EDRP: Discrete Random Processes
Problem set 9

9.1 A continuous-time Markov chain has generator matrix

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 2 & 2 & -4 \end{bmatrix}.$$

Exhibit the transition matrix of the embedded chain and the holding time parameter for each state.

9.2 A Markov chain on $\{1, 2, 3, 4\}$ has nonzero transition rates

$$q_{12} = q_{23} = q_{31} = q_{41} = 1, \; q_{14} = q_{32} = q_{34} = q_{43} = 2.$$

(a) Exhibit the generator of the Markov chain.
(b) What are the holding time parameters?
(c) Find the transition matrix for the embedded Markov chain.

9.3 A three-state Markov chain has distinct holding time parameters q_a, q_b, and q_c. From each state, the process is equally likely to transition to the other two states. Exhibit the generator matrix.

9.4 A machine is subject to failures of types $i = 1, 2, 3$, at rates $\lambda_1 = 1/2$, $\lambda_2 = 1/3$, and $\lambda_3 = 1/4$. A failure of type i takes an exponential amount of time to repair, with rate $\mu_1 = 1/4$, $\mu_2 = 1/3$, and $\mu_3 = 1/2$. Let X_t denote the type of failure the machine is subject to at time t. Then $(X_t)_t$ is a continuous-time Markov chain with state space $\{0, 1, 2, 3\}$ (with 0 denoting no failure of any type). Assume that only one type of failure can occur at a time, and that after repair, the machine goes to state 0 before entering any other state.

(a) Find the generator of the Markov chain.
(b) Find the transition matrix for the embedded Markov chain.

9.5 A small art gallery has room to display up to three large oil paintings for sale. Customers come at times of a Poisson process with rate 2 per week to buy a painting and will buy one if at least one is available. When the gallery has only one painting left, it places an order for two more paintings. The order takes an exponentially distributed amount of time with mean one week to arrive. Of course, while the store is waiting for delivery, sales may reduce the inventory to one and then to 0. Let X_t denote the number of paintings displayed in the gallery in time t. Find the generator of the Markov chain $(X_t)_t$.

9.6 Let $(N_t)_{t \geq 0}$ be a Poisson process with parameter $\lambda = 1$. Define the process

$$X_t = N_t \mod 4, \quad t \geq 0.$$

Then, $(X_t)_{t \geq 0}$ is a continuous-time Markov chain on $\{0, 1, 2, 3\}$. Find the generator of $(X_t)_t$.

Answers

9.1 $q_1 = 1$, $q_2 = 2$, $q_3 = 4$,

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 2 & 2 & -4 \end{bmatrix}, \quad \widetilde{\mathbf{P}} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

9.2 $q_1 = q_4 = 3$, $q_3 = 5$, $q_2 = 1$,

$$\mathbf{Q} = \begin{bmatrix} -3 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 1 & 2 & -5 & 2 \\ 1 & 0 & 2 & -3 \end{bmatrix}, \quad \widetilde{\mathbf{P}} = \begin{bmatrix} 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 1/5 & 2/5 & 0 & 2/5 \\ 1/3 & 0 & 2/3 & 0 \end{bmatrix}$$

9.3

$$\mathbf{Q} = \begin{bmatrix} -q_a & q_a/2 & q_a/2 \\ q_b/2 & -q_b & q_b/2 \\ q_c/2 & q_c/2 & -q_c \end{bmatrix}$$

9.4

$$\mathbf{Q} = \begin{bmatrix} -13/12 & 1/2 & 1/3 & 1/4 \\ 1/4 & -1/4 & 0 & 0 \\ 1/3 & 0 & -1/3 & 0 \\ 1/2 & 0 & 0 & -1/2 \end{bmatrix}, \quad \widetilde{\mathbf{P}} = \begin{bmatrix} 0 & 6/13 & 4/13 & 3/13 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

9.5

$$\mathbf{Q} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

9.6

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$