Relativistic Dynamics, Part 1

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1 Relativity Foundations

Recall that for two observers in inertial frames traveling at a relative speed v, the relativistic "boost" factor is denoted

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

We define the *proper time* t_0 or τ between two events to be the time between the events as measured in a frame where the events occur in the same place. Then, t, the time interval observed in an inertial frame moving at relative speed v, is characterized by

$$t = \gamma t_0 = \gamma \tau.$$

Similarly, we define the *proper length* x_0 to be the length between two points as measured in a frame where the endpoints are measured simultaneously. Then, x, the length observed in an inertial frame moving at relative speed v, is characterized by

$$x = \frac{1}{\gamma} x_0.$$

Lengths perpendicular to the direction of motion are unchanged. Note that these relations mean that we have an *invariant*, which we will denote s, which is the same in all frames.

$$s = -(ct)^2 + x^2.$$

Since the speed of light scales times to lengths universally, we will typically use ct instead of t in order to match units.

2 Momentum

Consider a frame of reference E, in which 2 particles of mass m collide at the origin $(0, 0)$. A fast particle travels at a relativistic speed from the point (x, y), y very small compared to x and a slow particle at a Newtonian speed from $(0, -y)$. We time these two particles to collide such that the slow particle rebounds at its original speed in the opposite direction.

Consider another frame of reference R moving in the x direction with the same x-velocity as the fast particle. Then, symmetry dictates that the proper time between the release of the fast particle and the collision, τ_R, equals the proper time between the release of the slow particle and the collision, τ_E. However, the proper time τ_E is nearly the measurement of the same time in E, t_E. So,

$$m\frac{y}{\tau_E} = m\frac{y}{\tau_R}$$
$$m\frac{y}{t_E} = m\frac{y}{\tau_R}$$

But the left side is simply the momentum of the slow particle, meaning the right side must be the momentum of the fast particle. Generalizing,

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = \gamma m\frac{d\vec{r}}{dt}.$$

Note: This derivation is just one of many possible. See Exercise 1 for another derivation.

3 Four-Vectors

The concepts of relativity mandate that we not only consider space but *space-time*, a consideration that requires *4-vectors*. As the name implies, a 4-vector is a vector in four dimensions; 3 space, one time. Conventionally, the index is a superscript, for reasons that will be explained later. We define the *contravariant space-time displacement vector* x^μ for displacement $(\Delta t; \Delta x, \Delta y, \Delta z)$ as

$$x^0 = c\Delta t; \qquad x^1 = \Delta x, x^2 = \Delta y, x^3 = \Delta z.$$

In order to preserve the invariant $(ct)^2 - x^2$, we define the *covariant space-time displacement vector* x_μ identically to x^μ except that $x_0 = -x^0$. Using index notation, we can now define the *scalar product* of two vectors a, b as $a^\mu b_\mu$. Just as the dot product is invariant under rotation, the scalar product is invariant from any frame; i.e. under Lorentz transformation.

4 Four-Momentum

We have now established the basics to treat 4-momentum, p. In Newtonian mechanics, we simply apply the operator $m\frac{d}{dt}$ to the reference frame. However, this application essentially assumes the universality of time, which relativity has overthrown. The only "preference" that relativity leaves us is the co-moving frame and the proper time. So we define the 4-momentum p as

$$p^\mu = m\frac{dx^\mu}{d\tau}.$$

Note that this takes us back immediately to the momentum found in Section 2. Furthermore, it gives us a relativistic expression for the energy,

$$p^0 = m\frac{d}{d\tau}ct = \gamma mc.$$

Scaling this back to traditional energy units gives the more well-known

$$E = \gamma mc^2.$$

Also, $p^\mu p_\mu$ is invariant; in fact, it comes out to be $(mc)^2$. To link it back to usual ideas of momentum and energy as more separate entities, $E^2 - p^2c^2 = m^2c^4$.

Four-momentum is physically important because it is *conserved in all collisions.* We will analyze this in Part 2 to solve relativistic collision problems as well as discover strange properties of mass, which will be invariant but not conserved.