Special Relativity (Mathing ed.)

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Tenets of Special Relativity

In 1905, Albert Einstein demonstrated that the core principles of Newtonian mechanics hold only in the condition $v \ll c$ due to incompatibility with Maxwell's Equations at high speeds. The theory of **special relativity** accounts for these discrepancies of Newtonian relativity and dynamics. In order to properly investigate special relativity, we first lay out the following tenets:

1. All laws of physics are invariant in an inertial reference frame.

2. The speed of light is constant in all inertial reference frames.

3. The relativity velocity of A with respect to B is equal and opposite to the relative velocity of B with respect to A. In other words, $v_{AB} = -v_{BA}$.

Note the implications of the tenets above. By tenets one and two, we are unable to devise an experiment that determines if we are at rest or moving with constant speed. Also, note that velocity addition is not included. We are not able to assume Newtonian velocity addition, $v_{AB} + v_{BC} = v_{AC}$, because it would violate tenet number two.

Phenomena in Special Relativity

From the three tenets above, we are able to calculate the effects of two phenomena, time dilation and length contraction, at high speeds.

Time Dilation

Because the speed of light is constant in all inertial reference frames, consider a light clock composed of two parallel mirrors spaced a distance L apart. In a stationary reference frame, the time it takes for light to start at one mirror, bounce off of the other, and return to the first mirror is $\frac{2L}{c}$. This is the period T_0 of the light-clock.

Now consider the situation in which the mirrors move at a constant speed v parallel to their length. Let the period of the light-clock be T in this case. After half a period, the mirrors move a distance $\frac{vT}{2}$. Thus, the total distance the light travels in one period is $2\sqrt{L^2 + \frac{v^2T^2}{4}}$ by the

Pythagorean theorem. Solving for T,

$$\begin{aligned}
T &= \frac{2\sqrt{L^2 + \frac{v^2T^2}{4}}}{c} \\
T^2 &= \frac{4L^2 + v^2T^2}{c^2} \\
T &= \frac{2L}{c}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
&= T_0\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{aligned}$$

The expressions $\frac{v}{c}$ and $(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ show up frequently in special relativity, so they are given special symbols, namely $\beta = \frac{v}{c}$ and the **Lorentz factor** $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Note that $\beta \leq 1$ and $\gamma \geq 1$. This means the period of a clock increases in a moving reference frame, since $T = \gamma T_0 \geq T_0$, and thus run slower as a result of time dilation.

Length Contraction

Consider an spaceship with constant speed v approaching a stationary observer. By the third tenet of special relativity, the observer is approaching the spaceship at a speed v in its own reference frame. Still working in the frame of the spaceship, let the initial distance to the observer be L. Then, the time it takes for the observer to "reach" the spaceship is $T_0 = \frac{L}{v}$. In the stationary observer's frame, let the initial distance to the spaceship be L_0. Then, the time it takes for the spaceship to reach the observer is $T = \frac{L_0}{v}$. But by time dilation, we know that the clock for the stationary observer runs slower by a factor of γ, and thus $T = \gamma T_0$. Solving for L in terms of L_0,

$$\begin{aligned}
L &= vT_0 \\
&= \frac{vT}{\gamma} \\
&= \frac{L_0}{\gamma}
\end{aligned}$$

Although in this scenario we considered distance, the result is the same for physical objects, such as a yardstick. Since $\gamma \geq 1$, the length of an object contracts according to a moving outside observer. The longest possible observed length of an object, or its proper length, is the length of that object in a reference frame stationary with respect to it. Any observer moving parallel to the length of the object will see its length contracted. On the other hand, an observer moving perpendicular to the length of the object will see no length contraction. Try to devise a thought experiment that prohibits perpendicular length contraction.

Lorentz Transformation

We are able to generalize time dilation and length contraction into a set of equations known as the Lorentz transformations.

An **event** is an phenomena that can described by a position in space and time. For special relativity in one dimension, we use the space-time coordinates (t, x). Suppose that reference frame F' is moving with velocity v along the positive x axis with respect to reference frame

F. The Lorentz transformations allows us to determine the space-time coordinates (t', x') of an event in F' using the space-time coordinates (t, x) of the same event in F. The one dimensional Lorentz transformations are given by,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma(x - vt)$$

The Lorentz transformations can also be used to find the time and distance between two events $(\Delta t', \Delta x')$ in F' using the time and distance between the same two events $(\Delta t, \Delta x)$ in F.

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$
$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

Lorentz Invariant

The Lorentz invariant Δs is a property of two spacetime events that is independent of the reference frame in which the events are observed in. If two events are separated by a time Δt and distance Δx, then,

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2$$

The invariance of Δs can be proven by substituting the Lorentz transformations. This exercise is left for the reader.

The Lorentz invariant has several implications. First, if two events happen at the same time and place in one reference frame (this property is called coincident) , then they must be coincident in all reference frames. Secondly, if two events happen at the same time but at different places in one reference frame, then there is no reference frame in which the two events happen at the same place but different times. This idea is related to space-time separation.

Space-time separation

The **proper time** $\Delta\tau = \frac{\Delta s}{c}$ is the time between two events that occur at the same place. If the proper time exists, meaning $\Delta s^2 > 0$, the two events are said to be time-like separated.

The **proper distance** $L = \sqrt{-\Delta s^2}$ is the distance between two events that occur at the same place. If the proper distance exists, meaning $\Delta s^2 < 0$, the two events are said to be space-like separated.

The Lorentz invariant implies that two events can not be both time-like and space-like separated. There are nice visual ways to see these results using a Minkowski diagram.