EDRP: Discrete Random Processes
Problem set 7

7.1 Consider a branching process $(Z_n)_n$ whose offspring distribution is $\mathbf{a} = (0.2, 0.5, 0.3)$.

(a) Give the probability distribution of the random variable Z_2.
(b) Compute the mean and variance of Z_2.
(c) Compute the expected number of offsprings at the tenth generation.
(d) What is the probability of extinction of this population?

7.2 A branching process has offspring distribution $\mathbf{a} = (1/4, 1/4, 1/2)$. Find the following:

(a) μ,
(b) the pgf of $\mathbf{a}$,
(c) the extinction probability,
(d) the pgf of Z_2,
(e) the distribution of Z_2,
(f) $\mathbb{E}Z_2$ (compare it with μ^2).

7.3 Consider a branching process $(Z_n)_{n\geq 0}$ defined by the offspring distribution with $a_0 = 1/5$, $a_1 = 3/5$ and $a_2 = 1/5$. Compute:

(a) $\mathbb{P}(Z_2 > 0)$,
(b) $\mathbb{P}(Z_2 = 1)$,
(c) $\mathbb{P}\left(Z_1 = 2 \middle| Z_2 = 1\right)$.

7.4 A cell culture is started with one red cell at time 0. After one minute the red cell dies and two new cells are born according to the following rule:

- with probability 1/4 both new cells are red,
- with probability 1/12 both new cells are white,
- with probability 2/3 one red and one white cells are born.

The above procedure is repeated minute after minute for any red cell present in the culture. The white cells however can only live for one minute, and disappear after that time without reproducing. We assume that the cells behave independently.

(a) What is the probability that no white cells have been generated up to time n (including time n)?
(b) Compute the extinction probability of the whole cell culture.

Answers

7.1 (a) $\mathbb{P}(Z_2 = 0) = 0.312$, $\mathbb{P}(Z_2 = 1) = 0.31$, $\mathbb{P}(Z_2 = 2) = 0.261$, $\mathbb{P}(Z_2 = 3) = 0.09$, $\mathbb{P}(Z_2 = 4) = 0.027$

(b) $\mathbb{E}Z_2 = 1.21$, $\operatorname{Var} Z_2 = 1.1319$

(c) $\mathbb{E}Z_{10} = 2.59374$

(d) $2/3$

7.2 (a) $\mu = 5/4$

(b) $G(s) = \frac{1}{4} + \frac{1}{4}s + \frac{1}{4}s^2$

(c) $1/2$

(d) $G_2(s) = \frac{11}{32} + \frac{1}{8}s + \frac{9}{32}s^2 + \frac{1}{8}s^3 + \frac{1}{8}s^4$

(e) $\mathbb{P}(Z_2 = 0) = \frac{11}{32}$, $\mathbb{P}(Z_2 = 1) = \frac{1}{8}$, $\mathbb{P}(Z_2 = 2) = \frac{9}{32}$, $\mathbb{P}(Z_2 = 3) = \frac{1}{8}$, $\mathbb{P}(Z_2 = 4) = \frac{1}{8}$

(f) $\mathbb{E}Z_2 = \frac{25}{16}$

7.3 (a) $84/125$

(b) $51/125$

(c) $2/17$

7.4 (a)

$$\frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 \cdot \left(\frac{1}{4}\right)^4 \cdot \ldots \cdot \left(\frac{1}{4}\right)^{2^{n-1}}, \ n \geq 0$$

(b) $\frac{1}{3}$