Introduction to Quantum Mechanics 2

Shankar Balasubramanian and Arun Kannan

April 11, 2014

1 Introduction

This lecture is a continuation of the previous lecture given on quantum mechanics. Our purpose is to introduce quantum mechanics at a basic level. Likewise, we will not be introducing advanced topics like perturbation theory.

2 Rotation Operators

As the state of a particle progress through time, it may rotate. For example, a particle initially in the $|+z\rangle$ state can evolve to the $|+x\rangle$ state. Although this deals with time evolution, which we will not discuss here, we do observe that we need a rotation operator to change our states. We briefly mentioned it in the previous lecture (when we asked to calculate the eigenvectors and eigenvalues). The rotation operator should therefore represent the following, taking the ψ-state to the ϕ-state after a rotation by an angle θ around a certain directional vector $\mathbf{n}$. We will use a hat to denote an operator.

$$|+\phi\rangle = \hat{R}(\theta\mathbf{n})|+\psi\rangle$$

For a more concrete example, this would be true:

$$|+x\rangle = \hat{R}(\frac{\pi}{2}\mathbf{j})|+z\rangle$$

Of course, we would like to reverse the rotation operator, since we are interested in finding a similar bra equation. We invent the **adjoint operator**. This simply reverses the rotation performed by the rotation operator. It inverses our rotation operator (but is not necessarily its inverse). So we'll say:

$$\langle +\phi| = \langle +\psi|\hat{R}^\dagger(\theta\mathbf{n})$$

The reason we do this is to preserve the Hermitian inner product. Test it out. If the states are normalized, their Hermitian inner product with their bra-vector yields one. This requires for the adjoint operator to multiply the with rotation operator to produce 1. A matrix which satisfies $\hat{U}^\dagger\hat{U} = 1$ is called a

unitary matrix. The dagger notation represents the complex-conjugate transpose of a matrix. Any matrix that satisfies $\hat{H}^\dagger = \hat{H}$ is called a Hermitian matrix. It might seem arbitrary, but we use the complex-conjugate transpose for simplicity's sake. In fact, many of the arguments we use in introducing quantum mechanics are some what arbitrary, as we generally try to use the most convenient notation (for example, using Dirac notation instead of matrices).

We are yet to actually determine what the rotation operator is, besides using some special notation for it. Let's strive to determine a more exact description of this operator. We will first assume that a rotation operator by an infinitesimal angle takes the form of

$$\hat{R}(d\phi\mathbf{k}) = 1 - \frac{i}{\hbar}\hat{J}_z d\phi$$

where $\hat{J}_z$ is a sort of operator that physically manifests the state by an angle $d\phi$. It is known as the **generator of rotations** around the z-axis, and does exactly what its name would suggest—generate rotations around the z-axis. Once again, this form, including the weird factor in front of the operator, is chosen for mere convenience and also for a physical interpretation (which we'll see later). Since we have been working in the z-basis, we will be rotating about the z-axis, once again for simplicity's sake.

We will also introduce the adjoint operator, as mentioned earlier. It is the complex-conjugate transpose of our rotation matrix, so we'll have:

$$\hat{R}(d\phi\mathbf{k})\hat{R}(d\phi\mathbf{k}) = 1 + \frac{i}{\hbar}\hat{J}_z^\dagger d\phi$$

Let's see if this satisfies the unitary condition:

$$\hat{R}(d\phi\mathbf{k}) = \left(1 + \frac{i}{\hbar}\hat{J}_z^\dagger d\phi\right)\left(1 - \frac{i}{\hbar}\hat{J}_z d\phi\right)$$

$$1 + \frac{i}{\hbar}\left(\hat{J}_z^\dagger - \hat{J}_z\right) + O(d\phi^2) = 1$$

So we necessitate that $\hat{J}_z$ be Hermitian, or otherwise the required unitary condition is not satisfied. The second-order differential terms will vanish and can be disregarded.

Let us now say that we want to rotate the state by a finite angle ϕ, and not by the differential angle as seen above. We therefore will need to apply the differential rotation operator several times. We will thus call $d\phi = \lim_{N\to\infty} \frac{\phi}{N}$. However, we will make these infinitesimal rotations N times, and take the limit as N goes to infinity.

$$\lim_{N\to\infty}\left(1 - \frac{i}{\hbar}\hat{J}_z\left(\frac{\phi}{N}\right)\right)^N$$

We recognize this as an exponential, and we get

$$\hat{R}(d\phi\mathbf{k}) = e^{-\frac{i\hat{J}_z\phi}{\hbar}}$$

What happens if were to rotate the $|+z\rangle$ around the **k**-direction? That requires the use of our generator of rotations around the z-axis, but from physical intuition we would know that we would have the same state. We had also said earlier that multiplying a state by the factor $e^{i\delta}$ should have no effect on the state. This suggests that the $|+z\rangle$ could be an eigenvector of the generator of rotations. Let's check it out. We want to show that:

$$\hat{J}_z|+z\rangle = c|+z\rangle$$

We know from a purely physical perspective that:

$$\hat{R}(\phi\mathbf{k})|+z\rangle = c|+z\rangle$$

Using a Taylor series expansion for our definition of the rotation operator around the z-axis (which corresponds to the unit vector **k**), we have:

$$\hat{R}(\phi\mathbf{k})|+z\rangle = e^{-\frac{i\hat{J}_z\phi}{\hbar}}|+z\rangle$$

$$= \left[1 - \frac{i\phi\hat{J}_z}{\hbar} + \frac{1}{2!}\left(-\frac{i\phi\hat{J}_z}{\hbar}\right)^2 + \dots\right]|+z\rangle$$

So clearly if $\hat{J}_z$ weren't to produce a multiple of the $|+z\rangle$ state when acting on it, we would have other states that wouldn't cancel out due to the linear independent property of the coefficients of a series expansion. Therefore $|+z\rangle$ (and of course $|-z\rangle$, since we didn't consider the sign) is an eigenvector of $\hat{J}_z$. We leave it as a problem to show that the eigenvalues of $|+z\rangle$ and $|-z\rangle$ when acted on by $\hat{J}_z$ are $\frac{\hbar}{2}$ and $\frac{-\hbar}{2}$ respectively. This goes back to our physical significance earlier—we introduced a strange constant, which was $\frac{i}{\hbar}$, and so we need the units to work out, making sense that $\hat{J}_z$ has units of angular momentum.

In this section we only detailed the generator of rotations around the z-axis. There is a generator of rotation for each axis, and the states corresponding to that axis will be eigenvectors of that generator of rotation.

3 Projection Operators

You may have remembered us mentioning this funny looking quantity, which we said not to confuse with the Hermitian inner product

$$|\psi\rangle\langle\psi|$$

This is, in fact a matrix itself. If we expand a state in some basis, we can write the state as a linear combination of weights placed by each basis elements. We introduced these weights as probability amplitudes

$$|\psi\rangle = \sum_i |e_1\rangle\langle e_i|\psi\rangle$$

This can be further simplified

$$|\psi\rangle = \left(\sum_i |e_1\rangle\langle e_i|\right)|\psi\rangle$$

In order to maintain this equality, the sum must break down to the identity matrix

$$\hat{I} = \sum_i |e_1\rangle\langle e_i|$$

This is known as the **completeness relationship**. Each individual term in the sum is known as a **projection operator** because its action on the state yields the component of the state along that basis vector.

$$(|e_1\rangle\langle e_i|)\,|\psi\rangle = |e_1\rangle\langle e_i|\psi\rangle$$

4 Change of Basis

Sometimes, our measurements are restricted only to some arbitrary set of axes. Some operators are written in terms of unit vectors in a different basis, so to the experimenter, it would be prudent to find the operator as represented by his/her new set of bases. Let us assume that the operator is in one set of bases m_i. The elements of the matrix of this operator are

$$A^j_{m_i}$$

Now, say we were to express this operator in another basis (n_i). The elements of this matrix is

$$\langle n_i|A^j_{m_i}|n_j\rangle$$

Now, we will apply a trick: we will keenly insert an identity operator

$$\langle n_i|m_i\rangle\langle m_i|A^j_{m_i}|m_j\rangle\langle m_j|n_j\rangle$$

The inner products that result are a matrix of scalar elements that cover the entire basis. Because

$$\langle n_i|m_i\rangle = \langle m_j|n_j\rangle^\dagger$$

We can rewrite the matrix in the new basis as

$$A_n = \langle n_i|m_i\rangle A_m\langle m_j|n_j\rangle^\dagger$$

This may be familiar to you if you have taken linear algebra. Yet, this is a powerful tool if one would like to work with one set of bases.

5 Photon Polarizers

A polarizer is a device that works to filter out electromagnetic waves which are not of a specific orientation. Classically, we talk about the intensity of a wave emitted after polarization as the magnitude of the resulting electric field squared. In quantum mechanics, however, the logic does not reside with this "continuous medium". Instead, the intensity is represented as a probability that a photon would be transmitted. In this sense, the electric field of the individual photons has not diminished; it is just the fact that there are fewer photons that successfully pass through the polarizer. Based upon rotation matrices, we can define a basis for the polarizer as

$$|x'\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle$$

$$|y'\rangle = -\sin\phi|x\rangle + \cos\phi|y\rangle$$

This is the basis for linear polarizers: that is, the electric fields trace out a line. We also have the case of circular polarizers: where light rays travel in a circular path. Without proof, the basis for circular polarization is

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle)$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle)$$

If we consider a right-circular polarized state that is rotated by some phase angle ϕ,

$$|R'\rangle = \frac{1}{\sqrt{2}}(|x'\rangle + i|y'\rangle)$$

Upon simplification, we see that

$$|R'\rangle = e^{-i\phi}|R\rangle$$

Similarly

$$|L'\rangle = e^{i\phi}|L\rangle$$

Because rotating this state is just applying the Rotation Operator to the state, we see that

$$|R'\rangle = e^{-i\phi}|R\rangle = e^{-\frac{i\hat{J}_z\phi}{\hbar}}|R\rangle$$

$$|L'\rangle = e^{i\phi}|L\rangle = e^{-\frac{i\hat{J}_z\phi}{\hbar}}|L\rangle$$

This implies that

$$\hat{J}_z|R\rangle = \hbar|R\rangle$$

$$\hat{J}_z|L\rangle = -\hbar|L\rangle$$

This implies that photons are not spin-1/2 particles; they are **spin-1** particles

6 Problems

1. Verify that $\frac{\hbar}{2}$ is an eigenvalue of the eigenvector $|+z\rangle$ when acted on by $\hat{J}_z$. Repeat for $\frac{-\hbar}{2}$ and $|-z\rangle$.

2. What is the matrix representation of $\hat{J}_z$ in the z-basis? What about in the x-basis?

3. Find $\hat{R}(\theta\mathbf{j})|+z\rangle$. Since we are rotating around the y-axis and not the z-axis, we know that $\hat{R}(\theta\mathbf{j}) = e^{\frac{-i\hat{J}_y\theta}{\hbar}}$, based on a similar derivation to the one in the section. Evaluate this at $\theta = \frac{\pi}{2}$. Does this answer make sense?

4. A system of $N \to \infty$ ideal linear polarizers are arranged in a sequence, whereby each subsequent polarizer is staggered by a small angle $d\phi$. If the last polarizer is displaced an angle ϕ, determine the probability that a photon passing through the initial setup will be transmitted all the way through.

5. Assume that a state $|\psi\rangle = a|x\rangle + b|y\rangle$. Photons polarized to this state are absorbed by a black disk at a rate of N. Determine the magnitude and direction of the torque exerted on the disk.

6. A birefringent crystal is a special type of crystal such that the index of refractions along the x and y axes are different (n_x and n_y). Linearly polarized light of wavelength λ is polarized at an angle ϕ and moves along the crystal. If the crystal is a distance d long, determine the probability that the exiting photon will be right circularly polarized.