Circular Motion Problem

You have two point masses rotating about the center of mass of the two-mass system, and both are moving in circular orbits. One has a mass of M, and the other has a mass of m. The distance between them is R. Find the magnitude of the velocity of the mass M about the center of mass of the two mass system in terms of M, m, R, and any needed constants.

Answer

Let r_1 be the distance between the mass M and the center of mass of the system, and let r_2 be the distance between the mass m and the center of mass of the system, so that

$$r_1 + r_2 = R.$$

Write out Newton's Law for each mass, since gravity is the only force acting on each:

$$G\frac{mM}{R^2} = ma_m$$

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Since this is circular motion, the accelerations are centripetal and

$$a_m = \omega^2 r_1$$

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Now we have 5 equations and 5 variables– r_1, r_2, a_m, a_M, and ω. So we can solve for ω as

$$\omega = \sqrt{G\frac{m+M}{R^3}}$$

and for r_1 as

$$r_1 = \frac{Rm}{m+M}$$

There are clever and less clever ways to do this, but we can all do algebra in 5 minutes or less. The desired quantity, the magnitude of the orbital velocity, is just

$$v = \omega r_1 = \sqrt{\frac{Gm^2}{R(m+M)}}$$