Kinematics

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1 Vectors!

1.1 Definition

A vector is defined to have two properties: magnitude and direction. Magnitude is the size, or length, of a vector. Its units may vary. Direction is, simply put, the direction in which the vector is pointing.

1.2 Vector Components

In two or three dimensions, vectors can be broken down into components. Typically, these would be $\hat{x}$, $\hat{y}$, and $\hat{z}$ for the x, y, and z components of a vector. For instance, a two dimensional vector with magnitude 6 and direction 60° counterclockwise of the x-axis could be written as $3\hat{x} + 3\sqrt{3}\hat{y}$.

1.3 Vector Addition

Vectors can be added and subtracted component by component. Suppose vector $\mathbf{A} = A_x\hat{x} + A_y\hat{y}$ and vector $\mathbf{B} = B_x\hat{x} + B_y\hat{y}$. Their sum will be $\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y}$. For vector subtraction, treat it as $\mathbf{A} + (-\mathbf{B})$.

1.4 Dot Products

The dot product of two vectors is defined as $\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y$ or in three dimensions, $\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$. A neat property of dot products is that $AB\cos\theta = \mathbf{A} \cdot \mathbf{B}$ where $A = |\mathbf{A}|$ and $B = |\mathbf{B}|$ where θ is the angle between vectors $\mathbf{A}$ and $\mathbf{B}$ ($|\mathbf{r}|$ designates the magnitude of some vector $\mathbf{r}$.)

1.5 Notation

There are three different types of commonly accepted vector notation: engineering notation, ordered set notation, and matrix notation. We won't worry about matrix notation; that will be covered when you take linear algebra. Engineering notation is the form I have been using throughout this section: in other words, component form. Ordered set notation, which will be used later, is as follows: $\mathbf{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = \langle A_x, A_y, A_z\rangle$

2 Kinematics

2.1 Definitions

There are three basic vector quantities in kinematics: position, velocity, and acceleration. The position vector of a particle points from the origin to that particle's location in space. The position vector's components have units of length, e.g. meters, and the standard position vector can be written as $\mathbf{r} = \langle r_x, r_y, r_z\rangle$. Related to position is displacement, which is the difference in position between two points. The displacement of point B from point A is defined as $\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A$, and the magnitude of this vector is the distance between these two points.

The velocity of a particle is defined as the rate of change of its position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$. The velocity vector's components have units of length per time, e.g. meters per second, and the standard velocity vector can be written as $\mathbf{v} = \langle v_x, v_y, v_z\rangle$. The speed of a particle is the magnitude of its velocity vector.

The acceleration of a particle is defined as the rate of change of its velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$. The acceleration vector's components have units of length per time squared, e.g. meters per second squared, and the standard acceleration vector can be written as $\mathbf{a} = \langle a_x, a_y, a_z\rangle$.

2.2 Constant Acceleration

When a particle undergoes constant acceleration, its motion can be computed as follows:

$$\mathbf{a}(t) = \mathbf{a}$$

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_0^t \mathbf{a}(t)\,dt = \mathbf{v}_0 + \int_0^t \mathbf{a}dt = \mathbf{v}_0 + \mathbf{a}t$$

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}(t)\,dt = \mathbf{r}_0 + \int_0^t (\mathbf{v}_0 + \mathbf{a}t)\,dt = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

3 Problems

1. What is the vector pointing from the point $(1, -4, 3)$ to the point $(4, 0, -9)$. What is its magnitude? What is the angle between this vector and the vector $4\hat{x} - 4\hat{y} + 6\hat{z}$?

2. Consider a juggler who is juggling two identical balls such that each ball goes into the air as the other one reaches the peak. If the motion of each ball is identical, at what fraction of the maximum height of the balls do are they at the same elevation? Assume they are tossed from height $H = 0$.

3. Two particles are oscillating along the x-axis across the origin. One starts at time $t = 0$ and the other starts at time $t = t_0$. If the motion of the first particle can be described by the equation $x(t) = A \sin \omega t$ and the second particle's motion is identical the that of the first, what is the maximum distance between the particles? What is the maximum relative speed between them?

4. A cat is launched on flat ground with initial speed v_0 from ground level at an angle of $45°$. A dog is launched with the same initial speed v_0 from the same place at an angle of $60°$. When the dog hits the ground, it bounces with half of its speed when it hit the ground, and when the cat hits the ground, it bounces with one third the speed with which it hit the ground. If there is no air resistance, what will be the distance between the dog and cat when they finally come to rest?

5. For a given initial speed, at what angle should a ball be thrown so that it travels the maximum horizontal distance between its launch and return to the ground? Assume that the ground is horizontal, and that the ball is launched from ground level.

6. Consider a particle whose position can be described by $x = A \cos(\omega t)$ and $y = B \sin(\omega t)$. Describe the motion of this particle. What will be the acceleration vector as a function of time?