Special Relativity

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1 The Theory of Relativity

The theory of relativity is based upon two postulates that Einstein formulated in order to explain the results of the Michelson-Morley experiment:

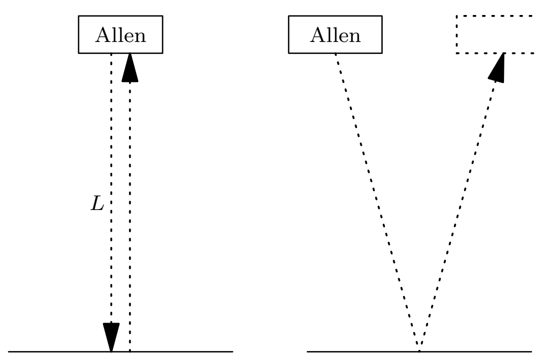
Postulate 1. No experiment can determine whether the observer is moving.

Suppose that Allen is on a spaceship above a planet and Sreenath is standing on the planet. From Allen's perspective, the spaceship he is on is not moving, while Sreenath (and the Earth) is the one moving. From Sreenath's perspective, however, Allen's spaceship is the one moving, and he is standing still.

Postulate 2. The speed of light is the same for all observers.

2 Time Dilation and the Twin Paradox

Suppose Allen fires a beam of light from his spaceship onto the planet directly underneath him from his perspective. From Allen's point of view, the time it takes for the beam to touch the planet and then come back to the spaceship is $\Delta t = \frac{2L}{c}$, where L is the distance from the spaceship to the planet.



From Sreenath's point of view, however, the beam must travel at a longer distance; Allen moves a distance of $\frac{1}{2}v\Delta t'$ during the time the beam reaches the planet, and so by the Pythagorean Theorem the length of the half-path is

$$L' = \sqrt{\left(\frac{1}{2}v\Delta t'\right)^2 + L^2}.$$

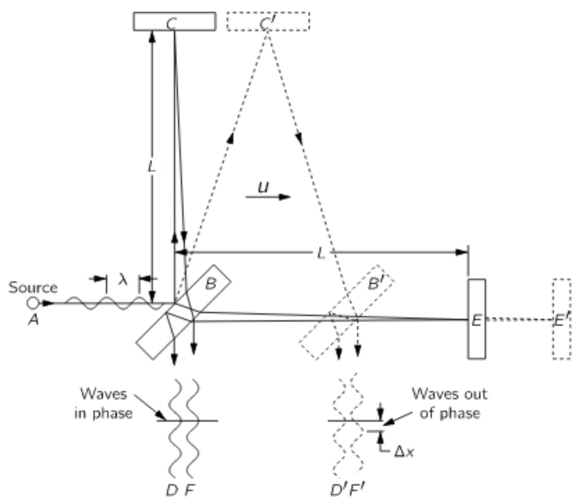
Since $\Delta t' = \frac{2L'}{c}$, we have that $\Delta t' = \frac{2L/c}{\sqrt{1-v^2/c^2}}$ (if we let $\gamma = (1-v^2/c^2)^{-1/2}$, then we can write $\Delta t' = \frac{2L}{c}\gamma$). In general, we have that $\Delta t' = \frac{\Delta t}{\sqrt{1-v^2/c^2}}$. As a result, from Sreenath's point of view, his internal clock runs faster than Allen's. As it is impossible to determine who is moving, from Allen's point of view, *his* internal clock runs faster than Sreenath's.

Suppose that Allen and Sreenath are identical twins. Allen decides to travel to Earth in a very high-speed journey while Sreenath stays back at their home planet. After arriving at Earth, Allen turns back and returns home. As Allen had traveled with a very high speed, Sreenath's internal clock had run faster than Allen's during the trip, and thus Sreenath had aged faster. This thought experiment is called the *twin paradox*, although it is not really a paradox; this is only a result of time dilation.

A natural question is why we had only considered Sreenath's perspective, and not Allen's: If Allen's internal clock would have run faster than Sreenath's from his point of view, then wouldn't Allen have aged more than Sreenath? The reason for this is because although position and velocity are relative, acceleration is not. When Allen had reached Earth, he decelerated so as to make his journey back home, causing Sreenath to age much faster during that period of deceleration.

3 Michelson-Morley Experiment

In 1887, Albert Michelson and Edward Morley developed an experiment that attempted to explain the motion of matter through the so-called *luminiferous ether*.



A beam of light would be split by the plate, and the splits would continue in mutually orthogonal directions until they bounced off the mirrors and returned to the plate, after which they would recombine. If the distance between the plate and the mirrors are exactly equal, the beams would combine at exactly the same time, and thus the resulting waves would be in phase. However, if the plate and the mirrors moved to the right as shown in the diagram above, the time it would take for one beam to go up and down would not equal the time the other beam would take to go right and left.

Let t_1 be the time it takes for light to go from plate B to mirror E, and let t_2 be the time it takes light to return. While the light is on its way from B to the mirror, the mechanism moves a distance vt_1, so light must travel a distance $L + vt_1 = ct_1$. Thus, $t_1 = \frac{L}{c-v}$. Similarly, we can calculate $t_2 = \frac{L}{c+v}$. The total time it takes for the light to return is therefore

$$t_1 + t_2 = \frac{2L/c}{1 - v^2/c^2}.$$

Now let t_3 be the time it takes for light to go from plate B to mirror C. During time t_3, the mechanism moves a distance of vt_3 while the light travels a distance ct_3. Thus $(ct_3)^2 = L^2 + (vt_3)^2$, or $t_3 = \frac{L}{\sqrt{c^2-v^2}}$. The total time it takes for the light to return is therefore

$$2t_3 = \frac{2L/c}{\sqrt{1-v^2/c^2}}.$$

Since $t_1 + t_2 \neq 2t_3$, the times for the beams of light to return are *not the same*.

When Michelson and Morley carried out the experiment, however, no time difference was detected; the waves were in phase despite the apparatus having been given a velocity. This null result baffled physicists, until Lorentz proposed that the length of the path that the beam split horizontally takes is contracted; he claimed that

$$L_{\parallel} = L_0\sqrt{1-u^2/c^2}$$

where $L_{\parallel}$ is the new length and u is the velocity of the body.

4 The Lorentz Transformation

When Maxwell publicized his equations of electrodynamics, the Michelson-Morley experiment gave a seemingly contradictory result, and as a result physicists cast the blame onto Maxwell's equations. Thus physicists attempted to correct Maxwell's equations so that they matched up with Galilean relativity. However, Lorentz noticed that the transformation (now called the *Lorentz transformation*)

$$\begin{aligned}
x' &= \frac{x-vt}{\sqrt{1-v^2/c^2}}, \\
y' &= y, \\
z' &= z, \\
t' &= \frac{t-\frac{vx}{c^2}}{\sqrt{1-v^2/c^2}}
\end{aligned}$$

instead matched up with Maxwell's equations.

5 Length Contraction

Suppose that a length $\Delta L = x_2 - x_1$ is measured in a moving reference. From the Lorentz transformation,

$$\Delta L' = x_2' - x_1' = \frac{x_2 - vt_2 - x_1 + vt_1}{\sqrt{1-v^2/c^2}},$$

which becomes $\Delta L' = \Delta L\sqrt{1-v^2/c^2}$ when $t_1 = t_2$. Thus, as objects move faster, an observer would see that the object would grow shorter in length parallel to the direction of its motion, in accordance to the formula given above.

6 Simultaneity and the Ladder Paradox

If one event occurs at point x_1 at time t_0 and another at point x_2 at the same time t_0, the corresponding times t_1' and t_2' differ by

$$t_2' - t_1' = \frac{v(x_1 - x_2)/c^2}{\sqrt{1 - v^2/c^2}}.$$

Therefore, even though an event seems to occur simultaneously to one observer, it may not seem simultaneous to another.

Consider a stationary garage with both front and back doors open and a ladder, when at rest, is too long to fit inside the garage. Allen takes the ladder and moves it at such a high velocity through the garage so that the ladder undergoes length contraction, and thus becomes shorter. Thus, there is a time at which the ladder is fully inside the garage, and we may shut the doors of the garage very briefly, so that the entire ladder is contained inside the garage.

However, let us suppose that we are observing this experiment from Allen's perspective. From his perspective, it is the ladder that is stationary and the garage that is moving at a high velocity, and so the garage must undergo length contraction instead. As a result, the ladder could not have fit inside the garage, as the garage is now significantly shorter than the ladder. This is the so-called *ladder paradox*.

The solution to this paradox is due to simultaneity being unclear in relativistic situations. In this case, from the perspective of the garage, the doors were shut simultaneously. From Allen's perspective, however, this does not mean that the doors have been shut simultaneously, as he is traveling at a very high velocity, and so the ladder did not need to fit inside the garage.

7 Equivalence of Mass and Energy

We show that the energy of a body is mc^2, where m is its mass. From the definition of power, we have that $\frac{dE}{dt} = Fv$. Thus

$$c^2 \frac{dm}{dt} = v \frac{d}{dt}(mv),$$

from which we multply $2m$ to both sides, which gives us

$$2mc^2 \frac{dm}{dt} = 2mv \frac{d}{dt}(mv) \implies c^2 \frac{d}{dt}(m^2) = \frac{d}{dt}(m^2v^2),$$

so $m^2c^2 = m^2v^2 + C$ for some constant C. If $v = 0$, then $m_0c^2 = C$, where m_0 is the rest mass of the object. As a result, $m^2c^2 = m^2v^2 + m_0^2c^2$, or $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$.

8 Problems

1. What is the speed of an electron which has a total energy of 1 MeV?
2. A particle of rest mass m_0 is traveling so that its total energy is just twice its rest mass energy. It collides with a stationary particle of rest mass m_0 to form a new particle. What is the rest mass of the new particle?
3. Two rockets named Allen and Sreenath depart from Earth at steady speeds of $0.6c$ in opposite directions, having synchronized clocks with each other and with Earth at departure. After one year as measured in Earth's reference frame, rocket Sreenath emits a light signal. At what times, in the reference frames of Earth and of rockets Allen and Sreenath, does rocket Allen receive the signal?