Rotational Kinematics

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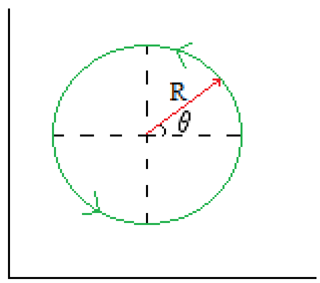
1. Circular Motion

Last week we took a look at the kinematics equations, which we use to model particles launched on the surface of the Earth. However, these equations are only true when we have a constant acceleration vector. The most important thing to understand is that acceleration is the time-derivative of velocity, and that velocity is the time-derivative of position. Another type of motion we are interested is circular motion. Circular and periodic motion and oscillations show up frequently, and the analysis of the corresponding forces plays an important role. However, for now, we'll stick to the basics.



Just like we can have position in space, we can have position on a circle as a function of time. Instead of using vectors however, I will use complex numbers here because they make everything far easier radian is dimensionless.

2. Rotational Quantities

We have three important rotational quantities, angular position, angular velocity, and angular acceleration. We denote these with $\vec{\theta}, \vec{\omega},$ and $\vec{\alpha}$, respectively, and are all vector quantities, and have the following relationships:

$$\vec{r} = \vec{\theta} \times \vec{R}$$

$$\vec{v} = \vec{\omega} \times \vec{R}$$

$$\vec{a} = \vec{\alpha} \times \vec{R}$$

They are also related to each other the same way position, velocity, and acceleration in kinematics are related. Deriving these equations will be left as a problem.

3. Describing Circular Motion with Constant Speed

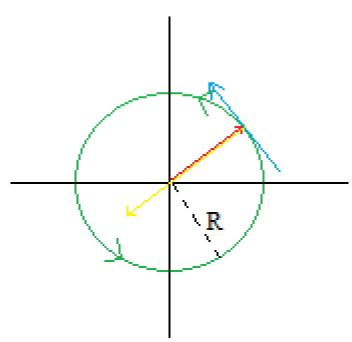
Once again, we are interested in motion with constant acceleration, however, what you will see here is completely different from normal kinematics. Since complex numbers are like vectors, except cooler and far more useful in the two dimensional case, we can use their properties we know that for a particle moving in some circle in the complex plane with uniform SPEED (not velocity) the position (which is now z, instead of r, it doesn't really matter) is given below:

$$z(t) = Rcos(\omega t) + iRsin(\omega t) = Re^{i\omega t}$$

Here, ωt represents the angle as a function of time, and it makes sense since an angular velocity times a time will give an angle. We need to be careful here, because this describes counterclockwise motion, which starts at the complex number R at time $t = 0$. If we are interested in clockwise motion, then we would need a negative sign in front of the i. If this is our position, then we know that our velocity is the time-derivative, which is:

$$v(t) = \frac{dz(t)}{dt} = i\omega Re^{i\omega t}$$

How do we interpret this? Basically, this is a complex number, which looks like a vector, and since it's the time-derivative of position, it points in the direction that the position is changing. However, upon closer



inspection it is obvious that this is basically $i\omega$ times our position. With respect to complex geometry, this is just a rotation by 90 degrees counterclockwise, since the angle of i is 90 degrees, and a dilation by a factor of ω, which isn't all that important for now. This means that our velocity is always perpendicular to our position. Ask yourself is that makes sense about motion on a circle. In order to find the acceleration, we take the derivative again:

$$a(t) = \frac{dv(t)}{dt} = -\omega^2 R e^{i\omega t}$$

Which tells us that the acceleration is opposite the direction of motion, and perpendicular to our velocity. The following diagram will clarify (let red represent the position vector, blue the velocity vector, and yellow the acceleration vector). By the way, these equations say that in this case, $|a(t)| = R\omega^2$, and since $R|\omega| = |v(t)|$, we also have $|a(t)| = \frac{|v(t)|^2}{R}$.

Problems:

1. Find equations for the non-constant acceleration case.
2. A particle is travelling around the unit circle in a counterclockwise fashion, starting at (1,0), with a magnitude of angular velocity 2. Find the following as a function of time:
    a. Acceleration
    b. Velocity
    c. Position
    d. Angular Acceleration
    e. Angular Position