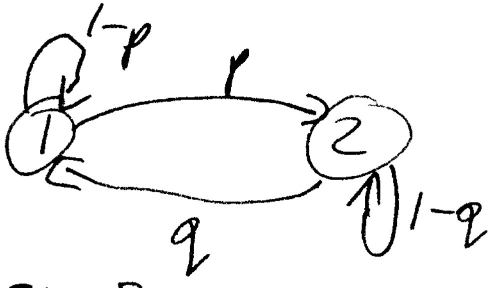
$$P = \begin{array}{c|cc} & ① & ② \\ \hline ① & 1-p & p \\ ② & q & 1-q \end{array}$$

$S = \{①, ②\}$



limiting a distribution

a probability distribution λ on S

$\forall i,j \in S \quad \lim_{n\to\infty} (P^n)_{ij} = \lambda_j \qquad \lambda = [\lambda_1 \quad \lambda_2] \qquad \lambda_1, \lambda_2 \geq 0$

$\lambda_1 + \lambda_2 = 1$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1 & \cdots & \lambda_n \end{bmatrix}$$

(a) $p+q=1 \Rightarrow \quad P = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix}$

$$P^2 = P \cdot P = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix} \cdot \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix} = \begin{bmatrix} (1-p)^2 + p(1-p) & (1-p)p + p^2 \\ (1-p)^2 + p(1-p) & (1-p)p + p^2 \end{bmatrix} =$$

$$= \begin{bmatrix} (1-p)[1-p+p] & p[1-p+p] \\ (1-p)[1-p+p] & p[1-p+p] \end{bmatrix} = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix} = P$$

$\forall n \quad P^n = P$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix}$$

the limiting distr. exists

$$\lambda = [1-p \quad p]$$

(b) $p+q \neq 1$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad c \in \mathbb{R}$$

$$cA = \begin{bmatrix} ca_{11} & c_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

$$\boxed{P^n = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + \frac{(1-p-q)^n}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}}$$

$\to 0$

$$\lim_{n\to\infty} P^n = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} = \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix}$$

$$\lambda = \left(\frac{q}{p+q} \quad \frac{p}{p+q}\right)$$

$\square^n$ $\quad \underline{p, q \in (0,1)}$

$$\underline{-1} < 1-p-q < \underline{1}$$

$$\lim_{n\to\infty} (1-p-q)^n =$$

$$\lim_{n\to\infty} \left(\frac{1}{7}\right)^n = 0$$

$$\lim_{n\to\infty} \left(-\frac{1}{7}\right)^n = 0$$

2.2

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$S = \{1, 2, 3, 4\}$

<u>stationary distribution</u>: (a row vector)

a prob. distr. on S

$\pi = (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4)$, $\pi_i \geq 0$, $\pi_1 + \ldots + \pi_4 = 1$

$$\pi P = \pi$$

$$\boxed{\pi}\ \boxed{P} = \boxed{\pi}$$

$x P = x \ |\cdot c$

$c x P = c x$

$(cx) P = (cx)$

$x := (1, x_2, x_3, x_4)$ $\quad \boxed{xP = x}$

$$\begin{cases} \frac{1}{2} + \frac{1}{4} x_3 = 1 \implies \boxed{x_3 = 2} \\ \frac{1}{4} + \frac{1}{2} x_2 + \frac{1}{4} x_3 + \frac{1}{4} x_4 = x_2 \\ \frac{1}{2} x_2 + \frac{1}{2} x_3 + \frac{1}{2} x_4 = x_3 \\ \frac{1}{4} + \frac{1}{4} x_4 = x_4 \implies \boxed{x_4 = \frac{1}{3}} \end{cases}$$

<u>3rd equation</u>:

$\frac{1}{2} x_2 + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{1}{3} = 2 \ |\cdot 6$

$3x_2 + 6 + 1 = 12$

$3x_2 = 5$ $\quad \boxed{x_2 = \frac{5}{3}}$

<u>2nd equation</u>

$\left(\frac{1}{4} + \frac{1}{2} \cdot \frac{5}{3} + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot \frac{1}{3} \stackrel{?}{=} \frac{5}{3} \right.$

$\ldots$ OK

$$x = \left(1,\ \tfrac{5}{3},\ 2,\ \tfrac{1}{3}\right) \Longrightarrow \boxed{xP = x}$$

$$1 + \tfrac{5}{3} + 2 + \tfrac{1}{3} = 5$$

$$\pi := \left(\tfrac{1}{5},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{15}\right) \Longrightarrow \left.\begin{array}{l} \pi P = \pi \\ \pi \text{ is a probability vector} \end{array}\right\} \Longrightarrow \pi \text{ is a stationary distribution}$$

$$\pi = (\pi_1 \quad \pi_2 \quad \pi_3)$$

$$x = (1 \quad x_2 \quad x_3) \qquad xP = x$$

$$x = (x_1 \quad x_2 \quad 1)$$