Kinematics

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1. What is Kinematics?

Kinematics is the study of the motion of particles. Generally, we consider particles launched on the surface of the Earth.

2. Acceleration, Velocity and Position

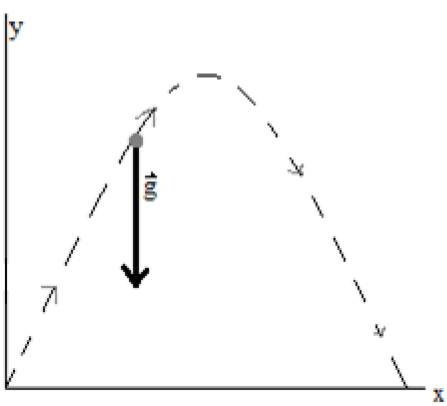
We know from our basic English vocabulary what position, velocity, and acceleration mean. But more specifically, we define position to be the place where an object is at with respect to a certain frame. For example, if the center of the Earth was (0, 0, 0), and I was on the moon, then my coordinate would be some point (x, y, z). But if I define where I was standing to be (0, 0, 0), then the center of the Earth would not be (0, 0, 0). This sounds simple, but choosing the right reference frame is important as it can greatly simplify problems. We then define velocity in this reference frame to be the change in position with respect to time (think derivative) and then acceleration to be the change in velocity with respect to time. This means acceleration is the first derivative with respect to time of velocity and the second derivative with respect to time of position.

All three of these are vector quantities. The best way to interpret this is that the position vector points to the object, and traces out its position as time passes. The velocity vector points in the direction that the position vector changes, and the acceleration points in the direction that the velocity vector changes. This may require vector calculus, but vector calculus that we will deal with isn't all too different from normal calculus and isn't too difficult. For now, if we want to take the derivative of a vector, we can just take the derivative of its components. Same thing goes for integration.

3. Gravity

Gravity is a force that acts on everything with a mass and is exerted by everything with a mass. The larger the masses, the larger the gravitational force between the two masses. Because the Earth is considerably larger than particles that we launch on the Earth, the mass of the particle becomes negligible and find that the acceleration due to gravity is effectively constant (we will see this later):

$$g = 9.81\frac{m}{s^2}$$



As far as choosing our reference frame goes, we generally restrict ourselves to two dimensions. This is because the path of the object is generally confined by two dimensions – the horizontal motion in the air, and the vertical motion in the air. Therefore we don't need to worry about a third dimension for depth. By the properties of vectors, we can see that both acceleration and velocity vectors will also be two-dimensional, as the derivative of any other component, which has magnitude 0, will also be 0 (this will be explained later). As a result, we are going to define our reference frame to have the starting point of the particle to be the origin, have gravity acting in the negative y-direction, which corresponds to $-\hat{\jmath}$, and horizontal motion being in the x-direction, which corresponds to $\hat{\imath}$. Therefore, we have the acceleration (a vector) due to gravity to be the following:

$$\vec{g} = -9.81\frac{m}{s^2}\hat{\jmath}$$

Remember that the direction of a vector is just as important as its magnitude, if not more. Also keep in mind this reference frame can be changed, as long as everything else changes with respect to the properties of the frame.

4. The Kinematics Equations

The kinematics equations will help us determine information about a moving particle. We will consider the 2-D case, as the 1-D case becomes very trivial from the 2-D equations, and the 3-D case isn't all that too useful for our purposes, as the motion of particles, as we will see, is generally confined to 2-dimensions. 4-D and higher are very abstract as there is no physical analogue that we can conceive and so they aren't very useful either.

Acceleration due to gravity, as mentioned before, can be assumed to constant. Is there any other acceleration? In the real world, yes – there is generally wind resistance that is a force that acts on the particle and friction, which is a force that acts on the particle between air molecules. Forces, as we will learn later, cause accelerations in the direction they act. However, for the purpose of kinematics, we will assume that the only force that is acting on the particle is gravity. We know that gravity acts in the negative y-direction, which corresponds to $-\hat{\jmath}$. Therefore, our acceleration vector is:

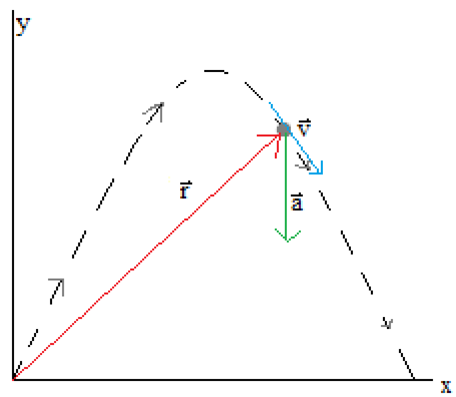
$$\vec{a} = \vec{g} = -g\hat{\jmath}$$

Realize that our acceleration is constant – this is critical to kinematics. The famous equations literally wouldn't be the same if we didn't have constant acceleration. In order to get velocity, we need to take an integral with respect to time.

$$\vec{v} = \int \vec{a}\, dt = \int -g\hat{\jmath} dt$$

In order to this integral, just take the integral of each of the components. Note that because the integral is linear, we can do this.

$$\vec{v} = v_x\hat{\imath} + (v_y - gt)\hat{\jmath}$$



You may be wondering where we got these supposedly random constants! These are actually just our constants of integration. Remember that our motion is two-dimensional, and that there is also an $\hat{\imath}$ component in the acceleration, but it has magnitude 0. The integral of 0 is just a constant. In reality, these constants have physical meaning – they are the initial velocity in that direction. This makes sense that there is constant velocity in the x-direction because there is no acceleration, or change in velocity, in that direction. Similarly, it makes sense that our y-direction velocity changes with respect to time. We can repeat the process to determine our position as a function of time.

$$\vec{r} = \int \vec{v}\, dt = \int v_x\hat{\imath} + (v_y - gt)\hat{\jmath}\, dt$$

$$\vec{r} = (r_x + v_xt)\hat{\imath} + \left(r_y + v_yt - \frac{gt^2}{2}\right)\hat{\jmath}$$

And there we have our three most important kinematics equations. Now the constants of integration that we've added for position represent initial position. Generally, we have our initial position to be at (0, 0), or (0, h), where 'h' represents the height that the particle starts at with respect to the ground.

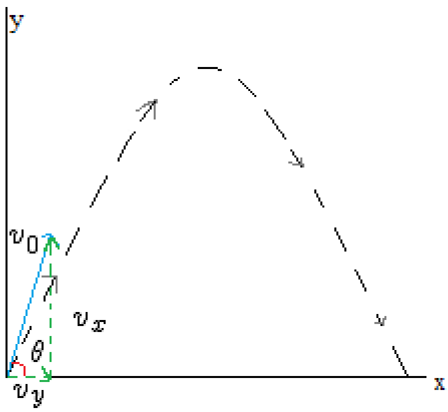
We also can relate the initial velocities in each direction, but we do have an idea of the magnitude of the velocity and the angle that we launched the particle at. This sounds resoundingly like a polar representation of velocity. Let θ represent the launch angle and v_0 represent the velocity with which the ball is launched. We can then write the following, which are all derived from a triangle and trigonometric relations:

$$v_x = v_0 \cos(\theta)$$

$$v_y = v_0 \sin(\theta)$$

$$v_0 = \sqrt{v_x^2 + v_y^2}$$

If these equations seem random or confusing, take a look at the following diagram.



We can then sum up what we have learned with the following equations:

$$\vec{a} = -g\hat{j}$$

$$\vec{v} = v_x\hat{\imath} + (v_y - gt)\hat{j}$$

$$\vec{r} = (r_x + v_0\cos(\theta)t)\hat{\imath} + \left(r_y + v_0\sin(\theta)t - \frac{gt^2}{2}\right)\hat{j}$$

<u>Problems</u>

For most of these questions, assume the conditions stated above.

1. Using the kinematics equations, find the height of the particle as a function of horizontal position (basically find y as a function of x). What is the shape of this graph?

2. What is the range a particle travels when it hits the ground, assuming its initial position is (0, 0), and y = 0 is the ground?

3. Find the maximum height of a particle launched from (0, 0).

4. Determine the angle that maximizes range (use the same scenario as question #2).

5. What is the time a particle travels in the air if it is launched under the same conditions in question #2?

6. A particle is launched straight upward. What is the total distance that it travels?