Gravitation-Field Analysis

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1 The Gravitational Force

We've referenced the gravitational acceleration g throughout our study of mechanics, but we haven't discussed its origins. Most scenarios we studied were on a small enough scale that we could consider the acceleration due to gravity constant, but that isn't the case for all the scenarios we're going to study. In this unit, we're going to take a look at scenarios where the gravitational field isn't constant. We'll start with a statement of Newton's Law of Gravitation.

2 Newton's Law of Gravitation

Newton's Law of Gravitation states that the force due to gravity on an object of mass m due to an object of mass M is

$$\vec{F}_g = \frac{-GMm}{r^2}\hat{r}$$

where

$$G \approx 6.67384 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$$

and r is the distance between the two objects. Note that this is an inverse square law.

3 The Shell Theorem

The gravitational field that acts on an object of mass m experiences as a result of an object of mass M is

$$\vec{g} = \frac{GM}{r^2}\hat{r}$$

Note that the field is independent of the mass m. The force is not.

The Shell theorem states that, for an inverse square law such as Newton's Law of Gravitation,

(1) A spherically symmetric body acts as a point mass on external bodies (e.g. the Earth, if we assume it to be spherically symmetric, acts as though it were a point mass on satellites and the moon).

(2) Spherically symmetric shells surrounding an object have no effect on that object (e.g. if an object is buried 5,000 meters deep into a sphere that has a radius of 10,000 meters, only the deepest 5,000 meters have any effect on the object).

4 Shell Theorem Problems

4.1 A solid sphere and Spherical shell

Let's take a look at the field at a distance r provided by a solid sphere of mass M and radius R. By the shell theorem, there are two circumstances:

(i) $r \geq R$

(ii) $r < R$

By statement (1) of the shell theorem, we can do (i) easily:

$$\vec{g} = \frac{-GM}{r^2}\hat{r}$$

Part (ii) is slightly more involved. The mass affecting the object is not M anymore; instead it depends on the volume enclosed by r. First we calculate the density of the sphere:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

And then the mass M' enclosed by a sphere of radius r:

$$M' = \rho V' = \frac{3M}{4\pi R^3} * \frac{4\pi r^3}{3} = \frac{Mr^3}{R^3}$$

And then the field:

$$\vec{g} = \frac{-GM'}{r^2}\hat{r} = \frac{-GMr}{R^3}\hat{r}$$

Note for condition (i), the magnitude of the field is linear! Also note that at $r = R$, formulas for both (i) and (ii) give the same result. Make sure you can plot the magnitude of the field as a function of r.

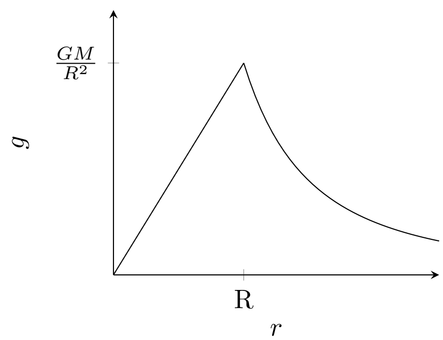

Figure 1: Gravitational field for solid sphere.

Now let's take a look at a solid spherical shell, like a tennis ball. The shell has inner radius R, outer radius $2R$, and mass M. There are three conditions:

(i) $r \leq R$

(ii) $R \leq r \leq 2R$

(iii) $r \geq 2R$

Part (iii) is relatively simple:

$$\vec{g} = \frac{-GM}{r^2}\hat{r}$$

Ans so is part (i), due to part (ii) of the shell theorem:

$$\vec{g} = \vec{0}$$

And now part (ii) We follow the procedure for the solid sphere problem. Density:

$$\rho = \frac{M}{\frac{4}{3}\pi((2R)^3 - R^3)} = \frac{3M}{4 * 7\pi R^3} = \frac{3M}{28\pi R^3}$$

Mass:

$$M' = \frac{3M}{28\pi R^3} * \frac{4}{3}\pi(r^3 - R^3) = \frac{M(r^3 - R^3)}{7\pi R^3}$$

And field:

$$\vec{g} = \frac{-GM'}{r^2}\hat{r} = \frac{-GM(r^3 - R^3)}{7r^2R^3}\hat{r}$$

On Desmos, the online graphing calculator, look carefully at the graph of the field's magnitude for condition (ii) for the range $0 \leq |\vec{g}| \leq \frac{GM}{R^2}$ and domain $R \leq r \leq 2R$, the graph appears linear and can be considered linear for this physically relevant domain and range. But if you zoom out, you'll realize that it is NOT linear. When graphing this on an assessment, you can *probably* treat it as linear, but confirm this with your teacher and know that it is not.