6.11 offspring distrib. is uniform on $\{0,1,2,3,4\}$

$q_0 = q_1 = q_2 = q_3 = q_4 = \frac{1}{5}$

$q_5 = q_6 = \dots = 0$

extinction prob.

$G(s) = s$ ← pgf of the offspring distr.

$G(s) = \frac{1}{5}(1+s+s^2+s^3+s^4)$

$\frac{1}{5}(1+s+s^2+s^3+s^4) = s$

$s^4+s^3+s^2-4s+1=0$

$s^4-s+s^3-s+s^2-s+1-s=0$

$s(s^3-1)+s(s^2-1)+s(s-1)+(1-s)=0$

$s(s-1)(s^2+s+1)+s(s-1)(s+1)+s(s-1)+(1-s)=0$

$(s-1)[s(s^2+s+1)+s(s+1)+s-1]=0$

$(s-1)[s^3+2s^2+3s-1]=0$

$s=1$ $\quad$ $s^3+2s^2+3s-1=0$

$s=0.275682\dots$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$s^3-1=(s-1)(s^2+s+1)$

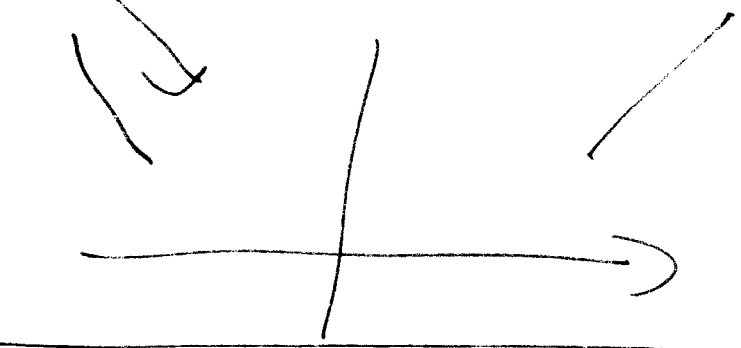
$s^2-1=(s-1)(s+1)$

$s^2+1=0$

$f(s)=s^3+2s^2+3s-1$

$\lim_{s\to\infty} f(s)=+\infty$

$\lim_{s\to-\infty} f(s)=-\infty$

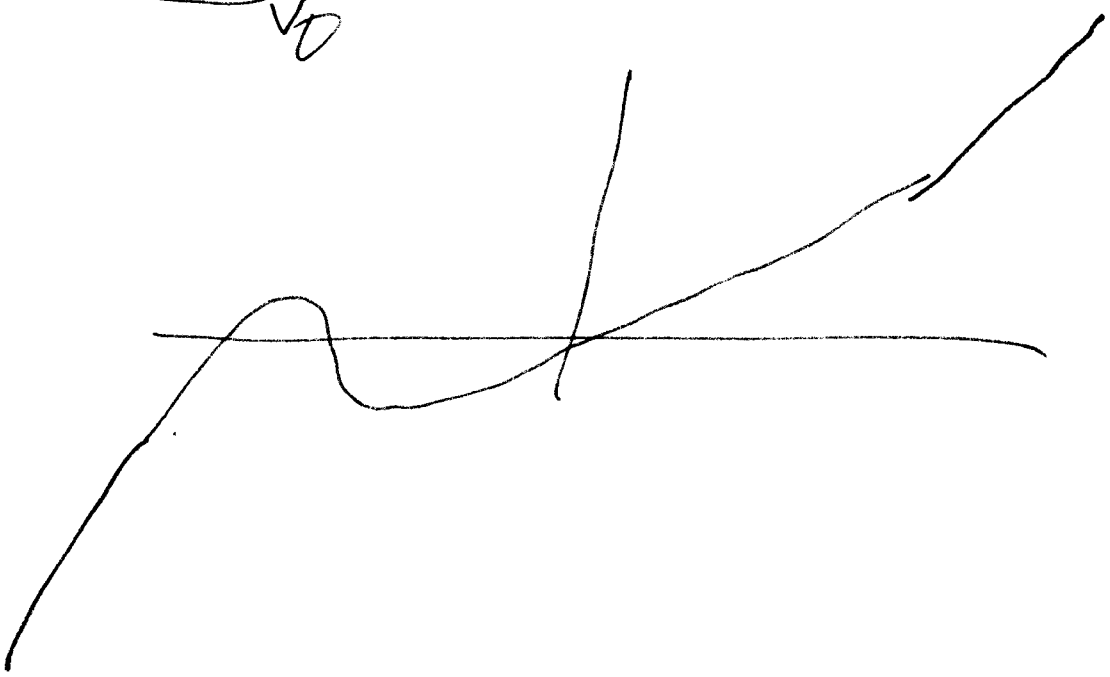


$s^4+1=0$

$\vee\vee$

0

$\vee 0$



6.12 $a_k=\left(\frac{1}{2}\right)^{k+1}$, $k=0,1,2,\ldots$

$a_0=\frac{1}{2}$, $a_1=\frac{1}{4}$, $a_2=\frac{1}{8}$, $\ldots$

$G(s)=s$

$G(s)=\sum_{k=0}^{\infty} a_k\cdot s^k=\frac{1}{2}+\frac{1}{4}s+\frac{1}{8}s^2+\ldots$

[geometrical series

$b_1+b_1q+b_1q^2+b_1q^3+\ldots=\frac{b_1}{1-q}$, $|q|<1$]

$b_1 = \frac{1}{2}$ $\quad q = \frac{1}{2}s$ $\quad |q| < 1$ $\quad \left|\frac{s}{2}\right| < 1$

$|s| < 2$

$$G(s) = \frac{\frac{1}{2}}{1 - \frac{1}{2}s} = \frac{1}{2-s}$$

$$\frac{1}{2-s} = s$$

$s(2-s) = 1$

$2s - s^2 = 1$

$s^2 - 2s + 1 = 0$

$(s-1)^2 = 0$

the extinction prob. $= 1$

(7.1) $a = (0.2, 0.5, 0.3)$

$\mu = 0 \cdot 0.2 + 1 \cdot 0.5 + 2 \cdot 0.3 = 1.1 > 1$ supercritical

$Z_2 \leftarrow G_2 \leftarrow$ the pgf of Z_2

$G_2(s) = G(G(s))$

$G(s) = 0.2 + 0.5s + 0.3s^2$

$= 0.2 + 0.5 \cdot G(s) + 0.3 \cdot (G(s))^2$

$= 0.2 + 0.5(0.2 + 0.5s + 0.3s^2) + 0.3 \cdot (0.2 + 0.5s + 0.3s^2)^2$

$= 0.312 + 0.31s + 0.261s^2 + 0.09s^3 + 0.027s^4$

Z_2	0	1	2	3	4
	0.312	0.31	0.261	0.09	0.027

$$\mathbb{E}Z_2 = 0 \cdot 0.312 + 1 \cdot 0.31 + 2 \cdot 0.261 + 3 \cdot 0.09 + 4 \cdot 0.027 =$$

mean of Z_2

$$\mathbb{E}Z_2 = \mu^2 = (1.1)^2$$

variance Z_2

$$\mathrm{Var}\, Z_2 = \mathbb{E}(Z_2)^2 - (\mathbb{E}Z_2)^2$$

$$\mathrm{Var}\, Z_n = \begin{cases} \cdots \\ \cdots \end{cases}$$

$$\mathbb{E}Z_{10} = \mu^{10} = (1.1)^{10}$$

$$0.2 + 0.5s + 0.3s^2 = s$$

$$0.3s^2 - 0.5s + 0.2 = 0$$

$ax^2 + bx + c = 0$

$$x_1 x_2 = \frac{c}{a} = \frac{0.2}{0.3} = \left(\frac{2}{3}\right) \in \text{the prob. of } \{t_1, t_2\}$$

7.3 $a_0 = \frac{1}{5}$, $a_1 = \frac{3}{5}$, $a_2 = \frac{1}{5}$ G - the pgf of the offspring d_1. [5/6]

$\mathbb{P}(Z_2 > 0)$

$\mathbb{P}(Z_2 = 1)$

$\mathbb{P}(Z_1 = 2 \mid Z_2 = 1)$

G_2 - the pgf of Z_2

$G_2(s) = G(G(s))$

$G(s) = \frac{1}{5} + \frac{3}{5}s + \frac{1}{5}s^2$

$G_2(s) = \frac{1}{5} + \frac{3}{5}\left(\frac{1}{5} + \frac{3}{5}s + \frac{1}{5}s^2\right) + \frac{1}{5}\left(\frac{1}{5} + \frac{3}{5}s + \frac{1}{5}s^2\right)^2$

$\mathbb{P}(Z_2 > 0) = 1 - \mathbb{P}(Z_2 = 0) = 1 - \left(\frac{1}{5} + \frac{3}{5}\cdot\frac{1}{5} + \frac{1}{5}\cdot\left(\frac{1}{5}\right)^2\right)$

$\mathbb{P}(Z_2 = 1) = \frac{3}{5}\cdot\frac{3}{5} + \frac{1}{5}\cdot\left(\frac{1}{5}\cdot\frac{3}{5}\cdot 2\right)$

$\mathbb{P}(Z_1 = 2 \mid Z_2 = 1) = \frac{\mathbb{P}(Z_2 = 1 \mid Z_1 = 2)\cdot\mathbb{P}(Z_1 = 2)}{\mathbb{P}(Z_2 = 1)}$, $\mathbb{P}(Z_1 = 2) = \frac{1}{5}$

$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(B \mid A)\cdot\mathbb{P}(A)$

$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\cdot\mathbb{P}(A)}{\mathbb{P}(B)}$

gen 0 → •

gen 1 →

$$\mathbb{P}(Z_1 = 2) = \frac{1}{5}$$

$$q_0 = \frac{1}{5}, \; q_1 = \frac{3}{5}, \; q_2 = \frac{1}{5}$$

$$\mathbb{P}(Z_2 = 1 \mid Z_1 = 2) = 2 \cdot \frac{3}{5} \cdot \frac{1}{5}$$

• • ← gen 1

• ← gen 2