Discrete Random Processes (EDRP)

Lecture 5

Poisson process

Counting processes

A Poisson process is a special type of **counting process**.

Given a stream of events that arrive at random times starting at $t = 0$, let N_t denote

the (random) number of arrivals that occur by time t, that is, the number of events in $[0, t]$.

Since N_t counts events in $[0, t]$, N_t is a random variable which is

- non-negative,
- integer-valued,
- and such that if $0 \leq s \leq t$, then $N_s \leq N_t$.

If $0 \leq s < t$, then $N_t - N_s$ is the number of events in the interval $(s, t]$.

Poisson distribution

In order to prepare for the definition of the Poisson process, we recall the **Poisson distribution** and derive some of its properties.

Definition 1

We say that X has a **Poisson distribution with parameter** $\lambda > 0$ (den. $X \sim \mathrm{Poisson}(\lambda)$), if

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} \exp(-\lambda), \quad k = 0, 1, 2, \ldots.$$

In order to convince ourselves that the above formula correctly defines a probability distribution, we need to check that:

- $\mathbb{P}(X = k) > 0$ for every $k = 0, 1, 2 \ldots,$
- $\sum_k \mathbb{P}(X = k) = 1.$

The mean of Poisson distribution

Recall the formula

$$\mathbb{E}X = \sum_{k} k \cdot \mathbb{P}(X = k)$$

giving the expected value of random variable X, when X has a discrete distribution. The formula implies that if $X \sim \mathrm{Poisson}(\lambda)$ then

$$\mathbb{E}X = \lambda.$$

Before we present a definition of Poisson process, recall that independence of discrete random variables X and Y means that

$$\mathbb{P}(X = k, Y = l) = \mathbb{P}(X = k) \cdot \mathbb{P}(Y = l) \quad \forall k, l.$$

Poisson process

Definition 2

A *Poisson process with parameter* λ is a counting process $(N_t)_{t \geq 0}$ with the following properties:

1. $N_0 = 0$ (we say that the process *starts at 0*),
2. for all $t > 0$, N_t has a Poisson distribution with parameter λt (denoted $N_t \sim \mathrm{Poisson}(\lambda t)$),
3. (**stationary increments**) for all $s, t > 0$, $N_{t+s} - N_s$ has the same distribution as N_t, that is

$$\mathbb{P}(N_{t+s} - N_s = k) = \mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

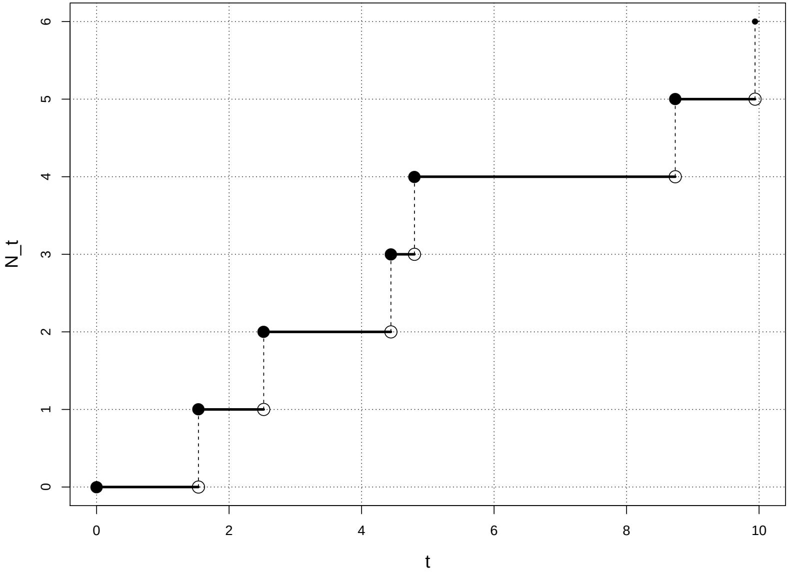
   for $k = 0, 1, 2, \ldots$,
4. (**independent increments**) for $0 \leq q < r \leq s < t$, $N_t - N_s$ and $N_r - N_q$ are independent random variables.

Remarks

- The stationary increments property says that the distribution of the number of arrivals in an interval only depends on the *length* of the interval.
- The independent increments property says that the numbers of arrivals on disjoint intervals are independent random variables.
- Since $N_t \sim \mathrm{Poisson}(\lambda t)$, $\mathbb{E}N_t = \lambda t$. That is, we expect about λt arrivals in t time units. This means that the **rate** (or **intensity**) of arrivals is

$$\frac{\mathbb{E}N_t}{t} = \lambda.$$

Poisson process - an example of a trajectory



Example 1

Let $(N_t)_t$ denote a Poisson process with parameter $\lambda = 2$.
Compute:

- $\mathbb{P}(N_3 = 4)$,
- $\mathbb{P}(N_2 > 1)$,
- $\mathbb{P}(N_3 = 1, N_5 = 1)$,
- $\mathbb{P}\left(N_5 = 2 \middle| N_3 = 1\right)$,
- $\mathbb{E}(N_7)$.

Exponential distribution

For a Poisson process with parameter λ, let X denote the time of the first arrival. Observe that

$$X > t \text{ if and only if there are no arrivals in } [0, t].$$

Thus,

$$\mathbb{P}(X > t) = \mathbb{P}(N_t = 0) = \exp(-\lambda t)$$

for $t > 0$. This means that X has an **exponential distribution with parameter** λ.

Exponential distribution - cont'd

Definition 3

A random variable X is said to have an **exponential distribution with rate** $\lambda > 0$ (den. $X \sim \mathrm{Exp}(\lambda)$), if its density is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The fact that the above function is the density of X means that if A is any subset of $\mathbb{R}$, then

$$\mathbb{P}(X \in A) = \int_A f(x)\mathrm{d}x = \int_{A \cap [0,\infty)} \lambda \exp(-\lambda x)\mathrm{d}x.$$

Exponential distribution - cont'd

Example 2

Compute $\mathbb{P}(X > t)$ for $t > 0$ if X has an exponential distribution with parameter λ.

The mean of exponential distribution

Recall the formula

$$\mathbb{E}X = \int_{\mathbb{R}} x \cdot f(x)\mathrm{d}x$$

giving the expected value of random variable X, when f is the density of X. The formula implies that if $X \sim \mathrm{Exp}(\lambda)$, then

$$\mathbb{E}X = \frac{1}{\lambda}.$$

Lack of memory property of exponential distribution

If $X \sim \mathrm{Exp}(\lambda)$ then for all $s, t > 0$

$$\mathbb{P}\left(X > t + s \middle| X > t\right) = \mathbb{P}(X > s).$$

An interpretation is the following. Suppose $X \sim \mathrm{Exp}(\lambda)$ denotes the waiting time for an event. The formula above says that if we've been waiting for t units of time then the probability we must wait s more units of time is the same as if we haven't waited at all.

Poisson process and exponential distribution - the link

The exponential distribution plays a central role in the Poisson process. Recall: for a Poisson process with parameter λ, if X is the time of the first arrival, then X has an *exponential distribution with parameter* λ.

What is true for the time of the first arrival is also true for the time between the first and second arrival, and for *all interarrival times*.

A Poisson process is a counting process for which interarrival times are *independent* and *identically distributed exponential random variables*.

Poisson process - a construction

Let

- $\rho_1, \rho_2, \ldots$ be a sequence of independent exponential random variables with parameter λ,
- $\tau_0 = 0$ and $\tau_n = \rho_1 + \ldots + \rho_n$ for $n \geq 1$.

For $t > 0$ let

$$N_t = \max\{n : \tau_n \leq t\},$$

with $N_0 = 0$.

Poisson process - a construction - cont'd

Some names:

- $\tau_1, \tau_2, \dots$ – the **arrival times** of the process $(N_t)_t$ (τ_k is the **time of the** k**th arrival**),
- $\rho_1, \rho_2, \dots$ – the **interarrival times** of the process $(N_t)_t$ ($\rho_k = \tau_k - \tau_{k-1}$ is the interarrival time between the $(k-1)$th and kth arrival).

Theorem 4

$(N_t)_{t \geq 0}$ is a Poisson process with parameter λ.

So a Poisson process is a counting process for which interarrival times are independent and identically distributed exponential random variables.

Example 3

Let $(N_t)_t$ be a Poisson process with intensity $\lambda = 2$. What is the probability that the time between the second and the third jump of the process does not exceed 1?