EDRP: Discrete Random Processes
Problem set 6

6.1 Find the pgf of a random variable X taking values $\{0, 1, 2, 5\}$ with $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = \mathbb{P}(X = 2)$ and $\mathbb{P}(X = 5) = 5/8$.

6.2 Let G be the pgf of X from the previous exercise. Compute $G^{(j)}(0)/j!$ for $j = 0, 1, 2, \ldots$.

6.3 Find the distribution of a random variable X if its pgf is $G_X(s) = 1 - p + ps$ and p is a parameter from $(0, 1)$.

6.4 Find the probability generating function of a Poisson random variable with parameter λ. Use the pgf to find the mean and variance of the Poisson distribution.

6.5 Suppose X_1 and X_2 are independent random variables uniformly distributed on $\{0, 1, 2\}$. Find the distribution of $X_1 + X_2$. Compute the pgfs of X_1, X_2, and $X_1 + X_2$.

6.6 Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$. Assume that X and Y are independent. Use pgfs to find the distribution of $X + Y$.

6.7 Consider a branching process with offspring distribution $\mathbf{a} = (a, b, c)$, where $a, b, c \geq 0$, and $a + b + c = 1$. Let $\mathbf{P}$ be the Markov transition matrix. Exhibit the first three rows of $\mathbf{P}$.

6.8 Give the probability generating function for an offspring distribution in which an individual either dies, with probability $1-p$, or gives birth to three children, with probability p. Find the mean and variance of the number of children in the fourth generation.

6.9 A branching process has offspring distribution with $a_0 = p$, $a_1 = 1 - p - q$, and $a_2 = q$. For what values of p and q is the process supercritical? In the supercritical case, find the extinction probability.

6.10 Consider a branching process with binomial (with parameters 2 and $1 - p$, where $0 < p < 1$) offspring distribution. Find the extinction probability.

6.11 Assume that the offspring distribution is uniform on $\{0, 1, 2, 3, 4\}$. Find the extinction probability.

6.12 Consider the offspring distribution defined by $a_k = (1/2)^{k+1}$, for $k \geq 0$. Find the extinction probability.

Answers

6.1 $G(s) = \frac{1}{8} + \frac{1}{8}s + \frac{1}{8}s^2 + \frac{5}{8}s^5$

6.2

$$\frac{G^{(j)}(0)}{j!} = \begin{cases} \frac{1}{8}, & j = 0, 1, 2 \\ \frac{5}{8}, & j = 5, \\ 0, & j \notin \{0, 1, 2, 5\} \end{cases}$$

6.3 $\mathbb{P}(X=0)=1-p$, $\mathbb{P}(X=1)=p$

6.4 $G_X(s)=\exp\left[-\lambda(1-s)\right]$, $\mathbb{E}X=G'_X(1)=\lambda$, $\operatorname{Var}X=G''_X(1)+G'_X(1)-\left(G'_X(1)\right)^2=\lambda$

6.5
- $\mathbb{P}(X_1+X_2=0)=\mathbb{P}(X_1+X_2=4)=1/9$, $\mathbb{P}(X_1+X_2=1)=\mathbb{P}(X_1+X_2=3)=2/9$, $\mathbb{P}(X_1+X_2=2)=3/9$,
- $G_{X_1}(s)=G_{X_2}(s)=\frac{1}{3}+\frac{1}{3}s+\frac{1}{3}s^2$
- $G_{X_1+X_2}(s)=G_{X_1}(s)G_{X_2}(s)=\frac{1}{9}\left(1+2s+3s^2+2s^3+s^4\right)$

6.6 $G_{X+Y}(s)=\exp\left[-(\lambda_1+\lambda_2)(1-s)\right]\Rightarrow X+Y\sim\operatorname{Poiss}(\lambda_1+\lambda_2)$

6.7
- 1st row: $1,0,0,\dots$
- 2nd row: $a,b,c,0,\dots$,
- 3rd row: $a^2,2ab,2ac+b^2,2bc,c^2,0,0,\dots$

6.8
- $G(s)=1-p+ps^3$,
- $\mathbb{E}Z_4=81p^4$,
- $$\operatorname{Var}Z_4=\begin{cases}8 & ,p=\frac{1}{3}\\ 9p(1-p)(3p)^3\frac{(3p)^4-1}{3p-1} & ,p\neq\frac{1}{3}\end{cases}$$

6.9 supercritical if $q>p$, extinction probability p/q

6.10 $\left(\frac{1-p}{p}\right)^2$

6.11 0.275682 (obtained numerically)

6.12 1