Non-Uniform Circular Motion, Practice Problems, and Other Errata

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September 2014

1 Vector Calculus in Polar Coordinates

In Cartesian coordinates, vector calculus is a straightforward (in fact, almost trivial) extension of the calculus of scalar functions. Given a function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we could always expand into components:

$$\vec{s}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

With this expansion, we could invoke the linearity of the derivative to easily find the time derivatives:

$$\begin{aligned}\dot{\vec{s}}(t) &= \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} + \dot{z}(t)\hat{k} \\ \ddot{\vec{s}}(t) &= \ddot{x}(t)\hat{\imath} + \ddot{y}(t)\hat{\jmath} + \ddot{z}(t)\hat{k}\end{aligned}$$

However, in polar coordinates, no such expansion is possible, since the basis vectors themselves change with position. In fact, as we see in Figure 1, $\hat{r} = \langle \cos\theta, \sin\theta \rangle$ and $\hat{\theta} = \langle -\sin\theta, \cos\theta \rangle$. To take the time derivative in polar coordinates, we write the point $(r(t), \theta(t))$ as $r(t)\langle \cos\theta(t), \sin\theta(t) \rangle$ and proceed to differentiate in the Cartesian coordinates:

$$\begin{aligned}\dot{\vec{s}}(t) &= \frac{d}{dt}[r(t)\langle \cos\theta(t), \sin\theta(t) \rangle \\ &= \dot{r}(t)\langle \cos\theta(t), \sin\theta(t) \rangle + r(t)\dot{\theta}(t)\langle -\sin\theta(t), \cos\theta(t) \rangle\end{aligned}$$

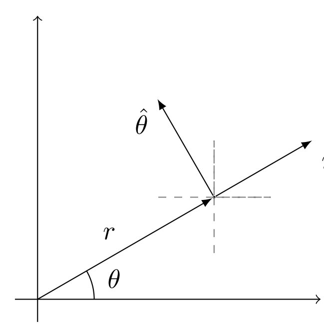

Figure 1: We see that $\hat{r} = \langle \cos\theta, \sin\theta \rangle$ and $\hat{\theta} = \langle -\sin\theta, \cos\theta \rangle$.

At this point, we recognize the two Cartesian vectors as $\hat{r}$ and $\hat{\theta}$. We can thus rewrite $\dot{\vec{s}}(t)$ more succinctly:

$$\dot{\vec{s}}(t) = \dot{r}(t)\hat{r} + r(t)\dot{\theta}(t)\hat{\theta}$$

If we return to Cartesian coordinates briefly and differentiate again, we obtain the acceleration:

$$\ddot{\vec{s}}(t) = \ddot{r}(t)\langle \cos\theta(t), \sin\theta(t) \rangle + 2\dot{r}(t)\dot{\theta}(t)\langle -\sin\theta(t), \cos\theta(t) \rangle + r(t)\ddot{\theta}(t)\langle -\sin\theta(t), \cos\theta(t) \rangle - r\dot{\theta}(t)^2\langle \cos\theta(t), \sin\theta(t) \rangle$$

Substituting again for $\hat{r}$ and $\hat{\theta}$, we obtain finally the acceleration in polar coordinates:

$$\ddot{\vec{s}}(t) = \left(\ddot{r}(t) - r(t)\dot{\theta}(t)^2\right)\hat{r} + \left(2\dot{r}(t)\dot{\theta}(t) + r(t)\ddot{\theta}(t)\right)\hat{\theta}$$

2 Problems

2.1 Introductory Problems

1. Describe the force required to linearly accelerate a particle around a circular track of radius r. (Do this first in a coordinate system moving with the particle, and then convert to a stationary frame with its origin at the center of the track).

2.2 Intermediate and Advanced Problems

1. A mass is connected to a string which passed through a hole in a table. The initial radius of the mass is R_0, and its initial velocity is v_0. An external agent pulls the rope from underneath the table at a constant velocity V. Determine the radius the mass swings in as a function of time, as well as the work the agent does as a function of time.

2. A mass m is swinging in a conical pendulum, attached to a string of length L and half angle θ. Unfortunately, the presence of a drag force $-bv$, applied tangential to the direction of motion of the mass, forces the mass to spiral inwards. At what velocity must the rope be pulled at so the mass remains at a constant radius from the central axis?

3. Consider a conical pendulum. The mass swings with a half angle θ, at a frequency Ω. However, the mass also swings conically with a half angle ϕ (in essence, a conical pendulum swinging as a conical pendulum). If the length of the string is L, determine Ω such that the above system functions. What does your result say about ϕ and θ?

4. A pendulum consists of a mass m suspended to a ceiling by a massless rope with a length L. The pendulum is displaced by an angle θ from the vertical. Determine at what point the mass will prefer to swing in a horizontal circle rather than oscillate (if perturbed). The small angle approximation does not necessarily have to hold.

3 Solving Problems by Using Constraint Equations

Often times, we are forced to consider complex systems, which operate using multiple pulleys and compounded mechanisms. It quickly becomes quite difficult to apply Newton's Laws to these situations without using a trick. The trick that we will detail uses a principle so fundamental, that it may not even appear to be useful.

Assume that we have a regular Atwood's machine (see Figure 2), but the pulley is accelerated upwards with a magnitude A.

The following diagram labels the height of each mass relative to the ground. We will use the fact that the length of the string remains constant. The length of the string is

$$L = (H - h_1) + (H - h_2) + \pi R \tag{1}$$

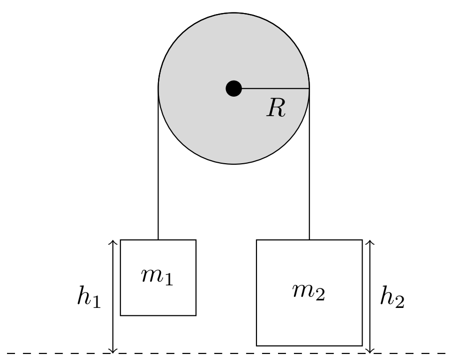

Figure 2: An Atwood machine, described below.

Since the length of the string is constant

$$\ddot{L} = 0 = 2\ddot{H} - \ddot{h}_1 - \ddot{h}_2 \tag{2}$$

$$0 = 2A - \ddot{h}_1 - \ddot{h}_2 \tag{3}$$

The idea that the length of string is conserved is very important, as the string constrains the motion of the masses, thus allowing us to obtain a constraint equation relating the accelerations of the masses.

3.1 Introductory Problems

1. Using Newton's Laws, determine the tension in the string in the example above. Don't forget to use the constraint equation!

2. Determine the acceleration of the masses in Figure 3 in terms of g.

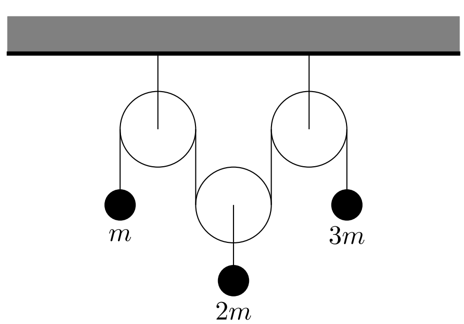

Figure 3: A pulley system with masses m, $2m$, and $3m$.