Lagrangian Lecture 1 Problem Solutions

Will Bunting

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1 Inverted Pendulum

The box oscillates with the function $x(t) = A\cos(\omega t)$. The pendulum is a rigid rod. Solve for the equation of motion of the mass m. Note that if the frequency ω is high enough the pendulum will not tip over.

If we make θ the angle between the bob and the vertical:

$$\begin{aligned}(x,y)_m &= (x + l\sin\theta, l\cos\theta)\\ (v_x, v_y)_m &= (\dot{x} + l\dot{\theta}\cos\theta, -l\dot{\theta}\sin\theta)\end{aligned}$$

The Lagrangian is then simply:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta) - mgl\cos\theta\\ &= \frac{1}{2}m(A^2\omega^2\sin^2\theta + l^2\dot{\theta}^2 - 2l\dot{\theta}A\omega\sin(\omega t)\cos\theta) - mgl\cos\theta\end{aligned}$$

Using the Euler-Lagrange equation on θ (we already know x!).

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\left(ml^2\dot{\theta} + ml\dot{x}\cos\theta\right) = -ml\dot{x}\sin\theta + mgl\cos\theta$$

$$l\ddot{\theta} + \ddot{x}\cos\theta = g\cos\theta$$

Substituting $x(t) = A\cos(\omega t)$ ...

$$\Rightarrow \ddot{\theta} - \left(\frac{g}{l} + \frac{A\omega^2}{l}\cos(\omega t)\right)\cos\theta = 0$$

2 Sliding Blocks

Two block slide as shown in the picture frictionlessly. They always remain in contact and the box with mass m cannot tip over (don't worry about torque). Solve for the equation of motion of both boxes.

The only tricky part about getting the Lagrangian for this system is the vertical component for the block with mass m. We see the vertical position of the mass must be $(x_1 + x_2)\tan\theta$, keeping the boxes constrained to remain in contact (this is where some of the trickiness comes when not using Lagrangians). So we get for the Lagrangian:

$$\mathcal{L} = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2 \tan^2\theta) + mg(x_1 + x_2)\tan\theta$$

Using the Euler-Lagrange equations on x_1 and x_2:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) = \frac{\partial \mathcal{L}}{\partial x_1} \quad \Rightarrow \quad M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta = mg\tan\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) = \frac{\partial \mathcal{L}}{\partial x_2} \quad \Rightarrow \quad m\ddot{x}_2 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta = mg\tan\theta$$

Notice the conservation of linear momentum in the horozontal direction derives from the subtraction of the above two equations ($\frac{d}{dt}(M\dot{x}_1 + m\dot{x}_2) = 0$). We will talk more about conserved quantities with Noether's Theorem.