8.1 $a = \left(\frac{1}{4}, \frac{3}{4}\right) \Rightarrow G(s) = \frac{1}{4} + \frac{3}{4}s$

(a) the pgf of the n-th gen. size Z_n

$G_1(s) = G(s) = \frac{1}{4} + \frac{3}{4}s$

$G_2(s) = G(G(s)) = \frac{1}{4} + \frac{3}{4}\left(\frac{1}{4} + \frac{3}{4}s\right) = \underline{\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \left(\frac{3}{4}\right)^2 s}$

Z_2	0	1
	$\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4}$	$\left(\frac{3}{4}\right)^2$

$G_3(s) = G_2(G(s)) = G(G(G(s))) =$

$= \frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \left(\frac{3}{4}\right)^2\left(\frac{1}{4} + \frac{3}{4}s\right)$

$= \frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \left(\frac{3}{4}\right)^2\cdot\frac{1}{4} + \left(\frac{3}{4}\right)^3 s$

$G_n(s) = \underbrace{\square}_{1-\left(\frac{3}{4}\right)^n} + \underbrace{\square}_{\left(\frac{3}{4}\right)^n}\cdot s$

$G_n(1) = 1$ because for any pgf H $H(1) = 1$.

$$\boxed{G_n(s) = 1 - \left(\frac{3}{4}\right)^n + \left(\frac{3}{4}\right)^n s}$$

(b) the distribution of Z_n

Z_n	0	1
	$1-\left(\frac{3}{4}\right)^n$	$\left(\frac{3}{4}\right)^n$

(c) the distr. of the total number of individuals up to gen. n.

φ_n – the pgf of the progeny up to time n

$$\boxed{\varphi_n(s) = s\,G(\varphi_{n-1}(s))}$$

$$\varphi_0(s) = s$$

$$\varphi_1(s) = s\,G(\varphi_0(s)) = s\,G(s) = s\left(\tfrac{1}{4} + \tfrac{3}{4}s\right) = \tfrac{1}{4}s + \tfrac{3}{4}s^2$$

$$\varphi_2(s) = s\,G(\varphi_1(s)) = s\,G\left(\tfrac{1}{4}s + \tfrac{3}{4}s^2\right) = s\left[\tfrac{1}{4} + \tfrac{3}{4}\left(\tfrac{1}{4}s + \tfrac{3}{4}s^2\right)\right]$$

$$= \tfrac{1}{4}s + \tfrac{3}{4}\cdot\tfrac{1}{4}s^2 + \left(\tfrac{3}{4}\right)^2 s^3$$

$$\varphi_3(s) = s\,G(\varphi_2(s)) = \ldots = \tfrac{1}{4}s + \tfrac{3}{4}\cdot\tfrac{1}{4}s^2 + \left(\tfrac{3}{4}\right)^2\cdot\tfrac{1}{4}s^3 + \left(\tfrac{3}{4}\right)^3 s^4$$

$$\boxed{\varphi_n(s) = \tfrac{1}{4}s + \tfrac{3}{4}\cdot\tfrac{1}{4}s^2 + \left(\tfrac{3}{4}\right)^2\cdot\tfrac{1}{4}s^3 + \ldots + \left(\tfrac{3}{4}\right)^{n-1}\cdot\tfrac{1}{4}s^n + \left(\tfrac{3}{4}\right)^n s^{n+1}}$$

T_n	1	2	3	…	n	$n+1$
					$\left(\tfrac{3}{4}\right)^{n-1}\cdot\tfrac{1}{4}$	$\left(\tfrac{3}{4}\right)^n$

(d) the distr. of the total progeny

φ – the pgf of the total progeny

$$\varphi(s) = s\,\underbrace{G(\varphi(s))}$$

$$G(s) = \frac{1}{4} + \frac{3}{4}s$$

$$\varphi(s) = s\cdot\left(\frac{1}{4} + \frac{3}{4}\varphi(s)\right)$$

$$\varphi(s) = \frac{1}{4}s + \frac{3}{4}\cdot s\cdot\varphi(s)$$

$$\varphi(s)\left(1 - \frac{3}{4}s\right) = \frac{1}{4}s$$

$$\underbrace{\varphi(s)} = \frac{\frac{1}{4}s}{1-\frac{3}{4}s} = \underbrace{\frac{s}{4-3s}}$$

$$\varphi(s) = \frac{s}{4-3s}$$

$$\cancel{\varphi(s) = \sum_{j=0}^{5} \alpha_j \cdot s^j}$$

$$\sum_{j=0}^{\infty} \alpha_j \cdot s^j$$

$$G(s) = \frac{1}{4} + \frac{3}{5}s$$

0	1
$\frac{1}{4}$	$\frac{3}{4}$

G_X – the pgf of a random variable X

$$\boxed{\mathbb{P}(X=j) = \frac{G_X^{(j)}(0)}{j!}}$$

$\varphi(s) = \boxed{\frac{s}{4-3s}}$ — the pgf of the ~~tot~~ total progeny T

$$\mathbb{P}(T=0) = \frac{\varphi^{(0)}(0)}{0!} = \frac{0}{1} = 0 \qquad \mathbb{P}(T \geq 1) = 1$$

~~with~~

$$\mathbb{P}(T=1) = \frac{\varphi'(0)}{1!} = \varphi'(0) = \frac{4}{16} = \frac{1}{4} \qquad a = \left(\frac{1}{4}, \frac{3}{4}\right)$$

$$\varphi'(s) = \frac{4-3s - s\cdot(-3)}{(4-3s)^2} = \frac{4}{(4-3s)^2} \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$\varphi(s) = \frac{s}{4-3s} \Rightarrow T$ has geometric distr. with ~~the~~ parameter $\frac{1}{4}$

(e) the expectation of the total progeny

$$\mathbb{E}T = \begin{cases} \frac{1}{1-\mu}, & \mu < 1 \\ +\infty, & \mu \geq 1 \end{cases} \qquad a = \left(\frac{1}{4}, \frac{3}{4}\right)$$

$$\mu = \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot 1 = \frac{3}{4}$$

$$\mathbb{E}T = \frac{1}{1-\frac{3}{4}} = \frac{1}{\frac{1}{4}} = 4.$$

(f) the extinction prob.

$\mu < 1$ $\Longrightarrow$ the extinction prob. $= 1$

subcritical case

(g) the distrib. of the time of extinction

$$\mathbb{P}(T=n) = G_n(0) - G_{n-1}(0)$$

$$G_n(s) = 1 - \left(\tfrac{3}{4}\right)^n + \left(\tfrac{3}{4}\right)^n s$$

$$\mathbb{P}(T=n) = 1 - \left(\tfrac{3}{4}\right)^n - \left(1 - \left(\tfrac{3}{4}\right)^{n-1}\right) = \left(\tfrac{3}{4}\right)^{n-1} - \left(\tfrac{3}{4}\right)^n =$$

$$= \left(\tfrac{3}{4}\right)^{n-1} \cdot \left(1 - \tfrac{3}{4}\right) =$$

$$= \left(\tfrac{3}{4}\right)^{n-1} \cdot \tfrac{1}{4}$$

$\mathbb{P}(T=1) = \frac{1}{4}$

$\mathbb{P}(T=2) = \frac{3}{4} \cdot \frac{1}{4}$

$\mathbb{P}(T=3) = \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$

$\vdots$

T has the geometr. distr. with parameter $\frac{1}{4}$

$X \sim \mathrm{geom}(p)$

$P(X=k) = (1-p)^{k-1} \cdot p, \quad k = 1, 2, \ldots$

$\mathbb{E}X = \frac{1}{p}$

h) the mean time of extinction

T – the time of ext.

$T \sim \text{geom}\left(\frac{1}{4}\right) \implies \mathbb{E}T = \frac{1}{\frac{1}{4}} = 4.$

8.2 $\quad a_k = \left(\frac{1}{2}\right)^{k+1}, \quad k \geq 0$ $\qquad$ $\ulcorner$geom

$$G(s) = \sum_{k=0}^{\infty} a_k \cdot s^k = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} \cdot s^k = \frac{1}{2} + \frac{1}{4}s + \frac{1}{8}s^2 + \ldots = \frac{\frac{1}{2}}{1-\frac{1}{2}s} = \frac{1}{2-s}.$$

geom. series

$a_1 = \frac{1}{2}$

$q = \frac{1}{2}s \qquad |q| < 1$

$\qquad\qquad |\frac{s}{2}| < 1$

$\frac{a_1}{1-q} \qquad |s| < 2$

(a) the extinction prob.

$s = G(s)$

$s = \frac{1}{2-s}$

$\ldots$

prob. of ext. $= 1$

$$\boxed{G_n(s) = \frac{n-(n-1)s}{n+1-ns}}$$

(c) T – the ~~of~~ time of extinction

$$\mathbb{P}(T=n) = G_n(0) - G_{n-1}(0) =$$

$$= \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2-(n+1)(n-1)}{n(n+1)} =$$

$$= \frac{n^2-(n^2-1)}{n(n+1)} = \frac{1}{n(n+1)}.$$

$$\boxed{\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}}$$

$$\mathbb{P}(T=1) + \mathbb{P}(T=2) + \dots = 1 \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots = 1$$

(d) the mean time of extinction

$$\mathbb{P}(T=n) = \frac{1}{n(n+1)}, \quad n = 1, 2, \dots$$

$$\mathbb{E}T = 1 \cdot \mathbb{P}(T=1) + 2 \cdot \mathbb{P}(T=2) + \dots$$

$$= \sum_{n=1}^{\infty} n \cdot \mathbb{P}(T=n) = \sum_{n=1}^{\infty} \cancel{n} \cdot \frac{1}{\cancel{n}(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = +\infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$