EDRP Lab 2

Version: 2020-03-28

Problem 1 - Matrix powers in R

R does not have a built-in command for taking matrix powers. Write a function `matrixpower` with two arguments `mat` and k that will take integer powers k of a matrix `mat`.

Problem 2 - Stationary distributions and eigenvectors

The stationary distribution of a Markov chain is related to the eigenstructure of the transition matrix. Namely, if π is the stationary distribution of a Markov chain (so it satisfies $\pi\mathbf{P} = \pi$), then π is a left eigenvector of $\mathbf{P}$ corresponding to eigenvalue $\lambda = 1$.

The R command `eigen(P)` returns the eigenvalues and eigenvectors of a square matrix P. These are given in a list with two components: `values` contains the eigenvalues, and `vectors` contains the corresponding eigenvectors stored as a matrix (write `eigen(P)$values` or `eigen(P)$vectors` to access eigenvalues or eigenvectors). If P is a stochastic matrix, an eigenvector corresponding to $\lambda = 1$ will be stored in the first column of the `vectors` matrix. The R command `t(P)` gives the transpose P^T of P.

1. Find an eigenvector for $\mathbf{P}^T$ corresponding to $\lambda = 1$ if

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}$$

2. Compute the stationary distribution by normalizing the eigenvector (normalization means dividing the entries by the sum of the entries)

Problem 3 - Stationary and limiting distributions for regular matrices

Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$

1. Check that $\mathbf{P}$ is a regular matrix by computing a sufficiently large power of $\mathbf{P}$. What is the smallest positive power of $\mathbf{P}$?
2. Find (if it exists) a limiting distribution of the chain.
3. Find (if it exists) a stationary distribution of the chain.

Problem 4 - Limiting distribution and proportion of time in each state

Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.2 & 0.4 & 0.3 \\ 0.4 & 0.0 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.0 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{bmatrix}$$

1. Compute $\mathbf{P}^{10}$.
2. Start the chain with the uniform distribution on states and run it for 1000000 steps.
3. Based on the previous point, compute the proportions of visits to each state (you can use `table` command in R). Compare them with the entries of $\mathbf{P}^{10}$.

Problem 5 - Simulation of an expected return time

Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

1. Start the chain at 1 and run it for 25 steps. Find the return time to 1.
2. Estimate the mean return time based on 10000 Monte Carlo simulations. That is, repeat the steps from the previous point 10000 times and compute the average of the obtained return times.

Question: what should you do in case the chain will not return to 1 in 25 steps? How does such situation affect your computation of the simulated mean return time?

3. Repeat the first two steps replacing state 1 with states 2 and 3.
4. Compute the stationary distribution and compare it with inverses of the estimated return times.

Problem 6 - Simulation of an expected time between visits to a state

Consider again a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

1. Start the chain at 1 and run it for 50 steps. Find the time between the first and the second visit to state 2.
2. Estimate the mean return time based on 10000 Monte Carlo simulations. Compare it with the results of the previous problem.