Electrodynamics and Electrostatics

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1 What is Electrodynamics?

As you have already been made aware of this year (if you attended the earlier lecture), the Gravitational Force is among the four major forces which govern physical objects. In addition to this somewhat unruly concept, there exists the Strong Nuclear Force, the aptly named Weak Nuclear Force, and the Electromagnetic Force, the latter of which we shall be concerning ourselves with.

Electrodynamics and Electrostatics are subdivisions of the larger study of the aforementioned force, and of the propagation and laws governing said force. Although the full extent of these concepts are beyond the scope of this lecture, electromagnetic forces dwarf those of gravitation, and are much more readily observed and utilized. Due to its dominance over the other interactions at the macroscopic level, the Electromagnetic Force has been investigated since antiquity, and much of the theory concerning it developed before the notion of a rigorously defined charged particle came into being.

2 The Point Charge and Electric Fields

Point charges have a given magnitude, occupy a point in space, and thus are convenient to work with. Given a set of charges in coordinate space, we can define the electric field, **E**, at any point save those a point charge occupies. This is represented graphically by field lines[1], which provide an convenient model for heuristics.

An important question is what we can do with these point charges, or contiguous objects. Much like gravity, there is a way to define the force exerted by one charge on another, such as with one mass on another. However, due to the existence of positive and negative charges, this force can either be repulsive or attractive. This is the general form[2], note the similarities to gravity. (r^{-2} dependence, and a proportionality constant)

$$\mathbf{F} = \frac{kQ_1Q_2}{r^2}\hat{r} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q_1Q_2}{r^2}\hat{r} \tag{1}$$

It's easy to see how this can be both attractive and repulsive dependent on the signs of Q_1 and Q_2. This law always holds, and can be used to determine the force from larger, non-point objects given the proper mathematical tools.

3 Distinctions and Other Relations

3.1 Superposition

As mentioned earlier, a useful concept is the analysis of the electric field at a point in space. This is defined in terms of the force observed on a motionless test-charge. Electricity, like gravitation, obeys the superposition principle, that is

$$\mathbf{E}_T = \sum_i \mathbf{E}_i = \sum_{i=1}^{N} \frac{kQ_i}{r^2}\hat{r} \tag{2}$$

This is very useful, and allows a variety of interesting and clever tricks for the simplification of an electric field due to a body of charge. Note however, that very quickly the number of discreet charges may become large, unduly large[3].

[1] By convention, an electric field line begins at a positive charge and ends at a negative charge or infinity.
[2] ϵ_0 is known verbosely as "the permittivity of free space"
[3] Charge is quantized, and so in technicality calculus is in this case only an approximation.

3.2 Too Much Superposition

When these charges become to numerous to efficiently write in the margins, the unfortunate[4] corollary is calculus. Fortunately, for well behaved objects, Maxwell conjured in his set of beautiful equations, that which is seen below.[5]

$$\|\nabla \times \mathbf{E}\| = \frac{\rho}{\epsilon_0} \tag{3}$$

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \tag{4}$$

These can be your friend, and they are always valid in well-behaved physical systems. However, they can be drastically simplified, and the testy calculus stripped away if you're willing to sacrifice generality. With these you can find the electric fields surrounding most basic geometric structures. The two forms are equivalent, and the reason why is yet another powerful tool in mathematics.

4 Magnetism

So, from earlier, you can determine from Coulomb's Law, that the force on a charged particle (q) can also be written in the form $\mathbf{F} = q\mathbf{E}$, where $\mathbf{E}$ is the electric field. However, when we introduce magnetism, it can be shown that $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$[6]. Where $\mathbf{B}$ is known as the Magnetic Field[7].

As such, the magnetic field surrounding a particle (note it must be moving to produce a non-trivial result), is defined as[8]

$$\mathbf{B} = \left(\frac{\mu_0}{4\pi}\right) \frac{q\mathbf{v} \times \mathbf{r}}{r^2} \tag{5}$$

Lovely, what can you tell about a magnetic field at certain points around a moving / still particle ? There are many major models for magnetism, among them the Amperian and the Gilbert, which deal with the existence of mono-poles. Magnetism, classically, is still somewhat poorly defined, but inextricably linked to its static counterpart.

4.1 Basic Geometries and Influence of Current

If we wiggle around the terms of the expression for $\mathbf{B}$, we can determine the Magnetic Field for any continuous distribution of flowing charge in which the charge is flowing in a manner in which no charge "bunches up", so to speak

$$\mathbf{B} = \sum_i \left(\frac{\mu_0}{4\pi}\right) \frac{q_i}{dt} \frac{d\ell \times \mathbf{r}}{r^2} \tag{6}$$

$$\mathbf{B} = \left(\frac{\mu_0}{4\pi}\right) \int_C \frac{I d\ell \times \hat{\mathbf{r}}}{\|r\|^2} \tag{7}$$

Where I is the current, or charge per time, and $d\ell$ is a displacement over the wire-like object. This entire expression is known as the Biot-Savart law.

5 Electric Potential

Although it has already been used regardless of your cognizance of it, electric potential is essential to understanding dynamic systems, while it is merely a curiosity in electrostatics. The notion of gain or loss of energy based on ones translation though space is not a new one, and has been shown extensively with gravitational forces. Voltage is a scalar, and it is given arbitrary, but convenient reference points, there is no V, only ΔV.

[4] Who am I kidding, Calculus is great.

[5] I prefer $\hat{n}dS$ over $d\mathbf{S}$, but physicists complain, and they have lasers.

[6] This is known as the Lorentz Force

[7] $\mathbf{H}$ has a similar meaning, and in a vacuum is proportional to $\mathbf{B}$, but this is more common, in fact, $\mathbf{B}$ is defined as the vector that makes the above law true.

[8] μ_0 is referred to as the "permeability of free space." $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$!

$$\Delta V = V_b - V_a = -\int_a^b \mathbf{E} \cdot d\vec{\ell} \tag{8}$$

Note that this field is logically distinct from the notion of potential energy, which has units J, and that V is instead defined as energy per unit charge, and is thusly "independent" of these charges (JC^{-1}). Positive charges wish to move from regions of higher to lower potential, whilst negative do the opposite. The nature of this force, as you already know, is to repel like charges, and attract different. This field, $\mathbf{E}$, with which you are familiar, is related to V under

$$\mathbf{E} = -\nabla V \tag{9}$$

This is an extremely important relationship, and the concept of the intertwining of scalar-potential and vector fields is strewn throughout physics. To be logical, and prevent perpetual motion machines of the first kind. Given this, we find that although the Electric Field may be discontinuous, the potential field is bounded cleverly to maintain its C^0 status.

6 A Brief Overview Dynamic Systems

As is introduced with above, the motion of charge or set of charges through space is not only an interesting phenomenon, but one which gives rise to an entire field of study (electrodynamics), and an entire subset of magnetism. Current, as we often associate naturally with the word, is a measure of a particular amount of substance, passing a given point in a given time, or

$$I = \frac{\Delta Q}{\Delta T} \tag{10}$$

This is a novel idea, and is most certainly true, but is poorly defined when speaking of actual, quantized charge carriers. Given uniform charges of q, each with a drift velocity v_d, a cross section of A, and a number density of n[9], the above expression can also be written as

$$I = nqAv_d \tag{11}$$

Perhaps the more interesting of the terms is the so aptly-named "drift-velocity", v_d, which, under normal conditions in wires, rests at a few millimeters per second. We can approximate this drift velocity given the time between collisions between charged particles, τ, and the acceleration experienced between said collisions. This acceleration from Coulomb's Law is given as $\mathbf{F} = q\mathbf{E}$. We know that, non-relativistically, $\mathbf{F} = m\mathbf{a}$, and so we arrive that[10]

$$v_d = \frac{eE}{m_e}\tau \tag{12}$$

And from this we can formulate a new expression for current.

$$I = \left[\frac{e^2 n\tau}{m_e}\right](AE) \tag{13}$$

$$I = \sigma A\frac{V}{\ell} \tag{14}$$

$$V = \left(\frac{\ell}{\sigma A}\right) I \tag{15}$$

$$V = RI \tag{16}$$

$$\tag{17}$$

We've skipped a few steps here, but σ is defined as the conductivity of a substance, with units $\Omega^{-1}m^{-1}$. The simplifications you have seen me make are the definitions of these new terms (loosely), and in this particular situation, condense down to what I hope you recognize as Ohm's Law.[11]

[9] charge units per volume.
[10] We're generalizing to electrons
[11] which only holds in "Ohmic" materials, ha!

We can also examine the concept of Magnetic Flux, as we did with the Electric Field, and its effect on the current in an object. In its simplest form, as is echoed by the Maxwell-Faraday Equation

$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t} = \oint_{\partial \Sigma} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\ell \tag{18}$$

7 Maxwell's Equations

At the base of classical electrodynamics lies a set of partial differential equations which define both how magnetic and electric fields are generated, and how they interact and alter each-other. Bear in mind, these may look rather confusing, but each represents a fundamental concept that it possible for you to understand without knowledge of the cumbersome notation. In fact, one of them you have already seen.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{19}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{20}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{21}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{22}$$

You have seen these symbols before, save **J**, which is the current density. Note the symmetry, as we convolute the del operator through both cross and dot products upon the Electric Vector Field, and the Magnetic Pseudo-Field. Also note that **J** is a vector, which is an interesting concept.[12]

7.1 Divergence

Here, the first two are seen to perform the dot product with ∇. In actuality, this notation is simply convenient, as ∇ itself is a very abstact concept were we assign the "vector"'s components to be the partial-derivatives with respect to space. When we use ∇ in this way, we find what is known as the *divergence* of the field; In other words

$$\operatorname{div} \mathbf{F}(p) \stackrel{\text{def}}{=} \lim_{V \to p} \oiint_S \frac{\mathbf{F} \cdot \mathbf{n}}{|V|}\, dS \tag{23}$$

This equates to finding the flux across the surface of a volume V as said volume shrinks around a point p. For electric fields, this is proportional to the charge density, while for magnetic fields, due to the non-existence of classical monopoles, it is zero.

7.2 Curl

The next operation, the cross-product with ∇ produces a vector, and hence when applied to a vector field, produces another field. This is also a slight abuse of notation for convenience, and can also be referred to as the *curl* of the field; In other words

$$(\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}} \stackrel{\text{def}}{=} \lim_{A \to 0} \left(\frac{1}{|A|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right) \tag{24}$$

When used in the two latter contexts, the Maxwell-Faraday Equation and Ampere's Circuital Law, the operator loosely represents the rotational tendencies of the field. This may be loosely interpreted in the M-F Equation to mean that a rotational electric field induces magnetic fields and allows analysis of induction and production of electromotive forces, or emf[13]. Note also that many of these laws, including MF and AC, are abstractions of other, more simple relations[14]. Take care when using them as often you do not need all of their impressive / accurate if not a little trying add-ons[15].

[12] $q = \iiint_S \mathbf{J} \cdot d\mathbf{S} dt$

[13] These are normally denoted by the symbol $\mathcal{E}$ and are beyond this lecture in their full nuances.

[14] In fact, these can also be generalized even further to Stokes' Law.

[15] For AC, see that if you distribute, you get a $\mu_0 \epsilon_0$.

8 Problems!

1. What is the electric flux through one side of a cube that has a single point charge of $3.00\mu C$ placed at its center? HINT: You do not need to integrate any equations to get the answer.

2. A uniformly charged, infinitely long line of negative charge has a linear charge density of λ and is located on the z-axis. A small positively charged particle that has a mass m and a charge q is in a circular orbit of radius R in the xy plane centered on the line of charge. Derive an expression for the speed of the particle. Obtain an expression for the period of the particle's orbit.

3. Given a parallel plate capacitor, derive an expression for the energy contained per unit volume if the total energy of a capacitor is $\frac{1}{2}QV$, and $Q = CV$. This is a general result. Use this to calculate the energy contained within a dielectric sheath (dielectric constant κ) of a sphere of charge Q, radius R_1, and dielectric radius R_2, $(R_1 \leq R_2)$.

4. Given a light, thin rod of length L, balanced at its center with a charge of $+2Q$ at the right end, and $+Q$ at the left, with each charge at equilibrium a distance h from a $+Q$ charge directly below said ends, determine the placement of a mass m on the rod to balance the system.

5. A current-carrying wire is bent into a closed semicircular loop of radius R that lies in the xy plane. The wire is in a uniform magnetic field that is in the $+z$ direction. Verify that the force acting on the loop is zero.

6. The hydrogen atom in its ground state can be modeled as a positive point charge of magnitude $+e$ (the proton) surrounded by a negative charge distribution that has a charge density (the electron) that varies with the distance from the center of the proton r as: $\rho(r) = -\rho_0 e^{\frac{-2r}{a}}$ (a result obtained from quantum mechanics). Calculate the value of ρ_0 needed for the hydrogen atom to be neutral. Calculate the electrostatic potential (relative to infinity) of this system as a function of the distance r from the proton.