Homework 3

Monday, September 6, 2021 11:06 PM

1) $\min \; x+y$
s.t. $x+y=1$
$x \le 0,\ y \le 0$

$\Rightarrow$ the solution to this problem is **NOT feasible**
there is no value of x or y that can satisfy the set of constraints

2) $\min \; (x \cdot \sin(x))^2$
s.t. $x \in \mathbb{R}$

a) The quadratic nature of the function ensures that $f(x) \geq 0$.
$\Rightarrow$ need to solve for where $f(x) = 0$.

$$(x \sin x)^2 = 0$$

① $x = 0$
② $\sin x = 0 \rightarrow x = \pi n,\ n \in \mathbb{Z}$

$\Rightarrow$ global minimum occurs when $x=0$ and $x = \pi n,\ n \in \mathbb{Z}$. The value of the objective function at these points is 0.

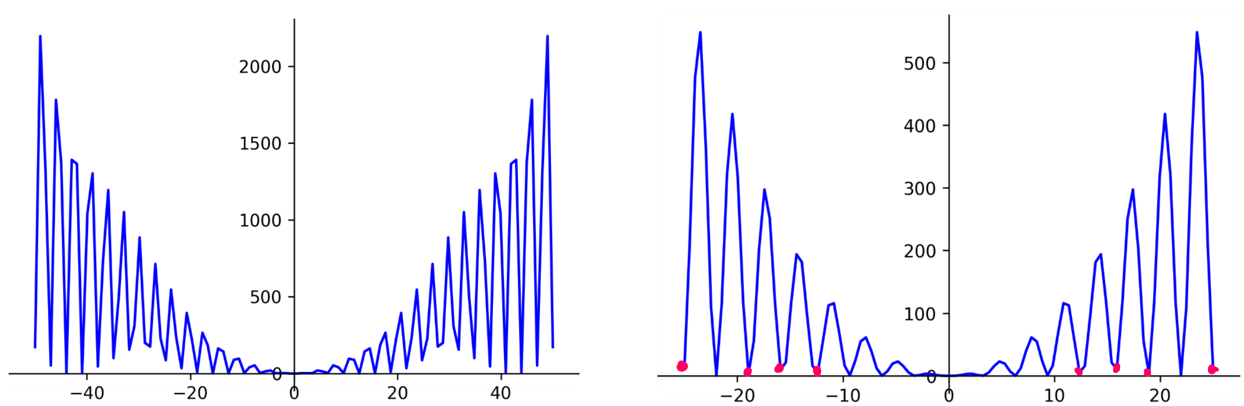
b) Yes, there are other local minimum solutions that are not part of the global minimum set.

$$\frac{df}{dx}(x \sin x)^2 = 2x\sin x\,(\sin x + x\cos x) = 0$$

$2x\sin x = 0$
$\Rightarrow \underbrace{x=0 \text{ or } x = \pi n,\ n \in \mathbb{Z}}_{\text{set of global min}}$

$\sin x + x\cos x = 0$
$\sin x = -x\cos x$

values of x that satisfy the equality $\sin x = -x\cos x$ are part of the set of solutions that are local min. but not global min. Sample points are indicated on the graph below.



c) To check if $f(x)$ is convex, I will check if $\frac{d^2f}{dx} \geq 0$

$$\frac{d^2f}{dx}(x\sin x)^2 = -2(x^2-1)\sin^2 x + 2x^2\cos^2 x + 8x\sin x\cos x$$

$$\left.\frac{d^2f}{dx}\right|_{x=2.25} = -9.7218$$

$\Rightarrow f(x)$ is NOT a convex function since $\frac{d^2f}{dx}$ is not positive for all values of x.

3) $\min \frac{1}{x}$

s.t $x \geq 0$

For this program to have an optimal solution, the following must be true:

① $f(x)$ is continuous → true (on the specified domain)

② X is closed and bounded → false.

↳ while $x \geq 0$ or $[0, \infty)$ is a closed interval, it is not bounded

⇒ This program does not have an optimal solution

4) $\min\ x + f(x)$

s.t $x \in \mathbb{R}$

$$f(x) = \begin{cases} 0, & -1 < x < 1 \\ 1, & x = 1 \\ 2, & x = -1 \\ +\infty, & x > 1 \text{ or } x < -1 \end{cases}$$

a) Let $g(x) = x + f(x) = \begin{cases} x, & -1 < x < 1 \\ x+1, & x = 1 \\ x+2, & x = -1 \\ x + \infty = \infty, & x > 1 \text{ or } x < -1 \end{cases}$

$g(x)$ is convex if:

$$g(\lambda a + (1-\lambda)b) \leq \lambda g(a) + (1-\lambda) g(b).$$

$g(\lambda a + (1-\lambda)b)$

$$= \begin{cases} \lambda a + (1-\lambda)b, & -1 < x < 1 \\ \lambda a + (1-\lambda)b + 1, & x = 1 \\ \lambda a + (1-\lambda)b + 2, & x = -1 \\ \lambda a + (1-\lambda)b + \infty, & x > 1 \text{ or } x < -1 \end{cases}$$

$$= \begin{cases} \lambda a + b - \lambda b, & -1 < x < 1 \\ \lambda a + b - \lambda b + 1, & x = 1 \\ \lambda a + b - \lambda b + 2, & x = -1 \\ \infty, & x > 1 \text{ or } x < -1 \end{cases}$$

$\lambda g(a) + (1-\lambda) g(b)$

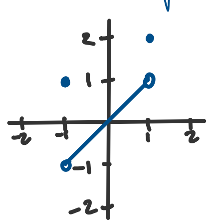
$$= \begin{cases} \lambda a + (1-\lambda)b, & -1 < x < 1 \\ \lambda(a+1) + (1-\lambda)(b+1), & x = 1 \\ \lambda(a+2) + (1-\lambda)(b+2), & x = -1 \\ \lambda(a+\infty) + (1-\lambda)(b+\infty) = \infty, & x > 1 \text{ or } x < -1 \end{cases}$$

$$= \begin{cases} \lambda a + b - \lambda b, & -1 < x < 1 \\ \lambda a + b - \lambda b + 1, & x = 1 \\ \lambda a + b - \lambda b + 2, & x = -1 \\ \infty, & x > 1 \text{ or } x < -1 \end{cases}$$

Comparing each piece of the resulting calculation, we can see that the inequality holds and LS = RS.

Thus, the objective function IS a convex function.

b) Below is a plot of the graph for $-1 \leq x \leq 1$. The min. value for $f(x)$ occurs as x approaches -1 from the right side.



There is no optimal solution because there are an infinite number of x values that will approach $f(x) = -1$ but will never be -1.

5) a) (Q) $v_Q = \min f(x)$ s.t. $g(x) \geq 0$

(P) $v_P = \min f(x)$ s.t $g(x) \geq 1$

$v_Q \leq v_P$ if (Q) is a relaxation of (P)

To determine if (P) is a relaxation of (Q):

1) $X \subseteq Y \Rightarrow$ TRUE
↳ $X = [1, \infty)$, $Y = [0, \infty)$

2) $f(x) \geq g(x) \Rightarrow$ TRUE
↳ in this case, the objective function in (P) and (Q) is the same so this condition holds

$\Rightarrow$ TRUE; since (Q) is a relaxation of (P), $v_Q \leq v_P$.

b) $\min f(x)^4$ s.t. $x \in X$

$v = x^*$, $f(x^*) = f(v) = 0$

Let $g(x) = f(x)^4$ and $w = f(x)$. The objective function can be rewritten as $g(w) = w^4$.

Since w^4 is strictly positive, we know that 0 is the global minimum of the objective function.

$\Rightarrow$ TRUE because the domain of w^4 is $[0, \infty)$

c) (P) $v_P = \max f(x)$ s.t. $g_i(x) \geq b$, $\forall i \in I$

(D) $v_D = \min \{ \mathcal{L}(\lambda) : \lambda \geq 0 \}$
$\mathcal{L}(\lambda) = \max \{ f(x) + \sum_{i \in I} \lambda_i (g_i(x) - b_i) \}$

$v_P \leq v_D$

(D) provides an upper bound on the original problem (P)

- $f(x) \leq f(x) + \underbrace{\sum_{i \in I} \lambda_i (g_i(\lambda) - b_i)}_{\text{this term is positive since } g_i(x) \geq b}$

- We minimize $L(\lambda)$ because we want to find the lowest possible upper bound for $f(x)$
  - ↳ it doesn't make sense to maximize the upper bound in a maximization problem, there would be no solution

⇒ TRUE because (D) provides an upper bound to the program and as a result, any V_P will be less than or equal to V_D.