Discrete Random Processes (EDRP)

Lecture 9

Branching processes - cont'd

Branching processes - a quick review

- i-th individual in a population produces a random number X_i of children according to the offspring distribution

$$\mathbf{a} = (a_0, a_1, \ldots).$$

- assumption: $X_1, X_2, \ldots$ - an i.i.d. sequence (with common distribution $\mathbf{a}$)
- Z_n - the size of the n-th generation (assumption: $Z_0 = 1$)
- a branching process - the sequence $(Z_n)_{n \in \mathbb{N}_0}$
- $Z_n = \sum_{i=1}^{Z_{n-1}} X_i$

Review - cont'd

- μ - the mean of the offspring distribution (=the average number of children produced by any individual),

$$\mu = \sum_k k a_k$$

- the mean size of the nth generation is $\mathbb{E}Z_n = \mu^n$, $n \in \mathbb{N}_0$

- $(Z_n)_n$ is
  - *subcritical* if $\mu < 1$,
  - *critical* if $\mu = 1$,
  - and *supercritical* if $\mu > 1$,

- if subcritical, then $\lim_n \mathbb{E}Z_n = 0$; if critical, then $\lim_n \mathbb{E}Z_n = 1$; if supercritical, then $\lim_n \mathbb{E}Z_n = +\infty$

A review - cont'd

- G - the probability generating function of the offspring distribution:
$$G(s) = \sum_k a_k s^k$$
- the probability of extinction is the smallest positive root of the equation
$$s = G(s)$$
- if subcritical or critical, the branching process goes extinct with probability 1
- if supercritical, there is a positive probability of eventual extinction

A review - cont'd

- G_n - the probability generating function of the nth generation size Z_n:

$$G_n(s) = \mathbb{E}\left(s^{Z_n}\right) = \sum_{k=0}^{\infty} s^k \mathbb{P}(Z_n = k)$$

- the pgf od Z_n is the n-fold composition of the offspring generating function

$$G_n(s) = \underbrace{G(\dots G(G(s))\dots)}_{n-\text{fold}}$$

Example 1

Let $\mathbf{a} = (1/2, 0, 1/2, 0, \ldots)$. *What is the probability that the number of individuals born in the third generation is 0?*

Time of extinction

Let

$$T = \min\{n : Z_n = 0\}$$

be the **time of extinction** for a branching process $(Z_n)_{n \in \mathbb{N}_0}$.

Theorem 1

For $n \geq 1$

$$\mathbb{P}(T = n) = G_n(0) - G_{n-1}(0).$$

Example 2

Assume that $\mathbf{a} = (1/2, 1/2, 0, \ldots)$. Find the distribution of the time of extinction. What is the average time of extinction?

Progeny

Let

$$T_n = Z_0 + Z_1 + \ldots + Z_n = 1 + Z_1 + \ldots + Z_n$$

be the **progeny** up to time n, that is the total number of individuals up through generation n.
Let

$$\phi_n(s) = \mathbb{E}\left(s^{T_n}\right)$$

be the pgf of T_n. For example,

$$\phi_0(s) = \mathbb{E}\left(s^{T_0}\right) = \mathbb{E}\left(s^{Z_0}\right) = \mathbb{E}\left(s^1\right) = s,$$

and

$$\phi_1(s) = \mathbb{E}\left(s^{T_1}\right) = \mathbb{E}\left(s^{Z_0+Z_1}\right) = \mathbb{E}\left(s^{1+Z_1}\right) = \mathbb{E}\left(s \cdot s^{Z_1}\right) = s\mathbb{E}\left(s^{Z_1}\right) = sG(s) = sG\left(\phi_0(s)\right).$$

Progeny - cont'd

It can be shown that ϕ_n satisfies the recurrence relation

$$\phi_n(s) = sG(\phi_{n-1}(s)) \quad \text{for } n = 1, 2, \dots \,.$$

This can be used to find the distribution of T_n.

Example 3

Consider a branching process whose offspring distribution has $a_0 = a_1 = 1/2$. Find the distribution of the total number of individuals up through generation n.

Total progeny

Since

$$T_n = Z_0 + Z_1 + \ldots + Z_n = 1 + Z_1 + \ldots + Z_n$$

is the total number of individuals up through generation n,

$$\lim_{n\to\infty} T_n =: T$$

is the **total progeny** of the branching process.

It is fairly easy to compute the mean total progeny:

Theorem 2

The expectation of total progeny of $(Z_n)_n$ is

$$\mathbb{E}T = \begin{cases} +\infty, & \mu \geq 1, \\ \frac{1}{1-\mu}, & \mu < 1. \end{cases}$$

Distribution of the total progeny

Theorem 3

The pgf

$$\phi(s) = \mathbb{E}\left(s^T\right)$$

of the total progeny T satisfies equation

$$\phi(s) = sG(\phi(s)).$$

One consequence of this is something we have already observed: in the subcritical case,

$$\mathbb{E}T = \frac{1}{1-\mu}.$$

Distribution of the total progeny - cont'd

In some cases, equation $\phi(s) = sG(\phi(s))$ can be used to find the distribution of the total progeny.

Example 4

Consider a branching process $(Z_n)_n$, whose offspring distribution has $a_0 = a_1 = 1/2$. Find the distribution of the total progeny of $(Z_n)_n$.

Branching processes with immigration

In a **branching process with immigration** with the offspring distribution **a**, a random number of immigrants W_n is independently added to the population at the n-th generation for $n = 1, 2, \ldots$. We assume that the immigrants reproduce according to the same offspring distribution **a**, independently of all other individuals.

Theorem 4

Let H_n be be the probability generating function of W_n. Then the pgf G_n of the size of the n-th generation satisfies the recursive equation

$$G_n(s) = G_{n-1}(G(s)) \cdot H_n(s), \;\; n = 1, 2, 3, \ldots.$$

Branching processes with immigration - cont'd

Example 5

Consider a branching process $(Z_n)_n$ *with immigration, whose offspring distribution has* $a_0 = a_1 = 1/2$*. Assume that at each generation, one immigrant individual is added to the population. Find the distribution of the number of individuals appearing in the second generation.*