

# Solutions Manual to Pattern Recognition and Machine Learning

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# 1 Introduction

## 1.1

To minimise

$$E(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^N (y(x_n, \mathbf{w}) - t_n)^2, \quad (1.1)$$

setting its derivative as zero gives

$$\mathbf{0} = \sum_{n=1}^N \frac{\partial y(x_n, \mathbf{w})}{\partial \mathbf{w}} (y(x_n, \mathbf{w}) - t_n). \quad (1.2)$$

Substituting

$$y(x_n, \mathbf{w}) = \sum_{j=0}^M w_j x_n^j \quad (1.3)$$

gives

$$0 = \sum_{n=1}^N x_n^i \left( \sum_{j=0}^M w_j x_n^j - t_n \right). \quad (1.4)$$

Therefore,

$$\sum_{j=0}^M A_{ij} w_j = T_i \quad (1.5)$$

where

$$\begin{aligned} A_{ij} &= \sum_{n=1}^N x_n^{i+j}, \\ T_i &= \sum_{n=1}^N x_n^i t_n. \end{aligned} \quad (1.6)$$

## 1.2

To minimise

$$\tilde{E}(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^N (y(x_n, \mathbf{w}) - t_n)^2 + \frac{\lambda}{2} \|\mathbf{w}\|^2, \quad (1.7)$$

setting its derivative as zero gives

$$\mathbf{0} = \sum_{n=1}^N \frac{\partial y(x_n, \mathbf{w})}{\partial \mathbf{w}} (y(x_n, \mathbf{w}) - t_n) + \lambda \mathbf{w}. \quad (1.8)$$

Substituting

$$y(x_n, \mathbf{w}) = \sum_{j=0}^M w_j x_n^j \quad (1.9)$$

gives

$$0 = \sum_{n=1}^N x_n^i \left( \sum_{j=0}^M w_j x_n^j - t_n \right) + \lambda w_i. \quad (1.10)$$

Therefore,

$$\sum_{j=0}^M \tilde{A}_{ij} w_j = T_i \quad (1.11)$$

where

$$\begin{aligned} \tilde{A}_{ij} &= \sum_{n=1}^N x_n^{i+j} + \lambda \delta_{ij}, \\ T_i &= \sum_{n=1}^N x_n^i t_n. \end{aligned} \quad (1.12)$$

### 1.3

Let  $a$ ,  $o$  and  $l$  be the events where an apple, orange and lime are selected respectively. The probability that an apple is selected is given by

$$p(a) = p(a|r)p(r) + p(a|b)p(b) + p(a|g)p(g). \quad (1.13)$$

Substituting  $p(a|r) = \frac{3}{10}$ ,  $p(r) = \frac{1}{5}$ ,  $p(a|g) = \frac{1}{2}$ ,  $p(r) = \frac{1}{5}$ ,  $p(a|g) = \frac{3}{10}$  and  $p(g) = \frac{3}{5}$  gives

$$p(a) = \frac{17}{50}. \quad (1.14)$$

If an orange is selected, the probability that it came from the geen box is given by

$$p(g|o) = \frac{p(g, o)}{p(o)}. \quad (1.15)$$

Here,

$$\begin{aligned} p(g, o) &= p(o|g)p(g), \\ p(o) &= p(o|r)p(r) + p(o|b)p(b) + p(o|g)p(g). \end{aligned} \quad (1.16)$$

Substituting  $p(o|r) = \frac{2}{5}$ ,  $p(r) = \frac{1}{5}$ ,  $p(o|b) = \frac{1}{2}$ ,  $p(b) = \frac{1}{5}$ ,  $p(o|g) = \frac{3}{10}$  and  $p(g) = \frac{3}{5}$  gives  $p(g, o) = \frac{9}{50}$  and  $p(o) = \frac{9}{25}$ . Therefore,

$$p(g|o) = \frac{1}{2}. \quad (1.17)$$

## 1.5

By the definition,

$$\text{var } f(x) = \text{E} (f(x) - \text{E}f(x))^2. \quad (1.18)$$

The right hand side can be written as

$$\text{E} ((f(x))^2 - 2f(x)\text{E}f(x) + (\text{E}f(x))^2) = \text{E} (f(x))^2 - (\text{E}f(x))^2. \quad (1.19)$$

Therefore,

$$\text{var } f(x) = \text{E} (f(x))^2 - (\text{E}f(x))^2. \quad (1.20)$$

## 1.6

By the definition,

$$\text{cov}(x, y) = \text{E}xy - \text{E}x\text{E}y, \quad (1.21)$$

where the right hand side can be written as

$$\int xyf(x, y)dxdy - \int xf(x)dx \int yf(y)dy. \quad (1.22)$$

If  $x$  and  $y$  are independent, by the definition

$$f(x, y) = f(x)f(y), \quad (1.23)$$

then

$$\int xyf(x, y)dxdy = \int f(x)dx \int f(y)dy. \quad (1.24)$$

Therefore,

$$\text{cov}(x, y) = 0. \quad (1.25)$$

## 1.7

Let

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx. \quad (1.26)$$

Then

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(x^2 + y^2)\right) dx dy. \quad (1.27)$$

By the transformation from Cartesian coordinates  $(x, y)$  to polar coordinates  $(r, \theta)$ , the right hand side can be written as

$$\int_0^{\infty} \int_0^{2\pi} \exp\left(-\frac{1}{2\sigma^2}r^2\right) \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} dr d\theta = 2\pi \int_0^{\infty} \exp\left(-\frac{1}{2\sigma^2}r^2\right) r dr. \quad (1.28)$$

By the transformation  $s = \frac{r}{\sigma}$ , the right hand side can be written as

$$2\pi\sigma^2 \int_0^{\infty} \exp\left(-\frac{1}{2}s^2\right) s ds = 2\pi\sigma^2 \left[-\exp\left(-\frac{1}{2}s^2\right)\right]_0^{\infty}. \quad (1.29)$$

Therefore,

$$I = (2\pi\sigma^2)^{\frac{1}{2}}. \quad (1.30)$$

By the definition,

$$\mathcal{N}(x|\mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right). \quad (1.31)$$

Then

$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) dx = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right) dx. \quad (1.32)$$

By the transformation  $t = x - \mu$ , the right hand side can be written as

$$(2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}t^2\right) dt = (2\pi\sigma^2)^{-\frac{1}{2}} I. \quad (1.33)$$

Therefore,

$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) dx = 1. \quad (1.34)$$

## 1.8

If  $x$  is under the Gaussian distribution, then

$$\mathbb{E}x = \int_{-\infty}^{\infty} x \mathcal{N}(x|\mu, \sigma^2) dx. \quad (1.35)$$

By the definition, the right hand side can be written as

$$(2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} x \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx. \quad (1.36)$$

By the transformation  $y = x - \mu$ , it can be written as

$$(2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (y + \mu) \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy. \quad (1.37)$$

Since

$$(2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} y \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy = 0, \quad (1.38)$$

and

$$(2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \mu \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy = \mu \int_{-\infty}^{\infty} \mathcal{N}(y|\mu, \sigma^2) dy, \quad (1.39)$$

we have

$$\mathbb{E}x = \mu. \quad (1.40)$$

By the definition,

$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) dx = 1 \quad (1.41)$$

can be written as

$$(2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx = 1. \quad (1.42)$$

Differentiating both sides with respect to  $\sigma^2$  gives

$$\begin{aligned} & (2\pi)^{-\frac{1}{2}} \left(-\frac{1}{2}\right) (\sigma^2)^{-\frac{3}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \\ & + (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{2} (\sigma^2)^{-2} (x-\mu)^2 \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx = 0. \end{aligned} \quad (1.43)$$

The left hand side can be written as

$$\begin{aligned}
 -\frac{1}{2}(\sigma^2)^{-1} \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) dx + \frac{1}{2}(\sigma^2)^{-2} \int_{-\infty}^{\infty} (x - \mu)^2 \mathcal{N}(x|\mu, \sigma^2) dx \\
 = -\frac{1}{2}(\sigma^2)^{-1} + \frac{1}{2}(\sigma^2)^{-2} \text{var} x.
 \end{aligned}
 \tag{1.44}$$

Therefore,

$$\text{var} x = \sigma^2.
 \tag{1.45}$$