Discrete Random Processes (EDRP)

Lecture 8

Branching processes - cont'd

Branching processes - a quick review

- i-th individual in a population produces a random number X_i of children according to the offspring distribution

$$\mathbf{a} = (a_0, a_1, \ldots).$$

- assumption: $X_1, X_2, \ldots$ - an i.i.d. sequence (with common distribution $\mathbf{a}$)
- Z_n - the size of the n-th generation (assumption: $Z_0 = 1$)
- a branching process - the sequence $(Z_n)_{n \in \mathbb{N}_0}$
- $Z_n = \sum_{i=1}^{Z_{n-1}} X_i$

Review - cont'd

- μ - the mean of the offspring distribution (=the average number of children produced by any individual),

$$\mu = \sum_k k a_k$$

- the mean size of the nth generation is $\mathbb{E} Z_n = \mu^n$, $n \in \mathbb{N}_0$
- $(Z_n)_n$ is
  - *subcritical* if $\mu < 1$,
  - *critical* if $\mu = 1$,
  - and *supercritical* if $\mu > 1$,
- if subcritical, then $\lim_n \mathbb{E} Z_n = 0$; if critical, then $\lim_n \mathbb{E} Z_n = 1$; if supercritical, then $\lim_n \mathbb{E} Z_n = +\infty$

Branching processes and probability generation functions

Let

$$G(s) = \sum_{k=0}^{\infty} s^k a_k$$

be the probability generating function of the offspring distribution.

Theorem 1

The probability of eventual extinction of a branching process is the smallest positive root of the equation

$$s = G(s).$$

Clearly, since the value of any pgf at 1 is 1, $s = 1$ satisfies the equation, but there could well be other solutions.

Some consequences of the theorem

Corollary 2

In the subcritical $\mu < 1$ case, the population goes extinct with probability 1.

Some consequences of the theorem - cont'd

Corollary 3

The population goes extinct with probability 1 *also for* $\mu = 1$, *even though for each generation the expected generation size is* $\mathbb{E}(Z_n) = \mu^n = 1$.

Some consequences of the theorem - cont'd

Corollary 4

For the supercritical case $\mu > 1$, the expected generation size Z_n grows without bound. However, the theorem gives that even in this case there is a positive probability of eventual extinction.

Example 1

Find the extinction probability for a branching process with offspring distribution $\mathbf{a} = (1 - p, 0, p)$*, where* $p \in (0, 1)$*.*

Probability of extinction revisited

The probability of eventual extinction of a branching process is the smallest positive root of the equation

$$s = G(s),$$

where G is the pgf of the offspring distribution.

The equation is the basis of a numerical, recursive method to approximate the extinction probability in the supercritical case.

Algorithm

To find the smallest root:

1. Initialize with $s_0 \in (0, 1)$.
2. Successively compute $s_n := G(s_{n-1})$ for $n \geq 1$.
3. Set $s = s_n$ for large n.

Example 2

Analyze the algorithm for

$$G(s) = \frac{1}{2} + \frac{1}{2}s.$$

Example 3

Analyze the algorithm for

$$G(s) = 1 - p + ps^2, \quad p \in (0, 1).$$

Convergence can be shown to be exponentially fast, so that n can often be taken to be relatively small (for example $n \approx 10-20$).

Example 4

Let $\mathbf{a} = (0.1, 0.1, 0.8, 0, 0, \ldots)$. *If* $s_0 = 0.5$ *then the first 11 values of* s_n *are:*

n	1	2	3	4	5	6
s_n	0.233	0.1667312	0.1389126	0.1293286	0.1263136	0.1253955

n	7	8	9	10	11
s_n	0.1251188	0.1250356	0.1250107	0.1250032	0.125001

Then it follows $s_{12} = 0.1250003$, $s_{13} = 0.1250001$ *and, starting from* s_{14}, *all the remaining* s_n *equal* $0.125 = 1/8$.

Example 5

If $\mathbf{a} = (0.1, 0.1, 0.8)$ *(again), and if* $s_0 = \mathbf{0.99}$*, then the first 22 values* $s_1, \ldots, s_{22}$ *generated by the algorithm, are:*

1	$s = 0.971465$	**12**	$s = 0.1308717$
2	$s = 0.9521419$	**13**	$s = 0.1267891$
3	$s = 0.9204736$	**14**	$s = 0.1255393$
4	$s = 0.8698647$	**15**	$s = 0.125162$
5	$s = 0.7923182$	**15**	$s = 0.1250486$
6	$s = 0.6814463$	**17**	$s = 0.1250146$
7	$s = 0.5396399$	**18**	$s = 0.1250044$
8	$s = 0.3869329$	**19**	$s = 0.1250013$
9	$s = 0.258467$	**20**	$s = 0.1250004$
10	$s = 0.1792908$	**21**	$s = 0.1250001$
11	$s = 0.1436452$	**22**	$s = 0.125$

Example 6

If $\mathbf{a} = (0.2, 0.2, 0.2, 0.2, 0.2)$ *and* $s_0 = 0.01$*, then the first 20 values* $s_1, \ldots, s_{20}$ *generated by the algorithm, are:*

1	$s = 0.2505486$	**11**	$s = 0.275681$
2	$s = 0.2665984$	**12**	$s = 0.2756817$
3	$s = 0.2723346$	**13**	$s = 0.275682$
4	$s = 0.2744399$	**14**	$s = 0.2756821$
5	$s = 0.27522$	**15**	$s = 0.2756822$
6	$s = 0.2755101$	**16**	$s = 0.2756822$
7	$s = 0.2756181$	**17**	$s = 0.2756822$
8	$s = 0.2756583$	**18**	$s = 0.2756822$
9	$s = 0.2756733$	**19**	$s = 0.2756822$
10	$s = 0.2756789$	**20**	$s = 0.2756822$

Size of the n-th generation

Consider a branching process $(Z_n)_{n\in\mathbb{N}_0}$ with offspring distribution

$$\mathbf{a} = (a_0, a_1, a_2, \dots).$$

Recall that by G we denote the probability generating function of $\mathbf{a}$, that is

$$G(s) = \sum_{k=0}^{\infty} s^k a_k.$$

For $n \geq 0$, let

$$G_n(s) = \mathbb{E}\left(s^{Z_n}\right) = \sum_{k=0}^{\infty} s^k \mathbb{P}(Z_n = k)$$

be the probability generating function of the nth generation size Z_n.

Size of the n-th generation - cont'd

Theorem 5

The pgf G_n of the nth generation size Z_n satisfies

$$G_n(s) = G_{n-1}\left(G(s)\right), \quad n \geq 1.$$

So the pgf of Z_n is the composition of the pgf of Z_{n-1} and the pgf of the offspring distribution.
Notice that

- since $Z_0 = 1$ (a deterministic start of the population), we have

$$G_0(s) = s,$$

- further,

$$G_1(s) = G_0\left(G(s)\right) = G(s)$$

  (this confirms that the distribution of Z_1 is the offspring distribution $\mathbf{a}$).

Size of the n-th generation - cont'd

In general, the pgf od Z_n is the n-fold composition of the offspring generating function:

$$
\begin{array}{rcl}
G_2(s) & = & G_1(G(s)) = G(G(s)), \\
G_3(s) & = & G_2(G(s)) = G(G(G(s))), \\
\dots & \dots & \dots \quad \dots \quad \dots \quad \dots, \\
G_n(s) & = & G_{n-1}(G(s)) = \underbrace{G(\dots G(G(s))\dots)}_{n-\text{fold}}.
\end{array}
$$

Size of the n-th generation - cont'd

Thus

$$G_n(s) = \underbrace{G(\dots G(G(s))\dots)}_{n-\text{fold}}.$$

How can this be useful?

- Knowing G_n, one can recover the distribution of the n-th generation size.
- From the recursive formula for G_n, one can quickly justify the fact that

$$\mathbb{E}Z_n = \mu^n.$$

Size of the n-th generation - cont'd

Example 7

Assume that $a_k = 1$ for some $k \in \mathbb{N}_0$ (that is, the number of children born to the ith individual is k). Find the distribution of the n-th generation size.

Size of the n-th generation - cont'd

Example 8

Assume that $\mathbf{a} = (1/2, 1/2, 0, \ldots)$. *Find the distribution of* Z_n.