The Conservation of Momentum with Changing Mass

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1 Introduction

Collisions, as we have seen, are governed by the conservation of momentum. However, collisions are not the only arena in which momentum is conserved. We define the net, external force as the rate of change of $p = m\vec{v}$. The $\vec{v}\frac{dm}{dt}$ term was neglected because we assumed that the system was not gaining or losing mass. In this lecture, we will not neglect this term. Such an area of study is often referred to as *mass transport*.

2 General Equation of Momentum Conservation

If we insert the definition of momentum into Newton's second law, we can expand the derivative into two terms:

$$\vec{F} = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

In most systems, the mass is constant and the second term of the right hand side vanishes. However, in some rare cases, both terms are important. Additionally, in a closed system (with no external forces acting), the left hand side is only the sum of internal forces, which (by Newton's third law) vanishes. Therefore, the law takes on this special form:

$$m\frac{dv}{dt} + v\frac{dm}{dt} = 0$$

This equation can be used for analyzing closed systems in which the masses of components may be changing wtih time. The canonical example is a rocket accelerating by ejecting exhaust. The rocket begins at rest with a mass m_0, some of which is in the form of fuel. It then begins burning its fuel and ejecting exhaust at a velocity v_e m/s (relative to the rocket). The task is to determine the velocity of the rocket as a function of time, using the conservation of momentum.

To solve this problem, all we need is the momentum conservation equation for the scenario. The first term is the familiar ma, and we identify m with the mass of the rocket at time t and a with its acceleration. The second term accounts for the change in momentum of the differential mass being ejected from the rocket. Thus, the velocity is $-v_e$ and the derivative is the rate at which exhaust accumulates outside the rocket, or equivalently, the negative rate of change of the rocket's mass. The equation becomes:

$$m\frac{dv}{dt} = -v_e\frac{dm}{dt}$$
$$\frac{1}{v_e}dv = -\frac{1}{m}dm$$

Integrating both sides, we arrive at an equation for the velocity of the rocket:

$$\int_{v_0}^{v_f} \frac{dv}{v_e} = -\int_{m_0}^{m_f} \frac{dm}{m}$$
$$v_f = v_0 + v_e \ln \frac{m_0}{m_f}$$

As a more challenging example, we can consider a snowball rolling down a hill. If the hill has an incline of θ, then the net external force has a magnitude of $mg\sin\theta$. Thus, the momentum "conservation" equation becomes

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg\sin\theta$$

For the sake of simplicity, we assume that the snowball accrues mass at a rate proportional to its momentum. If we call the proportionality α, then the differential equation becomes (upon cancelling the common factor of m)

$$\frac{dv}{dt} + \alpha v^2 = g\sin\theta$$

This equation is also separable, although not as simply as in the case of the rocket:

$$\frac{1}{g\sin\theta - \alpha v^2} dv = dt$$

To integrate the left hand side, we split it into partial fractions:

$$\frac{1}{2g\sin\theta}\int_0^{v_f}\left(\frac{1}{1+\sqrt{\frac{\alpha}{g\sin\theta}}v} + \frac{1}{1-\sqrt{\frac{\alpha}{g\sin\theta}}v}\right)dv = \int_0^t dt$$
$$\frac{1}{\sqrt{\alpha g\sin\theta}}\frac{1}{2}\left(\log\left(1+\sqrt{\frac{\alpha}{g\sin\theta}}v\right) - \log\left(1-\sqrt{\frac{\alpha}{g\sin\theta}}v\right)\right)\Bigg|_0^{v_f} = t$$
$$\tanh^{-1}\left(\sqrt{\frac{\alpha}{g\sin\theta}}v_f\right) = \sqrt{\alpha g\sin\theta}t$$
$$v_f = \sqrt{\frac{g\sin\theta}{\alpha}}\tanh\left(t\sqrt{\alpha g\sin\theta}\right)$$

3 Another class of examples

Consider another classic example of mass transport. Let us assume that a beam of particles is being directed at a wall at an intensity $\lambda = \frac{m}{d}$ with a velocity v. Determine the average force that the wall exerts on the beam of particles. The particles are spaced a distance d from each other.

Let us assume that one particle has just collided with the wall, and the subsequent particle is approaching the wall. We can consider the average force on the wall due to the collision of this particle with the wall. We know that

$$\frac{\Delta p_x}{\Delta t} = F_x$$

The collision is elastic, so the change in momentum is $2mv$. The time it takes for the particle to reach the wall is $t = \frac{d}{v}$. Substituting,

$$\frac{2mv}{d/v} = F_x = 2\lambda v^2$$

We can assume that this force is roughly constant if d is made particularly small. Otherwise, this result is a weak approximation.

Now, let us consider that this external force is not applied to the wall. Thus, the wall will move with a particular velocity as a function of time. At a particular moment when the wall's velocity is V, the relative velocity between the beam and the wall is $v - V$. Since the collision is elastic, we can work in the stationary reference frame of the wall. When the ball rebounds back, the velocity in this reference frame is given by $V - v$. The ball's velocity in the inertial reference frame is thus given by $2V - v$. The rate of change of momentum is the external force, which is thus

$$\frac{m((v-2V)-(-v))}{d/(v-V)} = F_x = 2\lambda(v-V)^2$$

Assuming that d is extremely small, and the mass of the wall (M) satisfies $M \gg \lambda d$, we use Newton's law to get

$$2\lambda(v-V)^2 = M\frac{dV}{dt}$$

Assuming an initial velocity of 0, we obtain as a solution

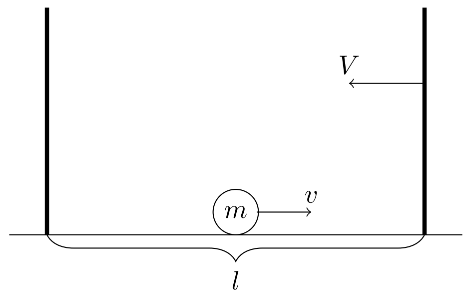
$$V(t) = v - \frac{1}{\frac{2\lambda t}{M} + \frac{1}{v}}$$

4 Problems

1. If a rocket releases mass at a rate $\alpha t + \beta$, determine the velocity of the rocket as a function of time, as well as when the rocket attains its maximum and/or minimum velocity.

2. Consider a rocket positioned tangentially at a distance r from the center of a merry-go-round. It starts releasing mass with a relative velocity u at a rate γ. If the rocket is prevented from moving in the radial direction, determine the rocket's angular motion as a function of time, as well as the normal force necessary for the rocket to remain at a radially constant distance.

3. A cylindrical rocket of diameter $2R$, mass M_R and containing fuel of mass M_F is coasting through empty space at velocity v_0. At some point the rocket enters a uniform cloud of interstellar particles with number density N (e.g., particles/m^3), with each particle having mass $m(<< M_R)$ and initially at rest. To compensate for the dissipative force of the particles colliding with the rocket, the rocket engines emit fuel at a rate $\frac{dm}{dt} = \gamma$ at a constant velocity u with respect to the rocket. Ignore gravitational effects between the rocket and cloud particles.
   (a) Assuming that the dissipative force from the cloud particles takes the form $F = -Av^2$, where A is a constant, derive the equation of motion of the rocket ($F = ma$) through the cloud as it is firing its engines.
   (b) What must the rocket's thrust be to maintain a constant velocity v_0?
   (c) Assuming that each cloud particle bounces off the rocket elastically, and collisions happen very frequently (i.e., collisions are continuous), prove that the dissipative force is proportional to v^2, and determine the constant A. Assume that the front nose-cone of the rocket has an opening angle of $\frac{\pi}{2}$.



4. A raindrop falls onto a homogeneous mass (essentially a cloud). All collisions are inelastic, as the mass picks up particles from the cloud. Assuming that the particle is spherical, determine the acceleration of the particle through the cloud (hint: it should be a multiple of g).

5. A "superball" of mass m bounces back and forth between two surfaces at a speed v_0. All collisions are perfectly elastic, and the environment has zero gravity. If one surface is moved toward the other at a speed $V \ll v$, the ball will speed up. Determine the average force on each wall when they are a distance x from each other. Assume that the initial distance between the walls is l.

6. A chain with length L and mass density σ is attached to a horizontal wall, and held at its highest position. It is then released. Determine the force that acts at the junction between the rope and the wall as a function of time.

7. (Requires a Computational Software) Repeat the last example using the fact that the net change in the kinetic energy after the collisions is given by Γ.