EDRP: Discrete Random Processes
Problem set 8

8.1 Consider a branching process $(Z_n)_n$ whose offspring distribution is $\mathbf{a} = (1/4, 3/4)$. Find:

(a) the pgf for the nth generation size Z_n,
(b) the distribution of Z_n,
(c) the distribution of the total number of individuals up through generation n,
(d) the distribution of the total progeny of $(Z_n)_n$,
(e) the expectation of the total progeny,
(f) the extinction probability,
(g) the distribution of the time of extinction,
(h) the mean time of extinction.

8.2 Consider the offspring distribution defined by $a_k = (1/2)^{k+1}$, for $k \geq 0$.

(a) Find the extinction probability.
(b) Show by induction that

$$G_n(s) = \frac{n - (n-1)s}{n + 1 - ns}.$$

(c) Find the distribution of the time of extinction.
(d) What is the expected value of the time of extinction?

8.3 Define an offspring distribution $\mathbf{a} = (a_0, a_1, a_2, \ldots)$, with

$$a_0 = \frac{1}{3}, \quad a_n = \frac{1}{2} \cdot \left(\frac{1}{4}\right)^{n-1}, \ n = 1, 2, 3, \ldots.$$

(a) Is the branching process with the offspring distribution $\mathbf{a}$ subcritical, critial, or supercritical?
(b) What is the probability that exactly one individual will be born in the second generation?
(c) What is the probability the process goes extinct by the second generation?

8.4 An infectious disease is spreading in the following way. At time 0, one person is infected. At each discrete unit of time, every person who has got infected "decides" how many people to infect according to the following mechanism. Each person flips a fair coin. If heads, they infect no one. If tails, they proceed to roll a fair die until 5 appears. The number of rolls needed determines how many people they will infect.

(a) After three generations, how many people, on average, have got infected?
(b) Find the probability that the infection-spreading process will stop after two generations.

(c) Find the probability that the infection-spreading process will eventually stop.

8.5 Consider a branching process with immigration, whose offspring distribution is

$$\mathbf{a} = (1/2, 1/2, 0, 0, \ldots).$$

Assume that a random number of immigrants W_n is independently added to the population at the n-th generation, with

$$\mathbb{P}(W_n = 0) = \mathbb{P}(W_n = 1) = \frac{1}{2}.$$

(a) Find the distribution of the number of individuals in the second generation.

(b) What is the probability the process goes extinct by the second generation?

8.6 Consider a branching process with immigration. Assume that the offspring distribution is

$$\mathbf{a} = (1-p, p, 0, \ldots),\ p \in (0,1),$$

and the immigration distribution is Poisson with parameter λ.

(a) Find the generating functions of the sizes of the first, second and third generation.

(b) Guess a formula for the pgf of the size of the n-th generation. Verify if it agrees with the recursion

$$G_{n+1}(s) = G_n\left(G(s)\right) H_n(s).$$

(c) Compute

$$\lim_{n\to\infty} G_n(s).$$

What can you conclude about the limiting distribution of generation size? How does it depend on the parameter p?

Answers

8.1 (a) $G_n(s) = 1 - (3/4)^n + s \cdot (3/4)^n$

(b) $\mathbb{P}(Z_n = 0) = 1 - (3/4)^n$, $\mathbb{P}(Z_n = 1) = (3/4)^n$

(c) $\mathbb{P}(T_n = 1) = 1/4$, $\mathbb{P}(T_n = 2) = (1/4) \cdot (3/4)$, $\mathbb{P}(T_n = 3) = (3/4)^2 \cdot (1/4), \ldots, \mathbb{P}(T_n = n) = (3/4)^{n-1} \cdot (1/4)$, $\mathbb{P}(T_n = n+1) = (3/4)^n$

(d) geometric distribution with parameter 1/4, that is $\mathbb{P}(T = n) = (3/4)^{n-1} \cdot (1/4), n = 1, 2, \ldots$

(e) 4

(f) 1

(g) geometric distribution with parameter 1/4

(h) 4

8.2 (a) 1

(b) -

(c) $\mathbb{P}(T=n)=\frac{1}{n(n+1)}, n=1,2,\ldots$

(d) $+\infty$

8.3 (a) subcritical

(b) 36/121

(c) 17/33

8.4 (a) $1+3+9+27=40$

(b) $\mathbb{P}(T=3)=3/154$

(c) 3/5

8.5 (a) $\mathbb{P}(Z_2=0)=9/32$, $\mathbb{P}(Z_2=1)=15/32$, $\mathbb{P}(Z_2=2)=7/32$, $\mathbb{P}(Z_2=3)=1/32$

(b) 9/32

8.6 (a)

$$\begin{aligned} G_1(s) &= (1-p+ps)\exp\left[-\lambda(1-s)\right] \\ G_2(s) &= (1-p^2+p^2s)\exp\left[-\lambda(1-s)(1+p)\right] \\ G_3(s) &= (1-p^3+p^3s)\exp\left[-\lambda(1-s)(1+p+p^2)\right] \end{aligned}$$

(b) $G_n(s)=(1-p^n+p^ns)\exp\left[-\lambda(1-s)(1+p+p^2+\ldots+p^{n-1})\right]$

(c)

$$\lim_{n\to\infty} G_n(s)=\exp\left[-\frac{\lambda(1-s)}{1-p}\right],$$

so the distribution of Z_n tends to a Poisson distribution with parameter $\lambda/(1-p)$