

# Solutions Manual to Mechanics

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# 1 The Equations of Motion

## 1.1

Let the field be a uniformly gravitational one of acceleration  $g$ . Let the system be a coplanar double pendulum with the strings of lengths  $l_1$  and  $l_2$  and the particles of masses  $m_1$  and  $m_2$ . Let  $\phi_1$  and  $\phi_2$  be the angles of the particles from the  $y$  axis. Then,

$$\mathbf{r}_1 = \begin{bmatrix} l_1 \sin \phi_1 \\ l_1 \cos \phi_1 \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} l_1 \sin \phi_1 + l_2 \sin \phi_2 \\ l_1 \cos \phi_1 + l_2 \cos \phi_2 \end{bmatrix}, \quad (1.1)$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} l_1 \dot{\phi}_1 \cos \phi_1 \\ -l_1 \dot{\phi}_1 \sin \phi_1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} l_1 \dot{\phi}_1 \cos \phi_1 + l_2 \dot{\phi}_2 \cos \phi_2 \\ -l_1 \dot{\phi}_1 \sin \phi_1 - l_2 \dot{\phi}_2 \sin \phi_2 \end{bmatrix}. \quad (1.2)$$

The Lagrangian is given by

$$L = \frac{1}{2}m_1\|\mathbf{v}_1\|^2 + mgr_{1y} + \frac{1}{2}m_2\|\mathbf{v}_2\|^2 + mgr_{2y}. \quad (1.3)$$

The right hand side can be written as

$$\begin{aligned} & \frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2 + m_1 g l_1 \cos \phi_1 \\ & + \frac{1}{2}m_2 \left( l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right) \\ & + m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2). \end{aligned} \quad (1.4)$$

Therefore,

$$\begin{aligned} L = & \frac{1}{2}(m_1 + m_2)l_1^2 \dot{\phi}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \\ & + (m_1 + m_2)gl_1 \cos \phi_1 + m_2 gl_2 \cos \phi_2. \end{aligned} \quad (1.5)$$

## 1.2

Let the field be a uniformly gravitational one of acceleration  $g$ . Let the system be a pendulum with the string of length  $l$  and the particles of mass  $m_1$  at the point of support which can move on the  $x$  axis and of mass  $m_2$  at

the end of the string. Let  $\phi$  be the angle of the second particle from the  $y$  axis. Then,

$$\mathbf{r}_1 = \begin{bmatrix} x \\ 0 \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} x + l \sin \phi \\ l \cos \phi \end{bmatrix}, \quad (1.6)$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \dot{x} + l\dot{\phi} \cos \phi \\ -l\dot{\phi} \sin \phi \end{bmatrix}. \quad (1.7)$$

The Lagrangian is given by

$$L = \frac{1}{2}m_1\|\mathbf{v}_1\|^2 + mgr_{1y} + \frac{1}{2}m_2\|\mathbf{v}_2\|^2 + mgr_{2y}. \quad (1.8)$$

The right hand side can be written as

$$\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi\right) + m_2gl\cos\phi. \quad (1.9)$$

Therefore,

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2\dot{x}l\dot{\phi}\cos\phi + m_2gl\cos\phi. \quad (1.10)$$

### 1.3

Let the field be a uniformly gravitational one of acceleration  $g$ . Let the system be a pendulum with the string of length  $l$  and the particle of mass  $m$ . Let  $\phi$  be the angle of the particle from the  $y$  axis.

(a)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \cos \gamma t \\ a \sin \gamma t \end{bmatrix}, \quad (1.11)$$

then

$$\mathbf{r} = \begin{bmatrix} l \sin \phi + a \cos \gamma t \\ l \cos \phi + a \sin \gamma t \end{bmatrix}, \quad (1.12)$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi} \cos \phi - a\gamma \sin \gamma t \\ -l\dot{\phi} \sin \phi - a\gamma \cos \gamma t \end{bmatrix}. \quad (1.13)$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \quad (1.14)$$

Therefore,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2 + mla\dot{\phi}\gamma\sin(\phi - \gamma t) + mg(l\cos\phi + a\sin\gamma t). \quad (1.15)$$

(b)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\cos\gamma t \\ 0 \end{bmatrix}, \quad (1.16)$$

then

$$\mathbf{r} = \begin{bmatrix} l\sin\phi + a\cos\gamma t \\ l\cos\phi \end{bmatrix}, \quad (1.17)$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi}\cos\phi - a\gamma\sin\gamma t \\ -l\dot{\phi}\sin\phi \end{bmatrix}. \quad (1.18)$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \quad (1.19)$$

Therefore,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2\sin^2\gamma t - mla\dot{\phi}\gamma\sin\gamma t\cos\phi + mgl\cos\phi. \quad (1.20)$$

(c)

If the point of support moves according to the law

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ a\cos\gamma t \end{bmatrix}, \quad (1.21)$$

then

$$\mathbf{r} = \begin{bmatrix} l\sin\phi \\ l\cos\phi + a\cos\gamma t \end{bmatrix}, \quad (1.22)$$

so that

$$\mathbf{v} = \begin{bmatrix} l\dot{\phi} \cos \phi \\ -l\dot{\phi} \sin \phi - a\gamma \sin \gamma t \end{bmatrix}. \quad (1.23)$$

The Lagrangian is given by

$$L = \frac{1}{2}m\|\mathbf{v}\|^2 + mgr_y. \quad (1.24)$$

Therefore,

$$\begin{aligned} L = & \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}ma^2\gamma^2 \sin^2 \gamma t + mla\dot{\phi}\gamma \sin \phi \sin \gamma t \\ & + mg(l \cos \phi + a \cos \gamma t). \end{aligned} \quad (1.25)$$

## 1.4

Let the field be a uniform gravitational one with the acceleration  $g$ . Let the system be two symmetric sets of two segments with the length  $a$  connected by a particle with the mass  $m_1$ , where the upper end of the system is fixed on the  $z$  axis and the lower end moves on the  $z$  axis with a particle with the mass  $m_2$ . Let the system rotate around the  $z$  axis with the angle  $\theta$  and with the constant angular velocity  $\Omega$ . Then,

$$\mathbf{r}_1 = \begin{bmatrix} a \sin \theta \cos \Omega t \\ a \sin \theta \sin \Omega t \\ a \cos \theta \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \\ 2a \cos \theta \end{bmatrix}, \quad (1.26)$$

so that

$$\mathbf{v}_1 = \begin{bmatrix} a \left( \dot{\theta} \cos \theta \cos \Omega t - \Omega \sin \theta \sin \Omega t \right) \\ a \left( \dot{\theta} \cos \theta \sin \Omega t + \Omega \sin \theta \cos \Omega t \right) \\ -a\dot{\theta} \sin \theta \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -2a\dot{\theta} \sin \theta \end{bmatrix}. \quad (1.27)$$

The Lagrangian is given by

$$L = m_1\|\mathbf{v}_1\|^2 + \frac{1}{2}m_2\|\mathbf{v}_2\|^2 + 2m_1gr_{1y} + m_2gr_{2y}. \quad (1.28)$$

Therefore,

$$\begin{aligned} L = & m_1a^2 \left( \dot{\theta}^2 + \Omega^2 \sin^2 \theta \right) + 2m_2a^2\dot{\theta}^2 \sin^2 \theta \\ & + 2(m_1 + m_2)ga \cos \theta. \end{aligned} \quad (1.29)$$

## 2 Conservation Laws

### 2.1

Let a particle of mass  $m$  move with velocity  $v_1$  leave a half-space of constant potential energy  $U_1$  and enter another half-space of constant potential energy  $U_2$ .