Jeonghyeon Woo

Math 301

HW 5

1. For all integers a, b, c, if $a^2|\mathrm{b}$ and $\mathrm{b}^3|c, then\ a^6|c$

Proof.

If $a^2|b$, then there is an integer x_1 that $a^2\mathrm{x}_1 = b$.

$$a^2x_1 = b$$

$$(a^2x_1)^3 = b^3$$

$$a^6x_1^3 = b^3$$

If $b^3|c$, then there is an integer x_2 that $b^3x_2 = c$.

By substitution, $a^6x_2x_1^3 = c$. Let $x_3 = x_2x_1^3$ which is an integer.

$$a^6x_3 = c$$

Therefore, $a^6|c$

2. For all $n \in \mathbb{Z}, n^2 + 5n + 4$ is even.

Proof.

Case 1: when n is odd, we can express $n = 2k + 1$

$$\begin{aligned} n^2 + 5n + 4 &= (2k + 1)^2 + 5(2k + 1) + 4 \\ &= 4k^2 + 4k + 1 + 10k + 5 + 4 \\ &= 4k^2 + 14k + 10 \\ &= 2(\mathrm{k}^2 + 7k + 5) \end{aligned}$$

$\mathrm{k}^2 + 7\mathrm{k} + 5$ should be an integer, so it is legal to express:

$$m = k^2 + 7k + 5$$

$\mathrm{n}^2 + 5n + 4 = 2\mathrm{m}$

Case 2: when n is even, we can express $n = 2k$

$$\begin{aligned} n^2 + 5n + 4 &= (2k)^2 + 5(2k) + 4 \\ &= 4k^2 + 10k + 4 \\ &= 2(2k^2 + 5k + 2) \end{aligned}$$

$2\mathrm{k}^2 + 5k + 2$ should be an integer, so it is legal to express:

$$m = 2\mathrm{k}^2 + 5k + 2$$

$n^2 + 5n + 4 = 2m$

Since all the cases of integer n agrees, $n^2 + 5n + 4$ is even.

3. For all real numbers $r, s, t, |\mathrm{r} + \mathrm{s} + \mathrm{t}| \leq |\mathrm{r}| + |\mathrm{s}| + |\mathrm{t}|$

Given: if $|r + \mathrm{s}| \leq |\mathrm{r}| + |\mathrm{s}|$ is true, then $rs \leq |rs|$ is also true

Proof.

$$|r + s + t| \leq |r| + |s| + |t|$$

$$|(r + s) + t| \leq (|r| + |s|) + |t|$$

$$|\mathrm{rs} + \mathrm{t}| \leq |\mathrm{rs}| + |\mathrm{t}|$$

$$\sqrt{(\mathrm{rs} + \mathrm{t})^2} \leq |\mathrm{rs}| + |\mathrm{t}|$$

$$(rs + t)^2 \leq (|rs| + |t|)^2$$

$$(rs)^2 + 2rst + t^2 \leq (rs)^2 + 2|rs||t| + t^2$$

$$\mathrm{rst} \leq |\mathrm{rs}||\mathrm{t}|$$

$$\mathrm{rst} \leq |\mathrm{rst}|$$

Thus, $|\mathrm{r} + \mathrm{s} + \mathrm{t}| \leq |\mathrm{r}| + |\mathrm{s}| + |\mathrm{t}|$ as the inequality is true.

4. For all $n, p \in \mathbb{Z}$, if np is even then either n is even or p is even.

Proof.

Contrapositive state:

For all $n, p \in \mathbb{Z}$, if n is odd and p is odd then np is odd.

Let $n = 2k_1 + 1$ and $p = 2k_2 + 1$

$np = (2k_1 + 1)(2\mathrm{k}_2 + 1)$

$np = 4k_1k_2 + 2k_1 + 2k_2 + 1$

$np = 2(2k_1k_2 + k_1 + k_2) + 1$

Since $2\mathrm{k}_1k_2 + k_1 + k_2$ is an integer,

let m is an integer that $m = 2k_1k_2 + k_1 + k_2$

$np = 2m + 1$

Since the contrapositive state is true, the given statement is also true.

Thus, for all $n, p \in \mathbb{Z}$, if np is even then either n is even or p is even.

5. For all integers p, q, if $3|(p-1)$ and $3|(q-1)$ then $3|(pq-1)$

Proof.

There are x_1 and x_2 that $3x_1 = p - 1$ and $3x_2 = q - 1$.

$p = 3x_1 + 1$

$q = 3x_2 + 1$

$pq = (3x_1 + 1)(3x_2 + 1)$

$\quad = 9x_1x_2 + 3x_1 + 3x_2 + 1$

$pq - 1 = 9x_1x_2 + 3x_1 + 3x_2$

$pq - 1 = 3(3x_1x_2 + x_1 + x_2)$

Since the result of $3x_1x_2 + x_1 + x_2$ is an integer, there is an integer x_3 such that $x_3 = 3x_1x_2 + x_1 + x_2$

$pq - 1 = 3x_3$

Thus, if $3|(p-1)$ and $3|(q-1)$ then $3|(pq-1)$.

6. For all integers p, q, if $pq \neq 1 \, mod \, 3$ then $p \neq 1 \, mod \, 3$ or $q \neq 1 \, mod \, 3$

Proof.

There exists p, q, such that $pq \neq 1 \, mod \, 3$ and $p = 1 \, mod \, 3$ and $q = 1 \, mod \, 3$.

$pq \neq 3n + 1$ for $n \in \mathbb{Z}$

$p = 3x_1 + 1$ for $x_1 \in \mathbb{Z}$

$q = 3x_2 + 1$ for $x_2 \in \mathbb{Z}$

$pq = 9x_1x_2 + 3x_1 + 3x_2 + 1$

$\quad = 3(3x_1x_2 + x_1 + x_2) + 1$

$3x_1x_2 + x_1 + x_2 \notin \mathbb{Z}$ #contradiction

Since negative of the given statement is false, the given statement is true.

Thus, if $pq \neq 1 \, mod \, 3$ then $p \neq 1 \, mod \, 3$ or $q \neq 1 \, mod \, 3$