

Solutions Manual to Mechanics

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1 The Equations of Motion

1.1

Let the field be a uniformly gravitational one of acceleration g . Let the system be a coplanar double pendulum with the strings of lengths l_1 and l_2 and the particles of masses m_1 and m_2 . Let ϕ_1 and ϕ_2 be the angles of the particles from the y axis. Then,

$$\mathbf{r}_1 = \begin{bmatrix} l_1 \sin \phi_1 \\ l_1 \cos \phi_1 \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} l_1 \sin \phi_1 + l_2 \sin \phi_2 \\ l_1 \cos \phi_1 + l_2 \cos \phi_2 \end{bmatrix}, \quad (1.1)$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} l_1 \dot{\phi}_1 \cos \phi_1 \\ -l_1 \dot{\phi}_1 \sin \phi_1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} l_1 \dot{\phi}_1 \cos \phi_1 + l_2 \dot{\phi}_2 \cos \phi_2 \\ -l_1 \dot{\phi}_1 \sin \phi_1 - l_2 \dot{\phi}_2 \sin \phi_2 \end{bmatrix}. \quad (1.2)$$

The Lagrangian is given by

$$L = \frac{1}{2}m_1\|\mathbf{v}_1\|^2 + m_1 g r_{1y} + \frac{1}{2}m_2\|\mathbf{v}_2\|^2 + m_2 g r_{2y}. \quad (1.3)$$

The right hand side can be written as

$$\begin{aligned} & \frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2 + m_1 g l_1 \cos \phi_1 \\ & + \frac{1}{2}m_2 \left(l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right) \\ & + m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2). \end{aligned} \quad (1.4)$$

Therefore,

$$\begin{aligned} L = & \frac{1}{2}(m_1 + m_2)l_1^2 \dot{\phi}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \\ & + (m_1 + m_2)g l_1 \cos \phi_1 + m_2 g l_2 \cos \phi_2. \end{aligned} \quad (1.5)$$

1.2

Let the field be a uniformly gravitational one of acceleration g . Let the system be a pendulum with the string of length l and the particles of mass m_1 at the point of support which can move on the x axis and of mass m_2 at

the end of the string. Let ϕ be the angle of the second particle from the y axis. Then,

$$\mathbf{r}_1 = \begin{bmatrix} x \\ 0 \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} x + l \sin \phi \\ l \cos \phi \end{bmatrix}, \quad (1.6)$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \dot{x} + l\dot{\phi} \cos \phi \\ -l\dot{\phi} \sin \phi \end{bmatrix}. \quad (1.7)$$

The Lagrangian is given by

$$L = \frac{1}{2}m_1\|\mathbf{v}_1\|^2 + m_1 g r_{1y} + \frac{1}{2}m_2\|\mathbf{v}_2\|^2 + m_2 g r_{2y}. \quad (1.8)$$

The right hand side can be written as

$$\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi} \cos \phi\right) + m_2gl \cos \phi. \quad (1.9)$$

Therefore,

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2\dot{x}l\dot{\phi} \cos \phi + m_2gl \cos \phi. \quad (1.10)$$

1.3

Let the field be a uniformly gravitational one of acceleration g . Let the system be a pendulum with the string of length l and the particle of mass m . Let ϕ be the angle of the particle from the y axis.

(a)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \cos \gamma t \\ a \sin \gamma t \end{bmatrix}, \quad (1.11)$$

then

$$\mathbf{r} = \begin{bmatrix} l \sin \phi + a \cos \gamma t \\ l \cos \phi + a \sin \gamma t \end{bmatrix}, \quad (1.12)$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi} \cos \phi - a\gamma \sin \gamma t \\ -l\dot{\phi} \sin \phi - a\gamma \cos \gamma t \end{bmatrix}. \quad (1.13)$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \quad (1.14)$$

Therefore,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2 + mla\dot{\phi}\gamma \sin(\phi - \gamma t) + mg(l \cos \phi + a \sin \gamma t). \quad (1.15)$$

(b)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \cos \gamma t \\ 0 \end{bmatrix}, \quad (1.16)$$

then

$$\mathbf{r} = \begin{bmatrix} l \sin \phi + a \cos \gamma t \\ l \cos \phi \end{bmatrix}, \quad (1.17)$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi} \cos \phi - a\gamma \sin \gamma t \\ -l\dot{\phi} \sin \phi \end{bmatrix}. \quad (1.18)$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \quad (1.19)$$

Therefore,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2 \sin^2 \gamma t - mla\dot{\phi}\gamma \sin \gamma t \cos \phi + mgl \cos \phi. \quad (1.20)$$

(c)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ a \cos \gamma t \end{bmatrix}, \quad (1.21)$$

then

$$\mathbf{r} = \begin{bmatrix} l \sin \phi \\ l \cos \phi + a \cos \gamma t \end{bmatrix}, \quad (1.22)$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi} \cos \phi \\ -l\dot{\phi} \sin \phi - a\gamma \sin \gamma t \end{bmatrix}. \quad (1.23)$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \quad (1.24)$$

Therefore,

$$\begin{aligned} L = & \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2 \sin^2 \gamma t + mla\dot{\phi}\gamma \sin \phi \sin \gamma t \\ & + mg(l \cos \phi + a \cos \gamma t). \end{aligned} \quad (1.25)$$

1.4

Let the field be a uniformly gravitational one of acceleration g . Let the system be two symmetric sets of two segments of length a connected by a particle of mass m_1 , where the lower end moves on the z axis with a particle of mass m_2 . Let θ be the angle of the particles in the middle from the z axis and let Ω be the constant angular velocity of the system around the z axis. Then,

$$\mathbf{r}_1 = \begin{bmatrix} a \sin \theta \cos \Omega t \\ a \sin \theta \sin \Omega t \\ a \cos \theta \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \\ 2a \cos \theta \end{bmatrix}, \quad (1.26)$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} a(\dot{\theta} \cos \theta \cos \Omega t - \Omega \sin \theta \sin \Omega t) \\ a(\dot{\theta} \cos \theta \sin \Omega t + \Omega \sin \theta \cos \Omega t) \\ -a\dot{\theta} \sin \theta \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -2a\dot{\theta} \sin \theta \end{bmatrix}. \quad (1.27)$$

The Lagrangian is given by

$$L = m_1\|\mathbf{v}_1\|^2 + \frac{1}{2}m_2\|\mathbf{v}_2\|^2 + 2m_1gr_{1y} + m_2gr_{2y}. \quad (1.28)$$

Therefore,

$$\begin{aligned} L = & m_1a^2(\dot{\theta}^2 + \Omega^2 \sin^2 \theta) + 2m_2a^2\dot{\theta}^2 \sin^2 \theta \\ & + 2(m_1 + m_2)ga \cos \theta. \end{aligned} \quad (1.29)$$

2 Conservation Laws

2.1

Let a particle of mass m move with velocity v_1 leave a half-space of constant potential energy U_1 and enter another half-space of constant potential energy U_2 . The Lagrangian is given by

$$L = \frac{1}{2}mv_1^2 + U_1. \quad (2.1)$$