Jeonghyeon Woo

Math 301

HW 8

1. $f(x) = \frac{1}{x^2+1}$ is injective and not surjective.

Assume the function is an injection.

$$f(x) = f(y)$$

$$\frac{1}{x^2+1} = \frac{1}{y^2+1}$$

$$x^2 + 1 = y^2 + 1$$

$$x^2 = y^2$$

$$x = y$$

Assume the function is a surjection. Let $x \in R^+$, such that $f(x) = 2$

$$\frac{1}{x^2+1} = 2$$

$$2x^2 + 2 = 1$$

$$2x^2 = -1$$

$$x^2 = -\frac{1}{2}$$

$$x = \sqrt{-\frac{1}{2}}$$

Contradiction.

Thus, the function is an injection and not a surjection.

2. $f: A \to B$ is injective construct $g: B \to A$ is surjective

Let x and y the variables.

$$f(x) = f(y)$$

Let $g: f(x) \to x$

$$g\big(f(y)\big) = y$$

$$(g \circ f)(y) = y$$

$$(g \circ f)(x) = y$$

$$g\big(f(x)\big) = y$$

G is indeed a surjection since for every f(x), there is g(f(x))=y

3. $A \subseteq B \text{ and } g: B \to A \text{ is injective, then } |A| = |B|$

Suppose B is an infinite set according to the given statement. A is also an infinite set since A may be an equal set to B that is an infinite set, so A is injective.

Since A and B are injective, $|A| = |B|$ by CBS

4. $|\mathbb{R}| = |\mathbb{R}^+|$

Let f and g, such that $f: \mathbb{R} \to \mathbb{R}^+$ and $g: \mathbb{R}^+ \to \mathbb{R}$

$\mathbb{R} \subset \mathbb{R}^+$, so g is injective. We know $\mathbb{R}$ is an infinite set by def, so f is injective.

$|\mathbb{R}| = |\mathbb{R}^+|$ by CBS

5. a. $f(W \cap X) \subseteq f(W) \cap f(X)$

$$W \cap X \subseteq X$$

$$W \cap X \subseteq W$$

$$f(W \cap X) \subseteq f(W)$$

$$f(W \cap X) \subseteq f(X)$$

$$f(W \cap X) \subseteq f(W) \cap f(W \cap X)$$

$$f(W \cap X) \subseteq f(W) \cap f(X)$$

Thus, the statement is true

b. $f(W \cup X) = f(W) \cup f(X)$

$$f(X) \cup f(W) = \{f(x): x \in X\} \cup \{f(x): x \in W\}\}$$

$$\{f(x): x \in X\} \cup \{f(x): x \in W\}\} = \{f(x): x \in X \cup W\}$$

$$f(W \cup X) = \{f(x): x \in X \cup W\}$$

$$f(X \cup W) = f(W) \cup f(X)$$

6. It has exactly the same process as in 5 but replace the f to f^{-1} and replace W to Z and X to Y.

7. $\text{if } f: A \to B \text{ and } X \subseteq A, \text{then } X \subseteq f^{-1}\left(f(X)\right)$

Suppose f is invertible.

1. $f: A \to B$
2. $f^{-1}: B \to A$
3. $X \subseteq A$
4. $X = f^{-1}\left(f(X)\right)$
5. $X \subseteq f^{-1}(f(X))$

4 is because X is a subset or a equal set of A, X is applicable for both f and f^{-1}.

5 is from the def of equal set.

Thus, the given statement is true

8. $f: A \to B$ is injective iff $X \subseteq A, f^{-1}\left(f(X)\right) \subseteq X$

$$f: A \to B$$

$$X \subseteq A$$

$$f: X \to f(X)$$

Let $y = f(x_1) = f(x_2)$ and $\{x_1, x_2\} = X$ for $X \subseteq A$

$$f^{-1}(y) = x_1$$

$$f^{-1}(y) = x_2$$

$$x_1 = x_2$$

$$x_1 \in X$$

$$f^{-1}(y) \in X$$

$$f^{-1}\left(f(X)\right) = X$$

$$f^{-1}\left(f(X)\right) \subseteq X$$

Thus, the statement is true.