The Conservation of Momentum with Changing Mass

Shankar Balasubramanian   Ross Dempsey

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1 Solutions

1. The initial energy is $\frac{1}{2}(5.0\,\mathrm{kg})\left(8.0\,\mathrm{m\,s^{-1}}\right)^2 = 160\,\mathrm{J}$. Before the collision, the 5.0 kg block travels 2 meters under the influence of a frictional force with magnitude $\mu mg = 17.5\,\mathrm{N}$. Therefore, it loses 35 Joules, and so immediately before the collision the energy is 125 Joules (and thus the velocity is $5\sqrt{2}$ m/s). After the collision, we can find the velocities of the masses by using the following formulas (which can be derived with some labor from the conservation of momentum and of energy):

$$v_1' = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} = \frac{5}{2}\sqrt{2}\,\mathrm{m\,s^{-1}}$$
$$v_2' = \frac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2} = \frac{5}{2}\sqrt{2}\,\mathrm{m\,s^{-1}}$$

   Thus, the initial energy of the larger block after the collision is $\frac{1}{2}(15.0\,\mathrm{kg})\left(\frac{5}{2}\sqrt{2}\mathrm{m\,s^{-1}}\right)^2 = 93.75\,\mathrm{J}$. The frictional force is $\mu mg = 52.5\,\mathrm{N}$, so the block travels $93.75/52.5 = 1.79\,\mathrm{m}$.

2. Note that momentum is conserved in both the x and y directions. Conservation of momentum in the x direction looks like

$$3m\frac{v_0}{2} = 4mv_x \tag{1}$$

   And in the y direction...

$$mv_0 = 4mv_y \tag{2}$$

   We immediately get $v_x = \frac{3}{8}v_0$ and $v_y = \frac{1}{4}v_0$. The final speed is thus $v_0\sqrt{\frac{9}{64} + \frac{1}{16}} = v_0\frac{\sqrt{13}}{8}$.

3. First, consider the problem of accelerating a mass enough so that it can complete a loop without falling. At the top of the loop, the gravitational force mg must be a centripetal force, so that it does not contribute to pulling the mass downwards. Thus, we have a minimum velocity at the top of the loop:

$$\frac{mv^2}{L} \geq mg$$
$$v \geq \sqrt{gL}$$

If the velocity at the top of the loop is to reach this threshold, then it must be even greater at the bottom of the loop. We can use the conservation of energy to determine the minimum velocity at the bottom:

$$\begin{aligned}\frac{1}{2}mv_f^2 + 2mgL &= \frac{1}{2}mv_0^2\\ \frac{5}{2}mgL &\leq \frac{1}{2}mv_0^2\\ v_0 &\geq \sqrt{5gL}\end{aligned}$$

Therefore, the velocity of the pendulum mass must be at least $\sqrt{5gL}$ after the collision. To find its post-collision velocity, we must use the conservation of momentum:

$$\begin{aligned}m_1 v_0 &= (m_1 + m_2)v_f\\ v_0 &= \frac{m_1 + m_2}{m_1}v_f\end{aligned}$$

We know that $v_f \geq \sqrt{5gL}$, so $v_0 \geq \frac{m_1+m_2}{m_1}\sqrt{5gL}$.

4. There is a fairly simple proof of this question which requires calculus. However, the method of Lagrange multipliers is far simpler. Plus, it helps to illustrate the importance of the method. We are given the conservation of momentum (for the solution, we will assume an arbitrary number of particles collide, but that they all move along the same axis. Generalizing for the 2D or 3D case is fairly straighforward once the methodology is in place).

$$P_0 = \sum_i m_i v_i \tag{3}$$

We also are given the “change in kinetic energy function”, which is denoted as

$$\Delta K = \sum_i \frac{1}{2}m_i v_i^2 - E_0 \tag{4}$$

ΔK is the quantity that we want to maximize, subject to the conservation of momentum equation. Thus, we define

$$\alpha = \sum_i \frac{1}{2}m_i v_i^2 - E_0 - \lambda\left(P_0 - \sum_i m_i v_i\right) \tag{5}$$

Taking partial derivatives with all non-fixed variables and setting the derivative to 0 give us the lagrange equations

$$\frac{\partial \alpha}{\partial v_i} = m_i v_i - \lambda m_i = 0 \tag{6}$$

This gives us $\lambda = v_i$. Since λ is a constant and it is equal to ALL v_i, then all the final velocities must be equal to each other. One can check to see that this clearly maximizes ΔK. If all the final velocities are equal, then all the masses must be moving as one constituent, which is exactly what an inelastic collision means.

5. One important thing to note is that BOTH energy and horizontal momentum are conserved. Vertical momentum is not conserved because of the presence of gravity. But, since all surfaces are frictionless and the small block slides onto the large block (rather than collides with it), there is not frictional or heat losses. First, the small mass will slide up until the small and large masses move as a single unit. This collision can effectively be treated as inelastic (but realize that the loss in kinetic energy occurs becasue the block is gaining gravitational potential energy). Thus, we have

$$mv_0 = (m+M)\bar{v} \rightarrow \bar{v} = \frac{mv_0}{m+M} \tag{7}$$

Using the conservation of energy, we can deduce the height

$$\frac{1}{2}mv_0^2 = \frac{1}{2}(m+M)\left(\frac{mv_0}{m+M}\right)^2 + mgh \tag{8}$$

$$h = \frac{v_0^2}{2g}\frac{M}{m+M} \tag{9}$$

After the block and mass move as a single unit, the mass descends down from the block, and exits with a particular velocity. Again, we can use the conservation of momentum (horizontal) and energy to solve the problem. From these two laws, we have

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2 \tag{10}$$

$$mv_0 = mv_m + Mv_M \tag{11}$$

This is clearly the set of equations for an elastic collision. Thus, we immediately get as our results

$$v_m = \frac{m-M}{m+M}v_0 \tag{12}$$

$$v_M = \frac{2m}{m+M}v_0 \tag{13}$$

6. See Lecture on Conservation of Momentum with Changing Mass for the solution to this problem (in this context however, the car is a "wall" and $\sigma \rightarrow \lambda$).

7. The most reliable way to solve this problem is to find the momentum before and after a differential time dt and set them equal. Before the time interval, the car has momentum mv and the incoming mass has momentum $u\,dm$. Afterwards, the momentum of the entire system is $(m+dm)(v+dv)$. Therefore, neglecting second-order differentials, we have

$$u\,dm + mv = (m+dm)(v+dv)$$
$$dm(u-v) = m\,dv$$

We can start by integrating this equation to obtain m in terms of v:

$$\int_{m_0}^{m} \frac{dm}{m} = \int_0^v \frac{dv}{u-v}$$
$$\ln\frac{m}{m_0} = \ln\frac{u}{u-v}$$
$$m = \frac{m_0 u}{u-v}$$

We can use this expression to eliminate the m term in our original equation. We also need to eliminate the dm term, which we can accomplish by determining dm/dt. The mass is emitted from the source at a rate of $\sigma\,\mathrm{kg\,s^{-1}}$ and a velocity of $u\,\mathrm{m\,s^{-1}}$, so it is distributed over space with a density $\frac{\sigma}{u}\,\mathrm{kg\,m^{-1}}$. The relative velocity of the incoming mass with the car is $u-v$, so the mass enters at a rate of

$$\frac{dm}{dt} = \frac{\sigma(u-v)}{u}$$

Now we can substitute this and the expression for m in the original equation:

$$\begin{aligned}\frac{\sigma(u-v)}{u}(u-v) &= \frac{m_0 u}{u-v}\frac{dv}{dt} \\ \frac{\sigma}{m_0 u^2} &= \frac{1}{(u-v)^3}\frac{dv}{dt}\end{aligned}$$

Finally we can separate and integrate to obtain v in terms of t:

$$\begin{aligned}\int_0^t \frac{\sigma\,dt}{m_0 u^2} &= \int_0^v \frac{1}{(u-v)^3}dv \\ \frac{\sigma t}{m_0 u^2} &= \frac{1}{2(u-v)^2} - \frac{1}{2u^2} \\ \frac{1}{(u-v)^2} &= \frac{2\sigma t + M}{M u^2} \\ v &= u\left(1 - \frac{1}{\sqrt{1+\frac{2\sigma t}{M}}}\right)\end{aligned}$$

8. As in the lecture, we get from the rocket equation that

$$v(t) = u\ln\frac{M_f}{M_i} = u\ln\frac{M_f}{M} \tag{14}$$

The momentum is

$$p = M_f u \ln\frac{M_f}{M} \tag{15}$$

This is maximized when $\frac{dp}{dM_f} = 0$, which occurs when $u\ln\frac{M_f}{M} + u = 0 \to M_f = \frac{M}{e}$. The kinetic energy is given by

$$K = \frac{1}{2}M_f v^2(t) = \frac{1}{2}M_f u^2 \ln^2\frac{M_f}{M} \tag{16}$$

This is maximized when $\frac{dp}{dM_f} = 0$, which occurs when $\frac{1}{2}u^2\ln^2\frac{M_f}{M} + u^2\ln\frac{M_f}{M} = 0 \to M_f = \frac{M}{e^2}$

9. By far the simplest way to approach this problem is to work in the center of mass frame. The velocity of the center of mass is given by

$$v_{CM} = \frac{M}{m+M}v_0$$

In the center of mass frame, the total momentum is clearly zero. Thus, defining v_1 as the velocity of M and other indexed variables likewise, we have

$$
\begin{aligned}
Mv_1 \cos\theta_1 + mv_2 \cos\theta_2 = 0 \\
Mv_1 \sin\theta_1 + mv_2 \sin\theta_2 = 0
\end{aligned}
$$

In addition, we have the conservation of energy.

$$
Mv_1^2 + mv_2^2 = M\left(1 - \frac{M}{M+m}\right)^2 v_0^2 + m\left(\frac{M}{M+m}\right)^2 = \frac{mM}{M+m}v_0^2
$$

If we square both of the momentum equations and add them, we obtain

$$
M^2v_1^2 + m^2v_2^2 + 2Mmv_1v_2\cos(\theta_1 - \theta_2) = 0
$$

The momentum equations clearly apply that $\theta_1 - \theta_2 = 180$ (the vectors are directly opposed), so the cosine is -1:

$$
\begin{aligned}
M^2v_1^2 + m^2v_2^2 - 2Mmv_1v_2 &= 0 \\
(Mv_1 - mv_2)^2 &= 0
\end{aligned}
$$

Therefore, we can substitute $v_1 = \frac{m}{M}v_2$ into the energy equation:

$$
\begin{aligned}
\frac{m^2}{M}v_2^2 + mv_2^2 &= \frac{mM}{M+m}v_0^2 \\
\frac{M+m}{M}v_2^2 &= \frac{M}{M+m}v_0^2 \\
v_2 &= \frac{M}{m+M}v_0
\end{aligned}
$$

This is a very useful result: the velocity of m in the center of mass frame is independent of the angle at which it departs. In addition, the vertical component of motion did not change upon transforming into the center of mass frame (since the initial velocity of M was purely horizontal), so the problem of maximizing the vertical velocity in the initial frame is equivalent to the identical maximization in the center of mass frame. Therefore, the optimal angle is clearly 90° below the horizontal in the CM frame. Transforming back to the lab frame is simply a matter of adding a velocity of $\frac{M}{M+m}v_0$ horizontally, which we see is equal to the vertical component, so the angle in the lab frame is 45°.