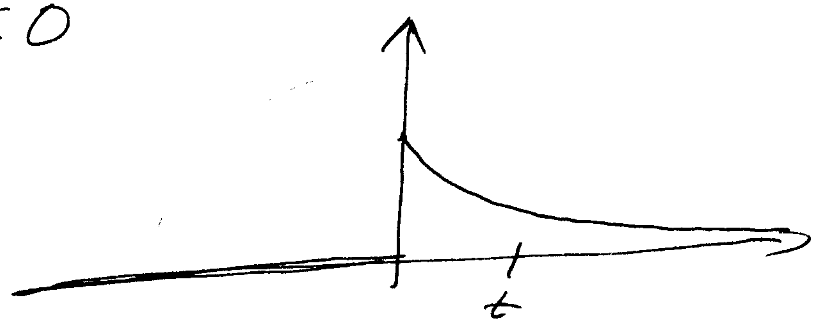
(4.1) $X \sim Exp(\lambda)$, $\lambda > 0$

$\forall A \subset \mathbb{R}$

$$\mathbb{P}(X \in A) = \int_A f(x)\,dx$$

← density of the exponential distn

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



1) $f \geq 0$

2) $\int_{\mathbb{R}} f(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}\,dx$

$= 1$

$X \sim Exp(\lambda)$

$$\mathbb{P}(X < 0) = \mathbb{P}(X \in \underbrace{(-\infty, 0)}_{A}) =$$

$$= \int_{-\infty}^{0} 0\,dx = 0$$

$X \sim Exp(\lambda)$, $\lambda > 0$, $s, t > 0$

$$\underbrace{\mathbb{P}(\overset{A}{X > t+s} \mid \overset{B}{X > t})} = e^{-\lambda s}$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(X > t+s \mid X > t) = \frac{\mathbb{P}(X > t+s, \cancel{X > t})}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > t)} = \bullet$$

$$\underbrace{\mathbb{P}(X > t)} = \int_t^{+\infty} \lambda e^{-\lambda x}\, dx = \lambda \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_t^{+\infty} = \cancel{\lambda\left(-\frac{1}{\lambda} e^{-\lambda t} + 0\right)}$$

$$= \lambda\left(0 - \left(-\frac{1}{\lambda} e^{-\lambda t}\right)\right) = \underbrace{e^{-\lambda t}}$$

$$\bullet = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}(X > s)$$

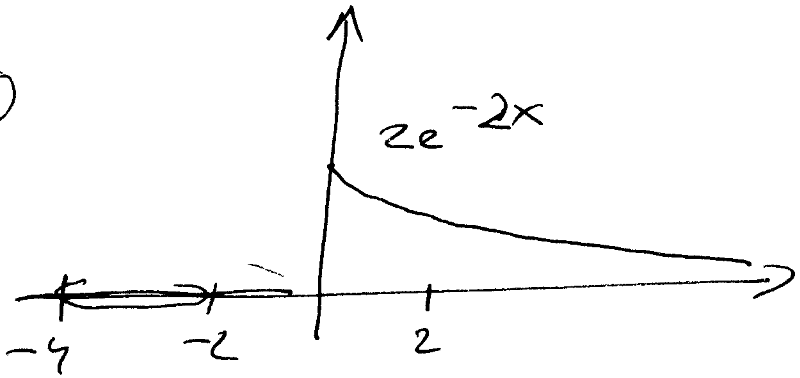
$$\boxed{\begin{array}{l} X \sim Exp(\lambda), \quad s, t > 0 \\ \qquad \mathbb{P}(X > t+s \mid X > t) = \mathbb{P}(X > s) \end{array}}$$

lack of memory property of the exponential distr.

(4.2) $X \sim Exp(2)$

$$\boxed{\forall A \subset \mathbb{R} \quad \mathbb{P}(X \in A) = \int_A f(x)\,dx}$$

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$$\mathbb{P}(X \in [-4,-2]) = \int_{-4}^{-2} f(x)\,dx = 0$$

$$\mathbb{P}(X > 2) =$$

$$= \int_2^{+\infty} f(x)\,dx = \int_2^{+\infty} 2e^{-2x}\,dx$$

$$= \ldots$$

$$\mathbb{P}(X \leq 1) = \int_0^1 2e^{-2x}\,dx = \ldots$$

$$\mathbb{P}(X > 5 \mid X > 1) = \mathbb{P}(X > 4) = \ldots$$

$X \sim Exp(\lambda)$

$$\mathbb{P}(X > t+s \mid X > t) = \mathbb{P}(X > s)$$

$\cancel{s=1}$ $\quad$ $t=1$

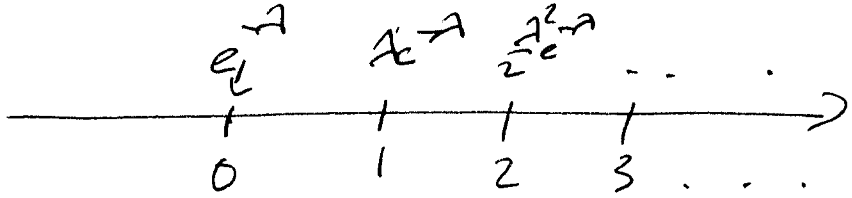
$\cancel{t=4}$ $\quad$ $s=4$

$$\int \lambda e^{-\lambda x}\,dx =$$

$$= \lambda \int e^{-\lambda x}\,dx =$$

$$= \lambda \left(-\frac{1}{\lambda} e^{-\lambda x} \right)$$

$\underbrace{X \sim \text{Poisson}(\lambda)}$, $\lambda > 0$



$$\boxed{\mathbb{P}(X = k) = \underbrace{\frac{\lambda^k}{k!} e^{-\lambda}} \quad k = 0, 1, 2, \ldots}$$

a discrete distribution

$\forall k = 0, 1, 2, \ldots \quad \mathbb{P}(X = k) > 0$

$$\sum_{k=0}^{\infty} \mathbb{P}(X = k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = 1$$

4.3 $\quad X \sim \text{Poisson}(1)$

$$\mathbb{P}(X < 3) = \mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) =$$

$$= e^{-1} + 1 \cdot e^{-1} + \frac{1^2}{2} e^{-1}$$

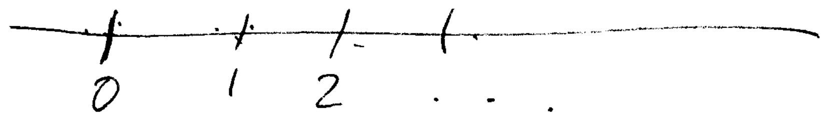
$\mathbb{P}(X \le 2) = \mathbb{P}(X < 3) = ?$

$\mathbb{P}(X > 0) = 1 - \mathbb{P}(X = 0) =$

$= 1 - e^{-1}$

$\mathbb{P}(X \ge 0) = 1$

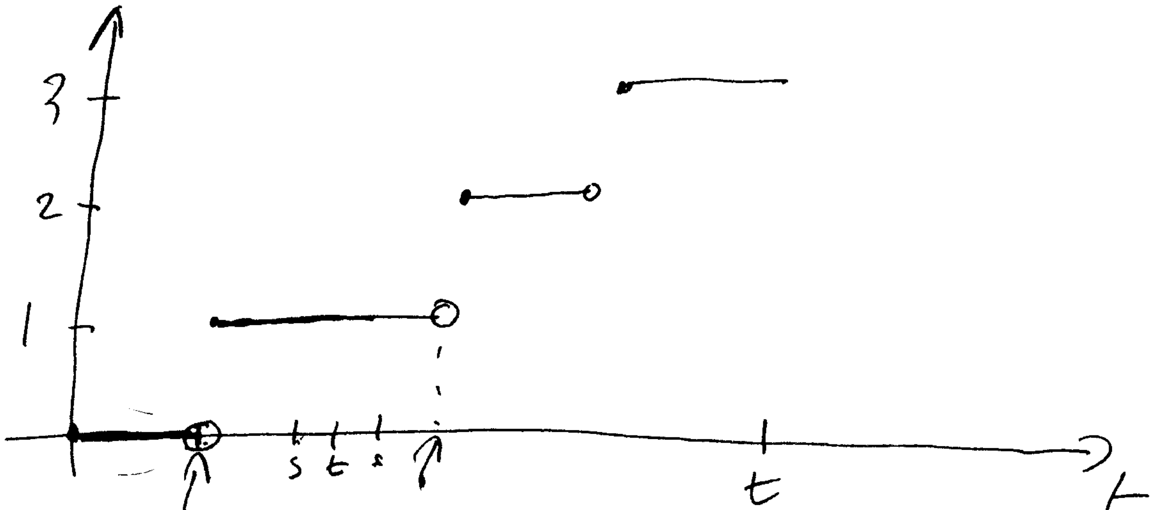
$\mathbb{P}(X = 2.5) = 0$



$$\mathbb{P}(X \in (2, 5)) = \mathbb{P}(X=3) + \mathbb{P}(X=4)$$

$$= \frac{1^3}{3!} e^{-3} + \frac{1^4}{4!} e^{-4}$$

$(N_t)_{t\geq 0}$ – a Poisson process with the parameter $\lambda > 0$



the time of the 1st event

$\mathbb{P}(N_0 = 0) = 1$

N_t – # of events up to time t

axiomatic

~~Th~~

1. $N_0 = 0$
2. it has independent increments

$0 \leq s < t < u$

$0 \quad s \quad t \quad u$

$N_t - N_s \,,\; N_u - N_t$ independent

increments

λ

3. $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$

$s < t$

$N_s \,,\, N_t$ not independent

$\left(\begin{matrix} N_s = 5 \\ s < t \end{matrix} \Rightarrow N_t \geq 5 \right)$

$$N_t - N_s \sim \text{Poisson}(\lambda(t-s)),$$
$$0 \le s < t$$

$s = 0$

$$N_t - \overset{\overset{0}{\|}}{N_0} \sim \text{Poisson}(\lambda t)$$

$$N_t \sim \text{Poisson}(\lambda t)$$

$$\mathbb{E} N_t = \lambda t$$

$$X \sim \text{Poisson}(\lambda)$$
$$\mathbb{E} X = \lambda$$