Discrete Random Processes (EDRP)

Lecture 6

Arrival times and gamma distribution

For a Poisson process, each arrival time τ_n is a sum of n independent and identically distributed exponential interarrival times. A sum of i.i.d. exponential variables has a **gamma distribution** (also called the **Erlang distribution** in queueing theory).

Theorem 1

For $n \in \mathbb{N}$, let τ_n be the time of the n-th arrival in a Poisson process with parameter λ. Then τ_n has a gamma distribution with parameters n and λ, that is, its density function is

$$f(t) = \begin{cases} 0, & t \leq 0, \\ \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}, & t > 0. \end{cases}$$

This allows to compute the probability that the n-th arrival occurs in the time interval between t_1 and t_2 for any $0 \leq t_1 < t_2 < \infty$.

Arrival times and gamma distribution - cont'd

Example 1

Let $(N_t)_t$ be a Poisson process with intensity $\lambda = 3$.

1. *What is the probability that the first arrival happens in the time interval $[0, 1]$?*
2. *What is the probability that the second arrival happens in the time interval $[1, 2]$?*

Arrival times and gamma distribution - cont'd

Since

$$\tau_n = \rho_1 + \ldots + \rho_n, \;\; n \geq 1,$$

where $\rho_1, \ldots, \rho_n$ are independent random variables, all exponentially distributed with the parameter λ, mean of n-th arrival time τ_n is

$$\mathbb{E}\tau_n = \frac{n}{\lambda}.$$

Minimum of independent exponential random variables

Results for the minimum of independent exponential random variables are particularly useful when working with the Poisson process.

Theorem 2

Let $X_1, \ldots, X_n$ be independent exponential random variables with respective parameters $\lambda_1, \ldots, \lambda_n$. If $M = \min(X_1, \ldots, X_n)$, then:

1. $M \sim Exp(\lambda_1 + \ldots + \lambda_n)$,
2. *for* $k = 1, \ldots, n$,

$$\mathbb{P}(M = X_k) = \frac{\lambda_k}{\lambda_1 + \ldots + \lambda_n}.$$

Example 2

Michael, Tom, and Agnes are at the front of three separate lines in the Biedronka supermarket waiting to be served. The serving times for the three lines follow independent Poisson processes with respective parameters 1, 2, and 3.

1. *Find the probability that Agnes is served first.*
2. *Find the probability that Michael is served before Agnes.*
3. *Find the expected waiting time for the first person served.*

Thinning

Thinning is a transformation of a Poisson process. It is a partitioning procedure for decomposing a Poisson process into several subprocesses.

Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. Assume that each arrival, independently of other arrivals, is marked as a *type k* event with probability p_k, for $k = 1, \ldots, n$, with $p_1 + \ldots + p_n = 1$. Let

$$N_t^{(k)} \text{ be the number of type } k \text{ events in } [0, t].$$

Theorem 3

$(N_t^{(k)})_{t\geq 0}$ is a Poisson process with parameter λp_k. Furthermore, the processes

$$\left(N_t^{(1)}\right)_{t\geq 0}, \ldots, \left(N_t^{(n)}\right)_{t\geq 0}$$

are independent.

Thinning - an example

Example 3

Consider a website that sells some products. Suppose the website has visitors arriving to it according to a Poisson process with rate λ. Suppose p percent of these visitors actually buy a product, which means that each visitor independently buys a product with probability p. Then from the preceding result,

- *the times of sales form a Poisson process with rate $p\lambda$,*
- *the visits without sales occur according to a Poisson process with rate $(1 - p)\lambda$.*

Thinning - another example

Before we state the next example, recall that independence of discrete random variables X and Y means that

$$\mathbb{P}(X = k, Y = l) = \mathbb{P}(X = k) \cdot \mathbb{P}(Y = l) \quad \forall k, l.$$

Example 4

Arthur catches fish at times of a Poisson process with rate 2 per hour. 40% of the fish are salmon, while 60% of the fish are trout. What is the probability he will catch exactly 1 salmon and 2 trout if he fishes for 2.5 hours?

Superposition

So one can view thinning as taking one Poisson process and *splitting it into two or more* by using an i.i.d. sequence. Going in the other direction and *adding up a lot of independent processes* is called **superposition**. When the independent Poisson processes are put together, the sum is Poisson with a rate equal to the sum of rates.

The reason for this is an analogous fact at the level of *Poisson random variables*.

Sum of independent Poissons

Theorem 4

If $X_1, \ldots, X_n$ are independent random variables and $X_i \sim Poisson(\lambda_i)$, then $X_1 + \ldots + X_n \sim Poisson(\lambda_1 + \ldots + \lambda_n)$.

Superposition

Theorem 5

Assume that

$$\left(N_t^{(1)}\right)_{t\geq 0}, \ldots, \left(N_t^{(n)}\right)_{t\geq 0}$$

are independent Poisson processes with respective parameters $\lambda_1, \ldots, \lambda_n$. *Let*

$$N_t = N_t^{(1)} + \ldots + N_t^{(n)}$$

for $t \geq 0$. *Then* $(N_t)_{t\geq 0}$ *is a Poisson process with parameter*

$$\lambda = \lambda_1 + \ldots + \lambda_n.$$

Superposition - an example

Example 5

Customers are waiting in three separate checkout lines in a Biedronka supermarket. The services in the lines are independent of each other and each of them follows a Poisson process with respective parameters 2, 3 and 4, where the time unit is minutes.

1. *Find the probability that no customer is going to be served in a 5 minutes period.*
2. *Three people have been served in a 5 minutes period. Find the probability that a customer from every line has been served in this period.*

Nonhomogeneous Poisson Process

In a Poisson process, arrivals occur at a constant rate, independent of time. However, for many applications this is an unrealistic assumption.

Real life situations with varying arrival rates can be modeled by a nonhomogeneous Poisson process with rate $\lambda = \lambda(t)$, which depends on t. Such a rate *function* is called the **intensity function**.

Nonhomogeneous Poisson Process - cont'd

Definition 6

A counting process $(N_t)_{t \geq 0}$ is a **nonhomogeneous Poisson process** with intensity function $\lambda(t)$, if

1. $N_0 = 0$,
2. for $t > 0$, N_t has a Poisson distribution with mean

$$\mathbb{E}N_t = \int_0^t \lambda(x)\mathrm{d}x,$$

3. for $0 \leq q < r \leq s < t$, $N_r - N_q$ and $N_t - N_s$ are independent random variables.

Nonhomogeneous Poisson Process - cont'd

- A nonhomogeneous Poisson process has independent increments, but not necessarily stationary increments.

- It can be shown that for $0 < s < t$, $N_t - N_s$ has a Poisson distribution with parameter

$$\int_s^t \lambda(x)\mathrm{d}x.$$

- For a nonhomogeneous Poisson process, the interarrival times are not exponential and they are not independent.

- If $\lambda(t) = \lambda$ for all t, one obtains an ordinary (homogeneous) Poisson process with intensity λ.

Nonhomogeneous Poisson Process - cont'd

Example 6

Customers arrive at a shop according to a nonhomogeneous Poisson process. The arrival rate increases linearly from 10 to 30 customers per hour between 8 a.m. and noon. The rate stays constant for the next 2 hours, and then decreases linearly down to 10 from 2 to 6 p.m. Find the probability that there are at least 40 customers in the shop between 11:30 a.m. and 1:30 p.m.