1 Ceva, Menalaus, Duality

1.1 Ceva

Problem 1.1.1. Let ABC be a triangle and let P be a point in its interior. Let X_1, Y_1, Z_1 be the intersections of AP, BP, CP, with BC, CA, AB. Let X_2, Y_2, Z_2, be the reflections of the points X_1, Y_1, Z_1 about the midpoints of BC, CA, AB. Prove that the lines AX_2, BY_2, CZ_2 are concurrent.

Solution. This is trivial by Ceva because $\dfrac{CA_1}{A_1B} = \dfrac{BA_2}{A_2C}$. □

Problem 1.1.2. Let ABC be a triangle and let P be a point in its interior. Let X_1, Y_1, Z_1 be the intersections of AP, BP, CP, with BC, CA, AB. Let the circumcircle of $X_1Y_1Z_1$ intersect segments BC, CA, AB again at X_3, Y_3, Z_3. Prove that AX_3, BY_3, CZ_3 are concurrent.

Solution. This is trivial by Power of a Point and Ceva. □

Problem 1.1.3. Let ABC be a triangle and let P be a point in its interior. Let X_1, Y_1, Z_1 be the intersections of AP, BP, CP, with BC, CA, AB. Let X_2, Y_2, Z_2, be the reflections of the points X_1, Y_1, Z_1 about the angle bisectors of $\angle CAB, \angle ABC, \angle BCA$. Prove that the lines AX_2, BY_2, CZ_2 are concurrent.

Solution. This is trivial by Trig Ceva in a method similar to Problem 1. □

Jacobi's Theorem. Let ABC be a triangle and let X, Y, Z be points in its plane such that $\angle ZAB = \angle YAC$, $\angle ZBA = \angle XBC$, and $\angle XCB = \angle YCA$. Then the lines AX, BY, CZ are concurrent.

Proof. Hooray for the creation of the Law of Sines. For the sake of clarity, let $\angle ZAB = \angle YAC = \alpha$, $\angle ZBA = \angle XBC = \beta$, $\angle XCB = \angle YCA = \gamma$. We know that

$$\frac{AZ}{\sin\angle ACZ} = \frac{CZ}{\sin\angle ZAC}.$$

Rearranging, we obtain

$$\sin\angle ACZ = \frac{AZ\sin\left(\angle CAB+\alpha\right)}{CW}.$$

Similarly, $\sin\angle ZCB = \dfrac{BZ\sin\left(\angle ABC+\beta\right)}{CZ}$. Thus

$$\frac{\sin\angle ACZ}{\sin\angle ZCB} = \frac{AZ\sin\left(\angle CAB+\alpha\right)}{BZ\sin\left(\angle ABC+\beta\right)}.$$

We now apply Law of Sines again on $\triangle ABZ$ to compute $\dfrac{AZ}{BZ} = \dfrac{\sin\beta}{\sin\alpha}$. Hence

$$\frac{\sin\angle ACZ}{\sin\angle ZCB} = \frac{\sin(\beta)\sin\left(\angle CAB+\alpha\right)}{\sin(\alpha)\sin\left(\angle ABC+\beta\right)}.$$

We are now done by Trig Ceva. □

Proof 2. Use the fact that AX, BX, CX concur at X. The rest of the proof is left as an exercise to the reader. □

Problem 1.1.4. Let ABC be a triangle with incentre I. Let the tangency points of the incircle with BC, CA, AB be D, E, F, respectively. Let X, Y, Z be points on lines ID, IE, IF, respectively, such that $IX = IY = IZ$ and X, Y, Z either all lie towards the interior of ABC or towards the exterior. Prove that the lines AX, BY, CZ are concurrent.

Solution. $XD = ZF$ and $BD = BF$ so $\triangle BXD \cong \triangle BZF$ or $\angle BXD = \angle BZF$. We get similar relations for the other two pairs of angles so we are done by Jacobi. □

Ceva for Convex Quadrilaterals. If $ABCD$ is a convex quadrilateral, then

$$\frac{\sin\angle DAC}{\sin\angle CAB}\cdot\frac{\sin\angle ABD}{\sin\angle DBC}\cdot\frac{\sin\angle BCA}{\sin\angle ACD}\cdot\frac{\sin\angle CDB}{\sin\angle BDA}=1.$$

Proof. Let the diagonals AC and BD intersect at P. We have the relations

$$\frac{DP}{PB}=\frac{DA}{AB}\cdot\frac{\sin\angle DAC}{\sin\angle CAB},$$

$$\frac{BP}{PD}=\frac{BC}{CD}\cdot\frac{\sin\angle BCA}{\sin\angle ACD},$$

$$\frac{AP}{PC}=\frac{AB}{BC}\cdot\frac{\sin\angle ABD}{\sin\angle DBC},$$

$$\frac{CP}{PA}=\frac{CD}{DA}\cdot\frac{\sin\angle CDB}{\sin\angle BDA}$$

so we are done. □

Problem 1.1.5. Let $ABCD$ be a convex quadrilateral with $\angle BAC=30^\circ$, $\angle CAD=20^\circ$, $\angle ABD=50^\circ$, and $\angle DBC=30^\circ$. If the diagonals intersect at P, prove that $PC=PD$.

Solution. Let $\angle DCP=\alpha$ and $\angle CDP=\beta$. By applying Ceva for Quads, we obtain

$$\frac{\sin\beta}{\sin\alpha}\cdot\frac{\sin 20^\circ\cdot\sin 50^\circ\cdot\sin 70^\circ}{\sin 30^\circ\cdot\sin 30^\circ\cdot\sin 80^\circ}=1.$$

But $\sin 20^\circ\cdot\sin 50^\circ\cdot\sin 70^\circ=\sin 30^\circ\cdot\sin 30^\circ\cdot\sin 80^\circ$ (just apply product to sum) so $\sin\alpha=\sin\beta$. It is easily computable that $\angle CPD=100^\circ$ so $\alpha,\beta\le 90^\circ$. This implies that $\alpha=\beta$ so $PC=PD$. □

Problem 1.1.6. Let $ABCD$ be a convex quadrilateral with $\angle DAC=\angle BDC=36^\circ$, $\angle CBD=18^\circ$, and $\angle BAC=72^\circ$. The diagonals intersect at P. Compute $\angle APD$.

Solution. Clearly $\angle BCD=126^\circ$. We obtain three crucial equalities:

$$\angle BCD+\frac{1}{2}\angle BAD=180^\circ,$$

$$\angle DAC=2\angle DBC,$$

$$\angle BAC=2\angle BDC$$

This implies that the circle centered at A with radius AB passes through C and D. It follows that $\angle ACB=54^\circ$ and $\angle APD=\angle BPC=180^\circ-\angle PCB-\angle PBC=108^\circ$. □

Solution 2. Consider the regular 10-gon $A_1A_2...A_{10}$. Let $A_1=D$, $A_3=A$, $A_5=B$, and $A_5A_{10}\cap A_1A_7=C$. We have now constructed $ABCD$ and this immediately implies that $AB=AC=AD$. □

1.2 Menalaus

Problem 1.2.1. Let ABC be a triangle and P be a point in its plane. Let A_1,B_1,C_1 be the intersections of AP,BP,CP with BC,CA,AB, respectively. Consider Let $X=A_1B_1\cap AB$, $Y=B_1C_1\cap BC$, $Z=C_1A_1\cap CA$. Prove that X,Y,Z are collinear.

Solution. This is trivial by Desargues' Theorem. □

Problem 1.2.2 (The Lemoine line)**.** Let ABC be a triangle and let A_1 be the intersection point of the tangent at A to the circumcircle of ABC with line BC. Similarly, define B_1 and C_1. Prove that A_1,B_1,C_1 are collinear.

Solution. This is trivial by Pascal on hexagon $AABBCC$. □

Problem 1.2.3 (USAMO 2012 #5)**.** Let ABC be a triangle and let P be a point in its interior. Let γ be a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB, respectively. Prove that A', B', C' are collinear.

Solution. We want that

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1.$$

We can rewrite this as a ratio of areas

$$\frac{[BPA']}{[A'PC]} \cdot \frac{[CPB']}{[B'PA]} \cdot \frac{[ACP']}{[C'PB]} = 1.$$

We can relate this to a ratio of sines because $[ABC] = \frac{1}{2}(AB)(BC)\sin\angle C$.

$$\frac{\sin\angle BPA'}{\sin\angle A'PC} \cdot \frac{\sin\angle CPB'}{\sin\angle B'PA} \cdot \frac{\sin\angle APC'}{\sin\angle C'PB} = 1.$$

Next, notice that $\angle BPA' = \angle B'PA$ or $\angle BPA' = 180° - \angle B'PA$, depending on the configuration of the points. The same holds true for the other two pairs of angles, so we're done. $\square$

Menelaus' Theorem for Regular n-gons. (one way only, though) Let p be a line that intersects the sides A_iA_{i+1} of the n-gon $A_1A_2\ldots A_n$ at the points M_i for all $1 \le i \le n$. Then

$$\prod_{i=1}^{n} \frac{A_iM_i}{M_iA_{i+1}} = 1.$$

Proof. Consider a parallel projection of the points A_i and M_i in the direction of p to a line q. Call the projections of A_i A_i'. Clearly all the M_i will project to a single point M. We then have

$$\prod_{i=1}^{n} \frac{A_iM_i}{M_iA_{i+1}} = \prod_{i=1}^{n} \frac{A_i'M}{MA_{i+1}} = 1.$$

$\square$

Note: There is a proof that uses induction. However, I am too lazy so that proof is left as an exercise to the reader.

Problem 1.2.4 (Van Aubel's Theorem)**.** Let ABC be a triangle and P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB at A', B', C', respectively. Prove that

$$\frac{AP}{PA'} = \frac{AC'}{C'B} + \frac{AB'}{B'C}$$

Solution. We quickly obtain the relations

$$\frac{AP}{PA'} = \frac{AB}{BD} \cdot \frac{\sin\angle ABP}{\sin\angle PBA'}$$

and

$$\frac{AB'}{B'C} = \frac{AB}{BC} \cdot \frac{\sin\angle ABB'}{\sin\angle B'BC'}$$

which implies

$$\frac{AB'}{B'C} = \frac{BB'}{BC} \cdot \frac{AP}{PD}.$$

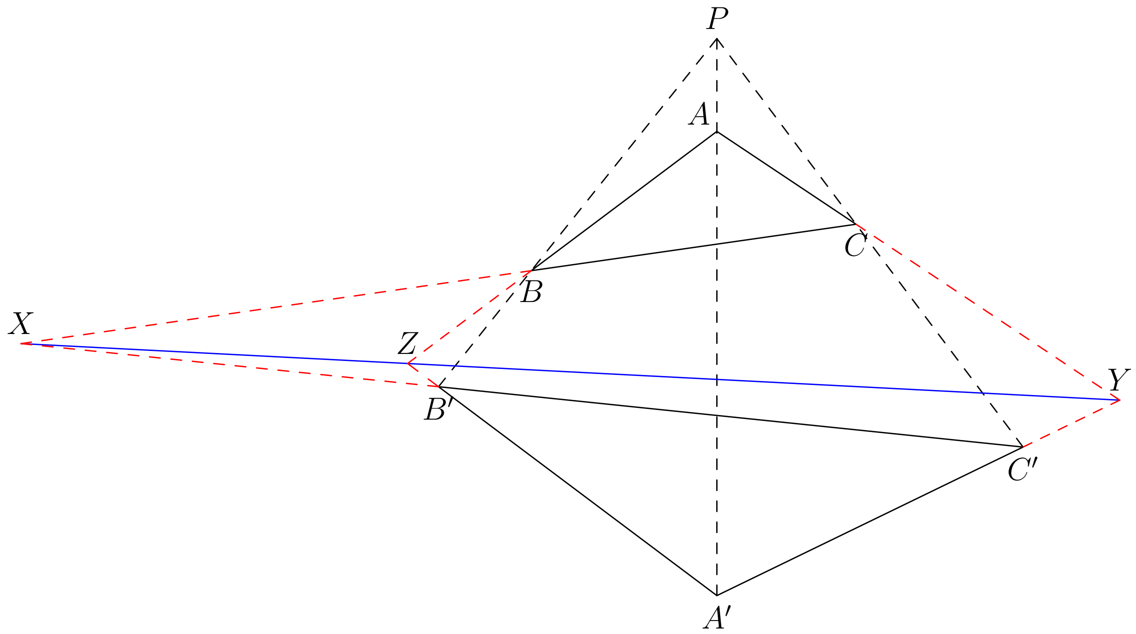
Likewise

$$\frac{AC'}{C'B} = \frac{B'C}{BC} \cdot \frac{AP}{PD}$$

and the result follows. $\square$

1.3 Duality

Desargues' Theorem. Two triangles ABC and $A'B'C'$ are perspective from a point if and only if they are perspective from a line.



Proof. First, we prove the "only if" part. Let P be the center of perspectivity. Let X, Y, Z be $BC \cap B'C', CA \cap C'A', AB \cap A'B'$, respectively. We apply Menelaus on $\triangle PBC$ with points X, B', C' to obtain

$$\frac{PB'}{B'B} \cdot \frac{BX}{XC} \cdot \frac{CC'}{C'P} = 1.$$

Similarly

$$\frac{PA'}{A'A} \cdot \frac{AY}{YC} \cdot \frac{CC'}{C'P} = 1$$

and

$$\frac{PB'}{B'B} \cdot \frac{BZ}{ZA} \cdot \frac{AA'}{A'P} = 1.$$

Then,

$$\frac{\frac{PB'}{B'B} \cdot \frac{BX}{XC} \cdot \frac{CC'}{C'P}}{\left(\frac{PA'}{A'A} \cdot \frac{AY}{YC} \cdot \frac{CC'}{C'P}\right)\left(\frac{PB'}{B'B} \cdot \frac{BZ}{ZA} \cdot \frac{AA'}{A'P}\right)} = \frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1.$$

Notice that Desargues' Theorem is self dual, which automatically implies that the converse is true. Anyways, here is a proof using the "only if" part of the theorem.

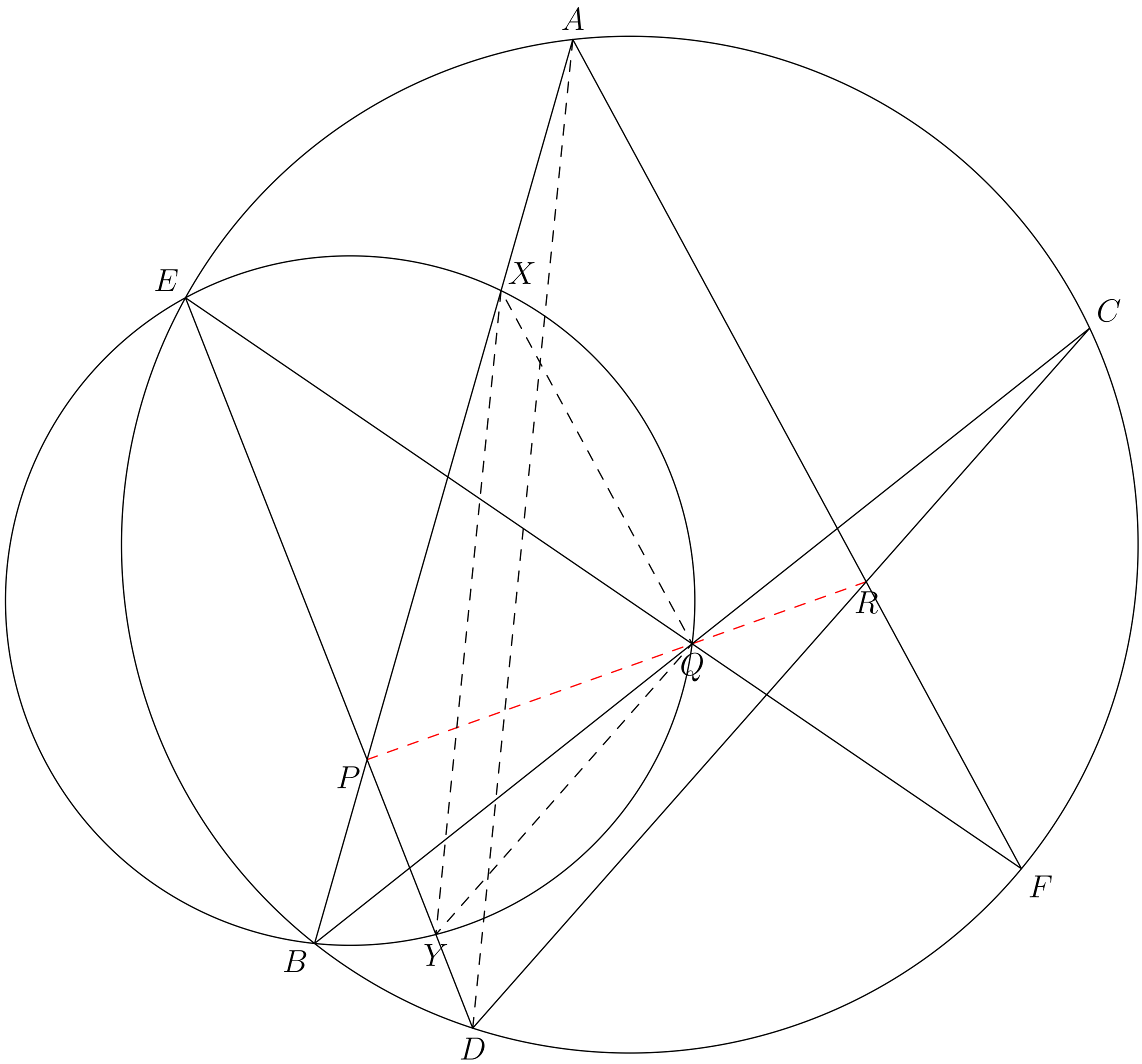
Let $P = BB' \cap CC'$. Consider the two triangles $BB'Z$ and $CC'Y$. Notice how these two triangles are persective from X. This implies that $BB' \cap CC', B'Z \cap C'Y, ZB \cap YC$ are collinear. This implies P, A, A' are collinear so $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a point. □

Pascal's Theorem. Let $ABCDEF$ be a hexagon inscribed in a conic. Then $AB \cap DE, BC \cap EF, CD \cap FA$ are collinear.

Proof. It is sufficient to prove the theorem for when $ABCDEF$ is inscribed in a circle because there exists a projectivity that will take it to all other conics.

Lemma. Let ω_1 and ω_2 be two circles intersecting at M and N and let AB be a chord of ω_1. Let AM and BN meet ω_2 at points C and D, respectively. Then $AB \parallel CD$.

Proof. We use directed angles mod π. We have $\angle CAB = 180^\circ - \angle MNB = \angle MND = 180^\circ - \angle ACD$ so $AB \parallel CD$. $\square$



Let ω_1 be the circumcircle of $ABCDEF$, $P = AB \cap DE, Q = BC \cap EF, R = CD \cap FA$, and ω_2 be the circumcircle of BEQ. Let X be the second intersection of AB with ω_2 and let Y be the second intersection of DE with ω_2. By the lemma, we have $AR \parallel AE \parallel XQ$, $DR \parallel DC \parallel YQ$, and $AD \parallel XY$. This implies that there is a homothety between triangles RAD and QXY. Hence, AX, DY, RQ concur at P which proves the collinearity of P, Q, R. $\square$

Problem 1.3.1. Let ABC be a triangle and let B_1, C_1 be points on the sides CA, AB, respectively. Let Γ be the incircle of ABC and let E, F be the tangency points of Γ with the sides CA, AB, respectively. Furthermore, draw the tangents from B_1 and C_1 to gamma which are different from the sides of ABC and take the tangency points with Γ to be Y and Z, respectively. Prove that the lines B_1C_1, EF, YZ are concurrent.

Solution. Let $P = EF \cap YZ$, $Q = EZ \cap FY$, $R = EY \cap FZ$. By Brokard's theorem, we have that PQR is self-polar. R lies on the polar EY of B_1, so B_1 lies on the polar of R (La Hire's Theorem). Similarly, C_1 lines on the polar of R, so P, Q, B_1, C_1 are collinear, as desired. $\square$

Solution 2. Using the same notation as above, use Pascal on $EFFYZZ$ to obtain that P, Q, C_1 are collinear. Pascal on $FEEZYY$ yields P, Q, B_1 are collinear, so we are done. $\square$

1.4 Problems

Problem 1.4.1. Let $ABCD$ be a trapezoid with $AD \parallel BC$. Let $P = AB \cap CD$ and $Q = AC \cap BD$. Let M, N be the midpoints of AD, BC, respectively. Prove that M, N, P, Q are collinear.

Solution. M and N are the two homothetic centers that take AD to BC. $\square$

Problem 1.4.2. Let D be the foot of the altutide from A of triangle ABC and let M, N be points on sides CA, AB such that the lines BM and CN intersect on AD. Prove that AD is the angle bisector of $\angle MDN$.

Solution. Apply Ratio Lemma. We obtain $\frac{AM}{MC} \cdot \frac{CD}{DB} \cdot \frac{BN}{NA} = \frac{\sin\angle ADM}{\sin\angle ADN} \cdot \frac{\sin\angle NDB}{\sin\angle MDC} = 1$. This is equivalent to $\sin\angle ADM \cos\angle ADN = \sin\angle ADN \cos\angle ADM$, or $\sin(\angle ADM - \angle ADN) = 0$. We also have $\angle ADM, \angle ADN < \pi$, so this implies that $\angle ADM = \angle ADN$, as desired. $\square$

Problem 1.4.3. Prove the converse of the previous problem.

Solution. It follows by reversing the steps of the previous proof. $\square$

Problem 1.4.4. Let ABC be a triangle and let A_1, B_1, C_1 be points on lines BC, CA, AB, respectively. Denote by G_A, G_B, G_C the centroids of triangles $AB_1C_1, BC_1A_1, CA_1B_1$, respectively. Prove that the lines AG_A, BG_B, CG_C are concurrent if and only if lines AA_1, BB_1, CC_1 are concurrent.

Solution. By the Ratio Lemma, we have

$$\left(\frac{AB_1}{AC_1} \cdot \frac{BC_1}{BA_1} \cdot \frac{CA_1}{CB_1}\right)\left(\frac{\sin\angle B_1AG_A}{\sin\angle G_AAC_1} \cdot \frac{\sin\angle C_1BG_B}{\sin\angle G_BBA_1} \cdot \frac{\sin\angle A_1CG_C}{\sin\angle G_CCB_1}\right) = 1$$

so $\frac{AB_1}{AC_1} \cdot \frac{BC_1}{BA_1} \cdot \frac{CA_1}{CB_1} = \frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$ if and only if $\frac{\sin\angle B_1AG_A}{\sin\angle G_AAC_1} \cdot \frac{\sin\angle C_1BG_B}{\sin\angle G_BBA_1} \cdot \frac{\sin\angle A_1CG_C}{\sin\angle G_CCB_1} = 1$. $\square$

Problem 1.4.5. Let ABC be a triangle and let D, E, F be any three points on the lines BC, CA, AB, respectively so that lines AD, BE, CF are concurrent. Let the line parallel to AB through E intersect line DF at Q and let the line parallel through D intersect line EF at T. Then, lines CF, DE, and QT are concurrent.

Solution. Let $FC \cap DT = X$ and $EF \cap BC = Y$. If $XD = XT$, we are done by Ceva on triangle FDT because $\frac{FQ}{QD} = \frac{FE}{ET}$. However, we know that $(B, C; D, Y) = -1$ so the pencil $F(B, C; D, Y) = F(\infty, X, D, T) = -1$ so X is the midpoint of D and T. $\square$

Problem 1.4.6. Let ABC be a triangle with $\angle A = \frac{\pi}{2}$, and let D be a point lying on the side AC. Denote by E the reflection of A over line BD, and by F the intersection point of CE with the perpendicular from D to BC. Prove that AF, DE, BC are concurrent.