Intro to Quantum Mechanics 1 Solutions

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March 2014

Other than a couple grammatical flaws, there is a mistake that we realized after giving the lecture. On the the fourth page of the lecture notes, the statement $|\langle +y|+x\rangle|^2 = \frac{1}{2}\cos(\delta-\gamma)$ should be $|\langle +y|+x\rangle|^2 = \frac{1}{2}\left(1-\cos(\delta-\gamma)\right)$

1 Solutions

1. Using the hint that was given, we need to set $\delta - \gamma = -\frac{\pi}{2}$. This means that

$$|+x\rangle = \frac{e^{i\delta_+}}{\sqrt{2}}|+z\rangle + \frac{e^{i\delta_-}}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{e^{i\gamma_+}}{\sqrt{2}}|+z\rangle + \frac{e^{i\gamma_-}}{\sqrt{2}}|-z\rangle$$

Setting $\gamma = -\frac{\pi}{2}$ (thus allowing $\delta = -\pi$, and one of the relative phases to zero, we get

$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{-i\pi}}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i\frac{\pi}{2}}}{\sqrt{2}}|-z\rangle$$

This gives

$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle$$

But, the question asks for the bra $\langle +x|$ and $\langle +y|$. We thus take the adjoint of both ket vectors, and we get:

$$\langle +x| = \frac{1}{\sqrt{2}}\langle +z| - \frac{1}{\sqrt{2}}\langle -z|$$

$$\langle +y| = \frac{1}{\sqrt{2}}\langle +z| + \frac{i}{\sqrt{2}}\langle -z|$$

2. The eigenvalues of the matrix must satisfy the relationship provided at the end of the lecture:

$$det\,(\mathbf{O} - o\mathbf{1}) = 0$$

Consequently, if $\mathbf{O}$ is the rotation matrix, and since o represents a corresponding eigenvalue, we have the following:

$$\begin{vmatrix} \cos\theta - o & -\sin\theta & 0 \\ \sin\theta & \cos\theta - o & 0 \\ 0 & 0 & 1 - o \end{vmatrix} = 0$$

Computing the determinant, we have:

$$(\cos\theta - o)(\cos\theta - o)(1 - o) + \sin^2\theta(1 - o) = 0$$

Inspection tells us that one eigenvalue is $o = 1$. We are then left with the following equation, in which the left-hand side is known as the characteristic polynomial:

$$(\cos\theta - o)(\cos\theta - o) + \sin^2\theta = 0$$

$$\cos^2\theta - 2o\cos\theta + o^2 + \sin^2\theta = 0$$

$$\begin{aligned} o^2 - 2o\cos\theta + 1 &= 0 \\ o &= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \\ &= \cos\theta \pm i\sin\theta \\ &= e^{\pm i\theta} \end{aligned}$$

There are no more eigenvalues as a polynomial of third degree has exactly three roots. Now the more tedious part is to determine the eigenvectors. Let us start with the eigenvalue $o = 1$:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This provides us with the following three equations:

$$a\cos\theta - b\sin\theta = a$$

$$a\sin\theta + b\cos\theta = b$$

$$c = c$$

Disregarding c for now and solving for a and b, we are left with the following:

$$\frac{a\sin\theta}{1 - \cos\theta} = b$$

This system is undetermined (it has two free variables), so we can just express the eigenvector in terms of the free variables. Generally we try to normalize it so that the condition $|a|^2 + |b|^2 + |c|^2 = 1$ is satisfied:

$$\begin{pmatrix} a \\ \frac{a\sin\theta}{1-\cos\theta} \\ c \end{pmatrix}$$

We repeat the process for the other eigenvalues:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = e^{i\theta} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This yields the following three equations:

$$a\cos\theta - b\sin\theta = ae^{i\theta}$$

$$a\sin\theta + b\cos\theta = be^{i\theta}$$

$$c = ce^{i\theta}$$

We can solve for the following:

$$\frac{a\left(\cos\theta - e^{i\theta}\right)}{\sin\theta} = b$$

$$-ia = b$$

The condition on c implies that $c = 0$, as $e^{i\theta}$ isn't necessarily 1. Unlike the previous eigenvector, which didn't have a unique normalization, we can normalize this one as it only has one free variable. Since $\sqrt{|a|^2 + |b|^2} = 1$, we can let $a = \frac{e^{i\pi/4}}{\sqrt{2}}$, making $b = \frac{e^{-i\pi/4}}{\sqrt{2}}$. We are left with the following eigenvector:

$$\begin{pmatrix} \frac{e^{i\pi/4}}{\sqrt{2}} \\ \frac{e^{-i\pi/4}}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Similar calculations for the last eigenvalue $e^{-i\theta}$ result:

$$\begin{pmatrix} \frac{e^{-i\pi/4}}{\sqrt{2}} \\ \frac{e^{i\pi/4}}{\sqrt{2}} \\ 0 \end{pmatrix}$$

If you did not get these exact eigenvalues, do not fret. Any constant times an eigenvector is another eigenvector. Therefore, with trigonometric functions and complex numbers, it can be difficult to find an appropriate constant that will result the above eigenvectors.

3. We will use the linear property of the inner-product and also the simple facts from the Stern-Gerlach experiments that $\langle +z| + z\rangle = \langle -z| - z\rangle = 1$ and $\langle -z| + z\rangle = \langle +z| - z\rangle = 0$ to solve this problem.

$$\langle +z| + n\rangle = \langle +z| \left(\cos\frac{\theta}{2}| + z\rangle + e^{i\phi}\sin\frac{\theta}{2}| - z\rangle \right)$$

$$= \langle +z| \cos\frac{\theta}{2}| + z\rangle \;\; + \langle +z| e^{i\phi}\sin\frac{\theta}{2}| - z\rangle$$

$$= \cos\frac{\theta}{2}\langle +z| + z\rangle \;\; + e^{i\phi}\sin\frac{\theta}{2}\langle +z| - z\rangle$$

$$= \cos\frac{\theta}{2}$$

$$\langle -z| + n\rangle = \langle -z| \left(\cos\frac{\theta}{2}| + z\rangle + e^{i\phi}\sin\frac{\theta}{2}| - z\rangle \right)$$

$$= \langle -z| \cos\frac{\theta}{2}| + z\rangle \;\; + \langle -z| e^{i\phi}\sin\frac{\theta}{2}| - z\rangle$$

$$= \cos\frac{\theta}{2}\langle -z| + z\rangle \;\; + e^{i\phi}\sin\frac{\theta}{2}\langle -z| - z\rangle$$

$$= e^{i\phi}\sin\frac{\theta}{2}$$

It is very simple to verify the sum of the magnitudes squared of these quantities.

$$1 = |\langle +z| + n\rangle|^2 + |\langle -z| + n\rangle|^2$$
$$1 = |\cos\theta|^2 + |e^{i\phi}\sin\theta|^2$$
$$1 = \cos^2\theta + \sin^2\theta$$

Which is obviously true and we have verified the identity.

4. We are trying to calculate the expected value of a measurement on the angular momentum in the z-direction of a $+n$ state. This is the sum of each angular momentum value possible times the probability of acquiring that value. We have:

$$\langle S_z\rangle = |\langle +z| + n\rangle|^2 \left(\frac{\hbar}{2}\right) + |\langle -z| + n\rangle|^2 \left(\frac{-\hbar}{2}\right)$$

Substituting in for $|{+n}\rangle$ with $|{+n}\rangle = \cos\frac{\theta}{2}|{+z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|{-z}\rangle$ and factoring out $\frac{\hbar}{2}$, we have:

$$= \frac{\hbar}{2}\left[\left|\langle +z|\left(\cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-z\rangle\right)\right|^2 - \left|\langle -z|\left(\cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}\right)\right|^2\right]$$

$$= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)$$

$$= \frac{\hbar}{2}\cos\theta$$

Now let's calculate the standard deviation. Using the derivation of standard value:

$$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$$

$$(\Delta A) = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

In this case, since the measurement we are considering is S_z, $A = S_z$. First we need to calculate $\langle S_z^2\rangle$.

$$\langle S_z^2\rangle = |\langle +z|+n\rangle|^2\left(\frac{\hbar}{2}\right)^2 + |\langle -z|+n\rangle|^2\left(\frac{-\hbar}{2}\right)^2$$

Factoring out $\left(\frac{\hbar}{2}\right)^2$ and substituting in for $|+n\rangle$, we have:

$$= \left(\frac{\hbar}{2}\right)^2\left[\left|\langle +z|\left(\cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-z\rangle\right)\right|^2 + \left|\langle -z|\left(\cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}\right)\right|^2\right]$$

$$= \left(\frac{\hbar}{2}\right)^2\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)$$

$$= \frac{\hbar^2}{4}$$

Substituting back into the formula, we have:

$$(\Delta S_z) = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2\cos^2\theta}{4}}$$

$$= \frac{\hbar}{2}\sin\theta$$

Which is really neat: $\langle S_z\rangle = \frac{\hbar}{2}\cos\theta$ and $(\Delta S_z) = \frac{\hbar}{2}\sin\theta$

5. This is again a trivial problem; we just plug and chug values into the formula for standard deviation and expected value. However, we are looking

for measurements of angular momentum in the **x**-direction. Thus, we must compute

$$\langle +x| + n\rangle = \cos(\frac{\theta}{2})\langle +x| + z\rangle + \sin(\frac{\theta}{2})e^{i\phi}\langle +x| - z\rangle$$

$$\langle -x| + n\rangle = \cos(\frac{\theta}{2})\langle -x| + z\rangle + \sin(\frac{\theta}{2})e^{i\phi}\langle -x| - z\rangle$$

Upon simplification, we get

$$\langle +x| + n\rangle = \frac{1}{\sqrt{2}}\cos(\frac{\theta}{2}) + \frac{1}{\sqrt{2}}\sin(\frac{\theta}{2})e^{i\phi}$$

$$\langle -x| + n\rangle = \frac{1}{\sqrt{2}}\cos(\frac{\theta}{2}) - \frac{1}{\sqrt{2}}\sin(\frac{\theta}{2})e^{i\phi}$$

Since the expected value requires a probability (not an amplitude)

$$|\langle +x| + n\rangle|^2 = \frac{1}{2}\left(1 + \sin(\theta)\cos(\phi)\right)$$

$$|\langle -x| + n\rangle|^2 = \frac{1}{2}\left(1 - \sin(\theta)\cos(\phi)\right)$$

The expected values is thus

$$\langle S_x\rangle = \frac{\hbar}{4}\left(1 + \sin(\theta)\cos(\phi)\right) - \frac{\hbar}{4}\left(1 - \sin(\theta)\cos(\phi)\right)$$

$$\langle S_x\rangle = \frac{\hbar}{2}\sin(\theta)\cos(\phi)$$

The expected value of the square of the angular momentum

$$\langle S_x^2\rangle = \frac{\hbar^2}{8}\left(1 + \sin(\theta)\cos(\phi)\right) + \frac{\hbar^2}{8}\left(1 - \sin(\theta)\cos(\phi)\right)$$

$$\langle S_x^2\rangle = \frac{\hbar^2}{4}$$

Thus, the standard deviation is

$$\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2}$$

$$\Delta S_x = \frac{\hbar}{2}\sqrt{1 - \sin(\theta)\cos(\phi)}$$

6. Let us assume that the state is evolved as

$$|\psi\rangle \to e^{i\delta}|\psi\rangle$$

This means that the probability of obtaining a value a_i is

$$|\langle a_i|\psi\rangle|^2 = |\langle a_i|e^{i\delta}|\psi\rangle|^2 = e^{i\delta}e^{-i\delta}|\langle a_i|\psi\rangle|^2 = |\langle a_i|\psi\rangle|^2$$

A simple 1-line proof which demonstrates the invariance of the inner product with the evolution of a state by a phase factor. This is very important, as such transformations are unitary and will play an important role in time evolution. It is a trivial matter to prove that the expected value is unchanged as well. The expected value is

$$\langle A\rangle = \sum_i a_i|\langle a_i|\psi\rangle|^2$$

Because the probability is invariant, the sum is invariant, rendering the expected value unchanged.

7. This problem is written in terms of Gaussian units, hence the c in the denominator. To convert back to metric units, we just set $c = 1$. This is preferable, as we do not need to worry about factors of c flying around. It is a simple exercise to prove that $F = \mu_z \frac{\partial B}{\partial z}$, as it is just a matter of taking the negative gradient of the magnetic dipole potential $-\vec{\mu}\cdot\vec{B}$. The magnetic dipole moment $\mu_z = \frac{kq}{2m}S_z = \frac{kq\hbar}{4m}$, because the spin angular momentum of the particle is $\hbar/2$ in the z-direction. According to Newton's Law:

$$F_z = \mu_z \frac{\partial B}{\partial z} = m\frac{dv_z}{dt}$$

$$F_z = \frac{kq\hbar}{4m}\frac{\partial B}{\partial z} = m\frac{dv_z}{dt}$$

As you may know, the particle will travel in a circle under the presence of the magnetic field. The velocity will thus change direction as the particle gets deflected. However, the velocity will remain constant as the magnetic field does no work on the particle. However, the z-component of the velocity vector will change with time. We will assume the circle to have a radius of curvature R for the moment. In this problem, we will measure θ to be the angle made by the vertical along the circle. Thus

$$v_z = v\sin(\theta)$$

$$\frac{dv_z}{dt} = v\cos(\theta)\frac{d\theta}{dt} + \sin(\theta)\frac{dv}{dt} = v\cos(\theta)\frac{d\theta}{dt}$$

because v is constant. $\frac{d\theta}{dt} = \frac{v}{R}$ because the particle is undergoing uniform circular motion. Thus

$$\frac{dv_z}{dt} = \frac{v^2}{R}\cos(\theta)$$

However, we have yet to find v^2. We do know that the atoms are emitted from a source at a temperature T. According to the kinetic theory of gases

$$U = \frac{1}{2}m\langle v\rangle^2 = \frac{3}{2}k_bT$$

where k_b is the Boltzmann constant. Plugging this result into our previous equation:

$$\frac{dv_z}{dt} = \frac{3k_bT}{mR}\cos(\theta)$$

Plugging this back into our Newton's Law equation, we get

$$\frac{kq\hbar}{4m}\frac{\partial B}{\partial z} = m\frac{dv_z}{dt} = \frac{3k_bT}{R}\cos(\theta)$$

Upon inspection of a drawing, it immediately becomes clear that $\cos(\theta) = 1 - \frac{z}{R}$, where z measures the z-coordinate of the particle. Our general differential equation is

$$\frac{\partial B}{\partial z} = \frac{12mk_bT}{kq\hbar R}\left(1 - \frac{z}{R}\right)$$

However, the deflection of the particle is tiny compared to the radius of curvature of the circle it moves along. We can thus neglect the last term to obtain

$$\frac{\partial B}{\partial z} = \frac{12mk_bT}{kq\hbar R}$$

However, we must find the radius of curvature. Let us assume that it has traveled a total angle ϕ before hitting the detector. Through simple trigonometry, it becomes obvious that

$$R\sin(\phi) = l$$

$$R(1 - \cos(\phi)) = d$$

We can solve this system of equations by eliminating ϕ. We can do so by rewriting $\sin(\phi)$ in terms of $\cos(\phi)$ through the Pythagorean identity. This gives us the following equation:

$$\left(1 - \frac{d}{R}\right)^2 = 1 - \left(\frac{l}{R}\right)^2$$

Because $d << R$, the binomial expansion gives us

$$1 - 2\frac{d}{R} = 1 - \left(\frac{l}{R}\right)^2$$

$$R = \frac{l^2}{2d}$$

Plugging this into our formula, we get our answer

$$\frac{\partial B}{\partial z} = \frac{24mk_bTd}{kq\hbar l^2}$$

Note that the more exact answer is

$$\frac{\partial B}{\partial z} = \frac{32mk_bTd}{kq\hbar l^2}$$

Because the average velocity of the molecules which hit the detector is larger than what we obtained. This can be attributed to the fact that the higher energy atoms will be more likely to reach the detector than the lower energy ones.