Discrete random processes

A basic probability refresher - part 1

We will first review some of the basic facts usually taught in a first course in probability, concentrating on the ones that are the most important in our course.

- The term **experiment** is used to refer to any process whose outcome is not known in advance.
- Two simple experiments are flip a coin, and roll a die.
- The **sample space** associated with an experiment is the set of all possible outcomes. The sample space is usually denoted by Ω, the capital Greek letter Omega.

Example 1

Flipping three coins leads to
$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

The goal of probability theory is to compute the probability of various events of interest. Intuitively, an **event** is a statement about the outcome of an experiment. Formally, an event is a subset of the sample space. An example for flipping three coins is "*two coins show heads*", or

$$\{HHT, HTH, THH\} \subset \Omega.$$

Two events are **disjoint** if their intersection is the empty set $\emptyset$.

Example 2

For flipping three coins, A= "*two coins show heads*", and B= "*three coins show tails*" are disjoint.

A **probability** is a way of assigning numbers to events that satisfies:

- For any event A, $0 \le \mathbb{P}(A) \le 1$, and $\mathbb{P}(\Omega) = 1$.
- For a finite or infinite sequence of disjoint events

$$\mathbb{P}\left(\bigcup_n A_n\right) = \sum_n \mathbb{P}(A_n).$$

Two basic properties that follow immediately from the definition of a probability are

- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

A **random variable** is a real-valued function defined on the sample space:

$$X : \Omega \to \mathbb{R}.$$

That is,the outcomes of a random variable are determined by a random experiment.

Example 3

Assume that three coins are flipped. Let X be the number of heads. Then, X is a random variable that takes values 0, 1, 2, or 3, depending on the outcome of the coin flips.

Example 4

We flip a coin until a head comes up. Let X be the number of filps. Then, X is a random variable that takes values $1, 2, 3, \ldots$, depending on the outcomes of the coin flips.

We describe the random variables by giving their **distributions**.

The distribution of a random variable X describes the set of values of X and their corresponding probabilities.

We write $\mathbb{P}(X = x)$ for the probability that X takes the value x, and $\mathbb{P}(X \leq x)$ for the probability that X takes a value less than or equal to x.

More generally, if $A \subset \mathbb{R}$, we write $\mathbb{P}(X \in A)$ for the probability that X takes a value that is contained in the set A.

Example 5

For the random experiment of flipping a fair coin three times, let X denote the number of heads that occur. Find the distribution of X.

Example 6

We flip a coin until a head comes up. Let X be the number of filps. Find the distribution of X.

Stochastic (random) processes

Definition 7

A **stochastic (random) process** is a collection of random variables $(X_t)_{t \in I}$. The set I is the **index set** of the process. The index t often represents time. For example

- $I = \{0, 1, 2, \ldots\} =: \mathbb{N}_0$ - **discrete time**, or
- $I = [0, \infty)$ - **continuous time**.

The random variables take values in a common **state space** S.

In other words, if $(X_t)_{t \in I}$ is a random process and Ω is a sample space for the random variables $(X_t)_t$, then

$$\forall t \in I \quad X_t : \Omega \to S.$$

Example of a stochastic process - a random walk

A random walker starts at the origin on the integer line. At each discrete unit of time the walker moves either right or left, with respective probabilities p and $1 - p$, independently of the past moves. This describes **a simple random walk in one dimension**.

An important point - the moves of the walker are assumed to be **independent**.

A basic probability refresher - part 2

Conditional probability of an event A given an event B (with $\mathbb{P}(B) > 0$ is

$$\mathbb{P}\left(A \middle| B\right) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Example 8

A fair die is rolled. What is the probability that the die shows a 6? What is the probability that the die shows a 6 if an even number appears on the die? What is the probability that the die shows a 6 if an odd number appears on the die?

Independent events

Two events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Events that are not independent are said to be **dependent**.
If $\mathbb{P}(B) > 0$ then the condition $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$ is equivalent to $\mathbb{P}\left(A \middle| B\right) = \mathbb{P}(A)$. In words, knowing that B occurs does not change the probability that A occurs.

Example 9

Roll two dice and let $A =$ "the first die is 4", $B_1 =$ "the second die is 2", $B_2 =$ "the sum of the two dice is 3", $B_3 =$ "the sum of the two dice is 9". Check independence of A and B_i, $i = 1, 2, 3$.

Independence of Discrete Random Variables

We say that **random variables** X and Y taking values in a subset S of integers, are **independent** if

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y) \quad \forall\, x, y \in S.$$

(The comma under $\mathbb{P}$ means that $X = x$ and $Y = y$ *simultaneously*.)
Random variables that are not independent are said to be **dependent**.

Example 10

We flip two symmetric coins. For $i = 1, 2$ let

$$X_i = \begin{cases} 1, & \text{if i-th coin comes up head,} \\ -1, & \text{if i-th coin comes up tail.} \end{cases}$$

Are X and Y independent?

Example of a stochastic process - a random walk - cont'd

A random walker starts at the origin on the integer line. At each discrete unit of time the walker moves either right or left, with respective probabilities p and $1-p$, independently of the past moves.

A formal description: let $X_1, X_2, \ldots$ - independent and identically distributed random variables with

$$X_k = \begin{cases} +1, & \text{with probability } p, \\ -1, & \text{with probability } 1-p, \end{cases}, \quad k \geq 1.$$

Set $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$ for $n \geq 1$. Then S_n is the random walks's position after n steps (a random variable).

In this example, $(S_n)_{n=0,1,2,\ldots}$ is a discrete time random process whose state space is $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Markov chains – first steps

A not very exciting game

Consider a game with a playing board consisting of squares numbered 1-10 arranged in a circle. A player starts at square 1. At each turn, the player rolls a die and moves around the board by the number of spaces shown on the face of the die.
Let X_k be the position of the player after k moves (with $X_0 = 1$).
For example, assume that the player successively rolls 2,1, and 4.
The first four positions are:

- $X_0 = 1$
- $X_1 = 3$
- $X_2 = 4$
- $X_3 = 8$.

Given this information, what can be said about the player's next location?

A not very exciting game - cont'd

Even though we know the player's full past history of moves, the only information relevant for predicting a future position is the most recent location X_3. Since $X_3 = 8$, then necessarily

$$X_4 \in \{9, 10, 1, 2, 3, 4\},$$

with equal probabilities, so

$$\mathbb{P}\left(X_4 = j \middle| X_0 = 1, X_1 = 3, X_2 = 4, X_3 = 8\right) = \mathbb{P}\left(X_4 = j \middle| X_3 = 8\right) = \frac{1}{6}$$

for $j = 9, 10, 1, 2, 3, 4$. Given the player's most recent location X_3, her/his future position X_4 is independent of past history X_0, X_1, X_2.

Markov chains

Definition 11

Let S be a discrete set. A **(discrete-time) Markov chain** with the state space S is a sequence of random variables $X_0, X_1, \ldots$, denoted as $(X_n)_{n\in\mathbb{N}_0}$, taking values in S such that for all $n = 0, 1, 2, \ldots$, and for all $x_0, x_1, \ldots, x_n, x_{n+1} \in S$

$$\mathbb{P}\left(X_{n+1} = x_{n+1} \middle| X_0 = x_0, \ldots, X_n = x_n\right) \\ = \mathbb{P}\left(X_{n+1} = x_{n+1} \middle| X_n = x_n\right).$$

This is the so called **Markov property** - for a Markov chain

the future, given the present, is independent of the past.

Time-homogeneity

Definition 12

A Markov chain $(X_n)_n$ is *time-homogeneous* if

$$\mathbb{P}\left(X_{n+1} = j \middle| X_n = i\right) = \mathbb{P}\left(X_1 = j \middle| X_0 = i\right)$$

for all $i, j \in S$ and $n = 0, 1, 2, \ldots$.

Unless stated otherwise,

the Markov chains in this course will be time-homogeneous.

It makes sense to define:

$$p_{ij} := \mathbb{P}\left(X_{n+1} = j \middle| X_n = i\right) = \mathbb{P}\left(X_1 = j \middle| X_0 = i\right)$$

- **one-step transition probability** of moving from state i to state j.

Example 13

As an example, we can consider the following random process with two states $S = \{1, 2\}$. We assume that for $n = 0, 1, 2, \ldots$

- $\mathbb{P}\left(X_{n+1} = 1 \middle| X_n = 1\right) = 0.2$,
- $\mathbb{P}\left(X_{n+1} = 2 \middle| X_n = 1\right) = 0.8$,
- $\mathbb{P}\left(X_{n+1} = 1 \middle| X_n = 2\right) = 0.5$,
- $\mathbb{P}\left(X_{n+1} = 2 \middle| X_n = 2\right) = 0.5$.