8.2.2 Public Key Encryption

For more than 2,000 years (since the time of the Caesar cipher and up to the 1970s), encrypted communication required that the two communicating parties share a common secret—the symmetric key used for encryption and decryption. One difficulty with this approach is that the two parties must somehow agree on the shared key; but to do so requires (presumably *secure*) communication! Perhaps the parties could first meet and agree on the key in person (for example, two of Caesar's centurions might meet at the Roman baths) and thereafter communicate with encryption. In a networked world, however, communicating parties may never meet and may never converse except over the network. Is it possible for two parties to communicate with encryption without having a shared secret key that is known in advance? In 1976, Diffie and Hellman [Diffie 1976] demonstrated an algorithm (known now as Diffie-Hellman Key Exchange) to do just that—a radically different and marvelously elegant approach toward secure communication that has led to the development of today's public key cryptography systems. We'll see shortly that public key cryptography systems also have several wonderful properties that make them useful not only for encryption, but for authentication and digital signatures as well. Interestingly, it has recently come to light that ideas similar to those in [Diffie 1976] and [RSA 1978] had been independently developed in the early 1970s in a series of secret reports by researchers at the Communications-Electronics Security Group in the United Kingdom [Ellis 1987]. As is often the case, great ideas can spring up independently in many places; fortunately, public key advances took place not only in private, but also in the public view, as well.

The use of public key cryptography is conceptually quite simple. Suppose Alice wants to communicate with Bob. As shown in Figure 8.6, rather than Bob and Alice

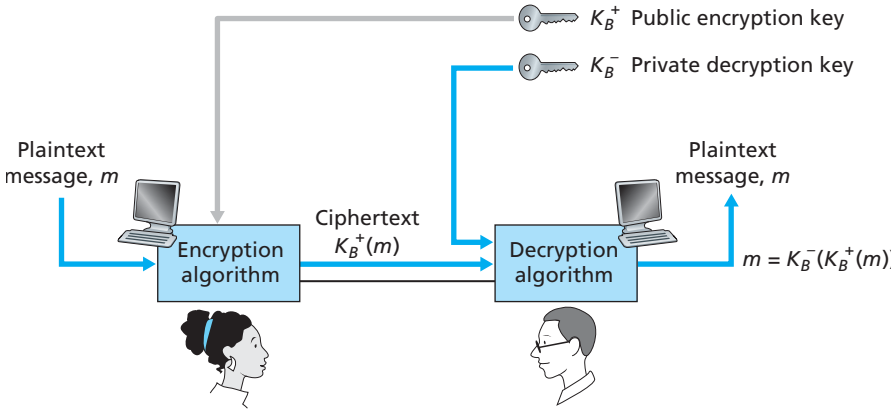

Figure 8.6 ♦ Public key cryptography

sharing a single secret key (as in the case of symmetric key systems), Bob (the recipient of Alice's messages) instead has two keys—a **public key** that is available to *everyone* in the world (including Trudy the intruder) and a **private key** that is known only to Bob. We will use the notation K_B^+ and K_B^- to refer to Bob's public and private keys, respectively. In order to communicate with Bob, Alice first fetches Bob's public key. Alice then encrypts her message, *m,* to Bob using Bob's public key and a known (for example, standardized) encryption algorithm; that is, Alice computes $K_B^+(m)$. Bob receives Alice's encrypted message and uses his private key and a known (for example, standardized) decryption algorithm to decrypt Alice's encrypted message. That is, Bob computes $K_B^-(K_B^+(m))$. We will see below that there are encryption/decryption algorithms and techniques for choosing public and private keys such that $K_B^-(K_B^+(m)) = m$; that is, applying Bob's public key, K_B^+, to a message, m (to get $K_B^+(m)$), and then applying Bob's private key, K_B^-, to the encrypted version of m (that is, computing $K_B^-(K_B^+(m))$) gives back m. This is a remarkable result! In this manner, Alice can use Bob's publicly available key to send a secret message to Bob without either of them having to distribute any secret keys! We will see shortly that we can interchange the public key and private key encryption and get the same remarkable result—that is, $K_B^-\,(_B{}^+(m)) = K_B^+\,(K_B^-(m)) = m$.

The use of public key cryptography is thus conceptually simple. But two immediate worries may spring to mind. A first concern is that although an intruder intercepting Alice's encrypted message will see only gibberish, the intruder knows both the key (Bob's public key, which is available for all the world to see) and the algorithm that Alice used for encryption. Trudy can thus mount a chosen-plaintext attack, using the known standardized encryption algorithm and Bob's publicly available encryption key to encode any message she chooses! Trudy might well try, for example, to encode messages, or parts of messages, that she suspects that Alice might send. Clearly, if public key cryptography is to work, key selection and encryption/decryption must be done in such a way that it is impossible (or at least so hard as to be nearly impossible) for an intruder to either determine Bob's private key or somehow otherwise decrypt or guess Alice's message to Bob. A second concern is that since Bob's encryption key is public, anyone can send an encrypted message to Bob, including Alice or someone *claiming* to be Alice. In the case of a single shared secret key, the fact that the sender knows the secret key implicitly identifies the sender to the receiver. In the case of public key cryptography, however, this is no longer the case since anyone can send an encrypted message to Bob using Bob's publicly available key. A digital signature, a topic we will study in Section 8.3, is needed to bind a sender to a message.

RSA

While there may be many algorithms that address these concerns, the **RSA algorithm** (named after its founders, Ron Rivest, Adi Shamir, and Leonard Adleman) has become almost synonymous with public key cryptography. Let's first see how RSA works and then examine why it works.

RSA makes extensive use of arithmetic operations using modulo-n arithmetic. So let's briefly review modular arithmetic. Recall that x mod n simply means the remainder of x when divided by n; so, for example, 19 mod 5 = 4. In modular arithmetic, one performs the usual operations of addition, multiplication, and exponentiation. However, the result of each operation is replaced by the integer remainder that is left when the result is divided by n. Adding and multiplying with modular arithmetic is facilitated with the following handy facts:

$$[(a \bmod n) + (b \bmod n)] \bmod n = (a + b) \bmod n$$
$$[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$$
$$[(a \bmod n) \cdot (b \bmod n)] \bmod n = (a \cdot b) \bmod n$$

It follows from the third fact that $(a \bmod n)^d \bmod n = a^d \bmod n$, which is an identity that we will soon find very useful.

Now suppose that Alice wants to send to Bob an RSA-encrypted message, as shown in Figure 8.6. In our discussion of RSA, let's always keep in mind that a message is nothing but a bit pattern, and every bit pattern can be uniquely represented by an integer number (along with the length of the bit pattern). For example, suppose a message is the bit pattern 1001; this message can be represented by the decimal integer 9. Thus, when encrypting a message with RSA, it is equivalent to encrypting the unique integer number that represents the message.

There are two interrelated components of RSA:

- The choice of the public key and the private key
- The encryption and decryption algorithm

To generate the public and private RSA keys, Bob performs the following steps:

1. Choose two large prime numbers, p and q. How large should p and q be? The larger the values, the more difficult it is to break RSA, but the longer it takes to perform the encoding and decoding. RSA Laboratories recommends that the product of p and q be on the order of 1,024 bits. For a discussion of how to find large prime numbers, see [Caldwell 2012].
2. Compute $n = pq$ and $z = (p - 1)(q - 1)$.
3. Choose a number, e, less than n, that has no common factors (other than 1) with z. (In this case, e and z are said to be relatively prime.) The letter e is used since this value will be used in encryption.
4. Find a number, d, such that $ed - 1$ is exactly divisible (that is, with no remainder) by z. The letter d is used because this value will be used in decryption. Put another way, given e, we choose d such that

$$ed \bmod z = 1$$

5. The public key that Bob makes available to the world, K_B^+, is the pair of numbers (n, e); his private key, K_B^-, is the pair of numbers (n, d).

The encryption by Alice and the decryption by Bob are done as follows:

- Suppose Alice wants to send Bob a bit pattern represented by the integer number m (with $m < n$). To encode, Alice performs the exponentiation m^e, and then computes the integer remainder when m^e is divided by n. In other words, the encrypted value, c, of Alice's plaintext message, m, is

$$c = m^e \bmod n$$

  The bit pattern corresponding to this ciphertext c is sent to Bob.

- To decrypt the received ciphertext message, c, Bob computes

$$m = c^d \bmod n$$

  which requires the use of his private key (n,d).

As a simple example of RSA, suppose Bob chooses $p = 5$ and $q = 7$. (Admittedly, these values are far too small to be secure.) Then $n = 35$ and $z = 24$. Bob chooses $e = 5$, since 5 and 24 have no common factors. Finally, Bob chooses $d = 29$, since $5 \cdot 29 - 1$ (that is, $ed - 1$) is exactly divisible by 24. Bob makes the two values, $n = 35$ and $e = 5$, public and keeps the value $d = 29$ secret. Observing these two public values, suppose Alice now wants to send the letters l, o, v, and e to Bob. Interpreting each letter as a number between 1 and 26 (with a being 1, and z being 26), Alice and Bob perform the encryption and decryption shown in Tables 8.2 and 8.3, respectively. Note that in this example, we consider each of the four letters as a distinct message. A more realistic example would be to convert the four letters into their 8-bit ASCII representations and then encrypt the integer corresponding to the resulting 32-bit bit pattern. (Such a realistic example generates numbers that are much too long to print in a textbook!)

Given that the "toy" example in Tables 8.2 and 8.3 has already produced some extremely large numbers, and given that we saw earlier that p and q should each be several hundred bits long, several practical issues regarding RSA come to mind.

Plaintext Letter	m: numeric representation	m^e	Ciphertext $c = m^e \bmod n$
l	12	248832	17
o	15	759375	15
v	22	5153632	22
e	5	3125	10

Table 8.2 ♦ Alice's RSA encryption, $e = 5$, $n = 35$

Ciphertext c	c^d	$m = c^d \bmod n$	Plaintext Letter
17	481968572106750915091509141182522307169 7	12	l
15	127834039403948858939111232757568359375	15	o
22	851643319086537701956194499721106030592	22	v
10	100000000000000000000000000000	5	e

Table 8.3 ♦ Bob's RSA decryption, $d = 29$, $n = 35$

How does one choose large prime numbers? How does one then choose e and d? How does one perform exponentiation with large numbers? A discussion of these important issues is beyond the scope of this book; see [Kaufman 1995] and the references therein for details.

Session Keys

We note here that the exponentiation required by RSA is a rather time-consuming process. By contrast, DES is at least 100 times faster in software and between 1,000 and 10,000 times faster in hardware [RSA Fast 2012]. As a result, RSA is often used in practice in combination with symmetric key cryptography. For example, if Alice wants to send Bob a large amount of encrypted data, she could do the following. First Alice chooses a key that will be used to encode the data itself; this key is referred to as a **session key**, and is denoted by K_S. Alice must inform Bob of the session key, since this is the shared symmetric key they will use with a symmetric key cipher (e.g., with DES or AES). Alice encrypts the session key using Bob's public key, that is, computes $c = (K_S)^e \bmod n$. Bob receives the RSA-encrypted session key, c, and decrypts it to obtain the session key, K_S. Bob now knows the session key that Alice will use for her encrypted data transfer.

Why Does RSA Work?

RSA encryption/decryption appears rather magical. Why should it be that by applying the encryption algorithm and then the decryption algorithm, one recovers the original message? In order to understand why RSA works, again denote $n = pq$, where p and q are the large prime numbers used in the RSA algorithm.

Recall that, under RSA encryption, a message (uniquely represented by an integer), m, is exponentiated to the power e using modulo-n arithmetic, that is,

$$c = m^e \bmod n$$

Decryption is performed by raising this value to the power d, again using modulo-n arithmetic. The result of an encryption step followed by a decryption step is thus

$(m^e \bmod n)^d \bmod n$. Let's now see what we can say about this quantity. As mentioned earlier, one important property of modulo arithmetic is $(a \bmod n)^d \bmod n = a^d \bmod n$ for any values a, n, and d. Thus, using $a = m^e$ in this property, we have

$$(m^e \bmod n)^d \bmod n = m^{ed} \bmod n$$

It therefore remains to show that $m^{ed} \bmod n = m$. Although we're trying to remove some of the magic about why RSA works, to establish this, we'll need to use a rather magical result from number theory here. Specifically, we'll need the result that says if p and q are prime, $n = pq$, and $z = (p - 1)(q - 1)$, then $x^y \bmod n$ is the same as $x^{(y \bmod z)} \bmod n$ [Kaufman 1995]. Applying this result with $x = m$ and $y = ed$ we have

$$m^{ed} \bmod n = m^{(ed \bmod z)} \bmod n$$

But remember that we have chosen e and d such that $ed \bmod z = 1$. This gives us

$$m^{ed} \bmod n = m^1 \bmod n = m$$

which is exactly the result we are looking for! By first exponentiating to the power of e (that is, encrypting) and then exponentiating to the power of d (that is, decrypting), we obtain the original value, m. Even *more* wonderful is the fact that if we first exponentiate to the power of d and then exponentiate to the power of e—that is, we reverse the order of encryption and decryption, performing the decryption operation first and then applying the encryption operation—we also obtain the original value, m. This wonderful result follows immediately from the modular arithmetic:

$$(m^d \bmod n)^e \bmod n = m^{de} \bmod n = m^{ed} \bmod n = (m^e \bmod n)^d \bmod n$$

The security of RSA relies on the fact that there are no known algorithms for quickly factoring a number, in this case the public value n, into the primes p and q. If one knew p and q, then given the public value e, one could easily compute the secret key, d. On the other hand, it is not known whether or not there *exist* fast algorithms for factoring a number, and in this sense, the security of RSA is not guaranteed.

Another popular public-key encryption algorithm is the Diffie-Hellman algorithm, which we will briefly explore in the homework problems. Diffie-Hellman is not as versatile as RSA in that it cannot be used to encrypt messages of arbitrary length; it can be used, however, to establish a symmetric session key, which is in turn used to encrypt messages.

8.3 Message Integrity and Digital Signatures

In the previous section we saw how encryption can be used to provide confidentiality to two communicating entities. In this section we turn to the equally important