(6.5) [116]

X_1	0	1	2
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

X_2	0	1	2
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$G_X(s) = \mathbb{P}(X=0) + s\cdot\mathbb{P}(X=1) + s^2\cdot\mathbb{P}(X=2) + \dots$$

$$\mathbb{P}(X_1 = a, X_2 = b) \overset{\text{indep.}}{=} \mathbb{P}(X_1 = a)\cdot\mathbb{P}(X_2 = b)$$

the distribution of $X_1 + X_2$

X_1+X_2	0	1	2	3	4
	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

$$\mathbb{P}(X_1+X_2=0) = \mathbb{P}(X_1=0, X_2=0) \overset{\text{ind.}}{=} \mathbb{P}(X_1=0)\cdot\mathbb{P}(X_2=0) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$$

$$\mathbb{P}(X_1+X_2=1) = \mathbb{P}(X_1=0, X_2=1) + \mathbb{P}(X_1=1, X_2=0) \overset{\text{ind}}{=} \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

$$\mathbb{P}(X_1+X_2=2) = \frac{3}{9}$$

$$\mathbb{P}(X_1+X_2=3) = \frac{2}{9}$$

$$\mathbb{P}(X_1+X_2=4) = \frac{1}{9}$$

$$G_{X_1}(s) = \frac{1}{3} + \frac{1}{3}\cdot s + \frac{1}{3}s^2 \Leftarrow \text{the pgf of } X_1$$

$$G_{X_2}(s) = \frac{1}{3} + \frac{1}{3}\cdot s + \frac{1}{3}s^2$$

X_1, X_2 – independent

$$G_{X_1+X_2}(s) = G_{X_1}(s)\cdot G_{X_2}(s) = \left(\frac{1}{3} + \frac{1}{3}s + \frac{1}{3}s^2\right)^2 = \left(\frac{1}{3}+\frac{1}{3}s\right)^2 + 2\left(\frac{1}{3}+\frac{1}{3}s\right)\cdot\frac{1}{3}s^2 + \frac{1}{9}s^4$$

$= \frac{1}{9} + \frac{2}{9}s + \frac{1}{9}s^2 + \frac{2}{9}s^2 + \frac{2}{9}s^3 + \frac{1}{9}s^4$

$= \underbrace{\frac{1}{9} + \frac{2}{9}s + \frac{3}{9}s^2 + \frac{2}{9}s^3 + \frac{1}{9}s^4}$ the pgf of $X_1 + X_2$

6.6

$X_1 \sim \text{Poiss}(\lambda_1)$

$X_2 \sim \text{Poiss}(\lambda_2)$

X_1, X_2 – <u>independent</u>

$\boxed{X \sim \text{Poiss}(\lambda) \quad \mathbb{P}(X=k) = \frac{\lambda^k}{k!}e^{-\lambda} \quad k = 0, 1, 2, \ldots}$

G_{X_1}, G_{X_2}

$G_{X_1}(s) = e^{\lambda_1(s-1)}$

$G_{X_2}(s) = e^{\lambda_2(s-1)}$

$G_{X}(s) = \sum_{k=0}^{\infty} s^k \cdot \mathbb{P}(X=k) =$

$= \sum_{k=0}^{\infty} s^k \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} =$

$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} = e^{-\lambda} \cdot e^{s\lambda}$

$= e^{\lambda(s-1)}$

$G_{X_1+X_2}(s) \overset{ind}{=} G_{X_1}(s) \cdot G_{X_2}(s)$

$= e^{\lambda_1(s-1)} \cdot e^{\lambda_2(s-1)}$

$= e^{(\lambda_1+\lambda_2)(s-1)}$

$\Rightarrow \boxed{X_1 + X_2 \sim \text{Poiss}(\lambda_1 + \lambda_2)}$

$\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \text{Taylor formula}}$

6.8 offspring distribution

$a = (1-p, 0, 0, p)$
a_0, a_1, a_2, a_3

$0 = a_4 = a_5 = a_6 = \dots$

the pgf of the of the offspring distr.

$$G(s) = \sum_{k=0}^{\infty} s^k \cdot a_k =$$
$$= (1-p) + p \cdot s^3$$

Y	0	1	2	3
	$1-p$	0	0	p

$\mu = 0 \cdot (1-p) + 1 \cdot 0 + 2 \cdot 0 + 3 \cdot p$
$\mu = \mathbb{E}Y$ $= \boxed{3p}$

Z_n – # of individuals born in the the n-th gen.

$\mathbb{E} Z_n = \mu^n$ μ – the mean of the offspring dist

σ^2 the variance of the offspring distr.

$$\operatorname{Var} Z_n = \begin{cases} \sigma^2 n, & \text{if } \mu = 1 \\ \sigma^2 \mu^{n-1} \cdot \frac{\mu^n - 1}{\mu - 1}, & \text{if } \mu \neq 1 \end{cases}$$

$\sigma^2 = \operatorname{Var} Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2$

$\mathbb{E} Z_4 = (3p)^4 = 81 p^4$

Y^2	0	1	4	9
	$1-p$	0	0	p

$$\operatorname{Var} Z_4 = \begin{cases} 2 \cdot 4, & p = \frac{1}{3} \\ \dots, & p \neq \frac{1}{3} \end{cases}$$

$\mathbb{E}Y^2 = 9p$

$\sigma^2 = 9p - (3p)^2 = 9p - 9p^2$
$= \boxed{9p(1-p)}$

(6.9) $a_0 = p\ ,\ a_1 = 1-p-q\ ,\ a_2 = q$ [4/6]

0	1	2
p	$1-p-q$	q

$\underbrace{\mu > 1}$

$$\mu = 0\cdot p + 1\cdot(1-p-q) + 2q$$
$$= 1-p-q+2q = 1 \cancel{q} - p + q$$

$\mu > 1$ $\quad 1-p+q > 1$

$$\boxed{q > p}$$

$$(a\cdot b)^2 = a^2 \cdot b^2$$

$$G(s) = p + (1-p-q)s + qs^2$$

$s = G(s)$

$$s = p + (1-p-q)s + q\cancel{*}s^2$$
$$\underbrace{qs^2 \cancel{*}(p+q)s + p = 0}$$

$a \neq 0$

$\longrightarrow ax^2 + bx + c = 0$

$\Delta = \underbrace{b^2 - 4ac}$

$$\Delta = (p+q)^2 - 4pq = p^2 + 2pq + q^2 - 4pq$$
$$= p^2 - 2pq + q^2 = (p-q)^2$$

$a = q$
$b = -(p+q)$
$c = p$

$\sqrt{\Delta} = \sqrt{(p-q)^2} =$

$b^2 - 4ac =$
$= (-(p+q))^2 - 4pq$
$= ((-1)\cdot(p+q))^2 - 4pq$

$\sqrt{\square}$ $\qquad \square \geq 0$

$\sqrt{(4-2)^2} = \sqrt{2^2} = 2$

$\sqrt{(2-4)^2} = \sqrt{4} = 2$

$(-2)^2 = 4$

$$\boxed{\sqrt{(p-q)^2} = p-q}$$

$p=2\ q=4$

$$\sqrt{A^2} = |A| = \begin{cases} A & , A \geq 0 \\ -A & , A < 0 \end{cases}$$

$$\sqrt{(p-q)^2} = |p-q| \Rightarrow$$

supercritical $q > p$.

$$\sqrt{\Delta} = q - p$$

$$a x^2 + bx + c = 0$$

$$x_1 = \frac{-b - \sqrt{\Delta}}{2a}$$

$$x_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

$$q s^2 - (p+q) s + p = 0$$

$$\sqrt{\Delta} = q - p$$

$$s_1 = \frac{p + q - (q-p)}{2q} = \frac{p}{q}$$

$$s_2 = \frac{p+q + (q-p)}{2q} = \underline{1}$$

$$\boxed{s = G(s)}$$

$s = 1 \leftarrow$ it is always a root

$$G(s) = \sum_{k=0}^{\infty} s^k \cdot a_k$$

$$1 = G(1) = \sum_{k=0}^{\infty} a_k$$

the extinction prob. $\frac{p}{q}$

[6/6]

$$\underbrace{q}_{a} s^2 \underbrace{-(p+q)}_{b} s + \underbrace{p}_{c} = 0$$

$\boxed{s=1 \text{ is a root}}$

$$\boxed{ax^2+bx+c=0}$$

$$\boxed{x_1+x_2=-\frac{b}{a}}$$

$$\boxed{x_1\cdot x_2=\frac{c}{a}}$$

<u>Viete</u> formulas

the second root is

$$\frac{c}{a}=\frac{p}{q}$$

6.10

~~after~~

$n \to 2$

$p \to 1-p$

$$a_0=\binom{2}{0}(1-p)^0 \cdot p^{2-0} = p^2$$

$$a_1=\binom{2}{1}(1-p)^1 \cdot p^{2-1} = 2\cdot(1-p)p$$

$$a_2=\binom{2}{2}(1-p)^2 \cdot p^{2-2} = (1-p)^2$$

$$1 \overset{?}{=} a_0+a_1+a_2 = p^2+2(1-p)p+(1-p)^2 = [p+(1-p)]^2 = 1^2 = 1$$

$X \sim bin(n,p)$

$$\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$$

$k=0,\dots,n$

$$a_0 = p^2$$
$$a_1 = 2(1-p)p$$
$$a_2 = (1-p)^2$$

$$\Rightarrow \quad G(s) = p^2 + 2(1-p)ps + (1-p)^2 s^2$$

$$G(s) = s$$