Kinematics Review Problem

A ball is launched with an initial velocity v_0 normal to a ramp inclined at an angle ϕ. The coefficient of restitution of the ramp is $\gamma < 1$. The ball will repeatedly bounce along the ramp for a finite distance. Assuming that the ramp is infinitely long, determine this distance.

Solution

Since the ramp is inclined, it would be judicious to tilt the reference frame. Undoubtedly, gravity is going to point in an unorthodox direction, but this can be resolved by decomposing the vector.

$$g_N = gcos(\phi)$$
$$g_T = gsin(\phi)$$

We can thus reformulate equations reflecting upon accelerations in both directions (normal and tangential to the plane). The time it takes for the particle to remain airborne is

$$-v_0 = v_0 - gcos(\phi)t$$
$$t = \frac{2v_0}{gcos(\phi)}$$

The coefficient of restitution has the effect of inverting the "normal velocity" and multiplying it by a dimensionless constant when the particle is about to strike the plane. The normal velocity of the particle when it leaves and approaches the plane are equal. However, the tangential velocity is subject to a constant acceleration, thus speeding it up. The velocity in the tangential direction at the nth bounce is defined as

$$v_n = v_{n-1} + gsin(\phi)t$$
$$v_n = v_{n-1} + gsin(\phi)(\frac{2v_{n-1,N}}{gcos(\phi)}) = v_{n-1} + 2v_{n-1,N}tan(\phi)$$

Now we impose the conditions that the coefficient of restitution dictate. After n bounces, the normal velocity will be reduced geometrically to $\gamma^n v_0$. The recursive sequence can thus be written as

$$v_n = v_{n-1} + 2\gamma^{n-1} v_0 tan(\phi)$$

This recursive sequence is explicitly a geometric series. Thus, it can be simplified to

$$v_n = 2v_0 tan(\phi)(1 + \gamma + \gamma^2 + ... + \gamma^{n-1}) = 2v_0 tan(\phi)\frac{1-\gamma^n}{1-\gamma}$$

The total distance traveled by the particle during the nth bounce is determined by its corresponding position equations.

$$d_n = v_n t + \frac{1}{2} g sin(\phi) t^2 = 2v_0 tan(\phi)\frac{1-\gamma^n}{1-\gamma}\frac{2v_{n,N}}{g cos(\phi)} + \frac{1}{2} g sin(\phi)(\frac{2v_{n,N}}{g cos(\phi)})^2$$

This can be further simplified to

$$d_n = v_n t + \frac{1}{2} g sin(\phi) t^2 = 2v_0 tan(\phi)\frac{1-\gamma^n}{1-\gamma}\frac{2v_0\gamma^n}{g cos(\phi)} + \frac{1}{2} g sin(\phi)(\frac{2v_0\gamma^n}{g cos(\phi)})^2$$

The total distance traveled by the particle is therefore the infinite sum of all of the "bounce distances"

$$D = \sum_{n=0}^{\infty} 2v_0 tan(\phi)\frac{1-\gamma^n}{1-\gamma}\frac{2v_0\gamma^n}{g cos(\phi)} + \frac{1}{2} g sin(\phi)(\frac{2v_0\gamma^n}{g cos(\phi)})^2$$

After exhaustive manipulation and simplification

$$D = \frac{2v_0^2 tan(\phi)}{g cos(\phi)(1-\gamma)^2}$$