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Dynamic Markov Bridges and Market Microstructure

Theory and Applications

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To our families
Emel, Christian and Alice.

Preface

During the course of our research on equilibrium models of asymmetric information in market microstructure theory, we have realised that one needed to apply techniques from different branches of stochastic analysis to treat these models with mathematical rigour. However, these subfields of stochastic analysis—to the best of our knowledge—are not presented in a single volume. This book intends to address this issue and provides one concise account of all fundamental theory that is necessary for studying such equilibrium models.

Equilibrium in these models can be viewed as an outcome of a game among asymmetrically informed agents. The less informed agents in these games endeavour to infer the information possessed by the agents with superior information. This obviously necessitates a good understanding of the stochastic filtering theory. On the other hand, the equilibrium strategy of an agent with superior information is to drive a commonly observed process to a given random variable without distorting the unconditional law of the process. Construction of such strategy turns out to be closely linked to the conditioning of Markov processes on their terminal value. Moreover, this construction needs to be admissible and adapted to the agent's filtration, which brings us to the study of stochastic differential equations (SDEs) representing Markov bridges. Therefore, an adequate knowledge of stochastic filtering, Markov bridges and SDEs is essential for a thorough analysis of asymmetric information models.

The aim of this book is to build this knowledge. Although there are many excellent texts covering various aspects of the aforementioned three fields, the standard assumptions in these literature are often too restrictive to be applied in the context of asymmetric information models. Driven by this need from applications we extend a lot of results known in the literature. Therefore, this book can also be viewed as a complementary text to the standard literature. Proofs of statements that already exist in the literature are often omitted and a precise reference is given.

This book assumes the reader has some knowledge of stochastic calculus and martingale theory in continuous time. Although familiarity with SDEs will make its reading more enjoyable, no prior knowledge on this subject is necessary. The exposition is largely self-contained, which allows it to be used as a graduate textbook on equilibrium models of insider trading.

The material presented here is divided into two parts. Part I develops the mathematical foundations for SDEs, static and dynamic Markov bridges, and stochastic filtering. Equilibrium models of insider trading and their analysis constitute the contents of Part II.

In Chap. 1 we present preliminaries of the theory of Markov process including the strong Markov property and the right continuity of the filtrations and introduce Feller processes. Naturally in this chapter we select the results that are necessary for the development of the theory of Markov bridges.

As proofs of the results presented in Chap. 1 will remain unaltered under an assumption of path continuity, we have refrained from assuming path regularity in that chapter. However, we will confine ourselves to diffusion processes for the rest of the book since the theory of SDE representation for general jump-diffusion bridges is yet to be developed.

Chapter 2 is devoted to stochastic differential equations and their relation with the local martingale problem. In particular standard results on the solutions of SDEs and comparison of one-dimensional SDEs have been extended to accommodate the exploding nature of the coefficients that are inherent to the SDEs associated with bridges.

Chapter 3 is an overview of stochastic filtering theory. Kushner–Stratonovich equation for the conditional density of the unobserved signal is introduced and uniqueness of its solution is proved using a suitable filtered martingale problem pioneered by Kurtz and Ocone [84].

Using the theory presented in Chaps. 1 and 2 we develop the SDE representation of Markov processes that are conditioned to have a prespecified distribution μ at a given time T in Chap. 4. Two types of conditioning are considered: weak conditioning refers to the case when μ is absolutely continuous with respect to the original law at time T whereas strong conditioning corresponds to the case when μ is a Dirac mass. We also discuss the relationship between such bridges and the enlargement of filtrations theory.

The bridges constructed in Chap. 4 are called static since their final bridge point is given in advance. Chapter 5 considers an extension of this theory when the final point is not known in advance but is revealed over time via the observation of a given process. To verify that the law of these dynamic bridges coincides with the law of the original Markov process when considered in their own filtration, we use techniques from Chap. 3.

Part II is concerned with the applications of the theory in Part I and starts with Chap. 6, which provides the description of the Kyle–Back model of insider trading as the underlying framework for the study of equilibrium in the chapters that follow. Chapter 6 also contains a proof in a general setting of the 'folk result' that one can limit insider's trading strategies to absolutely continuous ones. Chapter 7 presents an equilibrium in this framework when the inside information is dynamic in the absence of default risk. It also shows that equilibrium is not unique in this family of models. Chapter 8 studies the impact of default risk in the equilibrium outcome.

The book grew out of a series of paper with our long-term collaborator and colleague Luciano Campi, who has also read large portions of the first draft and suggested many corrections and improvements for which we are grateful. We also thank Christoph Czichowsky and Michail Zervos for their suggestion on various parts of the manuscript. This book was discussed at the Financial Mathematics Reading Group at the LSE and we are grateful to its participants for their input.

London, UK
October 2017

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Contents

Frequently Used Notation

A, A_t	Generator of a Markov process, see Sect. 2.1
$\mathscr{B}_t, \mathscr{B}_t^-$	For a given $T < \infty$, $\mathscr{B}_t$ is the history of the coordinate process on $C([0,T],\mathbf{E})$ up to time-t. $\mathscr{B}_t^-$ is the history of the coordinate process on $C([0,T),\mathbf{E})$ up to time-t. See the discussion preceding Theorem 4.2
$b\mathscr{G}$	The set of bounded and $\mathscr{G}$-measurable functions
$C(U,V), \mathbb{C}(U), \mathbb{C}$	$C(U,V)$ is the space of V-valued continuous functions on U, $\mathbb{C}(U) = C(U,\mathbb{R})$, $\mathbb{C} = \mathbb{C}(\mathbf{E}_\Delta)$. See also Remark 1.3 for the relationship between different subspaces of continuous functions
$\mathbb{C}_0(\mathbf{E}), \mathbb{C}_0$	$\mathbb{C}_0(\mathbf{E})$ is the space of $\mathbb{R}$-valued continuous functions on $\mathbf{E}$ vanishing at infinity, $\mathbb{C}_0 = \mathbb{C}_0(\mathbf{E}_\Delta)$
$\mathbb{C}_K(\mathbf{E}), \mathbb{C}_K$	$\mathbb{C}_K(\mathbf{E})$ is the space of $\mathbb{R}$-valued continuous functions on $\mathbf{E}$ with compact support, $\mathbb{C}_K = \mathbb{C}_K(\mathbf{E}_\Delta)$
$\mathbb{C}^n(\mathbf{E}), \mathbb{C}^n$	$\mathbb{C}^n(\mathbf{E})$ is the space of $\mathbb{R}$-valued, n-times continuously differentiable functions on $\mathbf{E}$, $\mathbb{C}^n = \mathbb{C}^n(\mathbf{E}_\Delta)$
$\mathbf{E}, \mathbf{E}_\Delta, \mathscr{E}$	$\mathbf{E}$ is locally compact separable metric space, $\mathbf{E}_\Delta$ is one-point compactification of $\mathbf{E}$, and $\mathscr{E}$ is the Borel sigma-algebra on $\mathbf{E}$
E^x	Expectation operator corresponding to P^x
ε_x	Measure with point mass at x
$\mathscr{F}_t^0, \mathscr{F}_t', \mathscr{F}^0$	$\mathscr{F}_t^0$ is the history of the underlying Markov process up to time-t, $\mathscr{F}_t'$ is its future evolution after time-t, and $\mathscr{F}^0 = \sigma(\mathscr{F}_t^0, \mathscr{F}_t')$
M_t^f	See the expression in (2.2)
$^\circ\eta$	Optional projection of η, see Definition 3.1

P^{μ}, P^{x}	For a given Markov process X with initial distribution μ, P^{μ} is the probability measure on $\mathscr{F}^0$ generated by X. $P^x = P^{\varepsilon_x}$. See Eq. (1.4) in this respect
π_t, $\pi_t f$	Conditional distribution of the signal. See Sects. 3.1 and 3.2
$P_t f$, $P_{s,t} f$	See Eq. (1.7) and Assumption 4.2 correspondingly
$P_{s,t}(\cdot, \cdot)$	Markov transition function, see Definition 1.2
θ	Shift operator, see Eq. (1.6)
U^{α}	α-potential operators, see Definition 1.5

Part I
Theory

Chapter 1
Markov Processes

1.1 Markov Property

Markov processes model the evolution of random phenomena whose future behaviour is independent of the past given their current state. In this section we will make this notion, i.e. *Markov property*, precise in a general context.

We start with a probability space $(\Omega, \mathscr{F}, P)$. Let $\mathbf{T} = [0, \infty)$ and $\mathbf{E}$ be a locally compact separable metric space. We will denote the Borel σ-field on $\mathbf{E}$ with $\mathscr{E}$. A particular example of $\mathbf{E}$, that appears very often in applications, is the d-dimensional Euclidean space $\mathbb{R}^d$.

For each $t \in \mathbf{T}$, let $X_t(\omega) = X(t, \omega)$ be a function from Ω to $\mathbf{E}$ such that it is $\mathscr{F}/\mathscr{E}$-measurable, i.e. $X_t^{-1}(A) \in \mathscr{F}$ for every $A \in \mathscr{E}$. Note that when $\mathbf{E} = \mathbb{R}$ this corresponds to the familiar case of X_t being a real random variable for every $t \in \mathbf{T}$. Under this setup $X = (X_t)_{t \in \mathbf{T}}$ is called a stochastic process.

We will now define two σ-algebras that are crucial in defining the Markov property. Let

$$\mathscr{F}_t^0 = \sigma(X_s; s \leq t); \qquad \mathscr{F}_t' = \sigma(X_u; u \geq t).$$

Observe that $\mathscr{F}_t^0$ contains the history of X until time t while $\mathscr{F}_t'$ is the future evolution of X after time t. Moreover, $(\mathscr{F}_t^0)_{t \in \mathbf{T}}$ is an increasing sequence of σ-algebras, i.e. a filtration. We will also denote the σ-algebra generated by the past and the future of the process with $\mathscr{F}^0$, i.e. $\mathscr{F}^0 = \sigma(\mathscr{F}_t^0, \mathscr{F}_t')$.

The answer to the question whether a given process possesses the Markov property crucially depends on the choice of filtration in a given model. Indeed, one can have a process with Markov property with respect to its natural filtration but fails to be Markov in a larger one. Thus, to define the Markov property for X we need to specify a filtration. Suppose that there exists a filtration $(\mathscr{F}_t)_{t \in \mathbf{T}}$ on our probability space such that $\mathscr{F}_t^0 \subset \mathscr{F}_t$, for all $t \in \mathbf{T}$, i.e. X is adapted to $(\mathscr{F}_t)$.

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Definition 1.1 $(X_t, \mathscr{F}_t)_{t\in\mathbf{T}}$ is a Markov process if

$$P(B|\mathscr{F}_t) = P(B|X_t), \qquad \forall t \in \mathbf{T}, B \in \mathscr{F}'_t. \tag{1.1}$$

The above well-known formulation of the Markov property states that given the current state of X at time t, the future of X is independent of the σ-algebra $\mathscr{F}_t$ of events including, but not limited to, the history of X until time t. The next theorem states two alternative and useful statements of the Markov property.

Theorem 1.1 *For $(X_t, \mathscr{F}_t)_{t\in\mathbf{T}}$ the condition (1.1) is equivalent to any of the following:*

i) $\forall t \in \mathbf{T}$, $B \in \mathscr{F}'_t$ *and* $A \in \mathscr{F}_t$,

$$P(A \cap B|X_t) = P(A|X_t)P(B|X_t);$$

ii) $\forall t \in \mathbf{T}$, $A \in \mathscr{F}_t$,

$$P(A|\mathscr{F}'_t) = P(A|X_t).$$

Proof First we show the implication (1.1) $\Rightarrow$ i). Let $t \in \mathbf{T}$, $B \in \mathscr{F}'_t$ and $A \in \mathscr{F}_t$. Then,

$$\begin{aligned} P(A \cap B|X_t) &= E\left[\mathbf{1}_A E\left[\mathbf{1}_B|\mathscr{F}_t\right]|X_t\right] \\ &= E[\mathbf{1}_A P(B|X_t)|X_t] = P(B|X_t)P(A|X_t). \end{aligned}$$

Next we show i) $\Rightarrow$ ii). Let $t \in \mathbf{T}$, $A \in \mathscr{F}_t$ and fix an arbitrary $B \in \mathscr{F}'_t$. Then, $P(A \cap B|X_t) = E[\mathbf{1}_B P(A|X_t)|X_t]$ implies that

$$E[\mathbf{1}_A\mathbf{1}_B] = E[\mathbf{1}_B P(A|X_t)],$$

which yields the claim.

To show ii) $\Rightarrow$ (1.1) consider $t \in \mathbf{T}$, $A \in \mathscr{F}_t$ and $B \in \mathscr{F}'_t$ and observe that

$$\begin{aligned} E[\mathbf{1}_B\mathbf{1}_A] &= E[\mathbf{1}_B P(A|X_t)] \\ &= E[P(B|X_t)P(A|X_t)] = E[P(B|X_t)\mathbf{1}_A]. \end{aligned}$$

$\square$

We summarise some alternative formulations of the Markov property in the next proposition, the proof of which is based on an application of the Monotone Class Theorem and is left to the reader. We will henceforth denote the set of bounded and $\mathscr{G}$-measurable functions with $b\mathscr{G}$.

Proposition 1.1 *For $(X_t, \mathscr{F}_t)_{t\in\mathbf{T}}$ the condition (1.1) is equivalent to any of the following:*

i) $\forall Y \in b\mathscr{F}'_t$

$$E[Y|\mathscr{F}_t] = E[Y|X_t];$$

ii) $\forall u \geq t,\ f \in b\mathscr{E}$,

$$E[f(X_u)|\mathscr{F}_t] = E[f(X_u)|X_t];$$

iii) $\forall u \geq t,\ f \in \mathbb{C}_K(\mathbf{E})$,

$$E[f(X_u)|\mathscr{F}_t] = E[f(X_u)|X_t],$$

where $\mathbb{C}_K(\mathbf{E})$ *is the set of continuous functions on* $\mathbf{E}$ *with compact support.*

1.2 Transition Functions

The Markov property (1.1) allows us to define for any $s < t$ a mapping $P_{s,t} : \mathbf{E} \times \mathscr{E} \mapsto [0, 1]$ such that

$$P(X_t \in A|X_s) = P_{s,t}(X_s, A),$$

and $P_{s,t}(x, \cdot)$ is a probability measure on $(\mathbf{E}, \mathscr{E})$ for each $x \in \mathbf{E}$. Moreover, whenever $s < t < u$, we have, due to (1.1) and the tower property of conditional expectations,

$$P(X_u \in A|X_s) = \int_{\mathbf{E}} P_{s,t}(x, dy)P_{t,u}(y, A), \qquad A \in \mathscr{E}.$$

The existence of a family of such mappings is the defining characterisation of Markov processes. In order to make this statement precise we need the following definition.

Definition 1.2 The collection $\{P_{s,t}(\cdot, \cdot); 0 \leq s < t < \infty\}$ is a Markov transition function on $(\mathbf{E}, \mathscr{E})$ if $\forall s < t < u$ we have

i) $\forall x \in \mathbf{E} : A \mapsto P_{s,t}(x, A)$ is a probability measure on $\mathscr{E}$;
ii) $\forall A \in \mathscr{E} : x \mapsto P_{s,t}(x, A)$ is $\mathscr{E}$-measurable;
iii) $\forall x \in \mathbf{E}, \forall A \in \mathscr{E}$ the following *Chapman–Kolmogorov* equation is satisfied:

$$P_{s,u}(x, A) = \int_{\mathbf{E}} P_{s,t}(x, dy)P_{t,u}(y, A).$$

As we discussed above, the Chapman–Kolmogorov equation is a manifestation of the Markov property. It can also be seen intuitively: when considering a journey from x to a set A in the interval $[s, u]$, the first part of the journey until time t is independent of the remaining part, in view of the Markov property, and the Chapman–Kolmogorov equation states just that!

Example 1.1 (Brownian Motion) $\mathbf{E} = \mathbb{R}$ and $\mathscr{E}$ is the Borel σ-algebra on $\mathbb{R}$. For real x and y and $t > s \geq 0$ put

$$p_{s,t}(x, y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(x-y)^2}{2(t-s)}\right),$$

and set

$$P_{s,t}(x, A) = \int_A p_{s,t}(x, y)\, dy, \qquad t > 0.$$

It can be directly verified that the collection $(P_{s,t})$ is a transition function, and the function $p_{s,t}(x, y)$ is called the *transition density*. Observe that the transition function is *time homogeneous*, i.e. $P_{s,t}(x, A)$ depends only on $t - s$ for fixed x and A. Also note that in this case *spatial homogeneity* holds, too; namely, $p_t(x, y)$ is a function of $x - y$ only.

For X a Markov process with conditional distribution defined by $P_{s,t}$, and initial distribution μ we can easily write for any $f \in \mathscr{E}^n$ and $0 \leq t_1 < \ldots < t_n$,

$$\begin{aligned} &E[f(X_{t_1}, \ldots, X_{t_n})] \qquad (1.2)\\ &= \int_{\mathbf{E}} \mu(dx_0) \int_{\mathbf{E}} P_{t_0,t_1}(x_0, dx_1) \ldots \int_{\mathbf{E}} P_{t_{n-1},t_n}(x_{n-1}, dx_n) f(x_1, \ldots, x_n) \end{aligned}$$

In particular, if f is the indicator function of $A_1 \times \ldots \times A_n$, then the above gives the finite dimensional joint distribution of the process.

Conversely, if one is given an initial distribution, μ, and a transition function $(P_{s,t})$, then one can construct a stochastic process on the canonical space of *all functions* from $\mathbf{T}$ to $\mathbf{E}$ whose joint distributions agree with the right-hand side of (1.2) by Kolmogorov's extension theorem. Thus, defining a Markov process via (1.1) is equivalent to the specification of a transition function.

Up to now the transition function $P_{s,t}$ has been assumed to be a strict probability kernel, namely, $P_{s,t}(x, \mathbf{E}) = 1$ for every $x \in \mathbf{E}$ and $s \in \mathbf{T}, t \in \mathbf{T}$. We will extend this by allowing

$$P_{s,t}(x, \mathbf{E}) \leq 1, \qquad \forall x \in \mathbf{E}, s \in \mathbf{T}, \text{ and } t \in \mathbf{T}.$$

Such a transition function is called *submarkovian* and appear naturally when studying Markov processes killed reaching a certain boundary. When the equality holds in above we say that the transition function is *(strictly) Markovian*. We can easily convert the former case into the latter as follows. We introduce a new $\Delta \notin E$ and set

$$\mathbf{E}_\Delta = \mathbf{E} \cup \{\Delta\}, \qquad \mathscr{E}_\Delta = \sigma(\mathscr{E}, \{\Delta\}).$$

The new point Δ may be considered as the *point at infinity* in the one-point compactification of $\mathbf{E}$. If $\mathbf{E}$ is already compact, Δ is nevertheless added as an isolated point. We can now define a new transition function $P'_{s,t}$ as follows for $s < t$ and $A \in \mathscr{E}$:

$$\begin{aligned} P'_{s.t}(x, A) &= P_{s.t}(x, A), \\ P'_{s,t}(x, \{\Delta\}) &= 1 - P_{s,t}(x, \mathbf{E}), \qquad \text{if } x \neq \Delta; \\ P'_{s,t}(\Delta, \mathbf{E}) &= 0, \qquad P'_{s,t}(\Delta, \{\Delta\}) = 1. \end{aligned}$$

It is easy to check that $P'_{s,t}$ is a Markovian transition function. Moreover, the above suggests that Δ is an 'absorbing state' (or 'trap'), and after an unessential modification of the probability space we can assume that

$$\forall \omega, \forall s \in \mathbf{T} : \{X_s(\omega) = \Delta\} \subset \{X_t(\omega) = \Delta \text{ for all } t \geq s\}.$$

Next we define the function $\zeta : \Omega \mapsto [0, \infty]$ by

$$\zeta(\omega) = \inf\{t \in \mathbf{T} : X_t(\omega) = \Delta\}, \tag{1.3}$$

where $\inf \emptyset = \infty$ by convention. Thus, $\zeta(\omega) = \infty$ if and only if $X_t(\omega) \in \mathbf{E}$ for all $t \in \mathbf{T}$. The random variable ζ is called the *lifetime* of X.

Observe that so far we have not defined $P_{t,t}$. There are interesting cases when $P_{t,t}(x, \cdot)$ is not the identity operator, then x is called a 'branching point'. However, we will not consider this case and maintain the assumption that

$$P_{t,t}(x, x) = 1, \qquad \forall x \in \mathbf{E}_\Delta, \forall t \in \mathbf{T}.$$

A particular case of Markov processes occurs when the transition function is time-homogeneous, i.e. for any $x \in \mathbf{E}_\Delta$, $A \in \mathscr{E}_\Delta$ and $s \leq t$

$$P_{s,t}(x, A) = P_{t-s}(x, A),$$

for some collection $P_t(\cdot, \cdot)$. In this case we say that X is a time-homogeneous Markov process. Conversely, if one is given a transition function $P_{s,t}$, then one can construct a time-homogenous Markov process, namely $(t, X_t)_{t \in \mathbf{T}}$ with a time-homogeneous transition function. This construction is straightforward and left to the reader.

For the sake of brevity in the rest of this section we will consider only time-homogeneous Markov processes and their transition functions of the form $P_t(\cdot, \cdot)$.

However, all the results, unless explicitly stated otherwise, will apply to time-inhomogeneous Markov processes.

For time-homogeneous Markov process Chapman–Kolmogorov equation translates into

$$P_{t+s} = P_t P_s,$$

thus, the family P_t forms a semigroup.

1.3 Measures Induced by a Markov Process

Consider a Markov process X with initial distribution $\mu = \varepsilon_x$ (the point mass at x). We define P^x to be the probability measure on $\mathscr{F}^0$ generated by X. This measure is also called the law of X when $X_0 = x$. We will denote the corresponding expectation operator with E^x. Thus, e.g. if $Y \in b\mathscr{F}^0$

$$E^x[Y] = \int_\Omega Y(\omega) P^x(d\omega).$$

If, in particular, $Y = \mathbf{1}_A(X_t)$, where $A \in \mathbf{E}_\Delta$,

$$E^x(Y) = P^x(X_t \in A) = P_t(x, A).$$

In the case that X_0 is random with distribution μ, to compute $P(X_t \in A)$, which coincides with

$$P^\mu(X_t \in A) = \int_{\mathbf{E}_\Delta} P^x(X_t \in A)\mu(dx), \tag{1.4}$$

we need to be able to evaluate the integral in the right-hand side of the above. This, of course, requires the function $x \mapsto P^x[Y]$ to be appropriately measurable and the following proposition establishes this fact.

Proposition 1.2 *For each $\Lambda \in \mathscr{F}^0$, the function $x \mapsto P^x(\Lambda)$ is $\mathscr{E}_\Delta$-measurable.*

Note that the above proposition holds when $\Lambda = X_t^{-1}(A)$ for some $A \in \mathscr{E}_\Delta$. The proof can be completed by an application of Dynkin's $\pi - \lambda$ Theorem A.1.

This allows us to define, for any probability measure μ on $\mathscr{E}_\Delta$, a new measure on $\mathscr{F}^0$ by setting

$$P^\mu(\Lambda) = \int_{\mathbf{E}_\Delta} P^x(\Lambda)\mu(dx), \qquad \Lambda \in \mathscr{F}^0.$$

The family of measures, P^x, yields yet another representation of the Markov property (1.1):

$$P(X_{s+t} \in A|\mathscr{F}_t) = P^{X_t}(X_s \in A) = P_s(X_t, A), \tag{1.5}$$

for any $A \in \mathscr{E}_\Delta$.

Next we want to demonstrate that (1.5) extends to sets more general than $[X_{s+t} \in A] = X_{s+t}^{-1}(A)$. To do so suppose there exists a 'shift' operator $(\theta_t)_{t\geq 0}$ such that for every t, $\theta_t : \Omega \mapsto \Omega$ and

$$(X_s \circ \theta_t)(\omega) = X_s(\theta_t(\omega)) = X_{s+t}(\omega) \tag{1.6}$$

holds. The role of the mapping θ_t is to chop off and discard the path of X before time t. Note that a shift operator θ exists trivially when Ω is the space of all functions from $\mathbf{T}$ to $\mathbf{E}_\Delta$ as in the construction of the Markov process by Kolmogorov's extension theorem. In this case

$$\theta_t(\omega) = X(t + \cdot, \omega)$$

where X is the coordinate process. The same is true when Ω is the space of all right continuous (or continuous) functions. Thus, from now on we will postulate the existence of a shift operator in our space, and use the implications of (1.6) freely.

Introduction of a shift operator allows us to write

$$X_{s+t}^{-1} = \theta_t^{-1} X_s^{-1}$$

and restate (1.5) as

$$P(\theta_t^{-1}(X_s^{-1}(A))|\mathscr{F}_t) = P^{X_t}(X_s^{-1}(A)).$$

With this notation the extension of (1.5) to more general sets is achieved by an application of Dynkin's $\pi - \lambda$ Theorem A.1, which yields the following proposition.

Proposition 1.3 *Consider $\Lambda \in \mathscr{F}^0$. Then $\theta_t^{-1}(\Lambda) \in \mathscr{F}_t'$, and*

$$P(\theta_t^{-1}\Lambda|\mathscr{F}_t) = P^{X_t}(\Lambda).$$

More generally, for all $Y \in b\mathscr{F}^0$

$$E[Y \circ \theta_t|\mathscr{F}_t] = E^{X_t}[Y].$$

Remark 1.1 Note that the implications of Proposition 1.3 hold true if P (resp. E) is replaced by P^μ (resp. E^μ). Observe that P^μ (in particular P^x) is only defined on the σ-field $\mathscr{F}^0$ as opposed to P, which is defined on $\mathscr{F}$. Later we will extend P^μ to a larger σ-algebra by completion.

Before proceeding further we give some examples of Markov processes which play an important role in applications.

Example 1.2 (Markov Chain)

$$\begin{aligned} \mathbf{E} &= \text{ any countable set, e.g. the set of integers.} \\ \mathscr{E} &= \text{ the set of all subsets of } E. \end{aligned}$$

If we write $p_{ij}(t) = P(X_t = j|X_0 = i)$ for $i \in \mathbf{E}$ and $j \in \mathbf{E}$, then we can define for any $A \subset \mathbf{E}$,

$$P_t(i, A) = \sum_{j \in A} p_{ij}(t).$$

Then, the conditions for P being a transition function are easily seen to be satisfied by P_t.

Example 1.3 (Poisson Process) $\mathbf{E} = \mathbb{N}$. For $n \in \mathbf{E}$ and $m \in \mathbf{E}$

$$P_t(n, \{m\}) = \begin{cases} 0 & \text{if } m < n, \\ \frac{e^{-\lambda t}(\lambda t)^{m-n}}{(m-n)!} & \text{if } m \geq n, \end{cases}$$

so that we can define a valid transition function on $(\mathbf{E}, \mathscr{E})$ where $\mathscr{E}$ is the set of all subsets of $\mathbf{E}$. Note the spatial homogeneity as in the Example 1.1.

Example 1.4 (Brownian Motion Killed at 0) Let $\mathbf{E} = (0, \infty)$, and define for $x > 0$ and $y > 0$

$$q_t(x, y) = p_t(x, y) - p_t(x, -y),$$

where p_t is as in defined in Example 1.1. Let for $A \in \mathscr{E}$

$$Q_t(x, A) = \int_A q_t(x, y)\, dy.$$

Then, it can be checked that Q_t is a *submarkovian* transition function. Indeed, for $t > 0$, $Q_t(x, E) < 1$. This is the transition density of a Brownian motion starting at a strictly positive value and killed when it reaches 0. In fact, $Q_t(x, E) = P^x(T_0 > t)$ where P^x is the law of standard Brownian motion starting at $x > 0$, and T_0 is its first hitting time of 0. As killed Brownian motion is a submarkovian process, there exists a finite lifetime associated with it. In this case

$$\zeta = \inf\{t > 0 : X_t = 0\}$$

as the probability of hitting ∞ in a finite time is 0. Note that the killed Brownian motion does not have the spatial homogeneity.

Example 1.5 (Three-Dimensional Bessel Process) $\mathbf{E} = (0, \infty)$. Note that the transition density q_t defined in Example 1.4 satisfies for any $x > 0$ and $y > 0$

$$\int_E y q_t(x, y)\, dy = x.$$

Now, if we define

$$p_t^{(3)}(x, y) = \frac{1}{x} q_t(x, y) y,$$

and let

$$P_t^{(3)}(x, A) = \int_A p_t^{(3)}(x, y)\, dy,$$

we can check that $P_t^{(3)}$ is a Markovian transition function, i.e. $P_t^{(3)}(x, E) = 1$ for all $t > 0$ and $x \in \mathbf{E}$. This is the transition density of three-dimensional Bessel process.

1.4 Feller Processes

1.4.1 Potential Operators

In this section we consider a time homogeneous Markov process $(X_t, \mathscr{F}_t)$ with transition function (P_t) and seek for a class of functions on $\mathbf{E}$ such that $(f(X_t), \mathscr{F}_t)$ is a supermartingale.

We will denote $E^x[f(X_t)]$ with $P_t f(x)$ for any $f \in b\mathscr{E}$. That is,

$$P_t f(x) = \int_{\mathbf{E}} P_t(x, dy) f(y). \tag{1.7}$$

Since P_t is a transition function we have $P_t f \in b\mathscr{E}$. Similarly, $P_t f \in \mathscr{E}_+$ for every $f \in \mathscr{E}_+$, where $\mathscr{E}_+$ is the class of positive (extended-valued) $\mathscr{E}$-measurable functions.

Definition 1.3 Let $f \in \mathscr{E}_+$ and $\alpha \geq 0$. Then f is α-superaveraging relative to P_t if

$$\forall t \geq 0 : f \geq e^{-\alpha t} P_t f. \tag{1.8}$$

If in addition we have

$$f = \lim_{t \downarrow 0} e^{-\alpha t} P_t f, \tag{1.9}$$

we say f is α-excessive.

Note that if we apply the operator $e^{-\alpha s} P_s$ to both sides of (1.8), we obtain

$$e^{-\alpha s} P_s f \geq e^{-\alpha(t+s)} P_s P_t f = e^{-\alpha(t+s)} P_{t+s} f,$$

thus, $e^{-\alpha t} P_t f$ is decreasing in t so that the limit in (1.9) exists.

Proposition 1.4 *If f is α-superaveraging and $f(X_t)$ is integrable for each $t \in \mathbf{T}$, then $(e^{-\alpha t} f(X_t), \mathscr{F}_t)$ is a P^x-supermartingale for every $x \in \mathbf{E}_\Delta$.*

Proof For $s \leq t$

$$f(X_s) \geq e^{-\alpha t} P_t f(X_s) = e^{-\alpha t} E^{X_s}[f(X_t)] = e^{-\alpha t} E^x[f(X_{t+s})|\mathscr{F}_s],$$

so that

$$e^{-\alpha s} f(X_s) \geq e^{-\alpha(s+t)} E^x[f(X_{t+s})|\mathscr{F}_s],$$

which establishes the proposition. □

We will next consider an important class of superaveraging functions.

Definition 1.4 We say that the transition function is Borelian if for any $A \in \mathscr{E}$

$$(t, x) \mapsto P_t(x, A)$$

is $\mathscr{B} \times \mathscr{E}$-measurable.

The above measurability condition is equivalent to the following:

$$\forall f \in b\mathscr{E} : (t, x) \mapsto P_t f(x)$$

is $\mathscr{B} \times \mathscr{E}$-measurable. Thus, if (X_t) is right-continuous, then the map

$$(t, x) \mapsto P_t f(x)$$

is right continuous in t, therefore it is $\mathscr{B} \times \mathscr{E}$-measurable. As we will very soon restrict our attention to right continuous X, we obtain results under the assumption that (P_t) is Borelian in the remainder of this section.

Definition 1.5 Let $f \in b\mathscr{E}$, $\alpha > 0$ and (P_t) be Borelian. Then, the α-potential of f is the function given by

$$\begin{aligned} U^\alpha f(x) &= \int_0^\infty e^{-\alpha t} P_t f(x)\, dt \\ &= E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dt. \end{aligned}$$

Note that the first integral in the definition is well defined due to the Borelian property of (P_t). The second equality follows from Fubini's theorem. Consequently, $U^\alpha f \in b\mathscr{E}$.

Observe that if we put the sup norm on the space of continuous functions, then the operator U^α becomes a bounded operator with operator norm $\frac{1}{\alpha}$. The family of operators $\{U^\alpha, \alpha > 0\}$ is also known as the *resolvent* of the semigroup (P_t).

Proposition 1.5 *Suppose (P_t) is Borelian. If $f \in b\mathscr{E}_+$, then $U^\alpha f$ is α-excessive.*

Proof

$$e^{-\alpha t} P_t(U^\alpha f) = \int_0^\infty e^{-\alpha(t+s)} P_{t+s} f\, ds = \int_t^\infty e^{-\alpha s} P_s f\, ds,$$

which is less than or equal to

$$\int_0^\infty e^{-\alpha s} P_s f\, ds = U^\alpha f,$$

and converges to $U^\alpha f$ as t converges to 0. □

Proposition 1.6 *Suppose (X_t) is progressively measurable and (P_t) is Borelian. For $f \in b\mathscr{E}_+$ and $\alpha > 0$, define*

$$Y_t = \int_0^t e^{-\alpha s} f(X_s)\, ds + e^{-\alpha t} U^\alpha f(X_t).$$

Then, $(Y_t, \mathscr{F}_t)$ is a progressively measurable P^x-martingale.

Proof Let

$$Y_\infty = \int_0^\infty e^{-\alpha s} f(X_s)\, ds.$$

Then,

$$E^x[Y_\infty] = U^\alpha f(x),$$

and

$$E^x[Y_\infty | \mathscr{F}_t] = \int_0^t e^{-\alpha s} f(X_s)\, ds + E^x\left[\int_t^\infty e^{-\alpha s} f(X_s)\, ds \middle| \mathscr{F}_t\right].$$

Moreover,

$$\begin{aligned} E^x\left[\int_t^\infty e^{-\alpha s} f(X_s)\, ds \middle| \mathscr{F}_t\right] &= E^x\left[\int_0^\infty e^{-\alpha(t+s)} f(X_{s+t})\, ds \middle| \mathscr{F}_t\right] \\ &= E^x\left[e^{-\alpha t}\int_0^\infty e^{-\alpha s} f(X_s \circ \theta_t)\, ds \middle| \mathscr{F}_t\right] \\ &= e^{-\alpha t} E^x[Y_\infty \circ \theta_t | \mathscr{F}_t] = e^{-\alpha t} E^{X_t}[Y_\infty] = e^{-\alpha t} U^\alpha f(X_t). \end{aligned}$$

Hence,

$$E^x[Y_\infty|\mathscr{F}_t] = \int_0^t e^{-\alpha s} f(X_s)\, ds + e^{-\alpha t} U^\alpha f(X_t).$$

The first term on the right side is progressively measurable being continuous and adapted. The second term is also progressively measurable since $U^\alpha f \in \mathscr{E}$, and X is progressively measurable. □

1.4.2 Definition and Continuity Properties

Let $\mathbb{C}$ denote the class of all continuous functions on $\mathbf{E}_\Delta$. Since $\mathbf{E}_\Delta$ is compact, each $f \in \mathbb{C}$ is bounded. Thus, we can define the usual sup-norm on $\mathbb{C}$ as follows:

$$\|f\| = \sup_{x \in \mathbf{E}_\Delta} |f(x)|.$$

Let $\mathbb{C}_0$ denote the space of continuous functions on $\mathbf{E}$ vanishing at infinity, i.e. for any $\varepsilon > 0$ there exists a compact $K \subset \mathbf{E}$ such that $\|f(x)\| < \varepsilon$ for all $x \in K^c$. This space can be made a subclass of $\mathbb{C}$ once we set $f(\Delta) = 0$ for any $f \in \mathbb{C}_0$. Denote by $\mathbb{C}_c$ the subclass of $\mathbb{C}_0$ having compact supports. It is easy to see that endowed with the sup-norm, $\mathbb{C}$ and $\mathbb{C}_0$ are Banach spaces and $\mathbb{C}_0$ is the completion of $\mathbb{C}_c$.

Definition 1.6 A Markov process X with a Borelian transition function P_t is called a Feller process if P_0 is the identity mapping, and

i) For any $f \in \mathbb{C}$, $P_t f \in \mathbb{C}$ for all $t \in \mathbf{T}$;
ii) For any $f \in \mathbb{C}$

$$\lim_{t \to 0} \|P_t f - f\| = 0. \tag{1.10}$$

A semigroup (P_t) satisfying this conditions is called Feller.

Remark 1.2 It is clear that ii) implies :

ii') For any $f \in \mathbb{C}, x \in \mathbf{E}_\Delta$,

$$\lim_{t \to 0} P_t f(x) = f(x). \tag{1.11}$$

In fact, these conditions are equivalent under i). For a proof of this result, see Proposition III.2.4 in [100].

Remark 1.3 It is common in the literature to state the Feller property for the class $\mathbb{C}_0(\mathbf{E})$. Indeed, we can replace $\mathbb{C}$ in above conditions with $\mathbb{C}_0(\mathbf{E})$ since $\mathbb{C}_0(\mathbf{E}) = \mathbb{C}_0$ and each member of $\mathbb{C}$ is the sum of a member of $\mathbb{C}_0$ plus a constant.

For the rest of this section we assume that (P_t) is Feller.

Theorem 1.2 *The function*

$$(t, x, f) \mapsto P_t f(x)$$

on $\mathbf{T} \times \mathbf{E}_\Delta \times \mathbb{C}$ *is continuous.*

Proof By triangle inequality

$$|P_t f(x) - P_s g(y)| \leq |P_t f(x) - P_t f(y)| + |P_t f(y) - P_s f(y)| + |P_s f(y) - P_s g(y)|.$$

Since $P_t f \in \mathbb{C}$, we have that the first term converges to 0 as $y \to x$. Since P is Markovian, $P_t f = P_s P_{t-s} f$. Thus,

$$|P_t f(y) - P_s f(y)| = |P_s P_{t-s} f(y) - P_s f(y)| \leq \|P_s\| \|P_{t-s} f - f\| = \|P_{t-s} f - f\|,$$

which converges to 0 as $s \uparrow t$. Finally, the last term is bounded by

$$\|P_s\| \|f - g\|,$$

which also converges to 0 as $g \to f$ in the sup norm. □

Our goal now is to demonstrate that for any given Feller process, X, there exists a càdlàg Feller process, $\tilde{X}$, such that $X_t = \tilde{X}_t$, P^x-a.s. for each $t \in \mathbf{T}$ and $x \in \mathbf{E}_\Delta$, and the transition functions of X and $\tilde{X}$ are identical. The first step is to show that X is stochastically continuous.

We denote by d the metric on $\mathbf{E}_\delta$ and by m the metric on $\mathbf{E}$. Note that convergence in m implies convergence in d. However, the reverse implications do not hold in general.

Proposition 1.7 *Let* $(X_t, \mathscr{F}_t)_{t \in \mathbf{T}}$ *be a Feller process. Then,* $(X_t)_{t \in \mathbf{T}}$ *satisfies Dynkin's stochastic continuity property. Namely, for any* $\varepsilon > 0$ *and* $t \in \mathbf{T}$

i)

$$\lim_{h \to 0} \sup_{x \in \mathbf{E}_\Delta} \sup_{s \in [(t-h) \vee 0, t+h]} P^x(d(X_s, X_t) > \varepsilon) = 0.$$

In particular, X is stochastically continuous, i.e. $\lim_{s \to t} P^x(d(X_t, X_s) > \varepsilon) = 0$ *for all* $\varepsilon > 0$.

ii) For any compact $K \subset \mathbf{E}$

$$\lim_{h \to 0} \sup_{x \in K} \sup_{s \in [0,h]} P^x(m(X_s, x) > \varepsilon) = 0.$$

Proof

i) We first show the statement for $t = 0$. Consider the function $f : \mathbb{R}_+ \mapsto [0, 1]$ defined by $f(y) = 1 - \frac{y}{\varepsilon}$ for y with $y \leq \varepsilon$ and 0 otherwise. Let

$\phi(x,\cdot) = f(d(x,\cdot))$ where d is the metric of the space $\mathbf{E}_\Delta$. Note that $\|\phi(x,y)-\phi(x,y')\| = \|\phi(y,x)-\phi(y',x)\| \le d(y,y')/\varepsilon$, thus for each $x \in \mathbf{E}_\Delta$, $\phi(x,\cdot)$ is continuous on $\mathbf{E}_\Delta$ with a compact support. Since $\mathbf{E}_\Delta$ is compact one can find a finite number of open balls with centres x_i and radius α with $\alpha < \frac{\varepsilon^2}{2}$ that cover $\mathbf{E}_\Delta$. Then, for any $x \in \mathbf{E}_\Delta$ there exists a j such that $x \in B_\alpha(x_j)$ and, thus,

$$\begin{aligned} P^x(d(X_s,x)>\varepsilon) &\le \phi(x,x) - E^x[\phi(x,X_s)] \\ &\le |\phi(x_j,x)-\phi(x,x)| + |\phi(x_j,x) - E^x[\phi(x_j,X_s)]| \\ &\quad + E^x[|\phi(x,X_s)-\phi(x_j,X_s)|] \\ &\le 2\frac{\alpha}{\varepsilon} + |\phi(x_j,x) - E^x[\phi(x_j,X_s)]|. \end{aligned}$$

The conclusion follows from Theorem 1.2 and the arbitrariness of α.

For $t > 0$ it suffices to observe that whenever $s < t$, we have

$$P^x(d(X_t,X_s)>\varepsilon) = E^x[P^{X_s}(d(X_{t-s},X_s)>\varepsilon)] \le \sup_{x\in\mathbf{E}_\Delta} P^x(d(X_{t-s},x)>\varepsilon).$$

A similar observation holds for $s > t$.

ii) The statement can be proved verbatim after substituting $\mathbf{E}_\Delta$ with K and d with m. □

Remark 1.4 Although in the proof above we have proved the convergence in probability for the measure P^x for $x \in \mathbf{E}_\Delta$, this also implies convergence in probability for P since for any $A \in \mathscr{F}^0$,

$$P(A) = \int_{\mathbf{E}_\Delta} \mu(dx) P^x(A).$$

To proceed with the path regularity the next basic result will be useful.

Lemma 1.1 *Let $f \in \mathbb{C}$. Then, $U^\alpha \in \mathbb{C}$ and*

$$\lim_{\alpha\to\infty} \|\alpha U^\alpha f - f\| = 0.$$

A class of functions defined in $\mathbf{E}_\Delta$ is said to *separate points* if for two distinct member x, y of $\mathbf{E}_\Delta$ there exists a function in that class such that $f(x) \ne f(y)$.

Let $\{O_n, n \in \mathbb{N}\}$ be a countable base of the open sets of $\mathbf{E}_\Delta$ and define

$$\forall x \in \mathbf{E}_\Delta : \varphi_n(x) = d(x, \bar{O}_n),$$

where d is the metric on $\mathbf{E}_\Delta$. Note that $\varphi_n \in \mathbb{C}$.

Proposition 1.8 *The following countable subset of* $\mathbb{C}$ *separates points.*

$$\mathbb{D} = \{U^\alpha \varphi_n : \alpha \in \mathbb{N}, n \in \mathbb{N}\}.$$

Proof For any $x \neq y$, there exists O_n such that $x \in \bar{O}_n$ and $y \notin \bar{O}_n$. Thus, $0 = \varphi_n(x) < \varphi_n(y)$. Since $\lim_{\alpha\to\infty} \|\alpha U^\alpha f - f\| = 0$, we can find a large enough α such that

$$|\alpha U^\alpha \varphi_n(x) - \varphi_n(x)| < \frac{1}{2}\varphi_n(y)$$

$$|\alpha U^\alpha \varphi_n(y) - \varphi_n(y)| < \frac{1}{2}\varphi_n(y).$$

This implies $U^\alpha \varphi_n(x) \neq U^\alpha \varphi_n(y)$. □

The following analytical lemma will help us prove that we can obtain a version of X with right and left limits.

Lemma 1.2 *Let* $\mathbb{D}$ *be a class of continuous functions from* $\mathbf{E}_\Delta$ *to* $\mathbb{R}$ *which separates points. Let* h *be any function on* $\mathbb{R}$ *to* $\mathbf{E}_\Delta$. *Suppose that* S *is a dense subset of* $\mathbb{R}$ *such that for each* $g \in \mathbb{D}$,

$$(g \circ h)|_S \text{ has right and left limits in } \mathbb{R}.$$

Then, $h|_S$ *has right and left limits in* $\mathbb{R}$.

Proposition 1.9 *Let* $(X_t, \mathscr{F}_t)_{t\in\mathbf{T}}$ *be a Feller process, and* S *be any countable dense subset of* $\mathbf{T}$. *Then for* P*-a.a.* ω, *the sample function* $X(\cdot, \omega)$ *restricted to* S *has right limits in* $[0, \infty)$ *and left limits in* $(0, \infty)$.

Proof Let g be a member of class $\mathbb{D}$ defined in Proposition 1.8 so that $g = U^k f$ for some $f \in \mathbb{C}$. Then, by Proposition 1.4, $\{e^{-kt}g(X_t), t \in \mathbf{T}\}$ is a P^x-supermartingale for any $x \in \mathbf{E}_\Delta$. Thus, by Theorem A.11 it has right and left limits when restricted to S except on a null set (relative to every P^x) which may depend on g. However, since $\mathbb{D}$ is countable, we can choose a null set that work for all $g \in \mathbb{D}$. Thus,

$$t \mapsto g(X(t, \omega))$$

has right and left limits when restricted to S. Lemma 1.2 now implies that X has the same property. □

In view of the above proposition we can define

$$\forall t \geq 0 : \tilde{X}_t(\omega) = \lim_{u\in S, u\downarrow t} X_u(\omega); \qquad \hat{X}_t = \lim_{s\in S, s\uparrow t} X_s(\omega) \tag{1.12}$$

for all ω for which these limits exist. The set of ω for which these limits cease to exist is a P^x-null set for every $x \in \mathbf{E}_\Delta$, on which $\tilde{X}$ and $\hat{X}$ can be defined arbitrarily. Note that $\tilde{X}$ is càdlàg and $\hat{X}$ is càglàd.

Theorem 1.3 *Suppose that* $(X_t, \mathscr{F}_t)$ *is a Feller process with semigroup* (P_t)*. Then,* $\tilde{X}$ *and* $\hat{X}$*, defined in (1.12), are* P^x*-modifications of* X *for any* $x \in \mathbf{E}_\Delta$*. Moreover, if* $(\mathscr{F}_t)$ *is augmented with the* P*-null sets, then both* $(\tilde{X}_t, \mathscr{F}_t)$ $(\hat{X}_t, \mathscr{F}_t)$ *become Feller processes with semigroup* (P_t)*.*

Proof Fix an arbitrary $x \in \mathbf{E}_\Delta$ and $t \in \mathbf{T}$. Consider any sequence $(u_n) \subset S$ such that $u_n \downarrow t$. Due to the previous proposition the limit $\lim_{n\to\infty} X_{u_n}$ exists P^x-a.s. and the stochastic continuity of X shown in Proposition 1.7 implies that

$$\lim_{n\to\infty} X_{u_n} = X_t,\ P^x - a.s.$$

However, the left-hand side of the above equals $\tilde{X}_t,\ P^x-$ a.s., i.e. $\tilde{X}$ is a P^x-version of X. Since this holds for all $x \in \mathbf{E}_\Delta$ simultaneously, $\tilde{X}$ is a P-version of X as well. Thus, $\tilde{X}$ becomes adapted to $(\mathscr{F}_t)$ once it is augmented with the P-null sets. Thus, for any $f \in \mathbb{C}$ we have

$$E[f(\tilde{X}_{s+t})|\mathscr{F}_t] = E[f(X_{s+t})|\mathscr{F}_t] = P_s f(X_t) = P_s f(\tilde{X}_t).$$

This shows that (P_t) is a transition function for $(\tilde{X}_t, \mathscr{F}_t)$, too. The same arguments apply to $\hat{X}$. □

1.4.3 Strong Markov Property and Right Continuity of Fields

The main objective of this section is to show the strong Markov property of a Feller process and the right-continuity of its appropriately augmented natural filtration. We assume that the Feller process has right continuous paths. We refer to Appendix A.3 for the relevant results regarding the optional and stopping times.

Theorem 1.4 *For each optional* T*, we have for each* $f \in \mathbb{C}$ *and* $u > 0$*:*

$$E[f(X_{T+u})|\mathscr{F}_{T+}] = P_u f(X_T).$$

Proof Observe that on $[T = \infty]$ claim holds trivially. Let

$$T_n = \frac{[2^n T]}{2^n}.$$

Then it follows from Lemma A.3 that (T_n) is a sequence of stopping times decreasing to T and taking values in the dyadic set $D = \{k2^{-n} : k \geq 1, n \geq 1\}$. Moreover, by Theorem A.5, we have

$$\mathscr{F}_{T+} = \bigwedge_{n=1}^{\infty} \mathscr{F}_{T_n}.$$

Thus, if $\Lambda \in \mathscr{F}_{T+}$, then $\Lambda_d := \Lambda \cap [T_n = d] \in \mathscr{F}_d$ for every $d \in D$. The Markov property applied at $t = d$ yields

$$\int_{\Lambda_d} f(X_{d+u})dP = \int_{\Lambda_d} P_u f(X_d)dP.$$

Thus, enumerating the possible values of T_n

$$\int_{\Lambda\cap[T<\infty]} f(X_{T_n+u})dP = \sum_{d\in D}\int_{\Lambda_d} f(X_{d+u})dP$$
$$= \sum_{d\in D}\int_{\Lambda_d} P_u f(X_d)dP = \int_{\Lambda\cap[T<\infty]} P_u f(X_{T_n})dP.$$

Since f and $P_t f$ are bounded and continuous, and X is right continuous, we obtain as $n \to \infty$

$$\int_{\Lambda\cap[T<\infty]} f(X_{T+u})dP = \int_{\Lambda\cap[T<\infty]} P_u f(X_T)dP.$$

□

A direct application of Dynkin's $\pi - \lambda$ Theorem A.1 to the conclusion of Theorem 1.4 yields

$$\forall Y \in b\mathscr{F}^0 : E[Y \circ \theta_T | \mathscr{F}_{T+}] = E^{X_T}[Y], \tag{1.13}$$

for all optional T.

Definition 1.7 The Markov process $(X_t, \mathscr{F}_t)_{t\in\mathbf{T}}$ is said to have the strong Markov property if (1.13) holds for each optional time T.

Thus, Theorem 1.4 is equivalent to the assertion that a Feller process with right continuous paths has the strong Markov property. Moreover, it also implies that if we consider this process after an optional time, then it is still strong Markov and has the same transition probabilities.

Corollary 1.1 *For each optional T, the process $(X_{T+t}, \mathscr{F}_{T+t})_{t\in\mathbf{T}}$ is a Markov process with (P_t) as transition semigroup. Moreover, it has the strong Markov property.*

We will now prove that the appropriately augmented filtration of a strong Markov process is right continuous. Observe that replacing T with t and shrinking $\mathscr{F}_t$ to $\mathscr{F}_t^0$ we may rewrite (1.13) as follows:

$$E^\mu[f(X_{t+s})|\mathscr{F}_{t+}^0] = P_s f(X_t) \tag{1.14}$$

since on $\mathscr{F}^0$ measures P and P^μ coincide. Let's denote the P^μ-completion of $\mathscr{F}^0$ (resp. $\mathscr{F}_t^0$) by $\mathscr{F}^\mu$ (resp. $\mathscr{F}_t^\mu$). Note that the following result does not require the Feller property.

Theorem 1.5 *Let $(X_t, \mathscr{F}_t)_{t\in\mathbf{T}}$ be a strong Markov process. Then the family $\{\mathscr{F}_t^\mu, t \in \mathbf{T}\}$ is right continuous.*

Proof Note that since $P_s f(X_t) \in \mathscr{F}_t^0$, any version of the conditional expectation in (1.14) belongs to $\mathscr{F}_t^\mu$ since $\mathscr{F}_t^\mu$ contains all P^μ-null sets. By a monotone class argument, we can then conclude that for any $Y \in b\mathscr{F}_t'$

$$E^\mu[Y|\mathscr{F}_{t+}^\mu] \in \mathscr{F}_t^\mu. \tag{1.15}$$

Let C be the class of sets A in $\mathscr{F}$ such that $\mathbf{1}_A$ satisfies (1.15). It is easy to check that C is a sub σ-field of $\mathscr{F}$. We have seen above that $\mathscr{F}_t' \in C$. Trivially, $\mathscr{F}_t^\mu \in C$, too. This means that $C \supset \sigma(\mathscr{F}_t^\mu, \mathscr{F}_t') = \mathscr{F}^\mu$. Since $\mathscr{F}_{t+}^\mu \subset \mathscr{F}^\mu$, we have that for any $A \in \mathscr{F}_{t+}^\mu$, $\mathbf{1}_A$ satisfies (1.15), i.e. $A \in \mathscr{F}_t^\mu$. Consequently, $\mathscr{F}_t^\mu = \mathscr{F}_{t+}^\mu$ since A was arbitrary and $\mathscr{F}_t^\mu \subset \mathscr{F}_{t+}^\mu$ by definition. □

The above theorem gives us a right continuous filtration for any fixed initial distribution μ, which clearly depends on the choice of μ. To get around this dependency, we will introduce a smaller σ-field

$$\widetilde{\mathscr{F}} = \bigwedge_\mu \mathscr{F}^\mu$$

and correspondingly for each $t \geq 0$

$$\widetilde{\mathscr{F}}_t = \bigwedge_\mu \mathscr{F}_t^\mu, \tag{1.16}$$

where μ ranges over all finite measures on $\mathscr{E}_\Delta$. By directly computing the intersections we see that

Corollary 1.2 *The family $(\widetilde{\mathscr{F}}_t)_{t\in\mathbf{T}}$ is right continuous, as well as the family $(\mathscr{F}_t^\mu)_{t\in\mathbf{T}}$ for each μ.*

We have seen in Proposition 1.2 that for any $Y \in b\mathscr{F}^0$, the function $x \mapsto E^x[Y]$ is $\mathscr{E}_\Delta$-measurable. Clearly, it will be too much to ask that this still holds when $Y \in b\widetilde{\mathscr{F}}$. In order to obtain the right measurability we need to enlarge the Borel field $\mathscr{E}_\Delta$ as follows:

$$\widetilde{\mathscr{E}} = \bigwedge_\mu \mathscr{E}^\mu$$

where μ ranges over all finite measures on $\mathscr{E}_\Delta$ and $\mathscr{E}^\mu$ is the completion of $\mathscr{E}_\Delta$ with μ-null sets. Then we have the following

Theorem 1.6 *If $Y \in b\widetilde{\mathscr{F}}$, then the mapping*

$$x \mapsto E^x[Y]$$

belongs to $\widetilde{\mathscr{E}}$. Also, for each μ and each T, optional time relative to $(\mathscr{F}_t^\mu)$, we have

$$Y \circ \theta_T \in \mathscr{F}^\mu$$

and

$$E^\mu[Y \circ \theta_T | \mathscr{F}_T^\mu] = E^{X_T}[Y].$$

We finish this section with the following important result called *Blumenthal's zero-one law*.

Theorem 1.7 *Let $\Lambda \in \widetilde{\mathscr{F}}_0$. Then, for each x we have $P^x(\Lambda) = 0$ or $P^x(\Lambda) = 1$.*

Proof First suppose that $\Lambda \in \mathscr{F}_0^0$. Then, $\Lambda = X_0^{-1}(A)$ for some $A \in \mathscr{E}$. Since $P^x(X_0 = x) = 1$, we have

$$P^x(\Lambda) = P^x(X_0^{-1}(A)) = \mathbf{1}_A(x),$$

which can only take the value 0 or 1. Now, if $\Lambda \in \widetilde{\mathscr{F}}_0$, then for any x, $\Lambda \in \mathscr{F}_0^{\varepsilon_x}$; thus, there exists some $\Lambda^x \in \mathscr{F}_0^0$ such that $(\Lambda \backslash \Lambda^x) \cup (\Lambda^x \backslash \Lambda)$ is a P^x-null set. This shows that $P^x(\Lambda) = P^x(\Lambda^x)$, which is either 0 or 1 as just proved. □

1.5 Notes

This chapter closely follows the first two chapters of Chung and Walsh [40] with the exception of Proposition 1.7, whose proof mimics the proof of Theorem 4.2 in [46]. The material covered in this chapter is standard and can also be found in classical texts such as Blumenthal and Getoor [27] and Sharpe [106]. It is limited to topics that we feel essential for the understanding of the sequel. Blumenthal and Getoor [27] is the definitive source for the potential theory of Markov processes. Sharpe [106], among other things, contains an account of Meyer's general theory of process and stochastic calculus in the context of Markov processes. The reader interested in h-transforms, Martin boundary and the time reversal for Markov process is referred to Chung and Walsh [40].

The Feller process that we use in this book is also called a Feller–Dynkin process, see, e.g. Chap. III of Rogers and Williams [101]. As there are several definitions for the Feller property in the literature, the reader should be careful when using results from several books.

Chapter 2
Stochastic Differential Equations and Martingale Problems

In this chapter we explore the well posedness of martingale problems of Stroock and Varadhan. The results of this chapter will be crucial for solving the filtering equations of Chap. 3. This well posedness will be obtained by establishing the relationship between solutions of martingale problems and stochastic differential equations (SDEs). Thus, our focus in this chapter will be the connection between SDEs and martingale problems. To formulate the martingale problem we first need to develop the notion of an infinitesimal generator.

2.1 Infinitesimal Generators

Definition 2.1 Let (P_t) be Fellerian and define the operator $A : \mathbb{C} \mapsto \mathbb{C}$ by

$$Af = \lim_{t \to 0} \frac{1}{t}(P_t f - f).$$

The domain of A is denoted with $\mathscr{D}(A)$ and it contains the functions $f \in \mathbb{C}$ for which the above limit exists and belongs to $\mathbb{C}$. The operator A as defined is said to be the *infinitesimal generator* of (P_t).

To get an intuitive grasp of the operator A observe that if $f \in \mathbb{C}$

$$E[f(X_{t+h}) - f(X_t)|\mathscr{F}_t] = P_h f(X_t) - f(X_t),$$

by the very definition of the Markov property. Thus, if $f \in \mathscr{D}(A)$ we may write

$$E[f(X_{t+h}) - f(X_t)|\mathscr{F}_t] = hAf(X_t) + o(h).$$

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This implies that A describes the movement of X over a very short period of time, which justifies the name infinitesimal generator.

The next proposition states some basic properties of an infinitesimal generator and gives some examples of functions in its domain.

Proposition 2.1 *Let (P_t) be Fellerian and A its generator.*

1. *If $f \in \mathbb{C}$ then $\int_0^t P_s f\, ds \in \mathscr{D}(A)$ and*

$$P_t f - f = A \int_0^t P_s f\, ds.$$

2. *If $f \in \mathscr{D}(A)$ and $t \geq 0$, then $P_t f \in \mathscr{D}(A)$ and*

$$\frac{d}{dt} P_t f = A P_t f = P_t A f.$$

3. *If $f \in \mathscr{D}(A)$ and $t \geq 0$, then*

$$P_t f - f = \int_0^t A P_s f\, ds = \int_0^t P_s A f\, ds.$$

Proof

1. Observe that

$$\begin{aligned} \frac{1}{h}(P_h - I) \int_0^t P_s f\, ds &= \frac{1}{h} \int_0^t (P_{s+h} f - P_s f)\, ds \\ &= \frac{1}{h} \left(\int_h^{t+h} P_s f\, ds - \int_0^t P_s f\, ds \right) \\ &= \frac{1}{h} \left(\int_t^{t+h} P_s f\, ds - \int_0^h P_s f\, ds \right) \end{aligned}$$

Since $s \mapsto P_s$ is continuous by Theorem 1.2, we have that the above converges to $P_t f - f \in \mathbb{C}$ as $h \to 0$. This proves that $\int_0^t P_s f\, ds \in \mathscr{D}(A)$ and that $A \int_0^t P_s f\, ds = P_t f - f$.

2. $P_t f \in \mathscr{D}(A)$ can be shown as above. In particular

$$A P_t f = \lim_{h \downarrow 0} \frac{P_{t+h} f - P_t f}{h} = \lim_{h \downarrow 0} P_t \left(\frac{P_h f - f}{h} \right) = P_t A f,$$

by Theorem 1.2. This shows that $t \mapsto P_t f$ has a right derivative which is equal to $P_t A f$. Moreover, the above also implies that $A P_t f = P_t A f$. In order to find the left derivative, consider

$$\lim_{h \downarrow 0} \frac{P_{t-h} f - P_t f}{-h} = \lim_{h \downarrow 0} \frac{P_t f - P_{t-h} f}{h}$$
$$= \lim_{h \downarrow 0} P_{t-h} \frac{P_h f - f}{h} = P_t A f$$

by, again, Theorem 1.2.

3. This is a direct consequence of 2. □

Corollary 2.1 *If A is the infinitesimal generator of a Feller semigroup (P_t), then $\mathscr{D}(A)$ is dense in $\mathbb{C}$ and A is a closed operator.*

Proof Since

$$f = \lim_{t \downarrow 0} \frac{\int_0^t P_s f \, ds}{t},$$

and $\int_0^t P_s f \, ds \in \mathscr{D}(A)$ by the previous proposition, we have that $\mathscr{D}(A)$ is dense in $\mathbb{C}$. To show that A is closed let $(f_n) \subset \mathscr{D}(A)$ and $f_n \to f$, $Af_n \to g$ in $\mathbb{C}$. However, $P_t f_n - f_n = \int_0^t P_s A f_n \, ds$ implies that

$$P_t f - f = \int_0^t P_s g \, ds$$

by letting n tend to ∞. Dividing both sides of above by t and letting $t \downarrow 0$, we obtain $Af = g$. □

Example 2.1 Let X be the linear Brownian motion. Then, using its transition density from Example 1.1 one can directly verify that $\mathscr{D}(A) \supset \mathbb{C}^2(\mathbf{E}_\Delta, \mathbb{R})$ and $Af = \frac{1}{2} f''$ for any $f \in \mathbb{C}^2(\mathbf{E}_\Delta, \mathbb{R})$, where $\mathbf{E}_\Delta$ is the one-point compactification of $\mathbb{R}$ with $\{-\infty, \infty\}$.

The next theorem provides the first glimpse into the martingale problem.

Theorem 2.1 *If $f \in \mathscr{D}(A)$, then the process*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s) \, ds$$

is a $(P^x, \mathscr{F}_t^0)$-martingale for any $x \in \mathbf{E}_\Delta$.

Conversely, if $f \in \mathbb{C}$ and there exists a function $g \in \mathbb{C}$ such that

$$f(X_t) - f(X_0) - \int_0^t g(X_s) \, ds$$

is a $(P^x, \mathscr{F}_t^0)$-martingale for every $x \in \mathbf{E}_\Delta$, then $f \in \mathscr{D}(A)$ and $Af = g$.

Proof Since f and Af are bounded, M^f is integrable. Moreover,

$$\begin{aligned} E^x\left[M_t^f \middle| \mathscr{F}_s^0\right] &= M_s^f + E^x\left[f(X_t) - f(X_s) - \int_s^t Af(X_r)\,dr \middle| \mathscr{F}_s^0\right] \\ &= M_s^f + E^{X_s}\left[f(X_{t-s}) - f(X_0) - \int_0^{t-s} Af(X_r)\,dr\right] \\ &= M_s^f + P_{t-s}f(X_s) - f(X_s) - \int_0^{t-s} P_r Af(X_s)\,dr \\ &= M_s^f \end{aligned}$$

by Proposition 2.1.

To show the converse statement observe that, taking expectation with respect to P^x, we have

$$P_t f(x) - f(x) - \int_0^t P_s g(x)\,ds = 0.$$

Thus,

$$\left\|\frac{P_t f - f}{t} - g\right\| = \left\|\frac{1}{t}\int_0^t (P_s g - g)\,ds\right\| \le \frac{1}{t}\int_0^t \|P_s g - g\|\,ds,$$

which goes to 0 as $t \to 0$. □

Loosely speaking the martingale problem is the inverse problem to the statement of the above theorem, i.e. given a generator A one needs to find a process X such that the statement of the theorem holds for a large class of f, which will be a subset of the space of functions defined below.

Definition 2.2 If X is a Markov process, then a Borel measurable function $f : \mathbf{E} \mapsto \mathbb{R}$ is said to belong to the domain $\mathbb{D}_A$ of the extended infinitesimal generator if there exists a Borel measurable function $g : \mathbf{E} \mapsto \mathbb{R}$ such that P^x-a.s. $\int_0^t |g(X_s)|\,ds < \infty$, for every t, and

$$\left(f(X_t) - f(x) - \int_0^t g(X_s)\,ds\right)_{t\ge 0}$$

is a $(P^x, \mathscr{F}_t^0)$-local martingale for any $x \in \mathbf{E}$.

Example 2.2 Consider three-dimensional Bessel process with transition density

$$p_t(x,y) = \frac{y}{x}\frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(x-y)^2}{2t}\right) - \exp\left(-\frac{(x+y)^2}{2t}\right)\right], \qquad x > 0, y \ge 0$$

from (1.5). This density can be extended to $x = 0$ by setting

$$p_t(0, y) = \sqrt{\frac{2}{\pi t^3}} y^2 \exp\left(-\frac{y^2}{2t}\right).$$

Note that three-dimensional Bessel process is the unique solution to

$$X_t = x + B_t + \int_0^t \frac{1}{X_s} ds.$$

In particular, $P^x \left(\int_0^t X_s^{-1} ds < \infty\right) = 1$ (see, e.g. Proposition 3.3.21 in [77]).

Suppose $f : [0, \infty) \mapsto [0, \infty)$ be twice continuously differentiable with a compact support such that $f(x) = x$ for $x \in [0, 1]$. Observe that the domain of this function can be extended to include ∞ by setting $f(\infty) = 0$. Note that this extended function is continuous on $\mathbf{E}_\Delta$. This f, however, does not belong to $\mathscr{D}(A)$. Indeed,

$$\lim_{t \to 0} \frac{P_t f(0)}{t} \geq \lim_{t \to 0} \int_0^1 \sqrt{\frac{2}{\pi t^5}} y^3 \exp\left(-\frac{y^2}{2t}\right) dy = \infty.$$

On the other hand, $f \in \mathbb{D}_A$. To observe that set $g(x) = \left\{\frac{f'(x)}{x} + \frac{1}{2} f''(x)\right\}$ and note that $|g(x)| < \frac{1}{x} + K$, for some constant K. Thus, by Ito's formula,

$$f(X_t) - f(x) - \int_0^t g(X_s)\, ds = \int_0^t f'(X_s)\, dB_s,$$

which is a martingale.

2.2 Local Martingale Problem

In this section we restrict our attention to state space $\mathbf{E} = \times_{i=1}^d [l_i, \infty)$ with the convention that if $l_i = -\infty$, then $[l_i, \infty) = \mathbb{R}$. The metric on $\mathbf{E}$ is assumed to be the Euclidean distance so that it is a Polish space. We would also consider a particular form of the generator, namely:

$$A_t = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, \cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t, \cdot) \frac{\partial}{\partial x_i}, \tag{2.1}$$

where a is a matrix field on $\mathbb{R}_+ \times \mathbf{E}$ and b is a vector field on $\mathbb{R}_+ \times \mathbf{E}$ such that

i) for all $i, j = 1, \ldots, d$, the maps $(t, x) \mapsto a_{ij}(t, x)$ and $(t, x) \mapsto b_i(t, x)$ are Borel measurable.
ii) For each (t, x) the matrix $a(t, x)$ is symmetric and nonnegative, i.e. for every $\lambda \in \mathbb{R}^d$

$$\sum_{i,j} a_{ij}(t, x)\lambda_i \lambda_j \geq 0.$$

Moreover, in many results that will be presented in this section we require a and b satisfy the following additional assumption.

Assumption 2.1

1. *For each $i, j = 1, \ldots, d$, the map $(t, x) \mapsto a_{ij}(t, x)$ is locally bounded on* $\mathbb{R}_+ \times \mathbf{E}$.
2. *For every $n \geq 1$ and any compact $K \subset \mathbf{E}$ there exists $C_n : (0, n) \mapsto \mathbb{R}_+$ with*

$$\int_0^n C_n(s)ds < \infty$$

such that either

i) $\max_{j=1,\ldots,d} b_j(t, x) \leq C_n(t)$ *for all $t \in (0, n)$ and $x \in K$ or*
ii) $\min_{j=1,\ldots,d} b_j(t, x) \geq -C_n(t)$ *for all $t \in (0, n)$ and $x \in K$*

holds.

Remark 2.1 The above assumption on b_i is a relaxation of the standard assumption that it is locally bounded. This in particular allows to study process with unbounded drifts such as the Bessel processes defined by the generator

$$Af = \frac{1}{2} f''(x) + \frac{\delta - 1}{2x} f'(x)$$

for $\delta \geq 0$ and on the state space $[0, \infty)$.

The aim of this section is to find conditions under which there exists a process with this generator. This process can be constructed in the canonical space of continuous functions from $\mathbb{R}_+$ to $\mathbf{E}$, i.e. $C(\mathbb{R}_+, \mathbf{E})$, equipped with the Skorokhod topology and endowed with the Borel σ-algebra, $\mathscr{B}$, as well as the canonical filtration $(\mathscr{B}_t)_{t \geq 0}$ defined by $\mathscr{B}_t = \sigma(X_s; s \leq t)$. It is well known that $\mathscr{B} = \bigvee_{t \geq 0} \mathscr{B}_t$ and it has a countable basis since the corresponding metric space is separable.

Definition 2.3 Let (A_t) be a generator given in (2.1), μ be a probability measure on $(\mathbf{E}, \mathscr{E})$, and $s \geq 0$ be a fixed time. For any $f \in \mathbb{C}^\infty(\mathbf{E})$ define

$$M_t^f := f(X_t) - f(X_s) - \int_s^t A_r f(X_r)\, dr, \tag{2.2}$$

where X is the coordinate process. A probability measure $P^{s,\mu}$ on $(C(\mathbb{R}_+, \mathbf{E}), \mathscr{B})$ is said to solve the local martingale problem for (A, μ) starting from s if

i) $P^{s,\mu} X_s^{-1} = \mu$, $P^{s,\mu}(X_r = X_s, r \leq s) = 1$,
ii) for any $i = 1, \ldots, d$,

$$P^{s,\mu}\left(\int_s^t \{a_{ii}(r, X_r) + |b_i(r, X_r)|\}\, dr < \infty\right) = 1, \qquad \forall t \geq s,$$

iii) for all $f \in \mathbb{C}^\infty(\mathbf{E})$, $\left((M_t^f)_{t \geq s}, (\mathscr{B}_t)_{t \geq s}\right)$ is a local martingale.

Note that the condition ii) yields that

$$P^{s,\mu}\left(\int_s^t |a_{ij}(r, X_r)| dr < \infty\right) = 1$$

for all $j = 1, \ldots, d$ since a is positive semidefinite. This in turn implies that for any $f \in \mathbb{C}^\infty(\mathbf{E})$ and $t \geq s$

$$P^{s,\mu}\left(\int_s^t |A_r f(X_r)|\, dr < \infty\right) = 1$$

by a standard localisation argument.

Remark 2.2 Note that the condition iii) implies that the local time of the solution at the boundary of $\mathbf{E}$ is identically 0. This is obviously satisfied when the boundary is not reached. However, other boundary behaviour is also possible such as instantaneous reflection as in the case of a squared Bessel process with dimension in (0, 2) (see Proposition XI.1.5 in [100]).

Remark 2.3 Observe that if Assumption 2.1 holds, then one only need to check the conditions i) and iii) in order to establish that a given measure is a solution.

Indeed, under Assumption 2.1 the integrals $\int_s^t b_i(u, X_u)\, du$ and $\int_s^t a_{ii}(u, X_u)\, du$ are well defined until $\tau_n = \inf\{t \geq s : \|X_t\| \geq n\}$ for any $n \geq 1$. Due to the continuity of X, $\tau_n \to \infty$ implying both of these integrals are well defined for all $t \geq s$ and

$$P^{s,\mu}\left(\int_s^t a_{ii}(u, X_u)\, du < \infty\right) = 1.$$

Since for any $i = 1, \ldots, d$

$$X^i - \int_s^\cdot b_i(r, X_r) dr$$

is a $P^{s,\mu}$-local martingale, it follows that

$$P^{s,\mu}\left(\left|\int_s^t b_i(u, X_u)\, du\right| < \infty\right) = 1.$$

Using the fact that b_i is locally bounded from above or below once more, one can show that the above further implies

$$P^{s,\mu}\left(\int_s^t |b_i(u, X_u)|\, du < \infty\right) = 1, \qquad t \geq s,$$

by using the localising sequence $(\tau_n)_{n\geq 1}$.

For a given generator there are many different ways of formulating the local martingale problem. These formulations will differ in classes of functions f for which the processes defined in (2.2) are martingales. One particularly useful way is to define the local martingale problem by only considering functions f belonging to a countable subset of $\mathbb{C}^\infty(\mathbf{E})$.

Proposition 2.2 *The set $\mathbb{C}_p^\infty$ defined by*

$$\mathbb{C}_p^\infty = \left\{p_n \in \mathbb{C}^\infty(\mathbf{E}) : p_n(x) = \Pi_{j=1}^d (x_j)^{\alpha_j},\ \textit{for some}\ \alpha \in \mathbb{N}^d\right\}$$

is countable. Suppose Assumption 2.1 is in force and there exists a probability measure $P^{s,\mu}$ on $(C(\mathbb{R}_+, \mathbf{E}), \mathscr{B})$ such that $\left((M_t^f)_{t\geq s}, (\mathscr{B}_t)_{t\geq s}\right)$ is a local martingale for all $f \in \mathbb{C}_p^\infty$, where M^f is given by (2.2). If, moreover, $P^{s,\mu}X_s^{-1} = \mu$ and $P^{s,\mu}(X_r = X_s, r \leq s) = 1$, then $P^{s,\mu}$ solves the local martingale problem for (A, μ) starting from s.

Proof The set $\mathbb{C}_p^\infty$ is clearly countable.

Next suppose there exists a $P^{s,\mu}$ as in the statement of the proposition. Then, for any $r \geq s, t \geq r$, we will have

$$E^{s,\mu}\left[\int_r^t \mathbf{1}_{[u\leq\tau_n]} \sum_{i=1}^d |b_i(u, X_u)|\, du\right] < \infty, \tag{2.3}$$

where $\tau_n = \inf\{t \geq s : X_t \notin \mathbf{E} \cap (-n, n)^d\} \wedge n$ is a stopping time. Observe that $\tau_n \to \infty$, $P^{s,\mu}$-a.s. as n tends to infinity since X is continuous. To demonstrate the statement consider $g(x) = \sum_{i=1}^d x_i$. Since g belongs to span($\mathbb{C}_p^\infty$), M^g is a local martingale with localising sequence $(S_m)_{m\geq 1}$. Suppose b_i is locally bounded from below for all i by an integrable function, i.e. there exists $C_n \geq 0$ such that $b_i(t, x) + C_n(t) \geq 0$ for all $x \in \mathbf{E} \cap [-n, n]^d$, $t \in [s, n)$, and all i. Thus,

$$
\begin{aligned}
&E^{s,\mu}\left[\int_r^t \mathbf{1}_{[u\le\tau_n\wedge S_m]}\sum_{i=1}^d |b_i(u,X_u)|\,du\right]\\
&= E^{s,\mu}\left[\int_r^t \mathbf{1}_{[u\le\tau_n\wedge S_m]}\sum_{i=1}^d |b_i(u,X_u)+C_n(u)-C_n(u)|\,du\right]\\
&\le E^{s,\mu}\left[\int_r^t \mathbf{1}_{[u\le\tau_n\wedge S_m]}\sum_{i=1}^d b_i(u,X_u)du\right]+2d\int_r^t C_n(u)du\\
&= E^{s,\mu}\left[M^g_{r\wedge\tau_n\wedge S_m}-M^g_{t\wedge\tau_n\wedge S_m}+g(X_{t\wedge\tau_n\wedge S_m})-g(X_{r\wedge\tau_n\wedge S_m})\right]+2d\int_r^t C_n(u)du\\
&\le 2d\left(n+\int_r^t C_n(u)du\right)<\infty.
\end{aligned}
$$

Since $S_m\to\infty$, $P^{s,\mu}$-a.s., we obtain (2.3) by the Monotone Convergence Theorem. Similar considerations show that (2.3) holds when b is locally bounded from above.

We next show that for any $g\in\mathrm{span}(\mathbb{C}_p^\infty)$, $(M^g_{t\wedge\tau_n})$ is a martingale. Indeed, for any $s\le r\le t$

$$
E^{s,\mu}\left[g(X_{t\wedge\tau_n\wedge S_m})-g(X_{r\wedge\tau_n\wedge S_m})-\int_r^t \mathbf{1}_{[u\le\tau_n\wedge S_m]}A_u g(X_u)du\,\middle|\,\mathscr{B}_r\right]=0,
$$

where (S_m) is the localising sequence for M^g. Observe that

$$
\int_r^t \mathbf{1}_{[u\le\tau_n\wedge S_m]}|A_u g(X_u)|du\le C_1\int_r^t \mathbf{1}_{[u\le\tau_n]}\sum_{i=1}^d|b_i(u,X_u)|\,du+C_2,
$$

where C_1 and C_2 are constants independent of m. Thus, the claim follows from the Dominated Convergence Theorem as $m\to\infty$ in view of (2.3).

Finally, we are ready to show that for any $f\in\mathbb{C}^\infty(\mathbf{E})$, M^f is a local martingale. If f is in the span of $\mathbb{C}_p^\infty$ this is true. If not, since polynomials are dense in $\mathbb{C}^2_K(\mathbf{E})$ for any compact K, for any $n\in\mathbb{N}$ there exists $(f_{k,n})_{k\ge1}\subset\mathrm{span}(\mathbb{C}_p^\infty)$ such that $f_{k,n}$ together with the first and second derivatives converge uniformly to f and its corresponding derivatives on $\mathbf{E}\cap[-n,n]^d$. For each $f_{k,n}$ by above we have

$$
E^{s,\mu}\left[f_{k,n}(X_{t\wedge\tau_n})-f_{k,n}(X_{r\wedge\tau_n})-\int_r^t \mathbf{1}_{[u\le\tau_n]}A_u f_{k,n}(X_u)\,du\,\middle|\,\mathscr{B}_r\right]=0,\ \forall s\le r\le t.
$$

Due to the aforementioned uniform convergence, we have that $f_{k,n}$s are uniformly bounded along with their first two derivatives. Thus,

$$\int_r^t \mathbf{1}_{[u \le \tau_n]} |A_u f_{k,n}(X_u)| \, du \le C_1 \int_r^t \mathbf{1}_{[u \le \tau_n]} \sum_{i=1}^d |b_i(u, X_u)| du + C_2,$$

where C_1 and C_2 do not depend on k. In view of (2.3) the Dominated Convergence Theorem yields

$$E^{s,\mu}\left[f(X_{t\wedge\tau_n}) - f(X_{r\wedge\tau_n}) - \int_{r\wedge\tau_n}^{t\wedge\tau_n} A_u f(X_u) \, du \middle| \mathscr{B}_r \right] = 0, \qquad \forall s \le r \le t.$$

Since $\tau_n \to \infty$, P^μ-a.s., M^f is a local martingale. □

Definition 2.4 A local martingale problem for a given generator A is said to be well posed if there exists a unique solution to the local martingale problem for (A, μ) starting from s for any probability measure μ on $(\mathbf{E}, \mathscr{E})$ and any $s \ge 0$.

If one has a solution $P^{s,x}$ for the local martingale problem (A, ε_x) starting from s for each $x \in \mathbf{E}$, then one may try to construct a solution to the local martingale problem for (A, μ) starting from s by setting $P^{s,\mu}(B) = \int_{\mathbf{E}} P^{s,x}(B)\mu(dx)$ for any $B \in \mathscr{E}$. However, this requires the measurability of the map $x \mapsto P^{s,x}(B)$ for any $B \in \mathscr{E}$ and $s \ge 0$. Under this condition the well posedness of the local martingale problem for A can be reduced to the existence and uniqueness of the local martingale problem for (A, ε_x) starting from s for each $x \in \mathbf{E}$ and $s \ge 0$.

Theorem 2.2 *Suppose that Assumption 2.1 holds and there exists a unique solution to the local martingale problem for (A, ε_x) starting from s for each $x \in \mathbf{E}$ and $s \ge 0$. Then, the mapping $x \mapsto P^{s,x}(B)$ is measurable for any $B \in \mathscr{B}$ and $s \ge 0$, and the local martingale problem for A is well posed.*

Proof Denote the space of probability measures on $(C(\mathbb{R}_+, \mathbf{E}), \mathscr{B})$ (resp. $(\mathbf{E}, \mathscr{E})$) with $\mathscr{P}(C(\mathbb{R}_+, \mathbf{E}))$ (resp. $\mathscr{P}(\mathbf{E})$). These spaces endowed with the Prokhorov metric are known to be complete and separable. Let $s \ge 0$ be fixed and consider $\mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E})) \subset \mathscr{P}(C(\mathbb{R}_+, \mathbf{E}))$ such that $P \in \mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E}))$ if and only if $P \in \mathscr{P}(C(\mathbb{R}_+, \mathbf{E}))$ and $P(X_r = X_s, r \le s) = 1$. Then, $\mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E}))$ is also complete and separable under the Prokhorov metric.

Observe that the map $x \mapsto \varepsilon_x$ is Borel measurable since it is continuous. Thus, if we can show that the map $\varepsilon_x \mapsto P^{s,x}$ is Borel measurable, this would imply the map $x \mapsto P^{s,x}$ is Borel measurable. Suppose we are able to demonstrate that the set of measures in $\mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E}))$ solving the local martingale problem (A, ε_x) starting from s is Borel for each $x \in \mathbf{E}$. Then, since the map $G : \mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E})) \mapsto \mathscr{P}(\mathbf{E})$ defined by $GP = PX_s^{-1}$ is continuous and its restriction to the aforementioned set of measures is one-to-one with range $\{\varepsilon_x : x \in \mathbf{E}\}$, its restriction has a measurable inverse, i.e. $\varepsilon_x \mapsto P^{s,x}$ is measurable. Thus, it remains to show that the set of measures solving the local martingale problem (A, ε_x) starting from s is Borel for each $x \in \mathbf{E}$.

To do so consider the collection of functions, H, on $C(\mathbb{R}_+, \mathbf{E})$ defined by

$$\eta = \left(f(X_{t_{n+1}\wedge\tau_m(f)}) - f(X_{t_n\wedge\tau_m(f)}) - \int_{t_n}^{t_{n+1}} \mathbf{1}_{[u\le\tau_m(f)]} A_u f(X_u)\, du \right) \Pi_{k=1}^n \mathbf{1}_{[X_{t_k}\in E_{k_i}]}$$

where $t_k \in \mathbb{Q} \cap [s, \infty)$, $f \in \mathbb{C}_p^\infty$, $\tau_m(f) := \inf\{t \ge s : |M_t^f| \ge m\}$, and E_k belong to the countable basis of $\mathscr{E}$. Clearly, the set of all such functions is countable. Note that when M^f is a local martingale, the sequence $(\tau_m(f))$ is a localising sequence under Assumption 2.1. This implies, in view of Proposition 2.2, that $P \in \mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E}))$ solves the local martingale problem for A starting from s if and only if

$$\int \eta dP = 0, \qquad \eta \in H.$$

Thus,

$$\mathscr{M}_A = \cap_{\eta\in H} \left\{ P \in \mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E})) : \int \eta dP = 0 \right\}$$

is the set of probability measures solving the local martingale problem for A. Note that for each $\eta \in H$ the set $\left\{P \in \mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E})) : \int \eta dP = 0\right\}$ is Borel.[1] Therefore, $\mathscr{M}_A$ is also Borel since H is countable. The set of measures solving the local martingale problem (A, ε_x) starting from s for some $x \in \mathbf{E}$ is given by $\mathscr{M}_A \cap D$, where $D = \{P \in \mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E})) : PX_s^{-1} = \varepsilon_x$ for some $x \in \mathbf{E}\}$. Thus, it suffices to show that D is Borel. Indeed,

$$D = \left\{ P \in \mathscr{P}_s(C(\mathbb{R}_+, \mathbf{E})) : \sum_{(i,j)\in J} PX_s^{-1}(B_i) PX_s^{-1}(B_j) = 0 \right\},$$

where $(B_i)_{i=1}^\infty$ is the countable base of $\mathscr{E}$ and $J = \{(i, j) : B_i \cap B_j = \emptyset\}$. Since J is countable it is evident that D is Borel. This completes the proof of the measurability of the map $x \mapsto P^{s,x}(B)$ for every $B \in \mathscr{B}$.

Consider any $\mu \in \mathscr{P}(\mathbf{E})$ and define $P^{s,\mu} = \int_E P^{s,x} \mu(dx)$. Then, $P^{s,\mu} X_s^{-1} = \mu$ and $P^{s,\mu}$ solves the local martingale problem (A, μ) starting from s. Thus, to show the well posedness, we need to show the uniqueness of the solution of martingale problem for (A, μ). Consider any $P \in \mathscr{M}_A$ such that $PX_s^{-1} = \mu$ and define $Q(B) = P[B|X_s]$ for any $B \in \mathscr{B}$. Observe that for any $f \in \mathbb{C}_p^\infty$

[1] If η is continuous, the map $\Gamma_\eta(P) := \int \eta dP$ is continuous, and therefore Borel. Since indicator functions of closed sets in metric spaces can be approximated by continuous functions by Urysohn's lemma, $\Gamma_{\mathbf{1}_A}$ is Borel for any closed subset of $C(\mathbb{R}_+, \mathbf{E})$. Finally, that Γ_η is Borel for any bounded η follows from a standard monotone class argument.

$$f(X_{t\wedge\tau_m(f)}) - f(X_s) - \int_s^{t\wedge\tau_m(f)} A_u f(X_u)\,du$$

is a $(P, (\mathscr{B}_t)_{t\geq s})$-martingale and therefore $\int \eta dQ = 0$ for all $\eta \in H$. Hence, $Q \in \mathscr{M}_A$. Moreover, $QX_s^{-1} = \varepsilon_{X_s}$. By the uniqueness of the solution to the local martingale problem (A, ε_x), $Q = P^{s,X_s}$. Since $P = EQ$, we have

$$P = EQ = EP^{s,X_s} = \int_E P^{s,x}\mu(dx) = P^{s,\mu}.$$

□

If the functions a and b do not depend on time, then $A_t = A$ for all $t \geq 0$. We say P^μ solves the time-homogeneous local martingale problem if $P^\mu = P^{0,\mu}$, where $P^{0,\mu}$ solves the local martingale problem for (A, μ) starting from 0. A time-homogeneous local martingale problem is said to be well posed if there exists a unique solution for the local martingale problem (A, μ) starting from 0 for every μ. However, as in the time inhomogeneous case, the time-homogeneous problem is going to be well posed if there exists a unique solution to the time-homogeneous local martingale problem (A, ε_x) for any $x \in \mathbf{E}$ in view of Theorem 2.2.

In the next theorem we will show that the well posedness of the time-homogeneous local martingale problem leads to the strong Markov property of its solutions. Note that this result does not impose any extra conditions on the coefficients of A as soon as the local martingale problem is well posed. Recall that $\mathscr{B}_{\tau+} = \{A \in \mathscr{B} : A \cap [\tau < t] \in \mathscr{B}_t, \ \forall t \geq 0\}$ for τ an optional time.

Theorem 2.3 *Suppose that the time-homogeneous local martingale problem for A is well posed. Then, for any P^μ-a.s. finite optional time τ, we have*

$$P^\mu[X \circ \theta_\tau \in \cdot|\mathscr{B}_{\tau+}] = P^\mu[X \circ \theta_\tau \in \cdot|X_\tau] = P^{X_\tau}[X \in \cdot], \ P^\mu\text{-a.s.}.$$

Proof Consider $F \in \mathscr{B}_{\tau+}$ with $P^\mu(F) > 0$ and define P_1, P_2 and P_3 on $\mathscr{B}$ by

$$P_1(B) = \frac{E^\mu[\mathbf{1}_F P^\mu[X \circ \theta_\tau \in B|X_\tau]]}{P^\mu(F)},$$

$$P_2(B) = \frac{E^\mu[\mathbf{1}_F P^\mu[X \circ \theta_\tau \in B|\mathscr{B}_{\tau+}]]}{P^\mu(F)}, \text{ and}$$

$$P_3(B) = \frac{E^\mu[\mathbf{1}_F P^{X_\tau}(B)]}{P^\mu(F)} \ B \in \mathscr{B}.$$

Observe that X_τ is $\mathscr{B}_{\tau+}$-measurable by Theorem A.7 and the continuity of X, thus, the statement of the theorem follows from the definition of conditional expectation if P_1, P_2 and P_3 coincide.

First, note that for any $\Gamma \in \mathscr{E}$

$$P_1(X_0 \in \Gamma) = \frac{E^\mu[\mathbf{1}_F \mathbf{1}_{X_\tau \in \Gamma}]}{P^\mu(F)}.$$

Similarly, $P_2(X_0 \in \Gamma) = P_3(X_0 \in \Gamma)\frac{E^\mu[\mathbf{1}_F \mathbf{1}_{X_\tau \in \Gamma}]}{P^\mu(F)}$. Thus, if we can demonstrate that under each P_i the process $M^f \circ \theta_\tau$ is a local martingale for all $f \in \mathbb{C}^\infty(\mathbf{E})$, we can conclude that P_is coincide since all of them solve the local martingale problem starting from 0 with the same initial distribution.

To do so fix $f \in \mathbb{C}^\infty(\mathbf{E})$ and let $T_m(f) = \inf\{t \geq 0 : |(M^f \circ \theta_\tau)_t| \geq m\}$ be a sequence of stopping times. Recall that $\tau_m(f) = \inf\{t \geq 0 : |M_t^f| \geq m\}$ and we have

$$\left(M^f \circ \theta_\tau\right)_{t \wedge T_m(f)} = M^f_{(\tau+t)\wedge\sigma_m(f)},$$

where $\sigma_m(f) = \inf\{t \geq \tau : |M_t^f| \geq m\}$ is optional by Corollary A.1. Moreover, it equals to $\tau + \tau_m(f) \circ \theta_\tau$.

For any $s \leq t$, we have $E^\mu[M^f_{(\tau+t)\wedge\sigma_m(f)} - M^f_{(\tau+s)\wedge\sigma_m(f)}|\mathscr{B}_{(\tau+s)\wedge\sigma_m(f)+}] = 0$. Indeed, on $[\sigma_m(f) = \tau]$ the equality is obvious. Moreover, $|M_r^f| < m$ for $r \in [\tau, \sigma_m(f)]$ on $[\sigma_m(f) > \tau]$. Thus, the claim follows from the Optional Sampling Theorem along with the fact that M^f is a continuous local martingale. Therefore, $\left(\left(M^f \circ \theta_\tau\right)_{t \wedge T_m(f)}\right)$ is a martingale under P_2 for all m; hence, $M^f \circ \theta_\tau$ is a local martingale under P_2 with the localising sequence $(T_m(f))_{m \geq 1}$. The same reasoning yields M^f is a local martingale under P_1 and P_3, too. □

Remark 2.4 When A is time inhomogeneous, one can create a time-homogeneous operator A^o simply via

$$A^o = \frac{d}{dt} + A.$$

In this case we can consider a new process $Y := (t, X)$ by adding time to our state space and look for the solutions of the *time-homogeneous* local martingale problem within the space of probability measures on $C(\mathbb{R}_+, \mathbb{R}_+ \times \mathbf{E})$. It can be directly verified that the solutions of the local martingale problem $(A^o, \varepsilon_s \times \varepsilon_x)$ are in one-to-one correspondence with those of the local martingale problem (A, ε_x) starting from s. Thus, if the time-inhomogeneous local martingale problem for A is well posed, so is the time-homogeneous local martingale problem for A^o. Theorem 2.3 can then be applied to deduce

$$P^{s,\mu}\left[X \circ \theta_\tau \in \cdot|\mathscr{B}_{\tau+}\right] = P^{s,\mu}\left[X \circ \theta_\tau \in \cdot|\tau, X_\tau\right],$$

where $P^{s,\mu}$ is the unique solution of the local martingale problem (A, μ) starting from s, and $\tau \geq s$. In the sequel the strong Markov property of the solutions of the time-inhomogeneous local martingale problem should be understood to be the strong Markov property of the space-time process (t, X).

If one is content with the Markov property of the solutions of the time-inhomogeneous local martingale problem, one can directly apply the arguments of Theorem 2.3 instead of enlarging the state space suggested in the remark above. Indeed, by fixing $s \leq t$, taking $\tau = t$, and replacing $\mathscr{B}_{\tau+}$ with $\mathscr{B}_t$, i.e.

$$P^{s,\mu}\left[X_{u+t} \in \cdot|\mathscr{B}_t\right] = P^{s,\mu}\left[X_{u+t} \in \cdot|X_t\right], \; P^{s,\mu}\text{-a.s.},$$

the conclusion will immediately follow if P_1 and P_2 are replaced by

$$\frac{E^{s,\mu}[\mathbf{1}_F P^{s,\mu}[X \circ \theta_t \in B|X_t]]}{P^{s,\mu}(F)}, \; B \in \mathscr{B},$$

and

$$\frac{E^{s,\mu}[\mathbf{1}_F P^{s,\mu}[X \circ \theta_t \in B|\mathscr{B}_t]]}{P^{s,\mu}(F)}, \; B \in \mathscr{B},$$

respectively.

If one follows the steps of the proof of Theorem 2.3 to prove the Markov property after making the necessary adjustments stated above, one will realise that one does not need the uniqueness of the solutions of the local martingale problem to establish the result. In fact, all that is needed is the uniqueness of the marginal distributions of solutions of the local martingale problem. This observation leads to the following result.

Proposition 2.3 *Suppose that for any $s \geq 0$ and distribution $\mu \in \mathscr{P}(\mathbf{E})$, we have*

$$P^{s,\mu}(X_t \in E) = Q^{s,\mu}(X_t \in E), \qquad \forall E \in \mathscr{E}, t \geq s, \tag{2.4}$$

where $P^{s,\mu}$ and $Q^{s,\mu}$ are two arbitrary solutions of the local martingale problem for (A, μ) starting at s. Then, X is a Markov process under any solution of the local martingale problem.

Inductively, the Markov property together with (2.4) yield that the solutions of the local martingale have the same finite dimensional distributions. Thus, (2.4) will imply that the local martingale problem for (A, μ) has at most one solution.

Proposition 2.4 *Suppose that (2.4) holds for all $\mu \in \mathscr{P}(\mathbf{E})$ and $s \geq 0$ whenever $P^{s,\mu}$ and $Q^{s,\mu}$ are two arbitrary solutions of the local martingale problem for (A, μ) starting from s. Then, for any $s \geq 0$, the local martingale problem for (A, μ) starting from s has at most one solution.*

Proof Fix a $\mu \in \mathscr{P}(\mathbf{E})$, $s \geq 0$, and consider two arbitrary solutions, $P^{s,\mu}$ and $Q^{s,\mu}$, of the local martingale problem (A, μ) starting from s. We need to show that for any $m \geq 1$

$$E^P\left[\prod_{k=1}^{m} f_k(X_{t_k})\right] = E^Q\left[\prod_{k=1}^{m} f_k(X_{t_k})\right] \tag{2.5}$$

for all choices of $t_k \in [s, \infty)$ and $f_k \in b\mathscr{E}_{++}$, where $P = P^{s,\mu}$ and $Q = Q^{s,\mu}$, which will in turn imply that the above holds for all $f_k \in b\mathscr{E}$. Then, a monotone class argument will yield that P and Q coincide on $\mathscr{B}$.

The proof will proceed by induction on m. Clearly, (2.5) holds for $m = 1$ by hypothesis. Suppose that it holds for all $m \leq n$ and fix $s \leq t_1 < t_2 < \ldots < t_n$ and $f_1, \ldots, f_n$. Define $Y_t = X_{t_n \vee t}$ for $t \geq 0$,

$$\tilde{P}(B) = \frac{E^P\left[\mathbf{1}_B \prod_{k=1}^{n} f_k(X_{t_k})\right]}{E^P\left[\prod_{k=1}^{n} f_k(X_{t_k})\right]}, \text{ and}$$

$$\tilde{Q}(B) = \frac{E^Q\left[\mathbf{1}_B \prod_{k=1}^{n} f_k(X_{t_k})\right]}{E^Q\left[\prod_{k=1}^{n} f_k(X_{t_k})\right]}, \;\; B \in \mathscr{B}.$$

Note that

$$\begin{aligned}\tilde{P}(X_{t_n} \in E) &= \frac{E^P\left[\mathbf{1}_{X_{t_n} \in E} \prod_{k=1}^{n} f_k(X_{t_k})\right]}{E^P\left[\prod_{k=1}^{n} f_k(X_{t_k})\right]} \\ &= \frac{E^Q\left[\mathbf{1}_{X_{t_n} \in E} \prod_{k=1}^{n} f_k(X_{t_k})\right]}{E^Q\left[\prod_{k=1}^{n} f_k(X_{t_k})\right]} = \tilde{Q}(X_{t_n} \in E),\end{aligned}$$

where the second equality is due to the induction hypothesis. Moreover, as in the proof of Theorem 2.3, one can demonstrate that, under $\tilde{P}$ and $\tilde{Q}$, $M^f(Y)$ is a local martingale for all $f \in \mathbb{C}^\infty(\mathbf{E})$, where

$$M_t^f(Y) := f(Y_t) - f(Y_{t_n}) - \int_{t_n}^{t} A_r f(Y_r)\, dr.$$

Since $\tilde{Q}(Y_r = Y_{t_n}, r \leq t_n) = \tilde{P}(Y_r = Y_{t_n}, r \leq t_n) = 1$ and $\tilde{P}(Y_{t_n} \in E) = \tilde{Q}(Y_{t_n} \in E)$ for all $E \in \mathscr{E}$, $\tilde{P}$ and $\tilde{Q}$ solve the same local martingale problem starting from t_n. Therefore, Y must have the same marginal distribution under either of these measures. That is, for any $t \geq 0$ and $f \in b\mathscr{E}$

$$\frac{E^P\left[f(X_{t+t_n}) \prod_{k=1}^{n} f_k(X_{t_k})\right]}{E^P\left[\prod_{k=1}^{n} f_k(X_{t_k})\right]} = \frac{E^Q\left[f(X_{t+t_n}) \prod_{k=1}^{n} f_k(X_{t_k})\right]}{E^Q\left[\prod_{k=1}^{n} f_k(X_{t_k})\right]}.$$

Due to the induction hypothesis this implies

$$E^P\left[f(X_{t+t_n})\prod_{k=1}^{n} f_k(X_{t_k})\right] = E^Q\left[f(X_{t+t_n})\prod_{k=1}^{n} f_k(X_{t_k})\right],$$

i.e. (2.5) holds for $m = n + 1$. □

Combining the above two propositions we arrive at the following corollary.

Corollary 2.2 *Suppose that for any $\mu \in \mathscr{P}(\mathbf{E})$ and $s \geq 0$ there exists a solution to the local martingale problem (A, μ) starting from s and that any two solutions satisfy (2.4). Then, the local martingale problem for A is well posed and its solutions have the strong Markov property.*

We have seen in Theorem 2.3 that well posedness of the local martingale problem implies the strong Markov property. Moreover, Theorem 2.2 has shown under a technical condition on the coefficients the measurability of the family $(P^{s,x})_{x\in\mathbf{E}}$ for each $s \geq 0$. On the other hand, if X is a time-homogeneous Feller process under $(P^x)_{x\in\mathbf{E}}$, then (P^x) is weakly continuous, rather than being merely measurable, even without the technical assumption.

Definition 2.5 A family of measures $(Q^x)_{x\in I}$, $I \subset \mathbb{R}^d$ on $(C(R_+, \mathbf{E}), \mathscr{B})$ is said to be weakly continuous if for any sequence $(x_n) \subset I$ such that $\lim_{n\to\infty} x_n = x \in I$, one of the following equivalent conditions holds:

1. For any bounded continuous function $f : C(R_+, \mathbf{E}) \mapsto \mathbb{R}$ $Q^{x_n}(f) \to Q^x(f)$, where $Q^x(f) = \int_{C(R_+,\mathbf{E})} f(\omega)Q^x(d\omega)$.
2. For any $A \in \mathscr{B}$ with $Q^x(\partial A) = 0$, we have $Q^{x_n}(A) \to Q^x(A)$.

Proposition 2.5 *Suppose that $\mathbf{E}$ is compact and the time-homogeneous local martingale problem, A, has a family of solutions $(P^x)_{x\in\mathbf{E}}$ under which X is a Feller process, then (P^x) is weakly continuous.*

Proof Let $x \in \mathbf{E}$ and $(x_n) \subset \mathbf{E}$ be sequence converting to x. The weak convergence of P^{x_n} to P^x can be verified by following a two-step procedure: first, the tightness of the sequence is established, and, second, finite-dimensional distributions of X under P^{x_n} are shown to converge to those under P^x (see Proposition A.2).

To show that (P^{x_n}) is tight we use Aldous' criterion (see Theorem A.4). First, observe that $(P^{x_n}X_0^{-1})$ are tight on $(\mathbf{E}, \mathscr{E})$ and converge to ε_x. Moreover, for any stopping time τ, $y \in \mathbf{E}$, and $s > 0$, we have

$$P^y(m(X_{\tau+s}, X_\tau) > \varepsilon) = E^y\left[P^{X_\tau}(d(X_s, X_0) > \varepsilon)\right] \leq \sup_{y\in\mathbf{E}} P^y(m(X_s, X_0) > \varepsilon),$$

where m is the metric of $\mathbf{E}$. The above together with Proposition 1.7 yields

$$\lim_{h\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathscr{T}}\sup_{s\in[0,h]} P^{x_n}(m(X_{\tau+s}, X_\tau) > \varepsilon) = 0,$$

where $\mathscr{T}$ is the class of stopping times taking finitely many values.

Thus, it remains to show the convergence of finite dimensional distributions. To this end let $r \in \mathbb{N}$ and $f : \mathbb{E}^r \mapsto \mathbb{R}$ be a continuous function with a compact support. If for all such functions we have

$$\lim_{n\to\infty} E^{x_n}\left[f(X_{t_1},\ldots,X_{t_r})\right] = E^x\left[f(X_{t_1},\ldots,X_{t_r})\right], \qquad \forall 0 \le t_1 \le \ldots \le t_r,$$

then this will complete the proof of that P^{x_n} converges weakly to P^x due to Proposition A.1. We will show this by induction on r. Clearly, the above holds for $r = 1$ by the Feller property. Suppose it holds for r and let $f = f^1 f^2$ with $f^1 : \mathbf{E}^r \mapsto \mathbb{R}$ and $f^2 : \mathbf{E} \mapsto \mathbb{R}$ being continuous functions with compact support. Then,

$$\begin{aligned}\lim_{n\to\infty} E^{x_n}[f^1(X_{t_1},\ldots,X_{t_r})f^2(X_{t_{r+1}})] &= \lim_{n\to\infty} E^{x_n} \\ &\quad [f^1(X_{t_1},\ldots,X_{t_m r})P_{t_{r+1}-t_r}f^2(X_{t_r})] \\ &= E^x[f^1(X_{t_1},\ldots,X_{t_r})P_{t_{r+1}-t_r}f^2(X_{t_r})],\end{aligned}$$

by the induction hypothesis since $f^1 P_{t_{r+1}-t_r} f^2$ is continuous (due to the Feller property) with a compact support.

By Stone–Weierstrass Theorem there exists a sequence of functions (f_k) of the form $\sum_{j=1}^{l_k} f_{k_j}^1 f_{k_j}^2$ such that $f_{k_j}^1 : \mathbf{E}^m \mapsto \mathbb{R}$ and $f_{k_j}^2 : \mathbf{E} \mapsto \mathbb{R}$ are continuous functions with compact support, and f_k converge uniformly to f. Fix an arbitrary $\delta > 0$ and choose K such that $\|f_k - f\| < \delta$ for all $k \ge K$. Then,

$$\begin{aligned}&\left|E^x\left[f(X_{t_1},\ldots,X_{t_{r+1}})\right] - E^{x_n}\left[f(X_{t_1},\ldots,X_{t_{r+1}})\right]\right| \\ &\le \left|E^x\left[f(X_{t_1},\ldots,X_{t_{r+1}})\right] - E^x\left[f^k(X_{t_1},\ldots,X_{t_{r+1}})\right]\right| \\ &\quad + \left|E^x\left[f^k(X_{t_1},\ldots,X_{t_{r+1}})\right] - E^{x_n}\left[f^k(X_{t_1},\ldots,X_{t_{r+1}})\right]\right| \\ &\quad + \left|E^{x_n}\left[f^k(X_{t_1},\ldots,X_{t_{r+1}})\right] - E^{x_n}\left[f(X_{t_1},\ldots,X_{t_{r+1}})\right]\right| \\ &\le 2\delta + \left|E^x\left[f^k(X_{t_1},\ldots,X_{t_{r+1}})\right] - E^{x_n}\left[f^k(X_{t_1},\ldots,X_{t_{r+1}})\right]\right|.\end{aligned}$$

Letting $n \to \infty$ yields the result by the arbitrariness of δ. □

The above proof obviously cannot be directly applied when $\mathbf{E}$ is not compact. However, one can consider the $\mathbf{E}_\Delta$, the one-point compactification of $\mathbf{E}$ and use the above proof to show the weak convergence of measures under the topology defined by d, the metric of $\mathbf{E}_\Delta$. Unfortunately, the convergence under this topology does not in general imply the convergence under the original weak topology defined by m, the metric of $\mathbf{E}$. Nevertheless, one can demonstrate the weak convergence of the family (P^{x_n}) when P^{x_n} is the law of the Feller process starting at x_n. The crux of the proof rests on the fact that the limit under the weak topology defined by d assigns 0 probability measure to the paths taking the value Δ. The proof requires techniques outside the scope of this text and can be found in Theorem 4.2.5 in [50].

Theorem 2.4 *Suppose that the time-homogeneous local martingale problem, A, has a family of solutions $(P^x)_{x\in\mathbf{E}}$ under which X is a Feller process, then (P^x) is weakly continuous.*

2.3 Stochastic Differential Equations

Let $\mathbf{E}$ be a state space as in Sect. 2.2. In this section we will introduce a class of stochastic differential equations with solutions in $\mathbf{E}$ and study its connection to the solution of the local martingale problem described in the previous section. In particular we will consider the following stochastic differential equation (SDE):

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)dW_t \tag{2.6}$$

where W is an d-dimensional Brownian motion, b is a $d \times 1$ drift vector and σ is a $d \times d$ dispersion matrix. Define

$$a_{ij}(t, x) = \sum_{k=1}^{r} \sigma_{ik}(t, x)\sigma_{kj}(t, x).$$

We suppose that a and b satisfy the conditions stated in Sect. 2.2.

Our first goal in this section is to establish the relationship between local martingale problems and weak solutions of stochastic differential equations.

2.3.1 Local Martingale Problem Connection

We will demonstrate in this section that under our standing assumptions the solution to a local martingale problem coincides with the weak solution of an SDE.

Definition 2.6 A weak solution starting from s of (2.6) is a triple (X, W), $(\Omega, \mathscr{F}, P)$, $(\mathscr{F}_t)_{t\geq 0}$, where

i) $(\Omega, \mathscr{F}, P)$ is a probability space, $(\mathscr{F}_t)$ is a filtration of sub-σ-fields of $\mathscr{F}$, and X is a process on $(\Omega, \mathscr{F}, P)$ with sample paths in $C([0,\infty), \mathbf{E})$ such that $P(X_r = X_s, r \leq s) = 1$;
ii) W is d-dimensional $(\mathscr{F}_t)$-Brownian motion and $X = (X_t)$ is adapted to $(\bar{\mathscr{F}}_t)$ where $\bar{\mathscr{F}}_t$ is the completion of $\mathscr{F}_t$ with P-null sets;
iii) For any $i = 1, \ldots, d$,

$$P\left(\int_s^t a_{ii}(u, X_u)\,du + \int_s^t |b_i(u, X_u)|\,du < \infty\right) = 1. \tag{2.7}$$

iv) X and W are such that

$$X_t = X_s + \int_s^t b(u, X_u)\, du + \int_s^t \sigma(u, X_u)\, dW_u, \; P\text{-a.s.}, \forall t \geq s.$$

The probability measure μ on $\mathscr{E}$ defined by $\mu(\Lambda) = P(X_s \in \Lambda)$ is called the *initial distribution* of the solution.

Note that Condition iii) implies that any weak solution is a continuous semimartingale.

Remark 2.5 Observe that for the reasons similar to the ones leading to Remark 2.3 if Assumption 2.1 holds, then one only need to check the conditions i), ii) and iv) in order to establish that a given process, X, is a weak solution.

Definition 2.7 We say that *uniqueness in the sense of probability law* holds for (2.6) if for all $s \geq 0$ and for any two weak solutions (X, W), $(\Omega, \mathscr{F}, P)$, $(\mathscr{F}_t)$ and $(\hat{X}, \hat{W})$, $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$, $(\hat{\mathscr{F}}_t)$ starting from s with the same initial distribution have the same law, i.e. for any $t_1, \ldots, t_n$ and $n \geq 1$, $P(X_{t_1} \in E_1, \ldots, X_{t_n} \in E_n) = \hat{P}(\hat{X}_{t_1} \in E_1, \ldots, \hat{X}_{t_n} \in E_n)$, where $E_i \in \mathscr{E}$.

The following is a direct consequence of Ito's formula

Proposition 2.6 *Let X be a weak solution of (2.6) starting from s, and consider the operator A defined by 2.1. Then for any $f \in \mathbb{C}^2(\mathbf{E})$*

$$M_t^f = f(X_t) - f(X_s) - \int_s^t A_u f(X_u)\, du$$

is a local martingale with the localising sequence $\tau_m(f) := \inf\{t \geq r : |M_t^f| \geq m\}$. If g is another function belonging to $\mathbb{C}^2(\mathbf{E})$, then

$$\langle M^f, M^g \rangle = \sum_{i,j} \int_s^t a_{ij}(u, X_u) \frac{\partial f}{\partial x_i}(X_u) \frac{\partial g}{\partial x_j}(X_u)\, du.$$

Furthermore, if f has compact support and a is locally bounded, then M^f is a martingale such that M_t^f is square integrable for each t.

Suppose that there exists a weak solution to (2.6) starting from s with initial distribution μ. Then, there exists a solution to the local martingale problem (A, μ) starting from s. Indeed, the weak solution X induces a probability measure, $P^{s,\mu}$, on $C(\mathbb{R}_+, \mathbf{E})$. In view of Proposition 2.6, this measure solves the local martingale problem for (A, μ) starting from s since the stopping times in the localising sequence of M^f are stopping times with respect to the natural filtration of X.

Given this observation, if we can show that the existence of a solution to the local martingale problem for (A, μ) starting from s implies the existence of a weak solution to (2.6) starting from s with initial distribution μ, then well posedness of the local martingale problem for A becomes equivalent to the existence and uniqueness of a weak solution to (2.6) starting from s with initial distribution μ. This is exactly the result of the next theorem.

Theorem 2.5 *Suppose there exists a solution, $P^{s,\mu}$, to the local martingale problem for (A, μ) starting from s. Then, there exists a weak solution to (2.6) starting from s with $P(X_s \in \Lambda) = \mu(\Lambda)$ for any $\Lambda \in \mathscr{E}$.*

Proof Let X be the coordinate process on $\Omega^o := C(\mathbb{R}_+, \mathbf{E})$, $(\mathscr{B}_t)_{t\geq 0}$ be the natural filtration of X, and $\mathscr{B} = \vee_{t\geq 0}\mathscr{B}_t$. Consider $(\Omega', \mathscr{F}', P')$ – a probability space rich enough to support d-dimensional $(\mathscr{F}'_t)$-Brownian motion W'. Define $\Omega = \Omega^o \times \Omega'$, $\mathscr{F} = \overline{\mathscr{B} \times \mathscr{F}'}$, $P = P^{s,\mu} \times P'$, $\mathscr{F}_t = \overline{\mathscr{B}_t \times \mathscr{F}'_t}$, where $\overline{\mathscr{G}}$ is the completion of the σ-algebra $\mathscr{G}$ with the P-null sets. Let $X(t, \omega^o, \omega') = \omega^o(t)$, $B(t, \omega^o, \omega') = W'(t, \omega')$, and for all $i, j = 1, \ldots, d$ define

$$
\begin{aligned}
M_i(t, \omega^o, \omega') &= \omega^o_i(t) - \omega^o_i(s) - \int_s^t b_i(u, \omega^o(u))\, du,\\
N_{ij}(t, \omega^o, \omega') &= \omega^o_i(t)\omega^o_j(t) - \omega^o_i(s)\omega^o_j(s)\\
&\quad - \int_s^t \Big\{a_{ij}(u, \omega^o(u)) + \omega^o_j(u) b_i(u, \omega^o(u)) + \omega^o_i(u) b_j(u, \omega^o(u))\Big\}\, du.
\end{aligned}
$$

Observe that both $(M(t, \omega^o, \omega'))_{t\geq s}$ and $(N(t, \omega^o, \omega'))_{t\geq s}$ are $(P, \mathscr{F}_t)$-local martingales. Moreover, direct computations show that $M_i M_j - \int_s^\cdot a_{ij}(u, \omega^o(u))\, du$ is also local martingale leading to

$$
\langle M_i, M_j\rangle_t = \int_s^t a_{ij}(u, \omega^o(u))\, du
$$

Next, we define W by

$$
W(t, \omega) = \int_s^t \rho(u, X(u, \omega)) dM(u, \omega) + \int_0^t \eta(u, X(u, \omega)) dB(u, \omega),
$$

where $\rho, \eta : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^d \otimes \mathbb{R}^d$ are Borel measurable and solve

$$
\rho\tilde{a}\rho^T + \eta\eta^T = I_d, \tag{2.8}
$$

$$
\tilde{\sigma}\eta = 0, \tag{2.9}
$$

$$
(I_d - \tilde{\sigma}\rho)\tilde{a}(I_d - \tilde{\sigma}\rho)^T = 0, \tag{2.10}
$$

with $\tilde{a}(t, x) = \mathbf{1}_{\{t\geq s\}}\mathbf{1}_{\{x\in\mathbf{E}\}} a(t, x)$ and $\tilde{\sigma}(t, x) = \mathbf{1}_{\{t\geq s\}}\mathbf{1}_{\{x\in\mathbf{E}\}}\sigma(t, x)$. (For the existence of ρ and η satisfying above, see, e.g. Lemma 5.3.2 in [50].)

Since X takes values in $\mathbf{E}$ under P, and M and B are independent by construction,

$$
\begin{aligned}
\langle W_i, W_j\rangle_t &= \sum_{k,l=1}^{d} \langle \int_s^{\cdot} \rho_{ik}(u, X(u,\omega))dM_k(u,\omega), \int_s^{\cdot} \rho_{jl}(u, X(u,\omega))dM_l(u,\omega)\rangle_t \\
&\quad + \sum_{k,l=1}^{d} \langle \int_0^{\cdot} \eta_{ik}(u, X(u,\omega))dB_k(u,\omega), \int_0^{\cdot} \eta_{jl}(u, X(u,\omega))dB_l(u,\omega)\rangle_t \\
&= \sum_{k,l=1}^{d} \int_0^t \rho_{ik}(u, X(u,\omega))\tilde{a}_{kl}(u, X(u,\omega))\rho_{jl}(u, X(u,\omega))du \\
&\quad + \sum_{k,l=1}^{d} \int_0^t \eta_{ik}(u, X(u,\omega))\delta_{kl}\eta_{jl}(u, X(u,\omega))du \\
&= \int_0^t (\rho\tilde{a}\rho^T + \eta\eta^T)_{ij}(u, X(u,\omega))du = \delta_{ij}t,
\end{aligned}
$$

where δ_{ij} is Kronecker's delta and the last equality is due to (2.8). Thus, W is a Brownian motion by Lévy's characterisation.

Next, observe that, since $\tilde{\sigma}\eta = 0$ by (2.9),

$$
\begin{aligned}
\int_s^t \sigma(u, X(u,\omega))dW(u,\omega) &= \int_s^t (\sigma\rho)(u, X(u,\omega))dM(u,\omega) \\
&= M(t,\omega) - \int_s^t (I_d - \sigma\rho)(u, X(u,\omega))dM(u,\omega) \\
&= M(t,\omega) = X(t,\omega) - X(s,\omega) - \int_s^t b(u, X(u,\omega))du, \qquad \forall t \geq s,
\end{aligned}
$$

where the third equality is a consequence of (2.10) since

$$
\begin{aligned}
&\langle \sum_{j=1}^{d} \int_s^{\cdot} (I_d - \sigma\rho)_{ij}(u, X(u,\omega))dM_j(u,\omega)\rangle_t \\
&\quad = \sum_{k,l=1}^{d} \int_s^t [(I_d - \tilde{\sigma}\rho)_{ik}\tilde{a}_{kl}(I_d - \tilde{\sigma}\rho)_{jl}](u, X(u,\omega))du.
\end{aligned}
$$

The conclusion follows since $X(r,\omega) = X(s,\omega)$ for all $r \leq s$, P-a.s. $\square$

Equivalence of local martingale problem and weak solutions is summarised in the following:

Corollary 2.3 *For any fixed $s \geq 0$ the existence of a solution $P^{s,\mu}$ to the local martingale problem for (A, μ) starting from s is equivalent to the existence of a weak solution (X, W), $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$, $(\hat{\mathscr{F}}_t)$ to (2.6) starting from s such that $\hat{P}(X_s \in \Lambda) = \mu(\Lambda)$ for any $\Lambda \in \mathscr{E}$. The two solutions are related by $P^{s,\mu} = \hat{P}X^{-1}$; i.e. X induces the measure $P^{s,\mu}$ on $(C(\mathbb{R}_+, \mathbf{E}), \mathscr{B})$.*

Moreover, $P^{s,\mu}$ is unique if and only if the uniqueness in the sense of probability law holds for the solutions of (2.6) starting from s with the initial distribution μ.

Because of the aforementioned connection between the local martingale problem and weak solutions of SDEs, it becomes apparent that instead of verifying the local martingale property of M^f when $f \in \mathbb{C}^\infty(\mathbf{E})$ we can focus on verifying the martingale property of M^f when $f \in \mathbb{C}^2_K(\mathbf{E})$. This observation leads to the following modification.

Definition 2.8 A probability measure $P^{s,\mu}$ on $(C(\mathbb{R}_+, \mathbf{E}), \mathscr{B})$ is called a solution of the martingale problem for (A, μ) starting from s if M^f is a martingale for every $f \in \mathbb{C}^2_K(\mathbf{E})$ and $P^{s,\mu}(X_s \in \Lambda) = \mu(\lambda)$ for any $\Lambda \in \mathscr{E}$ with $P^{s,\mu}(X_r = X_s, r \leq s) = 1$.

The martingale problem for A is said to be well posed if there exists a unique solution to martingale problem for (A, μ) starting from s for all $s \geq 0$ and $\mu \in \mathscr{P}(\mathbf{E})$.

The following is a direct consequence of Theorem 2.5 together with Proposition 2.6.

Proposition 2.7 *Suppose a is locally bounded. $P^{s,\mu}$ on $(C(\mathbb{R}_+, \mathbf{E}), \mathscr{B})$ is a solution of the martingale problem for (A, μ) starting from s if and only if it solves the local martingale problem for (A, μ) starting from s.*

2.3.2 Existence and Uniqueness of Solutions

Weak solution of an SDE discussed in the previous section has both the Brownian motion and the probability space as a part of solution. However, in many applications, one is often interested in the solution within a given probability space. This leads us to the notion of a strong solution.

Definition 2.9 Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a filtered probability space such that $\mathscr{F}_0$ contains all P-null sets. Suppose that this space is rich enough to support a d-dimensional Brownian motion, W, and an $\mathscr{F}_s$-measurable random variable, ξ. A strong solution to (2.6) starting from s relative to the pair (ξ, W) is a continuous stochastic process $X = (X_t)_{t\geq s}$ on $(\Omega, \mathscr{F}, P)$ such that

i) $X_t \in \mathscr{F}_t$ for all $t \geq s$,
ii) $P(X_s = \xi) = 1$,

iii) For any $i = 1, \dots, d$,

$$P\left(\int_s^t a_{ii}(u, X_u)\, du + \int_s^t |b_i(u, X_u)|\, du < \infty\right) = 1.$$

iv) X satisfies, P-a.s., for all $t \geq s$

$$X_t = \xi + \int_s^t b(u, X_u)du + \int_s^t \sigma(u, X_u)dW_u. \tag{2.11}$$

Remark 2.6 For the reasons similar to the ones leading to Remark 2.3 if Assumption 2.1 holds, then one only need to check the conditions i), ii) and iv) in order to establish that a given process, X, is a strong solution.

Definition 2.10 We say that pathwise uniqueness holds for (2.6) if for any $s \geq 0$ and fixed filtered probability space satisfying the conditions of Definition 2.9, $P(X_t = \tilde{X}_t, t \geq s) = 1$ whenever X and $\tilde{X}$ are the strong solutions of (2.6) starting from s relative to the same pair (ξ, W).

Remark 2.7 There seems to be no agreement with regard to the terminology describing various notions of uniqueness of strong solutions. Our definition agrees with what can be found in [50, 70], and [100]. On the other hand, the pathwise uniqueness defined in [77] corresponds to a seemingly stronger notion than the one defined above. However, as observed in Remark IV.1.3 in [70], these two notions are equivalent. Moreover, it can be directly observed as a consequence of Theorem 2.12.

The following examples give an important class of SDEs for which pathwise uniqueness holds.

Example 2.3 Suppose $\sigma(t, x) = \sigma(t)$ for all $x \in \mathbf{E}$, and $b_1(t, (x, x_2, \dots, x_d)) \geq b_1(t, (y, x_2, \dots, x_d)$ for all $t \geq 0$, $(x, x_2, \dots, x_d), (y, x_2, \dots, x_d) \in \mathbf{E}$ such that $x \leq y$. If X and Y are two strong solutions of (2.6) starting from s relative to (ξ, W) with $P(X_t^{(i)} = Y_t^{(i)}, t \geq s) = 1$ for all $i = 2, \dots, d$, then

$$P(X_t^{(1)} = Y_t^{(1)}, t \geq s) = 1.$$

Indeed, a direct application of Ito's formula yields

$$(X_t^{(1)} - Y_t^{(1)})^2 = 2\int_s^t (X_u^{(1)} - Y_u^{(1)})(b_1(u, X_u) - b_1(u, Y_u))du \leq 0.$$

In particular, pathwise uniqueness holds if $d = 1$.

Example 2.4 Let $\mathbf{E} = \mathbb{R}^d$. Suppose $\sigma(t, x) = \sigma(t)$ for all $x \in \mathbf{E}$ and it is locally bounded. Consider the linear SDE

$$X_t = \xi + \int_s^t \sigma(r)dW_r + \int_0^t \left\{b^0(r) + b^1(r)X_r\right\} dr,$$

where b^0 and b^1 are $d \times d$-matrices such that $\int_0^t |b^i_{jk}(s)|ds < \infty$ for all $t > 0$, $j, k = 1, \dots, d$, and $i = 0, 1$.

This SDE has pathwise uniqueness. Indeed, if X and Y are two strong solutions starting from s relative to (ξ, W), then

$$X_t - Y_t = \int_s^t \left\{ b^1(r)(X_r - Y_r) \right\} dr.$$

Note that $\bar{b}^1(r) := \max_{i,j=1,\dots,d} |b^1_{ij}(r)|$ is integrable since

$$\int_s^t \bar{b}^1(r)dr \le \sum_{i,j=1}^d \int_s^t |b^1_{ij}(r)|dr < \infty.$$

Thus,

$$\|X_t - Y_t\| \le \int_s^t \bar{b}^1(r)\|X_r - Y_r\|dr,$$

and Gronwall inequality yields $\|X_t - Y_t\| = 0$ for all $t > s$. Therefore, pathwise uniqueness holds.

In this case the solution is explicitly given by

$$X_t = \Phi(t)\left[\xi + \int_s^t \Phi^{-1}(r)b^0(r)dr + \int_s^t \Phi^{-1}(r)\sigma(r)dW_r\right],$$

where Φ is the unique solution to the equation

$$\dot{\Phi}(t) = b^1(t)\Phi(t), \; \Phi(s) = I.$$

This can be verified by integration by parts formula as soon as the Lebesgue and the stochastic integrals are well defined. This is indeed the case since Φ^{-1} is the unique absolutely continuous solution to the adjoint equation

$$\dot{y}(t) = -y(t)b^1(t), \; y(s) = I.$$

We refer the reader to Chap. III of [62] for the general theory of linear systems of ODEs with integrable coefficients.

The following classical results regarding the uniqueness and existence of solutions of SDEs can be found, e.g. in Chap. 5 of [50].

Theorem 2.6 *Suppose that for a given open set $U \subset \mathbf{E}$ the coefficients b and σ satisfy*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \le K_T\|x - y\|, \; 0 \le t \le T, \; x, y \in U.$$

Let X and Y be two strong solutions of (2.6) starting from s relative to (ξ, W). Define

$$\tau = \inf\{t \geq s : X_t \notin U \text{ or } Y_t \notin U\}.$$

Then, $P(X_{t\wedge\tau} = Y_{t\wedge\tau}, t \in [s, T]) = 1$. If there exists a sequence U_n increasing to $\mathbf{E} = \mathbb{R}^d$ such that each U_n satisfies the above condition for every $T > 0$, then pathwise uniqueness holds for (2.6).

The first statement of the above theorem follows from the proof of Theorem 5.3.7 in [50] by noting that our standing assumption on a and b guarantees that all integrals in the proof are well defined. The second conclusion follows from that $\tau_n \to \infty$, P-a.s., where τ_n is the first exit time from U_n.

The next theorem is Theorem 5.3.11 from [50]. The additional claim on uniqueness is a direct consequence of the theorem above.

Theorem 2.7 *Suppose that $\mathbf{E} = \mathbb{R}^d$ and the coefficients b and σ are locally bounded. Moreover, assume that for each $T > 0$ and $n \geq 1$, there exist constants K_T and $K_{T,n}$ such that*

$$\|b(t,x)-b(t,y)\|+\|\sigma(t,x)-\sigma(t,y)\| \leq K_{T,n}\|x-y\|, \ t \in [0,T], \ \|x\| \vee \|y\| \leq n,$$

$$\|\sigma(t,x)\|^2 \leq K_T(1+\|x\|^2), \ x \cdot b(t,x) \leq K_T(1+\|x\|^2), \ t \in [0,T], \ x \in \mathbf{E}.$$

Fix an $s \geq 0$ and let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the conditions in Definition 2.9 and assume $E\|\xi\|^2 < \infty$. Then, there exists a unique strong solution, $X = (X_t)_{t\geq s}$, to (2.6) starting from s on this space relative to (ξ, W). Moreover, for every $t \geq s$, $X_t \in \mathscr{F}_t^W \vee \sigma(\xi)$.

Remark 2.8 Under additional assumptions on the coefficients, one can obtain upper bounds on the second moment of the solution given by the previous theorem. More precisely, suppose in addition to the conditions of Theorem 2.7, $K_{T,n} = K_T$, and b satisfies

$$\|b(t,x)\|^2 \leq K_T(1+\|x\|^2), \ t \in [0,T], \ x \in \mathbf{E}.$$

Then (see, e.g. Theorem 5.2.9 and Problem 5.3.15 in [77]), there exists a $C < \infty$ depending only in K_T and T such that

$$E\|\sup_{s\leq r\leq t} X_r\|^2 \leq C(1+E\|\xi\|^2)e^{C(t-s)}, \qquad t \in [s,T].$$

Theorem 2.7 gives a *global* solution with values in $\mathbb{R}^d$ without explosion. However, in the applications, we are often interested in solutions to a given SDE up to its exit from a certain domain (if the domain is $\mathbb{R}^d$ the exit time is called *the explosion time*). Of course, one can use the solution provided by the previous theorem and stop it at its first exit time from the given domain. The issue with this approach is that it imposes conditions on the coefficients of the SDE outside the domain of interest. The next theorem allows us to remove this unnecessary assumption.

Theorem 2.8 *Suppose $(U_n)_{n\geq 1}$ is a sequence of open subsets of $\mathbb{R}^d$ with compact closure such that $cl(U_n) \subset U_{n+1}$ and let $U = \cup_{n\geq 1} U_n$. Suppose that b and σ satisfy the following for every $n \geq 1$:*

$$\sup_{t\in[0,T]} \{\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\|\} \leq K_{T,n}\|x - y\|, \; x, y \in cl(U_n),$$

$$\sup_{x\in U_n, t\in[0,T]} \{\|\sigma(t,x)\| + \|b(t,x)\|\} \leq K_{T,n},$$

where $K_{T,n}$ is a real constant. Fix an $s \geq 0$ and let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the conditions in Definition 2.9 and assume $\xi \in U_1$. Then, there exists an adapted process $X = (X_t)_{t\geq s}$ satisfying

$$\mathbf{1}_{[t<\tau^X]} X_t = \mathbf{1}_{[t<\tau^X]} \left(\xi + \int_s^t b(u, X_u)du + \int_s^t \sigma(u, X_u)dW_u \right),$$

where $\tau^X := \inf\{t \geq s : X_t \notin U\}$. Moreover, if Y is another adapted process such that

$$\mathbf{1}_{[t<\tau^Y]} Y_t = \mathbf{1}_{[t<\tau^Y]} \left(\xi + \int_s^t b(u, Y_u)du + \int_s^t \sigma(u, Y_u)dW_u \right),$$

with $\tau^Y := \inf\{t \geq s : Y_t \notin U\}$, then $P(\mathbf{1}_{[t<\tau^X]} X_t = \mathbf{1}_{[t<\tau^Y]} Y_t, t \geq s) = 1$.

Proof Let $\psi_n \in \mathbb{C}_K^\infty(\mathbb{R}^d)$ such that $\psi_n(x) \in [0, 1]$ for all $x \in \mathbb{R}^d$, $\psi_n(x) = 1$ for $x \in U_n$, and $\psi_n(x) = 0$ for $x \notin U_{n+1}$. Consider $\sigma_n = \psi_n\sigma$ and $b_n = \psi_n b$. It follows from direct calculations that σ_n and b_n satisfy the conditions of Theorems 2.6 and 2.7. Thus, there exists a unique solution, X^n, corresponding to the coefficients (σ_n, b_n). Let $\tau_n = \inf\{t \geq s : X_t^n \notin U_n \text{ or } X_t^{n+1} \notin U_n\}$. Since the coefficients (σ_n, b_n) and (σ_{n+1}, b_{n+1}) agree on U_n, we have for any $s \leq t \leq T$

$$\begin{aligned}
E\|X_{t\wedge\tau_n}^{n+1} - X_{t\wedge\tau_n}^n\| &\leq 4E\left[\int_s^{t\wedge\tau_n} \|b(u, X_u^{n+1}) - b(u, X_u^n)\|du\right]^2 \\
&\quad +4E\int_s^{t\wedge\tau_n} \|\sigma(u, X_u^{n+1}) - \sigma(u, X_u^n)\|^2 du \\
&\leq 4(t-s)E\int_s^{t\wedge\tau_n} \|b(u, X_u^{n+1}) - b(u, X_u^n)\|^2 du \\
&\quad +4E\int_s^{t\wedge\tau_n} \|\sigma(u, X_u^{n+1}) - \sigma(u, X_u^n)\|^2 du \\
&\leq 4(T+1)K_{T,n}^2 \int_s^t E\|X_{u\wedge\tau_n}^{n+1} - X_{u\wedge\tau_n}^n\|^2 du.
\end{aligned}$$

Thus, Gronwall inequality implies $P(X^n_{t\wedge\tau_n} = X^{n+1}_{t\wedge\tau_n}, t \in [s,T]) = 1$. Sending $T \to \infty$ yields $P(X^n_{t\wedge\tau_n} = X^{n+1}_{t\wedge\tau_n}, t \geq s) = 1$. Consequently, the process X defined by $X_{t\wedge\tau_n} = X^n_{t\wedge\tau_n}$ is well defined up to time $\tau = \lim_{n\to\infty}\tau_n$ as $(\tau_n)_{n\geq 1}$ is an increasing sequence of stopping times.

Moreover, we have

$$X_{t\wedge\tau_n} = \xi + \int_s^{t\wedge\tau_n} b(u, X_u)du + \int_s^{t\wedge\tau_n} \sigma(u, X_u)dW_u,$$

for every $n \geq 1$. Since $\tau = \inf\{t \geq s : X_t \notin U\}$, the claim follows. □

As mentioned earlier when $U = \mathbb{R}^d$ τ is the explosion time for the solution of the given SDE. If one is able to demonstrate that $\tau = \infty$, a.s., then one has a non-explosive solution. One way to verify that the solution never explodes is to show that the solution stays between two continuous processes with no explosion. The following result, Theorem VI.1.1 in [70], will be instrumental in finding such two processes.

Theorem 2.9 *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ be a filtered probability space which supports a one-dimensional Brownian motion, W. Suppose that on this space we are given adapted processes X^1, β^1, X^2 and β^2 such that*

$$X^i_t = X^i_0 + \int_0^t \sigma(s, X^i_s)dW_s + \int_0^t \beta^i_s ds, \qquad i = 1, 2,$$

which additionally satisfy with probability one

$$\begin{aligned} X^1_0 &\leq X^2_0, \\ \beta^1_t &\leq b_1(t, X^1_t), \qquad \forall t \in [0,T], \\ \beta^2_t &\geq b_2(t, X^2_t), \qquad \forall t \in [0,T], \end{aligned}$$

where $b_i : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ and $\sigma : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ are continuous.

Suppose further that there exists a strictly increasing function $\rho : [0,\infty) \mapsto \mathbb{R}$ such that $\rho(0) = 0$ with

$$\int_{0+} \frac{dx}{\rho^2(x)} = \infty,$$

and

$$|\sigma(t, x_1) - \sigma(t, x_2)| \leq \rho(|x_1 - x_2|), \qquad \forall (t, x) \in [0,T] \times \mathbb{R}^2.$$

Then, $P(X^1_t \le X^2_t, t \in [0,T]) = 1$ *if one of the following holds:*

i) $b_1(t,x) < b_2(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}$.
ii) $b_1(t,x) \le b_2(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}$ *and pathwise uniqueness holds on* $[0,T]$ *for at least one of the following SDEs:*

$$dX_t = b_i(t,X_t)dt + \sigma(t,X_t)dW_t, \qquad i = 1,2.$$

Note that if σ satisfies $|\sigma(t,x_1) - \sigma(t,x_2)| \le K|x_1 - x_2|$ on $[0,T] \times \mathbb{R}$, then $\rho(x) = Kx$ satisfies the conditions of the above theorem.

Observe that the above theorem places global conditions on the coefficients of the SDE, which might be undesirable. We will next present two results that relax this assumption.

Theorem 2.10 *Consider* $-\infty \le l < r \le \infty$ *and let* $\sigma : [0,T] \times (l,r) \mapsto \mathbb{R}$ *and* $b : [0,T] \times (l,r) \mapsto \mathbb{R}$ *be continuous functions. Assume that they are Lipschitz on every closed subinterval of* (l,r) *and suppose further that we are given* X^i, β^i, $i = 1,2$ *satisfying*

$$X^i_{t\wedge\tau_i} = X^i_0 + \int_0^{t\wedge\tau_i} \beta^i_u du + \int_0^{t\wedge\tau_i} \sigma(u, X^i_u)dW_u,$$

where $\tau_i := \inf\{t \ge 0 : X^i_t \notin (l,r)\}$*. Moreover, assume that the following hold with probability one*

$$\begin{aligned} l < X^1_0 &\le X^2_0 < r, \\ \beta^1_t &\le b(t,X^1_t), \qquad \forall t \in [0, T\wedge\tau_1], \\ \beta^2_t &\ge b(t,X^2_t), \qquad \forall t \in [0, T\wedge\tau_2]. \end{aligned}$$

Then, $P(X^1_{t\wedge\tau} \le X^2_{t\wedge\tau}, t \in [0,T]) = 1$*, where* $\tau = \tau_1 \wedge \tau_2$.

Proof Fix $\tilde{l} > l$ and $\tilde{r} < r$ and the pair (l_1, r_1) satisfying $\tilde{l} < l_1 < X^1_0 \le X^2_0 < r_1 < \tilde{r}$. Let $\psi \in \mathbb{C}^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on (l_1,r_1), $\psi \equiv 0$ outside $(\tilde{l},\tilde{r})$ and $\frac{\psi'}{\psi}$ is bounded on $[\tilde{l},\tilde{r}]$. Moreover, ψ is strictly monotone on $[\tilde{l}, r_1]$ and on $[r_1, \tilde{r}]$. Define

$$F(x) = \int_{-\infty}^x \psi(y)dy.$$

Observe that the range of F equals $[0,R]$ for some $R < \infty$. Thus, $F^{-1}(y)$ defined by $F^{-1}(y) = F^{-1}((y \vee 0) \wedge R)$ is well defined for $y \in \mathbb{R}$. Also note that $\psi(F^{-1}(0)) = \psi(F^{-1}(R)) = 0$. The same observation holds for the derivatives of ψ at $\tilde{l} = F^{-1}(0)$ and $\tilde{r} = F^{-1}(R)$ due to the L'Hospital rule and the boundedness of $\frac{\psi'}{\psi}$.

Consider $\tilde{\sigma}(t, y) = \psi(F^{-1}(y))\sigma(t, F^{-1}(y))$ and the process $\tilde{\beta}^i$ for $i = 1, 2$ defined by

$$\tilde{\beta}^i_t = \beta^i_t \psi(X^i_{t\wedge\tau_i}) + \frac{1}{2}\psi'(X^i_{t\wedge\tau_i})\sigma^2(t, X^i_{t\wedge\tau_i}).$$

Then, $Y^i_t = F(X^i_{t\wedge\tau_i})$ satisfies

$$Y^i_t = Y^i_0 + \int_0^t \tilde{\beta}^i_u du + \int_0^t \tilde{\sigma}(u, Y^i_u) dW_u.$$

In view of our assumptions on ψ we have

$$|x - y| = \left| \int_{F^{-1}(y)}^{F^{-1}(x)} \psi(z)dz \right| \geq \min\{\psi(F^{-1}(y)), \psi(F^{-1}(x))\} |F^{-1}(x) - F^{-1}(y)|. \tag{2.12}$$

Thus, supposing $\psi(F^{-1}(x)) \leq \psi(F^{-1}(y))$, we deduce

$$\begin{aligned}
|\tilde{\sigma}(t, y) - \tilde{\sigma}(t, x)| &\leq \psi(F^{-1}(x))|\sigma(t, F^{-1}(x)) - \sigma(t, F^{-1}(y))| \\
&\quad + \sigma(t, F^{-1}(y))|\psi(F^{-1}(y)) - \psi(F^{-1}(x))| \\
&\leq K\psi(F^{-1}(x))|F^{-1}(x) - F^{-1}(y)| \\
&\quad + \sup_{z\in(x\wedge y, x\vee y)} \sigma(t, F^{-1}(z)) \frac{\psi'(z)}{\psi(z)} |x - y| \\
&\leq \left(K + \sup_{z\in(x\wedge y, x\vee y)} \sigma(t, F^{-1}(z)) \frac{\psi'(z)}{\psi(z)} \right) |x - y|,
\end{aligned}$$

where K is a Lipschitz constant for σ and the last inequality follows from (2.12). Note that $\sup_{z\in(x\wedge y, x\vee y)} \sigma(t, F^{-1}(z)) \frac{\psi'(z)}{\psi(z)} < \infty$ due to the assumptions. Thus, $\tilde{\sigma}$ is Lipschitz on $[0, T] \times \mathbb{R}$.

Moreover, define

$$\tilde{b}(t, y) = b(t, F^{-1}(y))\psi(F^{-1}(y)) + \frac{1}{2}\psi'(F^{-1}(y))\sigma^2(t, F^{-1}(y)).$$

Arguments that led to the Lipschitz continuity of $\tilde{\sigma}$ can be applied to yield that of $\tilde{b}$ once we show that $\frac{\psi''}{\psi}$ is bounded. Indeed, on any interval strictly contained in $(\tilde{l}, \tilde{r})$ $\frac{\psi''}{\psi}$ is bounded due to our assumption on ψ. Thus, it suffices to show that $\frac{\psi''(z)}{\psi(z)}$ remains bounded when z approaches to the boundary. Observe that

$$\lim_{z\to\tilde{l}} \frac{\psi''(z)}{\psi(z)} = \lim_{z\to\tilde{l}} \frac{\psi''(z)}{\psi'(z)} \frac{\psi'(z)}{\psi(z)}$$

is finite in view of L'Hospital's rule.

Thus, one can apply Theorem 2.6 to show that the pathwise uniqueness holds for

$$dY_t = \tilde{b}(t, Y_t)dt + \tilde{\sigma}(t, Y_t)dW_t, \; i = 1, 2.$$

Hence, by Theorem 2.9

$$P(Y_t^1 \leq Y_t^2, t \in [0, T]) = 1 \tag{2.13}$$

since $Y_0^1 = F(X_0^1) \leq F(X_0^2) = Y_0^2$, $\tilde{\beta}_t^1 \leq \tilde{b}(t, Y_t^1)$, $\tilde{\beta}_t^2 \geq \tilde{b}(t, Y_t^2)$, and condition ii) holds.

Thus, $P(X^1_{t\wedge\tilde{\tau}} \leq X^2_{t\wedge\tilde{\tau}}, t \in [0, T]) = 1$, where $\tilde{\tau}_i := \inf\{t \geq 0 : X_t^i \notin (l_1, r_1)\}$ and $\tilde{\tau} = \tilde{\tau}_1 \wedge \tilde{\tau}_2$ since $Y^i_{t\wedge\tilde{\tau}} = F(X_{t\wedge\tilde{\tau}})$, and the fact that F is strictly increasing on $[l_1, r_1]$. The conclusion follows from the arbitrariness of l_1 and r_1. □

Theorem 2.11 *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ be a filtered probability space which supports a one-dimensional Brownian motion, W. Consider $-\infty \leq l < r \leq \infty$ and let I be the union of interval (l, r) and its endpoints when finite. Suppose that $b_i : [0, T] \times I \times \Omega \mapsto \mathbb{R}$ are measurable such that for any $x \in I$ the processes $(b_i(t, x, \omega))_{t\in[0,T]}$ are adapted. Moreover, $\sigma : [0, T] \times I \mapsto \mathbb{R}$ is a measurable function satisfying*

$$|\sigma(t, x_1) - \sigma(t, x_2)| \leq \rho(|x_1 - x_2|), \qquad \forall (t, x) \in [0, T] \times I^2,$$

where $\rho : [0, \infty) \mapsto \mathbb{R}$ is a strictly increasing function such that $\rho(0) = 0$ with

$$\int_{0+} \frac{dx}{\rho^2(x)} = \infty.$$

Assume that on this space we are given adapted I-valued processes X^1 and X^2 such that

$$X_t^i = X_0^i + \int_0^{t\wedge\nu} \sigma(s, X_s^i)dW_s + \int_0^{t\wedge\nu} b^i(s, X_s^i, \omega)ds, \qquad i = 1, 2,$$

where ν is an $(\mathscr{F}_t)$-stopping time.

Then, $P(X^1_{t\wedge\nu} \leq X^2_{t\wedge\nu}, t \in [0, T]) = 1$ if the following are satisfied:

1. *$X_0^1 \leq X_0^2$.*
2. *$b_1(t \wedge \nu, x, \omega) \leq b_2(t \wedge \nu, x, \omega)$, a.s. for any $(t, x) \in (0, T] \times I$.*
3. *At least one of the b_is satisfies*

$$|b(t \wedge \nu, x, \omega) - b(t \wedge \nu, y, \omega)| \leq K(t)|x - y| \quad \textit{for all } (t, x, y) \in (0, T] \times I^2, \tag{2.14}$$

where $K : (0, T) \mapsto \mathbb{R}_+$ is a measurable function satisfying

$$\int_0^T K(t)dt < \infty.$$

Proof Since

$$X^1_{t\wedge\nu} - X^2_{t\wedge\nu} = X^1_0 - X^2_0 + \int_0^{t\wedge\nu} \left\{\sigma(s, X^1_s) - \sigma(s, X^2_s)\right\} dW_s$$
$$+ \int_0^{t\wedge\nu} \left\{b_1(s, X^1_s, \omega) - b_2(s, X^2_s, \omega)\right\} ds,$$

we have

$$\int_0^{t\wedge\nu} \frac{\mathbf{1}_{[X^1_s > X^2_s]}}{\rho^2(X^1_s - X^2_s)} d[X_1 - X^2, X^1 - X^2]_s$$
$$= \int_0^{t\wedge\nu} \frac{\left\{\sigma(s, X^1_s) - \sigma(s, X^2_s)\right\}^2 \mathbf{1}_{[X^1_s > X^2_s]}}{\rho^2(X^1_s - X^2_s)} ds \le t \wedge \nu.$$

Then Lemma IX.3.3 in [100] implies $L^0(X) \equiv 0$, where $X_t = X^1_{t\wedge\nu} - X^2_{t\wedge\nu}$ and $L^0(X)$ is the local time of X at 0.

Let

$$T_n := \inf\left\{t \ge 0 : \int_0^{t\wedge\nu} \left\{\sigma^2(s, X^1_s) + \sigma^2(s, X^2_s)\right\} ds \ge n\right\},$$

with the convention that $\inf\emptyset = T$ and observe that $T_n \to T$ as $n \to \infty$. Thus, Ito-Tanaka formula (see, e.g. Theorem VI.1.2 in [100]) yields

$$\phi(t) := \mathbb{E}\left[X^+_{t\wedge T_n}\right] = \mathbb{E}\left[\int_0^{t\wedge\nu\wedge T_n} \mathbf{1}_{[X^1_s > X^2_s]} \left\{b_1(s, X^1_s, \omega) - b_2(s, X^2_s, \omega)\right\} ds\right].$$

If b_2 satisfies (2.14), then

$$\phi(t) \le \mathbb{E}\left[\int_0^{t\wedge\nu\wedge T_n} \mathbf{1}_{[X^1_s > X^2_s]} \left\{b_2(s, X^1_s, \omega) - b_2(s, X^2_s, \omega)\right\} ds\right]$$
$$\le \mathbb{E}\left[\int_0^{t\wedge\nu\wedge T_n} \mathbf{1}_{[X^1_s > X^2_s]} K(s)(X^1_s - X^2_s)^+ ds\right] \le \int_0^t K(s)\phi(s)ds.$$

Gronwall inequality (see Exercise 14 in Chap. V of [99]) yields $\phi \equiv 0$ and an application of Fatou's lemma implies

$$0 \ge \mathbb{E}\left[\lim_{n\to\infty} X^+_{t\wedge T_n}\right] = \mathbb{E}\left[X^+_t\right].$$

Therefore, $P(X^1_{t\wedge\nu} \le X^2_{t\wedge\nu}, t \in [0, T]) = 1$.

If b_1 satisfies (2.14), then

$$\begin{aligned}\phi(t) \le \mathbb{E}\left[\int_0^{t\wedge\nu\wedge T_n} \mathbf{1}_{[X_s^1>X_s^2]}\left\{b_1(s,X_s^1,\omega)-b_1(s,X_s^2,\omega)\right\}ds\right]\\ +\mathbb{E}\left[\int_0^{t\wedge\nu\wedge T_n} \mathbf{1}_{[X_s^1>X_s^2]}\left\{b_1(s,X_s^2,\omega)-b_2(s,X_s^2,\omega)\right\}ds\right]\\ \le \mathbb{E}\left[\int_0^{t\wedge\nu\wedge T_n} \mathbf{1}_{[X_s^1>X_s^2]}\left\{b_1(s,X_s^1,\omega)-b_1(s,X_s^2,\omega)\right\}ds\right]\\ \le \mathbb{E}\left[\int_0^{t\wedge\nu\wedge T_n} \mathbf{1}_{[X_s^1>X_s^2]}K(s)(X_s^1-X_s^2)^+ds\right],\end{aligned}$$

and the conclusion follows as in the previous case. □

The following theorem, due to Yamada and Watanabe, can be found in [77], Corollary 5.3.23, or in [70], Theorem IV.1.1. We denote the Wiener measure on $(C([0,\infty),\mathbb{R}^d),\mathscr{B})$ by $\mathbb{W}$.

Theorem 2.12 *Suppose that there exists a weak solution to (2.6) starting from some $r \ge 0$ with initial distribution μ, and pathwise uniqueness holds for (2.6). Then, there exists a $\mathscr{E}\otimes\mathscr{B}(C([r,\infty),\mathbb{R}^d))/\mathscr{B}(C([r,\infty),\mathbf{E}))$-measurable function, $F : \mathbf{E}\times C([r,\infty),\mathbb{R}^d) \mapsto C([r,\infty),\mathbf{E})$, which is also $\overline{\mathscr{E}\otimes\mathscr{B}_t(C([r,\infty),\mathbb{R}^d))}/\mathscr{B}_t(C([r,\infty),\mathbf{E}))$-measurable for all $t \ge r$, where $\overline{\mathscr{E}\otimes\mathscr{B}_t(C([r,\infty),\mathbb{R}^d))}$ corresponds to the completion of $\mathscr{E}\otimes\mathscr{B}_t(C([r,\infty),\mathbb{R}^d))$ with $\mu\times\mathbb{W}$-null sets, such that*

$$X_\cdot = F(X_r, W_\cdot),\ P\text{-a.s.}$$

Moreover, if a given filtered probability space, $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\ge0},P)$, is rich enough to support a Brownian motion and a random variable $\xi\in\mathscr{F}_r$ with distribution μ, then the process $F(\xi,W_\cdot)$ is the strong solution to (2.6) starting from r relative to (ξ,W).

As a consequence of this theorem we have the following

Corollary 2.4 *Pathwise uniqueness implies uniqueness in the sense of probability law.*

Combining the above corollary with Corollary 2.3 and Proposition 2.7 we obtain

Corollary 2.5 *Suppose Assumption 2.1 holds and there exists a unique strong solution to (2.6) starting from s relative to the pair (x,W) for all $s\ge0$ and $x\in\mathbf{E}$. Then, the martingale problem for A is well posed.*

The above result shows that existence and uniqueness of strong solutions for any deterministic initial condition imply the strong Markov property of solutions of (2.6) via Remark 2.4. Moreover, if we are willing to assume that the coefficients of the SDE are bounded and time-homogeneous, solutions will have the Feller property.

Theorem 2.13 *Suppose that $\mathbf{E} = \mathbb{R}^d$ and the coefficients b and σ are bounded and do not depend on time. Moreover, assume that there exists a constant K such that*

$$\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \leq K\|x - y\|, \; t \in [0, T], \; x, y \in \mathbf{E}.$$

Then the solutions of (2.6) have the Feller property.

Proof By Theorem 2.7 there exists a unique strong solution to (2.6) for any initial condition x. Corollary 2.5 together with Theorem 2.3 yields that the solutions are strong Markov. Let $(P_t)_{t \geq 0}$ be the associated transition semi-group and denote by $(X^x)_{x \in \mathbf{E}}$ the family of strong solutions to

$$X_t^x = x + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds$$

on a fixed filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$. Revuz and Yor show on p. 380 of [100] that

$$\mathbb{E}\left[\sup_{s \leq t} \|X_s^x - X_s^y\|^2\right] \leq a_t \|x - y\|^2$$

for some constant a_t that depends on t only. Since $P_t f(x) = \mathbb{E}[f(X_t^x)]$ the above inequality implies that $x \mapsto P_t f(x)$ is continuous for any $f \in \mathbb{C}_0$.

Next, we show that $P_t f \in \mathbb{C}_0$ whenever $f \in \mathbb{C}_0$. Indeed,

$$|P_t f(x)| \leq \sup_{y : \|y - x\| < r} |f(y)| + \|f\| \mathbb{P}(\|X_t^x - x\| > r)$$

and

$$\begin{aligned}
\mathbb{P}(\|X_t^x - x\| > r) &\leq r^{-2} \mathbb{E}[\|X_t^x - x\|^2] \\
&\leq 2r^{-2} \mathbb{E}\left[\left\|\int_0^t \sigma(X_s) dB_s\right\|^2 + \left\|\int_0^t b(X_s) ds\right\|^2\right] \\
&\leq 2k^2 r^{-2}(t + t^2)
\end{aligned}$$

where k is the uniform bound on σ and b. Thus, letting first x then r to infinity yields that $\lim_{x \to \infty} |P_t f(x)| = 0$.

Finally, $\lim_{t \to 0} P_t f(x) = \lim_{t \to 0} \mathbb{E}[f(X_t^x)] = f(x)$ by the continuity of the solutions of (2.6) and the boundedness of f. $\square$

Theorem 2.12 will be instrumental in establishing the existence of a strong solution when the coefficients of a given SDE do not satisfy the hypothesis of Theorems 2.7 and 2.8. In order to use this theorem one needs to show the existence of a weak solution.

One of the methods for establishing the existence of a weak solution is to exploit the relationship between the local martingale problems and SDEs given in Corollary 2.3. In the case of continuous coefficients, the result below uses this connection, together with a time-change technique, when $\mathbf{E} = \mathbb{R}^d$. It is a combination of Theorem IV.2.3 and Theorem IV.2.4 in [70].

Theorem 2.14 *Suppose that* $\mathbf{E} = \mathbb{R}^d$*, the coefficients* σ *and* b *are continuous and for each* $T > 0$ *there exists a constant* K_T *such that condition*

$$\|\sigma(t,x)\|^2 + \|b(t,x)\|^2 \leq K_T(1+\|x\|^2), \qquad t \in [0,T], x \in \mathbb{R}^d. \tag{2.15}$$

Then, for any $s \geq 0$ *and* $x \in \mathbb{R}^d$*, there exists a weak solution of (2.6) starting from* s *such that* $X_s = x$.

Remark 2.9 If the growth condition (2.15) is not satisfied, Theorem IV.2.3 in [70] shows the existence of a weak solution up to an explosion time, i.e. the time of exit from $\mathbb{R}^d$. The growth condition ensures via Theorem IV.2.4 that X_t has finite second moment and, therefore, the solution never exits $\mathbb{R}^d$.

In the more general case when $\mathbf{E}$ is not necessarily $\mathbb{R}^d$, continuous coefficients will still yield the existence of a weak solution until the exit time from the interior of $\mathbf{E}$. However, the growth condition will not be sufficient to ensure the existence of a global solution with paths in $\mathbf{E}$, i.e. the existence of a weak solution in the sense of Definition 2.6. The existence of a global solution can be demonstrated if one can show that the solution never exits the interior of $\mathbf{E}$. We will see a few examples when this can be done in the case of one-dimensional time homogeneous SDEs.

Another way of verifying the existence of a weak solution is to modify the drift of a given SDE so that the resulting SDE is known to have a solution. Then one can use this solution to construct a weak solution to the original SDE via an equivalent measure change. The next proposition makes this statement precise.

Proposition 2.8 *Let* (X, W), $(\Omega, \mathscr{F}, P)$, $(\mathscr{F}_t)_{t\geq 0}$ *be a weak solution to*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

starting from s *with initial distribution* $\mu = \varepsilon_x$*. Suppose that for a given measurable function* c*,* L *defined by* $L_t = 1$ *for* $t \leq s$ *and*

$$L_t = \exp\left(\sum_{i=1}^d \int_s^t c_i(u, X_u)dW_u^i - \frac{1}{2}\int_s^t \|c(u, X_u)\|^2 du\right), \qquad t > s,$$

is a uniformly integrable martingale. Then, (X, β), $(\Omega, \mathscr{F}, Q)$, $(\mathscr{F}_t)_{t\geq 0}$ *is a weak solution to*

$$dX_t = \{b(t, X_t) + \sigma(t, X_t)c(t, X_t)\}\, dt + \sigma(t, X_t)d\beta_t,$$

starting from s *with initial distribution* $\mu = \varepsilon_x$, *where* $\frac{dQ}{dP} = L_\infty$ *and*

$$\beta_t^i = W_t^i - \int_0^t \mathbf{1}_{\{u\geq s\}} c_i(u, X_u)du.$$

Proof Direct consequence of Girsanov theorem (see Theorem A.19). □

2.3.3 The One-Dimensional Case

The results presented in the previous sections can be significantly improved for one-dimensional time-homogeneous SDEs. In particular, we set $\mathbf{E} = [l, \infty)$, where $l \geq -\infty$ with the convention that $\mathbf{E} = \mathbb{R}$ when $l = \infty$, and look for the solutions of

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds, \qquad x > l.$$

In above b and σ are measurable and $\sigma > 0$ on (l, ∞). In this case Engelbert and Schmidt have shown the existence and uniqueness of a weak solution up to the exit time from (l, ∞) under an integrability condition, which turn out to be necessary when $b \equiv 0$. We refer the reader to [47–49] for a detailed analysis. The following theorem is a combination of Theorems 5.5.7 and 5.5.15 in [77].

Theorem 2.15 *Suppose*

$$\sigma^2(x) > 0, \quad x \in (l, \infty) \tag{2.16}$$

and

$$\forall x > l\ \exists \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(y)|}{\sigma^2(y)} dy < \infty. \tag{2.17}$$

Then, for any $x \in (l, \infty)$ *there exists a weak solution to*

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds, \qquad t < \zeta, \tag{2.18}$$

where $\zeta = \inf\{t \geq 0 : X_t = l \text{ or } \infty\}$. *Moreover, the solution is unique in law.*

To ensure the existence of a global solution to (2.18) one needs to demonstrate that $\zeta = \infty$, i.e. its solution never exits (l, ∞). This requires an understanding of the behaviour of X near l and ∞. For one-dimensional SDEs this behaviour is completely determined by the so-called *scale function*, s, and the *speed measure* on the Borel subsets of $(0, \infty)$, m, that are defined as follows:

$$s(x) = \int_c^x \exp\left(-2\int_c^y \frac{b(z)}{\sigma^2(z)}dz\right)dy, \tag{2.19}$$

$$m(dx) = \frac{2}{\sigma^2(x)s'(x)}dx, \tag{2.20}$$

where $c \in (l, \infty)$ is arbitrarily chosen. Note that s is well defined due to (2.17). Moreover, due to the arbitrariness of c it is unique only up to an affine transformation.

The left endpoint, l, is called *exit* if for some $z \in (l, \infty)$

$$\int_l^z m(a, z)s'(a)da < \infty,$$

and *entrance* if

$$\int_l^z (s(z) - s(a))m(da) < \infty.$$

Similarly, ∞ is called *exit* if for some $z \in (l, \infty)$

$$\int_z^\infty m(z, a)s'(a)da < \infty,$$

and *entrance* if

$$\int_z^\infty (s(a) - s(z))m(da) < \infty.$$

An endpoint which is both exit and entrance is called *non-singular* or *regular*. Thus, to avoid ambiguity we will call a boundary entrance (resp. exit) if it is entrance and not exit (resp. exit and not entrance). An endpoint which is neither exit nor entrance is called *natural*. Both exit and regular boundaries are reached from an interior point in finite time with positive probability. Entrance and natural boundaries, on the other hand, are inaccessible. The difference between an entrance and a natural boundary is that it is possible to start from the former but not from the latter. For further details and proofs on the boundary behaviour of one-dimensional diffusions see Sect. II.1 in Part I of [28] and Sect. 4.1 of [73].

In view of the above discussion we have the following corollary concerning the existence and uniqueness of global solutions.

Corollary 2.6 *Suppose conditions (2.16) and (2.17) hold. Then, for any $x \in (l, \infty)$ there exists a weak solution to*

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds$$

if ∞ is a natural boundary, and l is an entrance boundary when finite and natural otherwise. Moreover, the solution is unique in law.

Example 2.5 Consider $\mathbf{E} = [0, \infty)$ and the following SDE for *squared Bessel process*:

$$X_t = x + 2\int_0^t \sqrt{X_t}dB_t + \delta t, \tag{2.21}$$

for $\delta \geq 0$. The scale function and the speed measure are given by

$$\begin{aligned} &s(x) = \tfrac{1-x^{\frac{2-\delta}{2}}}{\delta-2},\ m(dx) = x^{\frac{\delta-2}{2}}dx, &&\delta \neq 2,\\ &s(x) = \log x,\ m(dx) = \tfrac{1}{2}dx, &&\delta = 2. \end{aligned}$$

Direct calculations show that 0 is an entrance boundary for $\delta \geq 2$ while ∞ is natural for all $\delta \geq 0$. Thus, when $\delta \geq 2$, there exists a unique weak solution to (2.21) for any $x > 0$ in view of Corollary 2.6.

When $0 < \delta < 2$, the right endpoint 0 is reached a.s.; however, the solution is immediately reflected into the interior of $\mathbf{E}$ (see Proposition XI.1.5 of [100]). This hints at the existence of a global solution despite the fact that 0 is not an entrance boundary. In fact, for these values of δ, 0 is a regular boundary and one can hope to construct a global solution by pasting the solutions that start from 0. However, this requires $P^0(T_0 > 0) = 1$, where P^0 is the law of the squared Bessel process starting from 0 and of dimension $0 \leq \delta < 2$. This is false since $P^0(T_0 > 0) = 0$. Indeed, for any $\varepsilon > 0$

$$P^0(T_0 < \varepsilon) = \lim_{x \to 0} P^x(T_0 < \varepsilon) = \int_{\frac{x^2}{2\varepsilon}}^{\infty} \frac{t^{-\frac{\delta}{2}}e^{-t}}{\Gamma\left(\frac{2-\delta}{2}\right)}dt,$$

where Γ is the gamma function and the last equality follows from (13) in [59]. Letting $\varepsilon \to 0$ yields the claim.

This example illustrates that Engelbert–Schmidt conditions, although very general, do not yield global solutions for (2.18) when $\mathbf{E} \neq \mathbb{R}$. We will show the existence of a unique strong solution for (2.21) for all $x \geq 0$ using the result of Yamada and Watanabe that establishes the existence of a unique strong solution as a product of weak existence and strong uniqueness. The following result gives strong uniqueness for one-dimensional SDEs under conditions of coefficients that are weaker than the usual Lipschitz condition imposed in the general case. We refer the reader to Sect. IX.3 of [100] for a proof.

Theorem 2.16 *Suppose*

$$|\sigma(x) - \sigma(y)|^2 \le \rho(|x - y|), \qquad \forall (x, y) \in \mathbf{E}^2, \tag{2.22}$$

where $\rho : \mathbf{E} \mapsto \mathbb{R}$ *satisfies*

$$\int_0^\varepsilon \frac{dx}{\rho(x)} = \infty, \qquad \textit{for some } \varepsilon > 0. \tag{2.23}$$

Then, if X^1 *and* X^2 *are two strong solutions of (2.18), so is* $X^1 \vee X^2$*. In particular, uniqueness in law implies pathwise uniqueness for (2.18).*

In the above theorem one has to first show weak uniqueness to obtain pathwise uniqueness for the SDE. However, when the drift coefficient is a constant, this step can be omitted.

Corollary 2.7 *Suppose that* σ *satisfies (2.22) and* $b(x) = b$ *for some* $b \in \mathbb{R}$ *for all* $x \in \mathbf{E}$*. Then, for any* $x \in \mathbf{E}$ *pathwise uniqueness holds for (2.18).*

Proof Let X^1 and X^2 be two strong solutions of (2.18). Then, Theorem 2.16 yields $X := X^1 \vee X^2$ is also a strong solution. Consider $Y := X - X^1$, which satisfies

$$Y_t = \int_0^t \{\sigma(X_s) - \sigma(X_s^1)\} dB_s, \qquad t \ge 0.$$

Clearly, Y is a nonnegative local martingale. Thus, it is a nonnegative supermartingale. Since it starts from 0, it has to stay at 0. Therefore, $Y = X^1 = X^2$. □

Corollary 2.8 *Consider* $\mathbf{E} = [0, \infty)$*. For any* $\delta \ge 0$ *and* $x \in \mathbf{E}$ *there exists a unique strong solution to (2.21).*

Proof Consider the auxiliary SDE on $\mathbb{R}$

$$X_t = x^+ + 2\int_0^t \sqrt{X_s^+} dB_s + \delta t. \tag{2.24}$$

Then, Theorem 2.14 implies the existence of a non-exploding weak solution to the above SDE. Moreover, Corollary 2.7 yields pathwise uniqueness for the above. Consequently, there exists a unique strong solution to (2.24) due to Theorem 2.12.

In particular, when $\delta = x = 0$, 0 is the unique strong solution. Hence, the comparison result of Theorem 2.9 establishes that the solution of (2.24) is nonnegative for any $x \in \mathbf{E}$ and $\delta \ge 0$. Thus, one can view (2.24) as an SDE on $\mathbf{E}$, which coincides with (2.21). □

When $l = -\infty$ and both $-\infty$ and ∞ are natural boundaries, Theorem 2.15 shows the existence and uniqueness of weak solutions to (2.18). This in turn implies the well posedness of the associated martingale problem via Corollary 2.3 and the strong Markov property of the solutions due to Theorem 2.3. Similarly, the Bessel processes are also strong Markov.

It must be noted, however, that the strong Markov property can be obtained under more general conditions. Corollary 8.1.2 in [50] states that solutions of (2.18) has the Feller property when the infinite boundaries are natural. The proof of this result is due to Mandl [89] and is based on the theory of Sturm–Liouville equations. More is known about the transition function, $(P_t)_{t\geq 0}$ of the solutions. McKean [92] has shown that the transition function admits a *symmetric* transition density with respect to the speed measure. More precisely, the following result holds.

Theorem 2.17 *Suppose that (2.16) and (2.17) hold, and infinite boundaries are natural. Then, the transition function (P_t) of the solutions of (2.18) satisfies the following:*

1. *There exists $p : \mathbb{R}_+ \times \mathbf{E}^2 \mapsto \mathbb{R}_+$ such that for any bounded and continuous f vanishing at exit or regular boundaries we have $P_t f(x) = \int_{\mathbf{E}} f(y)p(t,x,y)m(dy)$;*
2. *for each $t > 0$ and $(x, y) \in (l, \infty)^2$, $p(t, x, y) = p(t, y, x) > 0$;*
3. *for each $t > 0$ and $y \in (l, \infty)$, the maps $x \mapsto p(t, x, y)$ and $x \mapsto Ap(t, x, y)$ are continuous and bounded on (l, ∞);*
4. *$\frac{\partial}{\partial t} p(t, x, y) = Ap(t, x, y)$ for each $t > 0$ and $(x, y) \in (l, \infty)^2$.*

2.4 Notes

The material of Sect. 2.1 is borrowed from Chap. VII in Revuz and Yor [100].

Section 2.2 adapts the results and arguments in Chap. 4 of Ethier and Kurtz [50] to accommodate a more general drift term, which will be necessary for the study of Markov bridges in Chaps. 4 and 5. This relaxation is made precise in Assumption 2.1 and is an extension of the standard local boundedness hypothesis in the literature. While the ideas of proofs in this section mostly originate from [50], the proof of D being a Borel set in Theorem 2.2 is taken from Lemma 1.39 in Kallenberg [75] that allowed us to simplify the proof. The proof of Proposition 2.5 is a detailed version of the proof of Theorem 4.2 in El Karoui et al. [46].

The contents of Sect. 2.2 are standard and can be found in classical texts such as Ethier and Kurtz [50], Karatzas and Shreve [77], Revuz and Yor [100] and Ikeda and Watanabe [70] to name a few. Section 2.3.1 introduces weak solutions of an SDE following Chap. 5 of Karatzas and Shreve [77]. It also establishes the connection between the local martingale problem and weak solutions using the techniques of Ethier and Kurtz [50], while making necessary adjustments to allow for drifts that are not locally bounded.

The results on the existence and uniqueness of strong solutions in Sect. 2.3.2 are mainly from [50]. However, since Ethier and Kurz [50] are concerned with the solutions on the whole $\mathbb{R}^d$ and the applications that we will consider later in this book often need solutions up to an exit time from a subset of $\mathbb{R}^d$, we establish Theorem 2.8 that allows one to treat such cases. The comparison result for SDEs on $\mathbb{R}$ in Theorem 2.9 is from Ikeda and Watanabe [70]. However, it imposes global conditions on the coefficients, which are not desirable when one is only interested in a comparison up to an exit time. This issue is handled in Theorem 2.10 that imposes local conditions on the coefficients. Nevertheless Theorem 2.10 requires a local Lipschitz condition on the drift on every time interval $[0, t]$. Although this appears to be an innocent assumption, there are interesting examples of SDEs (see, e.g. the SDE (5.61)) that violate the local Lipschitz assumption at $t = 0$. Theorem 2.11 that addresses this issue is new and its proof uses techniques from Sect. IX.3 in Revuz and Yor [100]. Finally, the Feller property of solutions as established in Theorem 2.13 is taken from Revuz and Yor [100].

Section 2.3.3 starts with a summary of Engelbert–Schmidt conditions for the existence and uniqueness of weak solutions for one-dimensional SDEs. Our exposition follows Karatzas and Shreve [77]. It also contains a discussion of limitations of the Engelbert–Schmidt theory when the diffusion exhibits a certain behaviour near the boundaries. We illustrate this limitation using a Bessel SDE with dimension less than 2. Corollary 2.8 establishes the existence and uniqueness of a strong solution to such Bessel SDEs. The ideas developed therein can be used to circumvent the aforementioned limitations for more general cases as well.

Although our treatment of the martingale problem follows Ethier and Kurtz [50], its original formulation is due to Stroock and Varadhan [108]. We refer the reader to Stroock and Varadhan [109] for a thorough account of their approach and the connection between the martingale problem and PDEs. Section V.4 of Rogers and Williams [102] provides an excellent introduction to martingale problems for a reader familiar with the theory of SDEs.

The classical text for SDEs in the context of diffusion processes is Ikeda and Watanabe [70]. For the general theory of stochastic integration with respect to semimartingales and the resulting theory of SDEs we refer the reader to Protter [99] and Bichteler [23].

Ito and McKean [73] is the definitive source for the theory of one-dimensional diffusions beyond SDEs. Breiman [29], Karlin and Taylor [78] and Rogers and Williams [102] provide a more accessible account of the theory that is more suitable as a first read.

Chapter 3
Stochastic Filtering

In the second part of this book we will study an equilibrium in which market participants possess different information. Obviously, their rationality will bound them to infer the information of other players from their actions. To avoid the pathological cases this information ought to be contaminated, e.g. by noise trades. This corresponds to a stochastic filtering problem which we will formalise in this chapter.

3.1 General Equations for the Filtering of Markov Processes

Let $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\in[0,T]}, P)$ be a complete filtered probability space that supports an $m+n$-dimensional Brownian motion $\tilde{W}$ and a $\mathscr{G}_0$-measurable m-dimensional random variable Z_0. Consider the following system

$$d\begin{pmatrix} Z_t \\ X_t \end{pmatrix} = \begin{pmatrix} b(t, Z_t) \\ h(t, Z_t, X_t) \end{pmatrix} dt + \begin{pmatrix} \sigma^1(t, Z_t) & \sigma^2(t, Z_t) \\ 0 & I_n \end{pmatrix} d\tilde{W}_t,$$

where b and h are, respectively, m and n dimensional vector fields; σ^1 and σ^2 are, respectively, $m \times m$ and $m \times n$ matrix fields, and I_n is the $n \times n$ identity matrix.

If we denote by Y the vector $\begin{pmatrix} Z_t \\ X_t \end{pmatrix}$ then the above system can be written as

$$dY_t = \tilde{b}(t, Y_t)dt + \tilde{\sigma}(t, Y_t)d\tilde{W}_t, \tag{3.1}$$

where

$$\tilde{b}(t, y) = \begin{pmatrix} b(t, z) \\ h(t, y) \end{pmatrix} \qquad \tilde{\sigma}(t, y) = \begin{pmatrix} \sigma^1(t, z) & \sigma^2(t, z) \\ 0 & I_n \end{pmatrix}.$$

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For the rest of this chapter we assume there exists a unique strong solution to (3.1) starting from 0 relative to the pair $(Y_0, \tilde{W})$ where $Y_0 = \begin{pmatrix} Z_0 \\ x \end{pmatrix}$ for some $x \in \mathbb{R}^n$. The solution takes values in $\mathbf{E} = \mathbf{E}^1 \times \mathbf{E}^2$ where $\mathbf{E}^i$ are as defined in Sect. 2.2.

Let $\sigma(t, z) = [\sigma^1(t, z)\ \sigma^2(t, z)]$ and define

$$a_{ij}(t, z) = \sum_{l=1}^{m+n} \sigma_{il}(t, z)\sigma_{jl}(t, z), \qquad 1 \leq i, j \leq m,$$

$$\tilde{a}_{ij}(t, y) = \sum_{l=1}^{m+n} \tilde{\sigma}_{il}(t, y)\tilde{\sigma}_{jl}(t, y), \qquad 1 \leq i, j \leq m+n.$$

Thus, we can introduce the following differential operators associated with Z and Y:

$$A_t = \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(t, \cdot)\frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^{m} b_i(t, \cdot)\frac{\partial}{\partial z_i}, \tag{3.2}$$

$$\tilde{A}_t = \frac{1}{2}\sum_{i,j=1}^{m+n} \tilde{a}_{ij}(t, \cdot)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{m+n} \tilde{b}_i(t, \cdot)\frac{\partial}{\partial y_i}. \tag{3.3}$$

In the stochastic filtering theory Z corresponds to the unobserved signal and X is the observation process. The goal is to determine the conditional distribution of Z based on the observations of X. More precisely, if $\mathscr{F}_t^X = \mathscr{N} \vee \sigma(X_s; s \leq t)$, with $\mathscr{N}$ being the collection of P-null sets, then we are interested in $\pi_t f = E[f(Z_t)|\mathscr{F}_t]$, where $\mathscr{F}_t = \mathscr{F}_{t+}^X$ for a suitable class of test functions f. Since conditional expectation is defined only almost surely, the process πf might not be well defined. Thus, instead of looking at the conditional expectations for each t, we will work with optional projection of $(f(Z_t))_{t\in[0,T]}$.

Definition 3.1 Let η be a $(\mathscr{G}_t)$-adapted integrable process. The $(\mathscr{F}_t)$-optional projection of η is an $(\mathscr{F}_t)$-optional process, ${}^{\circ}\eta$, such that for any $(\mathscr{F}_t)$-stopping time τ

$$E[\eta_\tau|\mathscr{F}_\tau] = {}^{\circ}\eta_\tau.$$

Remark 3.1 The optional projection defined above exists and is unique up to indistinguishability (see Theorem IV.5.6 in [100]).

In view of the above remark whenever we write $(E[\eta_t|\mathscr{F}_t])$ we always choose the version which coincides with the $(\mathscr{F}_t)$-optional projection of η. In order to be able to identify the optional projection of $f(Z)$ for all $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$, we will need the following assumptions.

Assumption 3.1

i) The maps $\sigma_{ij}^k : [0, T] \times \mathbf{E}^1 \mapsto \mathbb{R}$ *are Borel measurable and locally bounded.*

ii) The maps $b_i : [0, T] + \times \mathbf{E}^1 \mapsto \mathbb{R}$ *are Borel measurable and*

$$E\left(\int_0^T \|b(t, Z_t)\| dt\right)^2 < \infty.$$

iii) The maps $h_i : [0, T] \times \mathbf{E} \mapsto \mathbb{R}$ *are Borel measurable and*

$$E\int_0^T \|h(t, Z_t, X_t)\|^2 dt < \infty.$$

The assumption i) ensures that whenever $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$, M^f is a martingale where

$$M_t^f = f(Z_t) - f(Z_0) - \int_0^t A_s f(Z_s) ds.$$

This allows us to define a local martingale problem on $C([0, T], \mathbf{E})$ for A. Observe that this local martingale problem can be made equivalent to the one considered in Sect. 2.2 as soon as one defines the coefficients of the operator A to be zero after T.

The next proposition will be very useful in the derivation of filtering equation.

Proposition 3.1 *Let* ξ *be* $(\mathscr{G}_t)$*-progressively measurable such that*

$$E\int_0^T |\xi_t| dt < \infty.$$

Then, for $\eta_t = \int_0^t \xi_s ds$,

$$^\circ\eta_t - \int_0^t {}^\circ\xi_s ds$$

is an $(\mathscr{F}_t)$*-martingale.*

Proof Let τ be an $(\mathscr{F}_t)$-stopping time (bounded by T). Then,

$$E^\circ\eta_\tau = E\eta_\tau = E\int_0^T \mathbf{1}_{[t\le\tau]}\xi_t dt = \int_0^T E\left[\mathbf{1}_{[t\le\tau]}{}^\circ\xi_t\right] dt = E\int_0^\tau {}^\circ\xi_t dt.$$

□

Corollary 3.1 *The* innovation process W^X *defined by*

$$W_t^X = X_t - \int_0^t E[h(s, Z_s, X_s)|\mathscr{F}_s] ds$$

is an n*-dimensional* $(\mathscr{F}_t)$*-Brownian motion.*

Proof Observe that $^\circ W$ is an $(\mathscr{F}_t)$-martingale. Thus, in view of Proposition 3.1 and the fact that $(\mathscr{F}_t)$-optional projection of X is itself, the process

$$X_t - \int_0^t E[h(s, Z_s, X_s)|\mathscr{F}_s]ds$$

is an $(\mathscr{F}_t)$-martingale. The claim follows from Lévy's characterisation since $[W^X, W^X]_t = [X, X]_t = I_n t$. □

Recall that for $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$ we have

$$f(Z_t) = f(Z_0) + \int_0^t A_s f(Z_s)ds + M_t^f.$$

Thus, taking the $(\mathscr{F}_t)$-optional projections of both sides yields

$$\pi_t f = \pi_0 f + \int_0^t E[A_s f(Z_s)|\mathscr{F}_s]ds + \text{ martingale} \tag{3.4}$$

since a_{ij} is locally bounded, f vanishes outside compacts, and $E\int_0^T \|b(t, Z_t)\|dt < \infty$ due to Assumption 3.1. Our goal for the rest of this section is to characterise the martingale in the above decomposition explicitly. The following theorem gives a characterisation of $(\mathscr{F}_t)$-martingales, which will be instrumental in achieving this goal.

Theorem 3.1 *Let M be a càdlàg d-dimensional $(\mathscr{F}_t)$-martingale. Then, there exists an $(\mathscr{F}_t)$-predictable matrix-valued process Φ such that*

$$M_t = M_0 + \int_0^t \Phi_s dW_s^X.$$

Moreover, if M is square integrable, then Φ satisfies

$$\int_0^t E\|\Phi_s\|^2 ds < \infty, \tag{3.5}$$

where

$$\|a\| = \sqrt{\sum_{i=1}^d \sum_{j=1}^m a_{ij}^2}$$

for a matrix a.

Proof Denote by h_t $h(t, Z_t, X_t)$ and by h_t^* the transpose of the vector h_t, and define

$$L_t = \exp\left(-\int_0^t {}^\circ h_s^* dW_s^X - \frac{1}{2}\int_0^t \|{}^\circ h_s\|^2 ds\right).$$

Let $\tau_k = \inf\{t > 0 : \int_0^t \|{}^\circ h_s\|^2 ds > k\}$ with the convention that infimum of the empty set is T. Note that $L_{t\wedge\tau_k}$ is a martingale as it satisfies the Novikov's condition. Moreover, due to Assumption 3.1, $\tau_k \to T$ as $k \to \infty$, P-a.s.,

Consider the probability measure P^k on $(\Omega, \mathscr{G})$ defined by $\frac{dP^k}{dP} = L_{\tau_k}$. By Girsanov's theorem the process

$$X_t^k = W_t^X + \int_0^{t\wedge\tau_k} {}^\circ h_s ds$$

is an n-dimensional $(P^k, (\mathscr{F}_t))$-Brownian motion. Set $\mathscr{F}_t^k = \mathscr{N} \vee \sigma(X_s^k; s \le t)$, with $\mathscr{N}$ being the collection of P-null sets. Then, every $(\mathscr{F}_t^k)$-martingale can be written as an integral with respect to X^k (see Theorem IV.3.43 in [99]).

Let $M_t^k = M_{t\wedge\tau_k} L_{t\wedge\tau_k}^{-1}$. Then, M^k is $((\mathscr{F}_t), P^k))$-martingale. If we can show M^k is $((\mathscr{F}_t^k), P^k))$-martingale, then there exists an $(\mathscr{F}_t^k)$-predictable process $\tilde{\Phi}^k$ such that

$$M_t^k = M_0 + \int_0^t \tilde{\Phi}_s^k dX_s^k.$$

Moreover, in view of the uniqueness of this representation, one has

$$\tilde{\Phi}_s^k = \tilde{\Phi}_s^l, \text{ if } s < \tau_k \text{ and } k < l. \tag{3.6}$$

Using integration by parts, and the fact that $X_{t\wedge\tau_k}^k = X_{t\wedge\tau_k}$ we have

$$M_{t\wedge\tau_k} = M_0 + \int_0^{t\wedge\tau_k} L_s \tilde{\Phi}_s^k dW_s^X,$$

since any continuous local martingale of finite variation has to be constant. Since $\tau_k \to T$ as $k \to \infty$, P-a.s.,

$$M_t = M_0 + \int_0^t \Phi_s dW_s^X,$$

where $\Phi_s = \tilde{\Phi}_s^k L_s$ for $s \le \tau_k$. Note that Φ is well defined due to (3.6) and it is also $(\mathscr{F}_t)$-predictable. When M is a square integrable martingale,

$$\int_0^t E\|\Phi_s\|^2 ds = E(M_t - M_0)^2 < \infty.$$

Thus, it remains to show that M^k is $((\mathscr{F}_t^k), P^k)$-martingale. First, observe that M^k is $((\mathscr{F}_{t\wedge\tau_k}), P^k))$-martingale, where the σ-algebra

$$\begin{aligned}\mathscr{F}_{t\wedge\tau_k} &= \{B \in \mathscr{F}_T : B \cap [t \wedge \tau_k \le s] \in \mathscr{F}_s,\ s \in [0,T]\}\\ &= \{B \in \mathscr{F}_t : B \cap [\tau_k \le s] \in \mathscr{F}_s,\ s \in [0,T]\}.\end{aligned}$$

Recall that $X^k_{t\wedge\tau_k} = X_{t\wedge\tau_k}$ and $(\mathscr{F}_t)$ and $(\mathscr{F}_t^k)$ are generated by X and X^k, respectively. Thus, it is reasonable to expect that

$$\mathscr{F}_{t\wedge\tau_k} = \mathscr{F}^k_{t\wedge\tau_k} = \{B \in \mathscr{F}_T : B \cap [t \wedge \tau_k \le s] \in \mathscr{F}^k_s,\ s \in [0,T]\}. \tag{3.7}$$

The inclusion $\mathscr{F}^k_{t\wedge\tau_k} \subset \mathscr{F}_{t\wedge\tau_k}$ is obvious. However, the reverse inclusion will only hold if τ_k is an $(\mathscr{F}_t^k)$-stopping time since, otherwise, the right-hand side of (3.7) ceases to be a σ-algebra. We will show that τ_k is indeed an $(\mathscr{F}_t^k)$-stopping time and the equality holds. That would imply that for any $(\mathscr{F}_t^k)$-stopping time τ we have $E^k[M^k_\tau] = E^k[M^k_{\tau_k\wedge\tau}] = E^k[M_0]$ by Doob's Optional Stopping Theorem since $\tau_k \wedge \tau$ is an $(\mathscr{F}^k_{t\wedge\tau_k})$-stopping time. Thus, M^k is an $((\mathscr{F}_t^k), P^k)$-martingale.

Define

$$\mathscr{F}^k_{\tau_k} = \sigma\{\eta_{\tau_k} : \eta \text{ any } (\mathscr{F}_t^k)\text{-optional process}\}.$$

Since $\mathscr{F}_t^k \subset \mathscr{F}_t$ for all $t \in [0,T]$, we have $\mathscr{F}^k_{\tau_k} \subset \mathscr{F}_{\tau_k}$. Then by Theorem A.16 τ_k is an $(\mathscr{F}_t^k)$-stopping time if and only if for all bounded $((\mathscr{F}_t^k), P^k)$ martingale N,

$$E^k[N_T | \mathscr{F}^k_{\tau_k}] = N_{\tau_k}.$$

Observe that by martingale representation theorem for Brownian filtrations, $N_t = N_0 + \int_0^t \psi_s dX^k_s$ with $E^k[(N_T - N_0)^2] = E^k \int_0^T \|\psi_s\|^2 ds$. Since X^k is also an $(\mathscr{F}^t)$-Brownian motion, this implies N is $((\mathscr{F}_t), P^k)$-martingale. Moreover, since τ_k is an $(\mathscr{F}_t)$-stopping time, it follows from Theorem A.16 that

$$N_{\tau_k} = E^k[N_T | \mathscr{F}_{\tau_k}].$$

The claim follows due to $\mathscr{F}^k_{\tau_k} \subset \mathscr{F}_{\tau_k}$.

Next, we need to show that $\mathscr{F}_{t\wedge\tau_k} \subset \mathscr{F}^k_{t\wedge\tau_k}$. To this end, consider

$$H_t = \int_0^t \|{}^\circ h_s\|^2 ds \text{ as well as } H_t^n = \sum_{i=0}^\infty \mathbf{1}_{[2^{-n}i,2^{-n}(i+1))}(t) H_{2^{-n}(i-1)},$$

and note that a.s. H^n converges to H uniformly in $t \in [0,T]$. Since $H_{2^{-n}(i-1)} \in \mathscr{F}^X_{2^{-n}i}$, there exists a random variable $P_i^n \in \sigma(X_s; s \le 2^{-n}i)$ such that $P_i^n = H_{2^{-n}(i-1)}$, a.s. Define

$$\bar{H}_t^n = \sum_{i=0}^{\infty} \mathbf{1}_{[2^{-n}i, 2^{-n}(i+1))}(t) P_i^n$$

and observe that $\bar{H}^n$ is $(\mathscr{F}_t^0)$-optional where $\mathscr{F}_t^0 = \sigma(X_s; s \le t)$. Moreover, $\bar{H}^n$ is indistinguishable from H^n, therefore $\bar{H} = \liminf_{n\to\infty} \bar{H}^n$ is indistinguishable from H. Hence, τ_k is indistinguishable from the $(\mathscr{F}_t^0)$-stopping time

$$\bar{\tau}_k = \inf\{t \ge 0 : \bar{H}_t \ge k\}.$$

Then, it follows from Lemma 2.32 of [13] that $\mathscr{F}^0_{t\wedge\bar{\tau}_k} = \sigma(X_{s\wedge\bar{\tau}_k}; s \le t)$. It is also easy to see that $\mathscr{F}_{t\wedge\bar{\tau}_k} = \mathscr{F}_{t\wedge\tau_k}$. Consequently,

$$\begin{aligned}
\mathscr{F}_{t\wedge\tau_k} = \mathscr{F}_{t\wedge\bar{\tau}_k} = \bigwedge_{u>t} \mathscr{F}^X_{u\wedge\bar{\tau}_k} &= \bigwedge_{u>t} \left(\sigma(X_{s\wedge\bar{\tau}_k}; s \le u) \vee \mathscr{N}\right) \\
&= \bigwedge_{u>t} \left(\sigma(X^k_{s\wedge\tau_k}; s \le u) \vee \mathscr{N}\right) \subset \bigwedge_{u>t} \mathscr{F}^k_{u\wedge\tau_k} = \mathscr{F}^k_{t\wedge\tau_k},
\end{aligned}$$

where the third equality is due to Theorem A.5.4, the fifth holds since $X_{\cdot\wedge\bar{\tau}_k} = X^k_{\cdot\wedge\tau_k}$, a.s., and the last equality follows from the right-continuity of $(\mathscr{F}^k_t)$ along with Theorem A.5.4. □

The next result is a consequence of the above representation theorem.

Corollary 3.2 *The collection of random variables* $\{\int_0^t \Phi_s dW_s^X : \Phi$ *is bounded and* $(\mathscr{F}_t)$*-predictable* $n \times 1$*-process*$\}$ *is dense in* $L^2(\mathscr{F}_t, P)$ *up to constants.*

Proof Let $\eta \in L^2(\mathscr{F}_t, P)$ be a one-dimensional random variable and define $M_s = E[\eta|\mathscr{F}_s]$ for $s \in [0, T]$. Then, M is a square integrable martingale and, therefore, in view of Theorem 3.1, there exists an $(\mathscr{F}_t)$-predictable process Φ with

$$\int_0^t E\|\Phi_s\|^2 ds < \infty$$

such that

$$\eta = M_t = E[\eta] + \int_0^t \Phi_s dW_s^X.$$

Thus, if we define $\Phi^k = (\Phi_1^k, \dots, \Phi_n^k)$ by $\Phi_i^k = (\Phi_i \wedge k) \vee -k$ for $i = 1, \dots, n$, we obtain

$$\lim_{k\to\infty} \int_0^t E\|\Phi_s^k - \Phi_s\|^2 = 0,$$

which yields $\int_0^t \Phi_s^k dW_s^X \to \int_0^t \Phi_s dW_s^X$ in $L^2(\mathscr{F}_t, P)$. □

The next theorem gives the equation for $\pi_t f$ and is the main result of this section.

Theorem 3.2 *Let $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$. Then, under Assumption 3.1 the following holds:*

$$\pi_t f = \pi_0 f + \int_0^t \pi_s A_s f ds + \int_0^t \left\{ E[h_s^* f(Z_s)|\mathscr{F}_s] - {}^\circ h_s^* \pi_s f + \pi_s (D_s f)^* \right\} dW_s^X, \tag{3.8}$$

where h_s denotes $h(s, Z_s, X_s)$, $D_s = \begin{pmatrix} D_s^1 \\ \vdots \\ D_s^n \end{pmatrix}$ is the operator defined by $D_s^i = \sum_{j=1}^m \sigma_{ji}^2(s,\cdot)\frac{\partial}{\partial z_j}$, and * stands for the matrix transpose.

Proof As it was already observed in (3.4) the process

$$\hat{M}_t = \pi_t f - \pi_0 f - \int_0^t \pi_s A_s f ds$$

is an $(\mathscr{F}_t)$-martingale. Moreover, it is square integrable by Assumption 3.1 and that f has compact support. Thus, there exists a n-dimensional $(\mathscr{F}_t)$-predictable vector process, Φ, such that

$$\hat{M}_t = \int_0^t \Phi_s^* dW_s^X,$$

with $E \int_0^t \|\Phi_s\|^2 ds < \infty$. Let λ be a bounded n-dimensional $(\mathscr{F}_t)$-predicable vector process and define $\xi_t = \int_0^t \lambda_s^* dW_s^X$. Since ξ is a square-integrable $(\mathscr{F}_t)$-martingale, we have

$$E[\hat{M}_t \xi_t] = \int_0^t E[\lambda_s^* \Phi_s] ds. \tag{3.9}$$

We will next compute $E[\hat{M}_t \xi_t]$ by considering $\xi_t \left(\pi_t f - \pi_0 f - \int_0^t \pi_s A_s f ds\right)$. First observe that $E[\xi_t \pi_0 f] = E[\xi_0 \pi_0 f] = 0$ since ξ is a martingale with $\xi_0 = 0$. Moreover,

$$\begin{aligned} E\left[\xi_t \left(\pi_t f - \pi_0 f - \int_0^t \pi_s A_s f ds\right)\right] &= E\left[\xi_t f(Z_t) - \int_0^t \xi_t \pi_s A_s f ds\right] \\ &= E\left[\xi_t f(Z_t) - \int_0^t \xi_s \pi_s A_s f ds\right] \\ &= E\left[\xi_t f(Z_t) - \int_0^t \xi_s A_s f(Z_s) ds\right]. \end{aligned}$$

Note that $\xi_t = \tilde{\xi}_t + \int_0^t \lambda_s^* (h_s - {}^\circ h_s)\, ds$ where $\tilde{\xi}_t = \int_0^t \lambda_s d\tilde{W}_s^X$ and $\tilde{W}^X = (0 \;\; I_n)\tilde{W}$ – a Brownian motion driving the observation X. Therefore,

$$E[\hat{M}_t \xi_t] = E\left[\tilde{\xi}_t f(Z_t) - \int_0^t \tilde{\xi}_s A_s f(Z_s) ds\right] + E\left[f(Z_t) \int_0^t \lambda_s (h_s - {}^\circ h_s)\, ds\right]$$
$$-E\left[\int_0^t \left\{\int_0^s \lambda_r (h_r - {}^\circ h_r)\, dr\right\} A_s f(Z_s) ds\right]. \tag{3.10}$$

On the other hand, since $\tilde{\xi}$, as well as M^f, is a square integrable $(\mathscr{G}_t)$-martingale with $\tilde{\xi}_0 = 0$,

$$\begin{aligned} &E\left[\tilde{\xi}_t f(Z_t) - \int_0^t \tilde{\xi}_s A_s f(Z_s) ds\right] \\ &= E\left[\tilde{\xi}_t \left(f(Z_t) - f(Z_0) - \int_0^t A_s f(Z_s) ds\right)\right] = E[\tilde{\xi}_t M_t^f] \\ &= E\left[\int_0^t \lambda_s^* D_s f(Z_s) ds\right] = E\left[\int_0^t \lambda_s^* \pi_s D_s f ds\right] \end{aligned} \tag{3.11}$$

We next compute the remaining terms in (3.10). Integration by parts formula applied to $f(Z_t) \int_0^t \lambda_s (h_s - {}^\circ h_s)\, ds$ yields

$$\begin{aligned} &E\left[f(Z_t) \int_0^t \lambda_s (h_s - {}^\circ h_s)\, ds - \int_0^t \left\{\int_0^s \lambda_r (h_r - {}^\circ h_r)\, dr\right\} A_s f(Z_s) ds\right] \\ &= E\left[\int_0^t \lambda_s^* f(Z_s) (h_s - {}^\circ h_s)\, ds\right] \\ &= E\left[\int_0^t \lambda_s^* E[f(Z_s) h_s - {}^\circ h_s f(Z_s) | \mathscr{F}_s] ds\right]. \end{aligned} \tag{3.12}$$

In the first equality above we have made use of the fact that $\int_0^t \left\{\int_0^s \lambda_r (h_r - {}^\circ h_r)\, dr\right\} dM_s^f$ is a $(\mathscr{G}_t)$-martingale. This is due to the square integrability condition (3.1), that σ is locally bounded and f has a compact support. Combining (3.9) with (3.10), (3.11) and (3.10) yields

$$\Phi_s^* = E[h_s^* f(Z_s) | \mathscr{F}_s] - {}^\circ h_s^* \pi_s f + \pi_s (D_s f)^*$$

in view of Corollary 3.2 since λ was an arbitrary bounded predictable process. $\square$

3.2 Kushner–Stratonovich Equation: Existence and Uniqueness

Intuitively, the knowledge of π_t for all $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$ amounts to the knowledge of the $\mathscr{F}_t$-conditional distribution of the signal. To study the evolution of this conditional distribution we look for an optional process, $(\pi_t(\omega, dz)$ taking values in $\mathscr{P}(\mathbf{E}^1)$ such $(\pi_t f)_{t\in[0,T]}$ is indistinguishable from $\left(\int_{\mathbf{E}^1} f(z)\pi_t(\omega, dz)\right)_{t\in[0,T]}$. It was shown by Yor [112] that such a process exists and is càdlàg in the sense that π_t converges to π_s as $t \downarrow s$ in the topology of weak convergence.

Proposition 3.2 *There exists càdlàg $\mathscr{P}(\mathbf{E}^1)$-valued $(\mathscr{F}_t)$-optional process $(\pi_t(\omega, dz))_{t\in[0,T]}$ such that for all $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$*

$$\pi_t f = \int_{\mathbf{E}^1} f(z)\pi_t(\omega, dz), \qquad t \in [0, T].$$

Moreover, for all but countably many t,

$$\pi_t f = E[f(Z_t)|\mathscr{F}_t^X].$$

It is clear that $(\pi_t(\omega, dz))$ solves (3.8) for all $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$. Hence, the first step in identifying the conditional distribution is to find a solution to (3.8). To be able to conclude that the obtained solution is in fact the desired distribution, we must demonstrate that this equation has a unique solution in the space of measure-valued processes.

This uniqueness will be established under some mild conditions on the operator $\tilde{A}$ following an approach introduced by Kurtz and Ocone [84]. The main ingredient of their approach is the so-called *filtered martingale problem*. Let $D([0, T], \tilde{\mathbf{E}})$ denote the space of càdlàg $\tilde{\mathbf{E}}$-valued paths, where $\tilde{\mathbf{E}}$ is a complete and separable metric space. As usual, the space $D([0, T], \tilde{\mathbf{E}})$ is endowed with Skorokhod topology. A process (μ, U) with sample paths in $D([0, T], \mathscr{P}(\mathbf{E}^1) \times \mathbf{E}^2)$ is called a solution to the filtered martingale problem for $\tilde{A}$ if μ is $(\mathscr{F}_{t+}^U)$-optional process and

$$\mu_t f(\cdot, U_t) - \int_0^t \mu_s \tilde{A}_s f(\cdot, U_s)ds$$

is an $(\mathscr{F}_{t+}^U)$-martingale under the probability measure induced by U for all $f \in \mathbb{C}_K^\infty(\mathbf{E})$, where

$$\mu_t f(\cdot, U_t) = \int_{\mathbf{E}^1} f(z, U_t)\mu_t(\omega, dz).$$

Observe that since (μ, U) is $(\mathscr{F}_{t+}^U)$-adapted, for each $t \in [0, T]$, there exists a Borel measurable function $F_t : D([0, T], \mathbf{E}^2) \mapsto \mathscr{P}(\mathbf{E}^1)$ such that

$$\mu_t = F_t(U) = F_t(U^s), \qquad \forall s > t,$$

where U^s denotes the process U stopped at time s. Since μ is càdlàg , $\mu_t = \mu_{t-}$, a.s., for all but countably many t, and, therefore, μ_t is $\mathscr{F}_t^U$-measurable. Hence, for such t, there exists a Borel measurable function $G_t : D([0, T], \mathbf{E}^2) \mapsto \mathscr{P}(\mathbf{E}^1)$ such that $G_t(U) = G_t(U^t) = F_t(U)$.

It can be easily verified that (π, X) is a solution to the filtered martingale problem for $\tilde{A}$. The above considerations yield that $\pi_t = F_t(X)$ for some measurable function F_t for each t, as well. Thus, if we can demonstrate that all solutions of the filtered martingale problem with the same initial condition share the same F_t for all $t \in [0, T]$, then $\pi_t = F_t(X)$.

A direct verification of the uniqueness of F_t is not straightforward. However, suppose that whenever (μ, U) is a solution of the filtered martingale problem for $\tilde{A}$, we have for each k and any choice of continuous and bounded $g^i : [0, T] \times \mathbf{E}^2 \mapsto \mathbb{R}_+$, $i = 1, \ldots, k$,

$$\begin{aligned} &E^U[\mu_t f(\cdot, U_t, \xi_t^0, \ldots, \xi_t^k)] = E[f(Z_t, X_t, \eta_t^0, \ldots, \eta_t^k)], \\ &f \in \mathbb{C}_K^\infty(\mathbf{E} \times [0, \infty)^{k+1}), \end{aligned} \tag{3.13}$$

where $\xi_t^0 = \eta_t^0 = t$ and

$$\xi_t^i = \int_0^t g^i(s, U_s)ds \qquad \eta_t^i = \int_0^t g^i(s, X_s)ds, \qquad i = 1, \ldots, k. \tag{3.14}$$

This implies in particular that $(U_t, \xi_t^0, \ldots, \xi_t^k)$ and $(X_t, \eta_t^0, \ldots, \eta_t^k)$ have the same distribution for all k and $t \in [0, T]$. Since k and g^i are arbitrary, this yields that $(U^t, \xi_t^0, \ldots, \xi_t^k)$ and $(X^t, \eta_t^0, \ldots, \eta_t^k)$ have the same law for all k and $t \in [0, T]$. Here, as before, U^t and X^t are the processes stopped at t.

By the definition of G_t, we have, for all but countably many t,

$$\begin{aligned} E^U[\mu_t f(\cdot, U_t, \xi_t^0, \ldots, \xi_t^k)] &= E^U[G_t(U) f(\cdot, U_t, \xi_t^0, \ldots, \xi_t^k)] \\ &= E^U[G_t(U^t) f(\cdot, U_t, \xi_t^0, \ldots, \xi_t^k)] \\ &= E[G_t(X^t) f(\cdot, X_t, \eta_t^0, \ldots, \eta_t^k)]. \end{aligned}$$

Since $G_t(X^t)$ is $\mathscr{F}_t^X$-measurable, the above implies $G_t(X^t) = \pi_t$ for t for which $\pi_t = \pi_{t-}$ and $\mu_t = \mu_{t-}$, a.s., in view of (3.13) and

$$\mathscr{F}_t^X = \sigma(X_t, \int_0^t g(s, X_s)ds : g \text{ positive, continuous and bounded function}) \vee \mathscr{N}.$$

Moreover, since π and μ are càdlàg , $\pi_t = F_t(X)$ for all t.

An obvious necessary condition for (3.13) to hold is that (μ, U) should satisfy

$$E^U[\mu_0 f(\cdot, U_0)] = E[f(Z_0, X_0)], \qquad f \in \mathbb{C}_K^\infty(\mathbf{E}). \tag{3.15}$$

To show (3.13) for $t > 0$ we follow Kurtz and Ocone [84] by fixing k and the functions $(g^i)_{i=1}^k$, and considering the auxiliary martingale problem for $\tilde{B^k}$, where

$$\tilde{B}_t^k f(z, x, v^1, \ldots, v^k) = \tilde{A}_t f_0(z, x) \Pi_{i=1}^k f_i(v^i) \\ + f_0(z, x) \sum_{i=1}^k g^i(x, t) f_i'(v^i) \Pi_{j \neq i} f_j(v^j),$$

for any f having the representation

$$f(z, x, v^1, \ldots, v^k) = f_0(z, x) \Pi_{i=1}^k f_i(v^i), \qquad f_0 \in \mathbb{C}_K^\infty(\mathbf{E}),\ f_i \in \mathbb{C}_K^1(\mathbb{R}_+).$$

We denote the set of linear combinations of functions with the above representation by $\mathscr{D}(\tilde{B}^k)$ and note that $\mathscr{D}(\tilde{B}^k)$ is an algebra that is dense in $\mathbb{C}_0(\mathbf{E} \times [0, \infty)^k)$ in view of Stone–Weierstrass Theorem. If (Z, X) is a solution to the martingale problem for $\tilde{A}$, then $(Z, X, \eta^1, \ldots, \eta^k)$ solves the martingale problem for $\tilde{B}^k$, where η^is are as defined in (3.14). In fact, more is true. Kurtz and Ocone have shown that the well posedness of the martingale problem for $\tilde{A}$ implies that the martingale problem for $\tilde{B}^k$ is well posed.

Well posedness of $\tilde{B}^k$ along with the fact that $\mathbf{E} \times [0, \infty)^k$ is locally compact and separable yield the following key result, which is Proposition 2.2 in [84].

Proposition 3.3 *Let $\tilde{B}^k$ be as above and suppose that the martingale problem for $\tilde{A}$ is well posed. If $(\nu_t)_{t \in [0,T]}$ and $(\mu_t)_{t \in [0,T]}$ are right continuous solutions of the forward equation*

$$\nu_t f = \nu_0 f + \int_0^t \nu_s \tilde{B}^k f ds, \qquad f \in \mathscr{D}(\tilde{B}) \tag{3.16}$$

such that $\nu_0 = \mu_0$, then $\nu_t = \mu_t$ for all $t \in [0, T]$.

In view of the above proposition suppose the martingale problem for $\tilde{A}$ is well posed and (μ, U) is a solution of the filtered martingale problem for $\tilde{A}$. Then, one can directly verify that $(\mu, U, \xi^1, \ldots, \xi^k)$ is a solution of the filtered martingale problem for $\tilde{B}^k$. Let $\nu_t \in \mathscr{P}(\mathbf{E}^1)$ satisfy

$$\nu_t f = E^U[\mu_t f(\cdot, U_t, \xi_t^1, \ldots, \xi_t^k)].$$

ν is clearly right continuous and is a solution of (3.16). Since (π, X) is a solution of the filtered martingale problem for $(\tilde{A})$, it now follows from Proposition 3.3 that (3.13) holds for any solution (μ, U) of the filtered martingale problem for $\tilde{A}$ satisfying (3.15).

Thus, we have shown under the hypothesis that the martingale problem for $\tilde{A}$ is well posed, that there exists a Borel measurable function $F_t : D([0,T], \mathbf{E}^2) \mapsto \mathscr{P}(\mathbf{E}^1)$ such that whenever (μ, U) is a solution of the filtered martingale problem for $\tilde{A}$ satisfying the initial condition (3.15), $\mu_t = F_t(U) = F_t(U^s)$, for any $s > t$. Kurtz and Ocone also provides a version of this result for the stopped filtered martingale problem, which will be instrumental in determining the conditional distribution of Z.

Corollary 3.3 *Suppose that the martingale problem for $\tilde{A}$ is well posed and (μ, U) is a process with sample paths in $D([0,T], \mathscr{P}(\mathbf{E}^1) \times \mathbf{E}^2)$ satisfying (3.15), that τ is an $(\mathscr{F}^U_{t+})$-stopping time, and that for each $f \in \mathbb{C}^\infty_K(\mathbf{E})$*

$$\mu_{t\wedge\tau} f(\cdot, U_{t\wedge\tau}) - \int_0^{t\wedge\tau} \mu_s \tilde{A}_s f(\cdot, U_s) ds$$

is an $(\mathscr{F}^U_{t+})$-martingale under the probability measure induced by U. Then, for each $t \in [0,T]$,

$$\mu_t \mathbf{1}_{[t<\tau]} = F_t(U)\mathbf{1}_{[t<\tau]}, \ a.s.,$$

where $F_t : D([0,T], \mathbf{E}^2) \mapsto \mathscr{P}(\mathbf{E}^1)$ is a Borel measurable function with $F_t(U) = F_t(U^s)$, for any $s > t$.

Our plan of identifying the conditional distribution of Z will be as follows. First, we introduce the *Kushner–Stratonovich equation* associated with (3.8), of which π will be a solution. Then, we will see that any solution of the Kushner–Stratonovich equation can be used to construct a solution to a stopped filtered martingale problem and conclude by applying the statement of Corollary 3.3.

Let μ be an $(\mathscr{F}_t)$-optional process with sample paths in $D([0,T], \mathscr{P}(\mathbf{E}^1))$ and suppose that μ solves the following Kushner–Stratonovich equation for all $f \in \mathbb{C}^\infty_K(\mathbf{E}^1)$:

$$\begin{aligned}\mu_t f = \pi_0 f + \int_0^t \mu_s A_s f ds \\ + \int_0^t \left\{\mu_s(h^*(s,\cdot,X_s) f(\cdot)) - \mu_s h^*(s,\cdot,X_s)\mu_s f + \mu_s (D_s f)^*\right\} dI^\mu_s,\end{aligned} \tag{3.17}$$

where

$$I^\mu_t = X_t - \int_0^t \mu_s h(s,\cdot,X_s) ds.$$

Note that π solves (3.17) and $E\left(\int_0^T \|\pi_s h(s,\cdot,X_s)\|^2 ds\right) < \infty$. Thus, it is reasonable to require the solution of the above Kushner–Stratonovich equation to satisfy the latter integrability property. However, one can achieve the characterisation of the conditional distribution in even larger class of measures as the following theorem states.

Theorem 3.3 *In addition to Assumption 3.1 suppose further that the martingale problem for $\tilde{A}$ is well posed, and μ is an $(\mathscr{F}_t)$-optional process with sample paths in $D([0,T],\mathscr{P}(\mathbf{E}^1))$ satisfying (3.17) for all $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$ such that $\int_0^T \|\mu_s h(s,\cdot,X_s)\|^2 ds < \infty$ a.s. Then, $\mu_t = \pi_t$, for $t < T$, a.s.*

Proof Let $\tilde{f} = f\psi$ for $f \in \mathbb{C}_K^\infty(\mathbf{E}^1)$ and $\psi \in \mathbb{C}_K^\infty(\mathbf{E}^2)$. The class of linear combinations of such functions is an algebra that separates points, thus it is dense in $\mathbb{C}_K^\infty(\mathbf{E})$ in view of Stone–Weierstrass theorem. Denoting the row vector of first partial derivatives of ψ with $\nabla\psi$, first observe that

$$\psi(X_t) = \psi(X_0) + \int_0^t \nabla\psi(X_s)dI_s^\mu + \int_0^t \left\{\nabla\psi(X_s)\mu_s h(s,\cdot,X_s) + \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\psi(X_s)\right\} ds.$$

Thus, integration by parts formula applied to $\psi(X)\mu f$ yields

$$\begin{aligned}\mu_t \tilde{f}(\cdot,X_t) &= \pi_0\tilde{f}(\cdot,X_0) + \int_0^t \mu_s\tilde{A}_s\tilde{f}(\cdot,X_s)ds \\ &\quad + q\int_0^t \left\{\mu_s f\nabla\psi(X_s) + \psi(X_s)\left[\mu_s\left(h^*(s,\cdot,X_s)f(\cdot) + (D_s f)^*(\cdot)\right)\right.\right. \\ &\quad \left.\left. -\mu_s h^*(s,\cdot,X_s)\mu_s f\right]\right\} dI_s^\mu.\end{aligned}$$

Next define $\delta_s = \mu_s h(s,\cdot,X_s) - \pi_s h(s,\cdot,X_s)$ and consider the stopping time

$$\tau_N = T \wedge \inf\{t \geq 0 : \int_0^t \|\delta_s\|^2 ds \geq N \text{ or } \int_0^t \|\mu_s h(s,\cdot,X_s)\|^2 ds \geq N\},$$

which converges to T, P-a.s., as $N \to \infty$ under the hypotheses of the theorem. We define a probability measure Q_N on $(\Omega,\mathscr{G})$ by

$$\frac{dQ_N}{dP} = \exp\left(\int_0^t \delta_s^* dW_s^X - \int_0^t \|\delta_s\|^2 ds\right),$$

where W^X is the innovation process defined in Corollary 3.1. Thus, under Q_N $(I_{t\wedge\tau_N}^\mu)_{t\in[0,T]}$ is a Brownian motion stopped at τ_N adapted to $(\mathscr{F}_t)$. Therefore, since f and ψ have compact supports and σ_{ij}s are locally bounded

$$\left(\mu_{t\wedge\tau_N}\tilde{f}(\cdot,X_{t\wedge\tau_N}) - \int_0^{t\wedge\tau_N}\mu_s\tilde{A}_s\tilde{f}(\cdot,X_s)ds\right)_{t\in[0,T]}$$

is a $(Q_N, \mathscr{F}_t)$-martingale. Hence, by Corollary 3.3, there exists an F_t, with the properties stated therein, such that

$$\mu_t \mathbf{1}_{[t<\tau_N]} = F_t(X)\mathbf{1}_{[t<\tau_N]} = \pi_t \mathbf{1}_{[t<\tau_N]}, \; Q_N\text{-a.s..}$$

Since $Q_N \sim P$, the above equality holds P-a.s., as well. Letting $N \to \infty$, yields the claim. □

It is unusual for the Kushner–Stratonovich equation to have an explicit solution. However, when the process Y is Gaussian, the conditional distribution of Z is also Gaussian and therefore is fully identified by its mean and variance.

Theorem 3.4 *Suppose $b(t,z) = b^0(t) + b^1(t)z$, $h(t,z,x) = h^0(t) + h^1(t)z + h^2(t)x$, and σ^1 and σ^2 are functions of time only. Assume further that b^i are integrable, h^i are square integrable and σ^i are bounded. Then, if Z_0 is Gaussian with mean m_0 and covariance matrix $\gamma(0)$, we have*

$$\pi_t(dz) = \frac{1}{\sqrt{(2\pi)^m|\gamma(t)|}} \exp\left(-\frac{1}{2}(z-m_t)^*\gamma(t)^{-1}(z-m_t)\right),$$

where $|\gamma(t)|$ is the determinant of $\gamma(t)$, and

$$\begin{aligned}
dm_t &= \left(b^0(t) + b^1(t)m_t\right)dt \\
&\quad + \left(\sigma^2(t) + \gamma(t)(h^1(t))^*\right)\left[dX_t - \left(h^0(t) + h^1(t)m_t + h^2(t)X_t\right)dt\right], \\
\frac{d\gamma(t)}{dt} &= (b^1(t) - \sigma^2(t)h^1(t))\gamma(t) + \gamma(t)(b^1(t) \\
&\quad - \sigma^2(t)h^1(t))^* + \sigma^1(t)(\sigma^1(t))^* - \gamma(t)(h^1(t))^*h_1(t)\gamma(t).
\end{aligned}$$

Proof It follows from Example 2.4 that there exists a unique strong solution to (3.1). Consequently, the martingale problem for $\tilde{A}$ is well posed due to Corollary 2.5. The fact that Assumption 3.1 is satisfied follows from Gronwall inequality and the assumed integrability of the coefficients. Indeed, using the fact that $(\sum_{i=1}^d a_i)^2 \leq C\sum_{i=1}^d a_i^2$ for some $C < \infty$, we obtain

$$\begin{aligned}
\mathbb{E}\|Z_t\|^2 &\leq K_1\left(1 + \sum_{i,j=1}^d \left(\int_0^t |b^1_{ij}(s)||Z^j_s|ds\right)^2\right) \\
&\leq K_2\left(1 + \int_0^t \bar{b}^1(s)\mathbb{E}\|Z_s\|^2 ds\right)
\end{aligned}$$

where $\bar{b}^1(s) := \max_{i,j=1,\ldots,d} b^1_{ij}(s)$ and we used Jensen's inequality in the second line. This implies $\sup_{t\leq T}\mathbb{E}\|Z_t\|^2 < \infty$.

Similarly,

$$\mathbb{E}\|X_t\|^2 \le K\left(1+\int_0^t \left(\bar{h}^1(s)\mathbb{E}\|Z_s\|^2+\bar{h}^2(s)\mathbb{E}\|X_s\|^2\right)ds\right),$$

implying $\sup_{t\le T}\mathbb{E}\|X_t\|^2 < \infty$. Hence, Assumption 3.1 is satisfied in view of the integrability properties of b^i and h^1.

Note that this choice of π satisfies Kushner–Stratonovich equation (3.17). Moreover, direct calculations give that $\pi_s h(s,\cdot,X_s) = m_s$, which can be shown to be square integrable via a similar application of Gronwall inequality. Thus it follows from Theorem 3.3 that π is the conditional density. □

3.3 Notes

The material in Sect. 3.1 is fairly standard and can be found in Bain and Crisan [13], Kallianour [76] and Liptser and Shiryaev [86]. The martingale representation in the observation filtration is proved in the literature under the square integrability condition (3.5) and is originally due to Fusijaki et al. [58]. Theorem 3.1 extends this representation by dropping this assumption. Our proof follows closely the lines of the proof of Theorem 8.3.1 in [76]. However, in order to accommodate a larger class of martingales, their arguments needed to be modified using some techniques borrowed from Lemma 2.32 in [13].

The main goal of Sect. 3.2 is to identify the conditional distribution of the signal. This is done via Kushner–Stratonovich equation once the uniqueness of its solutions is established. We follow the filtered martingale problem approach of Kurtz and Ocone [84]. Proposition 3.3 and Corollary 3.3 adapt the results of Kurtz and Ocone to the present setting. We then use these results in Theorem 3.3 to establish the uniqueness of solutions of the Kushner–Stratonovich equation by employing ideas from Theorem 4.1 in [84]. Kushner–Stratonovich equation is non-linear but can be transformed into a linear equation by considering the unnormalised conditional density. The resulting equation is called the Zakai equation, which was first introduced in [114]. The approach outlined in Sect. 3.2 can be used to show the uniqueness of the Zakai equation as well.

Finally, the linear filtering result is standard and our exposition is taken from Chap. 10 of Liptser and Shiryaev [86]. For a more general approach to linear filtering for Gaussian processes see Kallianpur [76].

The issue with the uniqueness of solutions of the Kushner–Stratonovich equation can also be tackled by establishing the uniqueness for the solutions of an associated stochastic partial differential equation (SPDE). However, this approach places extra conditions on the signal and observation processesthat are not satisfied in our applications, leading us to follow Kurtz and Ocone [84]. We refer the reader to Chap. 4 of [13] and the references therein for an exposition of the SPDE and other related approaches.

In this book we are only interested in the theory of stochastic filtering and its applications to the financial equilibrium theory discussed in the second part. However, Bain and Crisan [13] is an excellent source for the numerical methods in filtering. For the application of filtering to the statistics of diffusion processes with observable components the reader is referred to Liptser and Shiryaev [87].

Chapter 4
Static Markov Bridges and Enlargement of Filtrations

In the applications considered in the second part of this book, the rational agents in equilibrium trade so as to drive the demand for the traded to a certain level at a future date. A simple illustration of this phenomenon can be given via the well-known Brownian bridge process, which is the solution to the following SDE:

$$X_t = B_t + \int_0^t \frac{z - X_s}{1-s} ds, \qquad t \in [0, 1), \tag{4.1}$$

where B is a Brownian motion and z is a constant. The solution to the above equation is a Gaussian process and has the bridge property that $\lim_{t \to 1} X_t = z$, a.s. From the point of view of the financial applications we have in mind, B is the cumulative trades of the traders other than a particular strategic trader whose trading, which consists of the finite variation part of X, drives the total demand, X, to the level z at time 1. The law of the process X is that of a standard Brownian motion conditioned to be equal to z at time 1, hence the name Brownian bridge. In this chapter we will study analogous conditionings for a class of continuous Markov processes taking values in $\mathbb{R}^d$. Our basic tool in this study will be *Doob's h-transform*. We call the resulting conditioned process a static Markov bridge since the terminal value is known at time 0. We will also discuss the connection of Markov bridges to the theory of enlargement of filtrations.

4.1 Static Markov Bridges

When one conditions a standard Brownian motion starting at x to equal z at time 1, the resulting process, X, is a Markov process whose transition density can be found easily in view of the Markov property of Brownian motion. Indeed, the transition density of this Brownian bridge is given by

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$$p(t-s,v,y)\frac{p(1-t,y,z)}{p(1-s,v,z)}\,dy \quad \left(=\frac{P^x(X_s\in dv, X_t\in dy, X_1\in dz)}{P^x(X_s\in dv, X_1\in dz)}\right) \tag{4.2}$$

for $0 \le s < t \le 1$, where P^x is the law of Brownian motion starting at x and p represents the transition density of Brownian motion. It can be easily checked that the above satisfies the conditions of Definition 1.2; thus, Kolmogorov's extension theorem gives the existence of a stochastic process on the canonical space whose transition probabilities are defined by (4.2). Equation (4.1) and its solution construct such a process via a stochastic differential equation. Note that X is a time-inhomogeneous Markov process. This can be seen both from the SDE characterisation of (4.1) and from the form of the transition function.

Whenever s and t are strictly less than 1, one can immediately see from (4.2) an absolute continuity relationship between the laws of Brownian motion and the associated bridge process. Thus, if X is the coordinate process on the canonical space $C([0,1],\mathbb{R})$, and $P^{x\to z}_{0\to 1}$ is the law of Brownian bridge starting at x and ending up at z at time 1, then for any $t<1$ and a bounded Borel measurable function $F: C([0,t],\mathbb{R}) \mapsto \mathbb{R}$, we have

$$E^{x\to z}_{0\to 1}[F(X_s; s\le t)] = E^x\left[F(X_s; s\le t)\frac{p(1-t,X_t,z)}{p(1,x,z)}\right].$$

If we define h by $h(t,x)=p(1-t,x,z)$, then h is a space-time harmonic function, i.e. $h_t+\frac{1}{2}h_{xx}=0$, then $(h(t,X_t))_{t\in[0,1)}$ is a P^x-martingale since $p(1-t,x,z)$ is uniformly bounded when t is restricted to a compact interval in $[0,1)$. Therefore, for any $T<1$, we can define a probability measure, Q^T, on $\mathscr{F}_T$ by $\frac{dQ^T}{dP^x}|_{\mathscr{F}_T}=\frac{h(T,X_T)}{h(0,x)}$ and the Girsanov's theorem yields

$$X_t=\beta_t+\int_0^t \frac{z-X_s}{1-s}ds, \qquad t\in[0,T], \tag{4.3}$$

where $(\beta_t)_{t\in[0,T]}$ is a Q^T-Brownian motion. $(X_t)_{t\in[0,T]}$ under this new measure is called the *h-transform* of the original process and, in particular, it solves (4.3). We will see later that it is a Markov process with transition density

$$p(t-s,x,y)\frac{h(t,y)}{h(t,x)}.$$

Note that the above is nothing but (4.2). This procedure constructs a weak solution to (4.1) when time is restricted to $[0,T]$ for an arbitrary $T<1$. The natural question is whether it is possible to find a weak solution on $[0,1]$. Since T is chosen arbitrarily, we might hope to arrive at a single probability measure under which X satisfies (4.1) for all $t<1$ and $X_t\to z$ as $t\to 1$. Existence of such a measure can be, and will be, shown once the probability measures (Q^T) satisfy some consistency property.

In what follows we will follow this recipe for constructing stochastic differential equations solved by a Markov process conditioned to have a pre-specified distribution at a given terminal time.

To formulate the problem let $T^* \in \mathbb{R}_{++}$ denote the terminal time, redefine the index set of time $\mathbf{T}$ to be $\mathbf{T} = [0, T^*]$ and set, for any $s > 0$, $\mathbf{T}_s = [s, \infty) \cap \mathbf{T}$. To characterise Markov bridges in terms of SDEs we will extensively use the link between the solutions of local martingale problems and those of SDEs as studied in Chap. 2. This necessitates working with the canonical space of continuous paths. In particular, we consider $\Omega = C(\mathbf{T}, \mathbf{E})$, where $\mathbf{E}$ and the metric defined on it are as in Sect. 2.2. The space Ω is endowed with the locally uniform topology so that it is a Polish space. X denotes the coordinate process that generates the canonical filtration, $(\mathscr{B}_t)_{t \in \mathbf{T}}$, i.e. $\mathscr{B}_t = \sigma(X_s; s \leq t)$. Recall that $\mathscr{B}_{T^*}$ is countably generated since the corresponding metric space is separable.

Let A be the generator defined by

$$A_t = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t, \cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, \cdot) \frac{\partial}{\partial x_i},$$

where a is a matrix field and b is a vector field. We suppose the following.

Assumption 4.1

1. *Coefficients of A satisfy Assumption 2.1.*
2. *The local martingale problem for A is well posed.*

The above assumption implies in view of Theorem 2.3 and Remark 2.4 that X is a strong Markov process under any $P^{s,\mu}$ for all $s \geq 0$, where μ is a probability measure on $\mathbf{E}$. We will denote by $(P_{s,t})$ the transition function associated with the generator A.

Motivated by the discussion at the beginning of this chapter a Markov bridge starting at (s, ε_x) and ending at (T^*, μ) is the continuous process obtained by conditioning the original process X so that X_{T^*} has the law μ, where μ is a given probability measure on $(\mathbf{E}, \mathscr{E})$.

In what follows we will consider two types of conditioning: a) *weak conditioning* when μ is absolutely continuous with respect to $P_{s,T^*}(x, \cdot)$ and b) *strong conditioning* when $\mu = \varepsilon_z$ for some $z \in \mathbf{E}$. The weak conditioning can be obtained via an *h-function*. Indeed, since μ is absolutely continuous with respect to $P_{s,t}(x, \cdot)$, there exists a Radon–Nikodým derivative, H, so that

$$\mu(E) = \int_E H(y) P_{s,T^*}(x, dy), \forall E \in \mathscr{E}.$$

If we define the function $h : [0, T^*] \times \mathbf{E} \mapsto \mathbb{R}_+$ by

$$h(t, y) := \int H(z) P_{t,T^*}(y, dz),$$

then $\big((h(t, X_t)_{t\in\mathbf{T}_s}, (\mathscr{B}_t)_{t\in\mathbf{T}_s}\big)$ is a martingale under $P^{s,x}$. One can use this martingale to change the measure in the spirit of the h-transform discussed in the beginning of this chapter. Under mild conditions on h we will observe that the law of X under the new measure is identical to the one of the Markov bridge starting at (s, ε_x) and ending at (T^*, μ).

4.1.1 Weak Conditioning

This section is devoted to Markov bridges obtained from a weak conditioning. Following the discussion above we will construct such Markov bridges using h-functions.

Definition 4.1 We call a function $h : [0, T^*] \times \mathbf{E} \mapsto [0, \infty)$, an *h-function* if it is strictly positive on $[0, T^*) \times \mathbf{E}$, belongs to $\mathbb{C}^{1,2}([0, T^*) \times \mathbf{E})$, and $\big((h(t, X_t)_{t\geq 0}, (\mathscr{B}_t)_{t\geq 0}\big)$ is a martingale under every $P^{0,x}$.

Example 4.1 Let $T^* < \infty$ and suppose that $E \in \mathscr{E}$ is a set such that $(t, x) \mapsto P_{t,T^*}(x, E)$ belongs to $\mathbb{C}^{1,2}([0, T^*) \times \mathbf{E})$ and $P_{t,T^*}(x, E) > 0$ for all $t < T^*$ and $x \in \mathbf{E}$. Define h by $h(T^*, x) = \mathbf{1}_E$ and $h(t, x) = P_{t,T^*}(x, E)$. Clearly, $((h(t, X_t)_{t\in\mathbf{T}}, (\mathscr{B}_t)_{t\in\mathbf{T}})$ is a bounded martingale under every $P^{0,x}$. Moreover, $h \in \mathbb{C}^{1,2}([0, T^*) \times \mathbf{E})$ by assumption. Thus, h is an h-function that corresponds to the conditioning of the original process to be in the set E at time T^*.

Since h-functions lead to strictly positive martingales, one can use them to change the measure. The advantage of such measure changes is that it preserves the Markov property. The proof of this fact will be based on the following lemma.

Lemma 4.1 *Let $f \in \mathbb{C}^{1,2}([0, T^*] \times \mathbf{E})$. Then, $\Big((M^f_t)_{t\in\mathbf{T}_s}, (\mathscr{B}_t)_{t\in\mathbf{T}_s}\Big)$ is a local martingale for any solution $P^{s,x}$ of the local martingale problem for A, where*

$$M^f_t = f(t, X_t) - f(s, X_s) - \int_s^t \left\{\frac{\partial}{\partial u} f(u, X_u) + A_u f(u, X_u)\right\} du.$$

Proof We follow the lines of the proof of Theorem 4.2.1 (ii) in [109]. Consider $f \in \mathbb{C}^\infty_K([0, T^*] \times \mathbf{E})$ and $s \leq t_1 < t_2$. By stopping we may assume $(f(t_1, X_t) - f(t_1, X_s) - \int_s^t A_u f(t_1, X_u) du)$ is a martingale. Then,

$$\begin{aligned}
&E^{s,x}\left[f(t_2, X_{t_2}) - f(t_1, X_{t_1})\middle|\mathscr{B}_{t_1}\right]\\
&= E^{s,x}\left[f(t_2, X_{t_2}) - f(t_1, X_{t_2})\middle|\mathscr{B}_{t_1}\right] + E^{s,x}\left[f(t_1, X_{t_2}) - f(t_1, X_{t_1})\middle|\mathscr{B}_{t_1}\right]\\
&= E^{s,x}\left[\int_{t_1}^{t_2} \frac{\partial}{\partial u} f(u, X_{t_2})\, du\middle|\mathscr{B}_{t_1}\right] + E^{s,x}\left[\int_{t_1}^{t_2} A_u f(t_1, X_u)\, du\middle|\mathscr{B}_{t_1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= E^{s,x}\left[\int_{t_1}^{t_2}\left\{\frac{\partial}{\partial u}f(u,X_u)+\ A_u f(u,X_u)\right\}du\middle|\mathscr{B}_{t_1}\right]\\
&\quad +E^{s,x}\left[\int_{t_1}^{t_2}\left\{\frac{\partial}{\partial u}f(u,X_{t_2})-\frac{\partial}{\partial u}f(u,X_u)\right\}du\middle|\mathscr{B}_{t_1}\right]\\
&\quad +E^{s,x}\left[\int_{t_1}^{t_2}\{A_u f(t_1,X_u)-A_u f(u,X_u)\}\,du\middle|\mathscr{B}_{t_1}\right]\\
&= E^{s,x}\left[\int_{t_1}^{t_2}\left\{\frac{\partial}{\partial u}f(u,X_u)+\ A_u f(u,X_u)\right\}du\middle|\mathscr{B}_{t_1}\right]\\
&\quad +E^{s,x}\left[\int_{t_1}^{t_2}du\int_u^{t_2}\left(A_v\frac{\partial f}{\partial u}\right)(u,X_v)dv\middle|\mathscr{B}_{t_1}\right]\\
&\quad -E^{s,x}\left[\int_{t_1}^{t_2}dv\int_{t_1}^{v}\left(\frac{\partial}{\partial u}A_v f\right)(u,X_v)du\middle|\mathscr{B}_{t_1}\right]\\
&= E^{s,x}\left[\int_{t_1}^{t_2}\left\{\frac{\partial}{\partial u}f(u,X_u)+A_u f(u,X_u)\right\}du\middle|\mathscr{B}_{t_1}\right].
\end{aligned}
$$

The last two terms cancel since $A_v\frac{\partial f}{\partial u}=\frac{\partial}{\partial u}A_v f$ and the integrals are two double integrals of the same function over the same region, namely, $t_1\le u\le v\le t_2$. This shows that M^f is martingale. By a standard density argument we can now deduce M^f is a local martingale whenever f satisfies the hypothesis of the lemma. □

Theorem 4.1 *Suppose that the conditions of Assumption 4.1 hold and $T^*<\infty$. Let h be an h-function such that $h(T^*,\cdot)>0$ and $h\in\mathbb{C}^{1,2}([0,T^*]\times\mathbf{E})$. Define $P^{h;s,x}$ on $(\Omega,\mathscr{B}_{T^*})$ by $\frac{dP^{h;s,x}}{dP^{s,x}}=\frac{h(T^*,X_{T^*})}{h(s,x)}$. Then, $P^{h;s,x}$ is the unique solution of the local martingale problem for A^h starting from x at s, where*

$$A_t^h=A_t+\sum_{i.j=1}^{d}a_{ij}(t,x)\frac{\frac{\partial h}{\partial x_j}(t,x)}{h(t,x)}\frac{\partial}{\partial x_i}.$$

Consequently, X is a strong Markov process under every $P^{h;s,x}$ for s, T^* and $x\in\mathbf{E}$ and the associated transition function $(P^h_{s,t})$ is related to $(P_{s,t})$ via

$$P^h_{s,t}(x,A)=\frac{1}{h(s,x)}\int_E h(t,y)P_{s,t}(x,dy),\qquad x\in\mathbf{E},E\in\mathscr{E},t\in\mathbf{T}_s.\tag{4.4}$$

Proof Consider an $f\in\mathbb{C}^\infty(\mathbf{E})$ and let

$$M_t^f(h)=f(X_t)-f(X_s)-\int_s^t A_r^h f(X_r)\,dr.$$

Observe that

$$M_t^f(h) - M_t^f = -\sum_{i,j=1}^d \int_s^t a_{ij}(v, X_v) \frac{\frac{\partial h}{\partial x_j}(v, X_v)}{h(v, X_v)} \frac{\partial f}{\partial x_i}(X_v)\, dv. \tag{4.5}$$

Thus, if we let $\tau_n = T^* \wedge \inf\{t \geq s : |M_t^f(h)| \geq n\} \wedge \inf\{t \geq s : \|X_t\| \geq n\}$, then τ_n is a stopping time and $\tau_n \to T^*$, $P^{h;s,x}$-a.s. as $n \to \infty$ since M^f and X are continuous under $P^{s,x}$, and $P^{s,x} \sim P^{h;s,x}$. In particular, (4.5) also entails $(M_{t\wedge\tau_n}^f)$ is a bounded martingale under $P^{s,x}$. We will now see that $(M_{t\wedge\tau_n}^f(h))$ is a martingale under $P^{h;s,x}$. Indeed, for any $u \in [s, t]$ and $E \in \mathscr{B}_u$,

$$\begin{aligned}
&h(s, x)E^{h;s,x}\left[\left(M_{t\wedge\tau_n}^f(h) - M_u^f(h)\right)\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\right]\\
&\quad= E^{s,x}\left[h(t, X_t)\left(M_{t\wedge\tau_n}^f(h) - M_u^f(h)\right)\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\right]\\
&\quad= E^{s,x}\left[h(t, X_t)\left(M_{t\wedge\tau_n}^f - M_u^f\right)\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\right]\\
&\qquad -E^{s,x}\left[\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E \sum_{i,j=1}^d \int_u^{t\wedge\tau_n} h(t, X_t)a_{ij}(v, X_v)\frac{\frac{\partial h}{\partial x_j}(v, X_v)}{h(v, X_v)}\frac{\partial f}{\partial x_i}(X_v)\, dv\right]\\
&= E^{s,x}\left[h(t, X_t)\left(f(X_{t\wedge\tau_n}) - f(u, X_u)\right)\mathbf{1}_{[\tau>u]}\mathbf{1}_E\right]\\
&\quad -E^{s,x}\left[\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E h(t, X_t)\int_u^{t\wedge\tau_n} A_v f(X_v)\, dv\right]\\
&\quad -E^{s,x}\left[\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E \sum_{i.j=1}^d \int_u^{t\wedge\tau_n} a_{ij}(v, X_v)\frac{\partial h}{\partial x_j}(v, X_v)\frac{\partial f}{\partial x_i}(X_v)\, dv\right],
\end{aligned}$$

where the last equality follows from the martingale property of $h(t, X_t)$ under $P^{s,x}$. Letting $g = hf$, observing

$$\begin{aligned}
&E^{s,x}\left[h(t, X_t)\left(f(X_{t\wedge\tau_n}) - f(u, X_u)\right)\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\right]\\
&\quad= E^{s,x}\left[\left(g(t\wedge\tau_n, X_{t\wedge\tau_n}) - g(u, X_u)\right)\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\right]
\end{aligned}$$

by the same martingale argument, and utilising Lemma 4.1, we have

$$\begin{aligned}
&h(s, x)E^{h;s,x}\left[\left(M_{t\wedge\tau_n}^f(h) - M_u^f(h)\right)\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\right]\\
&\quad= E^{s,x}\Bigg[\Bigg(M_{t\wedge\tau_n}^g - M_u^g\\
&\qquad + \int_u^{t\wedge\tau_n}\left\{f(X_v)\frac{\partial h}{\partial u}(v, X_v) + A_v g(v, X_v)\right\} dv\Bigg)\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&-E^{s,x}\left[\int_u^{t\wedge\tau_n} h(t,X_t)A_v f(X_v)\,dv\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\right]\\
&-E^{s,x}\left[\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\sum_{i,j=1}^d\int_u^{t\wedge\tau_n} a_{ij}(u,X_u)\frac{\partial h}{\partial x_j}(v,X_v)\frac{\partial f}{\partial x_i}(X_v)\,dv\right]\\
=\ & E^{s,x}\left[\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\int_u^{t\wedge\tau_n}\left\{f(X_v)\frac{\partial h}{\partial u}(v,X_v)+A_v g(v,X_v)\right\}dv\right]\\
&-E^{s,x}\left[\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\int_u^{t\wedge\tau_n} h(v,X_v)A_v f(X_v)\,dv\right]\\
&-E^{s,x}\left[\mathbf{1}_{[\tau_n>u]}\mathbf{1}_E\sum_{i,j=1}^d\int_u^{t\wedge\tau_n} a_{ij}(u,X_u)\frac{\partial h}{\partial x_j}(v,X_v)\frac{\partial f}{\partial x_i}(X_v)\,dv\right]=0,
\end{aligned}
$$

where the second equality holds since $(M^g_{t\wedge\tau_n})$ is a bounded martingale due to the boundedness of $(M^f_{t\wedge\tau_n)}$ and the smoothness of h, and the last equality follows from the identity $\frac{\partial h}{\partial u}(v,x)+A_v h(v,x)=0$. As $\tau_n\to T^*$, $P^{h;s,x}$-a.s., we conclude that $P^{h;s,x}$ solves the local martingale problem. The uniqueness follows easily due to the one-to-one relationship between $P^{h;s,x}$ and $P^{s,x}$ since the local martingale problem for A is well posed.

The strong Markov property is a direct consequence of the well posedness of the local martingale problem for A^h via Theorem 2.3. The form of the transition function follows directly from the explicit absolute continuity relationship between the measures $P^{h;s,x}$ and $P^{s,x}$. □

The last theorem gives us a conditioning on the path space when h is strictly positive on $[0,T^*]\times\mathbf{E}$. The coordinate process after this conditioning is often referred to as the *h-path process* in the literature. Note that the h-function of Example 4.1 does not satisfy the conditions of the above theorem as $h(T^*,\cdot)$ is not strictly positive. This implies that we cannot use this theorem to condition the coordinate process to end up in a given set. However, $h(t,\cdot)$ is strictly positive and smooth for any $t<T^*$, which allows us to extend the results of the previous theorem to the case when $h(T^*,\cdot)$ does not satisfy the conditions.

For making this extension possible we first introduce a new canonical space $C([0,T^*),\mathbf{E})$ and $\mathscr{B}^-_t=\sigma(X_s;s\le t)$, $\mathscr{B}^-_{T^*}=\vee_{t<T^*}\mathscr{B}^-_t$, where X is the coordinate process on $C([0,T^*),\mathbf{E})$. The main difference between $\mathscr{B}^-_t$ and $\mathscr{B}_t$ is the measurable space on which they are defined. While the former is defined on the space of functions that are continuous on $[0,T^*)$, the latter is defined on the paths that are continuous on $[0,T^*]$. In particular, the functions that are divergent as $t\to T^*$ belong to the former but not the latter, which in turn implies $\mathscr{B}^-_{T^*}$ has more elements than $\mathscr{B}_{T^*}$. On the other hand, one can easily verify that there is a one-to-one correspondence between the members of $\mathscr{B}^-_t$ and those of $\mathscr{B}_t$ for $t<T^*$. In view of these observations the following fact can be established as a special case of Theorem 1.3.5 in [109].

Theorem 4.2 *Let (t_n) be an increasing sequence of deterministic times with $t_n < T^*$ for each n and suppose that for each n there exists a probability measure P^n on $(C([0, T^*), \mathbf{E}), \mathscr{B}^-_{t_n})$. If P^{n+1} agrees with P^n on $\mathscr{B}^-_{t_n}$ and $\lim_{n\to\infty} t_n = T^*$, then there exists a unique probability measure, P, on $(C([0, T^*), \mathbf{E}), \mathscr{B}^-_{T^*})$ that agrees with P^n on $\mathscr{B}^-_{t_n}$ for all $n \geq 0$.*

Corollary 4.1 *Suppose that the conditions of Assumption 4.1 hold and let h be an h-function. For any $s < T^*$ and $x \in \mathbf{E}$, there exists a unique probability measure $P^{h;s,x}$ on $(\Omega, \mathscr{B}_{T^*})$ which solves the local martingale problem for A^h, where*

$$A^h_t = A_t + \sum_{i.j=1}^{d} a_{ij}(t,x)\frac{\frac{\partial h}{\partial x_j}(t,x)}{h(t,x)}\frac{\partial}{\partial x_i}$$

on $[0, T] \times \mathbf{E}$ starting from x at s for any $T < T^*$. Consequently, X is a strong Markov process under every $P^{h;s,x}$ for $s < T^*$, $x \in \mathbf{E}$ and the associated transition function $(P^h_{s,t})$ is related to $(P_{s,t})$ via

$$P^h_{s,t}(x, E) = \frac{1}{h(s,x)} \int_E h(t,y) P_{s,t}(x, dy), \qquad x \in \mathbf{E},\ E \in \mathscr{E},\ t \in \mathbf{T}_s. \tag{4.6}$$

Proof Let $\hat{P}^{s,x}$ be the law of $(X_t)_{t\in[s,T^*)}$ under $P^{s,x}$, where X is the coordinate process of $C([0, T^*], \mathbf{E})$. $\hat{P}^{s,x}$ is a probability measure on $(C([0, T^*), \mathbf{E}), \mathscr{B}^-_{T^*})$ such that the corresponding coordinate process admits limits as $t \uparrow T^*$ with probability 1.

Using the function h as a measure change we can again obtain a sequence of measures $(Q^T)_{T<T^*}$ with the property that Q^T and Q^{t+T} agree on $\mathscr{B}^-_T$ for all $0 \leq t < T^* - T$. Theorem 4.2 now yields a probability measure, Q, on $(C([0, T^*), \mathbf{E}), \mathscr{B}^-_{T^*})$ that agrees with Q^T on $\mathscr{B}^-_T$ for $T < T^*$.

We will use this Q to construct the $P^{h;s,x}$ on $(C([0, T^*], \mathbf{E}), \mathscr{B}_{T^*})$. However, in order to do so, we need to establish that under Q the coordinate process, X, of $C([0, T^*), \mathbf{E})$ admits a limit as $t \uparrow T^*$. We will achieve this by showing that Q is absolutely continuous with respect to $\hat{P}^{s,x}$ on $\mathscr{B}^-_{T^*}$. Indeed, for any $E \in \mathscr{B}^-_t$ for some $t < T^*$, we have

$$Q(E) = \hat{E}^{s,x}\left[\frac{h(t, X_t)}{h(s,x)}\mathbf{1}_E\right] = \hat{E}^{s,x}\left[L_{T^*}\mathbf{1}_E\right], \tag{4.7}$$

where $0 \leq L_{T^*} = \lim_{t\to T^*} \frac{h(t,X_t)}{h(s,x)} = \frac{h(T^*,X_{T^*-})}{h(s,x)}$. The existence of this limit and the exchange of expectation and limit are justified since $(h(t, X_t))_{t\in[0,T^*)}$ is a positive uniformly integrable $\hat{P}^{s,x}$-martingale. Also note that $\hat{E}^{s,x}[L_{T^*}] = 1$.

Define

$$\lambda = \left\{E \in \mathscr{B}^-_{T^*} : Q(E) = \hat{E}^{s,x}\left[L_{T^*}\mathbf{1}_E\right]\right\}.$$

Clearly, λ satisfies the conditions of Dynkin's $\pi - \lambda$ Theorem A.1. Moreover,

$$\pi = \{E : E \in \mathscr{B}_t^- \text{ for some } t < T^*\} \subset \lambda$$

is closed under intersection. Thus, by Dynkin's $\pi - \lambda$ Theorem the equality (4.7) holds for all $E \in \mathscr{B}_{T^*}^-$ implying the claimed absolute continuity.

We now claim that the sequence (X_{t_n}) with $t_n \uparrow T^*$ is Q-a.s. Cauchy. Observe that on the set

$$E = \cap_{m\geq 1} \cup_{n\geq 1} \cap_{k\geq n} \left\{\omega : \|X_{t_k}(\omega) - X_{t_{k-1}}(\omega)\| < \frac{1}{m}\right\} \in \mathscr{B}_{T^*}^-,$$

the sequence is Cauchy. Moreover, $\hat{P}^{s,x}(E) = 1$ which together with (4.7) implies $Q(E) = 1$. Thus, $\lim_{t\to T^*} X_t$ exists, Q-a.s.

Next, define X^h by $X_t^h = X_t$ for $t < T^*$, and $X_{T^*}^h = \lim_{t\to T^*} X_t$. If we denote by $P^{h;s,x}$ the law of X^h, then it is easily seen that it is a probability measure on $(C([0, T^*], \mathbf{E}), \mathscr{B}_{T^*})$. Thus, we have shown in view of (4.7) that for any $E \in \mathscr{B}_t$

$$P^{h;s,x}(E) = E^{s,x}\left[\frac{h(t, X_t)}{h(s, x)}\mathbf{1}_E\right] \tag{4.8}$$

for all $s \leq t \leq T^*$.

To show the uniqueness assume that there exists another measure $\tilde{P}^{h;s,x}$ on $(C([0, T^*], \mathbf{E}), \mathscr{B}_{T^*})$ which solves the local martingale problem for A^h. Then, the restriction of this measure to $\mathscr{B}_T$ for $T < T^*$ solves the local martingale problem for A^h when solutions have sample paths in $C([0, T], \mathbf{E})$. However, this local martingale problem is well posed due to the one-to-one correspondence with the martingale problem for A when solutions have sample paths in $C([0, T], \mathbf{E})$ via Girsanov transform since $h(T, \cdot) > 0$. Thus, by Theorem 4.2 $\tilde{P}^{h;s,x}$ and $P^{h;s,x}$ agree on $(C([0, T^*], \mathbf{E}), \mathscr{B}_{T^*})$. This proves the well posedness of the local martingale problem for A^h and, therefore, via Theorem 2.3, the strong Markov property holds for X under $P^{h;s,x}$.

Finally, the representation for the transition function follows from (4.8). □

In view of the relationship between the solutions of the local martingale problem and the weak solutions of SDEs (Corollary 2.3) yields the following.

Corollary 4.2 *Let σ be a matrix field such that $a = \sigma\sigma^*$. Under the conditions of Corollary 4.1, there exists a unique weak solution to the following SDE with $P(X_{T^*} \in E) = \int_E \frac{h(T^*,y)}{h(s,x)} P_{s,T^*}(x, dy)$:*

$$X_t = x + \int_s^t \left\{b(X_u) + a(X_u)\frac{(\nabla h(u, X_u))^*}{h(u, X_u)}\right\} du + \int_s^t \sigma(X_u)\, dB_u.$$

Example 4.2 If we apply the above corollary to the h-function of Example 4.1 we end up with a conditioning on the coordinate process on the space of continuous paths. In view of (4.6) one can immediately see that $P^{h;s,x}(X_{T^*} \in E) = 1$.

4.1.2 Strong Conditioning

In this section we consider the problem of strong conditioning of a Markov process. Intuitively, one can see that such conditioning can be done via '$h(t,x) = P^{t,x}(X_T^* = z)$'. As $[X_T^* = z]$ is most likely a null set, the definition in the quotation marks shouldn't be taken too literally. However, it guides us how to proceed. Suppose that P_{t,T^*} admits a density $p(t,x;T^*,z)$ such that $(t,x) \mapsto p(t,x;T^*,z)$ is in $\mathbb{C}^{1,2}([0,T^*) \times \mathbf{E})$. Thus, if it is strictly positive it can be used as an h-function. The problem is that this function explodes at $t = T^*$, which is in fact the very reason why this conditioning works, so we cannot directly use Corollary 4.1. However, it can be applied locally, i.e. until times away from T^*, to produce a family of measures on the canonical space. If, additionally, the family of solutions to the local martingale problem is weakly continuous, e.g. Feller, it is possible to demonstrate that this family converges weakly to a probability measure on the canonical space yielding the bridge condition.

We will obtain the stochastic differential equation for the bridge process under two different sets of conditions. The first set of conditions will be handy when one can obtain bounds on the transition density of the process associated with the solution of the local martingale problem over the interval $[0,T^*]$, e.g. via Gaussian type estimates on the fundamental solution of the parabolic PDE $u_t = Au$.

The second set of assumptions can be seen as a relaxation of the first one in the case of a time-homogeneous local martingale problem whose solution has sample paths in $C([0,\infty),\mathbf{E})$. The proof in the latter case relies on a certain boundedness property of the potential density of X, which is generally satisfied in the one-dimensional case.

Assumption 4.2 *Suppose that Assumption 4.1 is satisfied, the family of solutions of the local martingale problem, P^x, is weakly continuous and $(P_{s,t})$ is a semi-group admitting a regular transition density $p(s,x;t,y)$ with respect to a σ-finite measure m_t on $(\mathbf{E},\mathscr{E})$, i.e. for any bounded and measurable f*

$$P_{s,t}f(x) = \int_{\mathbf{E}} f(y)p(s,x;t,y)m_t(dy), \qquad 0 \le s < t \le T^*.$$

Moreover, the transition density satisfies

$$\lim_{t\to u}\int_{B_r^c(z)} p(t,y;u,z)p(0,x;t,y)m_t(dy) = 0,\ \forall u > 0,\ r > 0, \tag{4.9}$$

the Chapman–Kolmogorov equations

$$p(s,x;u,y)=\int_{\mathbf{E}} p(s,x;t,z)p(t,z;u,y)m_t(dz),\ 0\le s<t\le T^*, \tag{4.10}$$

hold, and

$$\sup_{\substack{x\notin B_r(z)\\ t<T^*}} p(t,x;T^*,z)<\infty, \tag{4.11}$$

for every $z\in\mathbf{E}$ *and* $r>0$.

The boundedness assumption on the transition density as given in (4.11) is satisfied in many practical applications. In particular, if the coefficients b_i and a_{ij} are bounded, Hölder continuous and the matrix a is uniformly positive definite, then m_t becomes the Lebesgue measure for all t and $p(s,x;t,y)$ becomes the fundamental solution of the parabolic PDE, $u_t=Au$, and satisfies for some $k>0$

$$p(s,x;t,y)\le(t-s)^{-\frac{d}{2}}\exp\left(-k\frac{\|x-y\|^2}{t-s}\right),\qquad 0\le s<t\le T^*, \tag{4.12}$$

yielding the desired boundedness (see Theorem 11 in Chap. 1 of [57]). Also observe that this estimate implies (4.9), too. Moreover, $P_{s,t}f$ is a continuous function vanishing at infinity whenever f is continuous and vanishes at infinity, i.e. X is Feller.

Assumption 4.3 *Suppose that* a *and* b *do not depend on time,* $\mathbf{T}=[0,\infty)$, *Assumption 4.1 is satisfied, and the family of solutions of the local martingale problem,* P^x, *is weakly continuous. Moreover,* (P_t) *is a semi-group admitting a regular transition density* $p(t,x,y)$ *with respect to a* σ*-finite measure* m *on* $(\mathbf{E},\mathscr{E})$ *such that*

$$\lim_{t\to 0}\int_{B_r^c(z)} p(t,y,z)p(u-t,x,y)m(dy)=0,\ \forall u>0,\ r>0, \tag{4.13}$$

and the Chapman–Kolmogorov equations,

$$p(t,x,y)=\int_{\mathbf{E}} p(t-s,x,u)p(s,u,y)m(du),\ 0<s<t, \tag{4.14}$$

hold.

Furthermore, the α*-potential density,* u^α, *defined by*

$$u^\alpha(x,y):=\int_0^\infty e^{-\alpha t}p(t,x,y)dt,\qquad (x,y)\in\mathbf{E}\times\mathbf{E},$$

satisfies

$$\sup_{\alpha>0, x\in K} \alpha u^{\alpha}(x,y) < \infty \tag{4.15}$$

for any $y \in \mathbf{E}$ *and a compact set* $K \subset \mathbf{E}$ *such that* $y \notin K$.

Remark 4.1 Observe that u^{α} is the kernel of the α-potential operator, U^{α} from Definition 1.5. Indeed, a direct application of Fubini's theorem yields that for any $f \in C_b(\mathbf{E})$

$$U^{\alpha} f(x) = \int_0^{\infty} e^{-\alpha t} P_t f(x) dt = \int_{\mathbf{E}} u^{\alpha}(x,y) f(y) m(dy).$$

The condition (4.13) is satisfied when $p(s,x,y)$ is continuous on $(0,\infty)\times\mathbf{E}\times\mathbf{E}$ and there exists a right-continuous process, $\tilde{X}$ such that X and $\tilde{X}$ are in duality with respect to m. This would be the case if X were a strongly symmetric Borel right process (see Remark 3.3.5 in [91]) with continuous transition densities, in particular a one-dimensional regular diffusion without an absorbing boundary. Moreover, if X is a Feller process the laws (P^x) will be weakly continuous, too.

Assumption 4.3 is satisfied for a large class of one-dimensional regular diffusions on $[l,\infty)$ when l and ∞ are inaccessible boundaries. More precisely, suppose

$$A = \frac{1}{2} a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$$

with b continuous and a strictly positive and continuous on (l,∞). Recall that the scale function of this diffusion is given by

$$s(x) = \int_c^x \exp\left(-\int_c^y 2\frac{b(z)}{a(z)} dz\right) dy,$$

and the speed measure, $m(x)dx$, is determined by the function

$$m(x) = \frac{2}{s'(x)a(x)}.$$

Recall that l and ∞ are inaccessible if and only if

$$\int_c^{\infty} m((c,x)) s'(x) dx = \int_l^c m((x,c)) s'(x) dx = \infty. \tag{4.16}$$

This yields in view of Theorem 2.15 that there exists a unique weak solution to

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt,$$

and consequently the local martingale problem for A is well posed by Corollary 2.3. We will in fact require more and assume that the infinite boundaries are natural. This ensures that (P^x) is Feller (see Theorem 8.1.1 in [50]), and therefore weakly continuous.

Moreover, Theorem 2.17 gives that the semi-group P_t possesses a density $p(t, x, y)$ satisfying (4.14) with respect to the speed measure. Since $p(t, x, y) = p(t, y, x)$ for all $t > 0$ and $(x, y) \in (l, \infty)^2$ and is continuous on $(0, \infty) \times (l, \infty) \times (l, \infty)$, we obtain (4.13) in view of the continuity of X.

Also recall that $u^\alpha(x, y)$ is symmetric for each $\alpha > 0$ and is continuous on $(l, \infty) \times \mathbf{E}$ due to its construction and boundary behaviour (see (2.3) and Table 1 in [92]).

It remains to show (4.15). If $y \in \mathbf{E}$ is a natural boundary, $u^\alpha(x, y) = 0$ for all $\alpha > 0$ and $x \in \mathbf{E}\setminus\{y\}$. So, assume y is not a natural boundary.

First, note that for any continuous and bounded f,

$$\alpha \int_{\mathbf{E}} u^\alpha(x, z) f(z) m(dz) = \alpha \int_0^\infty e^{-\alpha t} E^x f(X_t) dt \to f(x), \tag{4.17}$$

as $\alpha \to \infty$ by the continuity of X.

Next, let K_1 and K_2 be two disjoint closed and bounded intervals contained in (l, ∞). It follows (see, e.g. Theorem 3.6.5 in [91]) from the strong Markov property of X that for $x \in K_1$ and $z \in K_2$ that

$$u^\alpha(x, z) = E^x \left[e^{-\alpha \tau_2}\right] u^\alpha(X_{\tau_2}, z),$$

where $\tau_2 = \inf\{t > 0 : X_t \in K_2\}$ and X_{τ_2} is deterministic and equals either the left or the right endpoint of K_2 depending on whether $x < z$ or not. Note that X_{τ_2} has the same value for all $x \in K_1$, and consequently

$$u^\alpha(x, z) \le u^\alpha(X_{\tau_2}, z), \forall x \in K_1.$$

Let K_3 be an arbitrary closed interval strictly contained in K_2. For any f with a support in K_3, we have

$$\sup_{x \in K_1} \int_0^\infty \alpha u^\alpha(x, z) f(z) m(dz) \le \int_0^\infty \alpha u^\alpha(X_{\tau_2}, z) f(z) m(dz).$$

This implies in view of Fatou's lemma and (4.17) that whenever $x_n \to x \in K_1$ and $\alpha_n \to \infty$, we have

$$\liminf_{n \to \infty} \alpha_n u^{\alpha_n}(x_n, z) = 0, \tag{4.18}$$

for m-a.a. z in K_2. Since m is absolutely continuous with respect to the Lebesgue measure in (l, ∞), we have that the above holds for a.a. z in K_2.

Now, suppose K is a compact set that does not contain y and there exists a convergent sequence (x_n) in K and a sequence of positive real numbers (α_n) such that $\lim_{n\to\infty}\alpha_n = \infty$ as well as

$$\lim_{n\to\infty} \alpha_n u^{\alpha_n}(x_n, y) = \infty. \tag{4.19}$$

In view of (4.18), there exists a $z \notin K \cup \{y\}$ such that $P^y(T_z < T_x) = 1$ for all $x \in K$, and

$$\liminf_{n\to\infty} \alpha_n u^{\alpha_n}(x_n, z) = 0. \tag{4.20}$$

On the other hand, using Theorem 3.6.5 in [91] once more and the fact that u^α is symmetric, we may write

$$\limsup_{n\to\infty} \frac{E^y\left[e^{-\alpha_n T_{x_n}}\right]}{E^z\left[e^{-\alpha_n T_{x_n}}\right]} = \limsup_{n\to\infty} \frac{u^{\alpha_n}(x_n, y)}{u^{\alpha_n}(x_n, z)} = \infty,$$

where the value of the limit follows from (4.19) and (4.20). However,

$$\frac{E^y\left[e^{-\alpha_n T_{x_n}}\right]}{E^z\left[e^{-\alpha_n T_{x_n}}\right]} = \frac{E^y\left[e^{-\alpha_n T_z} E^z\left[e^{-\alpha_n T_{x_n}}\right]\right]}{E^z\left[e^{-\alpha_n T_{x_n}}\right]} = E^y\left[e^{-\alpha_n T_z}\right] < 1$$

by the strong Markov property of X since $P^y(T_z < T_{x_n}) = 1$.

Theorem 4.3 *Let σ be a matrix field such that $\sigma\sigma^* = a$. Suppose that Assumption 4.2 is in force and fix $x \in \mathbf{E}$ and $z \in \mathbf{E}$ such that the following conditions hold:*

1. $m_{T^*}(\{z\}) = 0$.
2. $p(0, x; T^*, z) > 0$.
3. *For $h(t, y) = p(t, y; T^*, z)$ either $h \in \mathbb{C}^{1,2}([0, T^*) \times \mathbf{E})$ or $h \in \mathbb{C}^{1,2}([0, T^*) \times int(\mathbf{E}))$, $x \in int(\mathbf{E})$, and $P_{t,T^*}(x, int(\mathbf{E})) = 1$ for all $t < T^*$.*

Then there exists a weak solution on $[0, T^*]$ to

$$X_t = x + \int_0^t \left\{ b(u, X_u) + a(u, X_u) \frac{(\nabla h(u, X_u))^*}{h(u, X_u)} \right\} du + \int_0^t \sigma(u, X_u)\, dB_u, \tag{4.21}$$

the law of which, $P^{x\to z}_{0\to T^*}$, satisfies $P^{x\to z}_{0\to T^*}(\inf_{u\in[0,T]} h(u, X_u) = 0) = 0$ for any $T < T^*$, and $P^{x\to z}_{0\to T^*}(X_{T^*} = z) = 1$.

In addition, if $h(t, \cdot) > 0$ for all $t < T^*$, weak uniqueness holds for the above SDE. Moreover, X is a Markov process with transition function $(P^h_{s,t})$ defined by

$$P^h_{s,t}(x, E) = \int_E \frac{p(s, x; t, y) p(t, y; T^*, z)}{p(s, x; T^*, z)} m(dy),\ s < t < T^*, x \in \mathbf{E}, E \in \mathscr{E}.$$

Theorem 4.4 *Let σ be a matrix field such that $\sigma\sigma^* = a$. Suppose that Assumption 4.3 is in force and fix $x \in \mathbf{E}$ and $z \in \mathbf{E}$ such that the following conditions hold:*

1. $m(\{z\}) = 0$.
2. $p(T^*, x, z) > 0$.
3. *For $h(t, y) = p(T^* - t, y, z)$ either $h \in \mathbb{C}^{1,2}([0, T^*) \times \mathbf{E})$ or $h \in \mathbb{C}^{1,2}([0, T^*) \times int(\mathbf{E}))$, $x \in int(\mathbf{E})$, and $P_t(x, int(\mathbf{E})) = 1$ for all $t \leq T^*$.*
4. *$u^\alpha(x, z) < \infty$ for $\alpha > 0$,*
5. *either the map $t \mapsto p(t, x, y)$ is continuous on $(0, \infty)$ for every $y \in \mathbf{E}$, or for all $t > 0$ $p(t, x, y) > 0$, m-a.e. y.*

Then there exists a weak solution on $[0, T^*]$ to

$$X_t = x + \int_0^t \left\{ b(X_u) + a(X_u) \frac{(\nabla h(u, X_u))^*}{h(u, X_u)} \right\} du + \int_0^t \sigma(X_u)\, dB_u, \tag{4.22}$$

the law of which, $P^{x \to z}_{0 \to T^*}$, satisfies $P^{x \to z}_{0 \to T^*}(\inf_{u \in [0,T]} h(u, X_u) = 0) = 0$ for any $T < T^*$, and $P^{x \to z}_{0 \to T^*}(X_{T^*} = z) = 1$.

In addition, if $h(t, \cdot) > 0$ for all $t < T^*$, weak uniqueness holds for the above SDE. Moreover, X is a Markov process with transition function $(P^h_{s,t})$ defined by

$$P^h_{s,t}(x, E) = \int_E \frac{p(t-s, x, y)p(T^* - t, y, z)}{p(T^* - s, x, z)} m(dy),\ s < t < T^*, x \in \mathbf{E}, E \in \mathscr{E}.$$

Remark 4.2 Note that in both Theorems 4.3 and 4.4, $P^{x \to z}_{0 \to T^*}$ is related to the law of the original process via

$$P_{s,t}(y, E) = \int_{\mathbf{E}} P^{x \to z}_{0 \to T^*}(X_t \in E | X_s = y) P_{s,T^*}(y, dz).$$

That is, we can re-construct the original process by picking a bridge of length T^* with the endpoint chosen according to the time-T^* distribution of the original process. This justifies the term 'bridge'.

Remark 4.3 Condition 1) in both theorems is in fact not necessary. Indeed, if $m_{T^*}(\{z\}) > 0$, then $P^x(X_{T^*} = z) > 0$ due to Condition 2). This implies that we are in the setting of Example 4.2 and, therefore, Corollary 4.1 is applicable. If one, however, still wants to use the weak convergence techniques employed in the proof of these theorems, one can do so without the convergence result of Lemma 4.2 since M of the lemma is bounded by $1/m(\{z\})$ and (4.33) follows from (4.32) by the Dominated Convergence Theorem. Also note that whenever Condition 1) is violated, we do not need (4.9) or (4.13) to complete the proof either. Moreover, both (4.11) and (4.15) are automatically satisfied.

As we have seen before one-dimensional regular diffusions on $[l, \infty)$ with continuous coefficients for which l is inaccessible and ∞ is natural satisfy Assumption 4.3. Such a diffusion also satisfies the conditions of Theorem 4.4. Recall from Theorem 2.17 that p has the following properties:

1. For each $t > 0$ and $(x, y) \in (l, \infty)^2$, $p(t, x, y) = p(t, y, x) > 0$.
2. For each $t > 0$ and $y \in (l, \infty)$, the maps $x \mapsto p(t, x, y)$ and $x \mapsto Ap(t, x, y)$ are continuous and bounded on (l, ∞).
3. $\frac{\partial}{\partial t} p(t, x, y) = Ap(t, x, y)$ for each $t > 0$ and $(x, y) \in (l, \infty)^2$.

The boundedness of $Ap(t, x, y)$ for fixed t and $y \in (l, \infty)$ together with the fact that m has no atom implies via Theorem VII.3.12 in [100] that for each $y \in (l, \infty)$ the s-derivative $\frac{d}{ds} p(t, x, y)$ exists. Since s is differentiable, we have $\frac{d}{dx} p(t, x, y)$ exists.

Note that if $b \equiv 0$, $s(x) = x$, and the continuity of σ and Ap imply, once again by Theorem VII.3.12 in [100], that $p(t, x, y)$ is twice continuously differentiable with respect to x.

When b is not identically 0, consider the transformation $Y_t = s(X_t)$, which yields a one-dimensional diffusion on $s(\mathbf{E})$ with no drift and inaccessible boundaries. Then, Y possesses a transition density q with respect to its speed measure $\tilde{m}$. Moreover, it can be directly verified that $q(t, x, y) = p(t, s^{-1}(x), s^{-1}(y))$. By the previous discussion q is twice continuously differentiable with respect to x. Since $p(t, x, y) = q(t, s(x), s(y))$ and s is twice continuously differentiable, we deduce that p is twice continuously differentiable, as well. This shows that $p(\cdot, \cdot, y) \in \mathbb{C}^{1,2}((0, \infty) \times (l, \infty))$ for $y \in (l, \infty)$.

If l is finite but an entrance boundary, then $p(\cdot, \cdot, l) \in \mathbb{C}^{1,2}((0, \infty) \times (l, \infty))$ as well. Indeed, Chapman–Kolmogorov identity implies for $s < t$

$$p(t, y, l) = \int_l^\infty p(t - s, y, z) p(s, z, l) m(dz).$$

Since $p(s, \cdot, \cdot)$ is symmetric $\int_l^\infty p(s, z, l) m(dz) = 1$. Thus, the assertion follows from differentiating under the integral sign and the analogous properties for $p(t - s, y, z)$.

Thus, for $x \in (l, \infty)$ and $z \in (l, \infty)$ (resp. $z \in [l, \infty)$) if l is natural (resp. entrance) boundary, letting $h(t, y) = p(T^* - t, y, z)$, we see that the conditions of Theorem 4.4 are satisfied, and the Markov bridge from x to z is the unique weak solution of

$$X_t = x + \int_0^t \left\{ b(X_u) + a(X_u) \frac{p_x(T^* - u, X_u, z)}{p(T^* - u, X_u, z)} \right\} du + \int_0^t \sigma(X_u) dB_u. \tag{4.23}$$

The proofs of Theorems 4.3 and 4.4 differ only in one step. Thus, we will prove the common steps only once. As the proof of Theorem 4.3 does not depend on whether the coefficients of A are time dependent, we will demonstrate the result in the time-homogeneous case for the brevity of the exposition. For the proof we need the following lemmas.

Lemma 4.2 *Fix $x, z \in \mathbf{E}$ satisfying Conditions 1) and 3) of Theorem 4.4 as well as $t > 0$ such that $p(t, x, z) > 0$. Define $M_s = p(t-s, X_s, z)$ for $s < t$. Then, $(M_s)_{s\in[0,t)}$ is a continuous P^x-martingale. Moreover, if (4.13) is satisfied, then $M_t := \lim_{s\to t} M_s = 0$, P^x-a.s. and $(M_s)_{s\in[0,t]}$ is a P^x-supermartingale.*

Proof Observe that for any $0 \le s \le u < t$ we have

$$E^x[M_u|\mathscr{B}_s] = \int_{\mathbf{E}} p(t-u, y, z)p(u-s, X_s, y)m(dy) = p(t-s, X_s, z) = M_s,$$

and therefore M is a martingale on $[0, t)$ and Theorem A.12 yields that M_t exists, non-negative and $E^x[M_t] \le E^x[M_0]$, which in turn implies $(M_s)_{s\in[0,t]}$. Moreover, by Fatou's lemma we have

$$\begin{aligned}
E^x\left[M_t \mathbf{1}_{B^c_{\frac{1}{n}}(z)}(X_t)\right] &\le E^x\left[\liminf_{u\to t} M_u \mathbf{1}_{B^c_{\frac{1}{n+1}}(z)}(X_u)\right] \\
&\le \liminf_{u\to t} E^x\left[M_u \mathbf{1}_{B^c_{\frac{1}{n+1}}(z)}(X_u)\right] \\
&= \liminf_{u\to t} \int_{B^c_{\frac{1}{n+1}}(z)} p(t-u, y, z)p(u, x, y)m(dy) = 0,
\end{aligned}$$

where the first inequality is due to the continuity of X, and the last equality is due to (4.13). In view on non-negativity of $M_t \mathbf{1}_{B^c_{\frac{1}{n}}(z)}(X_t)$, we have

$$E^x\left[M_t \mathbf{1}_{B^c_{\frac{1}{n}}(z)}(X_t)\right] = 0.$$

Monotone convergence theorem implies

$$\begin{aligned}
E^x[M_t] = E^x\left[M_t \mathbf{1}_{[X_t \ne z]}\right] &= E^x\left[M_t \lim_{n\to\infty} \mathbf{1}_{B^c_{\frac{1}{n}}(z)}(X_t)\right] \\
&= \lim_{n\to\infty} E^x\left[M_t \mathbf{1}_{B^c_{\frac{1}{n}}(z)}(X_t)\right] = 0,
\end{aligned}$$

and since M_t is non-negative, $M_t = 0$, P^x-a.s. □

Lemma 4.3 *Consider $\varphi : (0, \infty) \times (0, \infty) \mapsto [0, \infty)$. Then*

(a) If $\varphi(t, \cdot)$ is increasing and either

(i)

$$\lim_{\alpha\to\infty} \alpha \int_0^t e^{-\alpha s}\varphi(t, s)ds = 0, \ \forall t > 0$$

or

(ii)

$$\lim_{\alpha\to\infty} \alpha \int_0^\infty \int_0^t e^{-\alpha s-\beta t}\varphi(t,s)dsdt = 0,\ \forall\beta > 0,$$

then $\varphi(t,0) := \lim_{\delta\to 0}\varphi(t,\delta) = 0$ *for almost every* $t > 0$.

(b) *If there exists a constant* K *such that* $\varphi(t,\delta) < k$ *for all* $\le \delta \le t$*, and* $\lim_{\delta\to 0}\varphi(t,\delta) = 0$*, then*

$$\lim_{\alpha\to\infty} \alpha \int_0^t e^{-\alpha s}\varphi(t,s)ds = 0,$$

Proof

(a) First, observe that, since $\varphi(t,\cdot)$ is increasing and non-negative, we have

$$0 \le \varphi(t,0) \le \varphi(t,\delta),\ \forall\delta \ge 0$$

and therefore

(i)

$$\varphi(t,0) = \lim_{\alpha\to\infty} \alpha \int_0^t e^{-\alpha s}\varphi(t,0)ds \le \lim_{\alpha\to\infty} \alpha \int_0^t e^{-\alpha s}\varphi(t,s)ds = 0.$$

(ii) Due to Fatou's Lemma and since $1 - e^{-\alpha t}$ is increasing in α

$$\begin{aligned} 0 \le \int_0^\infty e^{-\beta t}\varphi(t,0)dt &= \int_0^\infty \liminf_{\alpha\to\infty} \left(1 - e^{-\alpha t}\right) e^{-\beta t}\varphi(t,0)dt \\ &\le \lim_{\alpha\to\infty} \int_0^\infty \left(1 - e^{-\alpha t}\right) e^{-\beta t}\varphi(t,0)dt \\ &= \lim_{\alpha\to\infty} \alpha \int_0^\infty \int_0^t e^{-\alpha s-\beta t}\varphi(t,0)dsdt \\ &\le \lim_{\alpha\to\infty} \alpha \int_0^\infty \int_0^t e^{-\alpha s-\beta t}\varphi(t,s)dsdt = 0. \end{aligned}$$

Therefore, we have

$$\int_0^\infty e^{-\beta t}\varphi(t,0)dt = 0,$$

which implies that $\varphi(t,0) = 0$ for almost every $t > 0$.

(b) On the other hand, consider $\varepsilon > 0$. As $\lim_{\delta \to 0} \varphi(t, \delta) = 0$, there exists $\delta > 0$ such that $\varphi(t, s) < \varepsilon$ for all $s \leq \delta$. Then we will have

$$0 \leq \lim_{\alpha \to \infty} \alpha \int_0^t e^{-\alpha s} \varphi(t, s) ds \leq \lim_{\alpha \to \infty} \alpha \left[\int_\delta^t e^{-\alpha s} \varphi(t, s) ds + \varepsilon \int_0^\delta e^{-\alpha s} ds \right]$$
$$\leq \lim_{\alpha \to \infty} \left[K \left(e^{-\alpha \delta} - e^{-\alpha t} \right) + \varepsilon \left(1 - e^{-\alpha \delta} \right) \right] = \varepsilon.$$

The conclusion follows due to arbitrariness of ε.

□

Proof of Theorem 4.3 and Theorem 4.4 Let $h(t, x) = p(T^* - t, x, z)$ and define Q^T, on $(C([0, T^*], \mathbf{E}), \mathscr{B}_{T^*})$ by $\frac{dQ^T}{dP^x} = \frac{h(T, X_T)}{h(0, x)}$. First, we will show that (Q^T) converge weakly, as $T \to T^*$, to a probability measure, $P^{x \to z}_{0 \to T^*}$, on $(C([0, T^*], \mathbf{E}), \mathscr{B}_{T^*})$ such that $P^{x \to z}_{0 \to T^*}(X^*_T = z) = 1$. Usually this is achieved in two steps: 1) Verifying that the family of measures is tight. 2) Demonstrating that the finite-dimensional distributions of the coordinate process under Q^T converge to those under $P^{x \to z}_{0 \to T^*}$.

In view of Theorem A.3, since $Q^T(X_0 = x) = 1$ for all $T \in [0, T^*)$, the tightness will follow once we show that for any $c > 0$

$$\lim_{\delta \to 0} \limsup_{T \to T^*} Q^T(w(X, \delta, [0, T^*]) > 8c) = 0, \tag{4.24}$$

where

$$w(X, \delta, [S, T]) = \sup_{\substack{|s-t| \leq \delta \\ s,t \in [S,T]}} \|X_s - X_t\|.$$

We will first obtain some estimates on the modulus of continuity in a neighbourhood of T^*. To this end let $Z_\delta = w(X, \delta, [0, \delta])$ and observe that

$$[Z_\delta \circ \theta_{T^*-\delta} > 4c] \subset [Z_{T^*-T} \circ \theta_T > 2c] \cup [Z_{T-T^*+\delta} \circ \theta_{T^*-\delta} > 2c], \qquad \forall T > T^* - \delta. \tag{4.25}$$

To get an estimate on the probability of the left-hand side of the above, we will first consider the first set of the right-hand side.

$$\begin{aligned}
&Q^T(Z_{T^*-T} \circ \theta_T > 2c) \\
&= E^x \left[\mathbf{1}_{[Z_{T^*-T} \circ \theta_T > 2c]} \frac{p(T^* - T, X_T, z)}{p(T^*, x, z)} \right] \\
&= E^x \left[P^{X_T}(Z_{T^*-T} > 2c) \frac{p(T^* - T, X_T, z)}{p(T^*, x, z)} \mathbf{1}_{[X_T \in B_1(z)]} \right] \\
&\quad + E^x \left[P^{X_T}(Z_{T^*-T} > 2c) \frac{p(T^* - T, X_T, z)}{p(T^*, x, z)} \mathbf{1}_{[X_T \notin B_1(z)]} \right],
\end{aligned} \tag{4.26}$$

where the first equality is due to the definition of Q^T and the second is the Markov property.

Since $(p(T^* - t, X_t, z))_{t\in[0,T]}$ is a martingale, we have

$$E^x\left[P^{X_T}(Z_{T^*-T} > 2c)\frac{p(T^* - T, X_T, z)}{p(T^*, x, z)}\mathbf{1}_{[X_T\in B_1(z)]}\right] \leq \sup_{y\in B_1(z)} P^y(Z_{T^*-T} > 2c). \tag{4.27}$$

Observe that $\lim_{h\to 0} P^y(Z_h > 2c) = 0$. To see this note that the sets $[Z_h > 2c]$ are decreasing to a set in $\widetilde{\mathscr{F}}_0$ and therefore by Blumenthal's zero-one law, probability of the limiting set is either 0 or 1. If the limiting probability is 1, this implies that $P^y(Z_h > 2c) = 1$ for all h, which in turn means that in every neighbourhood of 0 there exists a time at which the value of the process is c away from its value at the origin. This contradicts the continuity of X, therefore $\lim_{h\to 0} P^y(Z_h > 2c) = 0$.

This observation allows us to conclude that for any compact subset, K, of $\mathbf{E}$

$$\lim_{h\to 0}\sup_{y\in K} P^y(Z_h > 2c) = 0. \tag{4.28}$$

Indeed, if the above fails, there exists a sequence of (y_n) and (h_n) such that $h_n \to 0$, $y_n \to y \in K$ with

$$\begin{aligned} 0 < \liminf_{n\to\infty} P^{y_n}(Z_{h_n} > 2c) &\leq \lim_{m\to\infty}\liminf_{n\to\infty} P^{y_n}(Z_{h_m} > 2c) \\ &= \lim_{m\to\infty} P^y(Z_{h_m} > 2c) = 0 \end{aligned}$$

by the weak continuity of the laws P^x, which is a contradiction. This, together with (4.27), implies that

$$\lim_{T\to T^*} E^x\left[P^{X_T}(Z_{T^*-T} > 2c)\frac{p(T^* - T, X_T, z)}{p(T^*, x, z)}\mathbf{1}_{[X_T\in B_1(z)]}\right] = 0. \tag{4.29}$$

The same limit holds for the second term of (4.26). Indeed,

$$\begin{aligned} &E^x\left[P^{X_T}(Z_{T^*-T} > 2c)\frac{p(T^* - T, X_T, z)}{p(T^*, x, z)}\mathbf{1}_{[X_T\notin B_1(z)]}\right] \\ &\leq E^x\left[\frac{p(T^* - T, X_T, z)}{p(T^*, x, z)}\mathbf{1}_{[X_T\notin B_1(z)]}\right] \\ &= \frac{1}{p(T^*, x, z)}\int_{B_1^c(z)} p(T^* - T, y, z)p(T, x, y)m(dy), \end{aligned}$$

which converges to 0 as $T \to T^*$ by (4.9) or (4.13).

Combining the above with (4.29) and (4.26) yields

$$\lim_{T\to T^*} Q^T(Z_{T^*-T}\circ\theta_T > 2c) = 0. \tag{4.30}$$

Next, we will show that $\lim_{\delta\to 0}\limsup_{T\to T^*} Q^T(Z_{T-T^*+\delta}\circ\theta_{T^*-\delta} > 2c) = 0$. Let

$$\tau^\delta := \inf\{t \geq 0 : \sup_{0\leq s\leq t} X_s - \inf_{0\leq s\leq t} X_s > 2c\} \wedge \delta \wedge T^*,$$

$$\tau_c = \inf\{t \geq 0 : X_t \notin B_{\frac{c}{2}}(X_0)\} \wedge \delta \wedge T^*,$$

where $\inf \emptyset = \infty$.

Observe that

$$[Z_{T-T^*+\delta}\circ\theta_{T^*-\delta} > 2c] = [T^*-\delta+\tau^\delta\circ\theta_{T^*-\delta} < T] \subset [T^*-\delta+\tau_c\circ\theta_{T^*-\delta} < T]. \tag{4.31}$$

Thus,

$$\begin{aligned}
&\lim_{T\to T^*} Q^T(Z_{T-T^*+\delta}\circ\theta_{T^*-\delta} > 2c)\\
&= \lim_{T\to T^*}\frac{E^x[\mathbf{1}_{[T^*-\delta+\tau^\delta\circ\theta_{T^*-\delta}<T]}p(T^*-T, X_T, z)]}{p(T^*, x, z)}\\
&= \lim_{T\to T^*}\frac{E^x[\mathbf{1}_{[T^*-\delta+\tau^\delta\circ\theta_{T^*-\delta}<T]}p(\delta-\tau^\delta\circ\theta_{T^*-\delta}, X_{T^*-\delta+\tau^\delta\circ\theta_{T^*-\delta}}, z)]}{p(T^*, x, z)}\\
&= \frac{E^x[\mathbf{1}_{[\tau^\delta\circ\theta_{T^*-\delta}<\delta]}p(\delta-\tau^\delta\circ\theta_{T^*-\delta}, X_{T^*-\delta+\tau^\delta\circ\theta_{T^*-\delta}}, z)]}{p(T^*, x, z)},
\end{aligned} \tag{4.32}$$

where the second equality follows from the Optional Stopping Theorem applied to the martingale $M = (p(T^*-t, X_t, z))_{t\in[0,T^*)}$, and the last is due to the Monotone Convergence Theorem.

Case 1 Suppose Assumption 4.2 holds. Note that the numerator in (4.32) can be rewritten as

$$\begin{aligned}
&E^x[\mathbf{1}_{[X_{T^*-\delta}\in B_{\frac{c}{4}}(z)]}\mathbf{1}_{[\tau^\delta\circ\theta_{T^*-\delta}<\delta]}M_{T^*-\delta+\tau^\delta\circ\theta_{T^*-\delta}}]\\
&\quad + E^x[\mathbf{1}_{[X_{T^*-\delta}\notin B_{\frac{c}{4}}(z)]}\mathbf{1}_{[\tau^\delta\circ\theta_{T^*-\delta}<\delta]}M_{T^*-\delta+\tau^\delta\circ\theta_{T^*-\delta}}]\\
&\quad \leq E^x[\mathbf{1}_{[X_{T^*-\delta}\in B_{\frac{c}{4}}(z)]}\mathbf{1}_{[\tau_c\circ\theta_{T^*-\delta}<\delta]}M_{T^*-\delta+\tau_c\circ\theta_{T^*-\delta}}] + E^x[\mathbf{1}_{[X_{T^*-\delta}\notin B_{\frac{c}{4}}(z)]}M_{T^*-\delta}]
\end{aligned}$$

Observe that under (4.11) we have $\mathbf{1}_{[X_{T^*-\delta}\in B_{\frac{c}{4}}(z)]}\mathbf{1}_{[\tau_c\circ\theta_{T^*-\delta}<\delta]}M_{T^*-\delta+\tau_c\circ\theta_{T^*-\delta}}$ and $\mathbf{1}_{[X_{T^*-\delta}\notin B_{\frac{c}{4}}(z)]}M_{T^*-\delta}$ uniformly bounded in δ. Thus, dominated convergence theorem yields

$$\lim_{\delta\to 0}\lim_{T\to T^*} Q^T(Z_{T-T^*+\delta}\circ\theta_{T^*-\delta} > 2c) = 0$$

in view of $P^x(\lim_{t\to T^*} M_t = 0) = 1$. Combining the above with (4.25) and (4.30) yields

$$\lim_{\delta\to 0}\lim_{T\to T^*} Q^T(Z_\delta \circ \theta_{T^*-\delta} > 4c) = 0. \tag{4.33}$$

Case 2 Suppose Assumption 4.3 holds. For all $t > 0$ and $\delta \le t$ define

$$\varphi(t,\delta,x) = E^x[\mathbf{1}_{[\tau^\delta\circ\theta_{t-\delta}<\delta]}p(\delta - \tau^\delta \circ \theta_{t-\delta}, X_{t-\delta+\tau^\delta\circ\theta_{t-\delta}}, z)].$$

Observe that for every $t > 0$ the map $\delta \mapsto \varphi(t,\delta,x)$ is increasing. Indeed, consider $M_s^t = p(t-s, X_s, z)$ and note that in view of Lemma 4.2

$$\varphi(t,\delta,x) = E^x[M^t_{t-\delta+\tau^\delta\circ\theta_{t-\delta}}].$$

The claim follows since the stopping times $t-\delta+\tau^\delta\circ\theta_{t-\delta}$ are decreasing in δ and M^t is a supermartingale on $[0,t]$.

Due to (4.32) we have

$$\lim_{T\to T^*} Q^T(Z_{T-T^*+\delta}\circ\theta_{T^*-\delta} > 2c) = \frac{\varphi(T^*,\delta,x)}{p(T^*,x,z)}. \tag{4.34}$$

Our next goal is to show that $\lim_{\delta\to 0}\varphi(T^*,\delta,x) = 0$. The first step towards this goal is to obtain that $\varphi(t,0,x) := \lim_{\delta\to 0}\varphi(t,\delta,x) = 0$ for almost every t. Since $\varphi(t,\cdot,x)$ is increasing for every $t > 0$, in view of Lemma 4.3 it is enough to show that

$$\lim_{\alpha\to\infty}\alpha\int_0^\infty\int_0^t e^{-\alpha s-\beta t}\varphi(t,s,x)dsdt = 0,\ \forall\beta > 0.$$

We will find an upper bound to this Laplace transform by considering the following decomposition.

$$\begin{aligned}\varphi(t,\delta,x) &= \int_{y\in B_{\frac{c}{4}}(z)} E^y[\mathbf{1}_{[\tau^\delta<\delta]}p(\delta-\tau^\delta, X_{\tau^\delta}, z)]p(t-\delta,x,y)m(dy)\\ &\quad+\int_{y\in B^c_{\frac{c}{4}}(z)} E^y[\mathbf{1}_{[\tau^\delta<\delta]}p(\delta-\tau^\delta, X_{\tau^\delta}, z)]p(t-\delta,x,y)m(dy)\\ &\le \int_{y\in B_{\frac{c}{4}}(z)} E^y[\mathbf{1}_{[\tau_c<\delta]}p(\delta-\tau_c, X_{\tau_c}, z)]p(t-\delta,x,y)m(dy) \quad (4.35)\\ &\quad+\int_{y\in B^c_{\frac{c}{4}}(z)} p(\delta,y,z)p(t-\delta,x,y)m(dy), \quad (4.36)\end{aligned}$$

where the inequality is due to Lemma 4.2.

Note that

$$\int_0^\infty \int_0^t e^{-\alpha\delta-\beta t} E^y[\mathbf{1}_{[\tau_c<\delta]} p(\delta-\tau_c, X_{\tau_c}, z)] p(t-\delta, x, y) d\delta dt$$
$$= E^y \int_{\tau_c}^\infty \int_{\tau_c}^t e^{-\alpha\delta-\beta t} p(\delta-\tau_c, X_{\tau_c}, z) p(t-\delta, x, y) d\delta dt$$
$$= E^y \left\{ e^{-(\alpha+\beta)\tau_c} \int_0^\infty \int_0^t e^{-\alpha\delta-\beta t} p(\delta, X_{\tau_c}, z) p(t-\delta, x, y) d\delta dt \right\}$$
$$= E^y \left\{ e^{-(\alpha+\beta)\tau_c} \int_0^\infty \int_\delta^\infty e^{-\alpha\delta-\beta t} p(\delta, X_{\tau_c}, z) p(t-\delta, x, y) dt d\delta \right\}$$
$$= E^y \left\{ e^{-(\alpha+\beta)\tau_c} \int_0^\infty e^{-(\alpha+\beta)\delta} p(\delta, X_{\tau_c}, z) d\delta \right\} \int_0^\infty e^{-\beta t} p(t, x, y) dt$$
$$= E^y \left[e^{-(\alpha+\beta)\tau_c} u^{\alpha+\beta}(X_{\tau_c}, z) \right] u^\beta(x, y).$$

Due to (4.15) $\sup_{\alpha>0, w\in\partial B_{\frac{c}{2}}(z)} \alpha u^{\alpha+\beta}(w, z) < \infty$. Moreover,

$$\int_{B_{\frac{c}{4}}(z)} u^\beta(x, y) m(dy) = \int_0^\infty e^{-\beta t} P^x(X_t \in B_{\frac{c}{4}}(z)) dt < \infty.$$

Thus, the Dominated Convergence Theorem yields

$$\lim_{\alpha\to\infty} \alpha \int_0^\infty \int_0^t e^{-\alpha\delta-\beta t} \int_{y\in B_{\frac{c}{4}}(z)}$$
$$\times E^y[\mathbf{1}_{[\tau_c<\delta]} p(\delta-\tau_c, X_{\tau_c}, z)] p(t-\delta, x, y) m(dy) d\delta dt$$
$$= \int_{y\in B_{\frac{c}{4}}(z)} E^y \left[\lim_{\alpha\to\infty} \alpha e^{-(\alpha+\beta)\tau_c} u^{\alpha+\beta}(X_{\tau_c}, z) \right] u^\beta(x, y) m(dy) = 0, \tag{4.37}$$

since $P^y(\tau_c = 0) = 0$ by the continuity of X.

Next we turn to (4.36). Note that

$$\int_0^t \alpha e^{-\alpha\delta} \int_{y\in B^c_{\frac{c}{4}}(z)} p(\delta, y, z) p(t-\delta, x, y) m(dy) d\delta$$
$$\leq p(t, x, z) \int_0^t \alpha e^{-\alpha\delta} d\delta \leq p(t, x, z).$$

Since

$$\int_0^\infty e^{-\beta t} p(t, x, z) dt = u^\beta(x, z) < \infty,$$

the Dominated Convergence Theorem yields

$$\begin{aligned}
&\lim_{\alpha\to\infty} \alpha \int_0^\infty \int_0^t e^{-\alpha\delta-\beta t} \int_{y\in B^c_{\frac{c}{4}}(z)} p(\delta, y, z) p(t-\delta, x, y) m(dy) d\delta dt \\
&= \int_0^\infty e^{-\beta t} \lim_{\alpha\to\infty} \alpha \int_0^t e^{-\alpha\delta} \int_{y\in B^c_{\frac{c}{4}}(z)} p(\delta, y, z) p(t-\delta, x, y) m(dy) d\delta dt.
\end{aligned}$$

Since $\int_{y\in B^c_{\frac{c}{4}}(z)} p(\delta, y, z) p(t-\delta, x, y) m(dy) \leq p(t, x, z)$ and converges to 0 as $\delta \to 0$ in view of (4.13), Lemma 4.3 yields

$$\lim_{\alpha\to\infty} \alpha \int_0^\infty \int_0^t e^{-\alpha\delta-\beta t} \int_{y\in B^c_{\frac{c}{4}}(z)} p(\delta, y, z) p(t-\delta, x, y) m(dy) d\delta dt = 0.$$

Combining the above with (4.37), (4.36) and (4.35) gives

$$\lim_{\alpha\to\infty} \alpha \int_0^\infty \int_0^t e^{-\alpha s-\beta t} \varphi(t, s, x) ds dt = 0, \ \forall \beta > 0.$$

Thus, $\varphi(t, 0, x) = 0$ for almost every $t > 0$.

Define $D := \{t > 0 : \varphi(t, 0, x) = 0\}$, let $S \in D$ and consider $s < S$. Then, by Chapman–Kolmogorov identity we have

$$\begin{aligned}
0 = \varphi(S, 0, x) &= \lim_{\delta\to 0} \varphi(S, \delta, x) \\
&= \lim_{\delta\to 0} \int \varphi(s, \delta, y) p(S-s, x, y) m(dy) \\
&= \int \varphi(s, 0, y) p(S-s, x, y) m(dy),
\end{aligned}$$

where the last equality is due to the Dominated Convergence Theorem since $\varphi(s, \delta, y) \leq p(s, y, z)$. Thus, $\varphi(s, 0, y) p(S-s, x, y) = 0$ for m-a.e. y, which implies $\varphi(s, 0, y) = 0$, m-a.e. if $p(t, x, y) > 0$ for all $t > 0$ and m-a.e. y. On the other hand, if $t \mapsto p(t, x, y)$ is continuous on $(0, \infty)$, the fact that D is dense in $(0, \infty)$ yields that for any $t > s$, $\varphi(s, 0, y) p(t-s, x, y) = 0$ for m-a.e. y. Therefore, under either assumption, we have

$$\varphi(T^*, 0, x) = \lim_{\delta \to 0} \int \varphi(s, \delta, y) p(T^* - s, x, y) m(dy)$$
$$= \int \varphi(s, 0, y) p(T^* - s, x, y) m(dy) = 0.$$

Combining the above with (4.25) and (4.30) yields

$$\lim_{\delta \to 0} \lim_{T \to T^*} Q^T(Z_\delta \circ \theta_{T^*-\delta} > 4c) = 0 \tag{4.38}$$

in view of (4.34).

The above as well as (4.33) imply that for any $\varepsilon > 0$ there exist $\hat{\delta}$ and $\hat{T} > T^* - \hat{\delta}$ such that for all $T > \hat{T}$ we have

$$Q^T(w(X, \hat{\delta}, [T^* - \hat{\delta}, T^*]) > 4c) < \frac{\varepsilon}{2}.$$

As $w(X, \delta, [u, v])$ is increasing in δ, we can conclude

$$Q^T(w(X, \delta, [T^* - \hat{\delta}, T^*]) > 4c) < \frac{\varepsilon}{2}, \qquad \forall \delta < \hat{\delta}, T > \hat{T}.$$

Since

$$Q^T(w(X, \delta, [0, T^*]) > 8c) \leq Q^T(w(X, \delta, [0, T^* - \hat{\delta}]) > 4c)$$
$$+ Q^T(w(X, \delta, [T^* - \hat{\delta}, T^*]) > 4c),$$

it remains to show that

$$\lim_{\delta \to 0} \limsup_{T \to T^*} Q^T(w(X, \delta, [0, T^* - \hat{\delta}]) > 4c) = 0.$$

However, for $T > \hat{T}$, one has $T^* - \hat{\delta} < T$, thus for such T

$$Q^T(w(X, \delta, [0, T^* - \hat{\delta}]) > 4c) = E^x\left[\mathbf{1}_{[w(X,\delta,[0,T^*-\hat{\delta}])>4c]} \frac{p(T^* - \hat{T}, X_{\hat{T}}, z)}{p(T^*, x, z)}\right].$$

Then, by the Dominated Convergence Theorem,

$$\lim_{\delta \to 0} \limsup_{T \to T^*} Q^T(w(X, \delta, [0, T^* - \hat{\delta}]) > 4c)$$
$$= E^x\left[\lim_{\delta \to 0} \mathbf{1}_{[w(X,\delta,[0,T^*-\hat{\delta}])>4c]} \frac{p(T^* - \hat{T}, X_{\hat{T}}, z)}{p(T^*, x, z)}\right] = 0.$$

To show the convergence of the finite dimensional distributions consider $[0, T^*)$ as a dense subset of $[0, T^*]$ and note that for any finite set $t_1, \ldots, t_k \subset [0, T^*)$ and a bounded continuous function $f : \mathbf{E}^k \mapsto \mathbb{R}$

$$\lim_{T \to T^*} E^{Q^T}[f(X_{t_1}, \ldots, X_{t_k})] = E^{Q^{t_k}}[f(X_{t_1}, \ldots, X_{t_k})],$$

which establishes the desired convergence.

Thus, the sequence of measures, (Q^T), has a unique limit point $P^{x \to z}_{0 \to T^*}$ on $(C([0, T^*], \mathbf{E}), \mathscr{B}_{T^*})$. Moreover, its restriction, Q^T, to $(C([0, T], \mathbf{E}), \mathscr{B}_T)$ for any $T < T^*$ is a solution to the local martingale problem for A^p on $[0, T] \times \mathbf{E}$. In particular, for any $E \in \mathscr{B}_T$, we have $P^{x \to z}_{0 \to T^*}(E) = E^x\left[\frac{h(T, X_T)}{h(0, x)} \mathbf{1}_E\right]$.

Next, we show the bridge condition. To this end pick $f \in \mathbb{C}^\infty_K(\mathbf{E})$, $\varepsilon > 0$ and consider $r > 0$ such that $\sup_{y \in B_r(z)} |f(y) - f(z)| < \varepsilon$. Thus,

$$\begin{aligned}
E^{x \to z}_{0 \to T^*}[f(X_{T^*})] &= \lim_{T \to T^*} E^{x \to z}_{0 \to T^*}[f(X_T)] = \lim_{T \to T^*} E^x\left[\frac{h(T, X_T)}{h(0, x)} f(X_T)\right] \\
&= f(z) + \lim_{T \to T^*} \frac{E^x\left[p(T^* - T, X_T, z)(f(X_T) - f(z))\right]}{p(T^*, x, z)} \\
&= f(z) + \lim_{T \to T^*} \int_{B_r(z)} \frac{p(T^* - T, y, z)p(T, x, y)}{p(T^*, x, z)} (f(y) - f(z))\, m(dy) \\
&\quad + \lim_{T \to T^*} \int_{B^c_r(z)} \frac{p(T^* - T, y, z)p(T, x, y)}{p(T^*, x, z)} (f(y) - f(z))\, m(dy).
\end{aligned}$$

Since f is bounded the second integral above converges to 0 in view of (4.9) or (4.13). Moreover,

$$\left| \int_{B_r(z)} \frac{p(T^* - T, y, z)p(T, x, y)}{p(T^*, x, z)} (f(y) - f(z))\, m(dy) \right| \leq \varepsilon,$$

which implies the bridge condition by the arbitrariness of ε.

Existence of a weak solution to (4.22) follows from Girsanov's Theorem. Indeed, since the martingale problem for A is well posed there exists a unique weak solution to

$$X_t = x + \int_0^t b(X_u)du + \int_0^t \sigma(X_u)\, dB_u,$$

by Corollary 2.3. With an abuse of notation, we denote the associated probability measure with P^x. The above considerations show that there exists a probability measure, $P^{x \to z}_{0 \to T^*}$, which is locally absolutely continuous with respect P^x in the sense that for any $t < T^*$ and $E \in \mathscr{F}_t$ $P^{x\to z}_{0\to T^*}(E) = E^x\left[\mathbf{1}_E \frac{h(t,X_t)}{h(0,x)}\right]$. Thus, an application of Girsanov's Theorem yields the conclusion.

To show that $\inf_{t\in[0,T]} h(t, X_t) > 0$, $P^{x\to z}_{0\to T^*}$-a.s. for any $T < T^*$ consider $T_n = \inf\{t \geq 0 : h(t, X_t) \leq \frac{1}{n}\}$ and observe that $P^{x\to z}_{0\to T^*}(T_n < t) = E^x\left[\mathbf{1}_{[T_n<t]} \frac{h(t,X_t)}{h(0,x)}\right] \leq \frac{1}{nh(0,x)} \to 0$ as $n \to \infty$.

To show the Markov property let $s < t < T^*$, f be bounded and measurable, and $E \in \mathscr{B}_s$. Then,

$$E^{x\to z}_{0\to T^*}[f(X_t)\mathbf{1}_E]=E^x[f(X_t)h(t, X_t)\mathbf{1}_E]=E^{x\to z}_{0\to T^*}\left[\frac{E^x[f(X_t)h(t, X_t)|X_s]}{h(s, X_s)}\mathbf{1}_E\right],$$

where we used the positivity of $h(t, X_t)$ under P^x for the second equality.

Similarly, if g is also bounded and measurable, we have in addition

$$E^{x\to z}_{0\to T^*}[f(X_t)g(X_s)] = E^{x\to z}_{0\to T^*}\left[\frac{E^x[f(X_t)h(t, X_t)|X_s]}{h(s, X_s)}g(X_s)\right].$$

This shows the Markov property and yields the representation of the transition function.

Finally, the weak uniqueness can be proved following the same steps as in the proof of Corollary 4.1.

Note that the starting point, x, of the SDE (4.21) (as well as of (4.22)) is fixed in Theorem 4.3 (resp. Theorem 4.4). If, additionally, the conditions of aforementioned theorems are satisfied by all $x \in \mathbf{E}$ and $h(t, \cdot) > 0$ for $t < T^*$, we can obtain the existence and uniqueness of weak solution to (4.39) for any $s < T^*$ and $x \in \mathbf{E}$. Thus, Corollary 2.3 implies well posedness of the local martingale problem which in turn yields the strong Markov property of its solutions by Theorem 2.3. More precisely, the following holds.

Corollary 4.3 *Suppose that the conditions of Theorem 4.3 or Theorem 4.4 hold for all $x \in \mathbf{E}$. Moreover, assume the following for all $x \in \mathbf{E}$:*

1. $h(t, x) > 0$ *for all* $t \in [0, T^*)$.
2. $h \in \mathbb{C}^{1,2}([0, T^*) \times \mathbf{E})$.

Then there exists a unique weak solution on $[s, T^*]$ to

$$X_t = x + \int_s^t \left\{b(u, X_u) + a(u, X_u)\frac{(\nabla h(u, X_u))^*}{h(u, X_u)}\right\} du + \int_s^t \sigma(u, X_u)\, dB_u, \tag{4.39}$$

the law of which, $P^{x\to z}_{s\to T^*}$, satisfies $P^{x\to z}_{s\to T^*}(\inf_{u\in[s,T]} h(u, X_u) = 0) = 0$ for any $T < T^*$, and $P^{x\to z}_{s\to T^*}(X_{T^*} = z) = 1$. Moreover, the solution has the strong Markov property.

To obtain strong solutions we need to impose stronger conditions on the transition density p and the coefficients a and b. These conditions will imply the pathwise uniqueness which in turn will lead to the existence of a strong solution in view of Yamada–Watanabe Theorem (see Theorem 2.12). The strict positivity of p in $\mathrm{int}(\mathbf{E})$, which we will require, is not too restrictive and it is always satisfied in the case of one-dimensional diffusions (see Theorem 2.17). On the other hand, the condition $p(t, y, y') > 0$ for all $t > 0$, $y \in \mathrm{int}(\mathbf{E})$ and $y' \in \partial(\mathbf{E})$ is more delicate and its fulfilment depends on the classification of the boundaries of the underlying diffusion.

Theorem 4.5 *Let σ be a matrix field such that $\sigma\sigma^* = a$ and fix $x \in int(\mathbf{E})$ and $z \in \mathbf{E}$ such that the hypotheses of Theorem 4.3 (resp. of Theorem 4.4) hold. Suppose, in addition, that for any closed set $C \subset int(\mathbf{E})$*

$$\|b(t, y) - b(t, y')\| + \|\sigma(t, y) - \sigma(t, y')\| \leq K_C \|y - y'\|, \; y, y' \in C, \; \text{and } t \leq T^*,$$

as well as $p(t, y; T^*, z) > 0$ for all $t \in [0, T^*)$ and $y \in int(\mathbf{E})$.

If $P^{0,x}(\inf\{t > 0 : \tilde{X}_t \notin int(\mathbf{E})\} < T^*) = 0$ when $\tilde{X}$ satisfies

$$\tilde{X}_t = x + \int_0^t b(u, \tilde{X}_u) du + \int_0^t \sigma(u, \tilde{X}_u)\, dB_u,$$

then there exists a unique strong solution, X, to (4.21) (resp. (4.22)). Moreover, $X_{T^*} = z$.

Proof We prove the result for the time-homogeneous case. Let X be a strong solution to (4.22) on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T^*]}, P)$ and $T < T^*$. Due to the form of $\mathbf{E}$ there exists an increasing sequence of open sets with compact closure, U_n, such that $\mathrm{cl}(U_n) \subset \mathrm{int}(\mathbf{E})$ and $\mathrm{int}(\mathbf{E}) = \cup_{n=1}^{\infty} U_n$. It is a simple matter to check that the coefficients of (4.22) satisfy the conditions of Theorem 2.6 on $[0, T] \times U_n$ for all $n \geq 1$. Thus, the conclusion of the theorem yields uniqueness up to $T \wedge \tau$, where $\tau = \inf\{t > 0 : X_t \notin \mathrm{int}(\mathbf{E})\}$.

Let $\nu_m = \inf\{t \geq 0 : h(t, X_t) \leq \frac{1}{m}\}$ and consider a measure $Q^{n,m}$ defined by

$$Q^{n,m}(E) = E\left[\frac{h(0, x)}{h\left(T \wedge \nu_m \wedge \tau_n, X_{T\wedge\nu_m\wedge\tau_n}\right)} \mathbf{1}_E\right], \qquad E \in \mathscr{F}_T.$$

Due to the choice of stopping times $Q^{n,m} \sim P$. Moreover,

$$W_t = B_t + \int_0^{t\wedge\tau_n\nu_m} \sigma^*(X_u) \frac{(\nabla h(u, X_u))^*}{h(u, X_u)} du$$

is a $Q^{n,m}$-Brownian motion. Let $\tilde{X}$ be the unique strong solution of

$$\tilde{X}_t = x + \int_0^t b(\tilde{X}_u) du + \int_0^t \sigma(\tilde{X}_u)\, dW_u. \tag{4.40}$$

Observe that X solves (4.40) under $Q^{n,m}$ until $T \wedge \nu_m \wedge \tau_n$, and thus $Q^{n,m}(X_{t\wedge\nu_m\wedge\tau_n} = \tilde{X}_{t\wedge\tilde{\nu}_m\wedge\tilde{\tau}_n}, t \in [0,T]) = 1$ by Theorem 2.6, where $\tilde{\tau}_n := T \wedge \inf\{t > 0 : \tilde{X}_t \notin U_n\}$, $\tilde{\nu}_m = \inf\{t \geq 0 : h(t, \tilde{X}_t) \leq \frac{1}{m}\}$. Hence,

$$
\begin{aligned}
P(\tau_n \wedge \nu_m < T) &= E^{Q^{n,m}}\left[\frac{h(T\wedge\tau_n\wedge\nu_m, X_{T\wedge\nu_m\wedge\tau_n})}{h(0,x)}\mathbf{1}_{[\tau_n\wedge\nu_m<T]}\right] \\
&= E^{Q^{n,m}}\left[\frac{h(T\wedge\tilde{\tau}_n\wedge\tilde{\nu}_m, \tilde{X}_{T\wedge\tilde{\nu}_m\wedge\tilde{\tau}_n})}{h(0,x)}\mathbf{1}_{[\tilde{\tau}_n\wedge\tilde{\nu}_m<T]}\right] \\
&= E^{x}\left[\frac{h(T\wedge\tilde{\tau}_n\wedge\tilde{\nu}_m, \tilde{X}_{T\wedge\tilde{\nu}_m\wedge\tilde{\tau}_n})}{h(0,x)}\mathbf{1}_{[\tilde{\tau}_n\wedge\tilde{\nu}_m<T]}\right] \\
&= E^{x}\left[\frac{h(T, \tilde{X}_T)}{h(0,x)}\mathbf{1}_{[\tilde{\tau}_n\wedge\tilde{\nu}_m<T]}\right] \\
&\leq E^{x}\left[\frac{h(T, \tilde{X}_T)}{h(0,x)}\left(\mathbf{1}_{[\tilde{\tau}_n<T]} + \mathbf{1}_{[\tilde{\nu}_m<T]}\right)\right],
\end{aligned}
$$

where P^x is the law of $\tilde{X}$. Note that the fourth equality is due to the fact that $(h(t, \tilde{X}_t))_{t\in[0,T]}$ is a martingale under P^x as a consequence of Chapman–Kolmogorov equation.

Next, observe that $\lim_{m\to\infty} \nu_m \geq \tau$, P-a.s. due to the strict positivity of h on $\mathrm{int}(\mathbf{E})$. Similarly, $\lim_{m\to\infty} \tilde{\nu}_m \geq \lim_{n\to\infty} \tilde{\tau}_n$, P^x-a.s. Furthermore, $\lim_{n\to\infty} \tilde{\tau}_n > T$, P^x-a.s. as $\tilde{X}$ stays in the interior under P^x. Hence, taking the limits in the above yields $P(\tau < T) = 0$ and thus establishes the pathwise uniqueness up to T. This together with the existence of a weak solution due to Theorem 4.4 implies the existence of a unique strong solution via Theorem 2.12 on $[0, T^*)$ as T was arbitrary.

Finally, since pathwise uniqueness implies uniqueness in law we conclude that the law of $(X_t)_{t\in[0,T*)}$ is given by $P^{x\to z}_{0\to T^*}$ obtained in Theorem 4.4 because of the continuity of the weak solution. Thus, we can uniquely define $X_{T^*} = \lim_{t\to T^*} X_t = z$. □

Remark 4.4 Using the arguments above we can also establish the existence and uniqueness of a strong solution of the SDE in Corollary 4.2 under the assumption that b and σ are locally Lipschitz.

Up to now we have been discussing the existence and the uniqueness of the local martingale problem associated with a bridge process described by (4.21) or (4.22) and the end condition $X_{T^*} = z \in \mathbf{E}$. We have shown that a solution, $P^{x\to z}_{0\to T^*}$, exists and is unique under fairly general assumptions on the transition density. It is also natural to ask whether a similar construction is possible when z is replaced by a random variable independent from the driving Brownian motion. Such a construction is related to enlargement of filtrations—a connection that we will explore in the next section. Moreover, the feasibility of this construction will be essential in the financial applications that we will consider in the second part of this book. In order to construct a solution to the bridge process with a random terminal condition using the family $\left(P^{x\to z}_{0\to T^*}\right)$ one obviously needs the following measurability result.

Proposition 4.1 *Fix $x \in \mathbf{E}$ and define*

$$D_x := \{z \in \mathbf{E} : (x, z) \textit{ satisfies the conditions of Theorem 4.3}\}.$$

For any $z \in D_x$ let $P^{x\to z}_{0\to T^*}$ be the corresponding measure. Then, for any $E \in \mathscr{B}_{T^*}$ the map $z \mapsto P^{x\to z}_{0\to T^*}(E)$ is $\mathscr{E} \cap D_x/\mathscr{B} \cap [0,1]$-measurable, where $\mathscr{B}$ is the Borel σ-algebra on $\mathbb{R}$ and $\mathscr{F} \cap E$ stands for the restriction of the σ-algebra $\mathscr{F}$ to E.

Proof Define

$$\lambda := \{E \in \mathscr{B}_{T^*} : z \mapsto P^{x\to z}_{0\to T^*}(E) \text{ is } \mathscr{E} \cap D_x/\mathscr{B} \cap [0,1]\text{-measurable}\}$$

and let $\pi = \{E \in \mathscr{B}_{T^*} : E \in \mathscr{B}_T \text{ for some } T < T^*\}$. Clearly, π is closed under intersections. The claim will follow from Theorem A.1 once we show that $\pi \subset \lambda$ and λ is a λ-system.

Clearly, $\Omega \in \lambda$ since $P^{x\to z}_{0\to T^*}(\Omega) = 1$. Let $E_n \subset E_{n+1}$ be a sequence of sets in λ. Since

$$P^{x\to z}_{0\to T^*}\left(\cup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P^{x\to z}_{0\to T^*}(E_n),$$

and the limit of measurable functions is measurable, $\cup_{n=1}^{\infty} E_n \in \lambda$. Moreover, let $E \subset F$ be two sets in λ. Then, $P^{x\to z}_{0\to T^*}(F\setminus E) = P^{x\to z}_{0\to T^*}(F) - P^{x\to z}_{0\to T^*}(E)$ is a measurable function implying that λ is indeed a λ-system.

To see that $\pi \subset \lambda$ consider $E \in \pi$, i.e. $E \in \mathscr{B}_T$ for some $T < T^*$. By the construction of $P^{x\to z}_{0\to T^*}$ in the proof of Theorem 4.3 we have

$$P^{x\to z}_{0\to T^*}(E) = \frac{E^{0,x}[p(T, X_T; T^*, z)\mathbf{1}_E]}{p(0, x; T^*, z)}.$$

The above is a continuous function as a mapping from $\mathbf{E}$ to $[0, 1]$ since $(p(T, X_T; T^*, z))_{z\in\mathbf{E}}$ is a uniformly integrable family by Theorem 16.14 in [25] and continuous in z. Thus, its restriction to D_x is measurable. □

Example 4.3 In case of one-dimensional Brownian motion,

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-(x-y)^2\right),$$

which vanishes nowhere. Thus, Theorem 4.5 implies

$$X_t = x + B_t + \int_0^t \frac{z - X_s}{T^* - s} dz$$

is a Brownian bridge from x to z on the interval $[0, T^*]$, where B is a standard Brownian motion.

Example 4.4 Recall from Examples 1.5 and 2.2 that three-dimensional Bessel process is a one-dimensional diffusion on $[0, \infty)$ with the infinitesimal generator $A = \frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$ and the transition density

$$\frac{1}{x} q_t(x, y) y, \qquad x > 0, y > 0,$$

where

$$q(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left(\exp\left(-(x-y)^2\right) - \exp\left(-(x+y)^2\right) \right).$$

The speed measure of three-dimensional Bessel process equals $m(dx) = 2x^2 dx$, thus the density of its transition function with respect to the speed measure is given by

$$p(t, x, y) = \frac{q_t(x, y)}{2xy}, \qquad x > 0, y > 0.$$

One can define $p(t, x, y)$ for $y = 0$ by taking limits:

$$p(t, x, 0) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right), \qquad x > 0.$$

The above density can be used to construct the three-dimensional Bessel bridge from $x > 0$. Indeed, since a three-dimensional Bessel process stays in $\mathbb{R}_{++}$ on $(0, \infty)$, a straightforward application of Theorem 4.5, which is applicable in view of the discussion following Remark 4.3, yields the existence of a unique strong solution to the corresponding SDE:

$$X_t = x + B_t + \int_0^t \left\{ \frac{1}{X_s} - \frac{X_s}{T^* - s} \right\} ds. \tag{4.41}$$

Moreover, the solution satisfies $X_{T^*} = 0$.

4.2 Connection with the Initial Enlargement of Filtrations

Theory of static Markov bridges is intimately linked to the initial enlargements of filtrations. We will illustrate this connection by considering diffusions satisfying Assumption 4.2 or 4.3. To ease the exposition we consider the one-dimensional case.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a filtered probability space supporting a one-dimensional Brownian motion B such that $(\mathscr{F}_t)_{t\geq 0}$ is right continuous and $\mathscr{F}_0$ contains all P-null sets. Consider the SDE

$$X_t = x + \int_0^t b(X_u)\,du + \int_0^t \sigma(X_u)\,dB_u, \tag{4.42}$$

where b and $\sigma > 0$ are continuous functions on (l, ∞). Under the assumption that l is inaccessible and ∞ is natural, there exists a unique weak solution to the above SDE with Feller property as observed before. These assumptions also imply that the conditions of Theorem 4.4 are satisfied for all $z \in (l, \infty)$ whenever $x \in (l, \infty)$.

Let us also assume that for any closed set $C \subset (l, \infty)$

$$\|b(y) - b(y')\| + \|\sigma(y) - \sigma(y')\| \leq K_C \|y - y'\|, \; y, y' \in C,$$

which implies the existence and uniqueness of a strong solution. We are interested in the $(\mathscr{G}_t)$-semimartingale decomposition of certain $(\mathscr{F}_t)$-adapted martingales when $\mathscr{G}_t = \sigma(X^*_T) \vee \mathscr{F}_t$ for some deterministic $T^* > 0$. Consider an $(\mathscr{F}_t)$-martingale M with

$$M_t = \int_0^t \vartheta_s \, dB_s, \text{ such that } \int_0^{T^*} E\vartheta_t^2 \, dt < \infty,$$

and observe that for any $f \in \mathbb{C}_K^\infty([l, \infty))$ we have

$$\begin{aligned} P_{T^*-t} f(X_t) &= \int_l^\infty f(y) p(T^* - t, X_t, y) m(dy) \\ &= P_{T^*} f(x) + \int_0^t \left(\int_l^\infty f(y) \frac{\partial p}{\partial x}(T^* - s, X_s, y) m(dy) \right) \sigma(X_s)\,dB_s. \end{aligned}$$

Indeed, since $\mathbb{P}(X_t \in (l, \infty) \text{ for } t \geq 0) = 1$, $\frac{\partial p}{\partial x}(T^* - s, X_s, y)$ is well defined and finite for all $y \in (l, \infty)$ and $s \in [0, T^*)$. Applying integration by parts formula yields

$$\begin{aligned} M_t P_{T^*-t} f(X_t) = M_s P_{T^*-s} f(X_s) &+ \int_s^t \left(\int_l^\infty f(y) \frac{\partial p}{\partial x}(T^* - u, X_u, y) m(dy) \right) \\ &\times \vartheta_u \sigma(X_u)\,du + N_t - N_s, \end{aligned}$$

where N is an $(\mathscr{F}_t)$-martingale by Theorem A.18.

Thus, if $A \in \mathscr{F}_s$ for some $s < T^*$

$$\begin{aligned} &E\left[(M_t - M_s) f(X_{T^*}) \mathbf{1}_A\right] \\ &\quad = E\left[(M_t - M_s) P_{T^*-t} f(X_t) \mathbf{1}_A\right] \end{aligned}$$

$$
\begin{aligned}
&= E\left[\int_s^t \left(\int_l^\infty f(y)\frac{\partial p}{\partial x}(T^*-u, X_u, y)m(dy)\right)\vartheta_u\sigma(X_u)\,du\mathbf{1}_A\right]\\
&= E\left[\int_s^t \left(\int_l^\infty f(y)\frac{\frac{\partial p}{\partial x}(T^*-u, X_u, y)}{p(T^*-u, X_u, y)}\right.\right.\\
&\qquad\left.\left.\times\ p(T^*-u, X_u, y)\,m(dy)\right)\vartheta_u\sigma(X_u)\,du\mathbf{1}_A\right]\\
&= E\left[\int_s^t f(X_{T^*})\frac{\frac{\partial p}{\partial x}(T^*-u, X_u, X_{T^*})}{p(T^*-u, X_u, X_{T^*})}\vartheta_u\sigma(X_u)\,du\mathbf{1}_A\right]\\
&= E\left[\int_s^t \frac{\frac{\partial p}{\partial x}(T^*-u, X_u, X_{T^*})}{p(T^*-u, X_u, X_{T^*})}\vartheta_u\sigma(X_u)\,du f(X_{T^*})\mathbf{1}_A\right],
\end{aligned}
$$

the third equality is due to the fact $\mathbb{P}(X_t \in (l, \infty) \text{ for } t \geq 0) = 1$ and $p(t, x, y) > 0$ for all $t > 0$ and $(x, y) \in (l, \infty)^2$ as discussed in the previous section.

The above calculations show that

$$
M_t - \int_0^t \frac{\frac{\partial p}{\partial x}(T^*-u, X_u, X_{T^*})}{p(T^*-u, X_u, X_{T^*})}\vartheta_u\sigma(X_u)\,du, \qquad t \leq T^*,
$$

is a $(\mathscr{G}_t)$-martingale provided the above Lebesgue integral is well defined, which holds, e.g. when

$$
\int_0^{T^*}\left|\frac{\frac{\partial p}{\partial x}(T^*-u, X_u, X_{T^*})}{p(T^*-u, X_u, X_{T^*})}\vartheta_u\sigma(X_u)\right| du < \infty,\ P\text{-a.s.}
$$

Thus, we have proved the following

Theorem 4.6 *Let $M = (M_t)_{t\geq 0}$ be an $(\mathscr{F}_t)$-martingale such that*

$$
M_t = \int_0^t \vartheta_s dB_s,\ \textit{and}\ \int_0^{T^*} E\vartheta_t^2\,dt < \infty.
$$

If

$$
\int_0^{T^*}\left|\frac{\frac{\partial p}{\partial x}(T^*-u, X_u, X_{T^*})}{p(T^*-u, X_u, X_{T^*})}\vartheta_u\sigma(X_u)\right| du < \infty,\ P\text{-a.s.},
$$

then for any $t \leq T^*$

$$
M_t = \tilde{M}_t + \int_0^t \frac{\frac{\partial p}{\partial x}(T^*-u, X_u, X_{T^*})}{p(T^*-u, X_u, X_{T^*})}\vartheta_u\sigma(X_u)\,du,
$$

where $\tilde{M}$ is a $(\mathscr{G}_t)$-martingale.

Remark 4.5 To generalise the above theorem to the multi-dimensional case one only needs to make sure that $\frac{\partial p}{\partial x}(T^* - u, X_u, y)$ as well as $\frac{\frac{\partial p}{\partial x}(T^*-u,X_u,y)}{p(T^*-u,X_u,y)}$ are well defined for m-a.e. y.

Consider $\vartheta = \sigma(X)$ and observe that the integral

$$\int_0^t \frac{\frac{\partial p}{\partial x}(T^* - u, X_u, X_{T^*})}{p(T^* - u, X_u, X_{T^*})}\sigma^2(X_u)\,du,$$

is well defined for all $t < T^*$ due to the continuity of σ. In view of Lévy's characterisation we thus obtain

$$X_t = x + \int_0^t \left\{ b(X_u) + \mathbf{1}_{[u \le T^*]}\sigma^2(X_u)\frac{\frac{\partial p}{\partial x}(T^* - u, X_u, X_{T^*})}{p(T^* - u, X_u, X_{T^*})} \right\} du + \int_0^t \sigma(X_u)d\beta_u, \tag{4.43}$$

where β is a $(\mathscr{G}_t)$-Brownian motion. In the Brownian case this reduces to the following example.

Example 4.5 Suppose X is a standard Brownian motion and $T^* = 1$. Then, in the enlarged filtration

$$X_t = \beta_t + \int_0^t \frac{X_1 - X_s}{1 - s} ds$$

since

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right).$$

We will next show that $((\Omega, \mathscr{F}, P), (\mathscr{G}_t)_{t\in[0,T^*]}, (X_t, \beta_t)_{t\in[0,T^*]})$ is a weak solution to an SDE describing the evolution of a Markov bridge with a random terminal condition.

Theorem 4.7 *Let μ be a Borel probability measure on $[l, \infty)$ such that $\mu(\{l\}) = 0$. Then, there exists a weak solution to*

$$\begin{aligned} X_t &= x + \int_0^t \left\{ b(X_u) + \sigma^2(X_u)\frac{p_x(T^* - u, X_u, Y_u)}{p(T^* - u, X_u, Y_u)} \right\} du + \int_0^t \sigma(X_u)\,dB_u, \\ Y_t &= Y_0, \end{aligned} \tag{4.44}$$

the law of which, $P^{x\to\mu}_{0\to T^*}$, satisfies i) $P^{x\to\mu}_{0\to T^*}(Y_0 \in dy) = \mu(dy)$ and $P^{x\to\mu}_{0\to T^*}(X_t \in (l, \infty), \forall t \in [0, T^*)) = 1$, ii) $P^{x\to\mu}_{0\to T^*}(\inf_{u\in[0,T]} p(T^* - u, X_u, Y_u) = 0) = 0$ for any $T < T^*$, and iii) $P^{x\to\mu}_{0\to T^*}(X_{T^*} = Y_0) = 1$. Moreover, if $(\tilde{X}, \tilde{Y})$ is another weak solution satisfying i), its law agree with that of (X, Y).

Proof Consider the product space[1] $\Omega = C([0, T^*), [l, \infty)) \times [l, \infty)$ equipped with the product σ-algebra $\mathscr{F} := \mathscr{B}^-_{T^*} \times \mathscr{B}$. Similarly, $\mathscr{F}_t := \mathscr{B}^-_t \times \mathscr{B}$ for $t \leq T^*$. We define the probability measure, P, on this space as the product measure $P^x \times \mu$, where P^x is the measure on $(C([0, T^*), [l, \infty)), \mathscr{B}^-_{T^*})$ induced by the solution of (4.42). Note that for a given $\omega \in \Omega$ we may write $\omega = (\omega_1, \omega_2)$ and define $X_t(\omega) = \omega_1(t)$ as well as $Y_0(\omega) = \omega_2$. Observe that the law of X is given by P^x and the distribution of Y_0 is μ.

For any $T < T^*$ define Q^T on $\mathscr{F}_{T^*}$ via

$$\frac{dQ^T}{dP} = \frac{p(T^* - T, X_T, Y_0)}{p(T^*, x, Y_0)}.$$

Let $Y_t = Y_0$ and

$$\beta_t := B_t - \int_0^t \sigma(X_u) \frac{p_x(T^* - u, X_u, Y_0)}{p(T^* - u, X_u, Y_0)} du.$$

Then, it follows from Girsanov's theorem that there exists a Q^T-Brownian motion, $(\beta_t)_{t \in [0,T]}$, such that

$$X_t = x + \int_0^t \left\{ b(X_u) + \sigma^2(X_u) \frac{p_x(T^* - u, X_u, Y_u)}{p(T^* - u, X_u, Y_u)} \right\} du + \int_0^t \sigma(X_u)\, d\beta_u.$$

We can also assume the existence of a probability measure Q on $\mathscr{F}_{T^*}$ that agrees with Q^T on $\mathscr{F}_T$ for $T < T^*$ by Theorem 4.2.

Thus, we have constructed a weak solution to (4.44) on $[0, T^*)$ such that $Q(Y_0 \in dy) = \mu(dy)$ and $Q(X_t \in (l, \infty), \forall t \in [0, T^*)) = Q\left(\cap_n [X_t \in (l, \infty), \forall t \in [0, T_n]]\right) = \lim_{n \to \infty} Q(X_t \in (l, \infty), \forall t \in [0, T_n]) = 1$ for any $T_n \uparrow T^*$.

Since $p(T^* - T, x, y) < \infty$ for all $(x, y) \in (l, \infty)^2$, a direct application of Girsanov's theorem shows the uniqueness in law of solutions to (4.44) on $[0, T^*)$.

Next, consider $F \in \mathscr{B}^-_T$ for $T < T^*$ and $G \in \mathscr{B}$ and observe that

$$Q^T(F \times G) = E\left[\mathbf{1}_{F \times G} \frac{p(T^* - T, X_T, Y_0)}{p(T^*, x, Y_0)}\right] = \int_G E^x\left[\mathbf{1}_F \frac{p(T^* - T, X_T, z)}{p(T^*, x, z)}\right] \mu(dz).$$

On the other hand, define $P^{x \to \mu}_{0 \to T^*}$ on Ω via

$$P^{x \to \mu}_{0 \to T^*}(F \times G) = \int_G P^{x \to z}_{0 \to T^*}(F) \mu(dz), \qquad F \in \mathscr{B}^-_{T^*},\ G \in \mathscr{B},$$

[1] For a more detailed explanation of this product space see the discussion following Theorem 4.1.

where $P^{x\to z}_{0\to T^*}$ is as defined in Theorem 4.4 and the integral is well defined due to the measurability established in Proposition 4.1.

Observe that for any $F \in \mathscr{B}^-_T$ we have

$$P^{x\to\mu}_{0\to T^*}(F \times G) = \int_G E^x \left[\mathbf{1}_F \frac{p(T^* - T, X_T, z)}{p(T^*, x, z)} \right] \mu(dz)$$

Thus, Q and $P^{x\to\mu}_{0\to T^*}$ agree on $\mathscr{F}_T$ and, hence, on $\mathscr{F}$ in view of Theorem 4.2. The remaining properties follow directly from Theorem 4.4. □

In the above theorem one can take μ to be the distribution of X_{T^*}. Thus, choosing $Y = X_{T^*}$ and $B = \beta$ one can see that X given by (4.43) is the unique weak solution of (4.44).

Remark 4.6 As one can easily notice from the above computations, this technique can be applied for enlarging the filtration with a random variable from a more general class. Let τ be a stopping time and η be an $\mathscr{F}_\tau$-measurable random variable. All we need is a smooth function, ρ, satisfying

$$\mathbf{1}_{[t<\tau]} P(\eta \in dy|\mathscr{F}_t) = \mathbf{1}_{[t<\tau]} \rho(t, X_t, y) dy. \tag{4.45}$$

If $(t, x) \mapsto \rho(t, x, y)$ is $\mathbb{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, then

$$M_t = \tilde{M}_t + \int_0^{t\wedge\tau} \frac{\frac{\partial \rho}{\partial x}(u, X_u, \eta)}{\rho(u, X_u, \eta)} \vartheta_u \sigma(X_u)\, du,$$

where $\tilde{M}$ is a $(\mathscr{G}_t)$-martingale with $\mathscr{G}_t = \sigma(\eta) \vee \mathscr{F}_t$, provided

$$\int_0^{t\wedge\tau} \left| \frac{\frac{\partial \rho}{\partial x}(u, X_u, \eta)}{\rho(u, X_u, \eta)} \vartheta_u \sigma(X_u) \right| du < \infty, \qquad \forall t \geq 0.$$

The manuscript [90] lists many examples for the application of this formula in a Brownian setting.

Example 4.6 Suppose $x > 0$, $\sigma = 1$, and $b = 0$, and let

$$T_0 = \inf\{t \geq 0 : X_t = 0\},$$

be the first time that X hits 0. It is well known that

$$\mathbf{1}_{[T_0>t]} P(T_0 \in du|\mathscr{F}_t) = \mathbf{1}_{[T_0>t]} \frac{X_t}{\sqrt{2\pi(u-t)^3}} \exp\left(-\frac{X_t^2}{2(u-t)}\right) du.$$

Thus, if $\mathscr{G}_t = \sigma(T_0) \vee \mathscr{F}_t$, then

$$X_t = x + \beta_t + \int_0^{t \wedge T_0} \left\{ \frac{1}{X_s} - \frac{X_s}{T_0 - s} \right\} ds,$$

where β is a $(\mathscr{G}_t)$-Brownian motion. Note that this is the same SDE as the one in (4.41), which is a manifestation of the fact that conditional on its first hitting time of 0 the Brownian motion has the same law as a three-dimensional Bessel bridge.

4.3 Notes

Section 4.1 follows very closely our work [35] with the exception of Proposition 4.1 that is needed to establish the connection between static bridges and the Doob–Meyer decomposition in enlarged filtrations presented in Sect. 4.2. Our treatment of enlargement of filtrations is borrowed from Mansuy and Yor [90] as well as Yor [113].

The strong conditioning considered in this chapter is well known in the context of Brownian and Bessel bridges, which have been studied extensively in the literature and found numerous applications (see, e.g. [19, 68, 97, 98, 105], and [107]). For a general right continuous strong Markov process [51] constructs a measure on the canonical space such that the coordinate process have the prescribed conditioning under a duality hypothesis. More recently, Chaumont and Uribe Bravo [38] performed the same construction without the duality hypothesis under the assumption that the semigroup (P_t) of the given process has continuous transition densities and $\|P_t - I\| \to 0$ as $t \to 0$, where $\|\cdot\|$ corresponds to the operator norm. Moreover, they have proven that if the original process is, in addition, self-similar, a path-wise construction of the Markov bridge can be performed. However, this construction is not adapted.

SDE representations resulting from strong or weak conditioning were also studied by Baudoin in [16–18]. However, Baudoin considers these SDEs on $[0, T^*)$ and, thus, continuous extension of the solution to $[0, T^*]$ was not the focus of his papers.

The strong solutions of (4.21) and (4.22) are related to the question of time reversal of diffusion processes. In particular, if $\sigma \equiv 1$ it is known that the time reversed process, (X_{T^*-t}) satisfies the above SDE weakly on $[0, T^*)$ under some mild conditions. The SDE representation for the reversed process was obtained by Föllmer in [52] using entropy methods in both Markovian and non-Markovian case, and by Hausmann and Pardoux [64] via weak solutions of backward and forward Kolmogorov equations. Later Millet et al. [93] extended the results of Hausmann and Pardoux by means of Malliavin calculus to obtain the necessary and sufficient conditions for the reversibility of diffusion property. This problem was also tackled with the enlargement of filtration techniques by Pardoux [95].

Finally, the weak conditioning when $T^* = \infty$ can be interpreted as penalisation on the canonical space (see [103] and [104] for a review of the topic).

Chapter 5
Dynamic Bridges

In this chapter we will extend the notion of a Markov bridge to the case when the final bridge condition or the length of the bridge is not known in advance but revealed via an observation of a related process. We will call such a process dynamic Markov bridge. We provide conditions under which such a process exists as a unique solution of an SDE. This construction will be fundamental in solving the Kyle–Back models considered in the second part of the book.

Suppose that B is a standard Brownian motion and Z is a standard normal random variable independent of B. Theorem 4.7 implies that the SDE

$$dX_t = dB_t + \frac{Z - X_t}{1 - t}\,dt$$

has a unique weak solution with $\lim_{t \to 1} X_t = Z$. Moreover, the distribution of Z conditional on $\mathscr{F}_t^X$ is normal with mean X_t and variance $1 - t$ in view of Theorem 3.4. Then Corollary 3.1 yields that X is a Brownian motion in its own filtration. Since the drift coefficient is Lipschitz for t away from 1, the strong uniqueness holds which in turn yields the existence of a unique strong solution to this equation.

Now, suppose that there exists another standard Brownian motion, β, independent of B and we are interested in constructing a process adapted to the joint filtration of B and β satisfying the following:

1. It ends up at β_1 at $t = 1$,
2. it is a Brownian motion in its own filtration,
3. its quadratic covariation with β is 0.

Observe that this construction is a generalisation of the above static bridge with random terminal condition. In particular, the last condition is analogous to V being independent of B.

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A naive approach would be to consider the following SDE:

$$dX_t = dB_t + \frac{\beta_t - X_t}{1-t}\,dt,$$

which has a solution given by

$$X_t = (1-t)\int_0^t \frac{\beta_s}{(1-s)^2}ds + (1-t)\int_0^t \frac{1}{1-s}dB_s.$$

It is easy to see that the first term above converges to β_1 while the second tends to 0. Also by construction the quadratic covariation of X with β is 0. However, X is not a Brownian motion in its own filtration. One way to see this is to use Theorem 3.4 in conjunction with Corollary 3.1. Another way is to observe that a process satisfying all of the conditions above cannot exist.

Indeed, if X is a process with properties 1)–3) as above, then

$$E[X_t\beta_t] = E[X_t E[\beta_1|\mathscr{F}_t^{B,\beta}]] = E[X_t\beta_1] = E[X_t X_1] = E[X_t^2] = t.$$

This implies $E(X_t - \beta_t)^2 = E[X_t^2] + E[\beta_t^2] - 2E[X_t\beta_t] = 0$, which violates the condition that $[X, \beta] = 0$. Hence, such a construction is not feasible.

To make the sought construction possible let us consider an alternative formulation by substituting β with a mean-zero Gaussian process, Z, with Z_1 having a standard normal distribution. To be more precise, let $Z_t = \beta_{V(t)}$ for some deterministic function V with $V(t) > t$ for all $t \in (0, 1)$ and $V(1) = 1$. Note that in the case $V(t) \le t$ for all $t \in [0, 1]$ the desired construction is still not possible since one would have $E(X_t - Z_t)^2 = V(t) - t \le 0$ by following the above arguments. We will see in this chapter that (under some other technical conditions on V) the solution to the SDE

$$dX_t = dB_t + \frac{Z_t - X_t}{V(t) - t}\,dt$$

is a Brownian motion in its own filtration and ends up at Z_1 at time 1. One can call this process a *dynamic Brownian bridge* due to the fact that the final bridge condition is not known in advance. In the first part of this chapter we will study construction of dynamic bridges in the context of one-dimensional Markov processes. The second part is devoted to the construction of a Brownian motion whose first hitting time of zero is determined by the first hitting time of a given process.

5.1 Dynamic Markov Bridges

Let A_t be the operator defined by

$$A_t = \frac{1}{2}\sigma^2(t)a(V(t), z)\frac{\partial^2}{\partial z^2},$$

where σ, a and V satisfy the following

Assumption 5.1 *Fix a real number* $c \in [0, 1]$. $\sigma : [0, 1] \mapsto \mathbb{R}_+$ *and* $a : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}_+$ *are two measurable functions such that:*

1. $V(t) := c + \int_0^t \sigma^2(u)du > t$ *for every* $t \in (0, 1)$, *and* $V(1) = 1$;
2. $\int_0^t \frac{1}{(V(s)-s)^2} ds < \infty$ *for all* $t \in [0, 1)$;
3. $\lim_{t\uparrow 1} \lambda^2(t)\Lambda(t)\log(\Lambda(t)) = 0$, *where* $\lambda(t) = \exp\left\{-\int_0^t \frac{1}{V(s)-s} ds\right\}$ *and* $\Lambda(t) = \int_0^t \frac{1+\sigma^2(s)}{\lambda^2(s)} ds$;
4. σ^2 *is bounded on* $[0, 1]$;
5. a *is strictly positive;*
6. $a \in \mathbb{C}^{1,2}([0.1] \times \mathbb{R})$ *and has enough regularity in order for the local martingale problem for* A *to be well posed.*

Remark 5.1 Notice that Assumption 5.1.2 is, in fact, an assumption on $\Lambda(t)$, since we always have that $\lim_{t\uparrow 1} \lambda(t) = 0$. Indeed, since $V(t)$ is increasing and $V(1) = 1$, we have that $V(t) \le 1$ for $t \in [0, 1]$ and therefore

$$\lambda(t) \le 1 - t \tag{5.1}$$

which leads to conclusion that $\lim_{t\uparrow 1} \lambda(t) = 0$. For another use of this assumption and further discussion see [44].

Although Assumption 5.1.3 seems to be involved, it is satisfied in many cases. For example, when $\lim_{t\uparrow 1} \Lambda(t) < \infty$ the condition is automatically satisfied due to above. Next, suppose that $\lim_{t\uparrow 1} \Lambda(t) = \infty$, σ is continuous in a vicinity of 1 and $\sigma(1) \neq 1$. Then, an application of de L'Hôpital rule yields

$$0 \le \lim_{t\uparrow 1} \frac{\Lambda(t)\log(\Lambda(t))}{\lambda^{-2}(t)} = \frac{1+\sigma^2(1)}{2} \lim_{t\uparrow 1} \frac{\log(\Lambda(t))}{(V(t)-t)^{-1}}.$$

Then note that since $\lim_{t\uparrow 1} \lambda^2(t)\Lambda(t) = 0$, $\Lambda(t) \le \lambda^{-2}(t)$ for t close to 1. Thus,

$$\begin{aligned} 0 \le \lim_{t\uparrow 1} \frac{\Lambda(t)\log(\Lambda(t))}{\lambda_t^{-2}} &\le \frac{1+\sigma^2(1)}{2} \lim_{t\uparrow 1} \frac{\log(\lambda^{-2}(t))}{(V(t)-t)^{-1}} \\ &= (1+\sigma^2(1)) \lim_{t\uparrow 1} \frac{\int_0^t \frac{1}{V(s)-s} ds}{(V(t)-t)^{-1}} = 0, \end{aligned}$$

after another application of de L'Hôpital rule since $\sigma^2(1) \neq 1$. This in particular shows that Assumption 5.1.3 is satisfied when σ is a constant.

The last assumption entails that if we are given a filtered probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\in[0,1]}, \mathbb{Q})$ which is large enough to support two independent standard Brownian motions, B and β, and a random variable $Z_0 \in \mathscr{G}_0$ with distribution, μ, and such that $\mathscr{G}_0$ contains the null sets of $\mathbb{Q}$, then there exists a unique strong solution to

$$Z_t = Z_0 + \int_0^t \sigma(s)a(V(s), Z_s)d\beta_s, \quad t \in (0,1]. \tag{5.2}$$

This is a direct consequence of results presented in Chap. 2. In particular, the equivalence between the well posedness of the local martingale problem and the existence and uniqueness of weak solution of the associated SDE as given in Corollary 2.3 yields a weak solution to (5.2), which can be used to construct a strong solution in view of Theorem 2.12 since the pathwise uniqueness holds due to Theorem 2.6.

In view of the above discussion we fix a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, \mathbb{Q})$ which is large enough to support two independent standard Brownian motions, B and β, and a random variable $Z_0 \in \mathcal{G}_0$ with distribution, μ. We further assume that $\mathcal{G}_0$ contains the null sets of $\mathbb{Q}$.

The rest of this section will be devoted to the construction of a process X satisfying the following three conditions under a suitable assumption on the probability measure μ:

C1 For every $(x,z) \in \mathbb{R}^2$ there exists a unique strong solution to the following system[1]

$$X_t = x + \int_0^t a(s, X_s)dB_s + \int_0^t \alpha(s, X_s, Z_s)ds, \qquad \text{for } t \in (0,1) \tag{5.3}$$
$$Z_t = z + \int_0^t \sigma(s)a(V(s), Z_s)d\beta_s, \qquad \text{for } t \in (0,1)$$

for some Borel measurable real valued function α.

C2 $P^{0,z}(\lim_{t\uparrow 1} X_t = Z_1) = 1$ and, therefore, $X_1 := Z_1$.

C3 $(X_t)_{t\in[0,1]}$ is a $P^{0,\mu}$-local martingale in its own filtration.

The main difficulty with the construction of the process X is that all the conditions C1–C3 have to be met simultaneously. To illustrate this point consider the simple case $a = \sigma = 1$.

If one allows the drift, α, in condition C1 to depend on Z_1, then

$$X_t = B_t + \int_0^t \frac{Z_1 - X_s}{1-s}\,ds \tag{5.4}$$

has a unique strong solution over $[0,1]$ with $X_1 = Z_1$ by Theorem 4.7 due to the independence of B and Z. This is the SDE for a Brownian bridge from 0 to Z_1

[1] Recall from Chap. 2 that this implies the existence and uniqueness of a solution, $P^{0,(x,z)}$, to the associated local martingale problem. In this chapter whenever we refer the solution of the local martingale problem starting from 0 we will drop the reference to the time component and write $P^{x,z}$ instead of $P^{0,(x,z)}$.

over the interval [0, 1]. As we have observed at the beginning of this chapter, it is a martingale in its own filtration.

We have also seen that replacing Z_1 with Z_t does not yield the desired construction. One may be tempted to think that a projection of the solution of (5.4) onto the filtration generated by X and Z could give us the construction that we seek. Note that the solution of (5.4) is adapted to the filtration $(\mathscr{F}_t^{B,Z})$ enlarged with Z_1. In this enlarged filtration Z has the decomposition

$$Z_t = \bar{\beta}_t + \int_0^t \frac{Z_1 - Z_s}{1-s}\, ds$$

where $\bar{\beta}$ is a standard Brownian motion adapted to this filtration (see Example 4.5) and independent of B. Comparison of the SDEs for X and Z reveals an inherent symmetry of these two processes. Thus, the semimartingale decomposition of these two processes with respect to $(\mathscr{F}_t^{X,Z})$ should have a symmetric structure, in particular if one is a martingale with respect to $(\mathscr{F}_t^{X,Z})$ so is the other. However, this is inconsistent with the structural assumption that $[X, Z] \equiv 0$ and $X_1 = Z_1$.

Theorem 4.7 constructs a static bridge with a random terminal condition assuming the existence and smoothness of the transition density of the underlying Markov process among some other technical conditions. As the dynamic bridge is an extension of a static one, one should expect that the SDE representation will still require the existence and smoothness of the transition density of Z. We will show the existence and smoothness of the transition density under further assumptions using well-known results from the theory of partial differential equations (PDEs).

To this end let us introduce the diffusion

$$d\xi_t = a(t, \xi_t)d\beta_t \tag{5.5}$$

and observe that the dynamics of Z can be obtained by applying the deterministic time change, V, to ξ. Therefore, there is a one-to-one correspondence between the transition densities of these two processes.

If a was bounded, Hölder continuous, and bounded away from 0, Corollary 3.2.2 Stroock and Varadhan [109] would imply that there exists a smooth transition density for ξ, which is a *fundamental solution* to the following PDE:

$$w_u(u, z) = \frac{1}{2}\left(a^2(u, z)w(u, z)\right)_{zz}. \tag{5.6}$$

However, as our assumptions on a are weaker, this theorem is not applicable. On the other hand, since a is continuous and strictly positive, the following function

$$K(t, x) := \int_0^x \frac{1}{a(t, y)} dy \tag{5.7}$$

is well defined and the transformation defined by $\zeta_t := K(t, \xi_t)$ will yield, via Ito's formula,

$$d\zeta_t = d\beta_t + b(t, \zeta_t)dt, \tag{5.8}$$

where

$$b(t, x) := K_t(t, K^{-1}(t, x)) - \frac{1}{2}a_z(t, K^{-1}(t, x)), \tag{5.9}$$

and K^{-1}, the inverse of K, is taken with respect to the space variable.

Assumption 5.2 *$b(t, x)$ is absolutely continuous with respect to t for each x, i.e. there exists a measurable function $b_t : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ such that*

$$b(t, x) = b(0, x) + \int_0^t b_t(s, x)\, ds,$$

for each $x \in \mathbb{R}$. Moreover, b, b_x and b_t are uniformly bounded on $[0, 1] \times \mathbb{R}$. Furthermore, b_x is Lipschitz continuous uniformly in t.

If this assumption is enforced, it follows from Theorem 10 in Chap. I of [57] that the fundamental solution, $\Gamma(t, x; u, z)$, to

$$w_u(u, z) = \frac{1}{2}w_{zz}(u, z) - (b(u, z)w(u, z))_z. \tag{5.10}$$

exists and is also the fundamental solution of

$$v_t(t, x) + b(t, x)v_x(t, x) + \frac{1}{2}v_{xx}(t, x) = 0 \tag{5.11}$$

by Theorem 15 in Chap. I of [57]. In particular, $\Gamma \in C^{1,2,1,2}$. Moreover, Corollary 3.2.2 in [109] shows that Γ is the transition density of ζ.

For the reader's convenience we recall the definition of fundamental solution, $\Gamma(t, x; u, z)$, of (5.11) as the function satisfying

1. For fixed (u, z), $\Gamma(t, x; u, z)$ satisfies (5.11) for all $u > t$;
2. For every continuous and bounded $f : \mathbb{R} \mapsto \mathbb{R}$

$$\lim_{t \uparrow u} \int_{\mathbb{R}} \Gamma(t, x; u, z) f(z) dz = f(x). \tag{5.12}$$

Since Γ is the fundamental solution of (5.10) as well, it also satisfies

$$\lim_{u \downarrow t} \int_{\mathbb{R}} \Gamma(t, x; u, z) f(x) dx = f(z). \tag{5.13}$$

The useful fact that the fundamental solution of (5.11) is the transition density follows from (5.12) and the Feynman–Kac representation of the solutions of (5.11) with bounded terminal conditions (see, e.g. Theorem 5.7.6 in [77]). The following proposition summarises the above discussion and other properties of the fundamental solution that will be useful in the sequel.

Proposition 5.1 *Under Assumptions 5.1 and 5.2 there exists a fundamental solution, $\Gamma \in C^{1,2,1,2}$, to (5.11) satisfying*

$$\Gamma(t,x;u,z) \leq C\frac{1}{\sqrt{u-t}}\exp\left(-k\frac{(x-z)^2}{2(u-t)}\right), \text{ and} \tag{5.14}$$

$$|\Gamma_x(t,x;u,z)| \leq C\frac{1}{u-t}\exp\left(-k\frac{(x-z)^2}{2(u-t)}\right), \tag{5.15}$$

for some positive C, depending on k, and any $k < 1$. Moreover, the function $G(t,x;u,z)$ defined by

$$G(t,x;u,z) := \Gamma(t,K(t,x);u,K(u,z))\frac{1}{a(u,z)}, \tag{5.16}$$

satisfies (5.6) for fixed (t,x) and it is the transition density of ξ, i.e.

$$G(t,x;u,z)dz = P(\xi_u \in dz|\xi_t = x) \qquad \text{for } u \geq t.$$

Furthermore, $G_x(t,x;u,z)$ exists and satisfies

$$\int_{\mathbb{R}} G_x(t,x;u,z)dz = 0 = \int_{\mathbb{R}} \Gamma_x(t,x;u,z)dz. \tag{5.17}$$

Proof Define $G(t,x;u,z)$ by (5.16) and observe that $G(t,x;u,z)$ for fixed (t,x) solves (5.6). Since by definition $\zeta_t = K(t,\xi_t)$ and K is strictly increasing

$$\begin{aligned}
G(t,x;u,z)\,dz &= \Gamma(t,K(t,x);u,K(u,z))\frac{1}{a(u,z)}\,dz \\
&= \Gamma(t,K(t,x);u,K(u,z))\,dK(u,z) \\
&= P(\zeta_u \in dK(u,z)|\zeta_t = K(t,x)) \\
&= P(K(u,\xi_u) \in dK(u,z)|K(t,\xi_t) = K(t,x)) \\
&= P(\xi_u \in dz|\xi_t = x),
\end{aligned}$$

which establishes that G is the transition density of ξ.

Moreover, Eqs. (6.12) and (6.13) following Theorem 11 in Chap. I of [57] give the estimates (5.14) and (5.15).

Note that $\int_{\mathbb{R}} G(t,x;u,z)dz = 1 = \int_{\mathbb{R}} \Gamma(t,x;u,z)dz$ since both G and Γ are transition densities. Thus, (5.17) will hold if one can interchange the derivative and the integral. This is justified since, due to (5.15), $|\Gamma_x| \leq k \exp\left(-c_1 z^2\right)$ when x is restricted to a bounded interval and where the constants k and c_1 do not depend on x and might depend on u and t. The result then follows from an application of the Dominated Convergence Theorem. □

Having established the existence of a smooth transition density we can now state the SDE for the dynamic bridge.

Theorem 5.1 *Suppose that Assumption 5.1 and 5.2 are satisfied and* $\mu(dz) = G(0,0;c,z)dz$*, where* $c \in [0,1]$ *is the real number fixed in Assumption 5.1 and* G *is given by (5.16). Let*

$$\alpha(t,x) = a^2(t,x)\frac{\rho_x(t,x,z)}{\rho(t,x,z)}, \tag{5.18}$$

where

$$\rho(t,x,z) := G(t,x;V(t),z). \tag{5.19}$$

Then, the conditions C1–C3 are satisfied.

We will give a proof of this result in the special case $a \equiv 1$ in Sect. 5.1.1 and a proof of the general result will be given in Sect. 5.1.2. However, we shall now state and prove two lemmata to show how the choice of the above drift term would imply condition C3, i.e. X is a local martingale in its own filtration. Before we can formulate them, we need to introduce the following notation.

Let $\mathscr{F} := \sigma(X_t, Z_t; t < 1)$ and define the probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$ by

$$\mathbb{P}(E) = \int_{\mathbb{R}} P^{0,z}(E)\mu(dz), \tag{5.20}$$

for any $E \in \mathscr{F}$. Of course, in order for this construction to make sense we need the condition C1 to be satisfied. This will be proved in Sect. 5.1.2 in Proposition 5.5; thus, the above probability space exists and is well defined. Moreover, the same proposition will show that under $\mathbb{P}$, $(X_t, Z_t)_{t<1}$ is a strong Markov process. Let $\mathscr{N}$ be the null sets of $\mathbb{P}$. Theorem 1.5 shows that the filtration $(\mathscr{N} \vee \mathscr{F}_t^{X,Z})_{t<1}$ is right-continuous. With an abuse of notation we shall still denote the σ-algebra generated by $\mathscr{F}$ and $\mathscr{N}$ with $\mathscr{F}$, and denote $\mathscr{N} \vee \mathscr{F}_t^Y$ with $\mathscr{F}_t^Y$ for any $(\mathscr{F}_t^{X,Z})_{t\leq 1}$-adapted process Y. Next, let $\widetilde{\mathscr{F}}_t^X := \cap_{1>u>t}\mathscr{F}_u^X$. We shall see in Remark 5.2 later that the $\mathscr{F}^X$ is right-continuous, i.e. $\mathscr{F}_t^X = \widetilde{\mathscr{F}}_t^X$. We say that $g_t : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is the conditional density of Z_t given $\mathscr{F}_t^X$, if g_t is measurable with respect to the product σ-algebra, $\mathscr{F}_t^X \times \mathscr{B}$ where $\mathscr{B}$ is the Borel σ-algebra of $\mathbb{R}$, and for any bounded measurable function f

$$\mathbb{E}[f(Z_t)|\mathscr{F}_t^X] = \int_{\mathbb{R}} f(z)g_t(\omega,z)\,dz,$$

where $\mathbb{E}$ is the expectation operator under $\mathbb{P}$. Note that due to Markov property of (X, Z), $\mathbb{E}^{\mathbb{Q}}[f(Z_1)|\mathscr{F}_t^X] = \mathbb{E}[f(Z_1)|\mathscr{F}_t^X]$. We will also often write $\mathbb{P}[Z_t \in dz|\mathscr{F}_t^X] = g_t(\omega, z)\, dz$ in order to refer to the conditional density property described above.

Lemma 5.1 *Suppose that Assumptions 5.1 and 5.2 are satisfied and $\mu(dz) = G(0, 0; c, z)dz$, where $c \in [0, 1]$ is the real number fixed in Assumption 5.1 and G is given by (5.16). Assume further that there exists a unique strong solution of*

$$X_t = \int_0^t a(s, X_s)dB_s + \int_0^t \alpha(s, X_s, Z_s)ds, \tag{5.21}$$

where α is as in (5.18). Let $U_t := K(V(t), Z_t)$ and $R_t := K(t, X_t)$, where K is defined by (5.7). Define

$$r(t, x, z) := \rho(t, K^{-1}(t, x), K^{-1}(V(t), z))a(V(t), K^{-1}(V(t), z)), \tag{5.22}$$

where ρ is given by (5.19). Then,

1. *$r(t, R_t, \cdot)$ is the conditional density of U_t given $\mathscr{F}_t^R$ iff $\rho(t, X_t, \cdot)$ is the conditional density of Z_t given $\mathscr{F}_t^X$.*
2. *$(r(t, R_t, \cdot))_{t\in[0,1)}$ is a weak solution to the following stochastic PDE:*

$$\begin{aligned} g_t(z) &= \Gamma(0, 0; c, z) + \int_0^t \sigma^2(s) \left\{-(b(V(s), z)g_s(z))_z + \frac{1}{2}(g_s(z))_{zz}\right\} ds \\ &+ \int_0^t g_s(z) \left(\frac{r_x(s, R_s, z)}{r(s, R_s, z)} - \int_{\mathbb{R}} g_s(z)\frac{r_x(s, R_s, z)}{r(s, R_s, z)}dz\right) dI_s^g, \end{aligned} \tag{5.23}$$

where

$$dI_s^g = dR_s - \left(\int_{\mathbb{R}} \left[\frac{r_x}{r}(s, R_s, z) + b(s, R_s)\right] g_s(z)dz\right) ds.$$

Proof Notice that since $K(t, \cdot)$ is strictly increasing, $\mathscr{F}_t^R = \mathscr{F}_t^X$ for every $t \in [0, 1)$ and there is a one-to-one correspondence between the conditional density of Z and that of U. More precisely,

$$\mathbb{P}[Z_t \in dz|\mathscr{F}_t^X] = \mathbb{P}[U_t \in dK(V(t), z)|\mathscr{F}_t^R].$$

Thus, if $\mathbb{P}[Z_t \in dz|\mathscr{F}_t^X] = \rho(t, X_t, z)\, dz$, then

$$\begin{aligned} \mathbb{P}[U_t \in dz|\mathscr{F}_t^R] &= \mathbb{P}[Z_t \in dK^{-1}(V(t), z)|\mathscr{F}_t^X] \\ &= \rho(t, X_t, K^{-1}(V(t), z))dK^{-1}(V(t), z) \\ &= \rho(t, X_t, K^{-1}(V(t), z))a(V(t), K^{-1}(V(t), z))\, dz \end{aligned}$$

$$= \rho(t, K^{-1}(t, R_t), K^{-1}(V(t), z))a(V(t), K^{-1}(V(t), z))\, dz$$
$$= r(t, R_t, z)dz$$

by (5.22). The reverse implication can be proved similarly.

In order to prove the second assertion observe that due to (5.16) and (5.22) we have

$$r(t, x, z) = \Gamma(t, x; V(t), z). \tag{5.24}$$

We have seen in Proposition 5.1 that $\Gamma(t, x; u, z)$ solves (5.10) for fixed (t, x) and it also solves (5.11) for fixed (u, z). Combining these two facts yields that r satisfies

$$\begin{aligned} r_t(t, x, z) + b(t, x)r_x(t, x, z) + \frac{1}{2}r_{xx}(t, x, z) \\ = -\sigma^2(t)(b(V(t), z)r(t, x, z))_z + \frac{1}{2}\sigma^2(t)r_{zz}(t, x, z). \end{aligned} \tag{5.25}$$

Using Ito's formula and (5.25), we get

$$\begin{aligned} r(t, R_t, z) = \Gamma(0, 0; c, z) \\ + \int_0^t \sigma^2(s) \left\{-(b(V(s), z)r(s, R_s, z))_z + \frac{1}{2}(r(s, R_s, z))_{zz}\right\} ds \\ + \int_0^t r_x(s, R_s, z)[dR_s - b(s, R_s)ds] \end{aligned}$$

Due to (5.17), $dI_s^g = dR_s - b(s, R_s)ds$ when $g_t(z) = r(t, R_t, z)$. By repeating this argument we arrive at the desired conclusion. □

Lemma 5.2 *Suppose that Assumptions 5.1 and 5.2 are satisfied and* $\mu(dz) = G(0, 0; c, z)dz$, *where* $c \in [0, 1]$ *is the real number fixed in Assumption 5.1 and* G *is given by (5.16). Assume further that there exists a unique strong solution of (5.21). If* $\rho(t, X_t, \cdot)$ *given by (5.19) is the conditional density of* Z_t *given* $\mathscr{F}_t^X$ *for every* $t \in [0, 1)$, *then* $(X_t)_{t\in[0,1)}$ *is a local martingale in its own filtration.*

Proof Let U and R be the processes defined in Lemma 5.1. It follows from an application of Ito's formula that $(X_t)_{t\in[0,1)}$ is a local martingale in its own filtration iff R satisfies

$$dR_t = dB_t^X + b(t, R_t)dt,$$

where B^X is an $\mathscr{F}^X$-Brownian motion.

Note that

$$R_t = B_t + \int_0^t \left\{b(s, R_s) + \frac{r_x(s, R_s, U_s)}{r(s, R_s, U_s)}\right\} ds.$$

Thus, Corollary 3.1 yields

$$dR_t = dB_t^X + \left\{ b(t, R_t) + \mathbb{E}\left[\frac{r_x(t, R_t, U_t)}{r(t, R_t, U_t)} \middle| \mathscr{F}_t^R \right] \right\} dt,$$

where B^X is an $\mathscr{F}^X$-Brownian motion since $\mathscr{F}_t^X = \mathscr{F}_t^R$. However, if $r(t, R_t, \cdot)$ is the conditional density of U_t,

$$\mathbb{E}\left[\frac{r_x(t, R_t, U_t)}{r(t, R_t, U_t)} \middle| \mathscr{F}_t^R \right] = \int_{\mathbb{R}} \Gamma_x(t, R_t; V(t), z) dz = 0$$

due to Proposition 5.1. Therefore, one has

$$dR_t = dB_t^X + b(t, R_t)dt.$$

□

Before we give a proof of Theorem 5.1, we will first investigate the Gaussian case, i.e. $a \equiv 1$.

5.1.1 Gaussian Case

Under the assumption $a \equiv 1$, Z becomes a Gaussian martingale and ξ is a Brownian motion whose transition density is $G(t, x; u, z) = \frac{1}{\sqrt{2\pi(u-t)}} \exp(-\frac{(x-z)^2}{2(u-t)})$. In this case, Eq. (5.21) reduces to

$$dX_t = dB_t + \frac{Z_t - X_t}{V(t) - t} dt. \tag{5.26}$$

This equation along with various properties of its solution is discussed in Danilova [44], Föllmer et al. [53] and Wu [111].

Theorem 5.2 *Suppose $a \equiv 1$ and ρ is given by (5.19). Then, Theorem 5.1 holds.*

The proof of the above theorem will be done in several steps, first of which being the following proposition.

Proposition 5.2 *Suppose $a \equiv 1$, Assumption 5.1 holds, and ρ is given by (5.19). Then, for any $s \in [0, 1)$ and $(x, z) \in \mathbb{R}^2$ there exists a unique strong solution to the system*

$$\begin{aligned} dZ_t &= \sigma(t) d\beta_t, \ Z_s = z; \\ dX_t &= dB_t + \frac{Z_t - X_t}{V(t) - t} dt, \ X_s = x. \end{aligned}$$

In particular, the condition C1 is satisfied. Moreover, $((X_t, Z_t))_{t\in[0,1)}$ *is strong Markov.*

Proof Since $\frac{z-x}{V(t)-t}$ is decreasing in x pathwise uniqueness holds due to Example 2.3. Moreover,

$$X_t = Z_t + \exp\left(-\int_s^t \frac{1}{V(r)-r}dr\right)$$
$$\times\left(x - z + \int_s^t \exp\left(\int_s^u \frac{1}{V(r)-r}dr\right)\{dB_u - \sigma(u)d\beta_u\}\right)$$

can be easily checked to satisfy (5.26). Since the coefficients of (5.26) satisfy Assumption 2.1, X is the unique strong solution in view of Remark 2.6. Moreover, Corollary 2.5 in conjunction with Remark 2.4 yield that (X, Z) has the strong Markov property. □

We next show that the solution to (5.26) satisfies condition C2 and then conclude this section with a proof of Theorem 5.2.

Lemma 5.3 *Suppose* $a \equiv 1$, *Assumption 5.1 holds, and define*

$$\varphi(t, x, z) = \frac{1}{\sqrt{2(\Lambda(t)+\ell)}}e^{\frac{(x-z)^2}{2\lambda^2(t)(\Lambda(t)+\ell)}} \tag{5.27}$$

for some $\ell > 0$. *Then,* $(\varphi(t, X_t, Z_t))_{t\in[0,1)}$ *is a positive* $P^{x,z}$*-supermartingale. Moreover, for any convergent sequence* (t_n, x_n, z_n) *such that* $\lim_{n\to\infty} t_n = 1$, *we have*

$$\lim_{n\to\infty}\varphi(t_n, x_n, z_n) = +\infty, \tag{5.28}$$

whenever $\lim_{n\to\infty}(x_n - z_n) \neq 0$.

Proof Direct calculations give

$$\varphi_t(t, x, z) + \frac{z-x}{V(t)-t}\varphi_x(t, x, z) + \frac{1}{2}\varphi_{xx}(t, x, z) + \frac{\sigma^2(t)}{2}\varphi_{zz}(t, x, z) = 0 \tag{5.29}$$

Thus, it follows from Ito's formula that $\varphi(t, X_t, Z_t)$ is a local martingale. Since it is obviously positive, it is a supermartingale.

In order to prove the convergence property let $(t_n, x_n, z_n) \to (1, x, z)$ and consider the following two cases:

- Case 1: $\lim_{t\uparrow 1}\Lambda(t) < +\infty$. Then, since due to the Remark 5.1 we have that $\lim_{t\uparrow 1}\lambda(t) = 0$, we obtain that

$$\lim_{n\to\infty}\varphi(t_n, x_n, z_n) = \lim_{n\to\infty}\frac{1}{\sqrt{2(\Lambda(t_n)+\ell)}}e^{\frac{(x_n-z_n)^2}{2\lambda^2(t_n)(\Lambda(t_n)+\ell)}} = +\infty \tag{5.30}$$

  if $x \neq z$.

- Case 2: $\lim_{t\uparrow 1} \Lambda(t) = +\infty$. In this case we will have, whenever $x \neq z$,

$$\lim_{n\to\infty} \log \varphi(t_n, x_n, z_n) = \lim_{n\to\infty} \log(2(\Lambda(t_n) + \ell)) \times \left[\frac{(x_n - z_n)^2}{2\lambda^2(t_n)(\Lambda(t_n) + \ell)\log(2(\Lambda(t_n) + \ell))} - \frac{1}{2} \right] = +\infty$$

where the last equality is due to Assumption 5.1.3. Indeed, the condition yields that

$$\lim_{t\uparrow 1} \lambda^2(t) 2(\Lambda(t) + \ell)\log(2(\Lambda(t) + \ell)) = 0$$

since when $\lim_{t\uparrow 1} \Lambda(t) = \infty$, $\lim_{t\uparrow 1} \frac{(\Lambda(t)+\ell)\log(2(\Lambda(t)+\ell))}{\Lambda(t)\log(\Lambda(t))} = 1$. □

Proposition 5.3 *Suppose $a \equiv 1$ and Assumption 5.1 holds. Then condition C2 is satisfied.*

Proof Let $M_t := \varphi(t, X_t, Z_t)$. Then, $M = (M_t)_{t\in[0,1)}$ is a positive $P^{0,z}$-supermartingale by the previous lemma. Due to the supermartingale convergence theorem, there exists an $M_1 \geq 0$ such that $\lim_{t\uparrow 1} M_t = M_1$, $P^{0,z}$-a.s. Using Fatou's lemma and the fact that M is a supermartingale, we have

$$M_0 \geq \liminf_{t\uparrow 1} E^{0,z}[M_t] \geq E^{0,z}[M_1] = E^{0,z}\left[\lim_{t\uparrow 1} \varphi(t, X_t, Z_t)\right],$$

where $E^{0,z}$ is the expectation operator with respect to $P^{0,z}$. Since M_0 is finite, $P^{0,z}(E) = 1$ where $E := \{\omega : \lim_{t\uparrow 1} \varphi(t, X_t(\omega), Z_t(\omega)) < \infty$ and Z is continuous.$\}$. Fix an $\omega \in E$. There exists a sequence (t_n), possibly depending on ω, converging to 1 such that $\lim_{n\to\infty} X_{t_n}(\omega) = \liminf_{t\to 1} X_t(\omega)$. Another application of the previous lemma now yields that $\liminf_{t\to 1} X_t(\omega) = Z_1(\omega)$. Similarly, $\limsup_{t\to 1} X_t(\omega) = Z_1(\omega)$. Since $P^{0,z}(E) = 1$, we have $P^{0,z}(\lim_{t\uparrow 1} X_t = Z_1) = 1$. □

Proposition 5.4 *Let $a \equiv 1$, $\mu(dz) = G(0, 0; c, z)dz$ where $c \in [0, 1]$ is the real number fixed in Assumption 5.1 and G is given by (5.16). Then,*

$$\mathbb{P}[Z_t \in dz|\mathscr{F}_t^X] = G(t, X_t; V(t), z)\, dz.$$

Proof It follows from Theorem 3.4 that the conditional distribution of Z_t given $\mathscr{F}_t^X$ is Gaussian and, thus, it suffices to find the conditional mean $\widehat{Z}_t$ and the variance γ_t in order to characterise the distribution completely. The same theorem yields

$$d\widehat{Z}_t = \frac{\gamma_t}{V(t) - t} \left\{ dX_t - \frac{\widehat{Z}_t - X_t}{V(t) - t} dt \right\}, \tag{5.31}$$

and

$$\frac{d\gamma_t}{dt} = \sigma^2(t) - \frac{\gamma_t}{V(t) - t}, \tag{5.32}$$

with the initial conditions $\widehat{Z_0} = \mathbb{E}[Z_0] = 0$ and $\gamma_0 = c$ due to the choice of μ. In particular, γ_t is deterministic. One can verify directly that $\gamma_t = V(t) - t$ satisfies (5.32) and the initial condition since $V(0) = c$ by Assumption 5.1. Thus, (5.31) yields

$$d(\widehat{Z}_t - X_t) = -\frac{\widehat{Z}_t - X_t}{V(t) - t} dt.$$

However, as $\widehat{Z_0} = X_0 = 0$, the unique solution to the above ODE is 0. Thus, $\widehat{Z}_t = X_t$, which in turn yields that the conditional distribution of Z_t is Gaussian with mean X_t and variance $V(t) - t$. Note that the density associated with this distribution is given by $G(t, X_t; V(t), \cdot)$ when $a \equiv 1$. □

Proof of Theorem 5.2 Propositions 5.2 and 5.3 establish that conditions C1 and C2 are satisfied. Finally, C3 is satisfied as well due to Proposition 5.4 in view of Lemma 5.2. □

5.1.2 The General Case

We now go back to proving Theorem 5.1. The proof is structured in several steps in the following way. We first show that there exists a strong solution, which is also Markov, to the system of SDEs given by and (5.3) on the time interval [0, 1). Then we show that $\lim_{t\uparrow 1} X_t$ exists and equals Z_1, $P^{0,z}$-a.s. implying that there is no explosion until time 1 so that the solution can be continuously extended to the whole interval [0, 1] and satisfies the bridge condition. Then we characterise the $\mathscr{F}_t^X$-conditional distribution of Z_t that satisfies (5.2) and identify it with $\rho(t, X_t, \cdot)$, which will in turn imply that X is a local martingale in its own filtration via Lemma 5.2. Finally, we provide an application of our method to the construction of Ornstein–Uhlenbeck bridges.

5.1.2.1 Existence of a Strong Solution on the Time Interval [0, 1) and the Bridge Property

Recall from Lemma 5.1 that $U_t = K(V(t), Z_t)$, $R_t = K(t, X_t)$ where K is defined in (5.7) and ρ is related to r via (5.22). Since K is strictly increasing, the existence and uniqueness of a strong solution to the system of SDEs given by (5.3) and the $P^{0,z}$-convergence of X_t to Z_1 are equivalent to the existence and uniqueness of a strong solution to the following system, which can be obtained by an application of Ito's formula,

$$\begin{aligned} dU_t &= \sigma(t)d\beta_t + \sigma^2(t)b(t, U_t)dt, \qquad U_0 = K(c, z), \\ dR_t &= dB_t + \left\{\frac{r_x(t, R_t, U_t)}{r(t, R_t, U_t)} + b(t, R_t)\right\}dt, \qquad R_0 = K(0, x), \end{aligned} \tag{5.33}$$

and convergence of R_t to U_1.

First observe that due to (5.24), we have $\frac{r_x(t,x,z)}{r(t,x,z)} + b(t, x) = \frac{\Gamma_x(t,x;V(t),z)}{\Gamma(t,x;V(t),z)} + b(t, x)$. Due to the assumptions on b and the properties of Γ this term will be locally Lipschitz provided Γ is bounded away from 0 on compacts. This in turn will show that C1 holds once we show that there is no explosion. The following lemma due to Aronson [8] gives such a lower bound.

Lemma 5.4 *There exist positive constants α_1, α_2, M_1 and M_2 such that*

$$M_1\, q(\alpha_1(u-t), x, z) \le \Gamma(t, x; u, z) \le M_2\, q(\alpha_2(u-t), x, z),$$

for all $(x, z) \in \mathbb{R}^2$ and $u > t$, where q is the transition density of Brownian motion, i.e.

$$q(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}.$$

Lemma 5.5 *Let $s \in [0, 1)$ and suppose that Assumptions 5.1 and 5.2 are satisfied. Then there exists a unique strong solution to the following system of SDEs for any $(x, z) \in \mathbb{R}^2$.*

$$\begin{aligned} dU_t &= \sigma(t)d\beta_t + \sigma^2(t)b(t, U_t)dt, \qquad U_s = K(V(s), z), \\ dR_t &= dB_t + \left\{\frac{r_x(t, R_t, U_t)}{r(t, R_t, U_t)} + b(t, R_t)\right\}dt, \qquad R_s = K(s, x), \end{aligned}$$

Proof Recall that $r(t, x, z) = \Gamma(t, x; V(t), z)$ (see (5.24)) and define

$$h(t, x; u, z) := \frac{\Gamma(t, x; u, z)}{q(u-t, x, z)}.$$

This yields that

$$\frac{r_x(t, x, z)}{r(t, x, z)} + b(t, x) = \frac{z-x}{V(t)-t} + \frac{h_x(t, x; V(t), z)}{h(t, x; V(t), z)} + b(t, x).$$

Recall from Proposition 5.2 that there exists a unique strong solution to

$$\begin{aligned} dU_t &= \sigma(t)d\beta_t, \qquad U_s = K(V(s), z), \\ dR_t &= dB_t + \frac{U_t - R_t}{V(t)-t}dt, \qquad R_s = K(s, x). \end{aligned}$$

Let $\bar{\mathbb{Q}} \sim \mathbb{Q}$ be the probability measure defined by

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} :=\mathscr{E}\left(\int_0^\cdot \sigma(s)b(s, U_s)d\beta_s\right)_1 \mathscr{E}\left(\int_0^\cdot \left\{b(s, R_s)+\frac{h_x(s, R_s; V(s), Z_s)}{h(s, R_s; V(s), Z_s)}\right\} dB_s\right)_1.$$

$\bar{\mathbb{Q}}$ is indeed equivalent to $\mathbb{Q}$ since b is bounded by assumption and, as we will see later in Proposition 5.6, $\frac{h_x(t,x;V(t),z)}{h(t,x;V(t),z)}$ is uniformly bounded on $[0, 1]\times\mathbb{R}\times[0, 1]\times\mathbb{R}$. This yields a weak solution to the system and also shows that the solution is unique in law.

To establish the statement of the lemma it remains to show the pathwise uniqueness for the system. Let (U^i, R^i) for $i = 1, 2$ be two strong solutions. Due to the already established uniqueness in law any strong solution does not explode. Moreover, it is clear that $U^1 = U^2$. In view of Ito-Tanaka formula

$$\begin{aligned} R_t^1 \vee R_t^2 = R_t^1 + \int_s^t \mathbf{1}_{[R_u^2 > R_u^1]} d(R^2 - R^1)_u + \frac{1}{2} L_t^0(R^2 - R^1) = K(s, x) + B_t - B_s \\ + \int_s^t \left\{\frac{r_x(u, R_u^1 \vee R_u^2, U_u)}{r(u, R_u^1 \vee R_u^2, U_u)} + b(u, R_u^1 \vee R_u^2)\right\} dt + L_t^0(R^2 - R^1), \end{aligned}$$

where $L^0(R^2 - R^1)$ is the local time of $R^2 - R^1$ at 0. However, $L^0(R^2 - R^1) \equiv 0$ since $R^2 - R^1$ is a continuous process of finite variation (see, e.g. Corollary VI.1.9 in [100]). This in turn shows that $(U^1, R^1 \vee R^2)$ is also a solution of the system. Due to the uniqueness in law we deduce that $R^1 \equiv R^2$ and that pathwise uniqueness holds. We can then conclude via Theorem 2.12.

Proposition 5.5 *Suppose that Assumptions 5.1 and 5.2 are satisfied. Then C1 and C2 hold. Moreover,* (X, Z) *has the strong Markov property.*

Proof That C1 holds is a consequence of Lemma 5.5 and the one-to-one relationship between (Z, X) and (U, R).

To show the limiting property let (U, R) be the unique strong solution of (5.33) and define a probability measure, $\bar{\mathbb{Q}}$ via

$$\begin{aligned} \frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} := \mathscr{E}\left(-\int_0^\cdot \sigma(s)b(s, U_s)d\beta_s\right)_1 \mathscr{E} \\ \times\left(-\int_0^\cdot \left\{b(s, R_s) + \frac{h_x(s, R_s; V(s), Z_s)}{h(s, R_s; V(s), Z_s)}\right\} dB_s\right)_1. \end{aligned}$$

Since b is bounded by assumption and $\frac{h_x}{x}$ is uniformly bounded by Proposition 5.6, $\bar{\mathbb{Q}} \sim \mathbb{Q}$. Moreover, (U, R) satisfy under $\mathbb{Q}$

$$\begin{aligned} dU_t &= \sigma(t)d\bar{\beta}_t, \qquad U_0 = K(c, z), \\ dR_t &= d\bar{B}_t + \frac{U_t - R_t}{V(t) - t}dt, \qquad R_0 = K(0, x). \end{aligned}$$

Proposition 5.3 yields that $\bar{Q}(\lim_{t\to 1} R_t = U_1) = 1$. Thus C2 holds due to the equivalence of $\bar{\mathbb{Q}}$ and $\mathbb{Q}$ and the one-to-relationship between (Z, X) and (U, R).

Due to Lemma 5.4 and Assumption 5.2, $\frac{r_x(t,x,z)}{r(t,x,z)} + b(t, x)$ is locally Lipschitz for $t \in [0, T]$ for any $T < 1$. Indeed, Lemma 5.4 gives a lower bound for r. This implies that $\frac{r_x}{r}(t, x, z)$ has locally bounded derivatives with respect to x and z since Γ has continuous second derivatives. Moreover, b is Lipschitz by assumption, thus the claim holds. Note that we require $T < 1$ so that $V(t) - t$ is bounded away from 0 for $t \in [0, T]$.

The strong Markov property of (X, Z) follows from Corollary 2.5 in conjunction with Remark 2.4. □

The following technical result will complete the proof of the above proposition.

Proposition 5.6 *Suppose Assumptions 5.1 and 5.2 are satisfied. Consider*

$$h(t, x; u, z) := \frac{\Gamma(t, x; u, z)}{q(u - t, x, z)}. \tag{5.34}$$

Then $\frac{h_x}{h} : [0, 1] \times \mathbb{R} \times [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ *is uniformly bounded.*

Proof In order to obtain estimates on the function h_x/h we will follow the approach employed in [14]. For this purpose define the martingale L by

$$dL_u = -L_u b(u, \zeta_u) d\beta_u, \qquad u \geq t$$

with $L_t = 1$ and let

$$I(u, z) := \int_0^z b(u, y)dy \qquad N_u := \int_t^u \left\{ I_t(s, \zeta_s) + \frac{1}{2} b_x(s, \zeta_s) + \frac{1}{2} b^2(s, \zeta_s) \right\} ds.$$

Recall that $\zeta_s = K(s, \xi_s)$ and $d\zeta_s = d\beta_s + b(s, \zeta_s)ds$. The above is well defined due to Assumption 5.2.

Then,

$$L_u^{-1} = \exp\{I(u, \zeta_u) - I(t, \zeta_t) - N_u\}$$

and a straightforward application of Girsanov's theorem yields the equivalent measure $\mathbb{W}_t^x$ under which ζ is standard Brownian motion starting at x at time t. Moreover,

$$\Gamma(t, x; u, z) = \exp(I(u, z) - I(t, x))\, \mathbb{W}_t^x[\exp(-N_u)|\zeta_u = z] q(u - t, x, z),$$

where the expectation operator is also denoted by $\mathbb{W}_t^x$, and $\mathbb{W}_t^x(\cdot|\zeta_u = z)$ corresponds to the regular conditional probability for $\mathcal{G}_u$ given ζ_u evaluated at z. Therefore, (5.34) becomes

$$h(t, x; u, z) = \exp(I(u, z) - I(t, x))\, \mathbb{W}_t^x[\exp(-N_u)|\zeta_u = z],$$

which in particular yields the differentiability of $\mathbb{W}_t^x[\exp(-N_u)|\zeta_u = z]$ with respect to x.

Observe that $\frac{\partial I(t,x)}{\partial x} = b(t,x)$, which is bounded. Therefore, the uniform boundedness of h_x/h will follow from that of

$$\frac{\frac{\partial}{\partial x}\mathbb{W}_t^x[\exp(-N_u)|\zeta_u = z]}{\mathbb{W}_t^x[\exp(-N_u)|\zeta_u = z]}.$$

If we can show

$$\frac{\partial}{\partial x}\mathbb{W}_t^x[\exp(-N_u)|\zeta_u = z] = -\mathbb{W}_t^x\left[\exp(-N_u)\frac{\partial N_u}{\partial x}\middle|\zeta_u = z\right], \tag{5.35}$$

$$\frac{\partial N_u}{\partial x} = \int_t^u \left\{b_t(s,\zeta_s) + \frac{1}{2}b_{xx}(s,\zeta_s) + b(s,\zeta_s)\,b_x(s,\zeta_s)\right\}ds, \tag{5.36}$$

then $\frac{\frac{\partial}{\partial x}\mathbb{W}_t^x[\exp(-N_u)|\zeta_u=z]}{\mathbb{W}_t^x[\exp(-N_u)|\zeta_u=z]}$ will be uniformly bounded since (5.36) is uniformly bounded in view of Assumption 5.2. Note that $b_{xx}(t,\cdot)$ exists almost everywhere since $b_x(t,\cdot)$ is absolutely continuous by Assumption 5.2 and bounded due to the uniform Lipschitz condition.

In order to prove (5.36) note that $\zeta_u = x + W_u$ for some Brownian motion with $W_t = 0$ under $\mathbb{W}_t^x$. Since the integrands are differentiable functions with bounded derivatives, this allows us to differentiate under the integral sign. Although derivative exists only almost everywhere, it is no problem since the law of Brownian motion is absolutely continuous with respect to the Lebesgue measure.

For the rest of the proof we will focus on establishing (5.35). To this end consider an infinitely differentiable $f : \mathbb{R} \mapsto \mathbb{R}$ with a compact support. Therefore, if differentiation inside the expectation is justified,

$$\begin{aligned}\frac{\partial}{\partial x}\mathbb{W}_t^x[\exp(-N_u)f(\zeta_u)] &= \mathbb{W}_t^x\left[\frac{\partial}{\partial x}\{\exp(-N_u)f(\zeta_u)\}\right] \\ &= -\mathbb{W}_t^x\left[\frac{\partial N_u}{\partial x}\exp(-N_u)f(\zeta_u)\right] + \mathbb{W}_t^x\left[\exp(-N_u)f'(\zeta_u)\right].\end{aligned} \tag{5.37}$$

As $\frac{\partial N_u}{\partial x}$, f and f' are bounded, we only need to show $\exp(-N_u)$ is bounded by an integrable random variable in order to justify the differentiation. Indeed, using Assumption 5.2 together with the definition of N_u, one can easily prove that $N_u \geq -K(m_u + |x| + k) \geq -K(m_1 + |x| + k)$ for some positive constants k, K, where $m_u = \max_{t\leq s\leq u}|W_s|$. Thus, $\exp(-N_u)$ is bounded from above by the random variable Ce^{Km_1} for some positive constant C, which may depend on x in a continuous fashion. Since $m_1 \leq \max_{t\leq s\leq 1} W_s - \min_{t\leq s\leq 1} W_s$, Cauchy-Schwartz inequality together with the fact that Brownian motion is symmetric yields

$$\mathbb{W}_t^x\left[e^{Km_1}\right] \leq \mathbb{W}_t^x\left[\exp(2K\max_{t\leq s\leq 1} W_s)\right].$$

By the reflection principle for Brownian motion $\max_{t\leq s\leq 1} W_s$ has the same law as $|W_{1-t}|$. Since the random variable $\exp(2K|W_{1-t}|)$ is integrable, we deduce that $\exp(-N_u)$ is bounded, uniformly in u, by an integrable random variable, which does not depend on x when x is restricted to a compact domain. This justifies the differentiation inside the expectation.

On the other hand,

$$\mathbb{W}_t^x\left[\frac{\partial}{\partial x}\mathbb{W}_t^x[\exp(-N_u)|\zeta_u]\,f(\zeta_u)\right] = \mathbb{W}_t^x\left[\frac{\partial}{\partial x}\left\{\mathbb{W}_t^x[\exp(-N_u)|\zeta_u]f(\zeta_u)\right\}\right] \\ -\mathbb{W}_t^{x,q}\left[\mathbb{W}_t^x[\exp(-N_u)|\zeta_u]f'(\zeta_u)\right].$$

Comparing above with (5.37) will yield (5.35) as soon as we show that

$$\frac{\partial}{\partial x}\mathbb{W}_t^x[\exp(-N_u)f(\zeta_u)] = \mathbb{W}_t^x\left[\frac{\partial}{\partial x}\left\{\mathbb{W}_t^x[\exp(-N_u)|\zeta_u]f(\zeta_u)\right\}\right]. \tag{5.38}$$

Following the above argument one can again show that $\mathbb{W}_t^x[\exp(-N_u)|\zeta_u]$ is bounded whenever (x, ζ_u) belongs to a bounded domain. Since f has a compact support, (5.38) will follow if $\frac{\partial}{\partial x}\mathbb{W}_t^x[\exp(-N_u)|\zeta_u]$ is bounded for fixed $u > t$ whenever (x, ζ_u) belongs to a bounded domain in $\mathbb{R}^2$. To see this note that

$$(u-t)\frac{\partial}{\partial x}\log\Gamma(t,x,u,z) = (u-t)\frac{\partial}{\partial x}\log h(t,x,u,z) + z - x \\ = -(u-t)b(t,x) + (u-t)\frac{\frac{\partial}{\partial x}\mathbb{W}_t^x[\exp(-N_u)|\zeta_u = z]}{\mathbb{W}_t^x[\exp(-N_u)|\zeta_u = z]} + z - x.$$

The claim follows from the boundedness of b and $\mathbb{W}_t^x[\exp(-N_u)|\zeta_u]$, (5.15) and Lemma 5.4. Thus, (5.35) holds and the proof is complete. □

5.1.2.2 Conditional Distribution of *Z*

We now turn to proving $\rho(t, X_t, \cdot)$ is the conditional density of Z_t given $\mathscr{F}_t^X$, which will in turn imply that the solution of (5.21) is a local martingale in its own filtration in view of Lemma 5.2. In order to find the conditional density of Z we will first find the conditional density of U given $(\mathscr{F}_t^R)$ and then use Lemma 5.1. Due to the one-to-one correspondence between R and X, we will use $\mathscr{F}^R$ and $\mathscr{F}^X$ interchangeably.

Next, fix a $T < 1$ and let $\mathbb{P}_T := \mathbb{P}|_{\mathscr{F}_T^{X,Z}}$ be the restriction of $\mathbb{P}$ to $\mathscr{F}_T^{X,Z}$. The reason for this restriction is due to the fact that the drift term in (5.33) is not defined at $t = 1$. Note that $\mathbb{P}[U_t \in dz|\mathscr{F}_t^R] = \mathbb{P}_T[U_t \in dz|\mathscr{F}_t^R]$ for $t \in [0, T]$ and, since T is arbitrary, this identity will allow us to obtain all the conditional distributions of Z_t for $t < 1$.

The remainder of this section is devoted to the proof of that

$$\mathbb{P}_T[U_t \in dz|\mathscr{F}_t^R] = r(t, R_t, z)\, dz,$$

where r is defined by (5.22).

Remark 5.2 Let $\overline{\mathbb{P}}_T$ be the absolutely continuous measure on the same space defined by the Radon–Nikodým derivative

$$L_T = \exp\left\{-\frac{1}{2}\int_0^T \left(\frac{r_x(s, R_s, U_s)}{r(s, R_s, U_s)} + b(s, R_s)\right)^2 ds \right.$$
$$\left. - \int_0^T \left(\frac{r_x(s, R_s, U_s)}{r(s, R_s, U_s)} + b(s, R_s)\right) dB_s\right\}.$$

Note that, under $\overline{\mathbb{P}}_T$, R is a Brownian motion independent of U. Let

$$\tau_n := \{t \geq 0 : \max\{|U_t|, |R_t|\} > n\}.$$

Then, it follows that

$$\frac{r_x(s \wedge \tau_n, R_{s\wedge\tau_n}, U_{s\wedge\tau_n})}{r(s \wedge \tau_n, R_{s\wedge\tau_n}, U_{s\wedge\tau_n})} + b(s \wedge \tau_n, R_{s\wedge\tau_n})$$

is bounded due to (5.34) and (5.6) as well as the representation (5.24). Thus, $\mathbb{P}_T$ and $\overline{\mathbb{P}}_T$ are equivalent when restricted to $\mathscr{G}_{T\wedge\tau_n}$. In particular,

$$\overline{\mathbb{E}}_T\left[\mathbf{1}_{[t<\tau_n]}\right] = \mathbb{E}_T\left[\mathbf{1}_{[t<\tau_n]} L_t\right]$$

for $t \leq T$. Since there is no explosion before time 1 for the system of SDEs for (U, R) under $\mathbb{P}$ and $\overline{\mathbb{P}}$, it follows from the monotone convergence theorem that $\mathbb{E}_T[L_t] = 1$ for any $t \leq T$. Thus, $\mathbb{P}_T$ and $\overline{\mathbb{P}}_T$ are equivalent. Since R is a Brownian motion under $\mathbb{P}_T$ and the natural filtration of a Brownian motion is right continuous, this in turn implies that $(\mathscr{F}_t^R)_{t\in[0,T]}$ is right continuous, too.

Recall from Chap. 3 that for any bounded and measurable f we denote by $(\pi_t f)_{t\in[0,T]}$ the $\mathscr{F}^X$-optional projection of $f(Z)$. Moreover, Proposition 3.2 gives the existence of a càdlàg $\mathscr{P}$-valued $\mathscr{F}^X$-optional process $(\pi_t(\omega, \cdot))_{t\in[0,T]}$ such that

$$\pi_t f = \int_{\mathbb{R}} f(z)\pi_t(\omega, dz), \qquad t \in [0, T],$$

where $\mathscr{P}$ is the set of probability measures on the Borel sets of $\mathbb{R}$ topologised by weak convergence.

Let's also recall the *innovation process*

$$I_t = R_t - \int_0^t \pi_s \kappa_s ds,$$

where $\kappa_s(z) := \frac{r_x(s,R_s,z)}{r(s,R_s,z)} + b(s, R_s)$. Note that the system (5.33) satisfies the conditions of Remark 2.8. Thus, both R and U have finite second moments, and consequently $\pi_s\kappa_s$ exists for all $s < 1$ since $\frac{r_x(s,x,z)}{r(s,x,z)} = \frac{z-x}{V(s)-s}$ +*a bounded function*, due to Proposition 5.6.

As we know from Chap. 3 that π solves the Kushner–Stratonovich equation (see Theorem 3.2):

$$\pi_t f = \pi_0 f + \int_0^t \pi_s(A_s f)ds + \int_0^t [\pi_s(\kappa_s f) - \pi_s\kappa_s\pi_s f]\, dI_s, \tag{5.39}$$

for all $f \in C_K^\infty(\mathbb{R})$. As, due to (5.23), $r(t, R_t, \cdot)dz$ solves this equation, it suffices to show that $r(t, R_t, \cdot)dz$ satisfies the conditions of Theorem 3.3 to establish that $r(t, R_t, \cdot)dz$ is the conditional density of U.

Theorem 5.3 *Suppose the conditions in Assumptions 5.1 and 5.2 hold. Then,*

$$\mathbb{P}_T[U_t \in dz|\mathscr{F}_t^R] = r(t, R_t, z)\, dz.$$

Proof The joint local martingale problem for (R, U) is well posed in view of Lemma 5.5 and Assumption 5.1 due to Corollary 2.5. Moreover, Assumption 3.1.i) is satisfied since σ is bounded; Assumption 3.1.ii) holds due to the boundedness of b; and, finally, Assumption 3.1.iii) follows from the square integrability of $(V(t) - t)^{-1}$ on $(0, T)$. Indeed, U is obviously square integrable. Moreover, since

$$\frac{r_x(t, x, z)}{r(t, x, z)} = \frac{z - x}{V(t) - t} + \frac{h_x(t, x; V(t), z)}{h(t, x; V(t), z)}$$

and $\frac{h_x}{h}$ is bounded due to Proposition 5.6, it follows that

$$\mathbb{E}[R_t^2] \le K\left(1 + \int_0^t \frac{\mathbb{E}[R_s^2]}{(V(s) - s)^2} ds\right)$$

since $\sup_{t\le 1} \mathbb{E}[U_t^2] < \infty$ and $\int_0^T \frac{1}{(V(s)-s)^2} ds < \infty$ for $T < 1$. Gronwall inequality now yields that $\sup_{t\le T} \mathbb{E}[R_t^2] < \infty$ for any $T < 1$. Invoking once again the square integrability of $(V(t) - t)^{-1}$ on $(0, T)$ establishes Assumption 3.1.iii).

Furthermore,

$$\int_{-\infty}^{\infty} \left(\frac{r_x(s, R_s, z)}{r(s, R_s, z)} + b(s, R_s)\right) r(s, R_s, z)dz = b(s, R_s)$$

in view of (5.17) and (5.24). The conclusion follows from Theorem 3.3. □

The following corollary is an immediate consequence of Lemma 5.1.

Corollary 5.1 *Suppose the conditions in Assumptions 5.1 and 5.2 hold. Then for any bounded measurable f*

$$\mathbb{E}[f(Z_t)|\mathscr{F}_t^X] = \int_{\mathbb{R}} f(z)\rho(t, X_t, z)\, dz.$$

Proposition 5.5 and the above corollary together with Lemma 5.2 complete the proof of Theorem 5.1.

5.1.2.3 Ornstein–Uhlenbeck Bridges

The boundedness of b assumed in this section is in fact not a necessary condition for the construction of a dynamic bridge. To see this consider an Ornstein–Uhlenbeck type process Z, i.e.

$$dZ_t = \sigma(t)d\beta_t - k\sigma^2(t)Z_t dt,$$

where $k > 0$ is a constant. Note that in this case the fundamental solution of (5.10) is given by

$$\Gamma(s, x; t, z) = q\left(\frac{1 - e^{-2k(t-s)}}{2k}, xe^{-k(t-s)}, z\right).$$

Let X be defined by $X_0 = 0$ and

$$dX_t = dB_t + \left\{2k\frac{Z_t - X_t e^{-k(V(t)-t)}}{e^{k(V(t)-t)} - e^{-k(V(t)-t)}} - kX_t\right\} dt,$$

for $t \in (0, 1)$.

Observe that for $T < 1$

$$\int_0^T \frac{1}{\left(e^{k(V(t)-t)} - e^{-k(V(t)-t)}\right)^2} dt$$
$$= \int_0^T \frac{(V(t) - t)^2}{\left(e^{k(V(t)-t)} - e^{-k(V(t)-t)}\right)^2} \frac{1}{(V(t) - t)^2} dt < \infty$$

since $\frac{V(t)-t}{e^{k(V(t)-t)} - e^{-k(V(t)-t)}}$ is bounded and $\frac{1}{V(t)-t}$ is square integrable on $[0, T]$. It follows from Example 2.4 that there exists a unique strong solution to the above. We claim that if Z_0 has a probability density given by $\Gamma(0, 0; c, \cdot)$, then X is an

Ornstein–Uhlenbeck process in its own filtration and $X_1 = Z_1$, $P^{0,z}$-a.s. under an appropriate modification of Assumption 5.1.3, which we will state later.

Indeed, using the method employed in the proof of Theorem 5.3 we have that $\Gamma(t, x; V(t), z)$ is the conditional density of Z_t given $\mathscr{F}_t^X$ for $t < 1$, and it remains to show that $X_1 = Z_1$, $P^{0,z}$-a.s.

Note that the above convergence will be obtained if one can find a continuous function $b(t)$ with $b(1) = 1$ such that $X_t - b(t)Z_t$ converges to 0 as $t \uparrow 1$. We will choose this new function so that $Y_t := X_t - b(t)Z_t$ defines a Markov process. It can be checked directly that if b satisfies the following ordinary differential equation

$$b'(t) + \gamma(t)b(t) = \theta(t), \tag{5.40}$$

where

$$\begin{aligned}
\gamma(t) &= \frac{e^{-c(V(t)-t)} + e^{c(V(t)-t)}}{e^{c(V(t)-t)} - e^{-c(V(t)-t)}} + c\sigma^2(t), \\
\theta(t) &= \frac{2c}{e^{c(V(t)-t)} - e^{-c(V(t)-t)}},
\end{aligned}$$

and c is the constant in Assumption 5.1, then Y satisfies the following SDE

$$dY_t = dB_t - b(t)\sigma(t)d\beta_t - c\frac{e^{-c(V(t)-t)} + e^{c(V(t)-t)}}{e^{c(V(t)-t)} - e^{-c(V(t)-t)}}Y_t dt. \tag{5.41}$$

The solution to (5.40) with the boundary condition $b(1) = 1$ is given by

$$b(t) = \frac{\int_0^t e^{\int_0^s \gamma(r)dr}\theta(s)ds}{e^{\int_0^t \gamma(r)dr}}.$$

In order to show Y_t converges to 0 as $t \uparrow 1$ consider the function φ defined by

$$\varphi(t, y) := \frac{1}{\sqrt{2(\Lambda(t) + \ell)}} e^{\frac{y^2}{2\lambda^2(t)(\Lambda(t)+\ell)}},$$

where $\ell > 0$ is a constant,

$$\begin{aligned}
\lambda(t) &:= \exp\left(-c\int_0^t \frac{e^{-c(V(s)-s)} + e^{c(V(s)-s)}}{e^{c(V(s)-s)} - e^{-c(V(s)-s)}}ds\right), \qquad \text{and} \\
\Lambda(t) &:= \int_0^t \frac{1 + b^2(s)\sigma^2(s)}{\lambda^2(s)}ds.
\end{aligned}$$

A direct application of Ito's formula gives that $\varphi(t, Y_t)$ is a positive local martingale, hence a super-martingale. If Assumption 5.1 holds with λ and Λ defined above, then we can repeat the proof of Proposition 5.3 using $\varphi(t, y)$ defined above to conclude that Y_t converges to 0 as $t \uparrow 1$.

5.2 Dynamic Bessel Bridge of Dimension 3

As discussed at the beginning of this chapter this section is devoted to the construction of a Brownian motion starting from 1 at time $t = 0$ and conditioned to hit the level 0 for the first time at a given random time.

More precisely, let Z be the deterministically time-changed Brownian motion $Z_t = 1 + \int_0^t \sigma(s) dW_s$ and let B be another standard Brownian motion independent of W. Let us denote by $V(t)$ the associated time-change, i.e. $V(t) = \int_0^t \sigma^2(s) ds$ for $t \geq 0$. Let τ be the first hitting time of Z of the level 0. In the spirit of the previous section our aim is to find an integrable process, α, adapted to the filtration generated by the pair (Z, B) such that the process X defined by $dX_t = dB_t + \alpha_t dt$ with $X_0 = 1$ satisfies the following two properties:

1. X hits level 0 for the first time at time $V(\tau)$;
2. X is a Brownian motion in its own filtration.

The above implies that such a process will be a Brownian motion that hits level 0 for the first time at $V(\tau)$, which necessitates that $V(\tau)$ should have the same distribution as the first hitting time of 0 by a standard Brownian motion starting at 1. It is easy to see that $V(\tau)$ does need to have such a distribution since $Z_t = \beta_{V(t)}$ for some Brownian motion, β, in view of a result that goes back to the works of Dambis and Dubins & Schwarz (see Theorem V.1.6 in [100]).

Recall from Example 4.6 that when conditioned on its first hitting time of 0, a standard Brownian motion behaves as a three-dimensional Bessel bridge. Since in our setting the first hitting time is revealed gradually rather than being known in advance the process X we seek to construct can be viewed as a dynamic version of a three-dimensional Bessel bridge.

The construction of such a process consists of two parts with varying difficulties. The easy part is the construction of this process after time τ. Since V is a deterministic function, the first hitting time of 0 is revealed at time τ. Thus, one can use Example 4.6 to conclude that for $t \in (\tau, V(\tau))$

$$dX_t = dB_t + \left\{ \frac{1}{X_t} - \frac{X_t}{V(\tau) - t} \right\} dt.$$

The difficult part is the construction of X until time τ. Thus, the challenge is to construct a Brownian motion which is conditioned to stay strictly positive until time τ using a drift term adapted to the filtration generated by B and Z.

5.2.1 Main Result

To formulate rigorously the problem let $(\Omega, \mathscr{H}, \mathbb{H} = (\mathscr{H}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. We suppose that $\mathscr{H}_0$ contains only the $\mathbb{P}$-null sets and there exist two independent $\mathbb{H}$-Brownian motions, B and W. Define the process

$$Z_t := 1 + \int_0^t \sigma(s) dW_s, \tag{5.42}$$

and assume for the rest of this section that σ satisfies the following assumption.

Assumption 5.3 *The function $\sigma : \mathbb{R}_+ \mapsto (0, \infty)$ is measurable, locally bounded and satisfies*

1. $V(t) > t$ for every $t > 0$, where

$$V(t) := \int_0^t \sigma^2(s) ds < \infty.$$

2. There exists an $\varepsilon > 0$ such that $\int_0^\varepsilon \frac{1}{(V(t)-t)^2}\, dt < \infty$.

Notice that under these assumptions, Z and W generate the same minimal filtration satisfying the usual conditions. Consider the following first hitting time of Z:

$$\tau := \inf\{t > 0 : Z_t = 0\}, \tag{5.43}$$

where $\inf \emptyset = \infty$ by convention. One can characterise the distribution of τ using the well-known distributions of first hitting times of a standard Brownian motion. To this end let

$$H(t, a) := \mathbb{P}[T_a > t] = \int_t^\infty \ell(u, a)\, du, \tag{5.44}$$

for $a > 0$ where

$$T_a := \inf\{t > 0 : B_t = a\}, \text{ and}$$
$$\ell(t, a) := \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right).$$

In view of the reflection principle for Brownian motion (see, e.g. Proposition III.3.7 in [100] and the discussion that follows it) we have

$$\mathbb{P}[T_a > t | \mathscr{H}_s] = \mathbf{1}_{[T_a > s]} H(t - s, a - B_s), \qquad s < t.$$

Thus, since V is deterministic and strictly increasing, $(Z_{V^{-1}(t)})_{t\geq 0}$ is a standard Brownian motion in its own filtration starting at 1, and consequently

$$\mathbb{P}[\tau > t|\mathscr{H}_s] = \mathbf{1}_{[\tau>s]}H(V(t) - V(s), Z_s). \tag{5.45}$$

Hence,

$$\mathbb{P}[V(\tau) > t] = H(t, 1),$$

for every $t \geq 0$, i.e. $V(\tau) = T_1$ in distribution. Here we would like to give another formulation for the function H in terms of the transition density of a Brownian motion killed at 0. Recall from Example 1.4 that this transition density is given by

$$q(t, x, y) := \frac{1}{\sqrt{2\pi t}}\left(\exp\left(-\frac{(x-y)^2}{2t}\right) - \exp\left(-\frac{(x+y)^2}{2t}\right)\right), \tag{5.46}$$

for $x > 0$ and $y > 0$. Then one has the identity

$$H(t, a) = \int_0^\infty q(t, a, y)\, dy. \tag{5.47}$$

The following is the main result of this section.

Theorem 5.4 *There exists a unique strong solution to*

$$X_t = 1 + B_t + \int_0^{\tau\wedge t} \frac{q_x(V(s) - s, X_s, Z_s)}{q(V(s) - s, X_s, Z_s)}\, ds + \int_{\tau\wedge t}^{V(\tau)\wedge t} \frac{\ell_a(V(\tau) - s, X_s)}{\ell(V(\tau) - s, X_s)}\, ds. \tag{5.48}$$

Moreover,

i) Let $\mathscr{F}_t^X = \mathscr{N} \bigvee \sigma(X_s; s \leq t)$, where $\mathscr{N}$ is the set of $\mathbb{P}$-null sets. Then, X is a standard Brownian motion with respect to $\mathbb{F}^X := (\mathscr{F}_t^X)_{t\geq 0}$;
ii) $V(\tau) = \inf\{t > 0 : X_t = 0\}$.

Before we proceed with the proof of this result let us explain why the assumption $V(t) > t$ is imposed for all $t > 0$.

First, observe that we necessarily must have $V(t) \geq t$ for any $t \geq 0$. This follows from the fact that if the construction in Theorem 5.4 is possible, then $V(\tau)$ is an $\mathscr{F}^{B,Z}$-stopping time since it is an exit time from the positive real line of the process X. Indeed, if $V(t) < t$ for some $t > 0$ so that $V^{-1}(t) > t$, then $[V(\tau) < t]$ cannot belong to $\mathscr{F}_t^{B,Z}$ since $[V(\tau) < t] \cap [\tau > t] = [\tau < V^{-1}(t)] \cap [\tau > t] \notin \mathscr{F}_t^Z$, and that τ is not $\mathscr{F}_\infty^B$-measurable.

We will next see, using arguments similar to the ones employed at the beginning of this chapter, that when $V(t) \equiv t$ construction of a dynamic Bessel bridge is not possible. Similar arguments will also show that $V(t)$ cannot be equal to t in an interval.

To this end consider any process $X_t = 1 + B_t + \int_0^t \alpha_s ds$ for some $\mathbb{H}$-adapted and integrable process α. Assume that X is a Brownian motion in its own filtration and that $\tau = \inf\{t : X_t = 0\}$ a.s. and fix an arbitrary time $t \geq 0$. The two processes $M_s^Z := \mathbb{P}[\tau > t|\mathscr{F}_s^Z]$ and $M_s^X := \mathbb{P}[\tau > t|\mathscr{F}_s^X]$, for $s \geq 0$, are uniformly integrable continuous martingales, the former for the filtration $\mathscr{F}^{Z,B}$ and the latter for the filtration $\mathscr{F}^X$. In this case, Doob's optional sampling theorem can be applied to any pair of finite stopping times, e.g. $\tau \wedge s$ and τ, to get the following:

$$\begin{aligned} M_{\tau\wedge s}^X &= \mathbb{E}[M_\tau^X|\mathscr{F}_{\tau\wedge s}^X] = \mathbb{E}[\mathbf{1}_{\tau>t}|\mathscr{F}_{\tau\wedge s}^X] \\ &= \mathbb{E}[M_\tau^Z|\mathscr{F}_{\tau\wedge s}^X] = \mathbb{E}[M_{\tau\wedge s}^Z|\mathscr{F}_{\tau\wedge s}^X], \end{aligned}$$

where the last equality is just an application of the tower property of conditional expectations and the fact that M^Z is martingale for the filtration $\mathscr{F}^{Z,B}$ which is bigger than $\mathscr{F}^X$. We also obtain

$$\mathbb{E}[(M_{\tau\wedge s}^X - M_{\tau\wedge s}^Z)^2] = \mathbb{E}[(M_{\tau\wedge s}^X)^2] + \mathbb{E}[(M_{\tau\wedge s}^Z)^2] - 2\mathbb{E}[M_{\tau\wedge s}^X M_{\tau\wedge s}^Z].$$

Notice that since the pairs (X, τ) and (Z, τ) have the same law by assumption, the random variables $M_{\tau\wedge s}^X$ and $M_{\tau\wedge s}^Z$ have the same law too. This implies

$$\mathbb{E}[(M_{\tau\wedge s}^X - M_{\tau\wedge s}^Z)^2] = 2\mathbb{E}[(M_{\tau\wedge s}^X)^2] - 2\mathbb{E}[M_{\tau\wedge s}^X M_{\tau\wedge s}^Z].$$

On the other hand, we can obtain

$$\mathbb{E}[M_{\tau\wedge s}^X M_{\tau\wedge s}^Z] = \mathbb{E}[M_{\tau\wedge s}^X \mathbb{E}[M_{\tau\wedge s}^Z|\mathscr{F}_{\tau\wedge s}^X]] = \mathbb{E}[(M_{\tau\wedge s}^X)^2],$$

which implies that $M_{\tau\wedge s}^X = M_{\tau\wedge s}^Z$ for all $s \geq 0$. Using the fact that

$$M_s^Z = \mathbf{1}_{\tau>s} H(t-s, Z_s), \quad M_s^X = \mathbf{1}_{\tau>s} H(t-s, X_s), \quad s < t,$$

one has

$$H(t-s, X_s) = H(t-s, Z_s) \quad \text{on } [\tau > s].$$

Since the function $a \mapsto H(u, a)$ is strictly monotone in a whenever $u > 0$, the last equality above implies that $X_s = Z_s$ for all $s < t$ on the set $[\tau > s]$. t being arbitrary, we have that $X_s^\tau = Z_s^\tau$ for all $s \geq 0$.

Thus, before τ, X and Z coincide, which contradicts the fact that B and Z are independent, so that the construction of a Brownian motion conditioned to hit 0 for the first time at τ is impossible. A possible way out is to assume that the signal process Z evolves faster than its underlying Brownian motion W, i.e. $V(t) \in (t, \infty)$ for all $t \geq 0$ as in our assumptions on σ.

5.2.2 Proof of the Main Result

First observe that in order to show the existence and uniqueness of the strong solution to the SDE in (5.48) it suffices to show these properties for the following SDE

$$Y_t = y + B_t + \int_0^{\tau \wedge t} \frac{q_x(V(s) - s, Y_s, Z_s)}{q(V(s) - s, Y_s, Z_s)} ds, \qquad y \geq 0, \tag{5.49}$$

and that $Y_\tau > 0$. Indeed, as we mentioned earlier, the drift term after τ is the same as that of a three-dimensional Bessel bridge from X_τ to 0 over the interval $[\tau, V(\tau)]$. Note that $V(\tau) = T_1$ in distribution implies that τ has the same law as $V^{-1}(T_1)$ which is finite since T_1 is finite and the function $V(t)$ is increasing to infinity as t tends to infinity. Thus τ is a.s. finite.

By Theorem 2.12 the existence and uniqueness of a strong solution of an SDE are equivalent to the existence of a weak solution and the pathwise uniqueness of strong solutions. Thus, if we can show the pathwise uniqueness for the system of (5.49) and (5.42), the existence of a weak solution to this system will imply the existence and uniqueness of a strong solution to this system.

In the sequel we will often work with a pair of SDEs defining (X, Z) where X is a semimartingale given by an SDE whose drift coefficient depends on Z. In order to simplify the statements of the following results, we will shortly write existence and/or uniqueness of the SDE for X, when we actually mean the corresponding property for the whole system (X, Z).

We start with demonstrating the pathwise uniqueness property.

Lemma 5.6 *Pathwise uniqueness holds for the system of (5.49) and (5.42).*

Proof It follows from direct calculations that

$$\frac{q_x(t,x,z)}{q(t,x,z)} = \frac{z-x}{t} + \frac{\exp\left(-\frac{2xz}{t}\right)}{1 - \exp\left(-\frac{2xz}{t}\right)} \frac{2z}{t}. \tag{5.50}$$

Moreover, $\frac{q_x(t,x,z)}{q(t,x,z)}$ is decreasing in x for fixed z and t. Thus, the claim follows from Example 2.3 and the SDE (5.42) having a unique strong solution due to Theorem 2.7.

The existence of a weak solution will be obtained in several steps. First we show the existence of a weak solution to a class of SDEs that will be instrumental in obtaining the solution to above SDE.

Proposition 5.7 *Consider a filtered probability space satisfying the usual conditions and rich enough to support two independent Brownian motions, β^1 and β^2. Let τ be a finite stopping time and suppose that there exists a family of processes, $(\xi^z)_{z \geq 0}$, satisfying the following for all $z \geq 0$:*

1. $\inf\{t > 0 : \xi_t^z \leq 0\} \geq \tau$, *a.s.*
2.

$$\xi_t^z = z + \int_0^t \kappa(s, \xi_s^z) d\beta_s^1 + \int_0^t \alpha(s, \xi_s^z) ds,$$

where κ and α are Lipschitz and satisfy a linear growth condition.

Then, for any $z \geq 0$ there exists a unique strong solution to

$$Y_t = y + \beta_t^2 + \int_0^{\tau \wedge t} f(V(s) - s, Y_s, \xi_s^z)\, ds \qquad y \geq 0, \tag{5.51}$$

where

$$f(t, x, z) := \frac{\exp\left(-\frac{2xz}{t}\right)}{1 - \exp\left(-\frac{2xz}{t}\right)} \frac{2z}{t}.$$

Moreover, $\inf\{t > 0 : Y_t \leq 0\} > \tau$, *a.s. Furthermore, if we denote by $Y^{y,z}$ the solution to (5.51), then* $\lim_{y \to 0} Y_t^{y,z} = Y_t^{0,z}$, *a.s., for all $t \geq 0$.*

The proof of this proposition will rely on the following lemma.

Lemma 5.7 *Let $(\xi^z)_{z \geq 0}$ be a family of processes satisfying the conditions of Proposition 5.7. For any $z > 0$ there exists a unique strong solution to*

$$dU_t = 2\sqrt{|U_t|} d\beta_t^2 + \left(2\sqrt{|U_t|} f(V(t) - t, \sqrt{|U_t|}, \xi_t^z) + 1\right) dt, \tag{5.52}$$

with $U_0 = y^2 > 0$ up to and including τ. Moreover, the solution is strictly positive on $[0, \tau]$.

Proof For the brevity of exposition we drop the superscript and write ξ instead of ξ^z.

Consider the measurable function $g : \mathbb{R}_+ \times \mathbb{R}^2 \mapsto [0, 1]$ defined by

$$g(t, x, z) = \begin{cases} \sqrt{|x|} f(t, \sqrt{|x|}, z), & \text{for } (t, x, z) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}_+ \\ 0, & \text{for } (t, x, z) \in \{0\} \times \mathbb{R} \times \mathbb{R}_+ \end{cases}, \tag{5.53}$$

and the following SDE:

$$d\tilde{U}_t = 2\sqrt{|\tilde{U}_t|} d\beta_t^2 + \left(2g(V(t) - t, |\tilde{U}_t|, \xi_t) + 1\right) dt. \tag{5.54}$$

Observe that g is jointly continuous on $[0, \infty) \times (0, \infty)^2$. If we can show the existence of a weak solution to (5.54) that stays strictly positive on $[0, \tau]$, then $U = (\tilde{U}_{t \wedge \tau})_{t \geq 0}$ is a strictly positive weak solution to (5.52) up to time τ.

It follows from Theorem 2.8 that there exists a unique strong solution to $(\tilde{U}, \xi)$ until

$$\bar{\tau} := \inf\{t \geq 0 : (\tilde{U}_t, \xi_t) \notin (0, \infty)^2\}.$$

Then, since the range of g is $[0, 1]$, it follows from Theorem 2.10 that the solution to (5.54) is sandwiched between zero- and three-dimensional squared Bessel processes. Thus, $\bar{\tau} = \inf\{t \geq 0 : \xi_t = 0 \text{ or } \tilde{U}_t = 0\}$. Define $\nu := \inf\{t \geq 0 : \tilde{U}_t = 0\}$. We will first show that $\nu \geq \tau$, a.s.

Let a and b be strictly positive numbers such that

$$\frac{ae^{-a}}{1 - e^{-a}} = \frac{3}{4} \qquad \text{and} \qquad \frac{be^{-b}}{1 - e^{-b}} = \frac{1}{2}.$$

As $\frac{xe^{-x}}{1-e^{-x}}$ is strictly decreasing for positive values of x, one has $0 < a < b$. Now define $R_t := \tilde{U}_{t \wedge \bar{\tau}}$ and the stopping time

$$I_0 := \inf\{0 < t \leq \bar{\tau} : \sqrt{R_t}\xi_t \leq \frac{V(t) - t}{2} a\},$$

where $\inf \emptyset = \bar{\tau}$ by convention. As $\sqrt{R_{\bar{\tau}}}\xi_{\bar{\tau}} = 0$ on $[\bar{\tau} < \infty]$, $\sqrt{R_0}\xi_0 = yz$, and $V(t) - t > 0$ for $t > 0$, we have that $0 < I_0 < \bar{\tau}$ on $[\bar{\tau} < \infty]$ by the continuity of (R, ξ) and V. In particular, $R_t > 0$ on the set $[t \leq I_0]$, and therefore $\nu \geq I_0$.

Note that $C_t := \frac{2\sqrt{R_t}\xi_t}{V(t)-t}$ is continuous on $(0, \infty)$ and $C_{I_0} = a$ on $[\bar{\tau} < \infty]$. Thus, $\bar{\tau} = \inf\{t > I_0 : C_t = 0\}$.

Consider the following sequence of stopping times:

$$J_n := \inf\{I_n \leq t \leq \bar{\tau} : C_t \notin (0, b)\}$$
$$I_{n+1} := \inf\{J_n \leq t \leq \bar{\tau} : C_t = a\}$$

for $n \in \mathbb{N} \cup \{0\}$, where $\inf \emptyset = \bar{\tau}$ by convention.

We next show that $\bar{\tau} = \lim_{n \to \infty} J_n$, a.s. As J_ns are increasing and bounded by $\bar{\tau}$, the limit exists and is bounded by $\bar{\tau}$. Suppose that $J := \lim_{n\to\infty} J_n < \bar{\tau}$ with positive probability, which in particular implies J is finite. Note that by construction we have $I_n \leq J_n \leq I_{n+1}$ and, therefore, $\lim_{n\to\infty} I_n = J$. Since C is continuous, one has $\lim_{n\to\infty} C_{I_n} = \lim_{n\to\infty} C_{J_n}$. However, as on the set $[J < \bar{\tau}]$ we have $C_{I_n} = a$ and $C_{J_n} = b$ for all n, we arrive at a contradiction. Therefore, $\bar{\tau} = J$.

Next, we will demonstrate that $\nu \geq \tau$. Suppose $\nu < \tau$ with positive probability. Then $\bar{\tau} = \nu < \tau < \infty$ on this set. We claim that on this set $C_{J_n} = b$ for all n, which will lead to a contradiction since then $b = \lim_{n\to\infty} C_{J_n} = C_{\bar{\tau}} = C_\nu = 0$. We will show our claim by induction.

1. For $n = 0$, recall that $I_0 < \bar{\tau}$ when $\bar{\tau} < \infty$. Also note that on $(I_0, J_0]$ the drift term in (5.54) is greater than 2 as $\frac{xe^{-x}}{1-e^{-x}}$ is strictly decreasing for positive

values of x and due to the choice of a and b. Therefore the solution to (5.54) is strictly positive on $(I_0, J_0]$ in view of Theorem 2.10 since a two-dimensional Bessel process is always strictly positive (see Example 2.5). Thus, $C_{J_0} = b$.

2. Suppose we have $C_{J_{n-1}} = b$. Then, due to the continuity of C, $I_n < \bar{\tau}$. For the same reasons as before, the solution to (5.54) is strictly positive on $(I_n, J_n]$, implying $C_{J_n} = b$.

Thus, we have shown that $\nu \geq \tau$. Now we turn to prove that $\nu > \tau$. The above arguments also show that a.s. $\tilde{U}$ is strictly positive in all the intervals $(I_n, J_n]$. That is, for any $\omega \in \Omega \setminus \mathcal{N}$, where $\mathcal{N}$ is a measurable set with $\mathbb{P}(\mathcal{N}) = 0$, we have C is continuous and $\tilde{U}_t(\omega) > 0$ for all $t \in (I_n, J_n]$ and $n \geq 0$. Fix an $\omega \in \{\omega' \in \Omega \setminus \mathcal{N} : \nu(\omega') = \tau(\omega')\}$ and observe that the continuity of C together with $C_\nu = 0$ on $[\nu < \infty]$ implies the existence of $N(\omega)$ such that $J_{N(\omega)}(\omega) > I_{N(\omega)}(\omega)$ and $J_n(\omega) = I_n(\omega) = \tau(\omega)$ whenever $n > N(\omega)$. Then, $I_{N(\omega)}(\omega) < \tau(\omega)$ and $J_{N(\omega)}(\omega) = \tau(\omega)$, which in turn implies $\tilde{U}_{\tau(\omega)}(\omega) > 0$ and leads to a contradiction.

Proof of Proposition 5.7 Pathwise uniqueness can be shown as in Lemma 5.6 using the monotonicity of f.

First, consider the case $y > 0$, $z > 0$ and let $U^{y,z}$ be a strong solution of (5.52). It follows from an application of Ito's formula that $\sqrt{U^{y,z}}$ is a strong solution of (5.51) on $[0, \tau]$ since $U^{y,z}$ is strictly positive on $[0, \tau]$. The global solution, denoted by $Y^{y,z}$, can now be easily constructed by the addition of $B_t - B_\tau$ after τ. This further implies that $Y^{y,z}$ is strictly positive on $[0, \tau]$ since $\sqrt{U^{y,z}}$ is clearly strictly positive.

We will next construct a solution of (5.51) by taking the limits of $Y^{y,z}$ as $z \to 0$ for a fixed $y > 0$. To do so let us first show that $U^{y,z_1} \geq U^{y,z_2}$ on $[0, \tau]$ whenever $0 < z_1 \leq z_2$. Recall from (5.53) that

$$dU_t^{y,z} = 2\sqrt{U_t^{y,z}}d\beta_t^2 + \left(2g(V(t) - t, U_t^{y,z}, \xi_t^z) + 1\right) dt$$

and observe that $\forall (t, x, z) \in (0, \infty) \times (\frac{1}{n}, \infty) \times [0, \infty)$

$$|g(V(t) - t, x, z) - g(V(t) - t, y, z)| \leq C_n \frac{\sqrt{z}}{V(t) - t}|x - y|, \tag{5.55}$$

for every n, where C_n is a constant independent of t and z. Let $\nu_m := \inf\{t \geq 0 : \xi_t^{z_1} \notin (0, m) \text{ or } \xi_t^{z_2} \notin (0, m)\}$, $\tau_n(z_i) := \inf\left\{t \geq 0 : U_t^{y,z_i} \leq \frac{1}{n}\right\}$, and $\tau_n = \tau_n(z_1) \wedge \tau_n(z_2)$. Thus, since g is decreasing in z for fixed $x \geq 0$ and $t > 0$, Theorem 2.11 yields $\inf_{t\in[0,T]} \left(U_{t\wedge\nu_m\wedge\tau_n}^{y,z_1} - U_{t\wedge\nu_m\wedge\tau_n}^{y,z_2}\right) \geq 0$ a.s. for any $T > 0$ since

$$\int_0^T \frac{1}{V(t) - t} dt < \infty$$

by Assumption 5.3. Note that $\lim_{n\to\infty} \tau_n > \tau$ and $\lim_{m\to\infty} \nu_m \geq \tau$. Thus, taking limits yields that $U^{y,z_1} \geq U^{y,z_2}$ on $[0, \tau]$ whenever $0 < z_1 \leq z_2$. This allows us to define $U^{y,0}_{t\wedge\tau} := \lim_{z\downarrow 0} U^{y,z}_{t\wedge\tau}$. Note that the limit is finite since $U^{y,z}$ is bounded from above by a three-dimensional squared Bessel process as observed above.

Recall g takes values in $[0, 1]$. Thus, the Dominated Convergence Theorem yields

$$\lim_{z\downarrow 0}\int_0^{t\wedge\tau} \left(2g(V(s)-s, U^{y,z}_s, \xi^z_s) + 1\right) ds = \int_0^{t\wedge\tau} \left(2g(V(s)-s, U^{y,0}_s, \xi^0_s) + 1\right) ds.$$

Similarly, since $U^{y,z}_{t\wedge\tau} \le U^{y,0}_{t\wedge\tau}$ for all $z > 0$, and $U^{y,0}$ is bounded by a three-dimensional squared Bessel process starting from y, the Dominated Convergence Theorem for stochastic integrals (see, e.g. Theorem 32 in Chap. IV of [99]) yields

$$\lim_{z\downarrow 0}\int_0^{t\wedge\tau} \sqrt{U^{y,z}_s}\, d\beta^2_s = \int_0^{t\wedge\tau} \sqrt{U^{y,0}_s}\, d\beta^2_s,$$

where the limit is taken in *ucp* (uniformly on compact time intervals in probability). This is in fact an a.s. limit since both $U^{y,z}_t$ and the ordinary integral converge a.s. Thus, we conclude that for every $t \ge 0$

$$U^{y,0}_{t\wedge\tau} = y + \int_0^{t\wedge\tau} 2\sqrt{U^{y,0}_s}\, d\beta^2_s + \int_0^{t\wedge\tau} \left(2g(V(s) - s, U^{y,0}_s, \xi^0_s) + 1\right) ds,$$

and in particular $(U^{y,0}_{t\wedge\tau})_{t\ge 0}$ is a continuous semimartingale for all $y > 0$. Applying Ito formula to $\sqrt{U^{y,0}}$ yields the solution to (5.51) for $z = 0$ in view of the strict positivity of $U^{y,z}$ on $[0, \tau]$ for $z > 0$ and the fact that $z \mapsto U^{y,z}$ is decreasing.

Our next goal is to define $U^{0,z}$ for any $z \ge 0$. We will do this by considering $\lim_{y\downarrow 0} U^{y,z}$ for a given $z \ge 0$. We will invoke Theorem 2.11 to first establish that $U^{y,z}$ is a decreasing sequence of processes as $y \downarrow 0$. Indeed, since g satisfies (5.55), Theorem 2.11 yields $\sup_{t\in[0,T]}\left(U^{x,z}_{t\wedge\nu_m\wedge\tau_n} - U^{y,z}_{t\wedge\nu_m\wedge\tau_n}\right) \le 0$, a.s., where $\nu_m := \inf\{t > 0 : \xi^z_t \notin (0, m)\}$, $\tau_n(x) := \inf\left\{t \ge 0 : U^{x,z}_t \le \frac{1}{n}\right\}$, and $\tau_n = \tau_n(x) \wedge \tau_n(y)$. Thus, $U^{0,z} = \lim_{y\downarrow 0} U^{y,z}$ is well defined. Same arguments of dominated convergence theorems as above yields that

$$U^{0,z}_{t\wedge\tau} = \int_0^{t\wedge\tau} 2\sqrt{U^{0,z}_s}\, d\beta^2_s + \int_0^{t\wedge\tau} \left(2g(V(s) - s, U^{0,z}_s, \xi^z_s) + 1\right) ds.$$

As one could expect we now want to conclude the existence of a solution to (5.51) for $y = 0$ by applying Ito's formula to $\sqrt{U^{0,z}}$ on $(0, \tau)$. This will be justified as soon as we show the strict positivity of $U^{0,z}$ on $(0, \tau)$. To achieve this let

$$\tau_n := \inf\left\{t \ge 0 : U^{0,z}_{t\wedge\tau} = \frac{1}{n}\right\},$$

with the convention that $\inf \emptyset = \tau$. Then, the arguments in Lemma 5.7 yield the strict positivity on $(\tau_n, \tau]$. Next define $S := \lim_{n\to\infty} \tau_n$. We claim that $S = 0$, a.s. Indeed, if this weren't the case, then $U^{0,z}$ would be identically 0 on $[0, S]$. However, a quick check of the drift coefficient of (5.52) reveals that 0 cannot be a solution on $[0, S]$ on the set $[S > 0]$.

Finally, the claim on the limit of $Y^{y,z}$ follows from the construction and the uniqueness of the solutions.

Proposition 5.8 *Let τ, β^2 and ξ^z be as in Proposition 5.7 and assume further that $(\xi^z_t)^2 \leq R^\delta(z)_{V(t)}$, for all $t \geq 0$, where $R^\delta(z)$ is a δ-dimensional squared Bessel process starting from z. Then, for any $y \geq 0$ and $z \geq 0$, there exists a unique strong solution to*

$$Y_t = y + \beta^2_t + \int_0^{\tau\wedge t} \frac{q_x(V(s)-s, Y_s, \xi_s)}{q(V(s)-s, Y_s, \xi^z_s)}\, ds, \tag{5.56}$$

which is strictly positive on $[0, \tau]$.

Proof For the brevity of the exposition we drop the superscript from ξ. Due to Proposition 5.7 there exists a unique strong solution, Y, of (5.51). Define $(L_t)_{t\geq 0}$ by $L_0 = 1$ and

$$dL_t = \mathbf{1}_{[\tau>t]} L_t \frac{Y_t - \xi_t}{V(t)-t}\, d\beta^2_t.$$

Observe that there exists a solution to the above equation since

$$\int_0^t \mathbf{1}_{[\tau>s]} \left(\frac{Y_s - \xi_s}{V(s)-s}\right)^2 ds < \infty, \text{ a.s. } \forall t \geq 0.$$

Indeed, since Y and ξ are well defined and continuous up to τ, we have $\sup_{s\leq\tau} |Y_s - \xi_s| < \infty$, a.s. and thus the above expression is finite in view of Assumption 5.3.2.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ denote the probability space of Proposition 5.7. If $(L_t)_{t\geq 0}$ is a true martingale, then for any $T > 0$, Q^T on $\mathscr{F}_T$ defined by

$$\frac{dQ^T}{dP_T} = L_T,$$

where P_T is the restriction of P to $\mathscr{F}_T$, is a probability measure on $\mathscr{F}_T$ equivalent to P_T. Then, by Girsanov Theorem (see Theorem A.19) under Q_T

$$Y_t = y + \beta^T_t + \int_0^{\tau\wedge t} \frac{q_x(V(s)-s, Y_s, \xi_s)}{q(V(s)-s, Y_s, \xi_s)}\, ds,$$

for $t \leq T$ where β^T is a Q^T-Brownian motion. Thus, Y is a weak solution to (5.56) on $[0, T]$. Therefore, due to Lemma 5.6 and Theorem 2.12, there exists a unique strong solution to (5.56) on $[0, T]$, and it is strictly positive on $(0, \tau]$ since Y has this property. Since T is arbitrary, this yields a unique strong solution on $[0, \infty)$ which is strictly positive on $(0, \tau]$.

Thus, it remains to show that L is a true martingale. Fix $T > 0$ and for some $0 \leq t_{n-1} < t_n \leq T$ consider

$$E\left[\exp\left(\frac{1}{2}\int_{t_{n-1}\wedge\tau}^{t_n\wedge\tau}\left(\frac{Y_t - Z_t}{V(t) - t}\right)^2 dt\right)\right], \tag{5.57}$$

where E denotes expectation with respect to P. As both Y and Z are positive until τ, $(Y_t - Z_t)^2 \leq Y_t^2 + Z_t^2 \leq R_t + Z_t^2$ by Theorem 2.9, where R is the three-dimensional squared Bessel process satisfying

$$R_t = y^2 + 2\int_0^t \sqrt{|R_s|}d\beta_s^2 + 3t.$$

Therefore, since R and Z are independent, the expression in (5.57) is bounded by

$$\begin{aligned} &E\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} R_t \upsilon(t)dt\right)\right] E\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} \xi_t^2 \upsilon(t)dt\right)\right] \\ &\leq E\left[\exp\left(\frac{1}{2}R_T^*\int_{t_{n-1}}^{t_n} \upsilon(t)dt\right)\right] E\left[\exp\left(\frac{1}{2}(\xi_T^*)^2\int_{t_{n-1}}^{t_n} \upsilon(t)dt\right)\right], \end{aligned} \tag{5.58}$$

where $Y_t^* := \sup_{s\leq t}|Y_s|$ for any càdlàg process Y and $\upsilon(t) := \frac{1}{(V(t)-t)^2}$. Recall that ξ is dominated by a time-changed δ-dimensional squared Bessel process and observe that we can assume that δ is a strictly positive integer in view of the comparison theorem for SDEs (see, e.g. Theorem 2.11). Since the time change is deterministic and R^δ is the square of the Euclidian norm of a δ-dimensional standard Brownian motion when δ is an integer, the above expression is going to be finite if

$$E^{y\vee 1}\left[\exp\left(\frac{1}{2}(\beta_{V(T)}^*)^2\int_{t_{n-1}}^{t_n} \upsilon(t)dt\right)\right] < \infty, \tag{5.59}$$

where β is a standard Brownian motion and E^x is the expectation with respect to the law of a standard Brownian motion starting at x. Indeed, it is clear that, by time change, (5.59) implies that the second expectation in the RHS of (5.58) is finite. Moreover, since R_T^* is the supremum over $[0, T]$ of a three-dimensional Bessel square process, it can be bounded above by the sum of three supremums of squared Brownian motions over $[0, V(T)]$ (remember that $V(T) > T$), which gives that (5.59) is an upper bound for the first expectation in the RHS of (5.58) as well.

In view of the reflection principle for standard Brownian motion (see, e.g. Proposition III.3.7 in [100]) the above expectation is going to be finite if

$$\int_{t_{n-1}}^{t_n} \upsilon(t)dt < \frac{1}{V(T)}. \tag{5.60}$$

However, Assumption 5.3 yields that $\int_0^T \upsilon(t)dt < \infty$. Therefore, we can find a finite sequence of real numbers $0 = t_0 < t_1 < \ldots < t_{n(T)} = T$ that satisfy (5.60). Since T was arbitrary, this means that we can find a sequence $(t_n)_{n\geq 0}$ with $\lim_{n\to\infty} t_n = \infty$ such that (5.57) is finite for all n. Then, it follows from Corollary 3.5.14 in [77] that L is a martingale. □

The above proposition establishes 0 as a lower bound to the solution of (5.56) over the interval $[0, \tau]$; however, one can obtain a tighter bound. Indeed, observe that $\frac{q_x}{q}(t, x, z)$ is strictly increasing in z on $[0, \infty)$ for fixed $(t, x) \in \mathbb{R}^2_{++}$. Moreover,

$$\frac{q_x}{q}(t, x, 0) := \lim_{z\downarrow 0} \frac{q_x}{q}(t, x, z) = \frac{1}{x} - \frac{x}{t}.$$

Therefore, $\frac{q_x}{q}(V(t) - t, Y_t, \xi_t^z) > \frac{q_x}{q}(V(t) - t, Y_t, 0) = \frac{1}{Y_t} - \frac{Y_t}{V(t)-t}$ for $t \in (0, \tau]$. Thus, a standard localisation argument applied together with Theorem 2.11 shows for every $y > 0$ that $\inf_{t\in[0,\tau)}(Y_t - r_t) \geq 0$, a.s., where r solves

$$r_t = y + \beta_t^2 + \int_0^t \left\{\frac{1}{r_s} - \frac{r_s}{V(s) - s}\right\} ds. \tag{5.61}$$

Before pushing the comparison result to $y = 0$ let us first establish that there exists a unique strong solution to the SDE above for all $y \geq 0$ and it equals in law to a scaled, time-changed three-dimensional Bessel process.

Proposition 5.9 *There exists a unique strong solution to (5.61) for any $y \geq 0$. Moreover, the law of r is equal to the law of $\tilde{\rho} = (\tilde{\rho}_t)_{t\geq 0}$, where $\tilde{\rho}_t = \lambda_t \rho_{\Lambda_t}$ where ρ is a three-dimensional Bessel process starting at y and*

$$\lambda_t := \exp\left(-\int_0^t \frac{1}{V(s) - s} ds\right),$$

$$\Lambda_t := \int_0^t \frac{1}{\lambda_s^2} ds.$$

In particular r is strictly positive on $(0, \infty)$.

Proof Note that $\frac{1}{x} - \frac{x}{t}$ is decreasing in x and, thus, pathwise uniqueness holds for (5.61). Thus, it suffices to find a weak solution for the existence and the uniqueness of strong solution by Theorem 2.12. Consider the three-dimensional Bessel process ρ which is the unique strong solution

$$\rho_t = y + \beta_t^2 + \int_0^t \frac{1}{\rho_s} ds, \qquad y \geq 0.$$

Such a solution exists since $\frac{1}{x}$ is decreasing, implying pathwise uniqueness, and a weak solution can be constructed by taking the square root of the solution in Example 2.5 with $\delta = 3$.

Therefore, $\rho_{\Lambda_t} = y + \beta^2_{\Lambda_t} + \int_0^{\Lambda_t} \frac{1}{\rho_s} ds$. Now, $M_t = \beta^2_{\Lambda_t}$ is a martingale with respect to the time-changed filtration with quadratic variation given by Λ. Integrating by parts we see that

$$d(\lambda_t \rho_{\Lambda_t}) = \lambda_t dM_t + \left\{ \frac{1}{\lambda_t \rho_{\Lambda_t}} - \frac{\lambda_t \rho_{\Lambda_t}}{V(t) - t} \right\} dt.$$

Since $\lambda_0 \rho_{\Lambda_0} = y$ and $\int_0^t \lambda_s^2 d[M, M]_s = t$, we see that $\lambda_t \rho_{\Lambda_t}$ is a weak solution to (5.61). This obviously implies the equivalence in law, which establishes the strict positivity in view of Example 2.5.

Proposition 5.10 *Fix $y \geq 0$ and let r be the unique strong solution to (5.61). Then,* $\inf_{t \in [0,\tau]}(Y_t - r_t) \geq 0$*, a.s., where Y is the unique strong solution of (5.56).*

Proof First suppose $y > 0$ and define $\tau_n := \inf\{t > 0 : r_t \wedge Y_t = \frac{1}{n}\}$. Then, Theorem 2.11 yields that $\inf_{t \in [0,\tau \wedge \tau_n]}(Y_t - r_t) \geq 0$, a.s. Taking limits and observing $\lim_{n \to \infty} \tau_n > \tau$, a.s., establishes the claim.

For any $y \geq 0$ denote the solutions of (5.56) and (5.61) by Y^y and r^y, respectively. Observe that $\frac{q_x(V(t)-t,x,z)}{q(V(t)-t,x,z)}$ satisfies the condition (2.14) on $(0, T] \times (\frac{1}{n}, \infty) \times (0, m)$ for any $n, m \geq 1$. Thus, a standard localisation argument shows $\inf_{t \in [0,\tau]}(Y_t^y - r_t^0) \geq 0$, a.s., for any $y > 0$. On the other hand, $Y_t^0 = \lim_{y \to 0} Y_t^y$ by Proposition 5.7, which in turn yields the claim in view of the continuity of Y^0 and r^0. □

Since Z is a time-changed Brownian motion and τ is its first hitting time of 0, Proposition 5.8 established the existence and uniqueness of a strong solution to (5.49) as well as its strictly positive on $[0, \tau]$. As the drift term in (5.48) after τ is the same as that of a three-dimensional Bessel bridge (see 4.41) from X_τ to 0 over $[\tau, V(\tau)]$, we have proved

Proposition 5.11 *There exists a unique strong solution to (5.48). Moreover, the solution is strictly positive in* $[0, \tau]$.

Recall from Example 4.4 that a Bessel bridge on an arbitrary interval $[0, T]$ is strictly positive on $(0, T)$ and converges to 0 at T. Thus we also have the following

Corollary 5.2 *Let X be the unique strong solution of (5.48). Then,*

$$V(\tau) = \inf\{t > 0 : X_t = 0\}.$$

Thus, in order to finish the proof of Theorem 5.4 it remains to show that X is a standard Brownian motion in its own filtration. We will achieve this result in several steps. First, we will obtain the canonical decomposition of X with respect to the minimal filtration, $\mathbb{G}$, satisfying the usual conditions such that X is $\mathbb{G}$-adapted and

τ is a $\mathbb{G}$-stopping time. More precisely, $\mathbb{G} = (\mathscr{G}_t)_{t \geq 0}$ where $\mathscr{G}_t = \cap_{u>t}\tilde{\mathscr{G}_u}$, with $\tilde{\mathscr{G}_t} := \mathscr{N} \bigvee \sigma(\{X_s, s \leq t\}, \tau \wedge t)$ and $\mathscr{N}$ being the set of $\mathbb{P}$-null sets. Then, we will initially enlarge this filtration with τ to show that the canonical decomposition of X in this filtration is the same as that of a Brownian motion starting at 1 in its own filtration enlarged with its first hitting time of 0. This observation will allow us to conclude that the law of X is the law of a Brownian motion.

In order to carry out this procedure we will use the following key result, the proof of which is deferred until the next section for the clarity of the exposition. We recall that

$$H(t,a) = \int_0^\infty q(t,a,y)dy,$$

where $q(t,a,y)$ is the transition density of a Brownian motion killed at 0.

Proposition 5.12 *Let X be the unique strong solution of (5.48) and $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be a bounded measurable function with a compact support contained in* $(0,\infty)$. *Then*

$$\mathbb{E}[\mathbf{1}_{[\tau>t]} f(Z_t)|\mathscr{G}_t] = \mathbf{1}_{[\tau>t]} \int_0^\infty f(z) \frac{q(V(t)-t, X_t, z)}{H(V(t)-t, X_t)}\, dz.$$

Using the above proposition we can easily obtain the $\mathbb{G}$-canonical decomposition of X.

Corollary 5.3 *Let X be the unique strong solution of (5.48). Then,*

$$M_t := X_t - 1 - \int_0^{\tau\wedge t} \frac{H_x(V(s)-s, X_s)}{H(V(s)-s, X_s)}\, ds - \int_{\tau\wedge t}^{V(\tau)\wedge t} \frac{\ell_a(V(\tau)-s, X_s)}{\ell(V(\tau)-s, X_s)}\, ds$$

is a standard $\mathbb{G}$-Brownian motion starting at 0.

Proof It follows from Corollary 3.1 and Lemma 5.10 in Sect. 5.2.4 that

$$X_t - 1 - \int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]} \frac{q_x(V(s)-s, X_s, Z_s)}{q(V(s)-s, X_s, Z_s)} \middle| \mathscr{G}_s\right] ds - \int_{\tau\wedge t}^{V(\tau)\wedge t} \frac{\ell_a(V(\tau)-s, X_s)}{\ell(V(\tau)-s, X_s)}\, ds$$

is a $\mathbb{G}$-Brownian motion. However, in view of Proposition 5.12,

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{[\tau>s]} \frac{q_x(V(s)-s, X_s, Z_s)}{q(V(s)-s, X_s, Z_s)} \middle| \mathscr{G}_s\right] \\
&= \mathbf{1}_{[\tau>s]} \int_0^\infty \frac{q_x(V(s)-s, X_s, z)}{q(V(s)-s, X_s, z)} \frac{q(V(s)-s, X_s, z)}{H(V(s)-s, X_s)}\, dz \\
&= \mathbf{1}_{[\tau>s]} \frac{1}{H(V(s)-s, X_s)} \int_0^\infty q_x(V(s)-s, X_s, z)\, dz
\end{aligned}$$

$$= \mathbf{1}_{[\tau>s]} \frac{1}{H(V(s)-s, X_s)} \frac{\partial}{\partial x} \int_0^\infty q(V(s)-s, x, z)\, dz \bigg|_{x=X_s}$$

$$= \mathbf{1}_{[\tau>s]} \frac{H_x(V(s)-s, X_s)}{H(V(s)-s, X_s)}.$$

□

We shall next find the canonical decomposition of X under $\mathbb{G}^\tau := (\mathscr{G}_t^\tau)_{t\geq 0}$ where $\mathscr{G}_t^\tau = \mathscr{G}_t \bigvee \sigma(\tau)$. Note that $\mathscr{G}_t^\tau = \mathscr{F}_{t+}^X \bigvee \sigma(\tau) \bigvee \mathscr{N}$. Therefore, the canonical decomposition of X under $\mathbb{G}^\tau$ would be its canonical decomposition with respect to its own filtration initially enlarged with τ. As we shall see in the next proposition it will be the same as the canonical decomposition of a Brownian motion in its own filtration initially enlarged with its first hitting time of 0.

Proposition 5.13 *Let X be the unique strong solution of (5.48). Then,*

$$X_t - 1 - \int_0^{V(\tau)\wedge t} \frac{\ell_a(V(\tau)-s, X_s)}{\ell(V(\tau)-s, X_s)}\, ds$$

is a standard $\mathbb{G}^\tau$-Brownian motion starting at 0.

Proof First, we will determine the law of τ conditional on $\mathscr{G}_t$ for each t. Let f be a test function. Then

$$\mathbb{E}\left[\mathbf{1}_{[\tau>t]} f(\tau) | \mathscr{G}_t\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{[\tau>t]} f(\tau) | \mathscr{H}_t\right] \bigg| \mathscr{G}_t\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{[\tau>t]} \int_t^\infty f(u)\sigma^2(u)\ell(V(u)-V(t), Z_t)\, du \bigg| \mathscr{G}_t\right]$$

$$= \mathbf{1}_{[\tau>t]} \int_t^\infty f(u)\sigma^2(u) \int_0^\infty \ell(V(u)-V(t), z) \frac{q(V(t)-t, X_t, z)}{H(V(t)-t, X_t)}\, dz\, du$$

$$= -\mathbf{1}_{[\tau>t]} \int_t^\infty f(u)\sigma^2(u) \int_0^\infty H_t(V(u)-V(t), z) \frac{q(V(t)-t, X_t, z)}{H(V(t)-t, X_t)}\, dz\, du$$

$$= -\mathbf{1}_{[\tau>t]} \int_t^\infty f(u)\sigma^2(u) \frac{\partial}{\partial s} \int_0^\infty \int_0^\infty q(s, z, y)\, dy$$
$$\times \frac{q(V(t)-t, X_t, z)}{H(V(t)-t, X_t)}\, dz \bigg|_{s=V(u)-V(t)} du$$

$$= -\mathbf{1}_{[\tau>t]} \int_t^\infty f(u)\sigma^2(u) \frac{\partial}{\partial s} \int_0^\infty \int_0^\infty$$
$$\times \frac{q(V(t)-t, X_t, z)}{H(V(t)-t, X_t)} q(s, z, y)\, dz\, dy \bigg|_{s=V(u)-V(t)} du$$

$$= -\mathbf{1}_{[\tau>t]} \int_t^\infty f(u)\sigma^2(u) \frac{\partial}{\partial s} \int_0^\infty \frac{q(V(t)-t+s, X_t, y)}{H(V(t)-t, X_t)} dy \bigg|_{s=V(u)-V(t)} du$$
$$= -\mathbf{1}_{[\tau>t]} \int_t^\infty f(u)\sigma^2(u) \frac{H_t(V(u)-t, X_t)}{H(V(t)-t, X_t)} du$$
$$= \mathbf{1}_{[\tau>t]} \int_t^\infty f(u)\sigma^2(u) \frac{\ell(V(u)-t, X_t)}{H(V(t)-t, X_t)} du.$$

Thus, $\mathbb{P}[\tau \in du, \tau > t|\mathscr{G}_t] = \mathbf{1}_{[\tau>t]}\sigma^2(u)\frac{\ell(V(u)-t,X_t)}{H(V(t)-t,X_t)} du$.

Then, it follows from Remark 4.6 that

$$M_t - \int_0^{\tau\wedge t} \left(\frac{\ell_a(V(\tau)-s, X_s)}{\ell(V(\tau)-s, X_s)} - \frac{H_x(V(s)-s, X_s)}{H(V(s)-s, X_s)} \right) ds$$

is a $\mathbb{G}^\tau$-Brownian motion. □

Corollary 5.4 *Let X be the unique strong solution of (5.48). Then, X is a Brownian motion with respect to $\mathbb{F}^X$.*

Proof It follows from Proposition 5.13 that $\mathbb{G}^\tau$- decomposition of X is given by

$$X_t = 1 + \mu_t + \int_0^{V(\tau)\wedge t} \left\{ \frac{1}{X_s} - \frac{X_s}{V(\tau)-s} \right\} ds,$$

where μ is a standard $\mathbb{G}^\tau$-Brownian motion vanishing at 0. Thus, X is a three-dimensional Bessel bridge from 1 to 0 of length $V(\tau)$. As $V(\tau)$ is the first hitting time of 0 for X and $V(\tau) = T_1$ in distribution, it follows from Example 4.6 that X has the same law as a standard Brownian motion starting at 1 in view of the uniqueness in law for the Bessel bridge SDE (4.41) established in Example 4.4.

Next section is devoted to the proof of Proposition 5.12.

5.2.3 Dynamic Bridge in Its Natural Filtration

Recall from Proposition 5.12 that we are interested in the conditional distribution of Z_t on the set $[\tau > t]$. To this end we introduce the following change of measure on $\mathscr{H}_t$. Let $\mathbb{P}_t$ be the restriction of $\mathbb{P}$ to $\mathscr{H}_t$ and define $\mathbb{P}^{\tau,t}$ on $\mathscr{H}_t$ by

$$\frac{d\mathbb{P}^{\tau,t}}{d\mathbb{P}_t} = \frac{\mathbf{1}_{[\tau>t]}}{\mathbb{P}[\tau > t]}.$$

Note that this measure change is equivalent to an h-transform on the paths of Z until time t, where the h-transform is defined by the h-function $H(V(t) - V(\cdot), \cdot)$, and H is the function defined in (5.44) (see Sect. 4.1 and in particular

Definition 4.1 for the definition and a further discussion of h-functions). Note also that $(\mathbf{1}_{[\tau>s]}H(V(t)-V(s),Z_s))_{s\in[0,t]}$ is a $(\mathbb{P},\mathbb{H})$-martingale as a consequence of (5.45). Therefore, we obtain via Girsanov's theorem that under $\mathbb{P}^{\tau,t}$ the pair (X,Z) satisfy for $s\leq t$

$$dZ_s=\sigma(s)d\beta_s^t+\sigma^2(s)\frac{H_x(V(t)-V(s),Z_s)}{H(V(t)-V(s),Z_s)}ds \tag{5.62}$$

$$dX_s=dB_s+\frac{q_x(V(s)-s,X_s,Z_s)}{q(V(s)-s,X_s,Z_s)}ds, \tag{5.63}$$

with $X_0=Z_0=1$ and β^t being a $\mathbb{P}^{\tau,t}$-Brownian motion. Moreover, a straightforward application of a Girsanov transform shows that the transition density of Z under $\mathbb{P}^{\tau,t}$ is given by

$$\mathbb{P}^{\tau,t}[Z_s\in dz|Z_r=x]=q(V(s)-V(r),x,z)\frac{H(V(t)-V(s),z)}{H(V(t)-V(r),x)}. \tag{5.64}$$

Thus, $\mathbb{P}^{\tau,t}[Z_s\in dz|Z_r=x]=p(V(t);V(r),V(s),x,z)$ where

$$p(t;r,s,x,z)=q(s-r,x,z)\frac{H(t-s,z)}{H(t-r,x)}. \tag{5.65}$$

Note that p is the transition density of the Brownian motion killed at 0 after the analogous h-transform where the h-function is given by $H(t-s,x)$.

Lemma 5.8 *Let $\mathscr{F}_s^{\tau,t,X}=\sigma(X_r;r\leq s)\vee\mathscr{N}^{\tau,t}$ where X is the process defined by (5.63) with $X_0=1$, and $\mathscr{N}^{\tau,t}$ is the collection of $\mathbb{P}^{\tau,t}$-null sets. Then the filtration $(\mathscr{F}_s^{\tau,t,X})_{s\in[0,t]}$ is right-continuous.*

Proof First observe that $(\mathscr{F}_{\tau_n\wedge s}^{\tau,t,X})_{s\in[0,t]}$, where $\tau_n:=\inf\{s>0:X_s=\frac{1}{n}\}$ is a right continuous filtration. This follows from the observation that X^{τ_n} is a Brownian motion under an equivalent probability measure, which can be shown using the arguments of Proposition 5.8 along with the identity (5.50) and the fact that $\frac{1}{X}$ is bounded up to τ_n. Thus, for each n one has

$$\begin{aligned}\mathscr{F}_{\tau_n}^{\tau,t,X}\cap\mathscr{F}_u^{\tau,t,X}&=\mathscr{F}_{\tau_n\wedge u}^{\tau,t,X}=\bigcap_{s>u}\mathscr{F}_{\tau_n\wedge s}^{\tau,t,X}\\&=\bigcap_{s>u}\left(\mathscr{F}_{\tau_n}^{\tau,t,X}\cap\mathscr{F}_s^{\tau,t,X}\right)=\left(\bigcap_{s>u}\mathscr{F}_s^{\tau,t,X}\right)\cap\mathscr{F}_{\tau_n}^{\tau,t,X}\end{aligned}$$

Indeed, since $\cup_n\mathscr{F}_{\tau_n}^{\tau,t,X}=\mathscr{F}_{\tau}^{\tau,t,X}$, letting n tend to infinity yields the conclusion. $\square$

The reason for the introduction of the probability measure $\mathbb{P}^{\tau,t}$ and the filtration $(\mathscr{F}_s^{\tau,t,X})_{s\in[0,t]}$ is that $(\mathbb{P}^{\tau,t},(\mathscr{F}_s^{\tau,t,X})_{s\in[0,t]})$-conditional distribution of Z can be characterised by a *Kushner–Stratonovich equation* which is well defined. Moreover, it gives us $(\mathbb{P}, \mathscr{G})$-conditional distribution of Z. Indeed, observe that $\mathbb{P}^{\tau,t}[\tau > t] = 1$ and for any set $E \in \mathscr{G}_t$, $\mathbf{1}_{[\tau>t]}\mathbf{1}_E = \mathbf{1}_{[\tau>t]}\mathbf{1}_F$ for some set $F \in \mathscr{F}_t^{\tau,t,X}$ (see Lemma 5.1.1 in [24] and the remarks that follow). Then, it follows from the definition of conditional expectation that for all $f \in \mathbb{C}_K^\infty(\mathbb{R}_+)$

$$\mathbb{E}\left[f(Z_t)\mathbf{1}_{[\tau>t]}|\mathscr{G}_t\right] = \mathbf{1}_{[\tau>t]}\mathbb{E}^{\tau,t}\left[f(Z_t)\middle|\mathscr{F}_t^{\tau,t,X}\right], \mathbb{P}-a.s. \tag{5.66}$$

Thus, it is enough to compute the conditional distribution of Z under $\mathbb{P}^{\tau,t}$ with respect to $(\mathscr{F}_s^{\tau,t,X})_{s\in[0,t]}$.

As usual let $\mathscr{P}$ be the set of probability measures on the Borel sets of $\mathbb{R}_+$ topologised by weak convergence. It follows from Proposition 3.2 that there exists a $\mathscr{P}$-valued càdlàg $\mathscr{F}^{\tau,t,X}$-optional process $\pi^t(\omega, dx)$ such that

$$\pi_s^t f = \mathbb{E}^{\tau,t}[f(Z_s)|\mathscr{F}_s^{\tau,t,X}]$$

for all $f \in \mathbb{C}_K^\infty(\mathbb{R}_+)$.

Let's also recall the *innovation process*

$$I_s = X_s - \int_0^s \pi_r^t \kappa_r dr$$

where $\kappa_r(z) := \frac{q_x(V(r)-r, X_r, z)}{q(V(r)-r, X_r, z)}$. Although it is clear that I depends on t, we don't emphasise it in the notation for convenience. Due to Lemma 5.10 $\pi_s^t\kappa_s$ exists for all $s \leq t$.

To characterise π^t as the solution of a Kushner–Stratonovich equation consider the operator A_s defined by

$$A_s\phi(x) = \frac{1}{2}\sigma^2(s)\frac{\partial^2\phi}{\partial x^2}(x) + \sigma^2(s)\frac{H_x}{H}(V(t) - V(s), x)\frac{\partial\phi}{\partial x}(x), \tag{5.67}$$

for all $\phi \in \mathbb{C}_K^\infty(\mathbb{R}_+)$ and $s < t$. We will see in Lemma 5.9 that the local martingale problem for A_s is well posed over the time interval $[0, t - \varepsilon]$ for any $\varepsilon > 0$. Therefore, it is well posed on $[0, t)$ and its unique solution is given by $(Z_s)_{s\in[0,t)}$ where Z is defined by (5.62). Moreover, Theorem 3.2 shows that π^t solves the following Kushner–Stratonovich equation:

$$\pi_s^t f = \pi_0^t f + \int_0^s \pi_r^t(A_r f)dr + \int_0^s \left[\pi_r^t(\kappa_r f) - \pi_r^t\kappa_r\pi_r^t f\right] dI_r, \tag{5.68}$$

for all $f \in \mathbb{C}_K^\infty(\mathbb{R}_+)$ (note that the conditions of Theorem 3.2 are satisfied due to Lemma 5.10).

Theorem 5.5 *Let $f \in \mathbb{C}_K^\infty(\mathbb{R}_+)$. Then,*

$$\pi_s^t f = \int_{\mathbb{R}_+} f(z) p(V(t); s, V(s), X_s, z)\, dz,$$

for $s < t$ where p is as defined in (5.65).

Proof Let $\rho(t; s, x, z) := p(V(t); s, V(s), x, z)$. Direct computations lead to

$$\begin{aligned} &\rho_s + \frac{H_x(V(t)-s, x)}{H(V(t)-s, x)}\rho_x + \frac{1}{2}\rho_{xx} \\ &= -\sigma^2(s)\left(\frac{H_x(V(t)-V(s), z)}{H(V(t)-V(s), z)}\rho\right)_z + \frac{1}{2}\sigma^2(s)\rho_{zz}. \end{aligned} \tag{5.69}$$

Define $m^t \in \mathscr{P}$ by $m_s^t f := \int_{\mathbb{R}_+} f(z)\rho(t; s, X_s, z)dz$. Then, using the above PDE and Ito's formula one can directly verify that m^t solves (5.68). Moreover, Proposition 5.14 shows that the joint local martingale problem is well posed. Theorem 3.3 yields the claim as soon as we verify, for any $\varepsilon > 0$,

$$\int_0^{t-\varepsilon} \mathbb{E}^{\tau,t}\left(\int_0^\infty \frac{q_x(V(s)-s, X_s, z)}{q(V(s)-s, X_s, z)} p(V(t); s, V(s), X_s, z)dz\right)^2 ds < \infty.$$

As

$$\frac{q_x(V(s)-s, x, z)}{q(V(s)-s, x, z)} = \frac{p_x(V(t); s, V(s), x, z)}{p(V(t); s, V(s), x, z)} + \frac{H_x(V(t)-s, x)}{H(V(t)-s, x)},$$

we have

$$\begin{aligned} &\int_0^\infty \frac{q_x(V(s)-s, X_s, z)}{q(V(s)-s, X_s, z)} p(V(t); s, V(s), X_s, z)dz \\ &= \int_0^\infty p_x(V(t); s, V(s), X_s, z)dz + \frac{H_x(V(t)-s, X_s)}{H(V(t)-s, X_s)}. \end{aligned}$$

Moreover, the integral vanishes for $s \le t - \varepsilon$ since

$$\int_0^\infty p_x(V(t); s, V(s), x, z)dz = \frac{\partial}{\partial x}\int_0^\infty p(V(t); s, V(s), x, z)dz,$$

where the interchange of integration and differentiation can be justified for any $s \le t - \varepsilon$.

Therefore,

$$\int_0^{t-\varepsilon} \mathbb{E}^{\tau,t} \left(\int_0^\infty \frac{q_x(V(s)-s, X_s, z)}{q(V(s)-s, X_s, z)} p(V(t); s, V(s), X_s, z) dz \right)^2 ds$$
$$= \int_0^{t-\varepsilon} \mathbb{E}^{\tau,t} \left(\frac{H_x(V(t)-s, X_s)}{H(V(t)-s, X_s)} \right)^2 ds,$$

which is finite due to Lemma 5.10. □

Now, we have all necessary results to prove Proposition 5.12.

Proof of Proposition 5.12 Note that as X is continuous, $\mathscr{F}_t^{\tau,t,X} = \bigvee_{s<t} \mathscr{F}_s^{\tau,t,X}$. Fix $r < t$ and let $E \in \mathscr{F}_r^{\tau,t,X}$. In view of (5.66) it suffices to show that for all $f \in \mathbb{C}_K^\infty(\mathbb{R}_+)$

$$\mathbb{E}^{\tau,t}[f(Z_t)\mathbf{1}_E] = \mathbb{E}^{\tau,t} \left[\int_{\mathbb{R}_+} f(z) \frac{q(V(t)-t, X_t, z)}{H(V(t)-t, X_t)} dz \, \mathbf{1}_E \right].$$

Since Z is continuous and f is bounded we have

$$\mathbb{E}^{\tau,t}[f(Z_t)\mathbf{1}_E] = \lim_{s\uparrow t} \mathbb{E}^{\tau,t}[f(Z_s)\mathbf{1}_E].$$

As s will eventually be larger than r, $\mathbf{1}_E \in \mathscr{F}_s^{\tau,t,X}$ for large enough s and, then, Theorem 5.5 and another application of the Dominated Convergence Theorem will yield

$$\lim_{s\uparrow t} \mathbb{E}^{\tau,t}[f(Z_s)\mathbf{1}_E] = \lim_{s\uparrow t} \mathbb{E}^{\tau,t} \left[\int_{\mathbb{R}_+} f(z) p(V(t); s, V(s), X_s, z) \, dz \, \mathbf{1}_E \right]$$
$$= \mathbb{E}^{\tau,t} \left[\lim_{s\uparrow t} \int_{\mathbb{R}_+} f(z) p(V(t); s, V(s), X_s, z) \, dz \, \mathbf{1}_E \right].$$

Since X is strictly positive until τ by Proposition 5.11, $\min_{s\le t} X_s > 0$. This yields that $\frac{1}{H(V(t)-s, X_s)}$ is bounded (ω-by-ω) for $s \le t$. Moreover, $q(V(s)-s, X_s, \cdot)$ is bounded by $\frac{1}{\sqrt{2\pi(V(s)-s)}}$. Thus, in view of (5.65),

$$p(V(t); s, V(s), X_s, z) \le \frac{K(\omega)}{\sqrt{V(s)-s}} H(V(t)-V(s), z),$$

where $K(\omega)$ is finite. Since $(V(s)-s)^{-1}$ can be bounded when s is away from 0, H is bounded by 1, and f has a compact support, it follows from the Dominated Convergence Theorem that

$$\lim_{s\uparrow t}\int_{\mathbb{R}_+} f(z)p(V(t); s, V(s), X_s, z)\,dz = \int_{\mathbb{R}_+} f(z)\frac{q(V(t)-t, X_t, z)}{H(V(t)-t, X_t)}\,dz,\ \mathbb{P}^{\tau,t}\text{−a.s.}$$

Hence

$$\mathbb{E}^{\tau,t}[f(Z_t)\mathbf{1}_E] = \mathbb{E}^{\tau,t}[\lim_{s\uparrow t} f(Z_s)\mathbf{1}_E] = \mathbb{E}^{\tau,t}\left[\int_{\mathbb{R}_+} f(z)\frac{q(V(t)-t, X_t, z)}{H(V(t)-t, X_t)}\,dz\,\mathbf{1}_E\right].$$

□

5.2.4 Local Martingale Problems and Some L^2 Estimates

In the next lemma we show that the local martingale problem related to Z as defined in (5.62) is well posed. Recall that A is the associated infinitesimal generator defined in (5.67). We will denote the restriction of A to $[0, t-\varepsilon]$ by A^ε.

Lemma 5.9 *For $\varepsilon > 0$ the local martingale problem for A^ε is well posed. Equivalently the SDE (5.62) has a unique weak solution for any nonnegative initial condition. Moreover, the solution is strictly positive on $(s, t-\varepsilon]$ for any $s \in [0, t-\varepsilon]$ and is bounded from above by $\rho_{V(t)}$, where ρ is a three-dimensional Bessel process starting at z at time s.*

Proof Observe that Assumption 2.1 is satisfied for A^ε. Thus it suffices to consider non-random initial conditions to establish the well posedness of the local martingale problem in view of Theorem 2.2.

Let $s \in [0, t-\varepsilon]$ and $z \in \mathbb{R}_+$. Then, direct calculations yield

$$dZ_r = \sigma(r)d\beta_r + \sigma^2(r)\left\{\frac{1}{Z_r} - Z_r\eta^t(r, Z_r)\right\}dr,\ \text{for } r \in [s, t-\varepsilon], \tag{5.70}$$

with $Z_s = z$, where

$$\eta^t(r, y) := \frac{\int_{V(t)-V(r)}^{\infty}\frac{1}{\sqrt{2\pi u^5}}\exp\left(-\frac{y^2}{2u}\right)du}{\int_{V(t)-V(r)}^{\infty}\frac{1}{\sqrt{2\pi u^3}}\exp\left(-\frac{y^2}{2u}\right)du}, \tag{5.71}$$

thus, $\eta^t(r, y) \in [0, \frac{1}{V(t)-V(t-\varepsilon)}]$ for any $r \in [0, t-\varepsilon]$ and $y \in \mathbb{R}_+$.

First, we show the uniqueness of the solutions to the local martingale problem. Suppose there exists a weak solution taking values in $\mathbb{R}_+$ to the SDE above. Thus, there exists $(\tilde{Z}, \tilde{\beta})$ on some filtered probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_r)_{r\in[0,t-\varepsilon]}, \tilde{P})$ such that

$$d\tilde{Z}_r = \sigma(r)d\tilde{\beta}_r + \sigma^2(r)\left\{\frac{1}{\tilde{Z}_r} - \tilde{Z}_r\eta^t(r, \tilde{Z}_r)\right\}dr, \qquad \text{for } r \in [s, t-\varepsilon],$$

with $\tilde{Z}_s = z$. Consider $\tilde{\rho}$ which solves

$$d\tilde{\rho}_r = \sigma(r)d\tilde{\beta}_r + \sigma^2(r)\frac{1}{\tilde{\rho}_r}dr, \tag{5.72}$$

with $\tilde{\rho}_s = z$. Note that this equation is the SDE for a time-changed three-dimensional Bessel process with a deterministic time change and the initial condition $\tilde{\rho}_s = z$. Therefore, it has a unique strong solution which is strictly positive on $(s, t-\varepsilon]$ (see Examples 2.2 and 2.5). Then, from Tanaka's formula (see Theorem 1.2 in Chap. VI of [100]) and the fact that the local time of $\tilde{\rho} - \tilde{Z}$ at 0 is identically 0 (see Corollary 1.9 in Chap. VI of [100]), we have

$$(\tilde{Z}_t - \tilde{\rho}_t)^+ = \int_0^t \mathbf{1}_{[\tilde{Z}_r > \tilde{\rho}_r]}\sigma^2(r)\left\{\frac{1}{\tilde{Z}_r} - \tilde{Z}_r\eta^t(r, \tilde{Z}_r) - \frac{1}{\tilde{\rho}_r}\right\}dr \leq 0,$$

where the last inequality is due to $\eta^t \geq 0$, and $\frac{1}{a} < \frac{1}{b}$ whenever $a > b > 0$. Thus, $\tilde{Z}_r \leq \tilde{\rho}_r$ for $r \in [s, t-\varepsilon]$.

Define $(L_r)_{r\in[0,t-\varepsilon]}$ by $L_0 = 1$ and

$$dL_r = -L_r\tilde{Z}_r\eta^t(r, \tilde{Z}_r)\,d\tilde{\beta}_r.$$

If $(L_r)_{r\in[0,t-\varepsilon]}$ is a strictly positive true martingale, then Q on $\tilde{\mathscr{F}}_{t-\varepsilon}$ defined by

$$\frac{dQ}{d\tilde{P}} = L_{t-\varepsilon},$$

is a probability measure on $\tilde{\mathscr{F}}_{t-\varepsilon}$ equivalent to $\tilde{P}$. Then, by Girsanov theorem

$$d\tilde{Z}_r = \sigma(r)d\tilde{\beta}_r^Q + \sigma^2(r)\frac{1}{\tilde{Z}_r}\,dr, \qquad \text{for } r \in [s, t-\varepsilon],$$

with $\tilde{Z}_s = z$, where $\tilde{\beta}^Q$ is a Q-Brownian motion. This shows that $(\tilde{Z}, \tilde{\beta}^Q)$ is a weak solution to (5.72). As (5.72) has a unique strong solution which is strictly positive on $(s, t-\varepsilon]$, any weak solution to (5.62) is strictly positive on $(s, t-\varepsilon]$ in view of the equivalence of the measures. Moreover, the equivalence of Q and $\tilde{P}$ establishes a one-to-one correspondence between the law of the solutions of (5.62) and (5.72) implying uniqueness in law for the solutions of (5.62).

Thus, to finish the proof of uniqueness in law we need to show that L is a strictly positive true martingale when $\tilde{Z}$ is a positive solution to (5.70). For some $0 \leq t_{n-1} < t_n \leq t - \varepsilon$ consider

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n}(\tilde{Z}_r\eta^t(r, \tilde{Z}_r))^2dr\right)\right]. \tag{5.73}$$

The expression in (5.73) is bounded by

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} \tilde{\rho}_r^2 \left(\frac{1}{V(t)-V(t-\varepsilon)}\right)^2 dr\right)\right]$$
$$\leq \mathbb{E}\left[\exp\left(\frac{1}{2}(\tilde{\rho}_r^*)^2 \frac{t_n - t_{n-1}}{(V(t)-V(t-\varepsilon))^2}\right)\right]$$

where $Y_t^* := \sup_{s \leq t} |Y_s|$ for any càdlàg process Y. Recall that $\tilde{\rho}$ is only a time-changed Bessel process where the time change is deterministic and, therefore, $\tilde{\rho}_r^2$ is the square of the Euclidian norm at time $V(r)$ of a three-dimensional standard Brownian motion, starting at $(z, 0, 0)$ at time $V(s)$. Thus, by using the same arguments as in Proposition 5.8, we get that the above expression is going to be finite if

$$E_{V(s)}^z\left[\exp\left(\frac{1}{2}(\beta_{V(t-\varepsilon)}^*)^2 \frac{t_n - t_{n-1}}{(V(t)-V(t-\varepsilon))^2}\right)\right] < \infty,$$

where β is a standard Brownian motion and E_s^x is the expectation with respect to the law of a standard Brownian motion starting at x at time s. In view of the reflection principle for standard Brownian motion (see, e.g. Proposition 3.7 in Chap. 3 of [100]) the above expectation is going to be finite if

$$\frac{t_n - t_{n-1}}{(V(t)-V(t-\varepsilon))^2} < \frac{1}{V(t-\varepsilon)}.$$

Clearly, we can find a finite sequence of real numbers $0 = t_0 < t_1 < \ldots < t_{n(T)} = T$ that satisfy above. Now, it follows from Corollary 3.5.14 in [77] that L is a martingale. It is strictly positive since calculations similar to the above yield

$$\int_s^{t-\varepsilon} (\tilde{Z}_r \eta^t(r, \tilde{Z}_r))^2 dr \leq (\tilde{\rho}_{t-\varepsilon}^*)^2 \frac{t-\varepsilon-s}{(V(t)-V(t-\varepsilon))^2} < \infty.$$

In order to show the existence of a nonnegative solution, consider the solution, $\tilde{\rho}$, to (5.72), which is a time-changed three-dimensional Bessel process, thus, nonnegative. Then, define $(L_r^{-1})_{r \in [0, t-\varepsilon]}$ by $L_0^{-1} = 1$ and

$$dL_r^{-1} = L_r^{-1} \tilde{\rho}_r \eta^t(r, \tilde{\rho}_r)\, d\tilde{\beta}_r.$$

Applying the same estimation to L^{-1} as we did for L yields that L^{-1} is a true martingale. Then, Q on $\tilde{\mathscr{F}}_{t-\varepsilon}$ defined by

$$\frac{dQ}{d\tilde{P}} = L_{t-\varepsilon}^{-1},$$

is a probability measure on $\tilde{\mathscr{F}}_{t-\varepsilon}$ under which $\tilde{\rho}$ solves

$$d\tilde{Z}_r = \sigma(r)d\tilde{\beta}_r^Q + \sigma^2(r)\left\{\frac{1}{\tilde{Z}_r} - \tilde{Z}_r\eta^t(r, \tilde{Z}_r)\right\} dr, \qquad \text{for } r \in [s, t-\varepsilon],$$

with $\tilde{Z}_s = z$ and $\tilde{\beta}^Q$ is a Q-Brownian motion. Therefore the SDE in (5.62) has a unique weak solution for any nonnegative initial condition. Moreover, the solution is strictly positive on $(s, t-\varepsilon]$. In view of Theorem 2.2 and Corollary 2.3 the local martingale problem A^ε is well posed. □

We are now ready to show that the joint local martingale problem for (X, Z) defined by the operator $\tilde{A}$ which is given by

$$\begin{aligned}\tilde{A}_s\phi(x,z) &= \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(x,z) + \frac{1}{2}\sigma^2(s)\frac{\partial^2\phi}{\partial z^2}(x,z) \\ &\quad + \frac{q_x}{q}(V(s)-s,x,z)\frac{\partial\phi}{\partial x}(x,z) + \sigma^2(s)\frac{H_z}{H}(V(t)-V(s),z)\frac{\partial\phi}{\partial z}(x,z)\end{aligned}$$

for all $\phi \in \mathbb{C}_K^\infty(\mathbb{R}_+^2)$. The restriction of $\tilde{A}$ to $[0, t-\varepsilon]$ will be denoted by $\tilde{A}^\varepsilon$.

Proposition 5.14 *For any $\varepsilon > 0$ the local martingale problem for $\tilde{A}^\varepsilon$ is well posed.*

Proof In view of Theorem 2.2 and Corollary 2.3 it suffices to show the existence and the uniqueness of weak solutions to the system of SDEs defined by (5.62) and (5.63) with the initial condition that $X_s = x$ and $Z_s = z$ for a fixed $s \in [0, t-\varepsilon]$. This follows from Proposition 5.8 since Z is bounded from above by a three-dimensional Bessel process in view of Lemma 5.9. □

Lemma 5.10 *Let (Z, X) be the unique strong solutions to (5.42) and (5.48). Then they solve the local martingale problem on the interval $[0, t)$ defined by (5.62) and (5.63) with the initial condition $X_0 = Z_0 = 1$. Moreover, under Assumption 5.3 we have*

i) $\mathbb{E}\left[\int_0^t \mathbf{1}_{[\tau>s]}\left(\frac{q_x}{q}(V(s)-s, X_s, Z_s)\right)^2 ds\right] < \infty.$

ii) $\mathbb{E}^{\tau,t}\left[\int_0^t \left(\frac{q_x}{q}(V(s)-s, X_s, Z_s)\right)^2 ds\right] < \infty.$

iii) $\mathbb{E}^{\tau,t}\int_0^{t-\varepsilon}\left|\frac{H_x}{H}(V(t)-s, X_s)\right|^2 ds < \infty$, *for any* $\varepsilon > 0$.

iv) $\mathbb{E}^{\tau,t}\left[\int_0^{t-\varepsilon}\sigma^2(s)\left|\frac{H_x}{H}(V(t)-V(s), Z_s)\right| ds\right]^2 < \infty$, *for any* $\varepsilon > 0$.

Proof Recall that $\frac{d\mathbb{P}^{\tau,t}}{d\mathbb{P}_t} = \frac{\mathbf{1}_{[\tau>t]}}{\mathbb{P}[\tau>t]}$ and that $\mathbb{E}^{\tau,t}$ denotes the expectation operator with respect to $\mathbb{P}^{\tau,t}$. Hence, under $\mathbb{P}^{\tau,t}$, (Z, X) satisfy (5.62) and (5.63) with the initial condition $X_0 = Z_0 = 1$, which implies that they solve the corresponding local martingale problem.

i) & ii) Note that

$$\begin{aligned}
&\mathbb{P}[\tau > t]\,\mathbb{E}^{\tau,t}\left[\int_0^t \left(\frac{q_x}{q}(V(s)-s, X_s, Z_s)\right)^2 ds\right] \\
&= \mathbb{E}\left[\mathbf{1}_{[\tau>t]}\int_0^t \left(\frac{q_x}{q}(V(s)-s, X_s, Z_s)\right)^2 ds\right] \\
&\le \mathbb{E}\left[\int_0^t \mathbf{1}_{[\tau>s]}\left(\frac{q_x}{q}(V(s)-s, X_s, Z_s)\right)^2 ds\right].
\end{aligned}$$

Thus, it suffices to prove the first assertion since $\mathbb{P}[\tau > t] > 0$ for all $t \ge 0$. Recall from (5.50) that

$$\frac{q_x(t,x,z)}{q(t,x,z)} = \frac{z-x}{t} + \frac{\exp\left(-\frac{2xz}{t}\right)}{1-\exp\left(-\frac{2xz}{t}\right)}\frac{2z}{t} = \frac{z-x}{t} + f\left(\frac{2xz}{t}\right)\frac{1}{x},$$

where $f(y) = \frac{e^{-y}}{1-e^{-y}}y$ is bounded by 1 on $[0,\infty)$. As $\int_0^t \frac{1}{(V(s)-s)^2}ds < \infty$ and $\sup_{s\in[0,t]}\mathbb{E}[Z_s^2] \le V(t)+1$, the result will follow once we obtain

$$\sup_{s\in[0,t]}\mathbb{E}[X_s^2\mathbf{1}_{[\tau>s]}] < \infty, \tag{5.74}$$

$$\mathbb{E}\left(\int_0^t \mathbf{1}_{[\tau>s]}\frac{1}{X_s^2}ds\right) < \infty. \tag{5.75}$$

1. To show (5.74) note that by Ito formula,

$$\begin{aligned}
\mathbf{1}_{[\tau>t]}X_t^2 = \mathbf{1}_{[\tau>t]}\Bigg(1 + 2\int_0^t X_s dB_s + 2\int_0^t \\
\times \left\{\frac{Z_sX_s - X_s^2}{V(s)-s} + f\left(\frac{2Z_sX_s}{V(s)-s}\right) + \frac{1}{2}\right\} ds\Bigg).
\end{aligned} \tag{5.76}$$

Observe that the elementary inequality $2ab \le a^2 + b^2$ implies

$$2\mathbf{1}_{[\tau>t]}\int_0^t X_s dB_s \le 1 + \left(\mathbf{1}_{[\tau>t]}\int_0^t X_s dB_s\right)^2 \le 1 + \left(\int_0^{\tau\wedge t} X_s dB_s\right)^2,$$

$$\text{and } 2\int_0^t \frac{Z_sX_s - X_s^2}{V(s)-s}ds \le \int_0^t \frac{Z_s^2 - X_s^2}{V(s)-s}ds \le \int_0^t \frac{Z_s^2}{V(s)-s}ds.$$

As f is bounded by 1, using the above inequalities and taking expectations of both sides of (5.76) yield

$$\mathbb{E}[\mathbf{1}_{[\tau>t]}X_t^2] \le 2 + \mathbb{E}\left(\int_0^t \mathbf{1}_{[\tau>s]}X_s dB_s\right)^2 + \int_0^t \frac{\mathbb{E}[Z_s^2]}{V(s)-s}ds + 3t$$
$$\le 2 + 3t + (V(t)+1)\int_0^t \frac{1}{V(s)-s}\,ds + \int_0^t \mathbb{E}\left(\mathbf{1}_{[\tau>s]}X_s^2\right)ds.$$

The last inequality obviously holds when $\int_0^t \mathbb{E}\left(\mathbf{1}_{[\tau>s]}X_s^2\right)ds = \infty$, otherwise, it is a consequence of Ito isometry. Let $T > 0$ be a constant, then for all $t \in [0, T]$ it follows from Gronwall's inequality that

$$\mathbb{E}[\mathbf{1}_{[\tau>t]}X_t^2] \le \left(2 + 3T + (V(T)+1)\int_0^T \frac{1}{V(s)-s}\,ds\right)e^T.$$

2. Now we turn to prove (5.75). In view of Proposition 5.10 we have $\mathbf{1}_{[\tau>s]}\frac{1}{X_s^2} \le \frac{1}{R_s^2}$ where R is the unique strong solution of (5.61). Thus, it is enough to show that $\int_0^t \mathbb{E}\left[\frac{1}{R_s^2}\right]ds < \infty$. Recall from Proposition 5.9 that the law of R_s is that of $\lambda_s\rho_{\Lambda_s}$ where ρ is a three-dimensional Bessel process starting at 1 and

$$\lambda_t = \exp\left(-\int_0^t \frac{1}{V(s)-s}\,ds\right),$$
$$\Lambda_t = \int_0^t \frac{1}{\lambda_s^2}\,ds.$$

Therefore, using the explicit form of the probability density of three-dimensional Bessel process (see Example 1.5) one has

$$\int_0^t \mathbb{E}\left[\frac{1}{R_s^2}\right]ds \le \int_0^t \mathbb{E}\left[\frac{1}{R_s^2}\mathbf{1}_{[R_s\le\sqrt[3]{\Lambda_s}]} + \Lambda_s^{-\frac{2}{3}}\right]ds$$
$$\le \int_0^t \lambda_s^{-2}\int_0^{\sqrt[3]{\Lambda_s}\lambda_s^{-1}} \frac{1}{y}q(\Lambda_s, 1, y)\,dy\,ds + 3\sqrt[3]{\Lambda_t}$$
$$= \int_0^t \lambda_s^{-2}\int_0^{\sqrt[3]{\Lambda_s}\lambda_s^{-1}} q_y(\Lambda_s, 1, y^*)\,dy\,ds + 3\sqrt[3]{\Lambda_t}$$

where the last equality is due to the Mean Value Theorem and $y^* \in [0, y]$. It follows from direct computations that $|q_y(t, 1, y)| \le \sqrt{\frac{2}{\pi e}}\frac{1}{t}$ for all $y \in \mathbb{R}$ and $t \in \mathbb{R}_+$. Therefore, we have

$$\int_0^t \mathbb{E}\left[\frac{1}{R_s^2}\right] ds \leq \sqrt{\frac{2}{\pi e}} \int_0^t \lambda_s^{-2} \int_0^{\sqrt[3]{\Lambda_s}\lambda_s^{-1}} \frac{1}{\Lambda_s}\, dy\, ds + 3\sqrt[3]{\Lambda_t}$$
$$= \sqrt{\frac{2}{\pi e}} \int_0^t \lambda_s^{-3} \Lambda_s^{-\frac{2}{3}}\, ds + 3\sqrt[3]{\Lambda_t}$$
$$\leq 3\left(\sqrt{\frac{2}{\pi e}}\lambda_t^{-1} + 1\right)\sqrt[3]{\Lambda_t}.$$

iii) Recall that

$$\frac{H_x}{H}(V(t)-s, X_s) = \frac{1}{X_s} - X_s \eta^t(V^{-1}(s), X_s),$$

where η^t is as defined in (5.71). Thus, due to the uniform boundedness of $\eta^t(V^{-1}(s), \cdot)$ for $s \leq t-\varepsilon$, it suffices to show that

$$\int_0^{t-\varepsilon} \mathbb{E}^{\tau,t}\left[X_s^2 + \frac{1}{X_s^2}\right] ds < \infty.$$

Indeed, in view of the absolute continuity relationship between $\mathbb{P}$ and $\mathbb{P}^{\tau,t}$ the above integral equals

$$\frac{1}{\mathbb{P}(\tau > t)} \int_0^{t-\varepsilon} \mathbb{E}\left[\mathbf{1}_{[\tau>t]} X_s^2 + \mathbf{1}_{[\tau>t]}\frac{1}{X_s^2}\right] ds$$
$$\leq \frac{1}{\mathbb{P}(\tau > t)} \int_0^{t-\varepsilon} \mathbb{E}\left[\mathbf{1}_{[\tau>s]} X_s^2 + \mathbf{1}_{[\tau>s]}\frac{1}{X_s^2}\right] ds,$$

which is finite due to (5.74) and (5.75).

iv) As above,

$$\frac{H_x}{H}(V(t)-V(s), Z_s) = \frac{1}{Z_s} - Z_s \eta^t(s, Z_s),$$

where η^t is as defined in (5.71). Fix an $\varepsilon > 0$ and note that

$$\int_0^{t-\varepsilon} \sigma^2(s) \left|\frac{H_x}{H}(V(t)-V(s), Z_s)\right| ds = \int_0^{V(t-\varepsilon)} \left|\frac{H_x}{H}(V(t)-s, Z_{V^{-1}(s)})\right| ds.$$

Next, consider the process $S_r := Z_{V^{-1}(r)}$ for $r \in [0, V(t))$. Then,

$$\mathbb{E}^{\tau,t}\left[\int_0^{t-\varepsilon}\sigma^2(s)\left|\frac{H_x}{H}(V(t)-V(s),Z_s)\right|ds\right]^2$$
$$=\mathbb{E}^{\tau,t}\left[\int_0^{V(t-\varepsilon)}\left|\frac{1}{S_s}-S_s\eta^t(V^{-1}(s),S_s)\right|ds\right]^2$$
$$\leq 2\left(\mathbb{E}^{\tau,t}\left[\int_0^{V(t-\varepsilon)}\frac{1}{S_s}ds\right]^2+\frac{V(t-\varepsilon)}{(V(t)-V(t-\varepsilon))^2}\int_0^{V(t-\varepsilon)}\mathbb{E}^{\tau,t}[S_s^2]ds\right). \tag{5.77}$$

Moreover, under $\mathbb{P}^{\tau,t}$

$$dS_s^2=(3-2S_s^2\eta^t(V^{-1}(s),S_s))ds+2S_s dW_s^t$$

for all $s < V(t)$ for the Brownian motion W^t defined by $W_s^t := \int_0^{V^{-1}(s)}\sigma^2(r)d\beta_r^t$. Thus,

$$\mathbb{E}^{\tau,t}[S_s^2]\leq 3s+1+\int_0^s\mathbb{E}^{\tau,t}[S_r^2]dr. \tag{5.78}$$

Hence, by Gronwall's inequality, we have $\mathbb{E}^{\tau,t}[S_s^2]\leq(3s+1)e^s$. In view of (5.77) to demonstrate iii) it suffices to show that

$$\mathbb{E}^{\tau,t}\left[\int_0^{V(t-\varepsilon)}\frac{1}{S_s}ds\right]^2<\infty.$$

However,

$$\left(\int_0^{V(t-\varepsilon)}\frac{1}{S_s}ds\right)^2=\left(S_{V(t-\varepsilon)}-S_0-W_{V(t-\varepsilon)}^t+\int_0^{V(t-\varepsilon)}\eta^t(V^{-1}(s),S_s)S_s ds\right)^2,$$

which obviously has a finite expectation due to (5.78) and the boundedness of $\eta^t(V^{-1}(s),\cdot)$ for $s\leq V(t-\varepsilon)$.

5.3 Notes

This chapter is based on our work that appeared in [32] and [34]. Section 5.1 is taken from [32] while Sect. 5.2 follows [34]. The proof of Lemma 5.7 has been modified to fill the gap in its original version.

Part II
Applications

Chapter 6
Financial Markets with Informational Asymmetries and Equilibrium

This chapter introduces the setup for the equilibrium models that extends, among others, the works of Kyle [85] and Back [9]. It also contains some key results that will be relevant for the characterisation of the equilibrium. Finally the equilibrium will be derived and discussed in Chaps. 7 and 8.

A vast number of financial assets change hands every day and, as a result, the prices of these assets fluctuate continuously. What drives the asset prices is the expectation of the assets' future payoffs and a price is set when a buyer and a seller agree on an exchange. At any given point in time, usually there are numerous agents in the market who are interested in trading. Some agents are individuals with relatively small portfolios while others are investments banks or hedge funds who are acting on behalf of large corporations or a collection of individuals. Obviously, these agents have different attitudes toward risk and do not have equal access to trading technologies, information or other relevant resources. Moreover, it is very rare that the cumulative demand of the buyers will be met by the total shares offered by the sellers without any excess supply. An imbalance of demand and supply is the rule rather than an exception in today's markets, which brings *liquidity risk* to the fore. These features of the modern markets challenge the conventional asset pricing theories which assume *perfect competition* and *no* liquidity risk.

Market microstructure theory provides an alternative to *frictionless* models of trading behaviour that assume perfect competition and free entry. To quote O'Hara [94], "[It] is the study of the process and outcomes of exchanging assets under explicit trading rules." Thus, market microstructure analyses how a specific trading mechanism or heterogeneity of traders affects the price formation, comes up with measures for market liquidity and studies the sensitivity of liquidity and other indicators of market behaviour on different trading mechanisms and information heterogeneity.

Although financial markets with informational asymmetries have been widely discussed in the market microstructure literature (see [30] and [94] for a review), the characterisation of the optimal trading strategy of an investor who possess superior

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information has been, until lately, largely unaddressed by the mathematical finance literature.

In recent years, with the development of enlargement of filtrations theory (see [90], models of the so-called *insider trading* had gained attention in mathematical finance as well (see e.g. [7, 21] and [96]). The salient assumptions of these models are that i) the informational advantage of the insider is a functional of the stock price process (e.g. the insider might know in advance the maximum value the stock price will achieve), and that ii) the insider does not affect the stock price dynamics. However, since the stock prices can be viewed as the discounted expectation of the future dividends conditional on market's information, which in particular depend on the total demand for the asset, the price dynamics being unaffected by the insider's trades is not a plausible assumption as they constitute part of the demand.

Thus, a more realistic model should allow the market's valuation change when new information arrives. In markets with asymmetrically informed traders this requires an equilibrium framework where trading is considered as a game among heterogeneously informed participants.

The canonical model of markets with asymmetric information, which will be the starting point of the financial models that will be covered in this book, is due to Kyle [85]. Kyle studies a market for a single risky asset whose price is determined in equilibrium. There are mainly three types of agents that constitute the market: a strategic risk-neutral informed trader with a private information regarding the future value of the asset, non-strategic noise traders, and a number of risk-neutral market makers competing for the net demand from the strategic and non-strategic traders. The key feature of this model is that the market makers cannot distinguish between the informed and uninformed trades and compete to fill the net demand by offering the best prices. The market makers learn from the order flow and they update their pricing strategies as a result of this learning mechanism.

Kyle's model is in discrete time and assumes that the noise traders follow a random walk and the future payoff of the asset has a normal distribution. This has been extended to a continuous time framework with general payoffs by Back [9]. In this extension the total demand of the noise traders is given by a Brownian motion, and the future payoff of the asset has a general continuous distribution while the informed trader still has the perfect information regarding the future value of the asset. Kyle's model and its continuous-time extension by Back have been further extended to allow multiple informed traders [11, 55] and to include default risk [31], where the authors have assumed that the insider knew the default time of the company.

Nevertheless, the assumption of perfect foresight into the future for the informed traders is unrealistic, and a more natural assumption would be that the informed trader has a private signal that converges to the future payoff of the asset. This assumption particularly makes sense when the informed trader is an investment bank with its own research facility. The earliest works in this vein are by Back and Pedersen [12] and Danilova [44], who have assumed that the informed trader received a continuous Gaussian signal as private information.

6.1 Model Setup

As in [9] we will assume that the trading will take place over the time interval [0, 1]. The time-1 value of the traded asset is given by $f(Z_1)$, which will become public knowledge at $t = 1$ to all market participants, where Z is a Markov process that solves

$$Z_t = Z_0 + \int_0^t \sigma(s)a(V(s), Z_s)d\beta_s \tag{6.1}$$

on a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q})$ satisfying the usual conditions, where f is a measurable function, β is a standard Brownian motion adapted to $(\mathcal{G}_t)$, a and σ are deterministic functions, and $V(t) = c + \int_0^t \sigma^2(s)ds$, for some positive constant c. We shall assume that Z admits a transition density. Moreover, f is assumed to be non-decreasing, which entails in particular that the larger the signal Z_1 the larger the value of the risky asset for the informed trader.[1]

We will also consider the possibility of default. The default will happen at some random time T_0 and, thus, the time-1 payoff of the asset is given by $D_1 f(Z_1)$, where $D_t := \mathbf{1}_{[T_0 > t]}$. Chapter 7 assumes no default risk, i.e. $T_0 = \infty$ while default will be a possibility in Chap. 8 and modelled in terms of a first hitting time for Z.

Three types of agents trade in the market. They differ in their information sets, and objectives, as follows.

- *Noise/liquidity traders* trade for liquidity reasons, and their total demand at time t is given by a standard $(\mathcal{G}_t)$-Brownian motion B independent of Z and D.
- *Market makers* observe only the default indicator process, D, and the total demand

  $$Y = \theta + B,$$

  where θ is the demand process of the informed trader. The admissibility condition imposed later on θ will entail in particular that Y is a semimartingale.

  They set the price of the risky asset via a *Bertrand competition* and clear the market. Similar to [10] we assume that the market makers set the price as a function of weighted total order process at time t, i.e. we consider pricing functionals $S\left(Y_{[0,t]}, t\right)$ of the following form

  $$S\left(Y_{[0,t]}, t\right) = D_t H\left(t, X_t\right), \qquad \forall t \in [0, 1) \tag{6.2}$$

  where X is the unique strong solution[2]

  $$dX_t = w(t, X_t)dY_t, \quad \forall t \in [0, 1), \; X_0 = 0 \tag{6.3}$$

[1] In fact we will assume f to be strictly increasing when there is no risk of default.

[2] Following Kurtz [83] X is a strong solution of (6.3) if there exists a measurable mapping, φ, such that $X := \varphi(Y)$ satisfies (6.3). Note that Y may jump due to the discontinuous in θ.

for some deterministic function $w(s, x)$ chosen by the market makers. Moreover, a pricing rule (H, w) has to be admissible in the sense of Definition 6.1. In particular, $H \in C^{1,2}$ and, therefore, S will be a semimartingale as well.

- *The informed trader (insider)* observes the price process $S_t = D_t H(t, X_t)$ where X is given by (6.3), and her private signal, Z. Since she is risk-neutral, her objective is to maximise the expected final wealth, i.e.

$$\sup_{\theta \in \mathscr{A}(H,w)} E^{0,z}\left[W_1^\theta\right], \text{ where} \tag{6.4}$$

$$W_1^\theta = (D_1 f(Z_1) - S_{1-})\theta_{1-} + \int_0^{1-} \theta_{s-} dS_s. \tag{6.5}$$

In above $\mathscr{A}(H, w)$ is the set of admissible trading strategies for the given pricing rule[3] (H, w), which will be defined in Definition 6.2. Moreover, $E^{0,z}$ is the expectation with respect to $P^{0,z}$, which is the probability measure on $\sigma(X_s, Z_s; s \leq 1)$ generated by (X, Z) with $X_0 = 0$ and $Z_0 = z$.

Thus, the insider maximises the expected value of her final wealth W_1^θ, where the first term on the right-hand side of Eq. (6.4) is the contribution to the final wealth due to a potential differential between the market price and the fundamental value at the time of information release, and the second term is the contribution to the final wealth coming from the trading activity.

Given the above market structure we can now precisely define the filtrations of the market makers and of the informed trader. As we shall require them to satisfy the usual conditions, we first define the probability measures that will be used in the completion of their filtrations.

First define $\mathscr{F} := \sigma(B_t, Z_t; t \leq 1)$ and let $Q^{0,z}$ be the probability measure on $\mathscr{F}$ generated by (B, Z) with $B_0 = 0$ and $Z_0 = z$. Observe that any $P^{0,z}$-null set is also $Q^{0,z}$-null due to the assumption that X is the unique solution of (6.3). and the probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$ by

$$\mathbb{P}(E) = \int_{\mathbb{R}} Q^{0,z}(E)\mathbb{Q}(Z_0 \in dz), \tag{6.6}$$

for any $E \in \mathscr{F}$.

While $Q^{0,z}$ is how the informed trader assign likelihood to the events generated by B and Z, $\mathbb{P}$ is the probability distribution of the market makers who do not observe Z_0 exactly. Thus, the market makers' filtration, denoted by $\mathscr{F}^M$, will be the right-continuous augmentation with the $\mathbb{P}$-null sets of the filtration generated by D and Y.

[3] Note that this implies the insider's optimal trading strategy takes into account the *feedback effect*, i.e. that prices react to her trading strategy.

On the other hand, since the informed trader knows the value of Z_0 perfectly, it is plausible to assume that her filtration is augmented with the $Q^{0,z}$-null sets. However, this will make the modelling cumbersome since the filtration will have an extra dependence on the value of Z_0 purely for technical reasons. Another natural choice is to consider the null sets that belong to every $Q^{0,z}$, i.e. the sets that are elements of the following

$$\mathscr{N}^I := \{E \subset \mathscr{F} : Q^{0,z}(E) = 0, \ \forall z \in \mathbb{R}\}. \tag{6.7}$$

These null sets will correspond to the *a priori* beliefs that the informed trader has about the model before she is given the private information about Z_0 and, thus, can be used as a good benchmark for comparison. Therefore we assume that the informed trader's filtration, denoted by $\mathscr{F}^I$, is the right continuous augmentation of the filtration generated by S, Z and D with the sets of $\mathscr{N}^I$.

We are finally in a position to give a rigorous definition of the rational expectations equilibrium of this market, i.e. a pair consisting of an *admissible* pricing rule and an *admissible* trading strategy such that: a) given the pricing rule the trading strategy is optimal, b) given the trading strategy, there exists a unique strong solution, X_t, of (6.3) over the time interval [0, 1), and the pricing rule is *rational* in the following sense:

$$D_t H(t, X_t) = S_t = \mathbb{E}\left[D_1 f(Z_1)|\mathscr{F}_t^M\right], \tag{6.8}$$

where $\mathbb{E}$ corresponds to the expectation operator under $\mathbb{P}$. To formalise this definition of equilibrium, we first define the sets of admissible pricing rules and trading strategies.

Definition 6.1 An *admissible pricing rule* is any pair (H, w) fulfilling the following conditions:

1. $w : [0, 1] \times \mathbb{R} \mapsto (0, \infty)$ is a function in $C^{1,2}([0, 1] \times \mathbb{R})$;
2. Given a Brownian motion, β, on some filtered probability space, there exists a unique strong solution to

$$d\tilde{X}_t = w(t, \tilde{X}_t)d\beta_t, \qquad \tilde{X}_0 = 0.$$

3. $H \in C^{1,2}([0, 1) \times \mathbb{R})$.
4. $x \mapsto H(t, x)$ is strictly increasing for every $t \in [0, 1)$;

Suppose that (H, w) is an admissible pricing rule and the market makers face the total demand, $Y = B + \theta$, where θ is an admissible trading strategy in the sense of Definition 6.2. Then the price set by the market makers is $S_t = H(t, X_t)$, where X is the unique strong solution of

$$dX_t = w(t, X_{t-})dY_t^c + dC_t + J_t, \quad X_0 = 0, \tag{6.9}$$

over the time interval [0, 1] on $(\Omega, \mathscr{F}, (\mathscr{F}_t^M), \mathbb{P})$. In above

$$\begin{aligned} dC_t &= \frac{1}{2} w_x(t, X_{t-}) w(t, X_{t-}) \left(d[Y, Y]_t^c - dt \right), \\ J_t &= K_w^{-1}(t, K_w(t, X_{t-}) + \Delta Y_t) - X_{t-}, \text{ and} \\ K_w(t, x) &= \int_0^x \frac{1}{w(t, y)} dy + \frac{1}{2} \int_0^t w_x(s, 0) ds. \end{aligned} \tag{6.10}$$

Note that in view of Definition 6.1 the set of admissible θ for which there exists a strong solution to (6.9) is not empty.

Remark 6.1 Observe that $K_w(t, \tilde{X}_t) = \beta_t$ by Ito's formula. Thus, the existence of a unique strong solution in Definition 6.1 implies $K_w(t, \cdot)$ has neither an upper nor a lower bound for every $t \in [0, 1]$. Indeed, since $K^{-1}(t, \beta_t)$ is the solution and $\mathbb{P}(\tau_x < t) > 0$ for all $t \in (0, 1]$ and $x \in \mathbb{R}$, where $\tau_x := \inf\{s \geq 0 : \beta_s = x\}$, existence of an upper or a lower bound implies that the solution explodes on (0, 1) with positive probability. Consequently, $K_w^{-1}(t, \cdot) : \mathbb{R} \mapsto \mathbb{R}$ for every $t \in [0, 1]$.

Remark 6.2 The strict monotonicity of H in the space variable implies H is invertible prior to time 1, thus, the filtration of the insider is generated by X and Z. Note that jumps of Y can be inferred from the jumps of X via (6.9) and the form of J. Moreover, since $K_w \in C^{1,2}$ under the hypothesis on w, an application of Ito's formula yields

$$dK_w(t, X_t) = dY_t^c - \frac{1}{2} w_x(t, X_{t-}) dt + K_w(t, X_t) - K_w(t, X_{t-}) + \frac{\partial}{\partial t} K_w(t, X_{t-}) dt.$$

Thus, one can also obtain the dynamics of Y^c by observing X. Hence, the natural filtrations of X and Y coincide. This in turn implies that $(\mathscr{F}_t^{S,Z}) = (\mathscr{F}_t^{B,Z})$, i.e. the insider has full information about the market. In the sequel we assume that the filtration $(\mathscr{F}_t^{B,Z})_{t \in [0,1]}$ is completed with the sets in $\mathscr{N}^I$, which yields its right continuity in view of Theorem 1.5. Note that the standing assumption on σ and a as well as the independent of B and Z imply that (B, Z) is a strong Markov process by Corollary 2.5 and Remark 2.4.

In view of the above one can take $\mathscr{F}_t^I = \mathscr{F}_t^{B,Z}$ for all $t \in [0, 1]$.

Definition 6.2 An $\mathscr{F}^{B,Z}$-adapted θ is said to be an admissible trading strategy for a given pair (H, w) if

1. θ is a semimartingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t^{B,Z}), Q^{0,z})$ for each $z \in \mathbb{R}$;
2. There exists a unique strong solution, X, to (6.9) over the time interval [0, 1] on $(\Omega, \mathscr{F}, (\mathscr{F}_t^{B,Z}), \mathbb{P})$, where $Y = B + \theta$;
3. and no doubling strategies are allowed, i.e. for all $z \in \mathbb{R}$

$$E^{0,z} \left[\int_0^1 D_t H^2(t, X_t) \, dt \right] < \infty. \tag{6.11}$$

The set of admissible trading strategies for the given pair (H, w) is denoted with $\mathscr{A}(H, w)$.

Remark 6.3 Observe that the jumps of θ are summable since θ is a semimartingale in a Brownian filtration and, thus, the local martingale in its decomposition is continuous. This in particular implies that Y is a semimartingale with summable jumps and the price process is well defined.

It is standard (see, e.g. [12, 39] or [111]) in the insider trading literature to limit the set of admissible strategies to absolutely continuous ones motivated by the result in Back [9]. We will prove this in Theorem 6.1 in a more general setting that the insider does not make any extra gain if she does not employ continuous strategies of finite variation.

In view of the above discussion we can limit the admissible strategies of the insider to the absolutely continuous ones denoted by $\mathscr{A}_c(H, w)$. Indeed, Theorem 6.1 shows that under a mild condition the value function of the insider is unchanged if the insider is only allowed to use absolutely continuous strategies. Moreover, even if these conditions are not met, Theorem 6.1 also demonstrates that an absolutely continuous strategy that brings the market price to the fundamental value of the asset is optimal within $\mathscr{A}$. In the models that we consider in the following chapters such a strategy will always exist and the equilibrium pricing rule will satisfy the conditions of Theorem 6.1. Consequently the restriction to $\mathscr{A}_c$ is without loss of generality.

Now we can formally define the market equilibrium as follows.

Definition 6.3 A couple $((H^*, w^*), \theta^*)$ is said to form an equilibrium if (H^*, w^*) is an admissible pricing rule, $\theta^* \in \mathscr{A}_c(H^*, w^*))$, and the following conditions are satisfied:

1. *Market efficiency condition:* given θ^*, (H^*, w^*) is a rational pricing rule, i.e. it satisfies (6.8).
2. *Insider optimality condition:* given (H^*, w^*), θ^* solves the insider optimisation problem:

$$\mathbb{E}[W_1^{\theta^*}] = \sup_{\theta \in \mathscr{A}_c(H^*, w^*)} \mathbb{E}[W_1^{\theta}].$$

6.2 On Insider's Optimal Strategy

This section assumes no default risk, i.e. $T_0 = \infty$, due to the fact that the optimality considerations for the informed trader require certain technicalities that needs to be addressed carefully as we will see in Chap. 8. However, in order not to obscure the main ideas with non-essential details, we merely emphasise here that the same result holds when there is a possibility of default and invite the reader to apply the necessary modifications after getting familiar with the proof of Proposition 8.1.

Before showing that the strategies with discontinuous paths or with paths of infinite variation are suboptimal let us informally deduce the Hamilton–Jacobi–Bellmann (HJB) equation associated with the value function of the insider assuming absolutely continuous trading strategies.

To this end assume that there is no default risk, i.e. $D \equiv 1$. Let (H, w) be any rational pricing rule and suppose that $d\theta_t = \alpha_t dt$. First, notice that a standard application of integration-by-parts formula applied to W_1^θ gives

$$W_1^\theta = \int_0^1 (f(Z_1) - S_s)\alpha_s\, ds. \tag{6.12}$$

Furthermore,

$$E^{0,z}\left[\int_0^1 (f(Z_1) - S_s)\alpha_s ds\right] = E^{0,z}\left[\int_0^1 (E^{0,z}[f(Z_1)|\mathscr{F}_s^I] - S_s)\alpha_s ds\right]. \tag{6.13}$$

Define the value, P, of the stock for the insider by

$$P_t := E^{0,z}[f(Z_1)|\mathscr{F}_t^I] = E^{0,z}[f(Z_1)|\mathscr{F}_t^Z] = F(t, Z_t), \tag{6.14}$$

for some measurable function $F : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ (due to the independence between Z and B and the Markov property of Z), where $(\mathscr{F}_t^Z)_{t\geq 0}$ corresponds to the natural filtration of Z augmented with the sets in $\mathscr{N}^I$, assuming that $f(Z_1)$ is integrable. Assuming more technical conditions if necessary one can deduce that $F \in C^{1,2}([0, 1] \times \mathbb{R})$ and satisfies

$$F_t(t, z) + \frac{1}{2}\sigma^2(t)a^2(V(t), z)F_{zz}(t, z) = 0. \tag{6.15}$$

In view of (6.12) and (6.13), insider's optimisation problem becomes

$$\sup_\theta E^{0,z}[W_1^\theta] = \sup_\theta E^{0,z}\left[\int_0^1 (F(s, Z_s) - H(s, X_s))\alpha_s ds\right]. \tag{6.16}$$

Recall that the market price is given by $H(t, X_t)$ with

$$dX_t = w(t, X_t)\left\{\alpha_t dt + dB_t\right\}.$$

Let us now introduce the value function of the insider:

$$\phi(t, x, z) := \operatorname{ess\,sup}_\alpha E^{0,z} \times \left[\int_t^1 (F(s, Z_s) - H(s, X_s))\alpha_s ds | X_t = x, Z_t = z\right], \quad t \in [0, 1].$$

Applying formally the dynamic programming principle, we get the following HJB equation:

$$
\begin{aligned}
0 = \sup_{\alpha} &\left([w(t,x)\phi_x + F(t,z) - H(t,x)]\,\alpha\right) \\
&+\phi_t + \frac{1}{2}w^2(t,x)\phi_{xx} + \frac{1}{2}\sigma^2(t)a^2(V(t),z)\phi_{zz}
\end{aligned}
\tag{6.17}
$$

Thus, for the finiteness of the value function and the existence of an optimal α we need

$$
w(t,x)\phi_x + F(t,z) - H(t,x) = 0 \tag{6.18}
$$

$$
\phi_t + \frac{1}{2}w^2(t,x)\phi_{xx} + \frac{1}{2}\sigma^2(t)a^2(V(t),z)\phi_{zz} = 0 \tag{6.19}
$$

Differentiating (6.18) with respect to x and since from (6.18) it follows that $\phi_x = \frac{H(t,x)-F(t,z)}{w(t,x)}$, we get

$$
w^2(t,x)\phi_{xx} = H_x(t,x)w(t,x) + (F(t,z) - H(t,x))w_x(t,x) \tag{6.20}
$$

Plugging (6.20) into (6.19) yields:

$$
\begin{aligned}
\phi_t &+ \frac{1}{2}\left(H_x(t,x)w(t,x) + (F(t,z) - H(t,x))w_x(t,x)\right) \\
&+\frac{1}{2}\sigma^2(t)a^2(V(t),z)\phi_{zz} = 0
\end{aligned}
\tag{6.21}
$$

Differentiating (6.18) with respect to z gives $\phi_{xz} = -\frac{F_z(t,z)}{w(t,x)}$ and therefore $\phi_{zzx} = -\frac{F_{zz}(t,z)}{w(t,x)}$. Thus, after differentiating (6.21) with respect to x we obtain:

$$
\begin{aligned}
\phi_{tx} &+ \frac{1}{2}\left(H_{xx}(t,x)w(t,x) + (F(t,z) - H(t,x))w_{xx}(t,x)\right) \\
&-\sigma^2(t)\frac{a^2(V(t),z)}{2w(t,x)}F_{zz}(t,z) = 0
\end{aligned}
\tag{6.22}
$$

Since differentiation (6.18) with respect to t gives

$$
\phi_{xt} = \frac{w_t(t,x)}{w^2(t,x)}(F(t,z) - H(t,x)) - \frac{1}{w(t,x)}(F_t(t,z) - H_t(t,x)),
$$

(6.22), in view of (6.15), implies

$$(H(t,x) - F(t,z))\left\{w_t(t,x) + \frac{w^2(t,x)}{2}w_{xx}(t,x)\right\}$$
$$= w(t,x)\left(H_t(t,x) + \frac{1}{2}w^2(t,x)H_{xx}(t,x)\right). \tag{6.23}$$

Since the right-hand side of (6.23) is not a function of z, we must have

$$w_t(t,x) + \frac{w^2(t,x)}{2}w_{xx}(t,x) = 0, \tag{6.24}$$

$$H_t(t,x) + \frac{1}{2}w^2(t,x)H_{xx}(t,x) = 0. \tag{6.25}$$

Thus, the last two equations seem to be necessary to have a finite solution to the insider's problem.

Remark 6.4 The system of Eqs. (6.24)–(6.25) also implies that the market makers can choose $H(t,x) = x$ as their pricing rule without loss of generality. Indeed, if the market makers choose a pricing rule (H, w) satisfying the above equations and the insider employs an absolutely continuous strategy, θ, then Ito formula directly yields

$$dS_t = \tilde{w}(t, S_t)dY_t,$$

where

$$\tilde{w}(t,x) = H_x(t, H^{-1}(t,x))w(t, H^{-1}(t,x)).$$

Direct calculations show that $\tilde{w}$ satisfies (6.24). Despite (H, w) being an over-parametrisation of the price process we nevertheless use it as a pricing rule since it allows us to consider price processes with or without weighting of the past signal in the same framework.

We shall next see that the Eqs. (6.24)–(6.25) also imply the insider must use continuous strategies of finite variation under a mild admissibility condition.

Theorem 6.1 *Suppose that*

$$E^{0,z}\left(f^2(Z_1)\right) < \infty, \ \forall z \in \mathbb{R}, \tag{6.26}$$

i.e. $f(Z_1)$ is square integrable for any initial condition of Z. Further assume that there is no default risk, i.e. $D \equiv 1$. Let (H, w) be an admissible pricing rule satisfying (6.24) and (6.25). Then $\theta \in \mathscr{A}(H, w)$ is an optimal strategy if

i) θ is continuous and of finite variation,
ii) and $H(1-, X_{1-}) = f(Z_1)$, $P^{0,z}$-a.s.,

where

$$X_t = \int_0^t w(s, X_s)\{dB_s + d\theta_s\}.$$

Moreover, if we further assume that f and H are bounded from below and

$$E^{0,z}\left[K_w^2(1, H^{-1}(1, f(Z_1)))\right] < \infty, \qquad \forall z \in \mathbb{R}, \tag{6.27}$$

then any $\theta \in \mathscr{A}(H, w)$, there exists a sequence of admissible absolutely continuous *strategies, $(\theta^n)_{n\geq 1}$, such that*

$$E^{0,z}\left[W_1^\theta\right] \leq \lim_{n\to\infty} E^{0,z}\left[W_1^{\theta^n}\right].$$

Proof Using Ito's formula for general semimartingales (see, e.g. Theorem II.32 in [99]) we obtain

$$\begin{aligned} dH(t, X_t) &= H_t(t, X_{t-})dt + H_x(t, X_{t-})dX_t + \frac{1}{2}H_{xx}(t, X_{t-})d[X, X]_t^c \\ &\quad + \{H(t, X_t) - H(t, X_{t-}) - H_x(t, X_{t-})\Delta X_t\} \\ &= H_x(t, X_{t-})w(t, X_{t-})dY_t^c + dFV_t, \end{aligned}$$

where FV is of finite variation. Therefore,

$$\begin{aligned} [\theta, S]_t^c &= \int_0^t H_x(s, X_{s-})w(s, X_{s-})d[Y^c, \theta]_s \\ &= \int_0^t H_x(s, X_{s-})w(s, X_{s-})\left\{d[B, \theta]_s + d[\theta, \theta]_s^c\right\}. \end{aligned} \tag{6.28}$$

Moreover, integrating (6.5) by parts (see Corollary 2 of Theorem II.22 in [99]) we get

$$W_1^\theta = f(Z_1)\theta_{1-} - \int_0^{1-} H(t, X_{t-}))d\theta_t - [\theta, H(\cdot, X)]_{1-} \tag{6.29}$$

since the jumps of θ are summable.

Consider the function

$$\Psi^a(t, x) := \int_{\xi(t,a)}^x \frac{H(t, u) - a}{w(t, u)} du + \frac{1}{2}\int_t^1 H_x(s, \xi(s, a))w(s, \xi(s, a))ds \tag{6.30}$$

where $\xi(t,a)$ is the unique solution of $H(t,\xi(t,a)) = a$. Direct differentiation with respect to x gives that

$$\Psi_x^a(t,x)w(t,x) = H(t,x) - a. \tag{6.31}$$

Differentiating above with respect to x gives

$$\Psi_{xx}^a(t,x)w^2(t,x) = w(t,x)H_x(t,x) - (H(t,x)-a)\,w_x(t,x). \tag{6.32}$$

Direct differentiation of $\Psi^a(t,x)$ with respect to t gives

$$\begin{aligned}
\Psi_t^a(t,x) &= \int_{\xi(t,a)}^x \frac{H_t(t,u)}{w(t,u)}du - \int_{\xi(t,a)}^x \frac{(H(t,u)-a)w_t(t,u)}{w^2(t,u)}du \\
&\quad -\frac{1}{2}H_x(t,\xi(t,a))w(t,\xi(t,a)) \\
&= \int_{\xi(t,a)}^x \frac{H_t(t,u)}{w(t,u)}du + \frac{1}{2}\int_{\xi(t,a)}^x (H(t,u)-a)dw_x(t,u) \\
&\quad -\frac{1}{2}H_x(t,\xi(t,a))w(t,\xi(t,a)) \\
&= \frac{1}{2}\left((H(t,x)-a)w_x(t,x) - H_x(t,x)w(t,x)\right)
\end{aligned} \tag{6.33}$$

where the second equality is due to (6.24) and the last one follows from integrating the second integral by parts twice and using (6.25). Combining (6.32) and (6.33) gives

$$\Psi_t^a + \frac{1}{2}w(t,x)^2\Psi_{xx}^a = 0. \tag{6.34}$$

Applying Ito's formula we deduce

$$\begin{aligned}
d\Psi^a(t,X_t) &= \Psi_t(t,X_{t-})dt + \frac{H(t,X_{t-})-a}{w(t,X_{t-})}\left(w(t,X_{t-})dY_t^c + dC_t\right) \\
&\quad +\frac{1}{2}\Psi_{xx}(t,X_{t-})w^2(t,X_{t-})d[Y,Y]_t^c + \Psi^a(t,X_t) - \Psi^a(t,X_{t-}) \\
&= (H(t,X_{t-})-a)\left(dY_t^c + \frac{1}{2}w_x(t,X_{t-})(d[Y,Y]_t^c - dt)\right) \\
&\quad +\frac{1}{2}\Psi_{xx}(t,X_{t-})w^2(t,X_{t-})\left(d[Y,Y]_t^c - dt\right) + \Psi^a(t,X_t) - \Psi^a(t,X_{t-}) \\
&= (H(t,X_{t-})-a)\,dY_t^c + \frac{1}{2}w(t,X_{t-})H_x(t,X_{t-})(d[Y,Y]_t^c - dt) \\
&\quad +\Psi^a(t,X_t) - \Psi^a(t,X_{t-})
\end{aligned}$$

where one to the last equality follows from (6.34) and the last one is due to (6.32).

The above implies

$$\begin{aligned}\Psi^a(1-,X_{1-}) &= \Psi^a(0,0) + \int_0^{1-} H(t,X_{t-})(dB_t + d\theta_t) - a(B_1+\theta_{1-}) \\ &\quad + \frac{1}{2}\int_0^{1-} w(t,X_{t-})H_x(t,X_{t-})(d[Y,Y]^c_t - dt) \\ &\quad + \sum_{0<t<1}\left\{\Psi^a(t,X_t) - \Psi^a(t,X_{t-}) - (H(t,X_{t-})-a)\Delta\theta_t\right\}\end{aligned}$$

Combining the above and (6.29) yields

$$\begin{aligned}&E^{0,z}\left[W_1^\theta\right] \\ &= E^{0,z}\Bigg[\Psi^{f(Z_1)}(0,0) - \Psi^{f(Z_1)}(1-,X_{1-}) - f(Z_1)B_1 + \int_0^{1-} H(t,X_{t-})dB_t \\ &\quad + \frac{1}{2}\int_0^{1-} w(t,X_{t-})H_x(t,X_{t-})(2d[B,\theta]_t + d[\theta,\theta]^c_t) \\ &\quad + \sum_{0<t<1}\left\{\Psi^{f(Z_1)}(t,X_t) - \Psi^{f(Z_1)}(t,X_{t-}) - (H(t,X_{t-}) - f(Z_1))\Delta\theta_t\right\} \\ &\quad - \int_0^1 H_x(s,X_{s-})w(s,X_{s-})\left\{d[B,\theta]_s + d[\theta,\theta]^c_s\right\} \\ &\qquad - \sum_{0<t<1}(H(t,X_t) - H(t,X_{t-}))\Delta\theta_t\Bigg] \\ &= E^{0,z}\Bigg[\Psi^{f(Z_1)}(0,0) - \Psi^{f(Z_1)}(1-,X_{1-}) - \frac{1}{2}\int_0^{1-} w(t,X_{t-})H_x(t,X_{t-})d[\theta,\theta]^c_t \\ &\quad + \sum_{0<t<1}\left\{\Psi^{f(Z_1)}(t,X_t) - \Psi^{f(Z_1)}(t,X_{t-}) - (H(t,X_t) - f(Z_1))\Delta\theta_t\right\}\Bigg] \\ &\le E^{0,z}\left[\Psi^{f(Z_1)}(0,0) - \Psi^{f(Z_1)}(1-,X_{1-})\right]\end{aligned}$$

since w is positive and H is increasing, and

$$\begin{aligned}&\Psi^a(t,X_t) - \Psi^a(t,X_{t-}) - (H(t,X_t)-a)\Delta\theta_t \\ &\quad = \int_{X_{t-}}^{X_t} \frac{H(t,u)-a}{w(t,u)}du - (H(t,X_t)-a)\Delta\theta_t \\ &\quad \le (H(t,X_t)-a)\int_{X_{t-}}^{X_t} \frac{1}{w(t,u)}du - (H(t,X_t)-a)\Delta\theta_t\end{aligned}$$

$$= (H(t, X_t) - a) (K_w(t, X_t) - K_w(t, X_{t-}) - \Delta\theta_t) = 0.$$

Note the inequality above becomes equality if and only if $\Delta\theta_t = 0$ due to the strict monotonicity of H. Moreover, $\Psi^{f(Z_1)}(1-, X_{1-}) \geq 0$ with an equality if and only if $H(1-, X_{1-}) = f(Z_1)$. Therefore, $E^{0,z}\left[W_1^{\theta}\right] \leq E^{0,z}\left[\Psi^{f(Z_1)}(0, 0)\right]$ for all admissible θs end equality is reached if and only if the following two conditions are met.

i) θ is continuous and of finite variation.
ii) $H(1-, X_{1-}) = f(Z_1)$, $P^{0,z}$-a.s.

Hence, the proof will be complete if one can find a sequence of absolutely continuous admissible strategies, $(\theta^n)_{n\geq 1}$ such that $\lim_{n\to\infty} E^{0,z}\left[W_1^{\theta^n}\right] = E^{0,z}\left[\Psi^{f(Z_1)}(0, 0)\right]$.

Define

$$M_t^m := E^{0,z}\left[K_w(1, H^{-1}(1, f(Z_1) \wedge m))|\mathscr{F}_t^Z\right]$$

and observe that M^m is independent of B for each $m \geq 1$. Consider the bridge process, Y^m, that starts at 0 and ends up at M_1^m at $t = 1$:

$$Y_t^m := B_t + \int_0^t \frac{M_s^m - Y_s^m}{1-s} ds.$$

The above SDE has a unique strong solution given by

$$Y_t^m = (1-t)\left(\int_0^t \frac{1}{1-s} dB_s + \int_0^t \frac{M_s^m}{(1-s)^2} ds\right),$$

and it is easy to check that the above converges a.s. to M_1^m using the continuity of M^m and L'Hospital rule since $(1-t)\int_0^t \frac{1}{1-s} dB_s$ is the Brownian bridge from 0 to 0 as in Example 4.3.

Next define the stopping times

$$\tau^{n,m} := \inf\{t : Y_t^m \geq n\}$$

with the convention that $\inf\emptyset = 1$, and introduce the sequence of trading strategies, $\theta^{n,m}$ given by

$$d\theta_t^{n,m} = \mathbf{1}_{[\tau^{n,m}\geq t]} \frac{M_t^m - Y_t^m}{1-t} dt - \mathbf{1}_{[\tau^{n,m}<t]} \frac{p_x}{p}(1-t, n+1-R_t, n+1) dt,$$

where p is the transition density of three-dimensional Bessel process with respect to its speed measure and R is the three-dimensional Bessel bridge from 1 to $n+1$ over $[\tau^{n,m}, 1)$ (see Example 4.4). In particular R satisfies on $[\tau^{n,m}, 1)$

$$dR_t = dB_t + \frac{p_x}{p}(1-t, R_t, n+1)dt, \quad R_{\tau^{n,m}} = 1.$$

Thus, the total demand process corresponding to $\theta^{n,m}$ satisfies

1. $\sup_{t\in[0,1]} Y^{n,m} \le n+1$, a.s.;
2. $Y^{n,m}\mathbf{1}_{[\tau^{n,m}=1]} = Y_1^m = K_w(1, H^{-1}(1, f(Z_1)\wedge m))$, a.s.;
3. $Y^{n,m}\mathbf{1}_{[\tau^{n,m}<1]} = 0$, a.s.

In view of Remark 6.1 as well as the continuity and the boundedness from below of $H(1, K_w^{-1}(1,\cdot))$ we deduce that $H(t, K_w^{-1}(t,\cdot))$ is bounded from below uniformly in t as it solves the backward heat equation. Therefore, $H(t, K_w^{-1}(t, Y_t^{n,m}))$ is bounded uniformly in t yielding $\theta^{n,m}$ admissible for each n and m.

Recall that since $\theta^{n,m}$ is absolutely continuous, we have

$$E^{0,z}[W^{\theta^{n,m}}] = E^{0,z}\left[\Psi^{f(Z_1)}(0,0) - \Psi^{f(Z_1)}(1-, K_w^{-1}(1, Y_1^{n,m}))\right].$$

On the other hand,

$$\begin{aligned}
&\Psi^{f(Z_1)}(1-, K_w^{-1}(1, Y_1^{n,m}))\\
&\quad\le (H(1, K_w^{-1}(1, Y_1^{n,m})) - f(Z_1))(Y_1^{n,m} - K_w(1, H^{-1}(1, f(Z_1))))\\
&\quad= -\mathbf{1}_{[\tau^{n,m}<1]}(H(1, K_w^{-1}(1,0)) - f(Z_1))K_w(1, H^{-1}(1, f(Z_1)))\\
&\qquad+\mathbf{1}_{[\tau^{n,m}=1, f(Z_1)>m]}(m - f(Z_1))(K_w(1, H^{-1}(1,m))\\
&\qquad-K_w(1, H^{-1}(1, f(Z_1)))).
\end{aligned}$$

Since $f(Z_1)K_w(1, H^{-1}(1, f(Z_1)))$ is integrable and $\mathbf{1}_{[\tau^{n,m}=1]}$ is increasing in n, applying the dominated convergence theorem for the first term and the monotone convergence to the second yield

$$\begin{aligned}
&\lim_{n\to\infty} E^{0,z}\left[\Psi^{f(Z_1)}(1-, K_w^{-1}(1, Y_1^{n,m}))\right]\\
&\le E^{0,z}\left[\mathbf{1}_{[f(Z_1)>m]}(m - f(Z_1))(K_w(1, H^{-1}(1,m)) - K_w(1, H^{-1}(1, f(Z_1))))\right]\\
&\le E^{0,z}\left[((f(Z_1)-m)^+)^2\right]\\
&\qquad\times E^{0,z}\left[\left(\left(K_w(1, H^{-1}(1, f(Z_1))) - K_w(1, H^{-1}(1,m))\right)^+\right)^2\right],
\end{aligned}$$

which converges to 0 as $m\to\infty$ in view of the dominated convergence theorem and the square integrability of $f(Z_1)$ and $K_w(1, H^{-1}(1, f(Z_1)))$. Hence,

$$\lim_{m\to\infty}\lim_{n\to\infty} E^{0,z}[W^{\theta^{n,m}}] = E^{0,z}\left[\Psi^{f(Z_1)}(0,0)\right],$$

i.e. the expected wealth corresponding to our sequence of admissible strategies converges to the upper limit of the value function.

6.3 Notes

The model introduced in this chapter is inspired by [32] and [33], which extends the models of [9] and [12] to allow general Markovian signals and default risk, respectively. We revised the definition of the pricing rule to allow the market makers to simultaneously price the jumps of and weight the demand process.

Characterisation of the optimal strategy of the insider as an absolutely continuous process driving the market price to its fundamental value goes back to Back [9]. Theorem 6.1 proves this characterisation in the more general setting of this chapter using techniques from [111] and [44]. The folk result that the value function of the insider remains the same when the set of admissible strategies is restricted to absolutely continuous strategy is first proved in [44] in a special case of the setup considered here. However, the result therein is not applicable to the setting of this chapter. Thus, in Theorem 6.1 we developed new techniques to prove this result in the present setting.

As the focus of this book is dynamic Markov bridges, we concentrate on the extension of the Kyle's model to the case of dynamically informed insider. However, there are numerous other extensions of Kyle's model available in the literature when the insider's signal is static.

The case of risk averse insider with exponential utility was first studied by Holden and Subrahmanyam [67] in discrete time. The authors characterised the equilibrium as the solution of a system of equations, which they were able to solve numerically. This model was brought to continuous time by Baruch [15], who limited the insider's strategies to absolutely continuous ones with speed of trading being an affine function of equilibrium price. To the best of our knowledge this is also the first paper that allowed the market maker to use pricing rules that go beyond being functions of solely time and demand process by applying a (deterministic) weighting to the demand. Later Cho [39] has shown that the affine strategy of Baruch is optimal in a larger class of absolutely continuous strategies. He also established the same characterisation of the optimal strategies as in Theorem 6.1.

Holden and Subrahmanyam [66] allow multiplicity of insiders having the same information trading in the market. They found via numerical analysis that the competition among the insiders lead to a faster revelation of their private information. In fact in the continuous time limit of their model the insiders would reveal their information immediately. This observation led to the study of the case when the insiders' private signals are not perfectly correlated. This was first done by Foster and Viswanathan [55] in discrete time and later extended to continuous time by Back et al. [11].

Although generalisation of the noise traders' demand process has attracted relatively less attention in the literature, there are nevertheless several works that address this issue. Volatility of the noise traders fluctuating randomly over time was considered in the recent work of Colin-Dufresne and Fos [41]. Biagini et al. [20] studied the Kyle's model when the noise demand follows a fractional Brownian motion. Corcuera et al. [42] in the context of noise demand process being a Lévy process concluded that an equilibrium cannot exist when a jump component is present. Indeed Çetin and Xing [37] have shown that the insider must follow a mixed strategy, i.e. apply an additional randomisation, in the equilibrium when the noise buy and sell orders follow Poisson processes.

Empirical studies suggest that a model with risk averse market makers is more realistic (see, e.g. [22, 26, 63, 69, 88]). However, to the best of our knowledge, there are only two papers that tackle this problem. Subrahmanyam [110] allowed the market makers to be risk averse in a one-period Kyle model. Çetin and Danilova [36] have shown that an equilibrium exists in the continuous time version of the Kyle model with risk averse market makers. Existence of an equilibrium with dynamic inside information and risk averse market makers is still an open question.

An alternative way of modelling insider trading is to assume that the insider is small enough not to affect the market prices. In this case the natural approach is via the enlargement of filtrations theory. There is a vast literature on this topic with varying assumptions on insider's utility function and the kind of information she possesses. As certain kinds of information, such as knowing the terminal stock price, can lead to arbitrage opportunities for the insider in the absence of price impact, a substantial part of this literature is devoted to the analysis of arbitrage opportunities for the insider (see, among others, [1, 2, 54, 60, 71, 72]). Another stream of literature within this framework studies the optimal portfolio choice for the insider in the absence of arbitrage opportunities. Of particular interest is the finiteness of utility, and the relationship between the utility of the additional information and the Shannon entropy (see [4–7, 43, 45, 65, 82, 96]). Finally, the works of Kohatsu-Higa and Ortiz-Latorre aim at building a bridge between the equilibrium and enlargement of filtrations approaches via the concept of weak Kyle–Back equilibrium that they have developed in [80] and [81].

Chapter 7
Kyle–Back Model with Dynamic Information: No Default Case

In this chapter we will illustrate how the dynamic bridge construction from Chap. 5 can be employed to solve the Kyle–Back model introduced in the previous chapter when there is no default risk.

That is, the private signal of the informed trader follows the Markov process, Z, given by (6.1). To be able to utilise the results of Chap. 5 we assume that a and σ satisfy Assumption 5.1 and the probability density of Z_0 is $G(0, 0; c, \cdot)$ with G given by (5.16). Additionally, we will require a to satisfy the following.

Assumption 7.1

$$a_t(t, z) + \frac{a^2(t, z)}{2} a_{zz}(t, z) = 0 \tag{7.1}$$

Remark 7.1 Note that if a satisfies (7.1), then (5.9) yields

$$b(t, x) = -\frac{1}{2} a_z(t, 0),$$

which is absolutely continuous. Therefore, Assumption 5.2 is automatically satisfied. In this case $U_t = K(V(t), Z_t)$, where K is defined in (5.7), is a Gaussian process.

Remark 7.2 PDE (7.1) admits many explicit solutions satisfying the properties listed in Assumption 5.1. Here are few examples.

(i) $a(t, z) = a_0$ for some constant $a_0 > 0$, which is the case already studied by Back and Pedersen [12];
(ii) $a(t, z) = \sqrt{k_1(z + k_2)^2 + k_3 e^{-k_1 t}}$, where k_1, k_2, k_3 are positive constants.
(iii) $a(t, z) = \frac{g(z)}{\sqrt{k_1 t + k_2}}$ where g is a strictly positive solution to $\frac{k_1}{g} = g''$.

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(iv) (Self-similar solution) $a(t, z) = y(z/\sqrt{t})$, where $y(x)$ satisfies $y^2 y_{xx} y_x x = 0$ and is strictly positive.

(v) (Generalised self-similar solution) $a(t, z) = e^{-2k_1 t} y(z e^{2k_1 t})$, where $y(x)$ satisfies

$$-\frac{1}{2} y^2 y_{xx} = 2k_1 x y_x - 2k_1 y$$

and is strictly positive.

Assumption 7.2 *$f : \mathbb{R} \mapsto \mathbb{R}$ is a non-decreasing and non-constant continuous function such that $\mathbb{E}[f^2(Z_1)] < \infty$. Moreover, f satisfies an exponential growth condition:*

$$|f(x)| \leq C e^{\frac{cK^2(1,x)}{2}} \textit{ for some } C > 0 \textit{ and } c < 1. \tag{7.2}$$

Remark 7.3 In what follows we shall use the exponential growth condition (7.2) only to show that the function $H(t, x) = \int_{\mathbb{R}} f(z) G(t, x; 1, z) dz$ is a classical solution of

$$H_t + \frac{1}{2} a^2(t, x) H_{xx} = 0, \quad H(1, x) = f(x).$$

The above follows from Sect. 1.7 in [57].

Thus, this assumption can be relaxed as long as one can demonstrate that H is a classical solution to the above problem.

Remark 7.4 Note that the models of Kyle [85] and Back [9] are included in our setting since $V \equiv 1$ and $a \equiv 1$ satisfy the above assumptions.

7.1 Existence of Equilibrium

We start with the following sufficient condition for (H, w^*, θ) to be an equilibrium.

Lemma 7.1 *A triplet (H^*, w^*, θ^*) where (H^*, w^*) is a pricing rule and $\theta^* \in \mathscr{A}$, is an equilibrium if it fulfils the following four conditions:*

1. *$H^*(t, x)$ satisfies the PDE $H_t^*(t, x) + \frac{1}{2} w^*(t, x)^2 H_{xx}(t, x) = 0$ for any $(t, x) \in [0, 1) \times \mathbb{R}$.*
2. *Weighting function satisfies $w_t^*(t, x) + \frac{w^*(t,x)^2}{2} w_{xx}^*(t, x) = 0$.*
3. *$Y_t^* = B_t + \theta_t^*$ is a standard BM in its own filtration.*
4. *$H^*(1, X_1^*) = f(Z_1)$, where X^* is the solution to $X_t = \int_0^t w(s, X_s) dY_s^*$ with $Y^* = B + \theta^*$.*

Proof Let (H^*, w^*, θ^*) be a triplet satisfying conditions 1 to 4 above. By Theorem 6.1, conditions 1, 2 and 4 imply that θ^* is optimal. On the other hand, 3 and 4 imply that the pricing rule (H^*, w^*) is rational.

Combining the lemma above and the bridge construction from Sect. 5.1, we can finally state and prove the main result of this chapter. We recall from Proposition 5.1 that there exists a fundamental solution, G, to $u_t = \frac{1}{2}(a^2 u)_{xx}$, which implies $P(Z_t \in dz | Z_s = x) = G(V(s), x; V(t), z)$ for $s < t$.

Theorem 7.1 *There exists an equilibrium* (H^*, w^*, θ^*)*, where*

(i) $H^*(t, x) = \int_{\mathbb{R}} f(y) G(t, x; 1, y) dy$ *and* $w^*(t, x) = a(t, x)$ *for all* $(t, x) \in [0, 1] \times \mathbb{R}$*;*
(ii) $\theta^*_t = \int_0^t \alpha^*_s ds$ *where* $\alpha^*_s = a(s, X_s) \frac{G_x(s, X_s; V(s), Z_s)}{G(s, X_s; V(s), Z_s)}$ *and the process* X *is the unique strong solution under* $\mathscr{F}^{B,Z}$ *of the following SDE:*

$$dX_t = a(t, X_t) dB_t + a(t, X_t)^2 \frac{G_x(t, X_t; V(t), Z_t)}{G(t, X_t; V(t), Z_t)} dt, \qquad X_0 = 0.$$

Proof Suppose (H^*, w^*) is an admissible pricing rule. A straightforward calculation shows that H^* as defined satisfies condition 1 of Lemma 7.1. Moreover, by Assumption 7.1, the weight $w^* = a$ clearly satisfies the property 2.

It follows from Theorem 5.1 that the process X is a local martingale in its own filtration with quadratic variation $d[X, X]_t = a(t, X_t)^2 dt$. Since the equilibrium demand, Y^*, is given by

$$dY^*_t = dB_t + \alpha^*_t dt = \frac{dX_t}{a(t, X_t)}, \qquad Y_0 = 0$$

in insider's filtration, Y^* is indeed a martingale under X's natural filtration and its quadratic variation is $[Y^*, Y^*]_t = t$. On the other hand, in view of Remark 6.2 the natural filtrations of X and Y^* coincide, giving that Y^* is a Brownian motion in its own filtration, i.e. condition 3 in Lemma 7.1 is fulfilled.

Applying Theorem 5.1 once more, we get that $X_1 = Z_1$ a.s. under all $P^{0,z}$, $z \in \mathbb{R}$. Thus, if we can show for any sequence $t_n \uparrow 1$ and $x_n \to x$ that $\lim_{n \to \infty} H(t_n, x_n) = f(x)$, we will have $H^*(1, X_1) = f(X_1) = f(Z_1)$, a.s. under $P^{0,z}$ for all $z \in \mathbb{R}$ and, therefore, condition 4 in Lemma 7.1 is satisfied. Note that, without loss of generality, we may assume (x_n) is a monotone sequence.

Indeed, due to the fact that G is a fundamental solution, one has

$$\lim_{t \uparrow 1} H^*(t, x) = f(x), \qquad \forall x \in \mathbb{R}. \tag{7.3}$$

Since $H^*(t, \cdot)$ is strictly increasing, the above implies that the sequence of functions $g_n(t) := H(t, x_n)$ is a monotone sequence of continuous functions on $[0, 1]$ as (x_n) is monotone. Thus, in view of Dini's theorem,

$$\lim_{n\to\infty}\sup_{t\in[0,1]}|g_n(t)-g(t)|=0,$$

where $g(t) = H(t,x)$ is continuous due to (7.3) and $x = \lim_{n\to\infty} x_n$. Therefore,

$$\lim_{n\to\infty}|H(t_n,x_n)-f(x)|\le\lim_{n\to\infty}|H(t_n,x_n)-H(t_n,x)|+\lim_{n\to\infty}|H(t_n,x)-f(x)|=0.$$

Thus, it remains to prove (H^*, w^*) is an admissible pricing rule and $\theta^* \in \mathscr{A}_c(H^*, w^*)$.

First we will show that (H^*, w^*) is an admissible pricing rule. As it is easily seen that $H^* \in C^{1,2}([0, 1) \times \mathbb{R})$ (see Remark 7.3), we only need to prove H^* is strictly increasing. Indeed, since $Y_t^* = K_a(t, X_t^*)$, we have

$$H^*(t,x)=\mathbb{E}[f(X_1^*)|X_t^*=x]=\mathbb{E}\left[\tilde{f}(Y_1^*)|Y_t^*=K_a(t,x)\right],$$

where $\tilde{f}(x) := f(K_a^{-1}(1, x))$. Recall that Y^* is a Brownian motion and, therefore, for any $x_1 > x_2$

$$\begin{aligned}&H^*(t,x_1)-H^*(t,x_2)\\&=\int_{\infty}(\tilde{f}(y+K_a(t,x_1))-\tilde{f}(y+K_a(t,x_2)))\frac{\exp\left(-\frac{y^2}{2(1-t)}\right)}{\sqrt{2\pi(1-t)}}dy\ge 0\end{aligned}$$

since $\tilde{f}$ is non-decreasing and $K_a(t,\cdot)$ is strictly increasing. Moreover, in view of the continuity of $\tilde{f}$, the above can vanish only if $\tilde{f}(y + K_a(t, x_1)) = \tilde{f}(y + K_a(t, x_2))$ for all y. However, this contradicts the assumption that f is not constant.

Finally, the admissibility of θ^* will follow once we establish that (6.11) holds. Note that

$$\mathbb{E}\left[\int_0^1 H^2(t,X_t)dt\right]=\int_{-\infty}^{\infty}G(0,0;c,z)E^{0,z}\left[\int_0^1 H^2(t,X_t)dt\right]dz$$

and the left-hand side of the above equation is bounded by $\mathbb{E}[H^2(1, X_1)] = \mathbb{E}[f^2(Z_1)]$ since $H^2(t, X_t)$ is a $\mathbb{P}$-submartingale. Consequently, left-hand side is finite due to Assumption 7.2. Since $G(0, 0; c,\cdot)$ is continuous and strictly positive by (5.16) and Lemma 5.4, we deduce that $E^{0,z}\left[\int_0^1 H^2(t, X_t)dt\right]$ is finite for a.a. $z \in \mathbb{R}$.

Next, define functions h^+ and h^- via

$$h^+(t,x)=H^2(t,x)\mathbf{1}_{[x\ge y(t)]}\qquad h^-(t,x)=H^2(t,x)\mathbf{1}_{[x\le y(t)]},$$

where $y(t)$ is the unique solution to $H(t, y(t)) = 0$. Observe that h^+ (resp. h^-) is continuous in (t, x) and increasing (resp. decreasing) in x.

Moreover, utilising (5.16) and Remark 7.1 we obtain

$$\alpha_t^* = \frac{K(V(t), Z_t) - K(t, X_t) + \frac{1}{2}\int_t^{V(t)} a_z(u, 0)du}{V(t) - t}.$$

On a single probability space consider the family $(Z^z, X(z))$ that solve

$$Z_t^z = z + \int_0^t \sigma(s)a(V(s), Z_s)dW_s^1$$

$$X_t(z) = \int_0^t a(s, X_s(z))dW_s^2$$
$$+ \int_0^t \frac{K(V(s), Z_s^z) - K(s, X_s(z)) + \frac{1}{2}\int_s^{V(s)} a_z(u, 0)du}{V(s) - s} ds$$

for independent Brownian motions, W^1 and W^2. Observe that the law of $X(z)$ is the same as that of X under $P^{0,z}$. Direct application of Theorem 2.10 yields that Z^z is increasing in z. Thus, Theorem 2.11, by means of a standard localisation argument, yields that $X(z)$ is also increasing in z. Therefore, g^+ (resp. g^-) is increasing (resp. decreasing) in z, where

$$g^{\pm}(z) = E^{0,z}\left(\int_0^1 h^{\pm}(t, X_t)dt\right).$$

Since $g^+ + g^-$ is finite a.e., the monotonicity of $g^{\pm}$ yields that they are finite everywhere.

7.2 On the Uniqueness of Equilibrium

As we shall see in this section the model introduced in this chapter does not have a unique equilibrium. This will be demonstrated via constructing an alternative optimal strategy for the insider in both dynamic and static cases. While the former relies only on the information flow generated by the signal and the price process, the latter requires an additional randomisation.

Let us assume that the information flow is dynamic, i.e. $V(0) < 1$, and (H^*, w^*) is a pricing rule as in Theorem 7.1. Then there exists a function $\tilde{V}$ that is not identically equal to V such that $\tilde{V} \leq V$, $\tilde{V}(0) = V(0)$, and satisfies the same conditions as V in Assumption 5.1. Next consider $\tilde{Z}_t = Z_{V^{-1}(\tilde{V}(t))}$ and observe that $\tilde{Z}$ is adapted to the filtration generated by Z. Moreover,

$$d\tilde{Z}_t = \tilde{\sigma}(t)a(\tilde{V}(t), \tilde{Z}_t)d\tilde{B}_t$$

for some Brownian motion $\tilde{B}$. Thus, choosing

$$\tilde{\alpha}_s^* = a(s, X_s)\frac{G_x(s, X_s; \tilde{V}(s), \tilde{Z}_s)}{G(s, X_s; \tilde{V}(s), \tilde{Z}_s)}$$

will also lead to an equilibrium. This observation indicates that the insider's value function is not affected by V as long as $V(0)$ remains the same. Indeed, inspecting the proof of Theorem 6.1 reveals that the value function of the insider is given by

$$E^{0,z}\left(\Psi^{f(Z_1)}(0, 0)\right),$$

which only depends on the distribution of Z_1 given Z_0.

When the information is static, the insider will need to apply an additional randomisation in order to carry out an alternative optimal strategy. Although in the derivation of the equilibrium above we assumed that the insider only observes her private signal and prices, in reality one can always construct an independent Brownian motion. In this case one can show that the conclusions of Theorem 6.1 continue to hold. In particular the insider strategy is optimal if it is absolutely continuous and drives the market prices to the fundamental value.

Using this observation one can extend the arguments in the dynamic case to show non-uniqueness of equilibrium in the static information case, as well. Indeed, if $V(0) = 1$, Z_0 has a probability density given by $G(0, 0; 1, \cdot)$, where G is the transition density of the diffusion

$$d\xi_t = a(t, \xi_t)d\beta_t,$$

for a given a. Suppose that the insider also observes a Brownian motion, $\tilde{\beta}$, independent of Z_0 and B. Then, the insider can choose any $\tilde{V}(t) := \int_0^t \tilde{\sigma}^2(s)ds$ satisfying Assumption 5.1 to construct a noisy signal

$$\tilde{Z}_t = \int_0^t \tilde{\sigma}(s)a(\tilde{V}(s), \tilde{Z}_s)d\tilde{\beta}_s + \int_0^t \tilde{\sigma}^2(s)a^2(\tilde{V}(s), \tilde{Z}_s)\frac{G_x(s, \tilde{Z}_s; 1, Z_0)}{G(s, \tilde{Z}_s; 1, Z_0)}ds.$$

Note that given $Z_0 = z$, $\tilde{Z}$ has the same distribution as $\xi_{\tilde{V}(\cdot)}$ conditioned on $\xi_0 = 0$ and $\xi_1 = z$. Thus, for (H^*, w^*) given by Theorem 7.1 the insider strategy

$$\tilde{\alpha}_s^* = a(s, X_s)\frac{G_x(s, X_s; \tilde{V}(s), \tilde{Z}_s)}{G(s, X_s; \tilde{V}(s), \tilde{Z}_s)}$$

is optimal since it is absolutely continuous and drives the market prices to their fundamental value. Moreover, it is simple to check that (H^*, w^*) is a rational pricing rule. Hence, $((H^*, w^*), \tilde{\theta}^*)$ is an equilibrium for any choice of $\tilde{V}$.

As one can see from the form of the alternative strategies proposed above, they are constructed by changing the speed, V, at which the insider reveals her private information. To be more precise the speed of revelation for the alternative strategies is slower than the one for the optimal strategy derived in Theorem 7.1. This independence of the insider's value function from the speed of revelation hints at value of static and dynamic information being the same.

To compare the value of static and dynamic information, one must consider an uninformed and risk-neutral investor at time 0 who is about to decide whether to purchase a particular private information at a given price. Obviously, as the investor is uninformed her information prior to making this decision is trivial. Thus, the decision will be based on the comparison of the expected profits resulting from the purchased information, and the expectation will be taken with respect to the trivial σ-algebra. This together with the above expression of the value function gives the ex-ante value of information as

$$\mathbb{E}\left(\Psi^{f(Z_1)}(0,0)\right),$$

which only depends on the distribution of Z_1 and not on the pattern of how it is being revealed to the insider. This leads to the immediate conclusion that the risk-neutral investor is indifferent between purchasing static or dynamic information.

This indifference might appear counterintuitive at first. However, it is clear that a necessary condition for the optimality is that the insider drives the market price to the fundamental value of the asset at the termination of the market since otherwise she wouldn't have used all her informational advantage. On the other hand, Theorem 6.1 demonstrates that this is also sufficient. Consequently, the only thing she strives to achieve is to make sure that the price converges to the fundamental value. This observation together with her risk-neutrality will lead her to value both types of information same since the variance of the signals does not affect her valuation.

7.3 Notes

This chapter closely follows [32]. The proof of the admissibility of the pricing rule had to be revised to accommodate unbounded fundamental values. This extension in particular allowed the models of Kyle [85] and Back [9] to be included in the framework of this chapter. Moreover, Theorem 7.1 recovers the results therein. In particular the equilibrium is given by $(H^*, 1, \theta^*)$, where

$$H^*(t,x) = \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(x-y)^2}{2(1-t)}\right) dy$$

and

$$\theta_t^* = \int_0^t \frac{Z_0 - X_s}{1-s} ds.$$

Models of insider trading provide price impact functions as equilibrium outcome and, therefore, have been used in the literature to study market liquidity. One of the standard measures for market liquidity is *Kyle's lambda*, λ, that measures the sensitivity of the prices to the change in the order flow. In [85] Kyle demonstrates that λ is constant and conjectures that there are no systematic changes to the market depth (that is, reciprocal of λ). To be more precise he argues that

> [...] neither increasing nor decreasing depth is consistent with behaviour by the informed trader which is "stable" enough to sustain an equilibrium. If depth ever increases, the insider wants to destabilise prices (before the increase in depth) to generate unbounded profits. If depth ever decreases, the insider wants to incorporate all of his private information into the price immediately.

In mathematical terms this amounts to λ being a true martingale.

In the model that is considered in this chapter

$$dS_t = H_x(t, X_t)w(t, X_t)dY_t = \lambda(t, Y_t)dY_t,$$

where $\lambda(t, y) := H_x(t, K_w^{-1}(t, y))w(t, K_w^{-1}(t, y))$ is the Kyle's lambda in this model. The fact that $K_w(t, X_t)$ is a Brownian motion and a direct application of Ito's formula yield that $(\lambda(t, Y_t))$ is a local martingale, hence a nonnegative supermartingale. If one further imposes additional differentiability assumptions on f (which are standard in the literature), $\lambda(t, Y_t)$ will be a true martingale. In view of Kyle's conjecture it will be interesting to find out whether there exists an equilibrium in which λ is a strict local martingale.

The other works in the literature that consider dynamic inside information model the insider's private signal as a Gaussian process. The first such example is due to Back and Pedersen [12], which in addition allowed for a time-dependent volatility of noise trading. Later Danilova [44] extended [12] by enlarging the set of admissible pricing rules. More recently, Foucault et al. [56] used a similar framework to model high-frequency trading as an interplay between the speed of information acquisition by the market maker and the insider.

Chapter 8
Kyle–Back Model with Default and Dynamic Information

In this chapter we will continue applying the dynamic bridge construction from Chap. 5 to solve the Kyle–Back model in the case of dynamic information and default risk.

Consider a defaultable claim issued by a company with no recovery and the payoff $f(1+\beta_1)$ in case of no-default, where β denotes the fundamental value process and is assumed to follow a standard Brownian motion with $\beta_0 = 0$, that is independent of B. The default time, T_0, is given by

$$T_0 := \inf\{t > 0 : 1 + \beta_t = 0\}.$$

If the insider observed β, then the results of the previous chapter suggest that the optimal strategy would be inconspicuous and the equilibrium price would converge to $f(1+\beta_1)\mathbf{1}_{[T_0>1]}$. However, this will imply $1+Y$ is a Brownian motion starting at 1 and hitting 0 for the first time at T_0.

Such an equilibrium cannot exist in view of the discussion following Theorem 5.4. Thus, to relax the assumption of static information while ensuring the existence of a solution, we allow the insider to look into the future, i.e. we assume that she observes $Z = (1+\beta_{V(t)})_{t\in[0,1]}$ where V is a continuous and increasing function with $V(0) = 0$, $V(t) > t$ for $t \in (0,1)$ and $V(1) = 1$. This is also consistent with the assumptions on the private signal of the insider in Chap. 6. Indeed, one can find another Brownian motion, B^Z, such that

$$Z_t = 1 + \int_0^t \sqrt{V'(s)}\,dB_s^Z \quad t \geq 0,$$

where V' is the left derivative of V. Thus, it has the same form as the private signal from the previous chapter when $a \equiv 1$ and $\sigma = \sqrt{V'}$. Note that $Z_1 = 1 + \beta_1$ and $T_0 = V(\inf\{t > 0 : Z_t = 0\})$.

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More precisely, we assume that the private signal Z satisfies (6.1) with $a \equiv 1$ and the firm's default time is given by the random time $V(\tau)$ where

$$\tau := \inf\{t > 0 : Z_t = 0\}. \tag{8.1}$$

Since $Z_0 = 1$, we assume Assumption 5.1 is in force with $c = 0$. Moreover, $Z_1 = 1 + \beta_1$ yields that the claim pays $f(Z_1)$ in case of no-default. As in the previous chapter, we impose the following assumption on f and σ.

Assumption 8.1 *$f : [0, \infty) \mapsto \mathbb{R}$ is a non-decreasing continuous function that is not identically 0. Moreover, it satisfies $\mathbb{E}\left[f^2(Z_1)\mathbf{1}_{[\tau>1]}\right] < \infty$ and*

$$|f(y)| \leq Ce^{\frac{cy^2}{2}}, \qquad \forall y \text{ for some } c < 1 \text{ and } C > 0. \tag{8.2}$$

Furthermore, σ satisfies the conditions in Assumption 5.1.

8.1 On Insider's Optimal Strategy

In order to determine the conditions for equilibrium we start with the optimality conditions for the insider, to establish which we need the following lemma.

Lemma 8.1 *Suppose that h is a nondecreasing right-continuous function such that $E\left[h^2(Z_1)\right] < \infty$. Let*

$$H(t, x) := \int_{-\infty}^{\infty} h(x + y)\frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{y^2}{2(1-t)}\right) dy, \qquad t < 1 \tag{8.3}$$

and $(\xi_n)_{n\geq 1}$ be a convergent sequence such that $\lim_{n\to\infty} H(t_n, \xi_n) = a$ for some a in the range of h or in the interval $(\inf_x h(x), \sup_x h(x))$, and some sequence $(t_n)_{n\geq 1} \subseteq [0, 1)$ converging to 1. Then,

$$\lim_{n\to\infty} \xi_n \in [X^a_{min}, X^a_{max}],$$

where

$$X^a_{min} := \inf\{x : h(x) \geq a\} \qquad \text{and} \qquad X^a_{max} := \sup\{x : h(x) \leq a\}.$$

Proof Suppose $\lim_{n\to\infty} \xi_n < X^a_{min}$. Then, there exists some ξ such that $\lim_{n\to\infty} \xi_n < \xi < X^a_{min}$. Since H is nondecreasing in x, one has

$$\lim_{n\to\infty} H(t_n, \xi_n) \leq \lim_{n\to\infty} H(t_n, \xi) = \frac{h(\xi) + h(\xi-)}{2} < a,$$

since $h(\xi) < a$ and h is non-decreasing. However, this is a contradiction.

On the other hand, if $\lim_{n\to\infty} \xi_n > X^a_{max}$, then there exists some ξ such that $\lim_{n\to\infty} \xi_n > \xi > X^a_{max}$. Then we again obtain a contradiction since

$$\lim_{n\to\infty} H(t_n, \xi_n) \geq \lim_{n\to\infty} H(t_n, \xi) = \frac{h(\xi) + h(\xi-)}{2} > a$$

due to $h(\xi-) > a$. Indeed, suppose $h(\xi-) \leq a$. Then, for any $x < \xi$ we have $h(x) \leq a$ leading to $X^a_{max} \geq \xi$, which contradicts the choice of ξ.

The following proposition builds upon the arguments used in the proof of Theorem 6.1 by working out the technicalities that arise due to the possibility of default. Thus, we will omit certain steps of the proof that closely follow the ones found therein.

Proposition 8.1 *Assume that h is a nonconstant function satisfying the conditions of Lemma 8.1 and an exponential growth condition as in (8.2). Define H on $[0, 1) \times \mathbb{R}$ by (8.3). Then, H is an admissible pricing rule satisfying*

$$H_t(t, x) + \frac{1}{2} H_{xx}(t, x) = 0. \tag{8.4}$$

Moreover, suppose the range of h contains that of f, and $0 \in (\inf_x h(x), \sup_x h(x))$. If $\theta^* \in \mathscr{A}_c(H)$ satisfies $\lim_{t\uparrow 1} H(V(\tau) \wedge t, X^*_{V(\tau)\wedge t}) = f(Z_1)\mathbf{1}_{[\tau>1]}$ a.s., where $X^* = B + \theta^*$, then θ^* is an optimal strategy, i.e. for all $\theta \in \mathscr{A}_c(H)$,

$$\mathbb{E}[W^\theta_1] \leq \mathbb{E}[W^{\theta^*}_1] = \mathbb{E}\left[\int_{\xi(0,a^*)}^1 (H(0, u)-a^*)du + \frac{1}{2}\int_0^{1\wedge V(\tau)} H_x(s, \xi(s, a^*))ds\right], \tag{8.5}$$

where $a^* = \mathbf{1}_{[\tau>1]} f(Z_1)$ and $\xi(t, a)$ is the unique solution of $H(t, \xi(t, a)) = a$ for all a in the range of h or in the interval $(\inf_x h(x), \sup_x h(x))$ and $t < 1$.

Proof The exponential bound on h allows us to differentiate under the integral sign, which leads to (8.4). Direct calculations also show that

$$H(t, x)-H(t, z) = \int_{-\infty}^{\infty} (h(x+y)-h(z+y)) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{y^2}{2(1-t)}\right) dy \geq 0.$$

Note that the above holds in equality if and only if $h(x + y) = h(z + y)$ for almost all y, and thus for all y due to the right continuity of h, which is a contradiction since h is not constant. Therefore, H is strictly increasing and, thus, an admissible pricing rule in the sense of Definition 6.1.

The rest of the proof can be split into two main steps.

Step 1. Fix an a in the range of h or in the interval $(\inf_x h(x), \sup_x h(x))$. Suppose that a is the maximum of h, then $\xi(t, a) = \infty$ for all $t \in [0, 1)$ since H is strictly increasing with supremum being equal to a. Similarly if a is the minimum of h, then $\xi(t,a) = -\infty$ for all $t \in [0,1)$. If a is neither the minimum nor the maximum, $\xi(t,a)$ exists since $H(t,x)$ is strictly increasing in x for each $t \in [0,1)$. Moreover, for all $a \in (\inf_x h(x), \sup_x h(x))$, the mapping $t \mapsto \xi(t,a)$ is uniformly bounded on the interval $[0,1]$ due to Lemma 8.1 and the continuity of H on $[0,1) \times \mathbb{R}$.

Define

$$\Phi^a(1,x) := \int_{X^a_{min}}^x (h(u)-a)du.$$

Observe that the above integral is finite whenever X^a_{min} is finite, which is the case unless $a = \min h$. On the other hand, if $a = \min h$, then $-\infty = X^a_{min} < X^a_{max} < \infty$ implying

$$\Phi^a(1,x) = \int_{X^a_{max}}^x (h(u)-a)du.$$

Next, consider the function

$$\begin{aligned}\Phi^a(t,x) &:= \int_{-\infty}^{\infty} \Phi^a(1,y)\frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{(x+y)^2}{2(1-t)}\right)dy \\ &= \int_{-\infty}^{\infty}\int_{X^a_{min}}^{y+x}(h(u)-a)\frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{y^2}{2(1-t)}\right)du\,dy,\end{aligned}$$

where

$$X^a_{min} = \inf\{x : h(x) \geq a\}.$$

Observe that $\Phi^a(1,\cdot)$ is a continuous function. Moreover, $0 \leq \Phi^a(1,x) \leq (h(x) - a)(x - x^*)$, where $x^* = X^a_{min}\mathbf{1}_{X^a_{min}>-\infty} + X^a_{max}\mathbf{1}_{X^a_{min}=-\infty}$. Thus, $\Phi^a(1,\cdot)$ has exponential growth and, therefore, $\Phi^a(t,x)$ is the unique classical solution of

$$u_t + \frac{1}{2}u_{xx} = 0,$$

with the boundary condition $u(1,\cdot) = \Phi^a(1,\cdot)$ (see Remark 7.3). In particular, Φ^a is jointly continuous and nonnegative on $[0,1]\times\mathbb{R}$.

Moreover, differentiating under the integral sign, which is justified by the exponential growth condition, for $t < 1$ yields

$$\begin{aligned}\Phi^a_x(t,x) &= \int_{-\infty}^{\infty}\Phi^a_+(1,x+y)\frac{\exp\left(-\frac{y^2}{2(1-t)}\right)}{\sqrt{2\pi(1-t)}}dy \\ &= \int_{-\infty}^{\infty}(h(x+y)-a)\frac{\exp\left(-\frac{y^2}{2(1-t)}\right)}{\sqrt{2\pi(1-t)}}dy = H(t,x)-a,\end{aligned}$$

where $\Phi^a_+(1,\cdot)$ denotes the right derivative of $\Phi^a(1,\cdot)$. Consequently, the minimum of Φ^a is achieved at $\xi(t,a)$.

Our next goal is to analyse the behaviour of $\Phi^a(t,\cdot)$ near $\xi(t,a)$.

First suppose that $|\xi(t,a)| = \infty$ so that a is either the minimum or the maximum of h. Then $\Phi^a(1,\cdot)$ is monotone and $\Phi^a(1,x)$ decreases to 0 as $x \to \xi(t,a)$. Thus, by the dominated convergence theorem,

$$\begin{aligned}
&\Phi^a(t,\xi(t,a))\\
&\quad := \lim_{x\to\xi(t,a)} \Phi^a(t,x)\\
&\quad = \lim_{x\to\xi(t,a)} \int_{-\infty}^{\infty} \Phi^a(1,x+y)\frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{y^2}{2(1-t)}\right)dy\\
&\quad = \int_{-\infty}^{\infty} \lim_{x\to\xi(t,a)} \Phi^a(1,x+y)\frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{y^2}{2(1-t)}\right)dy = 0. \qquad (8.6)
\end{aligned}$$

Next, suppose $\xi(t,a)$ is finite, Then both X^a_{min} and X^a_{min} are finite. Moreover, Lemma 8.1 yields $\lim_{t\uparrow 1}\xi(t,a) \in [X^a_{min}, X^a_{min}]$ and, therefore, finite. In view of the joint continuity of Φ^a we obtain

$$0 \le \lim_{t\uparrow 1}\Phi^a(t,\xi(t,a)) = \Phi^a(1,\lim_{t\uparrow 1}\xi(t,a)) = 0. \qquad (8.7)$$

Furthermore,

$$\begin{aligned}
&\Phi^a(t,x) - \Phi^a(t,\xi(t,a))\\
&\quad = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{y^2}{2(1-t)}\right)\int_{y+\xi(t,a)}^{y+x}(h(u)-a)du\,dy\\
&\quad = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{y^2}{2(1-t)}\right)\int_{\xi(t,a)}^{x}(h(u+y)-a)du\,dy\\
&\quad = \int_{\xi(t,a)}^{x}(H(t,u)-a)du. \qquad (8.8)
\end{aligned}$$

The above shows that $\int_{\xi(t,a)}^x (H(t,u)-a)du$ is well defined and finite even when $\xi(t,a)$ is infinite. Step 2. Consider for all $\nu < 1$

$$\Psi^{a,\nu}(t,x) := \int_{\xi(t,a)}^{x}(H(t,u)-a)du + \frac{1}{2}\int_t^\nu H_x(s,\xi(s,a))ds, \qquad t \le \nu.$$

Notice that both the integrals in the RHS are well defined: the first one is well defined for all values of $\xi(t,a)$ thanks to Step 1, and the second one is well defined due to the fact that $t \mapsto H_x(t,\xi(t,a))$ is uniformly bounded on $[0,\nu]$. Indeed, if a is neither the minimum nor the maximum of h, $\xi(t,a)$ is finite and continuous in

t. Since H_x is continuous on $[0, \nu] \times \mathbb{R}$, the claimed uniform boundedness follows. Otherwise, $\xi(t, a)$ is infinite for all t. Without loss of generality assume a is the maximum, i.e. $\xi(t, a) = \infty$. Then,

$$\begin{aligned} H_x(t,x) &= a \int_{X^a_{min}}^{\infty} \frac{y-x}{\sqrt{2\pi(1-t)^3}} \exp\left(-\frac{(x-y)^2}{2(1-t)}\right) dy \\ &\quad + \int_{-\infty}^{X^a_{min}} \frac{h(y)(y-x)}{\sqrt{2\pi(1-t)^3}} \exp\left(-\frac{(x-y)^2}{2(1-t)}\right) dy \\ &\leq \frac{a}{\sqrt{2\pi(1-t)}} e^{-\frac{(x-X^a_{min})^2}{2(1-t)}} + e^{-\frac{x^2}{2(1-t)}} \int_{-\infty}^{X^a_{min}} \frac{|y-x| e^{\frac{xy}{1-t}}}{\sqrt{2\pi(1-t)^3}} dy, \end{aligned}$$

which converges to 0 as $x \to \infty$. This proves the claim that $t \mapsto H_x(t, \xi(t, a))$ is uniformly bounded on $[0, \nu]$ for every $a \in \mathbb{R}$.

Moreover, the above shows that when a is either maximum or minimum $\Psi^{a,\nu}(t, x) = \Phi^a(t, x)$ in view of (8.7) and (8.8). In particular $\Psi^{a,\nu}$ solves the heat equation.

On the other hand, if a is neither the maximum nor the minimum, $\xi(t, a)$ is finite for all t. Thus, direct differentiation with respect to x gives that

$$\Psi^{a,\nu}_x(t,x) = H(t,x) - a. \tag{8.9}$$

Thus,

$$\Psi^{a,\nu}_{xx}(t,x) = H_x(t,x). \tag{8.10}$$

Similarly

$$\Psi^{a,\nu}_t(t,x) = \int_{\xi(t,a)}^{x} H_t(t,u)du - \frac{1}{2} H_x(t, \xi(t,a)) = -\frac{1}{2} H_x(t,x)$$

where in order to obtain the last equality we used (8.4). Combining this and (8.10) gives

$$\Psi^{a,\nu}_t + \frac{1}{2} \Psi^{a,\nu}_{xx} = 0.$$

Therefore from (8.9) and Ito's formula it follows that

$$\Psi^{a,\nu}(\nu, X_\nu) - \Psi^{a,\nu}(0,1) = \int_0^{\nu} (H(u, X_u) - a) dX_u,$$

and in particular, when $a := \mathbf{1}_{[\tau>1]} f(Z_1)$, $\nu(t) = t \wedge V(\tau)$,

$$\lim_{t\uparrow 1}\Big(\Psi^{a,\nu(t)}(t\wedge V(\tau),X_{t\wedge V(\tau)})-\Psi^{a,\nu(t)}(0,1)\Big)$$
$$=\int_0^{1\wedge V(\tau)}(H(t,X_t)-\mathbf{1}_{[\tau>1]}f(Z_1))dX_t. \tag{8.11}$$

By the admissibility properties of θ, in particular $d\theta_t=\alpha_t dt$, the insider's optimisation problem becomes

$$\sup_{\theta\in\mathscr{A}_c(H)}\mathbb{E}[W_1^\theta]$$
$$=\sup_{\theta\in\mathscr{A}_c(H)}\mathbb{E}\left[(f(Z_1)\mathbf{1}_{[\tau>1]}-S_{1\wedge V(\tau)-})\theta_{1\wedge V(\tau)}+\int_0^{V(\tau)\wedge 1-}\theta_s dS_s\right]$$
$$=\sup_{\theta\in\mathscr{A}_c(H)}\mathbb{E}\left[\int_0^{V(\tau)\wedge 1}(f(Z_1)\mathbf{1}_{[\tau>1]}-H(t,X_t))\alpha_t dt\right]$$

where we used integration-by-parts formula and the fact that $\theta_0=0$ to obtain the second equality. Due to above and (8.11), we have

$$\mathbb{E}[W_1^\theta]=-\mathbb{E}\left[\lim_{t\uparrow 1}\Big(\Psi^{a,\nu(t)}(t\wedge V(\tau),X_{t\wedge V(\tau)})-\Psi^{a,\nu(t)}(0,1)\Big)\right]. \tag{8.12}$$

Notice that all the Brownian integrals vanish due (6.11) in Definition 6.2 and

$$\mathbb{E}\left[\left(\int_0^{1\wedge\tau}f(Z_1)dB_t\right)^2\right]\le\mathbb{E}\left[f(Z_1)^2\right]\mathbb{E}[B_1^2]<\infty,$$

since Z and B are independent.

The conclusion follows from the fact that $\lim_{t\uparrow 1}\Psi^{a,\nu(t)}(t\wedge V(\tau),X_{t\wedge V(\tau)})$ is nonnegative and equals 0 if $\lim_{t\uparrow 1}H(V(\tau)\wedge t,X^*_{V(\tau)\wedge t})=f(Z_1)\mathbf{1}_{[\tau>1]}$. Indeed, observe that

$$\lim_{t\uparrow 1}\Psi^{a,\nu(t)}(t\wedge V(\tau),X_{t\wedge V(\tau)})=\lim_{t\uparrow 1}\int_{\xi(t\wedge V(\tau),a)}^{X_{t\wedge V(\tau)}}(H(t\wedge V(\tau),u)-a)du.$$

On the set $[V(\tau)\ge 1]$ we have

$$\lim_{t\uparrow 1}\Psi^{a,\nu(t)}(t\wedge V(\tau),X_{t\wedge V(\tau)})=\lim_{t\uparrow 1}\int_{\xi(t,a)}^{X_t}(H(t,u)-a)du$$
$$=\lim_{t\uparrow 1}\left\{\Phi^a(t,X_t)-\Phi^a(t,\xi(t,a))\right\}=\Phi^a(1,X_1),$$

which is nonnegative. Observe that we used (8.8) for the second equality above while (8.6) and (8.7) for the third one. Moreover, $\lim_{t\uparrow 1} H(t, X_t^*) = f(Z_1)$ implies in conjunction with Lemma 8.1 and the continuity of X^* that $X_1^* \in [X^a_{min}, X^a_{max}]$, a.s. However, this yields that $\Phi^a(1, X_1^*) = 0$.

On the set $[V(\tau) < 1]$,

$$\lim_{t\uparrow 1} \int_{\xi(t\wedge V(\tau),a)}^{X_{t\wedge V(\tau)}} (H(t\wedge V(\tau), u) - a)du = \int_{\xi(V(\tau),a)}^{X_{V(\tau)}} (H(V(\tau), u) - a)du,$$

which is nonnegative and equals 0 if $X_{V(\tau)} = \xi(V(\tau), a)$ due to the invertibility of H. Therefore, an insider trading strategy which gives $\lim_{t\uparrow 1} H(V(\tau)\wedge t, X^*_{V(\tau)\wedge t}) = f(Z_1)\mathbf{1}_{[\tau>1]}$ is optimal.

Remark 8.1 The same results as in Proposition 8.1 above apply when the initial insider's information $\mathscr{F}_0^I$ is not trivial provided one replaces expectations with conditional expectations given $\mathscr{F}_0^I$ in the statement as well as in its proof.

8.2 Existence of Equilibrium

Combining Proposition 8.1 and the dynamic Bessel bridge construction performed in Sect. 5.2, we can finally state and prove the main result of this chapter.[1] To this end define

$$P(t, z) := \int_0^\infty f(y)q(1 - V(t), z, y)dy \tag{8.13}$$

so that $\mathbf{1}_{[\tau>t]}P(t, Z_t)$ is the value of the defaultable claim for the insider at time t and the process $(\mathbf{1}_{[\tau>t]}P(t, Z_t))_{t\in[0,1]}$ is a martingale for the insider's filtration.

Note that Theorem 5.4 is not directly applicable since we impose $V(1) = 1$ for the time-change $V(t)$. Nonetheless, this theorem will be instrumental in proving the existence of equilibrium, which is stated below.

Theorem 8.1 *Under Assumption 8.1 there exists an equilibrium* (H^*, θ^*)*, where*

(i) $H^*(t, x) = P(V^{-1}(t), x)$ *where* $P(t, x)$ *is given by (8.13) for* $(t, x) \in [0, 1] \times \mathbb{R}$.

(ii) $\theta_t^* = \int_0^t \alpha_s^* ds$ *where*

$$\alpha_s^* = \frac{q_x(V(s) - s, X_s^*, Z_s)}{q(V(s) - s, X_s^*, Z_s)}\mathbf{1}_{[s\le\tau\wedge 1]} + \frac{\ell_a(V(\tau) - s, X_s^*)}{\ell(V(\tau) - s, X_s^*)}\mathbf{1}_{[\tau\wedge 1<s\le V(\tau)\wedge 1]} \tag{8.14}$$

[1] We invite the reader to review the notation of Sect. 5.2 at this point.

where the process X^* is the unique strong solution under insider's filtration $\mathscr{F}^{X,Z}$ of the following SDE:

$$X_t = 1 + B_{V(\tau)\wedge t} + \int_0^{\tau\wedge t} \frac{q_x(V(s)-s, X_s, Z_s)}{q(V(s)-s, X_s, Z_s)} ds + \int_{\tau\wedge t}^{V(\tau)\wedge t} \frac{\ell_a(V(\tau)-s, X_s)}{\ell(V(\tau)-s, X_s)} ds. \tag{8.15}$$

Moreover, $V(\tau) = \inf\{t \in [0,1] : X_t^* = 0\}$, where $\inf \emptyset = 1$ by convention, and one has $\lim_{t\uparrow 1} X_t^* = Z_1$ on the set of non-default $[\tau > 1]$. As a consequence, $V(\tau)$ is a predictable stopping time in the market filtration $\mathscr{F}^M$.

Furthermore, the expected profit of the insider is

$$\mathbb{E}\left[\int_{\xi(0,a^*)}^1 (H^*(0,u) - a^*)du + \frac{1}{2}\int_0^{1\wedge V(\tau)} H_x^*(s, \xi(s, a^*))ds\right], \tag{8.16}$$

where $a^* = \mathbf{1}_{[\tau>1]} f(Z_1)$ and $\xi(t,a)$ is the unique solution of $H(t, \xi(t,a)) = a$ for all $a \geq 0$.

Proof Observe that

$$H^*(t,x) = \int_{-\infty}^{\infty} \tilde{f}(x+y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{y^2}{2(1-t)}\right) dy,$$

where

$$\tilde{f}(y) = \begin{cases} f(y), & y \geq 0; \\ -f(-y), & y < 0. \end{cases}$$

Thus, since $\tilde{f}$ satisfies the conditions of Proposition 8.1, H^* is an admissible pricing rule that satisfies (8.4). As the process $(\mathbf{1}_{[\tau>t]} P(t, Z_t))_{t\in[0,1]}$ is a martingale for the insider's filtration, $(D_t H^*(t, X_t^*))_{t\in[0,1]}$ will be an $\mathscr{F}^M$-martingale as soon as we show that $(X_t^*)_{t\in[0,1]}$ is a Brownian motion stopped at $V(\tau)$ in its own filtration, where $V(\tau)$ is the first time that it hits 0. This will further imply that $\mathscr{F}^{X^*} = \mathscr{F}^M$ and that H^* is a rational pricing rule in the sense of (6.8).

To do so, we prove that there exists a unique strong solution of (8.15) on $[0,1)$, X^*, satisfying the following properties:

1) $\lim_{t\uparrow 1} X^*_{t\wedge V(\tau)} = 0$ a.s. on $[\tau < 1]$,
2) $\lim_{t\uparrow 1} X_t^* = Z_1$ a.s. on the set $[\tau > 1]$,
3) $(X_t^*)_{t\in[0,1]}$ is a Brownian motion stopped at $V(\tau)$ in its own filtration.

This will establish (H^*, θ^*) as an equilibrium in view of Proposition 8.1, where $h(x)$ is chosen to be equal to $\tilde{f}(x)$. Indeed, for $x \geq 0$, we have $\lim_{t\to 1} H^*(t,x) = f(x)$ since $\tilde{f} = f$ on $\mathbb{R}_+$ and f is square integrable and continuous. Thus, the arguments used in the proof of Theorem 7.1 can be repeated to obtain that $\lim_{n\to\infty} H^*(t_n, x_n) = f(x)$ for any sequence $t_n \to 1$ and $0 < x_n \to x > 0$. This implies, in view of 1) and 2), that $\lim_{t\uparrow 1} H^*(t \wedge V(\tau), X_{t\wedge V(\tau)}) = \mathbf{1}_{[\tau>1]} f(Z_1)$. In particular θ^* admissible since $(H^2(t, X_t^*)D_t)$ is a submartingale and f is square integrable. Moreover, the expected profit of the insider is given by (8.16) due to (8.5).

Due to Theorem 5.4 there exists unique strong solution, X^*, to (8.15) on $[0, T]$ for any $T < 1$. Note that although the assumptions of Theorem 5.4 are not satisfied, one can set $\sigma \equiv 1$ after T to obtain a modified V that satisfies the conditions therein. Since the modified V agrees with the original V on $[0, T]$, we deduce the existence and uniqueness of strong solution on $[0, 1)$.

Moreover, $V(\tau) = \inf\{t > 0 : X_t^* = 0\}$ on the set $[\tau < 1]$, so that property 1) above is satisfied.

On the non-default set $[\tau > 1]$, which is the same as $[V(\tau) > 1]$, the SDE (8.15) becomes

$$X_t = 1 + B_t + \int_0^t \frac{q_x(V(s) - s, X_s, Z_s)}{q(V(s) - s, X_s, Z_s)}\, ds, \quad t \in [0, 1).$$

The function $\frac{q_x(t,x,z)}{q(t,x,z)}$ appearing in the drift above can be decomposed as follows

$$\frac{q_x(t, x, z)}{q(t, x, z)} = \frac{z - x}{t} + b(t, x, z) := \frac{z - x}{t} + \frac{\exp\left(\frac{-2xz}{t}\right)}{1 - \exp\left(\frac{-2xz}{t}\right)} \frac{2z}{t}.$$

We want to prove that $\lim_{t\uparrow 1} X_t^* = Z_1$ a.s. on the set $[\tau > 1]$. Consider the process

$$R_t := X_t^* - \lambda(t) \int_0^t b(V(s) - s, X_s^*, Z_s) \frac{ds}{\lambda(s)},$$

where

$$\lambda(t) = \exp\left(-\int_0^t \frac{ds}{V(s) - s}\right).$$

Direct calculations give that, on $[\tau > 1]$, $dR_t = \frac{Z_t - R_t}{V(t) - t} dt + dB_t$, thus we can apply Proposition 5.3 to conclude that R_t goes to Z_1 as $t \uparrow 1$ a.s. on the set of non-default $[\tau > 1]$. To deduce from it that $X_t \to Z_1$ a.s. on $[\tau > 1]$ when $t \uparrow 1$, we have to show

$$\lim_{t\uparrow 1} \lambda(t) \int_0^t b(V(s) - s, X_s^*, Z_s) \frac{ds}{\lambda(s)} = 0,$$

a.s. on $[\tau > 1]$.

Observe that $b(V(t)-t, X_t^*, Z_t) = g\left(\frac{2X_t^* Z_t}{V(t)-t}\right)\frac{1}{X_t^*}$, with $g(u) = \frac{ue^{-u}}{1-e^{-u}}$, and $g(u) \in [0,1]$ for all $u \in [0,+\infty]$. Since on the set $[\tau > 1]$ we have $\inf_{t\in[0,1]} Z_t > 0$ and $\inf_{t\in[0,1]} X_t^* \geq 0$ (as, due to Theorem 5.4, $\inf_{t\in[0,T]} X_t^* > 0$ for any $T < 1$), we obtain $\inf_{t\in[0,1]} b(V(t)-t, X_t^*, Z_t) \geq 0$ on $[\tau > 1]$, and therefore the following two cases are possible:

- Case 1: $\lim_{t\uparrow 1}\int_0^t b(V(s)-s, X_s^*, Z_s)\frac{ds}{\lambda(s)} < \infty$. Then, since

$$0 \leq \lim_{t\uparrow 1}\lambda(t) \leq \lim_{t\uparrow 1}\exp\left(-\int_0^t \frac{ds}{1-s}\right) = 0,$$

  we are done.
- Case 2: $\lim_{t\uparrow 1}\int_0^t b(V(s)-s, X_s^*, Z_s)\frac{ds}{\lambda(s)} = \infty$. Since both $\lambda(t)$ and $\int_0^t b(V(s)-s, X_s^*, Z_s)\frac{ds}{\lambda(s)}$ are differentiable for fixed ω in $[\tau > 1]$, we can use de l'Hôpital's rule to get:

$$\lim_{t\uparrow 1}\lambda(t)\int_0^t b(V(s)-s, X_s^*, Z_s)\frac{ds}{\lambda(s)}ds = \lim_{t\uparrow 1}(V(t)-t)b(V(t)-t, X_t^*, Z_t) = 0 \tag{8.17}$$

a.s. on the set of non-default $[\tau > 1]$ provided $\limsup_{t\uparrow 1} b(V(t)-t, X_t^*, Z_t) < \infty$ a.s.

Since $b(V(t)-t, X_t^*, Z_t) = g\left(\frac{2X_t^* Z_t}{V(t)-t}\right)\frac{1}{X_t^*}$ with g being a bounded function on $[0,+\infty]$, to show that $\limsup_{t\uparrow 1} b(V(t)-t, X_t^*, Z_t) < \infty$, it is sufficient to demonstrate that $\liminf_{t\uparrow 1} X_t^* > 0$ on $[\tau > 1]$.

To prove it, consider two processes $\hat{X}$ and Y which follow

$$d\hat{X}_t = \left[\frac{Z_t - \hat{X}_t}{V(t)-t} + g\left(\frac{2\hat{X}_t Z_t}{V(t)-t}\right)\frac{1}{\hat{X}_t}\right]\mathbf{1}_{[\tau>t]}dt + dB_t, \quad t \in [0,1),$$

$$dY_t = \frac{Z_t - Y_t}{V(t)-t}\mathbf{1}_{[\tau>t]}dt + dB_t, \quad t \in [0,1].$$

The process $\hat{X}$ is well defined and is strictly positive for all $t \in [0,1)$ due to the Theorem 5.4. Moreover, for all $t \in [0,1)$ we have $Y_t\mathbf{1}_{[\tau>1]} = R_t\mathbf{1}_{[\tau>1]}$ and $\hat{X}_t\mathbf{1}_{[\tau>1]} = X_t^*\mathbf{1}_{[\tau>1]}$ and therefore it is sufficient to show that $\liminf_{t\uparrow 1}\hat{X}_t > 0$ on $[\tau > 1]$.

Observe that

$$d(Y_t - \hat{X}_t) = \left[\frac{\hat{X}_t - Y_t}{V(t)-t} - g\left(\frac{2\hat{X}_t Z_t}{V(t)-t}\right)\frac{1}{\hat{X}_t}\right]\mathbf{1}_{[\tau>t]}dt,$$

and g and $\hat{X}$ are strictly positive. Thus by Ito-Tanaka formula

$$
\begin{aligned}
(Y_t - \hat{X}_t)^+ &= \int_0^t \mathbf{1}_{[Y_s > \hat{X}_s]} \left[\frac{\hat{X}_s - Y_s}{V(s) - s} - g\left(\frac{2\hat{X}_s Z_s}{V(s) - s} \right) \frac{1}{\hat{X}_s} \right] \mathbf{1}_{[\tau > s]} ds \\
&\leq \int_0^t \mathbf{1}_{[Y_s > \hat{X}_s]} \left[\frac{\hat{X}_s - Y_s}{V(s) - s} \right] \mathbf{1}_{[\tau > s]} ds \leq 0
\end{aligned}
$$

since the local time of $Y - \hat{X}$ at 0 is identically 0 (see, e.g. Corollary 1.9 in Chap. VI of [100]).

Thus, on the set $[\tau > 1]$ we have

$$
\liminf_{t \uparrow 1} X_t^* = \liminf_{t \uparrow 1} \hat{X}_t \geq Y_1 = R_1 = Z_1 > 0
$$

as required.

Recall that $\mathbb{P}[\tau = 1] = 0$, therefore, one has $V(\tau) = \inf\{t \in [0, 1] : X_t^* = 0\}$, where $\inf \emptyset = 1$ by convention. This makes $V(\tau)$ a stopping time with respect to $\mathscr{F}^{X^*}$ and yields that $\mathscr{F}^{X^*} = \mathscr{F}^M$.

To complete the proof we need to show that X^* is a Brownian motion in its own filtration, stopped at $V(\tau)$. Notice first that Theorem 5.4 implies in particular that X^* is a Brownian motion in its own filtration over each interval $[0, T]$ for every $T < 1$, i.e. $(X^*_{t \wedge V(\tau)})_{t \in [0,1)}$ is a Brownian motion. As we have seen, $\lim_{t \uparrow 1} X^*_{t \wedge V(\tau)} = Z_{1 \wedge \tau}$; thus, it follows from Fatou's lemma that $(X^*_{t \wedge V(\tau)})_{t \in [0,1]}$ is a supermartingale. In order to obtain the martingale property over the whole interval $[0, 1]$, it suffices to show that $\mathbb{E}[X^*_{1 \wedge V(\tau)}] = 1$. However, since $X^*_{1 \wedge V(\tau)} = Z_{1 \wedge \tau}$, this follows from the fact that Z is a martingale. In view of Lévy's characterisation, we conclude that X^* is a Brownian motion in its own filtration, stopped at $V(\tau)$. □

In the next section we will compare properties of the equilibriums with dynamic versus static private information. However, we will analyse the effect of default on the market parameters such as volatility and market depth in the next remark.

Remark 8.2 In order to understand the effect of default risk on the market parameters, let us choose an absolutely continuous pay-off $f(x)$ which is null on $(-\infty, 0]$. We have seen in the theorem above that the equilibrium price of this defaultable claim is given by

$$
\begin{aligned}
H^*(t, x) = &\int_0^\infty f(y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-x)^2}{2(1-t)} \right) dy \\
&- \int_{-\infty}^0 f(-y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-x)^2}{2(1-t)} \right) dy
\end{aligned}
$$

while Theorem 7.1 gives the price of this claim in the absence of default as

$$\widehat{H}(t,x) = \int_0^\infty f(y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-x)^2}{2(1-t)}\right) dy.$$

It is clear from above that $H^*(t,x)$ is smaller than $\widehat{H}(t,x)$. Kyle's λ, which measures the market depth, is given by $1/H_x$ where H is the equilibrium price in the corresponding market. Direct calculations show that

$$\widehat{H}_x - H_x^* = -\int_{-\infty}^0 f'(-y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-x)^2}{2(1-t)}\right) dy < 0.$$

Thus, the market depth decreases with default risk. This is quite intuitive, since the presence of default risk increases market makers' informational disadvantage thereby causing them to decrease the market depth to compensate for the additional risk and informational disadvantage. Similarly default risk also increases the volatility of the log-returns of prices. Indeed, the above comparisons lead to $H_x^*/H^* > \widehat{H}_x/\widehat{H}$.

8.3 Comparison of Dynamic and Static Private Information

In this section we compare the expected profits of the insider in the cases of dynamic and static private information. By static private information we mean that the insider knows τ and Z_1 in advance. In order to do the comparison, we first need to obtain the equilibrium and the associated expected profit when the private information is static.

Recall that the proof of Proposition 8.1 did not depend on the type of the private information; therefore, the optimality conditions for the insider with a static information are still described by it after replacing expectations with conditional expectations (see Remark 8.1).

Theorem 8.2 *Suppose that the insider observes τ and Z_1 at time 0. Then, under Assumption 8.1 there exists an equilibrium (H^*, θ^*), where*

(i) $H^*(t,x) = P(V^{-1}(t), x)$ *where* $P(t,x)$ *is given by (8.13) for* $(t,x) \in [0,1] \times \mathbb{R}_+$.
(ii) $\theta_t^* = \int_0^t \alpha_s^* ds$ *for* $t \in [0, 1 \wedge V(\tau)]$*, where*

$$\alpha_s^* = \frac{q_x(1-s, X_s^*, Z_1)}{q(1-s, X_s^*, Z_1)} \mathbf{1}_{[\tau>1]} + \frac{\ell_a(V(\tau)-s, X_s^*)}{\ell(V(\tau)-s, X_s^*)} \mathbf{1}_{[\tau\le 1]} \tag{8.18}$$

where the process X^* is the unique strong solution under insider's filtration $\mathscr{F}^{X,Z_1,\tau}$ of the following SDE:

$$X_t^* = 1 + B_{V(\tau)\wedge t} + \int_0^{V(\tau)\wedge t} \times \left\{ \frac{q_x(1-s, X_s^*, Z_1)}{q(1-s, X_s^*, Z_1)} \mathbf{1}_{[\tau>1]} + \frac{\ell_a(V(\tau)-s, X_s^*)}{\ell(V(\tau)-s, X_s^*)} \mathbf{1}_{[\tau\le 1]} \right\} ds. \quad (8.19)$$

Moreover, $V(\tau) = \inf\{t \in [0,1] : X_t^* = 0\}$, where $\inf \emptyset = 1$ by convention, and one has $\lim_{t\uparrow 1} X_t^* = Z_1$ on the set of non-default $[\tau > 1]$. As a consequence, $V(\tau)$ is a predictable stopping time in the market filtration $\mathscr{F}^M$.

Furthermore, the expected profit of the insider is

$$\int_{\xi(0,a^*)}^1 (H^*(0,u) - a^*)du + \frac{1}{2}\int_0^{1\wedge V(\tau)} H_x^*(s, \xi(s,a^*))ds, \quad (8.20)$$

where $a^* = \mathbf{1}_{[\tau>1]} f(Z_1)$ and $\xi(t,a)$ is the unique solution of $H(t,\xi(t,a)) = a$ for all $a \ge 0$.

Proof Since the optimality conditions for the insider are still described by Proposition 8.1, the proof will follow the same lines as the proof of Theorem 8.1 once we show that there exist unique strong solution of (8.19) on $[0,1)$, X^*, satisfying the following properties:

1) $\lim_{t\uparrow 1} X^*_{t\wedge V(\tau)} = 0$ a.s. on $[\tau < 1]$,
2) $\lim_{t\uparrow 1} X_t^* = Z_1$ a.s. on the set $[\tau > 1]$,
3) $(X_t^*)_{t\in[0,1]}$ is a Brownian motion stopped at $V(\tau)$ in its own filtration.

To see this consider a Brownian motion β, in a possibly different probability space, with $\beta_0 = 1$ and $T_0 = \inf\{t > 0 : \beta_t = 0\}$. Let $(\mathscr{G}_t)_{t\ge 0}$ be the minimal filtration satisfying usual conditions and to which β is adapted and $\mathscr{G}_0 \supset \sigma(\beta_1, T_0)$. Direct calculations show that

$$\begin{aligned}
&\mathbb{P}[T_0 \in du, T_0 > 1, \beta_1 \in dy | \mathscr{F}_t^\beta] \\
&\quad = \mathbf{1}_{[1\wedge T_0 > t]} \ell(u-1, y) q(1-t, \beta_t, y)\, dy\, du + \mathbf{1}_{[T_0 > t \ge 1]} \ell(u-t, \beta_t)\, du \\
&\mathbb{P}[T_0 \in du, T_0 \le 1, \beta_1 \in dy | \mathscr{F}_t^\beta] \\
&\quad = \mathbf{1}_{[T_0 > t]} \frac{1}{\sqrt{2\pi(1-u)}} \exp\left(-\frac{y^2}{2(1-u)}\right) \ell(u-t, \beta_t)\, dy\, du \\
&\qquad + \mathbf{1}_{[T_0 \le t]} \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-\beta_t)^2}{2(1-t)}\right) dy
\end{aligned}$$

Thus, it follows from the discussion in Remark 4.6 that

$$\beta_t = 1 + \tilde{\beta}_t + \int_0^{t\wedge 1\wedge T_0} \left\{ \frac{q_x(1-s,\beta_s,\beta_1)}{q(1-s,\beta_s,\beta_1)} \mathbf{1}_{[T_0>1]} + \frac{\ell_a(T_0-s,\beta_s)}{\ell(T_0-s,\beta_s)} \mathbf{1}_{[T_0\le 1]} \right\} ds$$
$$+ \int_{t\wedge 1\wedge T_0}^{t\wedge T_0} \frac{\ell_a(T_0-s,\beta_s)}{\ell(T_0-s,\beta_s)} ds + \int_{t\wedge 1\wedge T_0}^{t\wedge 1} \frac{\beta_1-\beta_s}{1-s} ds,$$

where $\tilde{\beta}$ is a $\mathscr{G}$-Brownian motion independent of β_1 and T_0. Observe that $Z_{V^{-1}(t)}$ is a standard Brownian motion starting at 1 with $V(\tau)$ as its first hitting time of 0. Moreover, the SDE satisfied by β until $T_0 \wedge 1$ is the same as (8.19) until time 1 since $(\beta_1, T_0, \tilde{\beta})$ has the same law as $(Z_1, V(\tau), B)$ due to $V(1) = 1$. Therefore, the law of $(X^*_{t\wedge V(\tau)\wedge 1})_{t\ge 0}$ is the same as that of $(\beta_{t\wedge T_0\wedge 1})_{t\ge 0}$ since the solution of the SDE for β has strong uniqueness. In particular, properties 1), 2) and 3) above are satisfied. □

Comparing (8.16) with (8.20) yields that the values of static and dynamic information are the same. This is to be expected as we observed this phenomenon before in the case of no-default (see Sect. 7.2).

The same phenomenon is also responsible for the fact that the price of information does not depend on V, which also manifests itself in expression (8.16) since the distribution of $V(\tau)$ is the same as that of the first hitting time of 0 by a Brownian starting at 1. In fact, it is easy to observe that the static information is the limiting case of dynamic ones characterised by an increasing sequence of functions V^n with $\lim_{n\to\infty} V^n(t) = 1$ for all $t \in (0, 1]$.

The value of information, (8.16), can be computed more explicitly as the following proposition shows.

Proposition 8.2 *Suppose f is invertible with $f(0) = 0$ and satisfies Assumption 8.1. Then (8.16) becomes*

$$\mathbb{E}[W_1^{\theta^*}] = \mathbb{E}[F(|1+\beta_1|)] - \mathbb{E}\left[F\left(|\beta_1|\sqrt{1-V(\tau)}\right)\mathbf{1}_{[\tau<1]}\right] - F(1)\mathbb{P}[\tau > 1]$$
$$+\mathbb{E}\left[\{(Z_1-1)f(Z_1) - (F(Z_1)-F(1))\}\mathbf{1}_{[\tau>1]}\right], \tag{8.21}$$

where β is a standard Brownian motion independent of B^Z with $\beta_0 = 0$, and $F(z) := \int_0^z f(y)\,dy$.

Proof For any $a \ge 0$ let

$$g(a) := \int_{\xi(0,a)}^1 (H^*(0,u)-a)du + \frac{1}{2}\int_0^{1\wedge V(\tau)} H_x^*(s,\xi(s,a))ds.$$

Since $\xi(s,0) = 0$ for any $s \ge 0$,

$$g(0) = \int_0^1 H^*(0,u)du + \frac{1}{2}\int_0^{1\wedge V(\tau)} H_x^*(s,0)ds.$$

We will first compute the first term in the equation above.

$$\begin{aligned}
&\int_0^1 H^*(0,u)du \\
&= \int_0^1 \int_0^\infty f(y)q(1,u,y)\,dy\,du = -\int_0^\infty F(y)\int_0^1 q_y(1,u,y)\,du\,dy \\
&= \int_0^\infty F(y)\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(y-1)^2}{2}} + e^{-\frac{(y+1)^2}{2}}\right)dy - 2\int_0^\infty F(y)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\
&= \mathbb{E}[F(|1+\beta_1|)] - \mathbb{E}[F(|\beta_1|)].
\end{aligned}$$

In particular, this implies $\mathbb{E}[F(|1+\beta_1|)] - \mathbb{E}[F(c|\beta_1|)] > 0$ for any constant $0 \le c \le 1$ since F is increasing.

The second term is given by

$$\begin{aligned}
&\frac{1}{2}\int_0^{1\wedge V(\tau)} H_x^*(s,0)ds \\
&= \int_0^{1\wedge V(\tau)}\int_0^\infty f(y)\ell(1-s,y)\,dy\,ds = \int_0^\infty f(y)\int_0^{1\wedge V(\tau)} \ell(1-s,y)\,ds\,dy \\
&= \int_0^\infty f(y)\mathbb{P}[T_y < 1]\,dy - \int_0^\infty f(y)\mathbb{P}[T_y < 1 - 1\wedge V(\tau)|\tau]\,dy \\
&= \int_0^\infty f(y)\left\{\mathbb{P}[|\beta_1| > y] - \mathbb{P}[|\beta_{1-1\wedge V(\tau)}| > y|\tau]\right\}dy \\
&= \mathbb{E}[F(|\beta_1|)] - \mathbb{E}[F(\sqrt{1-1\wedge V(\tau)}|\beta_1|)|\tau],
\end{aligned}$$

where in the one to the last equality, we used the reflection principle and the last equality follows from the scaling property of Brownian motion. Thus,

$$g(0) = \mathbb{E}[F(|1+\beta_1|)] - \mathbb{E}[F(\sqrt{1-1\wedge V(\tau)}|\beta_1|)|\tau]. \tag{8.22}$$

Next, observe that

$$g'(a) = \xi(0,a) - 1 + \frac{1}{2}\int_0^{1\wedge V(\tau)} H_{xx}^*(s,\xi(s,a))\xi_a(s,a)\,ds.$$

Differentiating the equality $H^*(s,\xi(s,a)) = a$ with respect to s and a yields

$$H_s^* + H_x^*\xi_s = 0, \text{ and } H_x^* = \frac{1}{\xi_a},$$

and using the fact that $H_t^* + \frac{1}{2}H_{xx}^* = 0$, we get $\frac{1}{2}H_{xx}^*\xi_a = \xi_s$. Therefore,

$$g'(a) = \xi(0,a) - 1 + \int_0^{1\wedge V(\tau)} \xi_s(s,a)\,ds = \xi(1\wedge V(\tau),a) - 1,$$

and thus

$$g(a) = \mathbb{E}[F(|1+\beta_1|)] - \mathbb{E}[F(\sqrt{1-1\wedge V(\tau)}|\beta_1|)|\tau] + \int_0^a (\xi(1\wedge V(\tau),u)-1)du.$$

Plugging $a = \mathbf{1}_{[\tau>1]}f(Z_1)$ into above yields

$$\begin{aligned} g(a) &= \mathbb{E}[F(|1+\beta_1|)] - \mathbb{E}[F(\sqrt{1-1\wedge V(\tau)}|\beta_1|)|\tau] \\ &\quad + \mathbf{1}_{[\tau>1]}\int_0^{f(Z_1)} (f^{-1}(u)-1)du \\ &= \mathbb{E}[F(|1+\beta_1|)] - \mathbb{E}[F(\sqrt{1-1\wedge V(\tau)}|\beta_1|)|\tau] \\ &\quad + \mathbf{1}_{[\tau>1]}\{(Z_1-1)f(Z_1) - F(Z_1)\}. \end{aligned}$$

Taking the expectation of above, it is easy to see that the conclusion holds.

Below are some explicit examples where we can compute the value of information.

Example 8.1 In the case of defaultable stock, $f(x) = x$. Then the value of information becomes

$$\begin{aligned} \mathbb{E}[W_1^{\theta^*}] &= \frac{1}{2}\Big(\mathbb{E}\left[(1+\beta_1)^2\right] - \mathbb{E}\left[\beta_1^2\right]\mathbb{E}\left[(1-V(\tau))\mathbf{1}_{[\tau<1]}\right] \\ &\quad + \mathbb{E}\left[(Z_1^2 - 2Z_1)\mathbf{1}_{[\tau>1]}\right]\Big) \\ &= \mathbb{P}[V(\tau)\geq 1] + \mathbb{E}\left[V(\tau)\mathbf{1}_{[V(\tau)<1]}\right] = \mathbb{E}[V(\tau)\wedge 1]. \end{aligned}$$

According to the last equality above, the longer the defaultable stock is traded, the higher is the insider's expected profit. Such a result is to be expected since the insider can speculate on her private information only when the market operates.

Although Proposition 8.2 requires f to be invertible, one can still carry out similar calculation to obtain the value of the information when f fails this condition.

Example 8.2 Consider the defaultable zero-coupon bond with payoff $f \equiv 1$. Then, observe that (8.22) is still valid as we did not use the conditions on f to obtain it. Moreover, in this case a^* takes values in $\{0, 1\}$. Thus, it remains to calculate $g(1)$.

First, observe that H is bounded by 1 and strictly increasing, thus, $\xi(t, 1) = \infty$ and $H_x(t, \xi(t, 1)) = 0$. Therefore,

$$\begin{aligned} g(1) &= \int_{\infty}^{1} (H^*(0, u) - 1) du \\ &= \int_{1}^{\infty} \int_{-\infty}^{\infty} (1 - sgn(y + u)) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy du = \sqrt{\frac{2}{\pi e}} - 2\mathbb{P}[\beta_1 < -1]. \end{aligned}$$

Thus, the value of information is

$$\mathbb{E}[W_1^{\theta^*}] = \mathbb{E}\left[g(0)\mathbf{1}_{[\tau<1]} + g(1)\mathbf{1}_{[\tau>1]}\right] = \sqrt{\frac{2}{\pi e}} - \sqrt{\frac{2}{\pi}}\mathbb{E}\left[\sqrt{1 - V(\tau)}\mathbf{1}_{[V(\tau)<1]}\right].$$

The last expectation on the RHS is an indicator of how far, on average, is the default time from the defaultable bond's maturity in the case of default before maturity. As in the previous example, the larger is that expectation, i.e. the larger is the average distance between market's default time and the maturity, the lower is the value of the private information.

8.4 Notes

This chapter is largely taken from [33] and is an extension of [31] to the case of dynamic information. To the best of our knowledge these are the only two papers that deal with default in the context of Kyle model.

The modelling of default time in this chapter can be interpreted in terms of economic and recorded default times – τ and $V(\tau)$, respectively. It is documented that these two notions of default do not necessarily coincide and the latter is typically later than the former (see Guo et al. [61]).

Appendix A

A.1 Dynkin's $\pi - \lambda$ Theorem

Theorem A.1 *Let S be an arbitrary space and π a class of subsets of S which is closed under intersection. Let λ be a class of subsets of S such that $S \in \lambda$ and $\pi \subset \lambda$. Furthermore, suppose that λ has the following properties:*

i) if $A_n \in \lambda$ and $A_n \subset A_{n+1}$ for $n \geq 1$, then $\cup_{n=1}^{\infty} A_n \in \lambda$;
ii) if $A \subset B$ and $A \in \lambda$, $B \in \lambda$, then $B \setminus A \in \lambda$.

Then, $\sigma(\pi) \subset \lambda$.

A.2 Weak Convergence of Measures

Proposition A.1 *Let $(\mathbf{E}, d)$ be a separable locally compact metric space and (P_n) be a sequence of measures on $(\mathbf{E}, \mathscr{E})$. Then, if for all $f \in \mathbb{C}_K(\mathbf{E})$, $\int_{\mathbf{E}} f dP_n \to \int_{\mathbf{E}} f dP$ for some measure P on $(\mathbf{E}, \mathscr{E})$, then P_n converges weakly to P.*

Definition A.1 Let $\mathscr{P}(\mathbf{E})$ denote the space of probability measures on $(\mathbf{E}, \mathscr{E})$. Then, a subset A of $\mathscr{P}(\mathbf{E})$ is called tight if for any $\varepsilon > 0$ there exists a compact subset of $\mathbf{E}$, K, such that $P(K) \geq 1 - \varepsilon$ for all $P \in A$.

Theorem A.2 (Prokhorov's Theorem) *A subset A of $\mathscr{P}(\mathbf{E})$ is relatively weakly compact if and only if it is tight.*

Let $T^* \in \mathbb{R}_{++} \cup \{\infty\}$ and $\mathbf{E}$ be a Polish space with metric m. If $T^* < \infty$, define $\Omega = C([0, T^*], \mathbf{E})$, otherwise $\Omega = C([0, \infty), \mathbf{E})$. We endow Ω with the Skorokhod topology and denote by X the coordinate process. The canonical filtration $(\mathscr{B}_t)_{t \in [0, T^*)}$ is defined via $\mathscr{B}_t = \sigma(X_s; s \leq t)$ for $t < T^*$, and $\mathscr{B}_{T^*} = \vee_{t < T^*} \mathscr{B}_t$. Recall that $\mathscr{B}_{T^*}$ is countably generated since the corresponding

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metric space is separable. We will write $\mathscr{B}_{t\geq s}$ for $(\mathscr{B}_t)_{t\in[s,T^*]}$ (resp. $(\mathscr{B}_t)_{t\in[s,T^*)}$) when $T^* < \infty$ (resp. $T^* = \infty$). The following theorem can be found, e.g. in [74].

Theorem A.3 *Let Ω be as above and (P_n) be a sequence of probability measures on $(\Omega, \mathscr{B}_{T^*})$. Then, (P_n) is tight if and only if the following two conditions hold:*

1. *The sequence $(P_n X_0^{-1})$ is tight on $(\mathbf{E}, \mathscr{E})$.*
2. *For any $N \in \mathbb{N}$*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P_n(w(X, \delta, [0, N \wedge T^*]) > \varepsilon) = 0, \qquad \forall \varepsilon > 0,$$

where

$$w(X, \delta, [u, v]) = \sup_{\substack{|t-r|<\delta \\ t,r\in[u,v]}} \|X_t - X_r\|$$

is the modulus of continuity on $[u, v]$.

Theorem A.4 (Theorem 1 in [3]) *Let Ω be as above and (P_n) be a sequence of probability measures on $(\Omega, \mathscr{B}_{T^*})$. Then, (P_n) is tight if the following two conditions hold:*

1. *The sequence $(P_n X_0^{-1})$ is tight on $(\mathbf{E}, \mathscr{E})$.*
2. *For any $N \in \mathbb{N}$*

$$\lim_{h\to 0} \limsup_{n\to\infty} \sup_{\tau\in\mathscr{T}_N} \sup_{s\in[0,h]} P_n(m(X_{(\tau+s)\wedge T^*}, X_\tau) > \varepsilon) = 0, \qquad \forall \varepsilon > 0,$$

where $\mathscr{T}_N$ is the class of stopping times taking finitely many values and bounded by N.

Proposition A.2 *Let Ω be as above and (P_n) be a sequence of probability measures on $(\Omega, \mathscr{B}_{T^*})$. Then P_n converges weakly to some P on $(\Omega, \mathscr{B}_{T^*})$ if and only if (P^n) is tight and there exists a dense subset, D, of $[0, T^*)$ such that*

$$\lim_{n\to\infty} E^{P_n}[f(X_{t_1}, \dots, X_{t_r})] = E^P[f(X_{t_1}, \dots, X_{t_r})],$$

for all $r \in \mathbb{N}$, bounded and continuous $f : \mathbf{E}^r \mapsto \mathbb{R}$, and $t_i \in D$.

A.3 Optional Times

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbf{T}}, P)$ be a filtered probability space.

Definition A.2 The function $T : \Omega \mapsto [0, \infty]$ is called an optional time relative to $(\mathscr{F}_t)$ if for any $t \in \mathbf{T}$, $[T < t] \in \mathscr{F}_t$. If, on the other hand, $[T \leq t] \in \mathscr{F}_t$ for all $t \in \mathbf{T}$, T is called a stopping time relative to $(\mathscr{F}_t)$.

Observe that since we have *not* assumed that the filtration is right-continuous the notion of optional time is different than that of a stopping time in general. Define

$$\mathscr{F}_\infty = \bigvee_{t \geq 0} \mathscr{F}_t;$$

$$\forall t \in (0, \infty) : \mathscr{F}_{t-} = \bigvee_{s \in [0,t)} \mathscr{F}_s;$$

$$\forall t \in [0, \infty) : \mathscr{F}_{t+} = \bigwedge_{s \in (t,\infty)} \mathscr{F}_s.$$

Clearly, $\mathscr{F}_{t-} \subset \mathscr{F}_t \subset \mathscr{F}_{t+}$. We will say that a filtration $(\mathscr{F}_t)$ is right-continuous if $\mathscr{F}_t = \mathscr{F}_{t+}$ for all $t \geq 0$. The following proposition shows the relationship between optional and stopping times.

Proposition A.3 *T is an optional time relative to $(\mathscr{F}_t)$ if and only if it is a stopping time relative to $(\mathscr{F}_{t+})$.*

As an immediate corollary of this proposition we see that optional times and stopping times are the same notions when the filtration is right-continuous.

Proposition A.4 *Suppose that X is a right-continuous adapted process, S is an optional time relative to $(\mathscr{F}_t)$, and A is an open set. Let*

$$T = \inf\{t \geq S : X_t \in A\}.$$

Then, T is an optional time relative to $(\mathscr{F}_t)$.

Proof Using the right continuity of X, we have

$$[T < t] = \cup_{s \in \mathbb{Q} \cap [0,t)} [X_s \in A] \cap [S < s] \in \mathscr{F}_t,$$

since S is optional and X is adapted and right-continuous. □

The following results show that the class of optional times is stable under taking limits or addition.

Lemma A.1 *If $(T_n)_{n \geq 1}$ are optional, so are* $\sup_{n \geq 1} T_n$, $\inf_{n \geq 1} T_n$, $\limsup_{n \to \infty} T_n$ *and* $\liminf_{n \to \infty} T_n$.

Corollary A.1 *Suppose that X is an adapted continuous process, S is an optional time relative to $(\mathscr{F}_t)$, and A is a closed set. Let*

$$T = \inf\{t \geq S : X_t \in A\}.$$

Then, T is an optional time relative to $(\mathscr{F}_t)$. It is a stopping time if S is.

Proof Consider $A_n := \{x \in \mathbf{E} : d(x, A) < \frac{1}{n}\}$ and let $T_n := \inf\{t \geq S : X_t \in A_n\}$. Since A_n is open, T_n is an optional time for any n. Since X is continuous $T = \lim_{n\to\infty} T_n$, which is an optional time by above.

Moreover, we have

$$[T \leq t] = ([S \leq t] \cap [X_t \in A]) \cup \cap_{n\geq 1}[T_n < t] \in \mathscr{F}_t,$$

if S is a stopping time.

Lemma A.2 *$T + S$ is an optional time if T and S are optional times. It is a stopping time if one of the following holds:*

i) $S > 0$ *and* $T > 0$*;*
ii) $T > 0$ *and* T *is a stopping time.*

For any stopping time T we can define the σ-algebra of *events up to time* T as

$$\mathscr{F}_T = \{A \in \mathscr{F}_\infty : A \cap [T \leq t] \in \mathscr{F}_t, \forall t \in [0, \infty)\}.$$

An analogous definition applies for an optional time.

Definition A.3 Let T be an optional time. Then,

$$\mathscr{F}_{T+} = \{A \in \mathscr{F}_\infty : A \cap [T < t] \in \mathscr{F}_t, \forall t \in (0, \infty]\}.$$

$\mathscr{F}_{T+}$ can be shown to be a σ-algebra. Moreover,

$$\mathscr{F}_{T+} = \{A \in \mathscr{F}_\infty : A \cap [T \leq t] \in \mathscr{F}_{t+}, \forall t \in [0, \infty)\}.$$

Observe that if T is an optional time $\mathscr{F}_T$ is not necessarily defined. However, for a stopping time T both $\mathscr{F}_T$ and $\mathscr{F}_{T+}$ are well defined, and $\mathscr{F}_T \subset \mathscr{F}_{T+}$.

Theorem A.5

1. *If T is optional, then $T \in \mathscr{F}_{T+}$.*
2. *If S and T are optional such that $S \leq T$, then $\mathscr{F}_{S+} \subset \mathscr{F}_{T+}$. If, moreover, T is a stopping time such that $S < T$ on $[S < \infty]$, then $\mathscr{F}_{S+} \subset \mathscr{F}_T$.*
3. *If $(T_n)_{n\geq 1}$ are optional times such that $T_n \geq T_{n+1}$ and $T = \lim_{n\to\infty} T_n$, then*

$$\mathscr{F}_{T+} = \bigwedge_{n=1}^{\infty} \mathscr{F}_{T_n+}.$$

4. *If (T_n) is a sequence of stopping times decreasing to T with $T < T_n$ on $[T < \infty]$, for each $n \geq 1$, then*

$$\mathscr{F}_{T+} = \bigwedge_{n=1}^{\infty} \mathscr{F}_{T_n}.$$

Theorem A.6 *Let S and T be two optional times. Then,*

$$[S \le T], \qquad [S < T], \text{ and } \qquad [S = T]$$

belong to $\mathscr{F}_{S+} \wedge \mathscr{F}_{T+}$.

Lemma A.3 *Let T be an optional time and consider a sequence of random times $(T_n)_{n \ge 1}$ defined by*

$$T_n = \frac{[2^n T]}{2^n},$$

where $[x]$ is the smallest integer larger than $x \in [0, \infty)$, with the convention $[\infty] = \infty$. Then (T_n) is a decreasing sequence of stopping times with $\lim_{n \to \infty} T_n = T$. Moreover, for every $\Lambda \in \mathscr{F}_{T+}$, $\Lambda \cap [T_n = \frac{k}{2^n}] \in \mathscr{F}_{\frac{k}{2^n}}$, $k \ge 1$.

Theorem A.7 *Suppose X is adapted to $(\mathscr{F}_t)_{t \ge 0}$ and has right limits. Then, for T optional, we have*

$$X_{T+} \mathbf{1}_{[T < \infty]} \in \mathscr{F}_{T+}.$$

Theorem A.8 *Suppose $(X_t, \mathscr{F}_t)$ is a right-continuous Markov process. Let S be an optional time relative to $(\mathscr{F}_t^0)$ and T be optional relative to $(\mathscr{F}_t)$. Then,*

$$T + S \circ \theta_T,$$

where θ is the shift operator, is an optional time relative to $(\mathscr{F}_t)$.

The random variable $T + S \circ \theta_T$ has a nice interpretation when S is the first entrance time of X into some set A. Suppose that

$$S = \inf\{t \ge 0 : X_t \in A\}$$

for some open set A. Then

$$T + S \circ \theta_T = \inf\{t \ge T : X_t \in A\}.$$

In other words, it becomes the first entrance time of the same process after time T.

A.4 Regular Conditional Probability

The following definition and theorem are taken from Sect. I.3 of Ikeda and Watanabe [70].

Definition A.4 Let $(\Omega, \mathscr{F}, \mathbb{P})$be a probability space and $\mathscr{G}$ a sub-σ-field of $\mathscr{F}$. A function $Q : \Omega \times \mathscr{F} \mapsto [0, 1]$ is called a regular conditional probability for $\mathscr{F}$ given $\mathscr{G}$ if

i) For each $\omega \in \Omega$, $Q(\omega; \cdot)$ is a probability measure on $(\Omega, \mathscr{F})$,
ii) For each $E \in \mathscr{F}$ the mapping $\omega \mapsto Q(\omega, E)$ is $\mathscr{G}$-measurable,
iii) For each $E \in \mathscr{F}$ $Q(\omega, E) = \mathbb{P}[E|\mathscr{G}](\omega)$, $\mathbb{P}$-a.e. $\omega \in \Omega$.

Suppose that whenever $Q'(\omega, E)$ is another function with these properties there exists a null set $N \in \mathscr{G}$ such that $Q(\omega, E) = Q'(\omega, E)$ for all $E \in \mathscr{F}$ and $\omega \in \Omega \backslash N$. We then say that the regular conditional probability for $\mathscr{F}$ given $\mathscr{G}$ is unique.

Theorem A.9 *Suppose Ω is a complete, separable metric space with the Borel σ-field $\mathscr{F} = \mathscr{B}(\Omega)$. Let P be a probability measure on $(\Omega, \mathscr{F})$ and $\mathscr{G}$ a sub-σ-field of $\mathscr{F}$. Then there exists a unique regular conditional probability Q for $\mathscr{F}$ given $\mathscr{G}$.*

When the sub-σ-field is generated by a random variable, the above can be reformulated to obtain the following:

Theorem A.10 *Suppose Ω is a complete, separable metric space with the Borel σ-field $\mathscr{F} = \mathscr{B}(\Omega)$. Let P be a probability measure on $(\Omega, \mathscr{F})$ and X be a measurable mapping from Ω into $(S, \mathscr{B}(S))$. Let PX^{-1} be the measure induced by the mapping X. Then there exists a function $Q : S \times \mathscr{F} \mapsto [0, 1]$ which is a regular conditional probability for $\mathscr{F}$ given X:*

i) For each $x \in S$, $Q(x; \cdot)$ is a probability measure on $(\Omega, \mathscr{F})$,
ii) for each $E \in \mathscr{F}$ the mapping $x \mapsto Q(x, E)$ is $\mathscr{B}(S)$-measurable,
iii) for each $E \in \mathscr{F}$ $Q(x; E) = P(A|X = x)$, PX^{-1}-a.e. $x \in S$.

If Q' is another function with these properties there exists a set $N \in \mathscr{B}(S)$ such that $PX^{-1}(N) = 0$ and $Q(x, E) = Q'(x, E)$ for all $E \in \mathscr{F}$ and $x \in \mathscr{B}(S)\backslash N$.

A.5 Brief Review of Martingale Theory

In this section we will collect some result from martingale theory which is used throughout the text. We emphasise that we *do not* assume that the reference filtration $(\mathscr{F}_t)$ is satisfying the usual conditions of right continuity and completeness unless explicitly stated otherwise. Note that the definitions of martingale, supermartingale and submartingale are indifferent to the absence of this assumption.

Theorem A.11 *Let S be a dense subset of* **T** *and define*

$$X_{t+} = \lim_{u \in S, u \downarrow t} X_u;$$

$$X_{t-} = \lim_{s \in S, s \uparrow t} X_s.$$

When X is a supermartingale, the limits above exist and are finite in a bounded interval.

Proposition A.5 *Suppose X is a supermartingale with right continuous paths. Then, its left limits exist everywhere in $(0, \infty)$, and it is bounded a.s. in each finite interval.*

The following is a well-known convergence theorem for supermartingales:

Theorem A.12 *Suppose that X is a right-continuous supermartingale, and either a) $X_t \geq 0$ for each t, or b) $\sup_{t\geq 0} \mathbb{E}|X_t| < \infty$. Then, $\lim_{t\to\infty} X_t$ exists a.s. and it is an integrable random variable.*

Corollary A.2 *Suppose that X is a right-continuous positive supermartingale. Then $X_\infty = \lim_{t\to\infty} X_t$ exists and $(X_t, \mathscr{F}_t)_{t\in[0,\infty]}$ is a supermartingale.*

Moreover, we have the following:

Theorem A.13 *Let X be a supermartingale. Then,*

$$\forall t \in [0, \infty) : X_t \geq \mathbb{E}[X_{t+}|\mathscr{F}_t].$$

Moreover, $(X_{t+}, \mathscr{F}_{t+})$ is a supermartingale. It is a martingale, if $(X_t, \mathscr{F}_t)$ is.

Proof Choose a sequence $(t_n) \subset S$ such that $t_n \downarrow t$ and consider the supermartingale $(X_s, \mathscr{F}_s)$ with the index set $\{t, \ldots, t_n, \ldots, t_1\}$. Then, it follows from the discrete martingale theory that (X_s) is a uniformly integrable sequence. Thus, by taking the limit of the inequality $X_t \geq \mathbb{E}[X_{t_n}|\mathscr{F}_t]$ we obtain $X_t \geq \mathbb{E}[X_{t+}|\mathscr{F}_t]$.

Next let $\Lambda \in \mathscr{F}_{t+}$ and $u_n > u > t_n > t$ such that $(u_n) \subset S$, $(t_n) \subset S$ and $u_n \downarrow u$, $t_n \downarrow t$. Since $\mathscr{F}_{t+} \subset \mathscr{F}_{t_n} \subset \mathscr{F}_{u_n}$, one has

$$\mathbb{E}[\mathbf{1}_\Lambda X_{u_n}] \leq \mathbb{E}[\mathbf{1}_\Lambda X_{t_n}], \forall n.$$

Using the aforementioned uniform integrability, we obtain

$$\mathbb{E}[\mathbf{1}_\Lambda X_{u+}] \leq \mathbb{E}[\mathbf{1}_\Lambda X_{t+}],$$

i.e. $(X_{t+}, \mathscr{F}_{t+})$ is a supermartingale. The case of martingale is handled similarly.

Without imposing some regularity conditions on the processes, it is almost impossible to go further with computations. Most of the time right continuity of paths is a desirable condition. The following theorem will state in particular that one can always have a good version of a martingale. However, it needs some conditions on the probability space and the filtration.

Recall that we say X and Y are versions of each other if

$$\forall t \in [0, \infty) : \mathbb{P}(X_t = Y_t) = 1.$$

Consequently, if $\mathbf{S} \subset \mathbf{T}$ is any countable set,

$$\mathbb{P}(X_t = Y_t, \forall t \in \mathbf{S}) = 1,$$

implying that X and Y have the same finite-dimensional distributions.

Note that when the probability space is complete and the filtration is augmented with the $\mathbb{P}$-null sets, if X is adapted, Y is adapted, too. Moreover, if X is a supermartingale, so is Y.

Theorem A.14 *Suppose that the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is complete and the filtration $(\mathscr{F}_t)$ satisfies the usual conditions. If $(X_t, \mathscr{F}_t)$ is a supermartingale, then the process (X_t) has a right-continuous version iff*

$$t \mapsto \mathbb{E}[X_t]$$

is right-continuous.

Theorem A.15 (Doob's Optional Sampling) *Suppose that $(X_t)_{t\in[0,\infty]}$ is a right-continuous supermartingale, and let $S \leq T$ be two optional times relative to $(\mathscr{F}_t)$. Then*

$$\mathbb{E}[X_T | \mathscr{F}_{S+}] \leq X_S.$$

If S is a stopping time, one can replace $\mathscr{F}_{S+}$ with $\mathscr{F}_S$. If moreover, X is a martingale, inequality becomes equality.

The following result, due to Knight and Maisonneuve [79], can be viewed as a converse to this statement and is a useful characterisation of stopping times.

Theorem A.16 *Suppose that the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is complete and the filtration $(\mathscr{F}_t)$ satisfies the usual conditions. Then τ is a stopping time if and only if for any bounded martingale, M,*

$$\mathbb{E}[M_\infty | \mathscr{F}_\tau] = M_\tau.$$

Definition A.5 A potential is a right continuous positive supermartingale X such that $\lim_{t\to\infty} \mathbb{E}[X_t] = 0$.

Observe that for a potential X, $X_\infty = \lim_{t\to\infty} X_t = 0$. However, this does not necessarily imply that

$$\lim_{n\to\infty} \mathbb{E}[X_{T_n}] = 0,$$

for a sequence of optional times (T_n) with $\lim_{n\to\infty} T_n = \infty$.

Theorem A.17 *Let X be a potential. Then for any sequence of optional times (T_n) with $\lim_{n\to\infty} T_n = \infty$,*

$$\lim_{n\to\infty} \mathbb{E}[X_{T_n}] = 0,$$

iff the set $\{X_T : T \text{ is optional}\}$ is uniformly integrable.

If a potential X satisfies any of the equivalent conditions in the above theorem, it is said to be of *Class D*.

Theorem A.18 *Suppose that M and N are square integrable martingales, i.e. $\sup_t EM_t^2 < \infty$. Then, there exists a predictable process of integrable variation, denoted by $\langle M, N\rangle$, such that $MN - \langle M, N\rangle$ is a martingale of Class D.*

Theorem A.19 *Assume Q is a locally absolutely continuous probability measure with respect to P, i.e. there exists a density process, Z, such that for any t and $A \in \mathscr{F}_t$, $Q(A) = E^P[\mathbf{1}_A Z_t]$. Suppose that Z is continuous. Then, for any continuous P-local martingale M, the process*

$$M' = M - \int_0^{\cdot} \frac{1}{Z_t} d[M, Z]_t,$$

is a Q-local martingale.

References

1. Acciaio, B., Fontana, C., Kardaras, C.: Arbitrage of the first kind and filtration enlargements in semimartingale financial models. Stochastic Processes and their Applications **126**(6), 1761–1784 (2016)
2. Aksamit, A., Choulli, T., Deng, J., Jeanblanc, M.: No-arbitrage up to random horizon for quasi-left-continuous models. Finance and Stochastics pp. 1–37 (2017)
3. Aldous, D.: Stopping times and tightness. II. Ann. Probab. **17**(2), 586–595 (1989)
4. Amendinger, J., Becherer, D., Schweizer, M.: A monetary value for initial information in portfolio optimization. Finance Stoch. **7**(1), 29–46 (2003)
5. Amendinger, J., Imkeller, P., Schweizer, M.: Additional logarithmic utility of an insider. Stochastic Process. Appl. **75**(2), 263–286 (1998)
6. Ankirchner, S., Dereich, S., Imkeller, P.: The Shannon information of filtrations and the additional logarithmic utility of insiders. Ann. Probab. **34**(2), 743–778 (2006)
7. Ankirchner, S., Imkeller, P.: Finite utility on financial markets with asymmetric information and structure properties of the price dynamics. Ann. Inst. H. Poincaré Probab. Statist. **41**(3), 479–503 (2005)
8. Aronson, D.G.: Bounds for the fundamental solution of a parabolic equation. Bull. Amer. Math. Soc. **73**, 890–896 (1967)
9. Back, K.: Insider trading in continuous time. Review of Financial Studies **5**, 387–409 (1992)
10. Back, K., Baruch, S.: Information in securities markets: Kyle meets Glosten and Milgrom. Econometrica **72**, 433–465 (2004)
11. Back, K., Cao, C.H., Willard, G.A.: Imperfect competition among informed traders. The Journal of Finance **55**(5), 2117–2155 (2000)
12. Back, K., Pedersen, H.: Long-lived information and intraday patterns. Journal of Financial Markets (1), 385–402 (1998)
13. Bain, A., Crisan, D.: Fundamentals of stochastic filtering, *Stochastic Modelling and Applied Probability*, vol. 60. Springer, New York (2009)
14. Baldi, P., Caramellino, L.: Asymptotics of hitting probabilities for general one-dimensional pinned diffusions. Ann. Appl. Probab. **12**(3), 1071–1095 (2002). doi:10.1214/aoap/1031863181. URL http://dx.doi.org/10.1214/aoap/1031863181
15. Baruch, S.: Insider trading and risk aversion. Journal of Financial Markets **5**(4), 451–464 (2002)
16. Baudoin, F.: Conditioned stochastic differential equations: theory, examples and application to finance. Stochastic Process. Appl. **100**, 109–145 (2002). URL https://doi.org/10.1016/S0304-4149(02)00109-6

U. Çetin, A. Danilova, *Dynamic Markov Bridges and Market Microstructure*,
Probability Theory and Stochastic Modelling 90,
https://doi.org/10.1007/978-1-4939-8835-8

17. Baudoin, F.: Conditionnement de fonctionnelles browniennes et applications à la modélisation d'anticipations sur les marchés financiers. Ph.D. thesis, Paris 7 (2002)
18. Baudoin, F.: Modeling anticipations on financial markets. In: Paris-Princeton Lectures on Mathematical Finance, 2002, *Lecture Notes in Math.*, vol. 1814, pp. 43–94. Springer, Berlin (2003)
19. Bertoin, J., Pitman, J.: Path transformations connecting Brownian bridge, excursion and meander. Bull. Sci. Math. **118**(2), 147–166 (1994)
20. Biagini, F., Hu, Y., Meyer-Brandis, T., Øksendal, B.: Insider trading equilibrium in a market with memory. Mathematics and Financial Economics **6**(3), 229 (2012)
21. Biagini, F., Øksendal, B.: A general stochastic calculus approach to insider trading. Appl. Math. Optim. **52**(2), 167–181 (2005). doi:10.1007/s00245-005-0825-2. URL http://dx.doi.org/10.1007/s00245-005-0825-2
22. Biais, B., Glosten, L., Spatt, C.: Market microstructure: A survey of microfoundations, empirical results, and policy implications. Journal of Financial Markets **8**(2), 217–264 (2005)
23. Bichteler, K.: Stochastic integration with jumps, *Encyclopedia of Mathematics and its Applications*, vol. 89. Cambridge University Press, Cambridge (2002). doi:10.1017/CBO9780511549878. URL http://dx.doi.org/10.1017/CBO9780511549878
24. Bielecki, T.R., Rutkowski, M.: Credit risk: modelling, valuation and hedging. Springer Finance. Springer-Verlag, Berlin (2002)
25. Billingsley, P.: Probability and measure, third edn. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York (1995)
26. Bjønnes, G.H., Rime, D.: Dealer behavior and trading systems in foreign exchange markets. Journal of Financial Economics **75**(3), 571–605 (2005)
27. Blumenthal, R.M., Getoor, R.K.: Markov processes and potential theory. Pure and Applied Mathematics, Vol. 29. Academic Press, New York-London (1968)
28. Borodin, A.N., Salminen, P.: Handbook of Brownian motion—facts and formulae, second edn. Probability and its Applications. Birkhäuser Verlag, Basel (2002). doi:10.1007/978-3-0348-8163-0. URL http://dx.doi.org/10.1007/978-3-0348-8163-0
29. Breiman, L.: Probability, *Classics in Applied Mathematics*, vol. 7. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1992). doi:10.1137/1.9781611971286. URL http://dx.doi.org/10.1137/1.9781611971286. Corrected reprint of the 1968 original
30. Brunnermeier, M.K.: Asset pricing under asymmetric information: Bubbles, crashes, technical analysis, and herding. Oxford University Press on Demand (2001)
31. Campi, L., Çetin, U.: Insider trading in an equilibrium model with default: a passage from reduced-form to structural modelling. Finance Stoch. **11**(4), 591–602 (2007)
32. Campi, L., Çetin, U., Danilova, A.: Dynamic Markov bridges motivated by models of insider trading. Stochastic Processes and their Applications **121**(3), 534–567 (2011)
33. Campi, L., Çetin, U., Danilova, A.: Equilibrium model with default and dynamic insider information. Finance and Stochastics **17**(3), 565–585 (2013). doi:10.1007/s00780-012-0196-x. URL https://doi.org/10.1007/s00780-012-0196-x
34. Campi, L., Cetin, U., Danilova, A., et al.: Explicit construction of a dynamic Bessel bridge of dimension 3. Electronic Journal of Probability **18** (2013)
35. Çetin, U., Danilova, A.: Markov bridges: SDE representation. Stochastic Processes and their Applications **126**(3), 651–679 (2016)
36. Çetin, U., Danilova, A.: Markovian Nash equilibrium in financial markets with asymmetric information and related forward–backward systems. The Annals of Applied Probability **26**(4), 1996–2029 (2016)
37. Çetin, U., Xing, H.: Point process bridges and weak convergence of insider trading models. Electronic Journal of Probability **18** (2013)
38. Chaumont, L., Uribe Bravo, G.: Markovian bridges: weak continuity and pathwise constructions. Ann. Probab. **39**(2), 609–647 (2011)
39. Cho, K.H.: Continuous auctions and insider trading: uniqueness and risk aversion. Finance Stoch. **7**(1), 47–71 (2003)

40. Chung, K.L., Walsh, J.B.: Markov processes, Brownian motion, and time symmetry, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 249, second edn. Springer, New York (2005). doi:10.1007/0-387-28696-9. URL http://dx.doi.org/10.1007/0-387-28696-9
41. Collin-Dufresne, P., Fos, V.: Insider trading, stochastic liquidity, and equilibrium prices. Econometrica **84**(4), 1441–1475 (2016)
42. Corcuera, J.M., Farkas, G., Di Nunno, G., Øksendal, B.: Kyle-Back's model with Lévy noise. Preprint series in pure mathematics, Mathematical Institute, University of Oslo, Norway (2010)
43. Corcuera, J.M., Imkeller, P., Kohatsu-Higa, A., Nualart, D.: Additional utility of insiders with imperfect dynamical information. Finance Stoch. **8**(3), 437–450 (2004)
44. Danilova, A.: Stock market insider trading in continuous time with imperfect dynamic information. Stochastics **82**(1–3), 111–131 (2010). doi:10.1080/17442500903106614. URL http://dx.doi.org/10.1080/17442500903106614
45. Danilova, A., Monoyios, M., Ng, A.: Optimal investment with inside information and parameter uncertainty. Mathematics and Financial Economics **3**(1), 13–38 (2010)
46. El Karoui, N., Lepeltier, J.P., Millet, A.: A probabilistic approach to the reduite in optimal stopping. Probab. Math. Statist. **13**(1), 97–121 (1992)
47. Engelbert, H.J., Schmidt, W.: Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations. I. Math. Nachr. **143**, 167–184 (1989). doi:10.1002/mana.19891430115. URL http://dx.doi.org/10.1002/mana.19891430115
48. Engelbert, H.J., Schmidt, W.: Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations. II. Math. Nachr. **144**, 241–281 (1989). doi:10.1002/mana.19891440117. URL http://dx.doi.org/10.1002/mana.19891440117
49. Engelbert, H.J., Schmidt, W.: Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations. III. Math. Nachr. **151**, 149–197 (1991). doi:10.1002/mana.19911510111. URL http://dx.doi.org/10.1002/mana.19911510111
50. Ethier, S.N., Kurtz, T.G.: Markov processes: Characterization and convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York (1986)
51. Fitzsimmons, P., Pitman, J., Yor, M.: Markovian bridges: construction, Palm interpretation, and splicing. In: Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992), *Progr. Probab.*, vol. 33, pp. 101–134. Birkhäuser Boston, Boston, MA (1993)
52. Föllmer, H.: Time reversal on Wiener space. In: Stochastic processes—mathematics and physics (Bielefeld, 1984), *Lecture Notes in Math.*, vol. 1158, pp. 119–129. Springer, Berlin (1986). doi:10.1007/BFb0080212. URL http://dx.doi.org/10.1007/BFb0080212
53. Föllmer, H., Wu, C.T., Yor, M.: Canonical decomposition of linear transformations of two independent Brownian motions motivated by models of insider trading. Stochastic Process. Appl. **84**(1), 137–164 (1999). doi:10.1016/S0304-4149(99)00057-5. URL http://dx.doi.org/10.1016/S0304-4149(99)00057-5
54. Fontana, C., Jeanblanc, M., Song, S.: On arbitrages arising with honest times. Finance and Stochastics **18**(3), 515–543 (2014)
55. Foster, F.D., Viswanathan, S.: Strategic trading when agents forecast the forecasts of others. The Journal of Finance **51**(4), 1437–1478 (1996). URL http://www.jstor.org/stable/2329400
56. FOUCAULT, T., HOMBERT, J., ROŞU, I.: News trading and speed. The Journal of Finance **71**(1), 335–382 (2016). doi:10.1111/jofi.12302. URL http://dx.doi.org/10.1111/jofi.12302
57. Friedman, A.: Partial differential equations of parabolic type. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1964)
58. Fujisaki, M., Kallianpur, G., Kunita, H.: Stochastic differential equations for the non linear filtering problem. Osaka J. Math. **9**, 19–40 (1972). URL http://projecteuclid.org/euclid.ojm/1200693535
59. Göing-Jaeschke, A., Yor, M.: A survey and some generalizations of Bessel processes. Bernoulli **9**(2), 313–349 (2003). doi:10.3150/bj/1068128980. URL http://dx.doi.org/10.3150/bj/1068128980

60. Grorud, A., Pontier, M.: Insider trading in a continuous time market model. International Journal of Theoretical and Applied Finance **01**(03), 331–347 (1998)
61. Guo, X., Jarrow, R.A., Lin, H.: Distressed debt prices and recovery rate estimation. Review of Derivatives Research **11**(3), 171 (2009). doi:10.1007/s11147-009-9029-2. URL https://doi.org/10.1007/s11147-009-9029-2
62. Hale, J.K.: Ordinary differential equations. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney (1969). Pure and Applied Mathematics, Vol. XXI
63. Hansch, O., Naik, N.Y., Viswanathan, S.: Do inventories matter in dealership markets? evidence from the London stock exchange. The Journal of Finance **53**(5), 1623–1656 (1998)
64. Haussmann, U.G., Pardoux, É.: Time reversal of diffusions. Ann. Probab. **14**(4), 1188–1205 (1986)
65. Hillairet, C.: Comparison of insiders' optimal strategies depending on the type of side-information. Stochastic Processes and their Applications **115**(10), 1603–1627 (2005)
66. Holden, C.W., Subrahmanyam, A.: Long-lived private information and imperfect competition. The Journal of Finance **47**(1), 247–270 (1992)
67. Holden, C.W., Subrahmanyam, A.: Risk aversion, imperfect competition, and long-lived information. Economics Letters **44**(1), 181–190 (1994)
68. Hsu, P.: Brownian bridges on Riemannian manifolds. Probab. Theory Related Fields **84**(1), 103–118 (1990)
69. Huang, R.D., Stoll, H.R.: The components of the bid-ask spread: A general approach. The Review of Financial Studies **10**(4), 995–1034 (1997)
70. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes, *North-Holland Mathematical Library*, vol. 24, second edn. North-Holland Publishing Co., Amsterdam (1989)
71. Imkeller, P.: Random times at which insiders can have free lunches. Stochastics: An International Journal of Probability and Stochastic Processes **74**(1–2), 465–487 (2002)
72. Imkeller, P., Pontier, M., Weisz, F.: Free lunch and arbitrage possibilities in a financial market model with an insider. Stochastic Process. Appl. **92**(1), 103–130 (2001)
73. Itô, K., McKean Jr., H.P.: Diffusion processes and their sample paths. Springer-Verlag, Berlin (1974). Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125
74. Jacod, J., Shiryaev, A.: Limit theorems for stochastic processes, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 288, second edn. Springer-Verlag, Berlin (2003)
75. Kallenberg, O.: Foundations of modern probability, second edn. Probability and its Applications (New York). Springer-Verlag, New York (2002). doi:10.1007/978-1-4757-4015-8. URL http://dx.doi.org/10.1007/978-1-4757-4015-8
76. Kallianpur, G.: Stochastic filtering theory, *Applications of Mathematics*, vol. 13. Springer-Verlag, New York-Berlin (1980)
77. Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus, *Graduate Texts in Mathematics*, vol. 113, second edn. Springer-Verlag, New York (1991)
78. Karlin, S., Taylor, H.M.: A second course in stochastic processes. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London (1981)
79. Knight, F.B., Maisonneuve, B.: A characterization of stopping times. Ann. Probab. **22**(3), 1600–1606 (1994)
80. Kohatsu-Higa, A., Ortiz-Latorre, S.: Weak Kyle-Back equilibrium models for max and argmax. SIAM Journal on Financial Mathematics **1**(1), 179–211 (2010)
81. Kohatsu-Higa, A., Ortiz-Latorre, S.: Modeling of financial markets with inside information in continuous time. Stoch. Dyn. **11**(2–3), 415–438 (2011)
82. Kohatsu-Higa, A., Sulem, A.: Utility maximization in an insider influenced market. Mathematical Finance **16**(1), 153–179 (2006)
83. Kurtz, T.: The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. Electron. J. Probab. **12**, 951–965 (2007). doi:10.1214/EJP.v12-431. URL http://dx.doi.org/10.1214/EJP.v12-431

84. Kurtz, T.G., Ocone, D.L.: Unique characterization of conditional distributions in nonlinear filtering. Ann. Probab. **16**(1), 80–107 (1988)
85. Kyle, A.: Continuous auctions and insider trading. Econometrica **53**, 1315–1335 (1985)
86. Liptser, R.S., Shiryaev, A.N.: Statistics of random processes. I, *Applications of Mathematics (New York)*, vol. 5, expanded edn. Springer-Verlag, Berlin (2001). General theory, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability
87. Liptser, R.S., Shiryaev, A.N.: Statistics of random processes. II, *Applications of Mathematics (New York)*, vol. 6, expanded edn. Springer-Verlag, Berlin (2001). Applications, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability
88. Madhavan, A., Smidt, S.: An analysis of changes in specialist inventories and quotations. The Journal of Finance **48**(5), 1595–1628 (1993)
89. Mandl, P.: Analytical treatment of one-dimensional Markov processes. Die Grundlehren der mathematischen Wissenschaften, Band 151. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague; Springer-Verlag New York Inc., New York (1968)
90. Mansuy, R., Yor, M.: Random times and enlargements of filtrations in a Brownian setting, *Lecture Notes in Mathematics*, vol. 1873. Springer-Verlag, Berlin (2006)
91. Marcus, M.B., Rosen, J.: Markov processes, Gaussian processes, and local times, vol. 100. Cambridge University Press (2006)
92. McKean Jr., H.P.: Elementary solutions for certain parabolic partial differential equations. Trans. Amer. Math. Soc. **82**, 519–548 (1956)
93. Millet, A., Nualart, D., Sanz, M.: Integration by parts and time reversal for diffusion processes. Ann. Probab. **17**(1), 208–238 (1989)
94. O'Hara, M.: Market microstructure theory, vol. 108. Blackwell Cambridge, MA (1995)
95. Pardoux, É.: Grossissement d'une filtration et retournement du temps d'une diffusion. In: Séminaire de Probabilités, XX, 1984/85, *Lecture Notes in Math.*, vol. 1204, pp. 48–55. Springer, Berlin (1986). doi:10.1007/BFb0075711. URL http://dx.doi.org/10.1007/BFb0075711
96. Pikovsky, I., Karatzas, I.: Anticipative portfolio optimization. Adv. in Appl. Probab. **28**(4), 1095–1122 (1996)
97. Pitman, J., Yor, M.: Bessel processes and infinitely divisible laws. In: Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980), *Lecture Notes in Math.*, vol. 851, pp. 285–370. Springer, Berlin (1981)
98. Pitman, J., Yor, M.: A decomposition of Bessel bridges. Z. Wahrsch. Verw. Gebiete **59**(4), 425–457 (1982)
99. Protter, P.E.: Stochastic integration and differential equations, *Stochastic Modelling and Applied Probability*, vol. 21. Springer-Verlag, Berlin (2005). Second edition. Version 2.1, Corrected third printing
100. Revuz, D., Yor, M.: Continuous martingales and Brownian motion, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 293, third edn. Springer-Verlag, Berlin (1999)
101. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes, and martingales. Vol. 1. Cambridge Mathematical Library. Cambridge University Press, Cambridge (2000). doi:10.1017/CBO9781107590120. URL http://dx.doi.org/10.1017/CBO9781107590120. Foundations, Reprint of the second (1994) edition
102. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes, and martingales. Vol. 2. Cambridge Mathematical Library. Cambridge University Press, Cambridge (2000). doi:10.1017/CBO9781107590120. URL http://dx.doi.org/10.1017/CBO9781107590120. Itô calculus, Reprint of the second (1994) edition
103. Roynette, B., Vallois, P., Yor, M.: Some penalisations of the Wiener measure. Jpn. J. Math. **1**(1), 263–290 (2006). doi:10.1007/s11537-006-0507-0. URL http://dx.doi.org/10.1007/s11537-006-0507-0
104. Roynette, B., Yor, M.: Penalising Brownian paths, *Lecture Notes in Mathematics*, vol. 1969. Springer-Verlag, Berlin (2009). doi:10.1007/978-3-540-89699-9. URL http://dx.doi.org/10.1007/978-3-540-89699-9

105. Salminen, P.: Brownian excursions revisited. In: Seminar on stochastic processes, 1983 (Gainesville, Fla., 1983), *Progr. Probab. Statist.*, vol. 7, pp. 161–187. Birkhäuser Boston, Boston, MA (1984)
106. Sharpe, M.: General theory of Markov processes, *Pure and Applied Mathematics*, vol. 133. Academic Press, Inc., Boston, MA (1988)
107. Shorack, G.R., Wellner, J.A.: Empirical processes with applications to statistics. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York (1986)
108. Stroock, D.W., Varadhan, S.R.S.: Diffusion processes with continuous coefficients. I and II. Comm. Pure Appl. Math. **22**, 345–400, 479–530 (1969). doi:10.1002/cpa.3160220304. URL http://dx.doi.org/10.1002/cpa.3160220304
109. Stroock, D.W., Varadhan, S.R.S.: Multidimensional diffusion processes. Classics in Mathematics. Springer-Verlag, Berlin (2006). Reprint of the 1997 edition
110. Subrahmanyam, A.: Risk aversion, market liquidity, and price efficiency. The Review of Financial Studies **4**(3), 417–441 (1991)
111. Wu, C.T.: Construction of Brownian motions in enlarged filtrations and their role in mathematical models of insider trading. Ph.D. thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät II (1999)
112. Yor, M.: Sur les théories du filtrage et de la prédiction. In: Séminaire de Probabilités, XI, pp. 257–297. Lecture Notes in Math., Vol. 581. Springer, Berlin (1977)
113. Yor, M.: Some aspects of Brownian motion. Part II. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (1997). URL https://doi.org/10.1007/978-3-0348-8954-4. Some recent martingale problems
114. Zakai, M.: On the optimal filtering of diffusion processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11**, 230–243 (1969). doi:10.1007/BF00536382. URL http://dx.doi.org/10.1007/BF00536382

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https://doi.org/10.1007/978-1-4939-8835-8