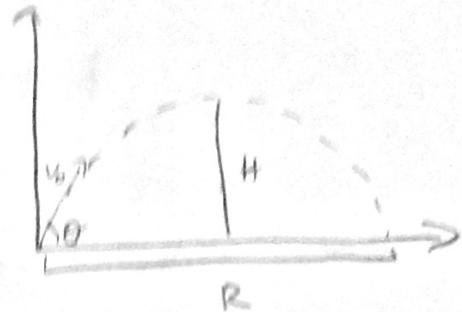


1. Let's find the range first. The projectile, since it starts at the ground, has a time-parametrization of its motion as follows:

$$(1) \quad x(t) = v_0 \cos \theta t$$

$$(2) \quad y(t) = v_0 \sin \theta t - \frac{1}{2} g t^2$$



When the projectile has travelled the range, it is easy to see that the above equations satisfy the following equation:

$$R = v_0 \cos \theta t$$

$$0 = v_0 \sin \theta t - \frac{1}{2} g t^2$$

Since $t \neq 0$ (since at $t=0$, the projectile hasn't moved), we can divide by t and solve for it in the second equation. Remember we aren't allowed to use t because it isn't given in the problem.

$$t = \underbrace{\frac{2v_0 \sin \theta}{g}}_{\text{Time of flight}} \quad R = v_0 \cos \theta t$$

Plugging in, we solve for R . Using simple trig identities, we have:

$$R = v_0 \cos \theta \left(\frac{2v_0 \sin \theta}{g} \right) = \frac{v_0^2 2 \sin \theta \cos \theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$$

Now let's find the maximum height. We know that by our second equation,

$$(1) H = v_0 \sin \theta t - \frac{1}{2} g t^2$$

This time t is different from the one we used to determine R , obviously. In order to determine it, we look to our velocity equation in the y -direction, which is just the derivative with respect to time of $y(t) = v_0 \sin \theta t - \frac{1}{2} g t^2$:

$$(3) \frac{dy}{dt} = v_y(t) = v_0 \sin \theta - gt$$

When the projectile reaches its apex, its velocity is 0 in the y -direction (not necessarily in the x though). One can verify this by tossing a pencil in the air. The velocity is 0 when the object starts to change direction, which is when it reaches its maximum height.

$$0 = v_0 \sin \theta - gt$$

$$t = \frac{v_0 \sin \theta}{g}$$

$$H = v_0 \sin \theta t - \frac{1}{2} g t^2$$

Solving for H:

$$H = \frac{V_0^2 \sin \theta}{2g}$$

This means:

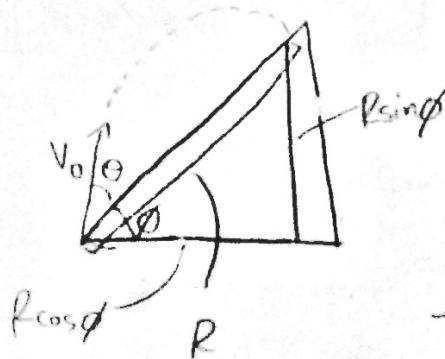
$$\frac{H}{R} = \frac{\left(\frac{V_0^2 \sin^2 \theta}{2g}\right)}{\left(\frac{V_0^2 \sin 2\theta}{g}\right)} = \frac{\left(\frac{\sin^2 \theta}{2}\right)}{2 \sin \theta \cos \theta} = \boxed{\frac{\tan \theta}{4}}$$

Notice how it doesn't depend on V_0 or gravity!

When $H=R$, $\frac{H}{R}=1$, and so:

$$1 = \frac{\tan \theta}{4} \rightarrow \boxed{\theta = \arctan(4)}$$

2. It's best to draw a picture first.



Realize that this problem is as if the cannonball were launched with angle $(\theta + \phi)$ and we are trying to maximize R by choosing an appropriate value of θ . Then let's find R as a function of θ .

But upon closer inspection, we see that this range is travelled on the hill when the (x, y) position is $(R\cos\phi, R\sin\phi)$, with the standard assumption that our reference frame has our launch point to be $(0, 0)$.

Using our kinematics equations:

$$R\cos\phi = v_0 \cos(\theta + \phi) t$$

$$R\sin\phi = v_0 \sin(\theta + \phi) t - \frac{gt^2}{2}$$

Solving for t in the first and plugging into the second:

$$\frac{R\cos\phi}{v_0 \cos(\theta + \phi)} = t$$

$$R\sin\phi = v_0 \sin(\theta + \phi) \left[\frac{R\cos\phi}{v_0 \cos(\theta + \phi)} \right] - \frac{g}{2} \left[\frac{R\cos\phi}{v_0 \cos(\theta + \phi)} \right]^2$$

$$\sin\phi = \frac{\sin(\theta + \phi) \cos\phi}{\cos(\theta + \phi)} - \frac{Rg \cos^2\phi}{2v_0^2 \cos^2(\theta + \phi)}$$

Multiplying through by $2v_0^2 \cos^2(\theta + \phi)$, as we are interested in finding R as a function of θ (ϕ and v_0 are constant).

we are left with:

$$2v_0^2 \cos^2(\theta + \phi) \sin\phi = 2v_0^2 \sin(\theta + \phi) \cos(\theta + \phi) \cos\phi = R \sin\phi$$

And furthermore solving for R:

$$R = \frac{2v_0^2 \sin(\theta + \phi) \cos(\theta + \phi) \cos\phi}{g \cos^2\phi} = \frac{2v_0^2 \cos^2(\theta + \phi) \sin\phi}{g \cos^2\phi}$$

$$= \frac{v_0^2}{g \cos^2\phi} \left[\sin(2\theta + 2\phi) \cos\phi - 2 \cos^2(\theta + \phi) \sin\phi \right]$$

$$= \frac{v_0^2}{g \cos^2\phi} \left[\sin(2\theta + 2\phi) \cos\phi - (1 + \cos(2\theta + 2\phi)) \sin\phi \right]$$

$$= \frac{v_0^2}{g \cos^2\phi} \left[\sin(2\theta + 2\phi) \cos\phi - \cos(2\theta + 2\phi) \sin\phi - \sin\phi \right]$$

$$R = \frac{v_0^2}{g \cos^2\phi} \left[\sin(2\theta + \phi) - \sin\phi \right] \quad \begin{matrix} \text{(ask yourself:} \\ \text{does this make} \\ \text{sense when } \phi = 0, \\ \text{ie. no hill ?)} \end{matrix}$$

R is clearly maximized when $\sin(2\theta + \phi)$ is maximized, which is when $1 = \sin(2\theta + \phi)$.

That means $\frac{\pi}{2} = 2\theta + \phi$. This means R is maximized when:

$$\theta = \frac{\pi}{4} - \frac{\phi}{2}$$

3. Centrifuges are used in chemistry to separate compounds using rotation. We know that since our centrifuge is moving with constant speed (the centrifuge is rotating with a constant acceleration magnitude in a circle), the following equation is true:

$$a = \frac{v^2}{R} = \omega^2 R \quad (\text{as } v = R\omega)$$

centrifugal acceleration

Then we have:

$$kg = \frac{|v|^2}{R} \rightarrow |v| = \boxed{\sqrt{Rkg}}$$

Although R isn't given in the problem, this is a mistake. We NEED the radius of motion to solve the problem.

The relationship between the angular frequency f and the angular velocity ω is:

$$\omega = 2\pi f \rightarrow f = \frac{\omega}{2\pi}$$

To solve for f :

$$f = \frac{\omega}{2\pi} = \boxed{\frac{\sqrt{\frac{a}{R}}}{2\pi}} = \boxed{\frac{1}{2\pi} \sqrt{\frac{kg}{R}}}$$

4. This problem is very similar to the previous one, and also really easy, as they are just formulas we should know.

$V = \omega R$, so $\boxed{\omega R}$ is the answer

$a = \omega^2 R$, so $\boxed{\omega^2 R}$ is the answer

5. Taking derivatives is how we solve this question.

$$x(t) = R\sin\omega t + R\omega t$$

$$v_x(t) = \frac{dx}{dt} = \boxed{R\omega\cos\omega t + R\omega}$$

$$a_x(t) = \frac{d^2x}{dt^2} = \boxed{-R\omega^2\sin\omega t}$$

$$y(t) = R\cos\omega t + R$$

$$v_y(t) = \frac{dy}{dt} = \boxed{-R\omega\sin\omega t}$$

$$a_y(t) = \frac{d^2y}{dt^2} = \boxed{-R\omega^2\cos\omega t}$$

not really necessary
for F=ma or
AP Physics, but useful

To find the tangential and normal components of the acceleration vector, we first realize that they must add to the acceleration vector.

$$\vec{a}(t) = -R\omega^2 \sin(\omega t) \hat{i} + -R\omega^2 \cos(\omega t) \hat{j} = \vec{a}_N + \vec{a}_T$$

We want to find a_N and a_T , which are given by the following formulas:

$$= \vec{a}_N \hat{N} + \vec{a}_T \hat{T}$$

unit normal vector unit tangent vector

$$a_N = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|}$$

$$= \begin{vmatrix} R\cos(\omega t) + R\omega & -R\sin(\omega t) \\ -R\omega^2 \sin(\omega t) & -R\omega^2 \cos(\omega t) \\ i & j & k \end{vmatrix}$$

$$\sqrt{(R\cos(\omega t) + R\omega)^2 + (-R\omega \sin(\omega t))^2}$$

$$= \frac{|-R\omega^3 \cos^2 \omega t - R\omega^3 \cos(\omega t) - R\omega^3 \sin^2 \omega t|}{\sqrt{2R^2 \omega^2 + 2R^2 \omega^2 \cos \omega t}}$$

$$= \frac{R^3 \omega^3 (1 + \cos \omega t)}{\sqrt{2} \cdot R\omega \sqrt{1 + \cos \omega t}}$$

$$= \boxed{\frac{\sqrt{2}}{2} R\omega^2 \sqrt{1 + \cos \omega t}}$$

$$a_T = \frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|}$$

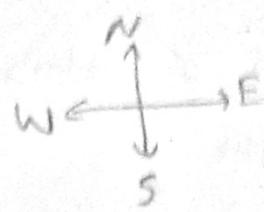
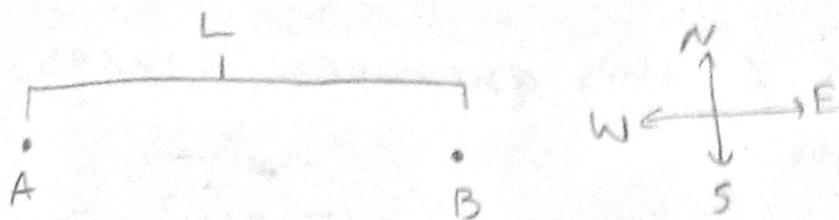
$$= \frac{-(R\cos(\omega t) + R\omega) R\omega^2 \sin(\omega t) - R\omega^3 \sin(\omega t) \cos(\omega t)}{\sqrt{(R\cos(\omega t) + R\omega)^2 + (-R\omega \sin(\omega t))^2}}$$

$$= \frac{R^2 \omega^3 \sin(\omega t) - 2R^2 \omega^3 \sin(\omega t) \cos(\omega t)}{\sqrt{2} \cdot R\omega \cdot \sqrt{1 + \cos \omega t}}$$

$$= \frac{R\omega^2 (\sin(\omega t) - \sin(2\omega t))}{\sqrt{2} \sqrt{1 + \cos \omega t}}$$

$$= \boxed{\frac{\sqrt{2}}{2} R\omega^2 \left(\frac{\sin \omega t - \sin 2\omega t}{\sqrt{1 + \cos \omega t}} \right)}$$

6.



This problem is relative motion. Going east in the first part of the trip, since both velocities are constant, we have:

$$t = \left| \frac{L}{|\vec{v}| + |\vec{u}|} \right|$$

And going west:

$$t = \left| \frac{-L}{-\vec{v} + \vec{u}} \right| = \left| \frac{L}{|\vec{v}| - |\vec{u}|} \right|$$

$|\vec{v}| = |\vec{u}|$,
they are
in opposite
directions

So our total time is:

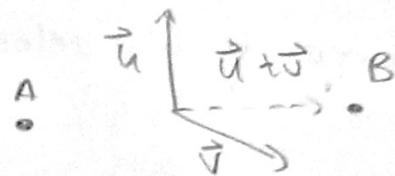
$$\begin{aligned} t &= \frac{L}{|\vec{v}| + |\vec{u}|} + \frac{L}{|\vec{v}| - |\vec{u}|} \\ &= \frac{L(\vec{v} + \vec{u}) + L(\vec{v} - \vec{u})}{(|\vec{v}| + |\vec{u}|)(|\vec{v}| - |\vec{u}|)} \end{aligned}$$

$$= \frac{2L\vec{v}}{|\vec{v}|^2 - |\vec{u}|^2}$$

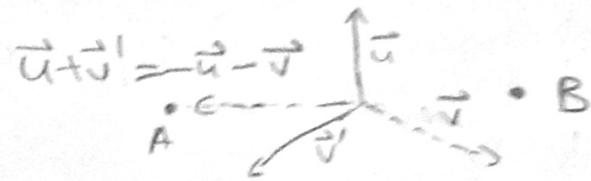
$$= \boxed{\frac{2Lu}{v^2 - u^2}}$$

The second part of this question requires more geometric intuition.

Going east:



Going west:



We know that the total time is:

\vec{v}' is the same velocity as the east velocity v , except a negated x -component

$$t = \left| \frac{L}{\vec{u} + \vec{v}} \right| + \left| \frac{L'}{\vec{u} - \vec{v}'} \right|$$

$$= \left| \frac{2L}{\vec{u} + \vec{v}} \right| = \frac{2L}{|\vec{u} + \vec{v}|}$$

$$|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2\vec{u} \cdot \vec{v} \\ = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v}$$

$$|\vec{u} + \vec{v}| = \sqrt{|\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v}}$$

$$t = \frac{2L}{\sqrt{|\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v}}}$$

$$= \frac{2L}{\sqrt{|\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot (-\vec{u})}}$$

$$= \frac{2L}{\sqrt{|\vec{v}|^2 + |\vec{u}|^2 - 2\vec{u} \cdot \vec{u}}} = \frac{2L}{\sqrt{|\vec{v}|^2 - |\vec{u}|^2}} =$$

Geometrically, $\vec{u} \cdot \vec{v}$ is the dot product of \vec{u} and whatever is parallel to \vec{u} in \vec{v} (which is just $-\vec{u}$!)

$$\boxed{\frac{2L}{\sqrt{|\vec{v}|^2 - |\vec{u}|^2}}}$$

The solutions to this problem set were written by me, Arun Kannan. If you notice any problems with the solutions or have any questions, please contact me at 2015akannan @tjhsst.edu.