Compton Scattering: Solution

Call the momentum of the electron after the collision p_e, and call the momentum of the photon before and after the collision p_1 and p_2, respectively. Conservation of linear momentum gives

$$p_1 = p_e + p_2 \tag{1}$$

Keep in mind that we know the magnitude of both $p_1 = \frac{h}{\lambda}$ and $p_2 = \frac{h}{\lambda'}$, where λ' denotes the photon's wavelength after scattering. We also know that the angle between p_1 and p_2 is $\pi - \theta$, which means that $p_1 \dot{p_2} = p_1 p_2 cos(\theta)$. Conservation of energy implies that

$$E_{initial} = cp_1 + m_e c^2 = cp_2 + \sqrt{m_e^2 c^4 + c^2 p_e^2} = E_{final} \tag{2}$$

Also keep in mind that we care nothing at all about the magnitude of direction of p_e, so might as well get rid of it. In Eqn. 1, subtract p_2 from both sides and dot each side by itself to get

$$p_1^2 + p_2^2 - 2p_1 p_2 \cos(\pi - \theta) = p_e^2 \tag{3}$$

Plug this into Eqn. 2 to get

$$p_1 + m_e c = p_2 + \sqrt{m_e^2 c^2 + p_1^2 + p_2^2 + 2p_1 p_2 \cos(\theta)} \tag{4}$$

(I cancelled out the c's.) Moving terms and squaring gives

$$2p_1 p_2 \cos(\theta) = -2p_1 p_2 - 2m_e c p_2 + 2m_e c p_1 \tag{5}$$

(I cancelled out some terms that were the same on both sides, as well.) Substituting $p_1 = \frac{h}{\lambda}$ and $p_2 = \frac{h}{\lambda'}$ and rearranging some more gives the final answer

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos(\theta)) \tag{6}$$