Taylor Approximations

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1 Theorem

For any function $f(z)$ analytic within a disk $|z - z_0| \leq R$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for every z within the disk, and the series converges. I'll repeat this taylor expansion equation:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f^{(3)}(z_0)}{3!}(z - z_0)^3 + ...$$

How did Taylor come up with this outlandish polynomial? The answer is that in order to approximate a function, he constructed a polynomial whose derivatives of all orders are equal to that of the function in question. You may verify for yourself by differentiating both sides any arbitrary number of times that indeed their derivatives of all orders are equal.

2 Several Common Expansions

Your garden-variety calculus textbooks contain a good number of them. Here we present several that we may have encountered over the years.

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 ...$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} ...$$

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + ax + \frac{a(a-1)}{2} x^2 + ...$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} ...$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} ...$$

You already learned or will learn this in your math class. In doing physics, of the greatest interest to us is the question of x being really small, therefore hopefully allowing us to safely dispose of the later terms of the sequence that contain high powers of x. Typically, we just work with the first-order approximation, meaning we deal only with the first power of x, assuming the higher powers are negligibly small.