Homework 2

Monday, September 6, 2021 11:06 PM

Question 1

a) $\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3$

b) $\sum_{t=1}^{3} \frac{x^{2t}}{t!} = \frac{x^{2(1)}}{1!} + \frac{x^{2(2)}}{2!} + \frac{x^{2(3)}}{3!} = x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

c) $\sum_{i=1}^{3} \sum_{j=1}^{i} x^{i+j} = x^{1+1} + x^{2+1} + x^{2+2} + x^{3+1} + x^{3+2} + x^{3+3}$

$= x^2 + x^3 + x^4 + x^4 + x^5 + x^6$

$= x^2 + x^3 + 2x^4 + x^5 + x^6$

d) $\sum_{i=1}^{2} \sum_{j=2}^{4} (x_{ij} - y_i) = (x_{12} - y_1) + (x_{13} - y_1) + (x_{14} - y_1) + (x_{22} - y_2)$

$+ (x_{23} - y_2) + (x_{24} - y_2)$

$= x_{12} + x_{13} + x_{14} + x_{22} + x_{23} + x_{24} - 3y_1 - 3y_2$

e) $\sum_{k=-1}^{1} (2k+1) x_{k-1}^2 = (2(-1)+1) x_{-1-1}^2 + (2(0)+1) x_{0-1}^2 + (2(1)+1) x_{1-1}^2$

$= -x_{-2}^2 + x_{-1}^2 + 3x_0^2$

f) $\sum_{n=2}^{4} \sum_{m=n}^{n+2} x_n y_m = x_2 y_2 + x_2 y_3 + x_2 y_4 + x_3 y_3 + x_3 y_4 + x_3 y_5$

$+ x_4 y_4 + x_4 y_5 + x_4 y_6$

Question 2

a) $n = 3 \times 1$

b) $2x - y = 2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} - \begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}-1\\2\\5\end{bmatrix}$

c) $x^T y = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix} = 3(1) + 2(2) + 3(1)$

$= 10$

d) $\|x-y\|_2 = \sqrt{(1-3)^2 + (2-2)^2 + (3-1)^2} = \sqrt{(-2)^2 + 0^2 + 2^2}$

$= \sqrt{8}$

e) $\|x-y\|_1 = |1-3| + |2-2| + |3-1| = |-2| + |0| + |2| = 4$

f) $\|x-y\|_\infty = \max_{1 \le i \le n} |x_i - y_i| = \max_{1 \le i \le n} \begin{bmatrix}-2\\0\\2\end{bmatrix}$

$= \max(|-2|, 0, |2|)$

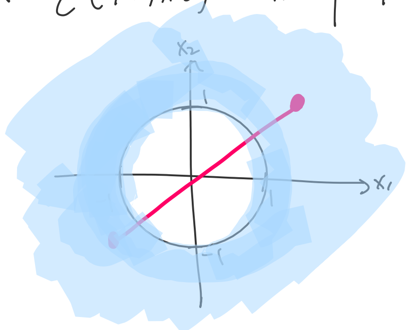
$= 2$

g) $x^T A y = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & -2 & 3\\-2 & 3 & -1\\3 & -1 & 2\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6 & 1 & 7\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}$

$= 27$

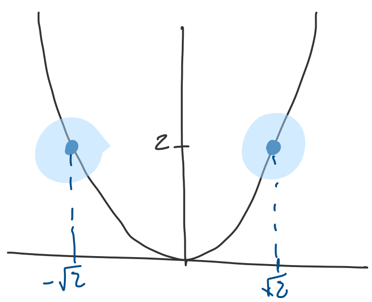
Question 3

a) $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \geq 1\}$ $\Rightarrow$ NOT a convex set



$\rightarrow$ a line segment connecting two points in the set does not lie entirely in the set (shaded area)

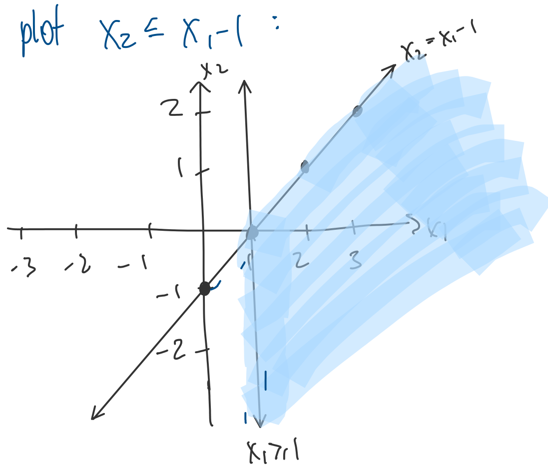
b) $X = \{x \in \mathbb{R} \mid x^2 = 2\}$ $\Rightarrow$ NOT a convex set



$\Rightarrow$ a line segment connecting the two points will leave the set.

c) $X = \{(x_1, x_2) \mid \dfrac{x_2}{x_1 - 1} \leq 1,\ x_1 \geq 1\}$ $\Rightarrow$ IS a convex set

$\hookrightarrow$ plot $x_2 \leq x_1 - 1$:



$\Rightarrow$ any line segment drawn in $\mathbb{R}^2$ space in shaded area will be contained within the set

Question 4

a) $\min \{ x_1^2 + 2x_2^2 \mid x_1 \leq 0,\ x_2 \leq 0 \}$

① check if objective function is convex

$f(x_1, x_2) = x_1^2 + 2x_2^2$

$\frac{df}{dx_1} = 2x_1 \quad , \quad \frac{df}{dx_2} = 4x_2$

} For any $x_1 < 0$ and $x_2 < 0$, the function's partial derivatives are negative

$\Rightarrow$ objective function is NOT a convex function and thus, the program is not a convex program

b) $\min \{ x_1 \cdot x_2 \mid x_1^2 + x_2^2 \leq 1 \}$

① check if objective function is convex

$f(x_1, x_2) = x_1 x_2$

$\frac{df}{dx_1} = x_2 \quad , \quad \frac{df}{dx_1} = x_1$

} Similar to (a), the partial derivatives for this objective function are negative for any $x_1 < 0$ and $x_2 < 0$

$\Rightarrow$ objective function is not a convex function so this is not a convex program

c) $\min\left\{ \sum_{i=1}^{n} \frac{x_i^{i}}{i!} \;\middle|\; \sum_{i=1}^{n} x_i \geq 5 \right\}$

① Check if objective function is convex

$$\sum_{i=1}^{n} \frac{x_i^{i}}{i!} = \frac{x_1}{1!} + \frac{x_2^2}{2!} + \frac{x_3^2}{3!} + \dots + \frac{x_n^n}{n!}$$

↳ proving each term is convex will be enough to determine if the function is convex since the sum of convex functions is also a convex function

$$\frac{d}{dx_1}(x_1) = 1 \quad , \quad \frac{d^2}{dx_1} x_1 = 0$$

$$\frac{d}{dx_2}\left(\frac{x_2^2}{2!}\right) = \frac{2x_2}{2} = x_2 \quad , \quad \frac{d^2}{dx_2}\left(\frac{x_2^2}{2!}\right) = 1$$

$$\frac{d}{dx_3}\left(\frac{x_3^3}{3!}\right) = \frac{3x_3^2}{3!} = \frac{3x_3^2}{2} \quad , \quad \frac{d^2}{dx_3}\left(\frac{x_3^3}{3!}\right) = 3x_3$$

$$\vdots$$

$$\frac{df}{dx_n}\left(\frac{x_n^n}{n!}\right) = \frac{n\,x_n^{n-1}}{n!} \quad , \quad \frac{d^2}{dx_n}\left(\frac{x_n^n}{n!}\right) = \frac{n(n-1)\,x_n^{n-2}}{n!} = \frac{x_n^{n-2}}{(n-2)!}$$

$\Rightarrow \frac{d^2}{dx_3} < 0$ for any $x_3 < 0$; Thus, the program is NOT a convex program

Question 5

a) $\min \{ \sum_{i=1}^{n} |y_i - \hat{y}| \mid \hat{y} = ax+b,\ a \in \mathbb{R},\ b \in \mathbb{R} \}$

$\Rightarrow$ Nonlinear Program because the objective function is not linear with respect to variables we're trying to optimize (a, b)

b) $\min \{ \max(|y_i - \hat{y}|) \mid \hat{y} = ax+b,\ a \in \mathbb{R},\ b \in \mathbb{R} \}$

$\Rightarrow$ NLP, for the same reasons as (a)

c) $\min \{ \max(|y_i - \hat{y}|) \mid \hat{y} = ax^2 + bx + c,\ a \in \mathbb{R},\ b \in \mathbb{R},\ c \in \mathbb{R} \}$

$\Rightarrow$ NLP. also for the same reasons as (a)
Furthermore, I believe the constraint is not quadratic because it is quadratic with respect to x, not the variables we are trying to optimize (a, b, c)