}|\psi\rangle = |\psi'\rangle$$

There is a special vector for the matrix such that the transformation yields a multiple times the vector itself.

$$\mathbf{O}|\psi\rangle = o|\psi\rangle$$

This vector is called an **eigenvector** (or eigenket), and the scalar is called an **eigenvalue**. To find such eigen vectors, we sandwich the identity operator in the right hand side

$$\mathbf{O}|\psi\rangle = o\mathbf{1}|\psi\rangle$$

$$det\,(\mathbf{O} - o\mathbf{1}) = 0$$

Problems

1. What are $\langle -y|$ and $\langle -x|$ in vector notation? Hint: we assumed that $\delta - \gamma = +\frac{\pi}{2}$.

2. Determine the eigenvalues and eigenvectors of the rotation matrix shown.

3. Assume that a vector $|+n\rangle$ has the form $\cos\theta|+z\rangle + \sin\theta e^{i\phi}|-z\rangle$, which is expressed in terms of an orthonormal basis. Determine the inner products $\langle +z|+n\rangle$ and $\langle -z|+n\rangle$. Show that $|\langle +z|+n\rangle|^2 + |\langle -z|+n\rangle|^2 = 1$.

4. Determine the standard deviation and expected value of the angular momentum in the z-direction given the $|+n\rangle$ state.

5. Determine the standard deviation and expected value of the angular momentum in the x-direction given the $|+n\rangle$ state.

6. Show that the neither probability of attaining a result a_i nor the expectation value $\langle A\rangle$ is affected by an overall phase factor $e^{i\delta}$.

7. A classical mechanics problem: When a particle enters a inhomogeneous magnetic field, the force acting on it in the z direction is approximately $\mu_z \frac{\partial B}{\partial z}$, where $\mu_z = \frac{kq}{2mc} S_z$. Determine the field gradient required for a Stern-Gerlach magnet of length L that produces a separation distance $d << L$. Assume that the electrons are emitted from a source with a temperature T.