Jeonghyeon Woo

Math 301

HW 6

1. For any sets A and B, $A \subseteq B$ iff $A \cap B = A$
   1. $A \subseteq B$
   2. $x \in A \rightarrow x \in B$ by def. of subset
   3. $x \in A$
   4. $x \in B$ by implication 2,3
   5. $x \in A \cap B$ conjunction 3,4
   6. $A \cap B = A$ def. of set equality 3,5

   Put 6,5 at 1,2 and have 1 as result will also make sense. Therefore, 1 and 6 are equivalent.

2. For any sets A and B, $A \subseteq B$ iff $B^c \subseteq A^c$

Proof.

   1. $A \subseteq B$
   2. $x \in A \rightarrow x \in B$ def. of subset
   3. $x \notin B \rightarrow x \notin A$ contraposition
   4. $B^c \rightarrow A^C$ def. of complement
   5. $B^C \subseteq A^c$ def. of subset

   Reverse of the above step will show $B^c \subseteq A^c \rightarrow A \subseteq B$. Therefore, 1 and 5 are equivalent.

3. sets A and $B \subseteq U$, $A \cap B = A$ iff $A^c \cap B^c = B^c$

proof.

1. $A \cap B = A$
2. $A \subseteq B$
3. $A \cup B = B$
4. $(A \cup B)^c = B^c$
5. $A^c \cap B^C = B^c$

2 is from problem 1. 3 is by the theorem. 4 is from problem 2. 5 is by DM law. Reverse of above steps will also make sense. Therefore, 1 and 5 are equivalent.

4. $\forall a, b, c, d \in \mathbb{Z},\ a < b < c < d \rightarrow\ (a, b) \cap (c, d) = \emptyset$

negation: $\exists a, b, c, d \in \mathbb{Z}, a < b < c < d \text{ and } (a, b) \cap (c, d) \neq \emptyset$

Let $x \in (a, b)$ *and* $y \in (c, d)$

$a < x < b$ and $c < y < d$

Let $p = x \text{ and } p = y, \text{so } p \in (a, b)$ *and* $p \in (c, d)$

$a < p < b < c < p < d$

$p < p$

#contradiction

Thus, there is no intersection between the intervals

5. A) 1. $\left((A \cap B) \cup C\right)^c$

   2. $\left((A \cup C) \cap (B \cup C)\right)^c$ dist.

   3. $(A \cup C)^c \cup (B \cup C)^c$ DM.

   B) 1.$(A \cup B) \cup (A \cap B^c)$

   2. $A \cap (B \cup B^c)$ dist.

3. $A \cap U$ op. U and $\emptyset$

4. A op. U and $\emptyset$

6. A)

$\lfloor r \rfloor = \lceil r \rceil$

$$k \leq r < k + 1$$

$$l - 1 \leq r \leq l$$

Let $p = r = l$

$$p - 1 < p \leq r \leq p < p + 1$$

Thus, $p = r$, so $r \in \mathbb{Z}$

B)

Let k an integer for floor

$k \leq -r < k + 1$

$k \leq -r$ and $-r < k + 1$

$-(k + 1) < r$ and $r \leq -k$

$-k - 1 < r \leq -k$

Let $l = -k$

$l - 1 < r \leq l$

Thus, $\lfloor -r \rfloor = -\lceil r \rceil$

C)

Let l an integer for ceiling

$l - 1 < -s \leq l$

$-l + 1 > s \geq -l$

Let $k = -l$

$k \leq s < k + 1$

Thus, $\lceil -s \rceil = -\lfloor s \rfloor$

D)

$\lceil r+s \rceil \leq \lceil r \rceil + \lceil s \rceil$

Let $k = \lceil r \rceil, l = \lceil s \rceil, m = \lceil r+s \rceil$

$k - 1 < r \leq k$

$l - 1 < s \leq l$

$m - 1 < r + s \leq m$

$k + l - 2 < r + s \leq k + l$

Case1: $k + l - 1 \geq r + s$

$m < k + l$ since $k + l - 1 \leq m < k + l$

Case2: $k + m - 1 < r + s$

$r + s \leq k + l$

$r + s \leq m$

so, $m = k + l$

Thus, $\lceil r+s \rceil \leq \lceil r \rceil + \lceil s \rceil$

E)

$\lceil r-s \rceil \leq \lceil r \rceil - \lfloor s \rfloor$

$\lceil r-s \rceil \leq \lceil r \rceil + \lceil -s \rceil$

Let $a = -s$

$\lceil r+a \rceil \leq \lceil r \rceil + \lceil a \rceil$, and it is true as shown in part D

7. $\forall n \in \mathbb{Z}, n \geq 1, \sum_{i=1}^{n} 2i = n^2 + n$

Proof.

When i=1,

$2(1) = 1^2 + 1$ is true

When $i \geq 1$,

$$
\begin{aligned}
(i+1)^2 + (i+1) &= 2(i+1) + i^2 + i \\
&= 2i + 2 + i^2 + i \\
&= i^2 + 2i + 1 + i + 1 \\
&= (i+1)^2 + (i+1)
\end{aligned}
$$

Thus, $\sum_{i=1}^{n} 2i = n^2 + n$ by induction.