Discrete Random Processes (EDRP)

Lecture 11

Continuous-time Markov chains - cont'd

A quick review

Recall: if $(X_t)_{t\geq 0}$ is a continuous-time Markov chain on a discrete state space S, then for states $i, j \in S$, and $s, t \geq 0$,

- $p_i(t) = \mathbb{P}(X_t = i)$,
- $p_{i,j}(t) = \mathbb{P}\left(X_{t+s} = j \middle| X_s = i\right) = \mathbb{P}\left(X_t = j \middle| X_0 = i\right)$,
- $\mathbf{P}(t) = \left[p_{i,j}(t)\right]_{i,j}$
- the Chapman-Kolmogorov equation: $\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s)$,
- the holding time T_i at state i has an exponential distribution with some parameter q_i
- ignoring time, we get the embedded chain $(Y_n)_{n\in\mathbb{N}_0}$, where Y_n is the nth state visited by $(X_t)_{t\geq 0}$,
- the transition matrix for the embedded chain is $\tilde{\mathbf{P}} = \left[p_{i,j}\right]_{i,j}$

A quick review - cont'd

- the jump (or transition) rates: $q_{i,j} = p'_{i,j}(0)$
- the generator of $(X_t)_{t \geq 0}$ - a matrix $\mathbf{Q}$ such that
  - $(\mathbf{Q})_{i,j} = q_{i,j}$ for $i \neq j$,
  - $(\mathbf{Q})_{i,i} = -q_i$.

From the transition rates (= from the generator $\mathbf{Q}$), we can obtain the holding time parameters and the embedded chain transition probabilities:

- $\sum_{j \neq i} q_{i,j} = q_i$
- $p_{i,j} = q_{i,j}/q_i$, if $i \neq j$ ($p_{i,i} = 0$)

Example - the generator for a Poisson process with parameter λ

Another view on the dynamics of CTMC

There is a nice and useful intepretation of continuous-time Markov chain's dynamics. Central to this interpretation is the notion of the **exponential alarm clocks**.

- Imagine that for each state i, there are independent alarm clocks associated with each of the states that the process can visit after i.

- If j can be hit from i, then the alarm clock associated with (i, j) will ring after an exponentially distributed length of time with parameter $q_{i,j}$.

Another view on the dynamics of CTMC - cont'd

- When the process hits a state, say i, the clocks are started simultaneously.
- The first alarm that rings determines the next state to visit.
- If the (i, j) clock rings first, then the process immediately moves to j, and a new set of exponential alarm clocks are started, with transition rates $q_{j,1}$, $q_{j,2}, \ldots$
- Again, the first alarm that rings determines the next state hit, and so on.

Another view on the dynamics of CTMC - cont'd

Why does it make sens?

Consider the process started in $i \in S$. The clocks are started, and the first one that rings determines the next transition. The time of the first alarm is the minimum of independent exponential random variables with parameters $q_{i,1}$, $q_{i,2}, \dots$. Recall:

Theorem 1

Let $X_1, \dots, X_n$ be independent exponential random variables with respective parameters $\lambda_1, \dots, \lambda_n$. If $M = \min(X_1, \dots, X_n)$, then:

1. $M \sim Exp(\lambda_1 + \dots + \lambda_n)$,
2. *for* $k = 1, \dots, n$,

$$\mathbb{P}(M = X_k) = \frac{\lambda_k}{\lambda_1 + \dots + \lambda_n}.$$

Another view on the dynamics of CTMC - cont'd

Since the time of the first alarm is the minimum of independent exponential random variables with parameters $q_{i,1}$, $q_{i,2}, \ldots$, it has an exponential distribution with parameter

$$\sum_{k \neq i} q_{i,k}.$$

That is, the chain stays at i for an exponentially distributed amount of time with parameter $\sum_{k \neq i} q_{i,k}$. From the discussion of holding times, the exponential length of time that the process stays in i has parameter q_i . So,

$$q_i = \sum_{k \neq i} q_{i,k}.$$

The interpretation is that the rate that the process leaves state i is equal to the sum of the rates from i to each of the next states.

Another view on the dynamics of CTMC - cont'd

From $i \in S$, the chain moves to j if the (i, j) clock rings first, which occurs with probability

$$\frac{q_{i,j}}{\sum_{k \neq i} q_{i,k}} = \frac{q_{i,j}}{q_i}.$$

Thus, the embedded chain transition probabilities are

$$p_{i,j} = \frac{q_{i,j}}{\sum_{k \neq i} q_{i,k}} = \frac{q_{i,j}}{q_i}.$$

An example - students at the dean's office

A line of students is forming at the dean's office.

- It takes the office an exponentially distributed amount of time to service each student, at the rate of one student every 5 minutes.
- Students arrive at the office and get in line according to a Poisson process at the rate of one student every 4 minutes.
- Line size is capped at 4 people. If an arriving student finds that there are already 4 people in line, then they try again later.
- As soon as there is at least one person in line, the office starts assisting the first available student. The arrival times of the students are independent of the office's service time.

Students at the dean's office - cont'd

Let X_t be the number of students in line at time t. Then, $(X_t)_{t \geq 0}$ is a continuous-time Markov chain on $\{0, 1, 2, 3, 4\}$.

- If there is no one in line, then the size of the line increases to 1 when a student arrives.
- If there are 4 people in line, then the number decreases to 3 when the office finishes assisting the student they are meeting with.
- If there are 1, 2, or 3 students in line, then the line size can either decrease or increase by 1.
  - If a student arrives at the dean's office before the office has finished serving the student being helped, then the line increases by 1.
  - If the office finishes serving the student being helped before another student arrives, the line decreases by 1.

Students at the dean's office - cont'd

Imagine that when there is 1 person in line, two exponential alarm clocks are started:

- one for student arrivals, with rate $1/4$,
- the other for the offices service time, with rate $1/5$.

What happens next?

- If the arrival time clock rings first, the line increases by one.
- If the service clock rings first, the line decreases by one.
- The same dynamics hold if there are 2 or 3 people in line.

Students at the dean's office - cont'd

- What are the holding time parameters for $(X_t)_t$?
- Find the embedded chain transition matrix.
- Find the generator matrix for $(X_t)_t$.

Kolmogorov equations

The transition function $\mathbf{P}(t)$ can be computed from the generator $\mathbf{Q}$ by solving a system of differential equations.

Theorem 2

A continuous-time Markov chain with transition function $\mathbf{P}(t)$ *and infinitesimal generator* $\mathbf{Q}$ *satisfies the* **forward Kolmogorov equation**

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}.$$

Equivalently, for all states i and j,

$$p'_{i,j}(t) = \sum_k (\mathbf{P}(t))_{i,k}(\mathbf{Q})_{k,j} = -p_{i,j}(t)q_j + \sum_{k \neq j} p_{i,k}(t)q_{k,j}.$$

Kolmogorov equations - cont'd

Theorem 3

The transition function and the generator satisfy also the **backward Kolmogorov equation***:*

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t).$$

Equivalently, for all states i and j,

$$p'_{i,j}(t) = \sum_k (\mathbf{Q})_{i,k}(\mathbf{P}(t))_{k,j} = -q_i p_{i,j}(t) + \sum_{k \neq i} q_{i,k} p_{k,j}(t).$$

Example - Poisson process

Example – two-state process

Matrix Exponential

The Kolmogorov backward equation

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$$

is a matrix equation, which looks strikingly similar to the nonmatrix ordinary differential equation

$$p'(t) = qp(t),$$

where p is a differentiable function and q is a constant. If one looks for p such that

$$p(0) = 1,$$

the nonmatrix equation has the unique solution

$$p(t) = e^{qt}.$$

Matrix Exponential - cont'd

Remarkably, the solution of the matrix equation

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$$

looks exactly the same:

$$\mathbf{P}(t) = e^{\mathbf{Q}t}, \ \ t \geq 0,$$

(observe that $\mathbf{P}(0) = \mathbf{I}$). How is $\exp(\mathbf{Q}t)$ defined?

Definition 4

Let $\mathbf{A}$ be a $k \times k$ matrix. **The matrix exponential** $\exp(A)$ is the $k \times k$ matrix

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \dots .$$

Matrix Exponential - cont'd

- The matrix $\exp(\mathbf{A})$ is well-defined as its defining series converges for all square matrices $\mathbf{A}$.

- The matrix exponential is the matrix version of the exponential function and reduces to the ordinary exponential function when $\mathbf{A}$ is a 1×1 matrix.

- The matrix exponential satisfies many familiar properties of the exponential function:
  - $\exp(\mathbf{0}) = \mathbf{I}$,
  - $\exp(\mathbf{A}) \exp(-\mathbf{A}) = \mathbf{I}$,
  - if $\mathbf{AB} = \mathbf{BA}$ then
  $$\exp(\mathbf{A} + \mathbf{B}) = e^{\mathbf{A}} e^{\mathbf{B}} = e^{\mathbf{B}} e^{\mathbf{A}}.$$

Solution of Kolmogorov equations

Theorem 5

For a continuous-time Markov chain with transition function $\mathbf{P}(t)$ *and infinitesimal generator* $\mathbf{Q}$*,*

$$\mathbf{P}(t) = \exp(t\mathbf{Q}) = \sum_{n=0}^{\infty} \frac{(t\mathbf{Q})^n}{n!}.$$

Some remarks:

- The transition function is difficult to obtain in closed form for all but the most specialized models.

- For applied problems, numerical approximation methods are often needed.

- Computing the matrix exponential is often numerically challenging.

Computing matrix exponential

Usually, computing $\exp(\mathbf{A})$ is hard. However, there are some exceptions. For example, if $\mathbf{A}$ is a diagonal matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & a_n \end{bmatrix},$$

then

$$\mathbf{A}^n = \begin{bmatrix} a_1^n & 0 & \dots & 0 \\ 0 & a_2^n & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & a_k^n \end{bmatrix}.$$

Computing matrix exponential - cont'd

$$\exp(A) = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} a_1^n & 0 & \dots & 0 \\ 0 & a_2^n & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & a_k^n \end{bmatrix} =$$

$$\begin{bmatrix} \sum_{n=0}^{\infty} a_1^n/n! & 0 & \dots & 0 \\ 0 & \sum_{n=0}^{\infty} a_2^n/n! & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \sum_{n=0}^{\infty} a_k^n/n! \end{bmatrix} =$$

$$\begin{bmatrix} \exp(a_1) & 0 & \dots & 0 \\ 0 & \exp(a_2) & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \exp(a_k) \end{bmatrix}.$$

Matrix diagonalization

Another case when the computation of the matrix exponential goes smoothly is the one of a diagonalizable matrix.

If the generator matrix $\mathbf{Q}$ of a Markov chain on n states is **diagonalizable**, that is if it has n linearly independent eigenvectors, than it can be written as

$$\mathbf{Q} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1},$$

where

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

is a diagonal matrix whose diagonal entries are the eigenvalues of $\mathbf{Q}$, and $\mathbf{S}$ is an invertible matrix whose columns are the corresponding eigenvectors.

Matrix diagonalization - cont'd

If the $\mathbf{Q}$ matrix is diagonalizable, then

$$\exp(t\mathbf{Q}) = \sum_{n=0}^{\infty} \frac{(t\mathbf{Q})^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\mathbf{SDS}^{-1})^n = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{SD}^n\mathbf{S}^{-1} =$$

$$= \mathbf{S}\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{D}^n\right)\mathbf{S}^{-1} = \mathbf{S}e^{t\mathbf{D}}S^{-1}.$$

Example 1

Find $\mathbf{P}(t)$ via diagonalization for the two-state chain, with the generator

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}.$$

Long-term behavior

For continuous-time Markov chains, limiting and stationary distributions are defined similarly as for discrete time.

Definition 6

A probability distribution ν is a **limiting distribution** of a continuous-time Markov chain if for all states i and j,

$$\lim_{t \to \infty} \mathbf{P}_{i,j}(t) = \nu_j.$$

Long-term behavior - cont'd

Definition 7

A probability distribution π is a **stationary distribution** if

$$\pi = \pi \mathbf{P}(t), \quad \text{for } t \geq 0.$$

That is, for all states j,

$$\pi_j = \sum_i \pi_i p_{i,j}(t), \quad \text{for } t \geq 0.$$

Long-term behavior - cont'd

A continuous-time Markov chain is **irreducible** if for all i and j, $p_{i,j}(t) > 0$ for some $t > 0$. However, it turns out that if $p_{i,j}(t) > 0$ for some $t > 0$, then $p_{i,j}(t) > 0$ for all $t > 0$.

The following fundamental limit theorem has many analogies with the discrete-time results.

Theorem 8

Let $(X_t)_{t \geq 0}$ be a finite, irreducible, continuous-time Markov chain with transition function $\mathbf{P}(t)$. Then, there exists a unique stationary distribution π, which is the limiting distribution. That is, for all j,

$$\lim_{t \to \infty} p_{i,j}(t) = \pi_j \quad \textit{for all } i.$$

Example – the stationary distribution for the two-state process

Stationary distribution and generator

The following result links the stationary distribution with the generator.

Theorem 9

A probability distribution π is a stationary distribution of a continuous-time Markov chain with generator $\mathbf{Q}$ if and only if

$$\pi \mathbf{Q} = \mathbf{0}.$$

That is, if

$$\sum_i \pi_i \mathbf{Q}_{i,j} = 0 \quad \forall j.$$

Example 2

Use Theorem 9 to find the stationary distribution of the general two-state chain.

Stationary distribution - an interpretation

The stationary probability π_j can be interpreted as the long-term proportion of time that the chain spends in state j. This is analogous to the discrete-time case in which the stationary probability represents the long-term fraction of transitions that the chain visits a given state.

Example 3

What is the long-term proportion of time that the two-state chain spends in each of its states?

Stationary distribution - an example

Example 4

A machine is always in one of three states: A, B, and C. It stays in A on average for 30 minutes at a time; in B on average for 1 hour; and in C for about 3 hours. After being in A, there is a 50–50 chance the machine will enter B or C. After B, there is a 50–50 chance it will go to A or C. And after C, it always goes to B. The machine is governed by a continuous-time Markov chain. What proportion of the day does the machine stay in C?