Lagrangian Mechanics

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Much of this lecture and problems has been taken from *Classical Mechanics* by Goldstein. I recommend that you check it out if you want a good book on advanced classical mechanics.

1 Forces, Systems, and Constraints

A *holonomic constraint* is any condition describing the allowed states of our system that can be expressed in terms of functions of the form $f(\mathbf{r}_1, \mathbf{r}_2, \ldots, t) = 0$. For example, if our system is a single bead moving along a wire, then the constraints are $y = 0$ and $z = 0$. If our system is a rigid body rotating in space, then for any two particles on the body, $(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0$. Notice that describing constraints in this way allows us to neglect the specific form of the constraint force; for example, we will not be calculating the force exerted by the bead on the wire, or the electromagnetic and chemical forces that are responsible for maintaining the rigidity of the body.

However, the disadvantage of introducing constraint equations is that now the equations of motion are coupled in weird ways. To resolve this, we introduce *generalized coordinates* to describe the system. N unconstrained particles have $3N$ degrees of freedom. If there are k constraining equations, then our set of coordinates should be $3N - k$ long. We write our transformation out as

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1(q_1, q_2, \ldots q_{3N-k}, t) \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N(q_1, q_2, \ldots q_{3N-k}, t) \end{aligned}$$

Generalized coordinates are useful because they are, well, general. You are free to choose whatever is the easiest way to represent a problem, such as representing position on a sphere using latitude and longitude rather than x, y, and z.

The solving the equations of motion for nonholonomic constraints can range from hard, in some cases, to impossible, in others. We will be neglecting them in this text.

2 Virtual Work and D'Alembert's Principle

I'd like to introduce the concept of an infinitesimal *virtual displacement*, $\delta\mathbf{r}$. A real displacement is the displacement observed by letting the system evolve for some infinitesimal amount of time. For example, the infinitesimal displacement of a falling apple a small amount downwards because of gravity. A virtual displacement is one made at constant time, and can be in any direction consistent with the constraints on the system. For the apple, possible virtual displacements include the apple displaced up, down, left, right, or any other direction. (If virtual displacements still don't make sense, just read on. They don't appear in our final result.)

By Newton's second law, we know $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$ so the dot products with virtual displacements must vanish as well.

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

Let's decompose the forces on each particle into applied forces, $\mathbf{F}_i^{(a)}$ and constraint forces, $\mathbf{f}_i$. This yields

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

If we make the further assumption that the net virtual work by constraint forces vanishes, then we get

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}) \cdot \delta \mathbf{r}_i = 0$$

This is known as D'Alembert's principle.

3 Euler-Lagrange

Now, let us transform into our generalized coordinates.

$$\begin{aligned}
\mathbf{r}_i &= \mathbf{r}_i(q_1, q_2, \ldots, q_n, t) \\
\mathbf{v}_i &= \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \\
\delta \mathbf{r}_i &= \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k
\end{aligned}$$

The latter two equations come from chain rule. Recall that virtual displacements occur at constant time, so we don't include a δt term. Also note

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_i \sum_k \mathbf{F_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k = \sum_k Q_k \delta q_k$$

where $Q_k = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$ denotes the *generalized force*. Similarly,

$$\sum_i \dot{\mathbf{p}} \cdot \delta \mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right]$$

Note that

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \text{ and } \frac{d}{dt} \left(\frac{\partial \mathbf{r_i}}{\partial q_j} \right) = \frac{\partial \mathbf{v}_i}{\partial q_j}$$

so we get

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt} \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right]$$

Letting $K = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v}$ denote the kinetic energy, we can rewrite D'Alembert's as

$$\sum \left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_j}\right) - \frac{\partial K}{\partial q_j} - Q_j \right] \delta q_j = 0$$

Now, suppose that we chose q_j so that they may be be changed independently. Then, each term of the above sum must separately vanish. Further suppose our forces are all conservative, so $\mathbf{F}_i = -\nabla_i V$. It follows that

$$Q_j = -\sum \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Combining the result with the assumption that the potential does not depend on generalized velocities, then we can rewrite in terms of the *Lagrangian*, $\mathcal{L}$, defined as $K - V$.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Now, note that any function of the form $\mathcal{L}_2(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}, t) + \frac{dF}{dt}$ where $F(q,t)$ is differentiable, then this new Lagrangian also satisfies the Euler-Lagrange equations. This is very useful for when we can't easily find coordinates that make our constraints vanish, so we can just add some Lagrange multipliers. Suppose our constraint is holonomic and can be written as $f(t, q_1, q_2, \ldots) = C$. Then, we replace Lagrangian as follows:

$$\mathcal{L}' = \mathcal{L} + \lambda[f(t, q_1, q_2, \ldots) - C]$$

and solve this, using the constraint to fix the value of λ at the end.

3.1 Principle of Stationary Action

It turns out that the above equation is a special case of a general result from the calculus of variations, saying that $\int f(t, y(t), \dot{y}(t))\, dt$ has a stationary value (independent of any small perturbations to the function $y(t)$) whenever $\frac{\partial f}{\partial y} - \frac{\partial f}{\partial \dot{y}} = 0$. For this reason, you'll often see people defining the *action* as the line integral of the Lagrangian, in which case Euler-Lagrange becomes the *principle of stationary action*.

4 Problems

Problem 1. Find the Lagrangian of a mass m moving in space under central force $F(r) = -dV/dr$.

Problem 2. Find the equations of motion of a spherical pendulum (a mass suspended by a taut string, without the assumption that all motion is planar).

Problem 3. Find the equations of motion of a double pendulum.

Problem 4. Suppose you pluck a guitar string with fixed endpoints, with transverse displacement being some function $y(x,t)$. Find the kinetic and potential energies, in terms of the density μ and tension F. Show that the kinetic and potential energies may be given as

$$K = \frac{\mu}{2}\int_0^l y_t^2\, dx$$
$$V = F\int_0^l (\sqrt{1+y_x^2}-1)dx$$

If $|y_x| << 1$, what can you say? Do you recognize the result?

Problem 5. Suppose a block m is held motionless on a frictionless plane of mass M and angle of inclination θ. The plane rests on a frictionless horizontal surface. The block is released. What are the accelerations of the objects?

Problem 6. Two masses m are connected by a massless string hung over two massless pulleys. The left mass moves vertically only, but the other mass can swing left and right in addition to moving up and down. Solve the equations of motion.

Problem 7. A pendulum consists of a mass m at the end of a massless stick of length l. The other end of the stick is made to oscillate vertically with a position given by $y(t) = A\cos(\omega t)$ where $A << l$. It turns out that if ω is large enough, and the pendulum is initially nearly upside-down, then suprisingly it will *not* fall over as time goes by. Instead, it will (sort of) oscillate back and forth around the vertical position.

Find the equation of motion for the angle of the pendulum. Explain why the pendulum doesn't fall over, and find the frequency of the back and forth motion.