(8.2) $a_k = \left(\frac{1}{2}\right)^{k+1}, \quad k \geq 0$

$a = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$ $\quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$

$\underset{a_0}{\shortparallel} \quad \underset{a_1}{\shortparallel} \quad \underset{a_2}{\shortparallel}$

$G(s) = \frac{1}{2-s}$ $\quad$ extinct. prob. $s = G(s)$

$s = 1$

the extinction prob. is 1

T – time of extinction

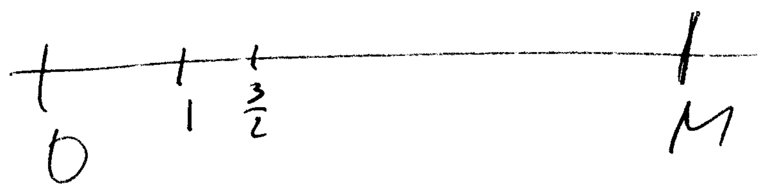
$\mathbb{P}(T = n) = \frac{1}{n(n+1)}, \quad n = 1, 2, \dots$

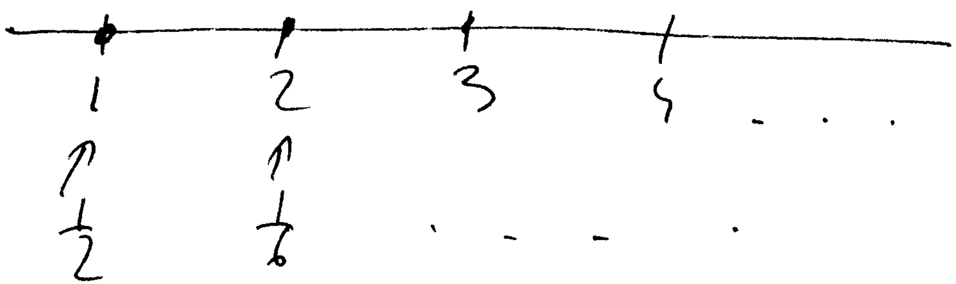
$\mathbb{E}T = \sum_{n=1}^{\infty} n \cdot \mathbb{P}(T = n) = \sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1}$

$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots = +\infty$

$M = 10^{100}$





$n \longrightarrow \frac{1}{n(n+1)}$

$ET = +\infty$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

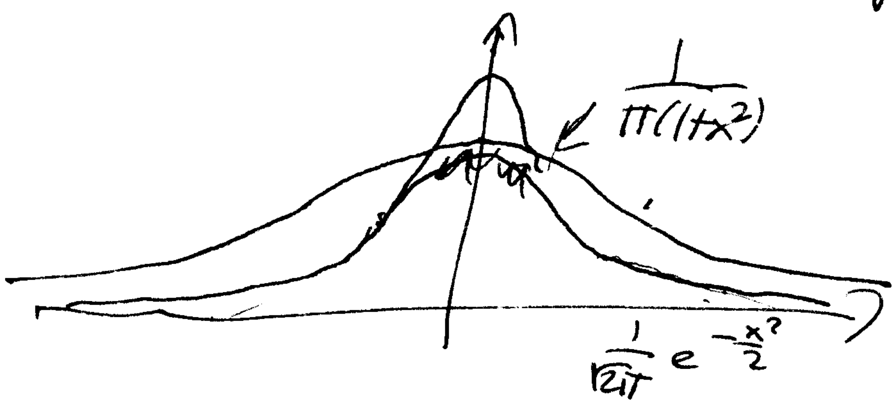
$$\sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = +\infty$$

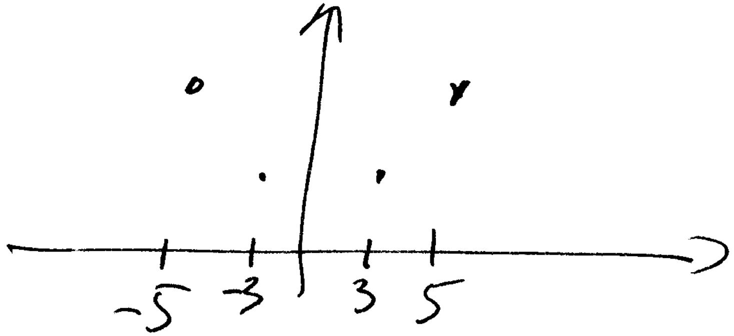
~~Cauchy~~ Cauchy distribution

$X \qquad f(x) = \frac{1}{\pi(1+x^2)}$ density

<u>even</u>

$f(x) = f(-x)$





$X \longrightarrow |X|$

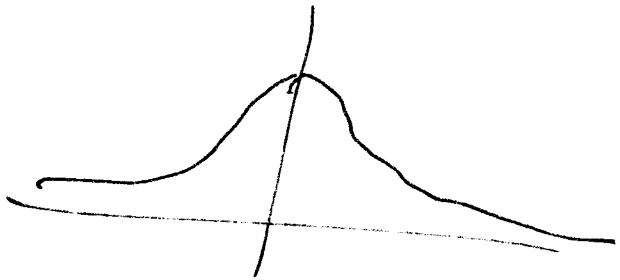
$$f(x) = \begin{cases} \frac{2}{\pi(1+x^2)}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

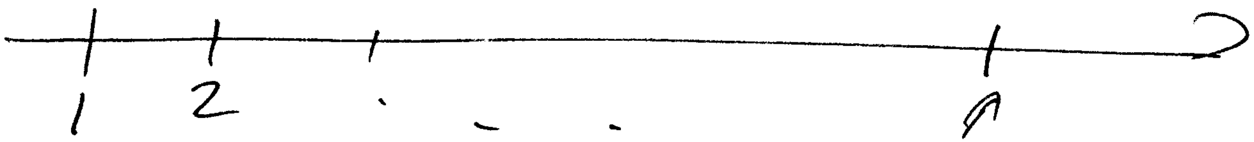
$\mathbb{E}|X| = +\infty$

$$\mathbb{E}|X| = \int_{\mathbb{R}} x \cdot f(x)\,dx = \int_{\mathbb{R}} x \cdot \frac{2}{\pi(1+x^2)}\,dx \approx \int \frac{1}{x} = +\infty$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$\mathbb{E}X = 0$





8.3 $a_0 = \frac{1}{3}$, $a_n = \frac{1}{2}\cdot\left(\frac{1}{4}\right)^{n-1}$, $n = 1, 2, \ldots$

$$\mu = \sum_{k=0}^{\infty} k\cdot a_k = \sum_{n=1}^{\infty} n\cdot\frac{1}{2}\cdot\left(\frac{1}{4}\right)^{n-1} = \frac{1}{2}\sum_{n=1}^{\infty} n\cdot\left(\frac{1}{4}\right)^{n-1}$$

$$= \frac{1}{2}\cdot\frac{16}{9} = \underline{\frac{8}{9}} < 1$$

subcritical

b) $G(s) = \frac{1}{3} + \sum_{n=1}^{\infty} s^n\cdot\frac{1}{2}\cdot\left(\frac{1}{4}\right)^{n-1} =$

$$= \frac{1}{3} + \frac{1}{2}s\sum_{n=1}^{\infty}\left(\frac{s}{4}\right)^{n-1} = \frac{1}{3} + \frac{1}{2}s\frac{1}{1-\boxed{\frac{s}{4}}} =$$

$$= \underbrace{\frac{1}{3}}\cdot 1 + \underbrace{\frac{2}{3}}\cdot\frac{3s}{4-s}$$

convex combination

$$\sum_{k=1}^{n} a_n b_n$$

$a_1, \ldots, a_n \geq 0$

$\sum_{k=1}^{n} a_k = 1$

G_n - the pgf of Z_n

$$G_1(s) = G(s) = \frac{1}{3} + \frac{2}{3}\cdot\frac{3s}{4-s}$$

$$G_2(s) = G(G(s)) = \frac{68+13s}{132-51s}$$

$$\boxed{\mathbb{P}(Z_2 = 1) = \frac{d}{ds}G_2(s)\Big|_{s=0}}$$

8.5

$a = (\frac{1}{2}, \frac{1}{2}) \Rightarrow G(s) = \frac{1}{2} + \frac{1}{2}s$ [5/5]

$\mathbb{P}(W_n = 0) = \mathbb{P}(W_n = 1) = \frac{1}{2} \Rightarrow H_n(s) = \frac{1}{2} + \frac{1}{2}s$

W_n	0	1
	$\frac{1}{2}$	$\frac{1}{2}$

$\to$ pgf

$G_n(s) = G_{n-1}(G(s)) \cdot H_n(s)$

$G_0(s) = s \Leftarrow$

$G_1(s) = \underbrace{G_0(G(s))} \cdot H_1(s) = G(s) \cdot H_1(s) = (\frac{1}{2} + \frac{1}{2}s)(\frac{1}{2} + \frac{1}{2}s)$

$= \frac{1}{4} + \frac{1}{2}s + \frac{1}{4}s^2$

$G_2(s) = G_1(G(s)) \cdot H_2(s) =$

$= \dots = \frac{9}{32} + \frac{15}{32}s + \frac{7}{32}s^2 + \frac{1}{32}s^3$

Z_2	0	1	2	3
	$\frac{9}{32}$	$\frac{15}{32}$	$\frac{7}{32}$	$\frac{1}{32}$

(b) $\mathbb{P}(T=2) = G_2(0) - G_1(0) = \frac{9}{32} - \frac{1}{4} = \frac{1}{32}$

prob. that the process goes extinct <u>in</u> the second generation

$\mathbb{P}(Z_2=0) = \frac{9}{32} \Leftarrow$ the prob. that the process goes extinct <u>by</u> the 2nd gen.