# Mechanics

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July 5, 2015

## 1 Kinematics

Kinematics is the study of the motion of particles without reference to the forces causing the motion. All of the methods of kinematics can be essentially reduced to integration.

### 1.1 Constant Acceleration

The most common problems handled by kinematics concern a particle with a uniform acceleration in some direction. For example, a particle near the surface of the earth will experience a downward acceleration of $g = 9.8\,\mathrm{m/s^2}$. In the general case, let the acceleration vector be $\vec{a}$. The velocity as a function of time can be found by integrating:

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}\,dt = \vec{v}_0 + \vec{a}t.$$

Another integration gives the position as a function of time:

$$\vec{x}(t) = \vec{x}_0 + \int_0^t (\vec{v}_0 + \vec{a}t)\,dt = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2.$$

These two equations are the most important kinematic equations for cases of constant acceleration. In some cases, it is helpful to have an equation not involving the time. We can derive such an equation from the above two.

$$\begin{aligned}
\vec{v}(t) &= \vec{v}_0 + \vec{a}t \\
|\vec{v}(t)|^2 &= |\vec{v}_0|^2 + 2\vec{v}_0 \cdot \vec{a}t + |\vec{a}|^2 t^2 \\
|\vec{v}(t)|^2 &= |\vec{v}_0|^2 + 2\vec{a} \cdot (\vec{x}(t) - \vec{x}_0) \\
|\vec{v}(t)|^2 &= |\vec{v}_0|^2 + 2\vec{a} \cdot \Delta\vec{x}
\end{aligned}$$

The final equation relates the change in the speed to the dot product of acceleration and displacement.

### 1.2 Non-constant Acceleration

In a case where the acceleration $\vec{a}(t)$ varies with time, we can follow the same process as before but leave the integrals unevaluated.

$$\begin{aligned}
\vec{v}(t) &= \vec{v}_0 + \int_0^t \vec{a}(t)\,dt \\
\vec{x}(t) &= \vec{x}_0 + \int_0^t \vec{v}(t)\,dt
\end{aligned}$$

If a higher-order derivative of the position is given (e.g. jerk, or $\frac{d^3\vec{x}}{dt^3}$), then integrating the appropriate number of times recovers the position.

## 1.3 Rotational Kinematics

In the case of motion around a circle of radius $R$, we can define kinematic quantities analogous to the linear ones:

$$x \to \theta$$
$$v \to \omega$$
$$a \to \alpha$$

In the case of uniform circular motion, where $\omega$ is constant, a particle experiences an inward centripetal acceleration of magnitude $v^2/R = \omega^2 R$. When the motion around the circle is nonuniform, the acceleration can be decomposed into centripetal and tangential components. In the case where $\alpha$ is constant, the tangential acceleration is $\alpha R$ and the centripetal acceleration is $(\alpha t)^2 R = \alpha^2 R t^2$. Since these are two perpendicular components of the acceleration, the total acceleration is $\alpha R\sqrt{1+\alpha^2 t^4}$.

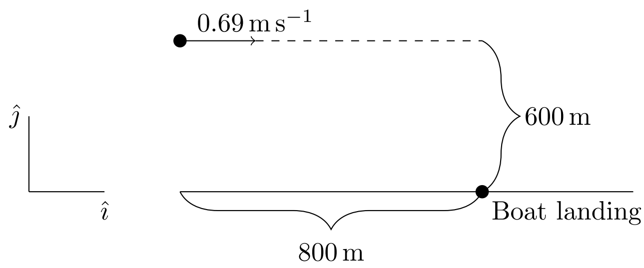
## 1.4 Worked Problems

The following problems, and other problems in this document, are from Mr. Hannum's 2013-14 physics tests.

1. Your copy of Tipler has fallen into a river and is floating way. It is being carried downstream by a current that is flowing with a uniform speed of $0.69\,\mathrm{m\,s^{-1}}$. Your book is 600 m from the shore and 800 m upstream from a boat landing when you set out in a rescue boat.

   (a) If your boat proceeds at its maximum speed of $5.56\,\mathrm{m\,s^{-1}}$ with respect to the water, what angle measured from the shore should you take?

   (b) How long will it take you to retrieve the most important book in your life?




The boat's velocity with respect to the water is $5.56\langle\cos\theta, \sin\theta\rangle$. Therefore, its absolute velocity is $5.56\langle\cos\theta, \sin\theta\rangle + \langle 0.69, 0\rangle$. Integrating the $y$-velocity, we get

$$y(t) = 5.56t\sin\theta$$
$$t = \frac{600}{5.56\sin\theta}$$

because the boat can only reach the Tipler book when it is 600 meters from the shore. The book's horizontal position is $0.69t$ and the boat's horizontal position is $800 + (5.56\cos\theta + 0.69)t$. Substituting for $t$, we can solve for $\theta$.

$$0.69t = 800 + (5.56\cos\theta + 0.69)t$$
$$0 = 800 + 5.56\cos\theta\frac{600}{5.56\sin\theta}$$
$$\cos\theta = -\frac{3}{4}$$
$$\theta \approx 138.6^\circ$$

Substituting this into the equation for $t$, we get $t = 163.15\,\mathrm{s}$.

2. Ball A is dropped from the top of a building at the same instant that ball B is thrown vertically upward from the ground. When the balls collide, they are moving in opposite directions, and the speed of A is twice the speed of B. At what fraction of the height of the building does the collision occur?

   The positions and velocities are easily found to be (taking upwards to be positive)

$$\begin{aligned} x_A &= h - \frac{1}{2}gt^2 \\ x_B &= v_0 t - \frac{1}{2}gt^2 \\ v_A &= -gt \\ v_B &= v_0 - gt \end{aligned}$$

   When the collision occurs, $x_A = x_B$, so $t = h/v_0$. Substituting into the velocity equations, we have

$$\begin{aligned} -g\frac{h}{v_0} &= -2\left(v_0 - g\frac{h}{v_0}\right) \\ \frac{3gh}{v_0} &= 2v_0 \\ 2v_0^2 &= 3gh \end{aligned}$$

   We can use this to determine the height at which the collision occurs:

$$\begin{aligned} x_A &= h - \frac{1}{2}g\left(\frac{h}{v_0}\right)^2 \\ &= h - \frac{gh^2}{2v_0^2} \\ &= h - \frac{g^2}{3gh} \\ &= \boxed{\frac{2}{3}h} \end{aligned}$$

# 2 Forces

## 2.1 Newton's Laws

All of the theory of Newtonian mechanics is contained in Newton's three laws. The first states the law of inertia, and the third states that every force has an opposite reaction force. These are relatively unimportant compared to the second law, the quantitative statement of Newtonian dynamics:

$$\vec{F} = m\vec{a}.$$

The force $\vec{F}$ is the net external force on a system, and $\vec{a}$ is the acceleration of its center of mass. For now, we will break up every system into components which can be treated as points, so that we do not need to deal with the notion of a center of mass.

## 2.2 Important Forces

Newton's second law tells us how forces interact with particles, but we still need to know what forces are at play. There are several important forces which commonly appear in problems.

### 2.2.1 Gravity

Objects near the surface of the earth experience a downward graviational force of magnitude $mg$. The implications of this have been dealt with in kinematics.

### 2.2.2 Normal Force

When two objects are in contact, they can exert normal forces on each other. These forces act as constraints to prevent one object from penetrating another. For example, a block of mass $m$ sitting on a table experiences a downward force $mg$, so the normal force from the table has the same magnitude directed upwards so that the block does not accelerate into the table.

### 2.2.3 Tension

A rope exerts a tension force $T$ at both ends. In an ideal rope (a massless and unbreakable rope), any force exerted on one end of the rope creates an opposing tension. The tension acts, in a sense, as a transfer mechanism.

With gravity and tension, we can solve the famous Atwood's Machine problem. Given masses $m_1$ and $m_2$ connected by a rope strung over a pulley, we wish to find the accelerations of both blocks.

Newton's second law gives us two equations, one for each block. Note that the positive direction corresponds to $m_1$ moving up and $m_2$ moving down. The accelerations are both $a$ because the rope cannot stretch, so they must move at the same rate.

$$
\begin{aligned}
T - m_1 g &= m_1 a \\
m_2 g - T &= m_2 a
\end{aligned}
$$

By adding these two equations, we can find that $a = \frac{m_2 - m_1}{m_1 + m_2} g$.

### 2.2.4 Springs

Ideal springs obey Hooke's law: $F = -kx$. In this equation, $x$ is the displacement of the spring from its natural length. The negative sign refers to the fact that the spring force is a restoring force; if the spring is extended beyond its natural length the force tends to compress the spring.

### 2.2.5 Friction

Two materials in contact have characteristic coefficients of static friction ($\mu_s$) and of kinetic friction ($\mu_k$). The kinetic coefficient is always less than the static coefficient. The static coefficient is used when the objects are not in relative motion, and the kinetic coefficient is used when they are. The frictional force is proportional to the normal force between the two objects:

$$F_f = \mu N$$

Its direction is opposite to the direction of motion (or "desired" motion, in the static case) of the object.

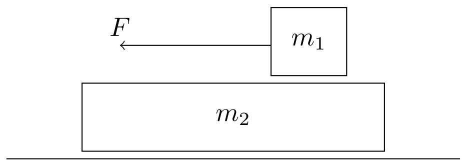
## 2.3 Simple Harmonic Motion

Consider a mass $m$ attached to one end of a spring that has its other end fixed. Newton's second law says $ma = -kx$. We can write this as a differential equation and solve:

$$
\begin{aligned}
\ddot{x} &= -\frac{k}{m} x \\
x &= A \cos\left(\sqrt{\frac{k}{m}} t + \phi\right)
\end{aligned}
$$

The amplitude $A$ and phase $\phi$ are determined by initial conditions.

Another system with simple harmonic motion is the pendulum, with a string of length $L$ holding a mass $m$. When the mass oscillates at small angles, so that $\sin x \approx x$, the angle oscillates so that $\theta(t) = A \cos\left(\sqrt{\frac{g}{L}} t + \phi\right)$. Therefore, the period of a pendulum is $2\pi\sqrt{\frac{L}{g}}$.



## 2.4 Worked Problems

1. In the figure above, a slab of mass $m_1$kg rests on a frictionless flor, and a block of mass $m_2$ rests on top of the slab. The surfaces between the block and the slab are rough, and governed by a static coefficient of friction $\mu_s$ and kinetic coefficient of friction $\mu_k$. A horizontal force $F$ pulls the block.

   (a) Determine the maximum force $F_{\text{max}}$ that can be applied to the block, such that the block and the slab move together.

   (b) If $5F_{\text{max}}$ is applied to the block determine the resulting accelerations of both the block and the slab.

   We can write three expressions of Newton's second law. First, in the horizontal direction for $m_1$ (left is positive),

   $$F - F_f = m_1 a_1$$

   In the vertical direction for $m_1$,

   $$F_N - m_1 g = 0$$

   In the horizontal direction for $m_2$,

   $$F_f = m_2 a_2$$

   The vertical direction for $m_2$ is unimportant since the floor is frictionless. We see from the vertical equation for $m_1$ that $F_N = m_1 g$. So, in the static case, we can substitute the maximum value of $F_f$ in the first and second equations:

   $$F - \mu_s m_1 g = m_1 a_1$$
   $$\mu_s m_1 g = m_2 a_2$$

   From this, we can easily solve for the accelerations:

   $$a_1 = \frac{F - \mu_s m_1 g}{m_1}$$
   $$a_2 = \frac{\mu_s m_1 g}{m_2}$$

   Setting these equal, we obtain

   $$F_{\text{max}} = \frac{m_1}{m_2}(m_1 + m_2)\mu_s g$$

   In part b, where the block is moving with respect to the slab, we can replace $\mu_s$ with $\mu_k$ in the equations for the accelerations. We then arrive at

   $$a_1 = \frac{5F_{\text{max}} - \mu_k m_1 g}{m_1} = g\left(\frac{5\mu_s(m_1 + m_2)}{m_2} - \mu_k\right)$$
   $$a_2 = \frac{\mu_k m_1 g}{m_2}$$

2. A red blood cell is traveling inside a straight segment of a blood vessel is subject to a drag force that is given by:

   $$F_D = k\sqrt{x}$$

   (a) Provide a differential equation that models the motion of the cell in the vessel.

   (b) Set up, but do not solve an integral that would provide te velocity of the cell as a function of position. You may assum ethat the cell has some initial velocity $v_0$ and starts from $x = 0$.

(c) Determine the ratio $\frac{x_{1/4}}{x_{1/2}}$ where $x_{1/2}$ is the distance the cell travels as its velocity drops from $v_0$ to $v_0/2$ and $x_{1/4}$ is defined analogously.

The answer to part a follows directly from Newton's second law:

$$\ddot{x} = -\frac{k}{m}\sqrt{x}$$

To convert the time dependence to a position dependence for part b, we need the chain rule:

$$\begin{aligned}
\frac{dv}{dt} &= -\frac{k}{m}\sqrt{x} \\
\frac{dv}{dx}\frac{dx}{dt} &= -\frac{k}{m}\sqrt{x} \\
v\frac{dv}{dx} &= -\frac{k}{m}\sqrt{x} \\
\int_{v_0}^{v} v\,dv &= \int_0^x -\frac{k}{m}\sqrt{x}\,dx
\end{aligned}$$

By carrying out the integrals in this equation, we can find the answer to part c.

$$\begin{aligned}
\int_{v_0}^{v_0/4} v\,dv &= \int_0^{x_{1/4}} -\frac{k}{m}\sqrt{x}\,dx \\
-\frac{15}{16}v_0^2 &= -\frac{2k}{3m}x_{1/4}^{3/2} \\
\frac{45v_0^2 m}{32k} &= x_{1/4}^{3/2}
\end{aligned}$$

In a similar fashion, we find that $x_{1/2}^{3/2} = \frac{9v_0^2 m}{8k}$. Therefore, the desired ratio is

$$\frac{x_{1/4}}{x_{1/2}} = \left(\frac{5}{4}\right)^{\frac{2}{3}}$$

# 3 Conservation Laws

Often only the initial and final states of a system are of interest, and not the states in the interim. In such cases, conservation laws are very useful.

## 3.1 Energy and Work

Work is defined by the line integral

$$W = \int_A^B \vec{F}\cdot d\vec{r}$$

The work, according to the Work-Energy theorem, is equal to the total change in kinetic energy:

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Some forces have a potential energy $U$ such that $F = -\frac{dU}{dx}$. For example, the gravitational potential is $U = mgh$ and the spring potential is $U = \frac{1}{2}kx^2$. In this case, we know from calculus that $W = U_i - U_f$. Combining this with the work-energy theorem, we obtain the conservation of energy:

$$\frac{1}{2}mv_i^2 + U_i = \frac{1}{2}mv_f^2 + U_f$$

This equatin only holds when a potential energy $U$ exists. Forces which have potential energies are called conservative forces. Friction is a notable non-conservative force for which the conservation of energy does not hold.

Power is defined as the time derivative of work, $dW/dt$, or equivalently, $\int F\,dv$.

## 3.2 Momentum

Momentum is defined as $\vec{p} = m\vec{v}$. Newton's second law is more accurately expressed using the momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

This can be expanded using the product rule, and as long as the mass remains constant (i.e. $dm/dt = 0$) it simplifies to the usual form of Newton's second law.

This equation means that force is the instantaneous rate of change of momentum. Thus, the total change in momentum divided by time elapsed is the average rate of change, or the average force. Total change in momentum is called the impulse $\vec{I}$. So, $\vec{I} = \vec{F}_{avg}t$.

In cases where there is no external force, $\vec{F} = d\vec{p}/dt = 0$ and so momentum is a constant. This is very important when handling collisions.

### 3.2.1 Inelastic Collisions

In an inelastic collision, two objects collide and stick together, continuing their motion with the same velocity. The only forces are the contact forces between the objects, and these are internal, so they cancel and do not contribute to any change in momentum of the system. Therefore, momentum is conserved, and we can set the initial momentum equal to the final momentum:

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}_f$$
$$\vec{v}_f = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

### 3.2.2 Elastic Collisions

In an elastic collision, energy is conserved. Such collisions are best handled in the center-of-mass frame of the system. If we use a frame moving with a velocity equal to that of the center of mass,

$$v_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2},$$

then the momentum is 0:

$$m_1v_{1,cm} + m_2v_{2,cm} = m_1v'_{1,cm} + m_2v'_{2,cm} = 0$$

Additionally, we have the conservation of energy

$$m_1v_{1,cm}^2 + m_2v_{2,cm}^2 = m_1(v'_{1,cm})^2 + m_2(v'_{2,cm})^2$$

We can substitute (according to the conservation of momentum) $v_{2,cm} = -\frac{m_1}{m_2}v_{1,cm}$ and similarly for the primed velocities to obtain

$$m_1v_{1,cm}^2 + m_2\left(-\frac{m_1}{m_2}v_{1,cm}\right)^2 = m_1(v'_{1,cm})^2 + m_2\left(-\frac{m_1}{m_2}v'_{1,cm}\right)^2$$
$$\left(m_1 + \frac{m_1^2}{m_2}\right)v_{1,cm}^2 = \left(m_1 + \frac{m_1^2}{m_2}\right)(v'_{1,cm})^2$$
$$v_{1,cm}^2 = (v'_{1,cm})^2$$

We can obtain a similar equation for $v_{2,cm}$. The only nontrivial solution is $v'_{1,cm} = -v_{1,cm}$ and $v'_{2,cm} = -v_{2,cm}$. Transforming back to the laboratory frame (using $v_{1,cm} = v_1 - v_{cm}$ and similar relations) to obtain

$$v'_1 = 2v_{cm} - v_1$$
$$v'_2 = 2v_{cm} - v_2$$

1. Inspired by the recent cold wind, Mr. Hannum sets out on his iceboat of mass $M$. Initially, he is stationary, however the wind picks up and starts to push the boat. Assuming there is no loss of energy due to sliding friction, the power delivered by the wind to the boat is given by:

$$P(t) = A\left(\frac{1}{2} - \frac{\cos(2Bt)}{2}\right)$$

   (a) Derive an expression for the velocity of the boat in terms of $t$, $A$, $B$, and $M$.

   (b) What is the average force of the wind pushing on the boat during the time interval $t = 0$ to $t = 2\pi/B$?

   First, we determine the kinetic energy by integrating the power function.

$$K(t) = \int_0^t P(t)\,dt = \frac{A}{2}\int_0^t 1 - \cos(2Bt), dt = \frac{At}{2} - \frac{1}{4B}\sin(2Bt)$$

   The velocity is then simply

$$v(t) = \sqrt{\frac{2}{M}K(t)} = \sqrt{\frac{At}{m} - \frac{1}{2MB}\sin(2Bt)}$$

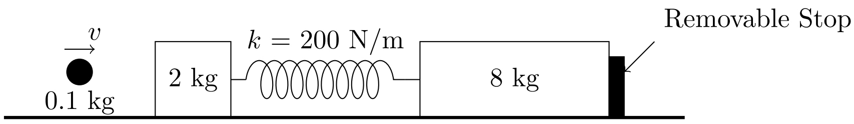
   To find the average force, we can determine the impulse.

$$I = M\left(v\left(\frac{2\pi}{B}\right) - v(0)\right) = \frac{2\pi AM}{B}$$

   The average force is then

$$F_{\text{avg}} = I/\Delta t = \sqrt{\frac{BAM}{2\pi}}$$

2. A 2 kg block and an 8 kg block are both attached to an ideal spring (for which $k$ =200 N/m) and are initially at rest on a horizontal frictionless surface, as shown in the diagram below.




   In an initial experiment, a 100 g ball of clay is thrown at the 2 kg block. The clay is moving horizontally with speed $v$ when it hits and sticks to the block. The 8 kg block is held still by a removable stop. As a result, the spring compresses a maximum distance of 0.4 m.

   (a) Calculate the energy stored in the spring at maximum compression.

   (b) Calculate the speed of the clay ball and the 2 kg block immediately after the clay sticks to the block but before the spring compresses significantly.

   (c) Calculate the initial speed $v$ of the clay.

   In a second experiment, an identical ball of clay is thrown at another identical 2 kg block, but this time the stop is removed so that the 8 kg block is free to move.

   (d) State whether the maximum compression of the spring will be greater than, equal to, or less tha 0.4 m. Explain your answer briefly.

   (e) What are the velocities of the two blocks at the instant the spring regains its original length?

The energy stored in the spring is

$$\frac{1}{2}kx^2 = \frac{1}{2}(200\,\mathrm{N\,m^{-1}})(0.4\,\mathrm{m})^2 = 16\,\mathrm{J}$$

From conservation of energy, this is the same as the initial kinetic energy of the ball and block. So,

$$\frac{1}{2}(2.1\,\mathrm{kg})(v')^2 = 16\,\mathrm{J}$$
$$v' = 3.90\,\mathrm{m\,s^{-1}}$$

From the conservation of momentum in the inelastic collision,

$$v = \frac{2.1}{1.1}v' = 82.0\,\mathrm{m\,s^{-1}}$$

In the second experiment, the maximum compression will be less than 0.4 m, because some of the energy will instead be transferred the motion 8 kg block.

The center of mass velocity is

$$v_{cm} = \frac{2.1 \cdot 3.90\,\mathrm{m\,s^{-1}}}{10.1} = 0.81\,\mathrm{m\,s^{-1}}$$

Therefore, the velocities following the elastic collision described are

$$v_1' = 2v_{cm} - v_1 = -2.28\,\mathrm{m\,s^{-1}}$$
$$v_2' = 2v_{cm} - v_2 = 1.62\,\mathrm{m\,s^{-1}}$$

# 4 Rotation

When dealing with objects that cannot be approximated as points, we must consider their rotation as well as their translation.

## 4.1 Moment of Inertia

The moment of inertia is the analog of mass in rotational dynamics. It is defined as

$$I = \int r^2\,dm = \int \rho r^2\,dV$$

Some of the most commonly used moments of inertia are

- Hoop: $MR^2$
- Disk: $\frac{1}{2}MR^2$
- Bar (rotating about center): $\frac{1}{12}ML^2$
- Hollow sphere: $\frac{2}{3}MR^2$
- Solid sphere: $\frac{2}{5}MR^2$

The parallel axis theorem says that shifting the axis from the center of mass by a distance $x$ adds $Mx^2$ to the moment of inertia. For example, the moment of inertia of bar rotating about its end is $\frac{1}{12}ML^2 + M(L/2)^2\frac{1}{3}ML^2$.

The perpendicular axis theorem applies to objects in the plane. If an object has all its mass in the $xy$-plane, then $I_z$ (the moment of inertia about the $z$-axis) is $I_x + I_y$. This is especially useful when an object is symmetric with respect to an interchange of $x$ and $y$, so that $I_x = I_y$. For example, a disk has all its mass in a plane, so $I_z = \frac{1}{2}MR^2 = I_x + I_y = 2I_x$. So, the disk has a moment of inertia of $\frac{1}{4}MR^2$ about an axis going through its diameter.

## 4.2 Dynamics

In rotation, the analog of force is the torque $\tau = \vec{r} \times \vec{F}$. The rotational form of Newton's second law is

$$\tau = I\alpha$$

With this equation, we can solve the Atwood's machine problem in the case of a pulley with friction mass $M$. In this case, the tension does not have to be the same on both sides (because of the friction). So, we can write Newton's law twice in linear form and once in rotational form:

$$\begin{aligned} T_1 - m_1 g &= m_1 a \\ m_2 g - T_2 &= m_2 a \\ T_2 R - T_1 R &= \left(\frac{1}{2} M R^2\right) \alpha \end{aligned}$$

The last of these equations, after substiuting $\alpha = a/R$, simplifies to $T_2 = T_1 + \frac{1}{2} M a$. Therefore, upon adding the first two equations, we have

$$\begin{aligned} (m_2 - m_1) g - \frac{1}{2} M a &= (m_1 + m_2) a \\ a &= \frac{(m_2 - m_1) g}{m_1 + m_2 + \frac{1}{2} M} \end{aligned}$$

### 4.2.1 Physical Pendulum

A pendulum can be formed by putting a pivot on a rigid body a distance $L$ from its center of mass. The period of this pendulum can be computed from its moment of inertia $I$ about this pivot:

$$t = 2\pi \sqrt{\frac{I}{mgL}}$$

## 4.3 Angular Momentum

The angular momentum of a particle is defined as $\vec{L} = \vec{r} \times \vec{p}$ and is related to torque in the same way that momentum is related to force:

$$\tau = \frac{d\vec{L}}{dt}$$

For a rigid body, the angular momentum is $\vec{L} = I\vec{\omega}$, and obeys the same equation. In the absence of net external torque (a somewhat more general condition than zero external force), angular momentum is conserved.

A spinning object with an external torque will experience a gyroscopic precession with frequency

$$\omega_p = \frac{mgr}{I_s \omega_s}$$

$I_s$ is the moment of inertia about the spin axis and $\omega_s$ is the spin angular velocity.

## 4.4 Rotational Energy

The rotational kinetic energy of a rigid body is defined in analogy to the linear kinetic energy of a particle:

$$K = \frac{1}{2} I \omega^2$$

## 4.5 Worked Problems

1. A system consists of two small disks, or masses $m$ and $2m$, attached to a rod by a string of negligible mass of length $3L$ so that mass $m$ rotates in a circle of radius $2L$ and mass $2m$ rotates in a circle of radius $L$. The rod is free to turn about a vertical axis through point P. The two disks rest on a rough horizontal surface; the coefficient of friction between the disks and the surfaces is $\mu$. At time $t = 0$, the rod has an initial counterclockwise angular velocity $\omega_0$ abuot P. The system is gradually brought to rest by friction. Develop expressions for the following quantities in terms of $\mu$, $m$, $L$, $g$, and $\omega_0$.

   (a) The initial angular momentum of the system about the axis through P.

   (b) The frictional torque acting on the system about the axis through P.

   (c) The time $t$ at which the system will cone to rest.

   The angular momentum is given by $I\omega$ or

   $$L = I\omega_0 = \left(2mL^2 + m(2L)^2\right)\omega_0 = 6mL^2\omega_0$$

   The torque is

   $$\tau = \sum \vec{F} \times \vec{r} = -(\mu mg)(2L)\hat{k} - (2\mu mg)(L)\hat{k} = -4\mu mgL\hat{k}$$

   The system comes to rest when the angular velocity vanishes:

   $$\begin{aligned} \omega_0 + \alpha t &= 0 \\ \omega_0 + \frac{I}{\tau} t &= 0 \\ \omega_0 - \frac{4\mu mgL}{6mL^2} t &= 0 \\ t = \frac{3mL^2\omega_0}{2\mu mgL} &= \frac{3L\omega_0}{2\mu g} \end{aligned}$$

2. A solid, uniform, spherical boulder of mass $M$ and radius $R$ starts from rest and rolls down a hill of height $H$. The top half of the hill is rough enough to cause the boulder to roll without slipping, but the lower half is covered with ice and there is no friction.

   (a) If the angle of the hill is $\theta$ to the horizontal, what is the initial angular acceleration of the boulder at the top of the hill?

   (b) What is the translation speed of hte boulder when it reaches the bottom of the hill.

   First, we need the frictional force on the boulder. We can find this by writing the rotational form of Newton's second law.

   $$\begin{aligned} F_f R &= I\alpha \\ F_f R^2 &= \left(\frac{2}{5}MR^2\right) a \\ F_f &= \frac{2}{5}Ma \end{aligned}$$

   We can then use this result in the linear form of Newton's second law.

   $$\begin{aligned} Ma &= Mg\sin\theta - F_f \\ \frac{7}{5}Ma &= Mg\sin\theta \\ a &= \frac{5}{7}g\sin\theta \end{aligned}$$

   It follows immediately that the angular acceleration is $\frac{5}{7R}g\sin\theta$.

To find the translational speed at the bottom of the hill, we can use the conservation of energy, noting that the angular speed is unchanged after the ball reaches the middle of the hill.

$$MgH = \frac{1}{2}Mv_b^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v_m}{R}\right)^2$$

To solve this equation for $v_b$, we first need to find $v_m$. We can determine $v_m$ from the conservation of energy at the middle of the hill.

$$\begin{aligned}
Mg\frac{H}{2} &= \frac{1}{2}Mv_m^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v_m}{R}\right)^2 \\
&= \frac{7}{10}Mv_m^2 \\
v_m^2 &= \frac{5}{7}gH
\end{aligned}$$

Substituting back into the equation at the bottom, we can solve for $v_b$:

$$\begin{aligned}
MgH &= \frac{1}{2}Mv_b^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\frac{5gH/7}{R^2} \\
&= \frac{1}{2}Mv_b^2 + \frac{1}{7}MgH \\
v_b &= \sqrt{\frac{12}{7}gH}
\end{aligned}$$