Introduction to Optics

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1 Introduction

In this lecture, we examine geometrical optics, which treats light as a series of linear rays, ignoring its wave or photon nature. We neglect phenomena like diffraction or interference in this lecture, although we will discuss these in the lectures on waves. These assumptions are useful in cases where the objects with which light is interacting are much larger than the wavelength of light, and geometrical optics can be used to understand many practical devices like lenses and mirrors.

As a side note, the most advanced theory of geometrical optics was developed by Hamilton, and it has important applications in forming the Hamiltonian formulation of mechanics. Here, however, we will work out simpler results in geometrical optics and apply them to basic systems.

2 The Propagation of Light

It turns out that there are two equivalent empirical principles on the propagation of light. One is Huygen's principle. We will not use this principle in the current lecture, but it will be useful for our future discussion of waves, specifically the single-slit experiment.

> **Huygen's Principle**: Each point on a wavefront serves as the source of spherical secondary wavelets that advance at the wavespeed for the propagating medium. The primary wavefront at a later time is the envelope of these wavelets.

The principle we will use to understand geometrical optics is Fermat's principle. It turns out that this is equivalent to Huygen's principle by a proof we will not go into here.

> **Fermat's Principle of Least Action**: When traveling between two points, light takes the path that requires the least time.

We can derive several other important laws from Fermat's principle.

3 Reflection and Refraction

3.1 Law of Reflection

Consider a light ray traveling from point A and reflecting off the surface shown to point B. The time taken to traverse this path is

$$t = \frac{1}{c}\left(\sqrt{x^2 + h_1^2} + \sqrt{(\ell - x)^2 + h_2^2}\right).$$

We minimize this by taking the derivative with respect to x and setting it equal to zero, finding that

$$\begin{aligned} \frac{x}{\sqrt{x^2+h_1^2}} - \frac{\ell - x}{\sqrt{(\ell-x)^2+h_2^2}} &= 0, \\ \sin\theta_1 &= \sin\theta_2, \\ \theta_1 &= \theta_2. \end{aligned} \tag{1}$$

Equation 1 is the **law of reflection**: angle of incidence is equal to angle of reflection.

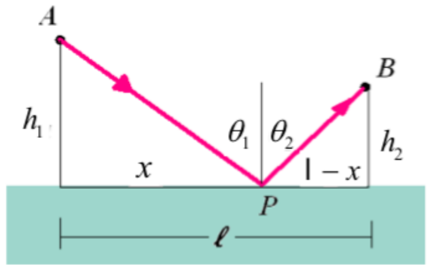

Figure 1: Setup for law of reflection derivation

3.2 Snell's Law

Light travels at different speeds in different media. This effect is described by the **index of refraction** n of the medium, which is defined so that the speed of light in the medium is $\frac{c}{n}$. When light travels from one medium to another, as a result of this change in speed, its path bends. We can derive a law for this bending, which is called **refraction** using Fermat's principle.

Consider a light ray originating from point A traveling toward point B through a change in medium, as shown in figure 2. The time it takes to traverse this path is

$$\frac{1}{c}\left(n_1\sqrt{x^2+h_1^2} + n_2\sqrt{(\ell-x)^2+h_2^2}\right).$$

As before, we set the derivative to zero to minimize the time.

$$\begin{aligned} n_1\frac{x}{\sqrt{x^2+h_1^2}} - n_2\frac{\ell - x}{\sqrt{(\ell-x)^2+h_2^2}} &= 0, \\ n_1\sin\theta_1 &= n_2\sin\theta_2. \end{aligned} \tag{2}$$

Equation 2 is known as **Snell's Law**.

3.3 Total Internal Reflection

When light travels from a medium with a higher index of refraction to one with a lower index of refraction, it refracts away from the normal. At some **critical angle** of incidence, the light will bend 90 degrees away from the normal, meaning that none of the light is transmitted to the second medium. At angles of incidence greater than this critical angle, a phenomenon called **total internal reflection** then takes place. All of the light is reflected back into the first medium.

Let the index of refraction of the first medium be n_1 and the index of refraction of the second be n_2. We

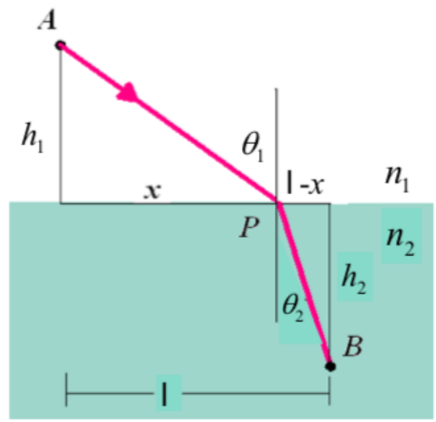

Figure 2: Setup for Snell's law derivation

can calculate the critical angle θ_c by Snell's law:

$$\begin{aligned} n_1 \sin\theta_c &= n_2 \sin 90^\circ, \\ \sin\theta_c &= \frac{n_2}{n_1}, \\ \theta_c &= \sin^{-1}\left(\frac{n_2}{n_1}\right). \end{aligned} \tag{3}$$

We can see that equation 3 implicitly contains the condition that $n_2 < n_1$, as sin must be less than 1. This, of course, is consistent with our conceptual understanding of total internal reflection.

One well-known application of total internal reflection is fiber optics. If light begins traveling parallel to an optical fiber's length, it will strike the walls of the fiber (given that the fiber does not twist severely) at angles larger than the critical angle, so no light will be lost. This allows for the use of fiber optics without massive energy losses.

3.4 Dispersion

The index of refraction of a material depends (slightly) on wavelength. This dependence is called **dispersion**. The most well-known example of dispersion is that of a glass prism. For such a prism, the angle of refraction is smaller for shorter wavelengths than for longer wavelengths. Thus, it disperses light into a rainbow.

4 Mirrors

In this section, we apply the principles of optics developed earlier in this lecture to one common optical device: **mirrors**. These take advantage of reflection.

4.1 Plane Mirrors

Let us examine figure 3, which depicts rays originating from a point source reflecting off a plane mirror. The paths of these rays are easy to determine, as we simply use the law of reflection (equation 11). Notice that they all seem to converge at a point behind the mirror, which is the image. In this case, it is a **virtual image**, because light does not actually emanate from that point.

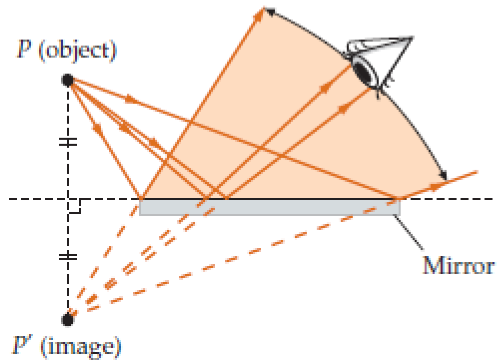

Figure 3: Reflection of rays from a plane mirror

In the figure, the image can be seen by an eye anywhere in the shaded region (which marks where there exists a line from the eye to the image which passes through the mirror). Note that the object does not need to be in front of the mirror for there to be an image visible from some point. It only has to be in front of the plane of the mirror.

Note also that the mirror exhibits **depth inversion**; that is, it reverses the front and back directions, as shown by figure 4. It transforms a right-handed coordinate system to a left-handed system.

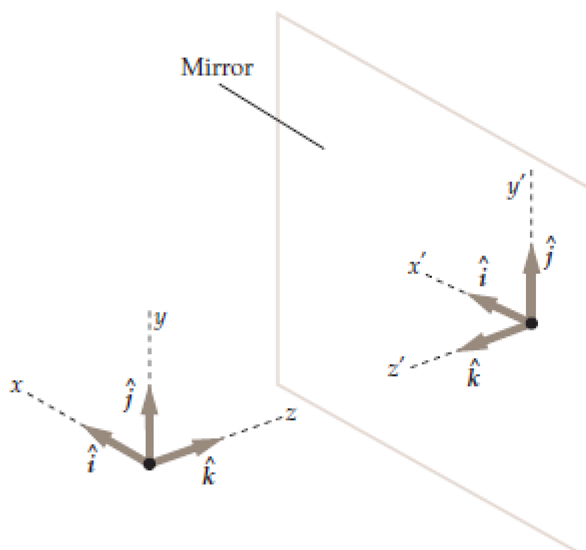

Figure 4: Depth inversion by a plane mirror

Finally, note that the mirror is the perpendicular bisector of the line between the object and its image. Thus, the distance from the mirror to the virtual image is equal to the distance from the object to the mirror. In addition, the mirror has a magnification of 1; the image size is equal to the object size. We can see this easily from figure 5. The ray from the top of the arrow perpendicular to the mirror is reflected straight back, so the image appears to have the same size as the object itself.

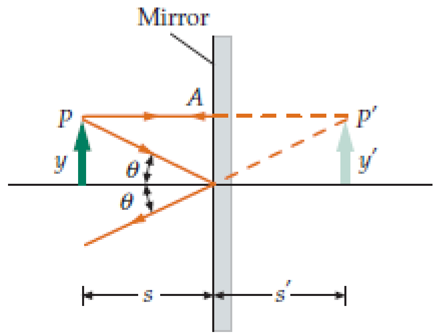

Figure 5: Image of an arrow formed by plane mirror

4.2 Spherical Mirrors

Having discussed plane mirrors, we can now analyze spherical mirrors, which are somewhat more interesting. There are two major kinds of spherical mirrors: concave and convex.

Consider the concave spherical mirror in figure 6. Reflected rays from the object intersect at P', forming a real image (unlike the virtual image formed by the plane mirror we studied earlier). We see that only rays that strike the mirror close to the axis are reflected through the image point, while rays far from the axis are reflected close to the image point but not exactly through it. This effect, which leads to blurring, is called **spherical aberration**, and the rays close to the axis are called **paraxial rays**.

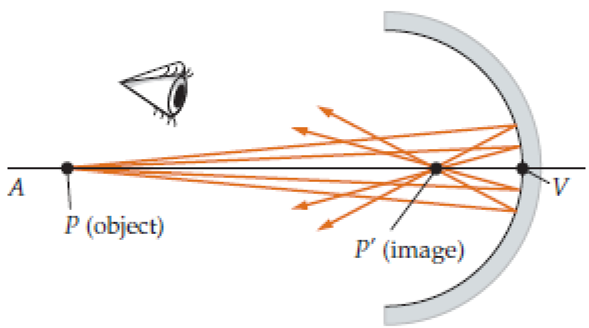

Figure 6: Reflection of rays from concave spherical mirror

We can derive an equation to describe image formation by this mirror. We do this by calculating the intersection point of the two rays emanating from an object point shown in figure 7. The bottom ray is chosen so that it reflects back on itself. This means that the tangent plane to the sphere at the point at which the ray hits the mirror is perpendicular to the ray, so the ray must pass through the center C of the sphere. Thus, r is the radius of curvature of the sphere, while s and s' are the distances from the object and the image to the center of the mirror, respectively. Using simple geometry, we find that

$$\alpha + \gamma = 2\beta.$$

We now use the small-angle approximation, as we assume the rays are paraxial. Then $\alpha \approx \ell/s$, $\beta \approx \ell/r$, and $\gamma \approx \ell/s'$, so

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{r}. \tag{4}$$

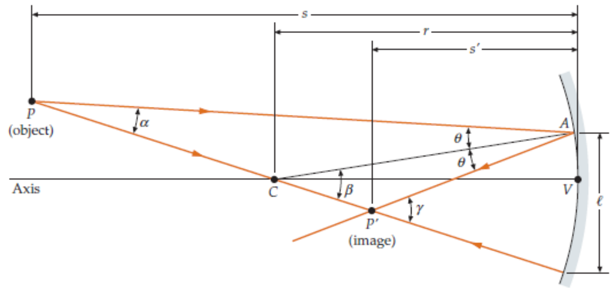

Figure 7: Geometric setup for mirror equation derivation

Observe that equation 4 contains no information related to the point A at which the ray hits the mirror. That implies that all paraxial rays emanating from the object point will intersect at the image point, as the equation applies to all of them.

When the object distance is large compared with the radius of curvature, the $\frac{1}{s}$ term in the equation is much smaller than the $\frac{2}{r}$ term, so we can neglect it. We then have that, as $s \to \infty$, the image distance approaches $\frac{1}{2}r$. We call this distance, the image distance of parallel rays coming in from infinity, the **focal length** f of the mirror. The plane on which these rays intersect is called the *focal plane*, and the intersection of the axis with the focal plane is called the **focal point** F. We can then write equation 4 in terms of the focal length, so that we have

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}. \tag{5}$$

Equation 5 is called the **mirror equation**.

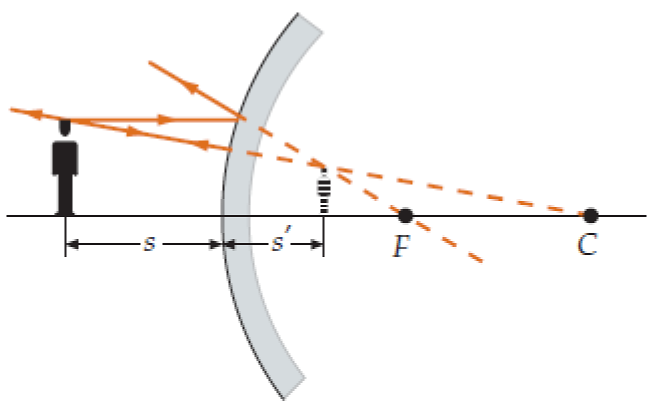

Figure 8: Convex mirror

The mirror equations applies to both concave mirrors, like the one in 7, and convex mirrors, like the one in 8. However, we must apply certain sign conventions. The object distance is by convention positive for objects on the incident-light side of the mirror, and the image distance is positive if the image is on the reflected-light side of the mirror (that is, if it is a real image). Note that, for mirrors, the incident-light and reflected-light sides are the same. In addition, the focal length is positive if the mirror is concave and negative if it is convex.

To conclude our treatment of spherical mirrors, we calculate the **lateral magnification** of such mirrors. This is the ratio of image height to object height. Consider figure 9. Using the fact that angle of incidence is equal to angle of reflection, it is clear by similar triangles that

$$\frac{y'}{y} = -\frac{s'}{s}, \tag{6}$$

where the negative sign comes from the fact that the image is inverted. Finally, we apply our spherical mirror

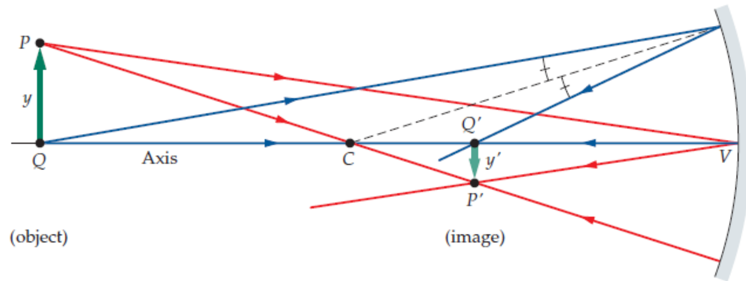

Figure 9: Geometry for showing the lateral magnification

equations to plane mirrors by using an infinite radius of curvature and therefore infinite focal length. Then equation 5 gives us that $s = -s'$, meaning that the image distance as the object distance, but the image is on the other side. Equation 6 then implies that plane mirrors have a magnification of 1, so the image has the same height as the object and is upright.

4.3 Ray Diagrams

To complete our treatment of mirrors, we discuss a useful technique for locating images. We can locate the image of an object by figuring out the intersection of the paths of three **principal rays**: the **parallel ray**, which is parallel to the axis, the **focal ray**, which passes through the mirror's focal point, and the **radial ray**, which passes through the center of curvature.The paths of these three rays constitute a **ray diagram**.

By the definition of the focus, the parallel ray will bounce off the mirror and pass through the focal point. By reversing this reasoning, it is simple to see that the focal ray will hit the mirror and be reflected off into a ray parallel to the axis. Finally, because the radial ray is drawn through the center of curvature, it will bounce off the mirror in the same way it would bounce off a plane mirror. The intersection of these three rays will give the image point.

Figure 10 depicts an example ray diagram. As you can see, it depicts the paths of the three principal rays, and the image is found at the point of intersection of these three. Ray diagrams like the example are useful in figuring out what sort of image forms. In the figure 10, the image is an inverted real image.

It is easy to prove using these diagrams that concave mirrors result in inverted real images for objects farther from the mirror than the focus and upright virtual images for objects closer to the mirror than the focus. We can also show that convex mirrors form upright virtual images for objects at any distance to the focus.

5 Lenses

We now apply our treatment of refraction to study **lenses**, another common optical device. To understand lenses, we must first understand the image that forms when paraxial rays are refracted by a spherical surface,

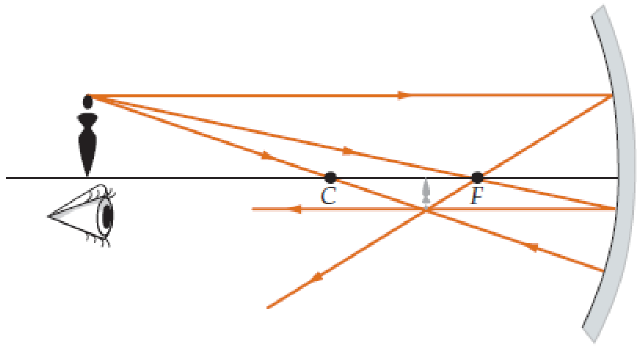

Figure 10: Example ray diagram

as shown in figure 11. Applying the small-angle approximation and Snell's law, we find that

$$n_1\theta_1 = n_2\theta_2.$$

From basic geometry, we have

$$\alpha + \beta = \theta_1$$

and

$$\theta_2 + \gamma = \frac{n_1}{n_2}\theta_1 + \gamma = \beta.$$

Then

$$\alpha + \beta = \frac{n_2}{n_1}(\beta - \gamma),$$
$$n_1\alpha + n_2\gamma = n_2\beta.$$

Using the small angle approximation, we have $\alpha \approx \frac{\ell}{s}$, $\beta \approx \frac{\ell}{r}$, and $\gamma \approx \ell s'$. Substituting these in, we find that

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r} \tag{7}$$

for refraction at a single surface.

It is easy to prove that the magnification for this refracting boundary is

$$\frac{y'}{y} = -\frac{n_1 s'}{n_2 s},$$

and we will leave the proof as an exercise.

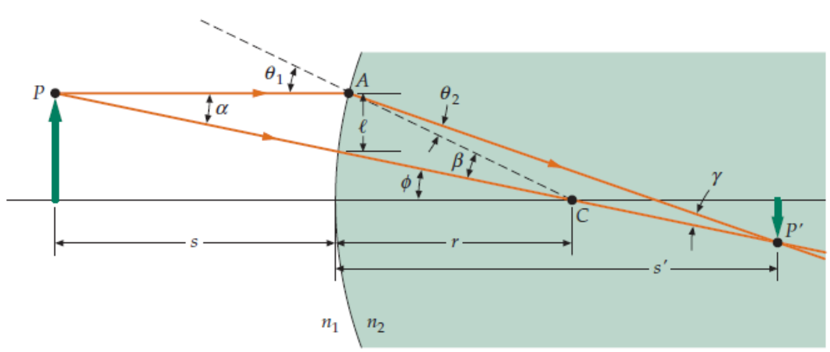

Figure 11: Refraction by a spherical surface

5.1 Thin Lenses

We can now analyze thin lenses. These consist of two refracting surfaces, as shown in figure 12. Let the radius of the first surface be r_1 and the radius of the second surface be r_2, and let the index of refraction of the lens be n. If the object is at a distance s from the center of the lens, we can find the distance s_1' of the image formed by refraction at the first surface by equation 7:

$$\frac{n_{\text{air}}}{s} + \frac{n}{s_1'} = \frac{n - n_{\text{air}}}{r_1}.$$

Note that s_1' here is negative, as the image is on the incident light side of the surface (it is virtual). The light

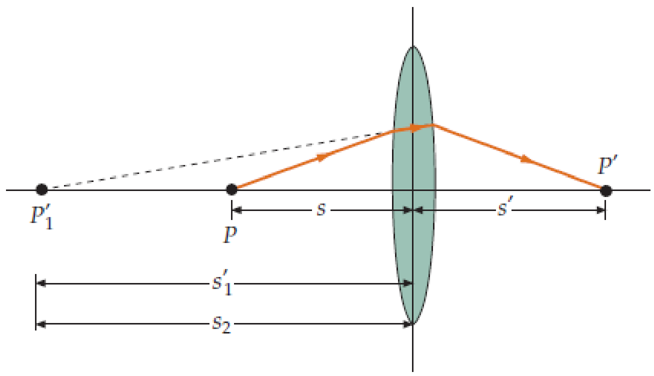

Figure 12: Thin lens

refracted from the first surface is then refracted again by the second surface. After being refracted by the first surface, the light looks like it is originating from the image point of that refraction, P_1', so we let the object distance of the light hitting the second surface be s_1'. Then the final image distance s' satisfies

$$\frac{n}{-s_1'} + \frac{n_{\text{air}}}{s'} = \frac{n_{\text{air}} - n}{r_2}.$$

We can add our two equations to find that

$$\frac{1}{f} = \left(\frac{n}{n_{\text{air}}} - 1\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{8}$$

where we define the focal length f to be the image distance when the object distance is infinite. Equation 8 is called the **lens-maker equation**. We also find the **lens equation**:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}. \tag{9}$$

Equation 9 is identical to the mirror equation (equation 5). However, the sign convention is different for lenses. The image distance is positive for when the image is on the refracted-light side of the lens and negative otherwise. That is, it is positive when the image is real. In addition, the focal length is determined by the radii of the refracting surfaces. In general, convex lenses like the one in figure 12 have positive focal lengths, whereas concave lenses like the one in figure 13 have negative focal lengths.

5.2 Ray Diagrams

Just as we can draw ray diagrams to determine image position for a mirror, we can use ray diagrams to find images for lenses. The three principal rays are the parallel ray, which will be directed toward the second focal point, the central ray, which is undeflected, and the focal ray, which is drawn through the first focal point

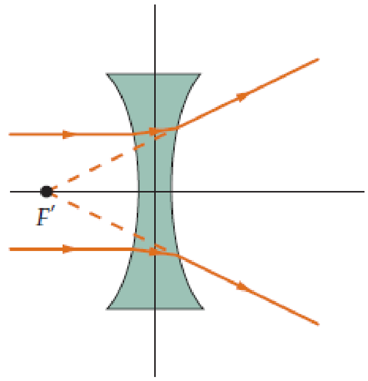

Figure 13: Concave lens

and will emerge parallel to the axis. These are shown in figure 14.

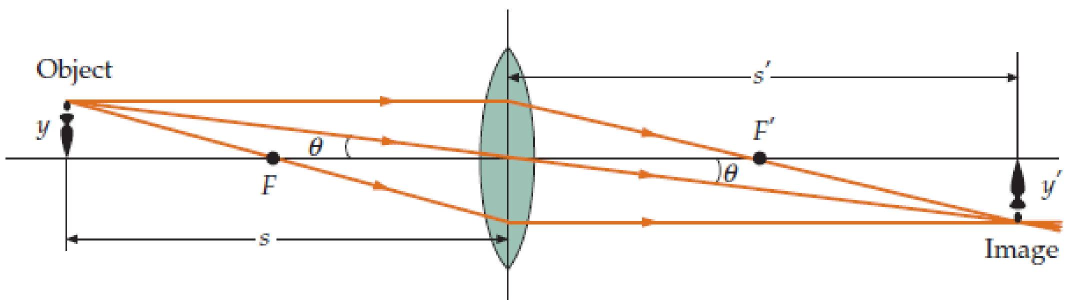

Figure 14: Ray diagram for lenses

It is straightforward to calculate the magnification of this lens from figure 14. By similar triangles, we have that

$$\frac{y'}{y} = -\frac{s'}{s}.$$

This is identical to the corresponding equation for mirrors (equation 6). As with mirrors, the image height is positive if the image is upright and negative if it is inverted.