1. Lenguajes

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Ejercicio 1. Sea $\Sigma = \{a, b\}$ un alfabeto. Hallar:

$$\Sigma^0, \quad \Sigma^1, \quad \Sigma^2, \quad \Sigma^*, \quad \Sigma^+, \quad |\Sigma|, \quad |\Sigma^0|$$

($|A|$ indica la cantidad de elementos de A).

$\Sigma^0 = \{\lambda\}, |\Sigma^0| = 1$
$\Sigma^1 = \Sigma = \{\alpha : \alpha \in \Sigma\} = \{a, b\}, |\Sigma| = 2$
$\Sigma^2 = \{ab : a, b \in \Sigma\} = \{aa, ab, ba, bb\}$
$\Sigma^* = \bigcup_{i \geq 0} \Sigma^i$
$\Sigma^+ = \bigcup_{i \geq 1} \Sigma^i$

Ejercicio 2. Decidir si, dado $\Sigma = \{a, b\}$, vale:

$$\lambda \in \Sigma, \quad \lambda \subseteq \Sigma, \quad \lambda \in \Sigma^+, \quad \lambda \in \Sigma^*, \quad \Sigma^0 = \lambda, \quad \Sigma^0 = \{\lambda\}$$

- $\lambda \in \Sigma$: No, porque λ es una palabra y Σ un alfabeto.
- $\lambda \subseteq \Sigma$: No, λ no es un conjunto.
- $\lambda \in \Sigma^+$: No, Σ^+ tiene solo de Σ^+ en adelante.
- $\lambda \in \Sigma^*$: ¡Sí!
- $\Sigma^0 = \lambda$: No, Σ^0 es un cjto. y λ una palabra.
- $\Sigma^0 = \{\lambda\}$: ¡Sí!

Ejercicio 3. Sea $\alpha = abb$ una cadena. Calcular:

$$\alpha^0, \quad \alpha^1, \quad \alpha^2, \quad \alpha^3, \quad \prod_{k=0,\dots,3} \alpha^k = \alpha^0.\alpha^1.\alpha^2.\alpha^3, \quad \alpha^r$$

$\alpha^0 = \lambda$, $\alpha^1 = abb$, $\alpha^2 = abbabb$, $\alpha^3 = abbabbabb$, $\prod_{k=0}^{3} \alpha^k = \lambda abbabbabbabbabbabb$, $\alpha^r = bba$

Ejercicio 4. Sean las cadenas $\alpha = abb$ y $\beta = acb$. Calcular:

$$\alpha\beta, \quad (\alpha\beta)^r, \quad \beta^r, \quad \beta^r\alpha^r, \quad \lambda\alpha, \quad \lambda\beta, \quad \alpha\lambda\beta, \quad \alpha^2\lambda^3\beta^2$$

- $\alpha\beta = abba$
- $(\alpha\beta)^r = bcabba$
- $\beta^r = bca$
- $\beta^r\alpha^r = bcabba$
- $\lambda\alpha = abb$
- $\lambda\beta = acb$
- $\alpha\lambda\beta = \alpha\beta$
- $\alpha^2\lambda^3\beta^2 = abbabbacbacb$

Ejercicio 5. Dado un alfabeto Σ, sean $\overbrace{x, y}^{\text{CARACTERES}} \in \Sigma$ y $\overbrace{\alpha, \beta}^{\text{PALABRAS}} \in \Sigma^*$. Demostrar que:

a. $|x.(y.\alpha)| = 2 + |\alpha|$
b. $|\alpha^r| = |\alpha|$
c. $|\alpha x \beta| = |x \alpha \beta|$
d. $|\alpha.\alpha| = 2|\alpha|$
e. $(\alpha.\beta)^r = \beta^r.\alpha^r$
f. $(\alpha^r)^r = \alpha$
g. $(\alpha^r)^n = (\alpha^n)^r$

($|\alpha|$ indica la longitud de la cadena α).

a) $|x.(y.\alpha)| = 2 + |\alpha|$ DEF $|\cdot|$
$1 + |y.\alpha| = 2 + |\alpha|$ DEF $|\cdot|$
$2 + |\alpha| = 2 + |\alpha|$

def. $|\cdot|$: $|\lambda| = 0$
$|x.\alpha| = 1 + |\alpha|$

b) $|\alpha^r| = |\alpha|$ Por ind. estructural en α

Caso base $\alpha = \lambda$:
$|\lambda^r| = |\lambda|$ DEF $\cdot^r$
$|\lambda| = 0$ DEF $|\cdot|$

Paso inductivo $\alpha = x\beta$
Vale $|\beta^r| = |\beta|$ para algún β. Qvq valga para $\alpha = x\beta$
$|(x\beta)^r| = |\beta^r.x|$ DEF $\cdot^r$
$= |\beta^r| + 1$ DEF $|\cdot|$
$= |\beta| + 1$ HI
$= |x\beta|$ DEF $|\cdot|$
$= |\alpha|$

c) $|\alpha x \beta| = |x \alpha \beta|$
$|\alpha x \beta| = |\alpha| + |x\beta|$ PROP
$= |\alpha| + |x| + |\beta|$ PROP
$= |x| + |\alpha| + |\beta|$ PROP SUMA
$= |x\alpha| + |\beta| = |x\alpha\beta|$ PROP

Demostramos por casos, haciendo inducción estructural sobre α:
1. Si $\alpha = \lambda$:
$|\lambda.\beta| = |\beta| = 0 + |\beta| = |\lambda| + |\beta|$
2. Si $\alpha = x.\alpha'$, suponemos que vale para α', y:
$|(x.\alpha').\beta| = |x.(\alpha'.\beta)|$ (def. .)
$= 1 + |\alpha'.\beta|$ (def. $|\bullet|$)
$= 1 + |\alpha'| + |\beta|$ (hip. ind.)
$= |x.\alpha'| + |\beta|$ (def. $|\bullet|$)
$= |\alpha| + |\beta|$ (def. α)

d) $|\alpha.\alpha| = 2|\alpha|$
$|\alpha.\alpha| = |\alpha| + |\alpha|$ PROP
$= 2|\alpha|$ FACTOR COMÚN

e) $(\alpha\beta)^r = \beta^r\alpha^r$
Por ind. estructural en α, con β fijo:

def. $\cdot^r$: $\lambda^r = \lambda$
$(x.\alpha)^r = \alpha^r.x$

CASO BASE: $\alpha = \lambda$
$(\lambda\beta)^r = \beta^r\lambda^r$ DEF $\cdot$, DEF $\cdot^r$
$\beta^r = \beta^r\lambda$ DEF $\cdot$
$\beta^r = \beta^r$

PASO INDUCTIVO: $\alpha = x\gamma$
Asumo $(\gamma\beta)^r = \beta^r\gamma^r$. Qvq vale $(\overbrace{x\gamma}^{\alpha}\beta)^r = \beta^r(\overbrace{x\gamma}^{\alpha})^r$

$(x\gamma\beta)^r = (\gamma\beta)^r x$ DEF. $\cdot^r$
$= \beta^r\gamma^r x$ HI
$= \beta^r(x\gamma)^r$ DEF $\cdot^r$
$= \beta^r\alpha^r$

f) $(\alpha^r)^r = \alpha$. Ind. est. en α

CB: $\alpha = \lambda$
$(\lambda^r)^r = \lambda^r = \lambda$

P.IND: $\alpha = x\beta$, $(\beta^r)^r = \beta$
Qvq $(\alpha^r)^r = \alpha \equiv ((x\beta)^r)^r = x\beta$

$(\alpha^r)^r = ((x\beta)^r)^r = (\beta^r x)^r$ DEF $\cdot^r$
$= x^r(\beta^r)^r$ DEF $\cdot^r$
$= x\ \beta$ HI y • x es de la forma $x.\lambda$. Luego, $(x.\lambda)^r = \lambda^r x = \lambda x = x$. Por lo tanto, $x = x^r$.
$= \alpha$

g) $(\alpha^r)^n = (\alpha^n)^r$ Ind. en n (en $\mathbb{N}_0$)

VOLVER A INTENTAR

CASO BASE: $n = 0$
$(\alpha^r)^0 = (\alpha^0)^r$
$\lambda = \lambda^r$ DEFs $\cdot^0$
$\lambda = \lambda$ DEF λ^r

$\alpha^{n+1} = \alpha^n.\alpha$
$= \alpha.\alpha^n$

PASO INDUCTIVO: Vale $(\alpha^r)^k = (\alpha^k)^r$ $k \in \mathbb{N}$
Qvq vale $(\alpha^r)^{k+1} = (\alpha^{k+1})^r$

$(\alpha^r)^{k+1} = \alpha^r(\alpha^r)^k$ DEF $\cdot^n$
$= \alpha^r(\alpha^k)^r$ HI
$= (\cancel{(\alpha^k)^r\alpha})^r$ DEF $\cdot^r$
$= ((\alpha^r)^k\alpha)^r$ DEF $\cdot^r$
$= \alpha^r(\alpha^r)^k$
$= (\alpha^r)^{k+1}$

Vale $(\alpha^r)^n = (\alpha^n)^r$ $n \in \mathbb{N}$
qvq valga $(\alpha^r)^{n+1} = (\alpha^{n+1})^r$

$(\alpha^r)^{n+1} \underset{\text{def.}^n}{=} \alpha^r.(\alpha^r)^n \underset{\text{HI}}{=} \alpha^r.(\alpha^n)^r \underset{\text{ej e)}}{=} (\alpha^n.\alpha)^r$
$= (\alpha.\alpha^n)^r \underset{\text{def.}^n}{=} (\alpha^{n+1})^r$

Acá hace falta usar que $\alpha^n.\alpha = \alpha.\alpha^n$ demostrado por Cami.

Ejercicio 6. Dar ejemplos de cadenas que pertenezcan a los siguientes lenguajes:

a. $\mathcal{L} = \{a^n b^n \mid n \geq 0\}$ ¿Qué son a y b? ab
b. $\mathcal{L} = \{a^n b^n \mid n \geq 1\}$ ab
c. $\mathcal{L} = \{a^n b^m \mid n \geq 1 \wedge m \geq 1\}$ ab
d. $\mathcal{L} = \{a^n b^m \mid n \geq 1 \wedge m \geq 0\}$ ab
e. $\mathcal{L} = \{a^n (ac)^p (bab)^q \mid n \geq 0 \wedge q = p + 2 \wedge p \geq 1\}$ acbabbabbab
f. $\mathcal{L} = \{a, b\}^3 \cap \Lambda$ $\varnothing$
g. $\mathcal{L} = \left\{\alpha\alpha^r \,\middle|\, \alpha \in \{a, b\}^+\right\}$ abba
h. $\mathcal{L} = \left\{\alpha \in \{a, b\}^+ \,\middle|\, \alpha = \alpha^r\right\}$ a

Ejercicio 7. Definir por comprensión los siguientes lenguajes:

a. $\mathcal{L}_1 = \{ab, aabb, aaabbb, \dots\} = \{a^n b^n : n \geq 1\}$
b. $\mathcal{L}_2 = \{aab, aaaabb, aaaaaabbb, \dots\} = \{a^n b^m : 2m = n \wedge n \equiv 0 \ (\text{mód } 2)\}$ Notar que, elevar a una potencia negativa no tiene sentido, y $n \equiv 0\ (2)$ elimina el caso $n = 1$.
c. $\mathcal{L}_3 = \{aaabccc, aaaabcccc, aaaaabccccc, \dots\} = \{a^n b^1 c^n : n \geq 3\}$

(donde el «crecimiento» en la cantidad de cada símbolo es lineal en cada caso).

Ejercicio 8. Dados $\mathcal{L}_1 = \{a, bc\}$, $\mathcal{L}_2 = \{aaa, bc\}$, y siendo $\Lambda = \{\lambda\}$, calcular:

a. $\mathcal{L}_1 \cup \mathcal{L}_2$
b. $\mathcal{L}_1 \cap \mathcal{L}_2$
c. $\mathcal{L}_1.\mathcal{L}_2$
d. $\mathcal{L}_1.(\mathcal{L}_2)^0$
e. $\mathcal{L}_1.(\mathcal{L}_2)^2$
f. $\mathcal{L}_1.(\mathcal{L}_2)^+$
g. $(\mathcal{L}_1.\mathcal{L}_2)^+$
h. $(\mathcal{L}_1.\mathcal{L}_2)^*$
i. $\mathcal{L}_1.\Lambda.\mathcal{L}_2$
j. $\mathcal{L}_1 \varnothing \mathcal{L}_2$
k. $(\mathcal{L}_1)^r$
l. $(\mathcal{L}_1.\mathcal{L}_2)^r$

a) $\mathcal{L}_1 \cup \mathcal{L}_2 = \{a, aaa, bc\}$
b) $\mathcal{L}_1 \cap \mathcal{L}_2 = \{bc\}$
c) $\mathcal{L}_1.\mathcal{L}_2 = \{aaaa, abc, bcaaa, bcbc\}$
d) $\mathcal{L}_1.\mathcal{L}_2^0 = \mathcal{L}_1.\Lambda = \mathcal{L}_1$
e) $\mathcal{L}_1.\mathcal{L}_2^2 = \mathcal{L}_1.(\mathcal{L}_2.\mathcal{L}_2) = \mathcal{L}_1.\{aaaaaa, aaabc, bcaaa, bcbc\}$
f) $\mathcal{L}_1.(\mathcal{L}_2)^+ =$
g) $(\mathcal{L}_1.\mathcal{L}_2)^+ = \{aaaa, bcaaa, abc, bcbc\}^+$
h) $(\mathcal{L}_1.\mathcal{L}_2)^* = \Lambda \cup \{aaaa, abc, bcaaa, bcbc\}^+$
i) $\mathcal{L}_1.\Lambda.\mathcal{L}_2 = \mathcal{L}_1.\mathcal{L}_2$
j) $\mathcal{L}_1.\varnothing.\mathcal{L}_2 = \varnothing$
k) $(\mathcal{L}_1)^r = \{a, cb\}$
l) $(\mathcal{L}_1.\mathcal{L}_2)^r = \{aaaa, cba, aaacb, cbcb\}$

Ejercicio 9. Determinar el complemento de los siguientes lenguajes, considerando los alfabetos indicados en cada caso.

a. $\mathcal{L} = \Lambda$ para $\Sigma = \{a, b\}$ Σ^+
b. $\mathcal{L} = \{\lambda, a\}$ para $\Sigma = \{a\}$ y $\Sigma = \{a, b\}$ $\bigcup_{i \geq 2} \Sigma^i$
c. $\mathcal{L} = \{b\alpha \mid \alpha \in \{a, b\}^*\}$ para $\Sigma = \{a, b\}$ $\Lambda \cup \{a\alpha : \alpha \in \{a, b\}^*\}$
d. $\mathcal{L} = \{a^{2n} \mid n \geq 0\}$ para $\Sigma = \{a\}$ y $\Sigma = \{a, b\}$ $\underbrace{\{a^{2n+1} : n \geq 0\}}_{\text{Para } \Sigma = \{a\}} \cup \{\alpha : |\alpha|_b \geq 1\}$
e. $\mathcal{L} = \{\alpha_1 b \alpha_2 \mid \alpha_1, \alpha_2 \in \{a, b\}^* \wedge |\alpha_1| > |\alpha_2|\}$ para $\Sigma = \{a, b\}$
$\Lambda \cup \{\alpha_1 b \alpha_2 : \alpha_1, \alpha_2 \in \{a, b\}^* \wedge |\alpha_1| > |\alpha_2|\} \cup \{a^n : n \geq 1\}$

Ejercicio 10. Sea $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$ lenguajes cualesquiera. Determinar si las siguientes afirmaciones son verdaderas o falsas. Si son verdaderas, demostrarlas. Si no, dar un contraejemplo.

a. $\mathcal{L}^+ \subseteq \mathcal{L}^+$. Trivialmente cierto.
b. $\mathcal{L}^+ \subsetneq \mathcal{L}^*$
c. $\mathcal{L}^n \mathcal{L}^m = \mathcal{L}^{n+m}$ para todo $n, m \geq 0$
d. $\mathcal{L}^n \subseteq \mathcal{L}^{n+1}$ para todo $n \geq 0$
e. $\mathcal{L}_1 \subseteq \mathcal{L}_2, n \geq 0 \Longrightarrow (\mathcal{L}_1)^n \subseteq (\mathcal{L}_2)^n$
f. $\mathcal{L}_1 \subseteq \mathcal{L}_2 \Longrightarrow (\mathcal{L}_1)^* \subseteq (\mathcal{L}_2)^*$
g. $(\mathcal{L}^*)^* = \mathcal{L}^*$
h. $(\mathcal{L}^+)^+ = \mathcal{L}^*$
i. $(\mathcal{L}^+)^* = \mathcal{L}^*$
j. $(\mathcal{L}_1 \cup \mathcal{L}_2)^* = (\mathcal{L}_1)^* \cup (\mathcal{L}_2)^*$
k. $(\mathcal{L}_1 \cap \mathcal{L}_2)^* = (\mathcal{L}_1)^* \cap (\mathcal{L}_2)^*$
l. $(\mathcal{L}^2)^* = \mathcal{L}^*$
m. $(\mathcal{L} \cup \mathcal{L}_2)^* = \mathcal{L}^*$
n. $(\mathcal{L}^n)^r = (\mathcal{L}^r)^n$ para todo $n \geq 0$
ñ. $(\mathcal{L}^*)^r = (\mathcal{L}^r)^*$

b) $\mathcal{L}^+ \subsetneq \mathcal{L}^*$ Verdadero

c) $\mathcal{L}^n \mathcal{L}^m = \mathcal{L}^{n+m}$ $\forall n, m \geq 0$