EDRP: Discrete Random Processes
Problem set 4

4.1 Let $X \sim \mathrm{Exp}(\lambda)$, where $\lambda > 0$, and $s, t > 0$. Show that

$$\mathbb{P}\left(X > t + s \big| X > t\right) = e^{-\lambda s}.$$

Deduce the lack of memory property of the exponential distribution.

Hint: observe that $\mathbb{P}(X > t + s, X > t) = \mathbb{P}(X > t + s)$*. Recall a formula for* $\mathbb{P}(X > t)$ *from the lectures.*

4.2 Suppose $X \sim \mathrm{Exp}(2)$. Compute $\mathbb{P}(X \in [-4, -2])$, $\mathbb{P}(X > 2)$, $\mathbb{P}(X \leq 1)$, $\mathbb{P}(X > 5|X > 1)$.

4.3 Let $X \sim \mathrm{Poisson}(1)$. Compute $\mathbb{P}(X < 3)$, $\mathbb{P}(X \leq 2)$, $\mathbb{P}(X > 0)$, $\mathbb{P}(X \geq 0)$, $\mathbb{P}(X \geq 1)$, $\mathbb{P}(X \in (2, 5))$.

4.4 Let $(N_t)_t$ be a Poisson process with parameter $\lambda = 1.5$. Compute:

(a) $\mathbb{P}(N_1 = 2, N_4 = 6)$,
(b) $\mathbb{P}(N_5 \leq 1, N_6 = 2)$,
(c) $\mathbb{P}(N_5 \geq 3, N_6 = 2)$,
(d) $\mathbb{P}\left(N_4 = 6 \big| N_1 = 2\right)$,
(e) $\mathbb{P}\left(N_1 = 2 \big| N_4 = 6\right)$.

4.5 Let $(N_t)_t$ be a Poisson process with parameter $\lambda = 2$. By τ_k denote the time of the k-th arrival ($k = 1, 2, \ldots$), and by $\rho_k = \tau_k - \tau_{k-1}$ - the interarrival time between the $(k-1)$th and kth arrival ($k = 1, 2, \ldots$), with $\tau_0 = 0$ (as in the construction of Poisson process).

Find the following:

(a) $\mathbb{E}(N_3 N_4)$
(b) $\mathbb{E}(\rho_3 \rho_4)$
(c) $\mathbb{E}(\tau_3 \tau_4)$

4.6 Calls are received at a company call center according to a Poisson process at the rate of five calls per minute.

(a) Find the probability that no call occurs over a 30-second period.
(b) Find the probability that exactly four calls occur in the first minute, and six calls occur in the second minute.
(c) Find the probability that 25 calls are received in the first 5 minutes and six of those calls occur in the first minute.

4.7 Starting at 9 a.m., customers arrive at a shop according to a Poisson process. On average, three customers arrive every hour.

(a) Find the probability that at least two customers arrive by 9:30 a.m.

(b) Find the probability that 10 customers arrive by noon and eight of them come to the shop before 11 a.m.

(c) If six customers arrive by 10 a.m., find the probability that only one customer arrives by 9:15 a.m.

Answers

4.1 -

4.2 $\mathbb{P}(X \in [-4, -2]) = 0$, $\mathbb{P}(X > 2) = \exp(-4)$, $\mathbb{P}(X \leq 1) = 1 - \exp(-2)$, $\mathbb{P}\left(X > 5 \middle| X > 1\right) = \exp(-8)$

4.3 $\mathbb{P}(X < 3) = \frac{5}{2}\exp(-1)$, $\mathbb{P}(X \leq 2) = \frac{5}{2}\exp(-1)$, $\mathbb{P}(X > 0) = 1 - \exp(-1)$, $\mathbb{P}(X \geq 0) = 1$, $\mathbb{P}(X \in (2, 5)) = \frac{5}{24}\exp(-1)$

4.4 (a) $\frac{(1.5)^2}{2!}\exp(-1.5) \cdot \frac{(4.5)^4}{4!}\exp(-4.5)$

(b) $\frac{9}{8}\exp(-9) + \frac{45}{4}\exp(-9)$

(c) 0

(d) $\frac{(4.5)^4}{4!}\exp(-4.5)$

(e) $\frac{5 \cdot 3^5}{4^6}$

4.5 (a) 54

(b) 1/4

(c) 3

4.6 (a) $\exp(-5/2)$

(b) $\frac{5^{10}}{4!6!}\exp(-10)$

(c) $\frac{5^{25}4^{19}}{6!19!}\exp(-25)$

4.7 (a) $1 - \frac{5}{2}\exp(-3/2)$

(b) $\frac{6^8 3^2}{8!2!}\exp(-9)$

(c) $\frac{729}{2048}$