**EDRP: Discrete Random Processes**
**Problem set 3**

3.1 A Markov chain has transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 3/4 & 0 & 1/4 \end{bmatrix}.$$

Find the set of all stationary distributions.

3.2 Determine which of the following matrices are regular:

$$\begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ p & 1-p \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0.25 & 0.5 & 0.25 \\ 1 & 0 & 0 \end{bmatrix}.$$

*Hint: you don't have to compute any powers of the matrices. Think about the possibilities of reaching states from other states.*

3.3 Assume that a Markov chain has transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

Find the limiting distribution.

3.4 Find all pairs of communicating states if

$$\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

3.5 Find the communication classes of a Markov chain with transition matrix

(a)

$$\mathbf{P} = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1/6 & 0 & 0 \\ 0 & 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\mathbf{P} = \begin{bmatrix} 1/6 & 1/3 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 1/4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 4/5 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

3.6 Consider a Markov chain with transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}.$$

Obtain a closed form expression for $\mathbf{P}^n$. Exhibit the matrix $\sum_{n=0}^{\infty} \mathbf{P}^n$ (some entries may be $+\infty$). Explain what this shows about the recurrence and transience of the states.

3.7 Consider a Markov chain with transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.2 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 & 0.9 \\ 0 & 0.2 & 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}$$

Identify the communication classes. Classify the states as recurrent or transient, and determine the period of each state.

3.8 Find the expected return times to all states $a, b, c$ of the Markov chain with transition matrix

$$\mathbf{P} = \begin{bmatrix} 2/5 & 3/5 & 0 \\ 1/5 & 1/2 & 3/10 \\ 1/10 & 7/10 & 2/10 \end{bmatrix}.$$

3.9 Find the expected return time to state $b$ for the Markov chain with transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 0 & 3/4 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

3.10 Suppose that weather in a city might be described by the following transition matrix for a weather Markov chain on $S = \{r, s, c\}$, where $r$, $s$ and $c$ denote *rain*, *snow* and *clear*, respectively:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{10} & \frac{4}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{bmatrix}$$

(a) What is the long-term probability that a given day will be a rainy day?

(b) Over a years time about how many days are expected to be clear days?

(c) In the long-term, what is the average number of days that will transpire between snowy days?

## Answers

3.1 There are infinitely many stationary distributions. They all are given by the formula

$$\pi = \begin{bmatrix} 1/(3+2a) & a/(3+2a) & 2/(3+2a) & a/(3+2a) \end{bmatrix}$$

with $a > 0$.

3.2 (a) regular (b) regular (c) regular

3.3 Limiting distribution exists (why?) and equals the unique stationary distribution $\pi = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$.

3.4 Pairs of communicating classes $\{1, 4\}, \{2, 3\}$.

3.5 (a) $\{1, 5\}, \{2, 3\}, \{4\}$, (b) $\{1, 5, 4\}, \{2\}, \{3, 6\}$.

3.6 1 - transient, 2 - recurrent,

$$\mathbf{P}^n = \begin{bmatrix} 1/2^n & 1 - 1/2^n \\ 0 & 1 \end{bmatrix}$$

3.7 communication classes:

- $\{1\}$, transient, period=1,
- $\{2, 6, 7\}$, recurrent, period=1,
- $\{3, 4\}$, transient, period=1,
- $\{5\}$, recurrent, period=1.

3.8 $a$ - 85/19, $b$ - 85/48, $c$ - 85/18

3.9 3

3.10 (a) 1/9 (b) 50.6944 (c) 4/3