8th Period Mechanics Problem Solution

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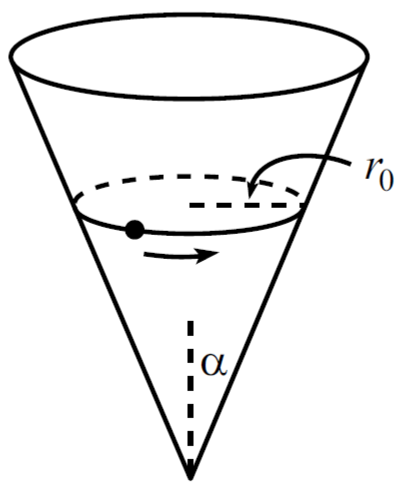
Motion Constrained to a Cone

Consider the motion of a mass m constrained to move in a cone as in the picture. The angle bisector of the tip is α. Determine the frequency of circular motion around the cone, ω. Then determine the frequency of small oscillations about the equilibrium position.



The Lagrangian for the system is

$$\mathcal{L} = \frac{1}{2}m\left(r^2\dot{\theta}^2 + \frac{\dot{r}^2}{\sin^2\alpha}\right) - \frac{mgr}{\tan\alpha}$$

varing the equation in θ gives conservation of angular momentum:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

with conserved angular momentum $L = mr^2\dot{\theta}$. We now vary r:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = \frac{m\ddot{r}}{\sin^2\alpha} = \frac{\partial\mathcal{L}}{\partial r} = mr\dot{\theta}^2 - \frac{mg}{\tan\alpha}$$

The angular frequency of a circular orbit with radius r_0 occurs when $\ddot{r} = \dot{r} = 0$ so we have:

$$\omega = \dot{\theta} = \sqrt{\frac{g}{r_0 \tan\alpha}}$$

To determine the frequency of smalloscillations in r about a circle of radius r_0 we write: $r(t) = r_0 + \delta(t)$. Then we rewrite the equation of motion obtained by varying r by substituting in the angular momentum and expand to first order in δ:

$$\ddot{r} = \frac{L^2}{m^2 r^3} \sin^2\alpha - g \sin\alpha \cos\alpha$$

$$\ddot{\delta} = \frac{L^2 \sin^2\alpha}{m^2} \frac{1}{r_0^3} \left(1 - \frac{\delta}{r_0}\right)^3 - g \sin\alpha \cos\alpha = \frac{L^2 \sin^2\alpha}{m^2} \frac{1}{r_0^3} \left(1 - \frac{3\delta}{r_0}\right) - g \sin\alpha \cos\alpha$$

Using the first of the two above equations with $\ddot{r} = 0$ we obtain:

$$\ddot{\delta} + \frac{3L^2 \sin^2\alpha}{m^2 r_0^4} \delta = 0$$

so we have the frequency of small oscillations as:

$$\Omega = \sqrt{\frac{3L^2 \sin^2\alpha}{m^2 r_0^4}} = \sqrt{\frac{3g}{r_0} \sin\alpha \cos\alpha}$$