Lagrangian and Hamiltonian Mechanics

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1 Introduction

After Isaac Newton's formulation of classical mechanics in the late 17th century, classical mechanics was reinvented in the 18th century by two mathematicians, Leonhard Euler (1707-1783) and Joseph-Louis Lagrange (1736-1813), in terms of variational principles. This approach essentially argued that particles do not follow certain paths because they are pushed around by external forces, but rather they choose the path that minimizes a quantity known as the *action*.

It turns out that this formulation of classical mechanics is extremely effective and useful in solving various complex problems. Its true power derives from the fact that these formulations are independent of any specific coordinate system. This lecture will serve as an introduction to specifically Lagrangian and Hamiltonian mechanics and look at their practical uses in solving problems.

2 Lagrangian Mechanics

2.1 Euler-Lagrange equations

Let us define a seemingly silly combination of kinetic (T) and potential energies (V) which will be known as the *Lagrangian*.

$$\mathcal{L} \equiv T - V$$

Assume we have a coordinate system $q_1, q_2, ..., q_n$. Now according to Lagrangian Mechanics, the laws of motion will be determined by the equations known as the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial q_m} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_m}$$

Now, understandably for those who haven't seen this expression before, this expression will strike you as odd. However, the point of this lecture is not to go deep into the underlying theory, but to apply these principles to problems. In order to not leave you completely in the dark, I will just say that these equations come from the idea of minimizing the *action* of the system, where the action is defined to be:

$$S \equiv \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)dt$$

2.2 Change of Coordinates

The true of power of Lagrangian mechanics is that the equations of motion are the same no matter what coordinate system we pick. We will now prove this using the Euler-Lagrange equations.

We want to show that if the Euler-Lagrange equations hold for a set of coordinates $(x_1, x_2, ..., x_n)$, then it will hold for another set of coordinates $(q_1, q_2, ..., q_n)$, where q_i is a function of x and t: $q_i = q_i(x_1, ..., x_n, t)$

So we have:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_m} = \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_m}$$

Let's rewrite the $\frac{\partial \dot{x}_i}{\partial \dot{q}_m}$ term. By chain rule, we have:

$$\dot{x}_i = \sum_{m=1}^{n} \frac{\partial x_i}{\partial q_m} \dot{q}_m + \frac{\partial x_i}{\partial t}$$

Therefore,

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_m} = \frac{\partial x_i}{\partial q_m}.$$

Substituting this in and applying the time derivative on both sides in the original expression we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) \frac{\partial x_i}{\partial q_m} + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}_i} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right)$$

We then notice that the right hand side of the equation simply becomes

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_m}\right) = \frac{\partial L}{\partial q_m}$$

and we are done.

2.3 Spring Pendulum

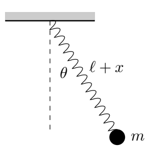

Figure 1

So we have the Euler-Lagrange Equations, but we haven't seen how these equations work in practice. A simple example involves the spring pendulum.

Consider a spring pendulum that has a mass m attached on the end. Assuming that the spring is aligned in a straight line and has an equilibrium length l, let's find the corresponding equations of motion.

Solution: The best set of coordinates to model this situation is x and θ, where x is the distance the spring has changed and θ is the angle from the vertical.

The kinetic and potential energies are easily expressed and are shown below:

$$T = \frac{1}{2}m(\dot{x}^2 + (l+x)^2\dot{\theta}^2)$$

$$V = -mg(l+x)\cos\theta + \frac{1}{2}kx^2$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + (l+x)^2\dot{\theta}^2) - mg(l+x)\cos\theta + \frac{1}{2}kx^2$$

Now, we just apply the Euler-Lagrange equation for both coordinates. In the x coordinate we have:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$$

$$m\ddot{x} = m(l+x)\dot{\theta}^2 + mg\cos\theta - kx$$

We immediately notice this to be the Newtonian $F = ma$ expression in the radial direction, which at least gives us some assurance that these formulas are indeed correct.

In the θ coordinate we get

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$m(l+x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta$$

which is simply the tangential $F = ma$ equation.

Hopefully this example assures you that the Lagrangian formulation agrees with the Newton's theory.

2.4 Small Oscillations

In many physical systems, particles undergo small oscillations around an equilibrium point. For one-dimensional motion, we know that the frequency of these small oscillations is:

$$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

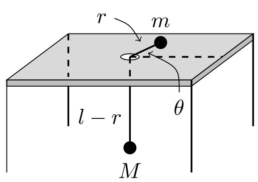

Figure 2

However, this formula does not hold for more complicated systems and thus we need an alternate method to help determine the frequency. This is where Lagrangian mechanics becomes extremely useful to us.

Consider the following situation: A mass m is free to slide on a frictionless table and is connected, via a string that passes through a hole in the table, to a mass M that hangs below. Assuming that M moves in a vertical line only and assuming that the string always remains taut, what is the frequency of the small oscillations (in the variable r) about this circular motion

Solution: Letting the string have length l, we can see that the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + Mg(l - r)$$

Applying the Euler-Lagrange Equations, we get the following equations of motion:

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$
$$(M + m)\ddot{r} = mr\dot{\theta}^2 - Mg$$

The first statement tells us that the quantity $mr^2\dot{\theta}$ (the angular momentum) is conserved. The second lets us know that the gravitational force and the centripetal acceleration accounts for the acceleration of the two masses.

Now for m to undergo circular motion it is clear that $\dot{r} = \ddot{r} = 0$. Also setting $L = mr^2\dot{\theta}$, we arrive at the following result:

$$Mg = \frac{L^2}{mr_0^3}$$

Finally, in order to determine the frequency, we just look at what happens when we perturb it from its equilibrium value of r_0. We can do this by considering $r(t) = r_0 + \delta(t)$. So we have:

$$\frac{1}{r^3} = \frac{1}{(r_0 + \delta)^3} = \frac{1}{r_0^3 + 3r_0^2\delta} = \frac{1}{r_0^3(1 + \frac{3\delta}{r_0})} \approx \frac{1}{r_0^3}\left(1 - \frac{3\delta}{r_0}\right)$$

Plugging this back into the second Euler-Lagrange equation of motion we see that:

$$(M + m)\ddot{\delta} \approx \frac{L^2}{mr_0^3}\left(1 - \frac{3\delta}{r_0}\right) - Mg$$

We notice at this stage that the terms not involving the δ cancel. So now we have:

$$\ddot{\delta} + (\frac{3L^2}{(M + m)mr_0^4})\delta \approx 0.$$

This is the standard simple-harmonic equation. The overall frequency is therefore:

$$\omega \approx \sqrt{\frac{3L^2}{(M + m)mr_0^4}}$$

3 Hamiltonian Mechanics

Although the Lagrangian equations of motion are very useful and are very applicable in problems, we are not at the end of the road yet. We can see that the Lagrangian function is quadratic in the velocities. However, William R. Hamilton discovered a remarkable transformation which renders the Lagrangian function linear in the velocities (however at the cost of doubling the number of mechanical variables and differential equations).

3.1 Development of the Hamiltonian

We can derive the Hamiltonian by applying a *Legendre transformation* on the Lagrangian. Legendre's transformation allows us to change a given function of a given set of coordinates into a new function of a new set of coordinates. The general scheme of the Legendre transformation can be thought of in three steps.

First, we introduce the "new variables," which we shall call the "momenta" and will be denoted by p_i:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Then we'll introduce the new function, which will be denoted by $\mathcal{H}$ and will be called the "total energy"

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}(q, \dot{q}, t)$$

Finally we'll express the new function $\mathcal{H}$ in terms of the new variables p_i for $\dot{q}_i$. Thus, we obtain

$$\mathcal{H} = \mathcal{H}(q, p, t)$$

3.2 Hamilton's equations of motion

The Lagrange equations of motion above directly imply Hamiltonian's *canonical* equations, which are shown below:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial \dot{q}_i}$$

We can see this through chain rule as shown below:

$$\frac{\partial \mathcal{H}}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial p_i} p_j + \dot{q}_i - \sum_{j=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i$$
$$\frac{\partial \mathcal{H}}{\partial \dot{q}_i} = \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial q_i} p_j - \frac{\partial \mathcal{L}}{\partial q_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_j}{\partial q_i} p_j = \dot{p}_i$$

3.3 Important Observation

Let us calculate $\frac{d\mathcal{H}}{dt}$:

$$\frac{d\mathcal{H}}{dt} = \sum_{i=1}^{n} \frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n} \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i + \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t}$$

Therefore if $\mathcal{H}$ is not *dependent* on time, then $\mathcal{H}$ will either be a constant of motion, a conserved quantity, or an integral of motion.

3.4 Symmetries and Conservation Laws

Lagrangian and Hamiltonian mechanics can also lead to the development of conservation laws. For example, consider the case where the Lagrangian does not depend on a certain coordinate. Then, clearly by the Euler-Lagrange equations: $\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = C$. Therefore, the Lagrangian will be constant in time with respect to that coordinate (a *cyclic* coordinate) and $\frac{\partial \mathcal{L}}{\partial \dot{q}_k}$ will be a conserved quantity. If q_k is not present in the Lagrangian it will not be present in the Hamiltonian either and the corresponding momentum will also be conserved.

3.5 Kepler Problem

Recall the standard Kepler problem for a mass m moving in an inverse-square central force field. Let's calculate the equations of motion

The Lagrangian, $T - V$, is expressed as:

$$\mathcal{L}(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{km}{r}$$

Therefore the generalized momenta are:

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$
$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

We can now write the Hamiltonian as:

$$\mathcal{H}(r, \theta, p_r, p_\theta) = \dot{r}p_r + \dot{\theta}p_\theta - \mathcal{L}(r, \dot{r}, \theta, \dot{\theta})$$
$$\mathcal{H} = \frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \left(\frac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4}\right) + \frac{km}{r}\right)$$
$$\mathcal{H} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{km}{r}$$

Finally we write the equations of motion:

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{km}{r^2}$$
$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mr^2} \qquad \dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = 0$$

We can see that the last equation implies that angular momentum is conserved.

3.6 Poisson Brackets

Let's end our discussion with Poisson Brackets. For two functions $u = (q_1, q_2, ...q_n; p_1, ..., p_n)$ and $v = (q_1, q_2, ...q_n; p_1, ..., p_n)$, we define their Poisson bracket to be

$$[u, v] := \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)$$

Some basic properties of these Poisson Brackets are shown below:

1. Skew-symmetry: $[v, u] = [u, v]$
2. Bilinearity: $[\lambda u + \mu v, w] = \lambda[u, w] + \mu[v, w]$
3. Jacobi's identity: $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$.

From these properties and the definition, we can also quickly notice the following results (assuming that all coordinates are independent from each other):

1. $\frac{\partial u}{\partial q_i} = [u, p_i]$
2. $\frac{\partial u}{\partial p_i} = -[u, q_i]$
3. $[q_i, q_j] = 0$
4. $[p_i, q_j] = 0$
5. $[q_i, p_j] = \delta_i j$

Another thing to note is the Poisson Bracket for canonical variables for a function f:

$$\frac{df(p,q,t)}{dt} = \frac{\partial f}{\partial t} + [f, \mathcal{H}]$$

This is easily seen through an application of the chain rule:

$$\frac{df(p,q,t)}{dt} = \frac{\partial f}{\partial q}\frac{dq}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial t}$$
$$\frac{df(p,q,t)}{dt} = \frac{\partial f}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial \mathcal{H}}{\partial q} + \frac{\partial f}{\partial t}$$
$$\frac{df(p,q,t)}{dt} = \frac{\partial f}{\partial t} + [f, \mathcal{H}]$$

However, what really makes Poisson brackets special is the fact that they are *invariant under canonical transformations*. All a canonical transformation really is a shift from one set of canonical coordinates to another set. In order words:

$$[U, V]_{Q,P} = [u, v]_{q,p}$$

Another useful thing about Poisson brackets is that we are potentially able to generate more constants of motion in a systematic manner. For $u = u(q, p, t)$ to be a constant of motion, then

$$\frac{du}{dt} = 0$$

This implies that

$$\frac{\partial u}{\partial t} = [\mathcal{H}, u]$$

However, if we assert that u is independent of time, then u must commute with the Hamiltonian operator. Now applying Jacobi's identity we see that:

$$[H, [u, v]] = -[u, [v, H]] - [v, [H, u]] = 0.$$

So we can see that $[u, v]$ is another constant of motion. The fact that we are able to generate constants of motions in a systematic manner gives Poisson brackets added relevance in solving physical problems.

4 Final Remarks

It is important to note that the material presented in this lecture is only scratching the surface of the variational principles of mechanics. There is a lot of theory behind this and I encourage you to take the time to look at some of the background work done to come to the results presented in this lecture. As much as I would like to extend this lecture to being a more theoretical based one, I think it is more useful learning how to set up problems and actually using them in order to build some sort of intuition of why they work.

One thing that I particularly regret is not covering the canonical transformations in Hamiltonian mechanics. But it rather difficult to cover this topic in just a 45 minute block and I would rather like to focus on things slowly and carefully, than rushing through. Regardless, within the next couple of weeks, I will be writing a separate set of notes that will cover these transformations in much more detail. So be on the lookout for those!

5 Problems

1. **Pendulum with a free support** A mass M is free to slide along a frictionless rail. A pendulum of length l and mass m hands from M (see Fig. 3). Find the equations of motion. For small oscillations, find the normal modes and their frequencies.

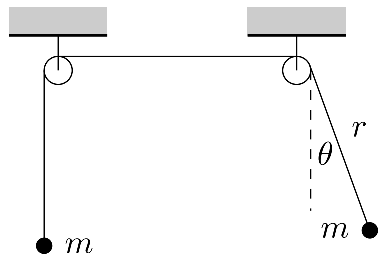

Figure 3

2. **Three falling sticks** Three massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at the ground. Thre are held such that the lower two sticks are vertical, and the upper one is tilted at a small angle ϵ with respect to the vertical. They are then released. At this instant, what are angular accelerations of the three sticks. Work in the approximation where ϵ is very small

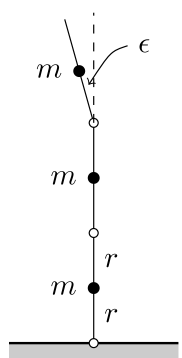

Figure 4

3. **Bead on a Straight Wire** Consider a bead sliding on a frictionless rigid straight wire lying along the x axis. The bead has mass m and is subject to a conservative force, with corresponding potential energy $U(x)$. Write down the Lagrangian and Lagrange's equations of motion. Find the Hamiltonian and Hamilton's equations, and compare the two approaches.

4. **Poisson Invariance Under Canonical Transformation** In section 3.6, we declared that the Poisson Brackets are invariant under canonical transformations. Show that this is indeed true.