Markov chains for the long term

Large powers of some stochastic matrices

Example 1

If

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix},$$

then

$$\mathbf{P}^2 = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix},$$

$$\mathbf{P}^4 = \begin{bmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332, \end{bmatrix},$$

and

$$\mathbf{P}^8 = \begin{bmatrix} 0.571457 & 0.428543 \\ 0.571391 & 0.428609. \end{bmatrix}$$

Large powers of some stochastic matrices - cont'd

Example 2

$$\mathbf{P} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}^5 = \begin{bmatrix} 0.440526 & 0.559474 \\ 0.372983 & 0.627017 \end{bmatrix}$$

$$\mathbf{P}^{10} = \begin{bmatrix} 0.402737 & 0.597263 \\ 0.398175 & 0.601825 \end{bmatrix}$$

$$\mathbf{P}^{20} = \begin{bmatrix} 0.400012 & 0.599988 \\ 0.399992 & 0.600008 \end{bmatrix}$$

$$\mathbf{P}^{30} = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}$$

Yet another example

Example 3

$$\mathbf{P} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{2}{3} & \frac{5}{24} \\ 0 & \frac{1}{6} & \frac{5}{6} \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}^5 = \begin{bmatrix} 0.362766 & 0.403402 & 0.233832 \\ 0.201701 & 0.384187 & 0.414112 \\ 0.093533 & 0.33129 & 0.575177 \end{bmatrix}$$

$$\mathbf{P}^{20} = \begin{bmatrix} 0.186603 & 0.365029 & 0.448369 \\ 0.182514 & 0.363839 & 0.453647 \\ 0.179347 & 0.362917 & 0.457735 \end{bmatrix}$$

$$\mathbf{P}^{70} = \begin{bmatrix} 0.181818 & 0.363636 & 0.454545 \\ 0.181818 & 0.363636 & 0.454545 \\ 0.181818 & 0.363636 & 0.454545 \end{bmatrix}$$

What does it mean for a Markov chain with transition matrix $\mathbf{P}$ and $\#S = N$ if there exists a limit of the form

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \end{bmatrix},$$

with $\lambda_i \geq 0$ and $\sum \lambda_i = 1$?

- For any initial distribution α,

$$\lim_{n\to\infty} \alpha\mathbf{P}^n = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots \lambda_N \end{bmatrix}.$$

- For any initial distribution, and for all j,

$$\lim_{n\to\infty} \mathbb{P}(X_n = j) = \lambda_j.$$

Limiting distribution

Thus we have seen that, in many cases, a Markov chain exhibits a long-term limiting behavior. The chain settles down to an equilibrium distribution, which is independent of its initial state.

Definition 1

Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with transition matrix $\mathbf{P}$.
A **limiting distribution** for the Markov chain is a probability distribution λ such that for all i and j

$$\lim_{n\to\infty}(\mathbf{P}^n)_{ij} = \lambda_j.$$

So, if $(X_n)_n$ has a limiting distribution $\lambda = (\lambda_j)_{j \in S}$, then ...

- ... if Λ is a stochastic matrix all of whose rows are equal to λ, then

$$\lim_{n \to \infty} \mathbf{P}^n = \Lambda.$$

- ... for any initial distribution α,

$$\lim_{n \to \infty} \alpha \mathbf{P}^n = \lambda.$$

- ... for any initial distribution, and for all j,

$$\lim_{n \to \infty} \mathbb{P}(X_n = j) = \lambda_j.$$

An important point – the limiting distribution does not always exist:

Example 4

If

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

then for large n

$$\mathbf{P}^{2n} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}, \qquad \mathbf{P}^{2n+1} = \begin{bmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{bmatrix}.$$

Some examples

Example 5

Let $(X_n)_{n\in\mathbb{N}_0}$ *be a Markov chain with transition matrix* $\mathbf{P}$*. Find, if it exists, a limiting distribution for* $(X_n)_{n\in\mathbb{N}_0}$*.*

1. $$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. $$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

Proportion of Time in Each State

The limiting distribution gives the long-term probability that a Markov chain hits each state.

It can also be interpreted as the long-term proportion of time that the chain visits each state.

To make this precise, let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with transition matrix $\mathbf{P}$. Assume that there exists limiting distribution λ for $(X_n)_{n\in\mathbb{N}_0}$.

Proportion of Time in Each State - cont'd

For $j \in S$ let

$$I_n = \begin{cases} 1, & \text{if } X_n = j \\ 0, & \text{otherwise.} \end{cases}$$

Then

- $\sum_{k=0}^{n-1} I_k$ is the number of visits to j in the first n steps (counting X_0 as the first step),
- $\frac{1}{n}\sum_{k=0}^{n-1} I_k$ is the proportion of time that chain visits j,
- $\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{k=0}^{n-1} I_k \,\middle|\, X_0 = i\right]$ is the long-term expected proportion of time that the chain (started in i) visits j.

Proportion of Time in Each State - cont'd

One can prove that for all $i, j \in S$,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n} \sum_{k=0}^{n-1} I_k \,\middle|\, X_0 = i \right] = \lambda_j.$$

Observe, in particular, that the long-term expected proportion of time that the chain (started in i) visits j, does not depend on the starting point.

Stationary distribution

Let's see what happens if we assign the limiting distribution of a Markov chain to be its initial distribution.

Example 6

For the two-state chain with

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix},$$

the limiting distribution can be shown (see Problem Sets) to be

$$\lambda = \left(\frac{q}{p+q}, \frac{p}{p+q} \right).$$

Find the distribution of X_1 if λ is the initial distribution.

Stationary distribution cont'd

A probability vector π that satisfies $\pi \mathbf{P} = \pi$ plays a special role in the theory of Markov chains.

Definition 2

Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with transition matrix $\mathbf{P}$.
A **stationary distribution** is a probability distribution π, which satisfies

$$\pi \mathbf{P} = \pi.$$

That is,

$$\pi_j = \sum_{i\in S} \pi_i P_{ij} \quad \forall j \in S.$$

Stationary distribution cont'd

The name **stationary** (other names for the stationary distribution are **invariant**, **steady-state**, and **equilibrium** distribution) comes from the fact that if the chain starts in its stationary distribution, then it stays in that distribution.

More precisely, if we assume that a stationary distribution π is the initial distribution, then $(X_n)_{n\in\mathbb{N}_0}$ is a sequence of identically distributed random variables.

A connection between the stationary and the limiting distributions

Theorem 3

Assume that π is the limiting distribution of a Markov chain with transition matrix $\mathbf{P}$. Then π is a stationary distribution.

The converse is not true - stationary distributions are not necessarily limiting distributions.

Example 7

- $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Finding the stationary distribution

Thus, there are Markov chains ...

- ... with more than one stationary distribution,
- ... with unique stationary distributions that are not limiting distributions,
- ... that do not have stationary distributions (this can happen only if the state space is infinite).

If one wants to find the stationary distribution, then the condition $\pi\mathbf{P} = \pi$ gives a system of linear equations. If $\mathbf{P}$ is a $k \times k$ matrix, the system has k equations and k unknowns. Since the rows of $\mathbf{P}$ sum to 1, the system will contain a redundant equation.

Finding the stationary distribution - cont'd

Example 8

Find, if they exist, the stationary distributions of the general two-state chain with

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}.$$

Finding the stationary distribution - cont'd

Example 9

Find, if they exist, the stationary distributions if

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

There are three natural questions to ask about invariant probability distributions for stochastic matrices:

- Does every Markov chain have a stationary distribution π?
- Is the stationary distribution *unique*?
- When can we conclude that

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix},$$

  and hence that for all initial distributions α

$$\lim_{n\to\infty} \alpha\mathbf{P}^n = \pi?$$

There is a large and important class of Markov chains with unique stationary distributions that are the limiting distribution of the chain.

Definition 4

- A matrix **M** is said to be **positive** if all the entries of **M** are positive. We write $\mathbf{M} > 0$. Similarly, write $x > 0$ for a vector x with all positive entries.

- A transition matrix **P** is said to be **regular** if some power of **P** is positive. That is, $\mathbf{P}^n > 0$ for some $n \geq 1$.

Example 10

If

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix},$$

then

$$\mathbf{P}^4 = \begin{bmatrix} 9/16 & 5/16 & 1/8 \\ 1/4 & 3/8 & 3/8 \\ 1/2 & 5/16 & 3/16 \end{bmatrix},$$

so $\mathbf{P}$ *is regular.*

Example 11

Is

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

a regular matrix?

Limit theorem for regular Markov chains

Theorem 5

A Markov chain whose transition matrix $\mathbf{P}$ *is regular has a limiting distribution, which is the unique, positive, stationary distribution of the chain.*