Surface Integrals

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1 Recall...

This was all in Arvind's lecture on vector calculus, but I don't assume that any of you read that, so here is what you need to know.

1. In physics we only work with three-dimensional vectors, which can be represented as $\langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. We also write it as $a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$.
2. The *dot product*, or inner product, of two vectors is $\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1b_1 + a_2b_2 + a_3c_3$.
3. The *cross product* of two vectors is $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.
4. The *del* is an operator represented as $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$. When applied to a vector function it gives the *gradient*.
5. The *divergence* of a vector function is $\nabla \cdot f$
6. The *curl* of a vector function is $\nabla \times f$
7. Line integrals, denoted $\int_C F \cdot dr$, represent the integral of a function F over an arbitrary curve. It can be evaluated (because of the chain rule) as $\int_C F \cdot \frac{dr}{dt} dt$ for any parameterization $r(t)$.

2 Surface Integrals

Well, we can integrate over an arbitrary curve, so it's quite natural for us to want to integrate over an arbitrary surface.

Let $r(u, v)$ be a parameterization of a surface, S. We want a method to evaluate $\iint_S f(r(u, v))\, dS$, were dS is a small rectangle on the surface. The area of these small rectangles can be approximated by $|r_u \times r_v|\Delta u \Delta v$, so we write $dS = |r_u \times r_v| dA$. Then $\iint_S f(r(u, v))\, dS = \iint_D f(r(u, v))|r_u \times r_v|\, dA$, where dA can be any of the standard formulas: $du\, dv$, $r\, dr\, d\theta$, etc.

When we're working over a vector field, we want a unique vector to dot the field vector with. However, if we take one tangent to the surface, we have infinitely many to choice from. However, if we take a normal, there are only two *orientations* of the surface. To take these integrals, we also require that there be exactly two orientations; nonorientable surfaces are quite annoying for calculus purposes. So, if we let $\vec{n}$ be the unit normal vector to S at a point, then we define the surface integral of F over S to be $\iint\limits_S F \cdot \vec{n}\, dS$, which can then be integrated as before.

3 Theorems

Yeah, so these can be quite annoying to evaluate. Luckily there are theorems that we can use! Here, ∂D represents the *boundary* of D. That is, the points for which every open neighborhood around the point contains points of D and points of D^c, the complement. More formally, it is $\overline{D} \cap \overline{D^c}$, where $\overline{D}$ is the topological closure of D. Don't worry about this - I just wanted to make sure you were reading :P

- Green's Theorem: $\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \oint_{\partial D} P\, dx + Q\, dy$
- Stokes' Theorem: $\iint\limits_S \text{curl } F \cdot \vec{n}\, dS = \oint_C F \cdot d\vec{r}$. Here, C is the one-dimensional boundary of S.
- Divergence Theorem: $\iiint\limits_E \text{div } F\, dV = \iint\limits_{\partial E} F \cdot \vec{n}\, dS$