Trig and Complex Exponentials

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This equation below is the only one you really need to derive everything else on this sheet. To derive it requires Taylor series or expressing one solution to a certain differential equation as a linear combination of its other solutions and then applying initial conditions. The formula is called Euler's Formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{1}$$

We introduce notation that will let us switch back and forth (to some extent) between trig and exponentials.

$$\cos(\theta) = Re(e^{i\theta})$$
$$\sin(\theta) = Im(e^{i\theta})$$

Another way to switch forth between trig and exponentials comes from linear combinations of $e^{i\theta}$ and $e^{-i\theta}$:

$$\cos(\theta) = \frac{e^{i\theta} + \exp^{-i\theta}}{2}$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Primarily, we are interested in adding waves, so that we're adding a bunch of sine or cosine terms together. As adding together trig terms is not always nice or easy to simplify, we often express sums of trig terms as sums of exponentials. Exponentials are better than trig in that they can form geometric series, and we do know how to sum a geometric series:

$$1 + z + ... + z^n = \frac{z^{n+1} - 1}{z - 1}$$

The most common such manipulation in wave optics (from Young's n-slit experiment) looks like this:

$$\begin{aligned}
answer &= 1 + \cos(\theta) + \cos(2\theta) + ... + \cos(n\theta) \\
&= 1 + Re(e^{i\theta}) + Re(e^{2i\theta}) + ...Re(e^{ni\theta}) \\
&= Re(1 + e^{i\theta} + e^{2i\theta} + ... + e^{ni\theta}) \\
&= Re(\frac{e^{(n+1)i\theta} - 1}{e^{i\theta} - 1}) \\
&= Re(\frac{e^{\frac{(n+1)i\theta}{2}}}{e^{\frac{i\theta}{2}}} \frac{e^{\frac{(n+1)i\theta}{2}} - e^{-\frac{(n+1)i\theta}{2}}}{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}) \\
&= Re(e^{\frac{n}{2}i\theta} \frac{\sin(\frac{n+1}{2}\theta)}{\sin(\frac{\theta}{2})}) \\
&= \cos(\frac{n}{2}\theta) \frac{\sin(\frac{n+1}{2}\theta)}{\sin(\frac{\theta}{2})}
\end{aligned}$$

Cool, no? You could do all the above in a few pages of trig identities, or you could do this... Obviously this Euler stuff is pretty powerful.