Solutions Manual to Mechanics

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1 The Equations of Motion

1.1

Let the field be a uniformly gravitational one of acceleration g. Let the system be a coplanar double pendulum with the strings of lengths l_1 and l_2 and the particles of masses m_1 and m_2. Let ϕ_1 and ϕ_2 be the angles of the particles from the y axis. Then,

$$\mathbf{r}_1 = \begin{bmatrix} l_1 \sin\phi_1 \\ l_1 \cos\phi_1 \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} l_1 \sin\phi_1 + l_2 \sin\phi_2 \\ l_1 \cos\phi_1 + l_2 \cos\phi_2 \end{bmatrix}, \tag{1.1}$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} l_1 \dot{\phi}_1 \cos\phi_1 \\ -l_1 \dot{\phi}_1 \sin\phi_1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} l_1 \dot{\phi}_1 \cos\phi_1 + l_2 \dot{\phi}_2 \cos\phi_2 \\ -l_1 \dot{\phi}_1 \sin\phi_1 - l_2 \dot{\phi}_2 \sin\phi_2 \end{bmatrix}. \tag{1.2}$$

The Lagrangian is given by

$$L = \frac{1}{2} m_1 \|\mathbf{v}_1\|^2 + mgr_{1y} + \frac{1}{2} m_2 \|\mathbf{v}_2\|^2 + mgr_{2y}. \tag{1.3}$$

The right hand side can be written as

$$\begin{aligned} &\frac{1}{2} m_1 l_1^2 \dot{\phi}_1^{\,2} + m_1 g l_1 \cos\phi_1 \\ &+ \frac{1}{2} m_2 \left(l_1^2 \dot{\phi}_1^{\,2} + l_2^2 \dot{\phi}_2^{\,2} + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right) \\ &+ m_2 g \left(l_1 \cos\phi_1 + l_2 \cos\phi_2 \right). \end{aligned} \tag{1.4}$$

Therefore,

$$\begin{aligned} L =& \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\phi}_1^{\,2} + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^{\,2} + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \\ &+ (m_1 + m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2. \end{aligned} \tag{1.5}$$

1.2

Let the field be a uniformly gravitational one of acceleration g. Let the system be a pendulum with the string of length l and the particles of mass m_1 at the point of support which can move on the x axis and of mass m_2 at the end of the string. Let ϕ be the angle of the second particle from the y axis. Then,

$$\mathbf{r}_1 = \begin{bmatrix} x \\ 0 \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} x + l \sin\phi \\ l\cos\phi \end{bmatrix}, \tag{1.6}$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \dot{x} + l\dot{\phi}\cos\phi \\ -l\dot{\phi}\sin\phi \end{bmatrix}. \tag{1.7}$$

The Lagrangian is given by

$$L = \frac{1}{2}m_1\|\mathbf{v}_1\|^2 + mgr_{1y} + \frac{1}{2}m_2\|\mathbf{v}_2\|^2 + mgr_{2y}. \tag{1.8}$$

The right hand side can be written as

$$\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi\right) + m_2gl\cos\phi. \tag{1.9}$$

Therefore,

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2\dot{x}l\dot{\phi}\cos\phi + m_2gl\cos\phi. \tag{1.10}$$

1.3

Let the field be a uniformly gravitational one of acceleration g. Let the system be a pendulum with the string of length l and the particle of mass m. Let ϕ be the angle of the particle from the y axis.

(a)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\cos\gamma t \\ a\sin\gamma t \end{bmatrix}, \tag{1.11}$$

then

$$\mathbf{r} = \begin{bmatrix} l\sin\phi + a\cos\gamma t \\ l\cos\phi + a\sin\gamma t \end{bmatrix}, \tag{1.12}$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi}\cos\phi - a\gamma\sin\gamma t \\ -l\dot{\phi}\sin\phi - a\gamma\cos\gamma t \end{bmatrix}. \tag{1.13}$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \tag{1.14}$$

Therefore,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2 + mla\dot{\phi}\gamma\sin(\phi - \gamma t) + mg(l\cos\phi + a\sin\gamma t). \tag{1.15}$$

(b)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\cos\gamma t \\ 0 \end{bmatrix}, \tag{1.16}$$

then

$$\mathbf{r} = \begin{bmatrix} l\sin\phi + a\cos\gamma t \\ l\cos\phi \end{bmatrix}, \tag{1.17}$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi}\cos\phi - a\gamma\sin\gamma t \\ -l\dot{\phi}\sin\phi \end{bmatrix}. \tag{1.18}$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \tag{1.19}$$

Therefore,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2\sin^2\gamma t - mla\dot{\phi}\gamma\sin\gamma t\cos\phi + mgl\cos\phi. \tag{1.20}$$

(c)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ a\cos\gamma t \end{bmatrix}, \tag{1.21}$$

then

$$\mathbf{r} = \begin{bmatrix} l\sin\phi \\ l\cos\phi + a\cos\gamma t \end{bmatrix}, \tag{1.22}$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi}\cos\phi \\ -l\dot{\phi}\sin\phi - a\gamma\sin\gamma t \end{bmatrix}. \tag{1.23}$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \tag{1.24}$$

Therefore,

$$\begin{aligned} L =& \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2\sin^2\gamma t + mla\dot{\phi}\gamma\sin\phi\sin\gamma t \\ &+ mg\left(l\cos\phi + a\cos\gamma t\right). \end{aligned} \tag{1.25}$$

1.4

Let the field be a uniformly gravitational one of acceleration g. Let the system be two symmetric sets of two segments of length a connected by a particle of mass m_1, where the lower end moves on the z axis with a particle of mass m_2. Let θ be the angle of the particles in the middle from the z axis and let Ω be the constant angular velocity of the system around the z axis. Then,

$$\mathbf{r}_1 = \begin{bmatrix} a\sin\theta\cos\Omega t \\ a\sin\theta\sin\Omega t \\ a\cos\theta \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \\ 2a\cos\theta \end{bmatrix}, \tag{1.26}$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} a\left(\dot{\theta}\cos\theta\cos\Omega t - \Omega\sin\theta\sin\Omega t\right) \\ a\left(\dot{\theta}\cos\theta\sin\Omega t + \Omega\sin\theta\cos\Omega t\right) \\ -a\dot{\theta}\sin\theta \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -2a\dot{\theta}\sin\theta \end{bmatrix}. \tag{1.27}$$

The Lagrangian is given by

$$L = m_1\|\mathbf{v}_1\|^2 + \frac{1}{2}m_2\|\mathbf{v}_2\|^2 + 2m_1gr_{1y} + m_2gr_{2y}. \tag{1.28}$$

Therefore,

$$\begin{aligned} L =& m_1a^2\left(\dot{\theta}^2 + \Omega^2\sin^2\theta\right) + 2m_2a^2\dot{\theta}^2\sin^2\theta \\ &+ 2(m_1 + m_2)ga\cos\theta. \end{aligned} \tag{1.29}$$

2 Conservation Laws

2.1

Let a particle of mass m move with velocity v_1 and leave a half-space of constant potential energy U_1 and enter another half-space of constant potential energy U_2. Let θ_1 and θ_2 be angles towards the border of the spaces before and after the entrance. The Lagrangian is given by

$$L = \frac{1}{2}mv_1^2 + U_1. \tag{2.1}$$

We have

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}. \tag{2.2}$$

We have

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}. \tag{2.3}$$

Then,

$$\frac{dL}{dt} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}, \tag{2.4}$$

so that

$$\frac{d}{dt}\left(L - \frac{\partial L}{\partial \dot{q}}\frac{dq}{dt}\right) = 0. \tag{2.5}$$