AP Physics Notes on Introductory Calculus

[Produced by D. Forbes, any and all errors in this document are owned by him.☺]

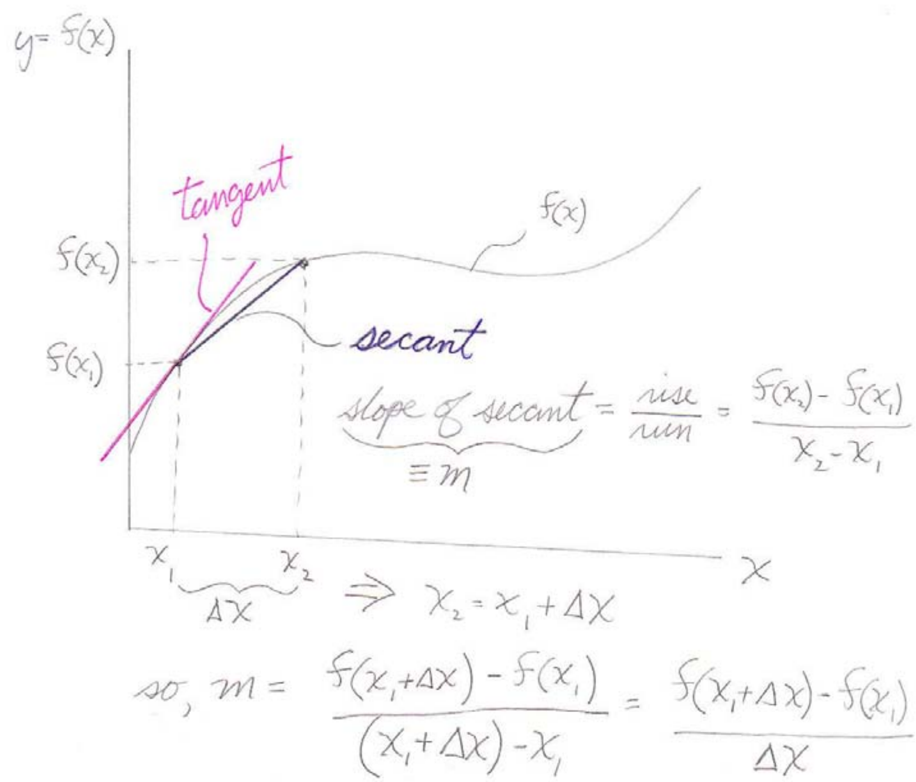
READ THESE BULLETS FIRST!

- These notes are primarily designed for students taking BC Calculus in September. However, ***ALL*** AP Physics students are responsible for this material, including those who will be enrolled in more advanced courses in the fall.
- These notes are not a substitute for a course in calculus, but are intended to provide entry-level calculus students with enough insight (conceptual understanding), nomenclature and notation, and practical examples and results, so that they will feel more comfortable during the 1st month-or-so of their AP Physics class.
- THERE IS NO NEED FOR YOU TO LEARN SINGLE-VARIABLE CALCULUS THIS SUMMER ON YOUR OWN! Don't purchase any texts, or attempt to work through any problems but what is contained in these notes. You will quickly gain a deeper understanding of calculus in your BC Calculus class as well as in classroom lectures, discussions and examples in our AP physics class[1].
- To use the links, you need Mathematica (available through TJ), or install the free reader CDF Player

Part I – Derivatives Of A Function Of One Variable

You are familiar with the slope of a line. Now we take that concept of slope and generalize it. The first ordinary derivative of a function in one variable, say $f(x)$, is itself a function that gives you the slope of $f(x)$ for any x.

What do we mean by the slope of a function that usually is a curve? Carefully examine the plot below.



[Note: in math texts, "Δx" is often called "h," and "x_1" is called "a."]

[1] Appendix D in Tipler covers some elementary calculus, but it is too terse – we can do better.

As x_2 gets closer to x_1, the secant looks more and more like a line tangent to $f(x)$ at the point x_1. [See this link: approximate the tangent with a secant [Make sure you have the CDF Player and that you fool around with the sliders.] If x_2 is getting closer and closer to x_1, then Δx is getting smaller and smaller. As Δx gets infinitesimally small, in other words when x_2 is basically on top of x_1, the secant essentially becomes the tangent to $f(x)$ at x_1. The slope of that tangent line is

$$m = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

The notation $\lim\limits_{\Delta x \to 0}$ simply means "in the limit as Δx goes to zero," and is the formal mathematical statement of the process described in the preceding paragraph. [Note: the limit must exist!]

The slope of the tangent line at a particular value, say x_1 (or a, or any other value) of the function f is the 1st derivative of f at x_1.

Here is an interactive example: 1st derivative of SQRT(x)

Notation: $\frac{df(x_1)}{dx} = f'(x_1)$ [the right side is said as "f prime of x_1."]

Now, if we let x_1 be *any* value of x, and with $y = f(x)$, we get

$$\frac{df(x)}{dx} = \frac{dy}{dx} = y' = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This is the first derivative of the function f with respect to x. The derivative will only exist if f is smooth and continuous – at least over the region that you wish to find the derivative. Why? Well, smooth so there are no sharp points. The slope of a sharp point is undefined. The requirement that f is be continuous should be clear; no function, no derivative.

Here are several **important** plots of functions and their 1st derivatives[2].

Now we see that the 1st derivative of f is itself a function giving the slope of f at any point (in an interval where f is differentiable). [This concept can be extended to functions of more than one variable, but we won't need to do that until the 2nd semester when we meet the *gradient* in our study of electricity & magnetism.]

How do you find these derivatives? Well, you literally just plug specific functions into the definition.

[2] The trigonometric functions, cos, sin, tan, and the exponential function e^x have very important derivatives, that for now, will be given to you. When using the definition of the derivative to develop their formulae, you end up with indeterminate forms of 0/0 which require more analysis than is appropriate in these notes.

EXAMPLE 1: Let $y = f(x) = x^2$, find y'

$$\text{let } f(x) = x^2 \longrightarrow \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}}$$

and, since $\lim_{\Delta x \to 0} \Delta x = 0$, we have $\frac{d(x^2)}{dx} = 2x = y'$

EXERCISE 1: Try this one. Let $y = f(x) = x^3$, find y'

EXAMPLE 2: This one is trickier.

let $f(x) = \frac{1-x}{2+x}$, find the 1ST derivative of $f(x)$ with respect to x [i.e., this means $\frac{d}{dx}\left(\frac{1-x}{2+x}\right) = ?$]

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\rightarrow \lim_{\Delta x \to 0} \frac{\frac{1-(x+\Delta x)}{2+(x+\Delta x)} - \frac{1-x}{2+x}}{\Delta x}$$ hmmmm... add the fractions!

$$\lim_{\Delta x \to 0} \frac{(2+x)(1-x-\Delta x) - (1-x)(2+x+\Delta x)}{\Delta x(2+x)(2+x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{2} - \cancel{2x} - 2\Delta x + \cancel{x} - \cancel{x^2} - \cancel{x\Delta x} - \cancel{2} - \cancel{x} - \Delta x + \cancel{2x} + \cancel{x^2} + \cancel{x\Delta x}}{\Delta x(2+x)(2+x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{-3\cancel{\Delta x}}{\cancel{\Delta x}(2+x)(2+x+\Delta x)} = -\frac{3}{(2+x)^2}$$

so, $\frac{d}{dx}\left(\frac{1-x}{2+x}\right) = -\frac{3}{(2+x)^2}$

EXERCISE 2: Try this one. Let $y = f(x) = \sqrt{x}$, find y' `[Hint: multiply by 1]`

Clearly, this method of using the definition directly will always work, but it can be somewhat involved. [And this *is* the method one uses in numerical (computing) techniques.] However, there are several elementary rules for computing derivatives that can be derived from the definition that will make your life much easier.

Rule 1: Constants What is $\frac{d(cx)}{dx}$; where c is *any* number – integer, real, even complex?

$$\lim_{\Delta x \to 0} \frac{(cx + c\Delta x) - cx}{\Delta x} = \lim_{\Delta x \to 0} \frac{c\cancel{\Delta x}}{\cancel{\Delta x}} = c$$

[Note: this can be extended to $\frac{d\left[cf(x)\right]}{dx} = c\frac{d\left[f(x)\right]}{dx}$]

Rule 2: The Power Rule [Probably the most used differentiation rule.]

What is $\frac{d(x^n)}{dx}$? Use the definition and expand the $(x + \Delta x)^n$ term with the *binomial theorem*:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

The objects in parenthesis are the *binomial coefficients*. One way to determine them is by the use of *Pascal's triangle*. However, in any case, they are numbers.

[Expand $(x + \Delta x)^4$ as an example. $(x+\Delta x)^4 = x^4 + 4x^3\Delta x + 6x^2\Delta x^2 + 4x\Delta x^3 + \Delta x^4$]

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[x^n + nx^{n-1}\Delta x^1 + \binom{n}{2}x^{n-2}\Delta x^2 + \ldots + \binom{n}{n-1}x^1\Delta x^{n-1} + \Delta x^n\right] - x^n}{\Delta x}$$

The 1ST and last terms, x^n, cancel. The Δx in the denominator cancels with a Δx in each remaining term.

$$\rightarrow \lim_{\Delta x \to 0} nx^{n-1} + \cancel{\binom{n}{2}x^{n-2}\Delta x} + \cancel{\ldots} + \cancel{\Delta x^{n-1}} = nx^{n-1}$$

so, $\frac{d(x^n)}{dx} = nx^{n-1}$

example $\frac{d(6x^5)}{dx} = 30x^4$

EXAMPLE 3: Look at what you needed to do in Exercise 2. Now let's use the power rule:

$$\sqrt{x} = x^{1/2} \rightarrow \frac{d\left(x^{1/2}\right)}{dx} = \tfrac{1}{2}x^{(1/2-1)} = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Rule 3: The Product Rule [Presented without proof – we'll do this one in class.]

Let f and g both be differentiable functions of x. Then $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$

EXAMPLE 4:

$$\frac{d}{dx}\left[4x^3\sqrt{x}\right] = 4x^3\,\frac{d}{dx}(\sqrt{x}) + \sqrt{x}\,\frac{d}{dx}(4x^3)$$

$$= \overset{2}{\cancel{4}}x^3\,\cancel{\tfrac{1}{2}}x^{-1/2} + \sqrt{x}\,12x^2$$

$$= \frac{2x^3}{\sqrt{x}} + 12x^2\sqrt{x}$$

[Note: There is also a *quotient rule*, but I have *never* had to use it, so I don't teach it☺]

Rule 4: The Chain Rule [Perhaps the most important rule?]

In calculus, the **chain rule** is a formula for computing the derivative of the composition of two or more functions. That is, if f is a function and g is a function, then the chain rule expresses the derivative of the composite function $f \circ g$ in terms of the derivatives of f and g. For example, the chain rule for $(f \circ g)(x)$ is

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

[The definition above is taken verbatim from http://en.wikipedia.org/wiki/Chain_rule -- ALWAYS CITE YOUR SOURCES!]

EXAMPLE 5: Acceleration is *defined* to be $\vec{a} = \frac{d\vec{v}}{dt}$. [See Example 7, below.] Let our discussion be defined to one linear dimension, say x, then $a = \frac{dv}{dt}$. But, what if you know the acceleration as a function of position, and not of time? The chain rule gives you $a(x) = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$, since the derivative of position with respect time is velocity. NOTE THAT $\frac{dv}{dx} \neq a$! ***Mastery of the chain rule generally requires significant practice!***

Here are several links to proofs of the chain rule:
http://math.rice.edu/~cjd/chainrule.pdf good, straight-forward discussion

http://www.youtube.com/watch?v=yAG2acoURtk a video that derives the chain rule in "real time." A different viewpoint.

EXAMPLE 6: 2nd Derivatives If you can do it once, why not do it again?

$$\frac{d^2[f(x)]}{dx^2} = \frac{d}{dx}\left\{\frac{d[f(x)]}{dx}\right\}; \quad \text{let } f(x) = x^3$$

$$\frac{d^2}{dx^2}(x^3) = \frac{d}{dx}(3x^2) = 6x$$

Check out this link: plots of tangent, 1st derivative, 2nd derivative

Very Important Ending Comments!

These notes have been written in the language of your math class; x is just some generic variable, and f is just some arbitrary function of x. **This is not the case in physics***!* x, y, z, and r carry the dimensions of length (units are nm, mm, cm, m, km, light years – whatever). The variable t stands for time (units are ns, μs, s, years …). We will need to define a variety of functions in terms of the derivatives of other functions.

EXAMPLE 7: Some kinematic definitions

- Velocity is the *rate of change* of displacement: $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$, where $\vec{r}$ is the three-dimensional position **vector**, and the dot is a shorthand for the time derivative. In one-dimension (motion back-and-forth along a line, say, in the x-direction) $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{x}}{\Delta t} = \frac{d\vec{x}}{dt} = \dot{x}$
- Acceleration is rate of change of velocity. In other words, it is the second derivative of displacement with respect to time: $\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = \ddot{\vec{r}}$ [See Example 5 above.]

Concerning *Dimensionally Incorrect Statements* [dis] – In math class, there is no problem writing statements such as sin(x), ln(t), e^x, tan(y), etc. However, in reality, these are all ***dis*** – x, y, and t all carry dimensions (length, length, and time), AND THE ARGUMENTS OF THESE FUNCTIONS MUST BE DIMENSIONLESS!

EXAMPLE 8: Some dimensionally correct statements

- $\ln\left(\frac{v_0}{v}\right)$ is dimensionally correct, since a speed divided by a speed is dimensionless.
- $e^{\frac{\tau}{t}}$ is dimensionally correct [τ, the Greek lower-case letter tau[3], has dimensions of time, so τ divided by t is dimensionless.]
- $\sin(\omega t) = \sin(2\pi ft)$ is dimensionally correct. ω, the Greek lower-case omega, is the *angular frequency* (also called the angular speed), and has dimensions of radians[4] per time. f is the *frequency*, and has dimensions of inverse time. So, $\omega t = 2\pi ft$, is a dimensionless argument.

Table of useful derivatives:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x$$

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

These are from the "cheat sheet" that College Board supplies to you when you take the AP exam in May. The first formula is a simplified form of the *chain rule* – you need it, and we will go over it the first week of class. We will, of course, need other derivatives besides this very short list. [Notice the absence of the Product Rule. Why, I wonder?]

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[3] You need to know the Greek alphabet – go here, and learn the alphabet.

[4] Radians are dimensionless. The definition of one radian, the angle subtended when the arc length equals the radius, gives $\theta = \frac{l}{r}$, which is dimensionless.