SOME RESULTS ON THE (p,q)- CALCULUS AND PARTIAL (p,q)-DIFFERENTIAL EQUATIONS

İnci Çakmak [1] **and Aynur Şahin** [2]

[1,2] Department of Mathematics, Sakarya University, 54050, Sakarya, Türkiye

incicakmak13113@gmail.com, ayuce@sakarya.edu.tr

Abstract

Recently, Jafari et al. (Rom. Journ. Phys. 59, 399-407, 2014) presented the reduced q-differential transform method for solving partial q-differential equations. In this paper, we define the concept of partial (p,q)-derivative for a multivariable function and generalize the method of Jafari et al. to partial (p,q)-differential equations. Also, we give some examples to discover the effectiveness and performance of the proposed method.

Mathematics Subject Classification: 05A30; 35A25; 35N20.

Keywords: q -calculus; (p,q)-calculus; Partial differential equation; Initial value problem.

1. INTRODUCTION AND PRELIMINARIES

The q-calculus (or recalled the quantum calculus) appeared as a connection between mathematics and physics (see [1-7]). It has many applications in different mathematical areas, such as number theory, combinatorics, orthogonal polynomials, and other sciences: quantum theory, mechanics, and theory of relativity. Further, there is the possibility of extension of the q -calculus to post-quantum calculus denoted by the (p,q) -calculus. When the case $p=1$, the (p,q)-calculus reduces to the q-calculus (see [8-10]).

The (p,q)-number is defined as

$$[n]_{p,q}=\frac{p^n-q^n}{p-q},\ \ p\neq q.$$

The (p,q)- binomial coefficients given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!},\ \ n\geq k$$

and the (p,q)-factorial given by

$$[n]_{p,q}!=[n]_{p,q}[n-1]_{p,q}\ldots[2]_{pq}[1]_{p,q},\ \ n\in\mathbb{N}$$

are also known from [10] and [11].

Let $f: \mathbb{R} \to \mathbb{R}$ then the (p,q)-derivative of a function f, denoted by $D_{p,q}f$, is defined as

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \qquad x \neq 0, \qquad p \neq q$$

and

$$(D_{p,q}f)(0) = f'(0),$$

provided that f is differentiable at 0.

Note that for $p = 1$, the (p,q)-derivative reduces to the q-derivative given by

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad x \neq 0, \; q \neq 1;$$

also for $q \to 1 = p$, the (p,q)-derivative reduces to the ordinary derivative named as $Df(x)$ if $f(x)$ is differentiable.

As with the q-derivative and the ordinary derivative, the action of applying the (p,q)-derivative of any function is a linear operator, see [10].

Example 1: (see [9,10]) The following equation holds

$$D_{p,q}(x^n) = \frac{(px)^n - (qx)^n}{px - qx} = [n]_{p,q} x^{n-1}, \; n \in \mathbb{Z}^+.$$

By induction on n, it follows from the (p,q)-analog of the Leibniz rule [8,9]

$$D_{p,q}^n (fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} D_{p,q}^k (f)(xp^{n-k}) D_{p,q}^{n-k}(g)(xq^k),$$

where $D_{p,q}^n = \frac{d_{p,q}^n}{d_{p,q}x^n}$.

Now the (p,q)-Taylor formula for polynomials is given as follows.

Theorem 1: (see [10]) For any polynomial $f(x)$ of degree N, and any number a , the following (p,q)-Taylor expansion holds

$$f(x) = \sum_{k=0}^{N} p^{-\binom{k}{2}} \frac{(D_{p,q}^k f)(ap^{-k})}{[k]_{p,q}\,!} (x \ominus a)_{p,q}^k$$

where $(x \ominus a)_{p,q}^k = (x-a)(px-aq) \ldots (p^{k-2}x - aq^{k-2})(p^{k-1}x - aq^{k-1})$.

Note that in the above equality, N can be taken to be ∞ with the condition that the infinite series obtained is convergent. Then, the formula becomes

$$f(x) = \sum_{k=0}^{\infty} p^{-\binom{k}{2}} \frac{(D_{p,q}^k f)(ap^{-k})}{[k]_{p,q}\,!} (x \ominus a)_{p,q}^k.$$

In the following parts of the paper, we calculus the (p,q)-derivatives of some functions and present the (p,q)-differential transform method for solving some partial (p,q)-differential equations. Our results provide a generalization of the results of Jafari et al. [12].

2. (p,q)-DERIVATIVES OF SOME STANDARD FUNCTIONS

Following the procedure for computing the (p,q)-derivative, we now obtain some results for the (p,q)-derivative of standard functions, such as $\sin x$, e^x and $\ln x$.

Let us recall that $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $\ln x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$.

2.1. (p,q)-Derivative of the Function $f(x) = \sin x$

By the definition of (p,q)-derivative, we have

$$D_{p,q}\sin x = \frac{\sin(px)-\sin(qx)}{(p-q)x}$$

$$= \frac{\left(px - \frac{(px)^3}{3!} + \frac{(px)^5}{5!} - \frac{(px)^7}{7!} + \cdots\right) - \left(qx - \frac{(qx)^3}{3!} + \frac{(qx)^5}{5!} - \frac{(qx)^7}{7!} + \cdots\right)}{(p-q)x}$$

$$= 1 - \frac{1}{3!}x^2\frac{p^3-q^3}{p-q} + \frac{1}{5!}x^4\frac{p^5-q^5}{p-q} - \frac{1}{7!}x^6\frac{p^7-q^7}{p-q} + \cdots$$

Using the fact that

$$\frac{p^r-q^r}{p-q} = p^{r-1} + p^{r-2}q + \cdots + pq^{r-2} + q^{r-1},$$

we obtain

$$D_{p,q}\sin x = 1 - \frac{1}{3!}x^2(p^2+pq+q^2) + \frac{1}{5!}x^4(p^4+p^3q+p^2q^2+pq^3+q^4) - \frac{1}{7!}x^6(p^6+p^5q+p^4q^2+p^3q^3+p^2q^4+pq^5+q^6) + \cdots$$

If we take $p=1$ and the limit as $q \to 1 = p$ in this equation, then we get the q-derivative and the ordinary derivative of the function $\sin x$

$$D_q\sin x = 1 - \frac{1}{3!}x^2(1+q+q^2) + \frac{1}{5!}x^4(1+q+q^2+q^3+q^4) - \frac{1}{7!}x^6(1+q+q^2+q^3+q^4+q^5+q^6) + \cdots$$

and

$$D\sin x = 1 - \frac{1}{3!}x^2(3) + \frac{1}{5!}x^4(5) - \frac{1}{7!}x^6(7) + \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \cos x,$$

respectively.

2.2. (p,q)-Derivative of the Function $f(x)=e^x$

We have by the definition of (p,q)-derivative

$$D_{p,q}e^x = \frac{e^{px}-e^{qx}}{(p-q)x} = \frac{e^x(\,e^{-x(1-p)}-e^{-x(1-q)})}{(p-q)x}.$$

Since

$$e^{-x(1-p)} = \sum_{n=0}^{\infty}\frac{\left(-x(1-p)\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n(1-p)^n x^n}{n!}$$

$$= 1-(1-p)x+\ (1-p)^2\frac{x^2}{2!}-(1-p)^3\frac{x^3}{3!}+\cdots+(-1)^k(1-p)^k\frac{x^k}{k!}+\cdots$$

and

$$e^{-x(1-q)} = \sum_{n=0}^{\infty}\frac{\left(-x(1-q)\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n(1-q)^n x^n}{n!}$$

$$=\ 1-(1-q)x+\ (1-q)^2\frac{x^2}{2!}-(1-q)^3\frac{x^3}{3!}+\cdots+(-1)^k(1-q)^k\frac{x^k}{k!}+\cdots,$$

then we have

$$D_{p,q}e^x = \frac{e^x}{(p-q)x}\Big[(p-q)x-[(1-q)^2-(1-p)^2]\frac{x^2}{2!}+[(1-q)^3-(1-p)^3]\frac{x^3}{3!}+\cdots+ (-1)^k[(1-q)^k-(1-p)^k]\frac{x^k}{k!}+\cdots\Big].$$

Using the fact that

$$\frac{(1-q)^r-(1-p)^r}{p-q} = (1-q)^{r-1}+(1-q)^{r-2}(1-p)+\cdots+(1-q)(1-p)^{r-2}+(1-p)^{r-1},$$

we obtain

$$\begin{aligned}D_{p,q}e^x = e^x\Bigg[&1-[(1-q)+(1-p)]\frac{x}{2!}+[(1-q)^2+(1-q)(1-p)+(1-p)^2]\frac{1}{3!}x^2\\&+\cdots+(-1)^k[(1-q)^{k-1}+(1-q)^{k-2}(1-p)+\cdots+(1-q)(1-p)^{k-2}\\&+(1-p)^{k-1}]\frac{x^{k-1}}{k!}+\cdots\Bigg].\end{aligned}$$

If we take $p=1$ and the limit as $q\to 1=p$ in this equation, then we get the q-derivative and the ordinary derivative of the function e^x

$$D_q e^x = e^x\left[1-(1-q)\frac{x}{2!}+(1-q)^2\frac{1}{3!}x^2+\cdots+(-1)^k(1-q)^{k-1}\frac{x^{k-1}}{k!}+\cdots\right]$$

and

$De^x = e^x,$

respectively.

2.3. (p,q)-Derivative of the Function $f(x) = \ln x$

By the definition of (p,q)-derivative, we have

$$D_{p,q} \ \ln x = \frac{\ln px - \ln qx}{(p-q)x}$$

$$= \frac{(px-1) - \frac{(px-1)^2}{2} + \frac{(px-1)^3}{3} + \cdots + (-1)^{k-1}\frac{(px-1)^k}{k} + \cdots}{(p-q)x}$$

$$- \frac{(qx-1) - \frac{(qx-1)^2}{2} + \frac{(qx-1)^3}{3} + \cdots + (-1)^{k-1}\frac{(qx-1)^k}{k} + \cdots}{(p-q)x}$$

$$= 1 - \tfrac{1}{2}[(px-1) + (qx-1)] + \tfrac{1}{3}[(px-1)^2 + (px-1)(qx-1) + (qx-1)^2] + \cdots$$

$$+ \frac{(-1)^{k-1}}{k}[(px-1)^{k-1} + (px-1)^{k-2}(qx-1) + \cdots + (qx-1)^{k-1}] + \cdots$$

If we take $p = 1$ and the limit as $q \to 1 = p$ in this equation, then we get the q-derivative and the ordinary derivative of the function $\ln x$

$$D_q \ \ln x = 1 - \frac{1}{2}[(x-1) + (qx-1)] + \frac{1}{3}[(x-1)^2 + (x-1)(qx-1) + (qx-1)^2] + \cdots$$

$$+ \frac{(-1)^{k-1}}{k}[(x-1)^{k-1} + (x-1)^{k-2}(qx-1) + \cdots + (qx-1)^{k-1}] + \cdots$$

and

$$D \ln x = \ 1 - \frac{1}{2}.2.(x-1) + \frac{1}{3}.3.(x-1)^2 + \cdots + \frac{(-1)^{k-1}}{k}.k.(x-1)^{k-1} + \cdots$$

$$= \textstyle\sum_{n=1}^{\infty}(-1)^{n-1}(x-1)^{n-1} = \sum_{n=0}^{\infty}(-1)^n (x-1)^n = \frac{1}{x},$$

respectively.

We remark that the (p,q)-derivatives of other standard functions can be obtained similarly.

3. PARTIAL (p,q)-DERIVATIVE OF A MULTIVARIABLE FUNCTION

Firstly, we define the partial (p,q)-derivative of a multivariable real continuous function $f(x_1, x_2, \ldots, x_n)$ for a variable x_i

$$D_{p,q,x_i} f(x) = \frac{(\varepsilon_{p,x_i} f)(x) - (\varepsilon_{q,x_i} f)(x)}{(p-q)x_i}, \qquad x_i \neq 0, \qquad p \neq q$$

$$\left[D_{p,q,x_i}f(x)\right]_{x_i=0} = \lim_{x_i\to 0} D_{p,q,x_i}f(x)$$

where $\left(\varepsilon_{p,x_i}f\right)(x) = f(x_1,x_2,\dots,px_i,\dots,x_n)$ and $\left(\varepsilon_{q,x_i}f\right)(x) = f(x_1,x_2,\dots,qx_i,\dots x_n)$.

After the definition of partial (p,q)-derivative, we can give the following theorem for the functions of two variables (compare with [6,12]).

Theorem 2: Suppose that the function $f(x,t)$ has continuous partial (p,q)-derivatives of all orders. Then

$$f(x,t) = \sum_{k=0}^{\infty} \frac{p^{-\binom{k}{2}}}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}f(x,t)\right)_{t=ap^{-k}}(t\ominus a)_{p,q}^k.$$

In this theorem, we set $a=0$, then

$$(t\ominus 0)_{p,q}^k = (t-0)(pt-0q)\dots(p^{k-2}t-0q^{k-2})(p^{k-1}t-0q^{k-1})$$

$$= t.pt\dots p^{k-2}t.p^{k-1}t = t^kp^{1+2+\cdots+k-1} = t^kp^{\binom{k}{2}}.$$

Hence, we obtain

$$f(x,t) = \sum_{k=0}^{\infty} \frac{1}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}f(x,t)\right)_{t=0} t^k.$$

Definition 1: Suppose that the function $u(x,t)$ has continuous partial (p,q)-derivatives of all orders. t-dimensional (p,q)-differential transform of function $u(x,t)$ is defined as follows:

$$U_k(x) = \frac{p^{-\binom{k}{2}}}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}u(x,t)\right)_{t=ap^{-k}}. \qquad (1)$$

In the equation (1), $u(x,t)$ is the original function and $U_k(x)$ is the transformed function.

Definition 2: The t-dimensional (p,q)-differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t\ominus a)_{p,q}^k. \qquad (2)$$

In fact, from the equations (1) and (2), we obtain

$$u(x,t) = \sum_{k=0}^{\infty} \frac{p^{-\binom{k}{2}}}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}u(x,t)\right)_{t=ap^{-k}}(t\ominus a)_{p,q}^k$$

which implies that the concept of t-dimensional (p,q)-differential transform is derived from (p,q)- Taylor formula given in Theorem 2.

In the following theorems, we set $a=0$ then we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}u(x,t)\right)_{t=0} t^k$$

and so

$$U_k(x) = \frac{1}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}u(x,t)\right)_{t=0}. \tag{3}$$

Theorem 3: If $W_k(x), U_k(x)$ and $V_k(x)$ are the t-dimensional (p,q)-differential transforms of functions $w(x,y), u(x,t)$ and $v(x,t)$ at point $t=0$ respectively, the following results hold.

(i) If $w(x,t) = \alpha u(x,t) \mp v(x,t)$ then $W_k(x) = \alpha U_k(x) \mp V_k(x)$ where α is a real constant.

(ii) If $w(x,y) = x^m t^n$ then $W_k(x) = x^m \delta(k-n)$ where

$$\delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

Proof: By the linearity of the (p,q)-derivative, we can easily obtain the result *(i)*.

(ii) By Definition 2 and the equation (3), we have

$$W_k(x) = \frac{1}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}w(x,t)\right)_{t=0}$$

$$= \frac{1}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k(x^m t^n)}{\partial_{p,q}t^k}\right)_{t=0}$$

$$= \frac{x^m}{[k]_{p,q}!}\left(\frac{\partial_{p,q}^k(t^n)}{\partial_{p,q}t^k}\right)_{t=0}$$

$$= \begin{cases} \dfrac{x^m}{[k]_{p,q}!}.[k]_{p,q}! = x^m, & k = n \\ \dfrac{x^m}{[k]_{p,q}!}.\ [n]_{p,q}[n-1]_{p,q}\dots[n-k-1]_{p,q}t^{n-k}\Big|_{t=0} = 0, & k < n \\ \dfrac{x^m}{[k]_{p,q}!}.0 = 0, & k > n \end{cases}$$

$$= x^m \delta(k-n).$$

Theorem 4: If $w(x,t) = \frac{\partial_{p,q}}{\partial_{p,q}x}u(x,t)$, then $W_k(x) = \frac{\partial_{p,q}}{\partial_{p,q}x}U_k(x)$.

Proof: $W_k(x) = \frac{1}{[k]_{p,q}!}\left[\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}\left(\frac{\partial_{p,q}}{\partial_{p,q}x}u(x,t)\right)\right]_{t=0}$

$$= \frac{1}{[k]_{p,q}!}\left[\frac{\partial_{p,q}}{\partial_{p,q}x}\left(\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}u(x,t)\right)\right]_{t=0}$$

$$= \frac{\partial_{p,q}}{\partial_{p,q}x}\left[\frac{1}{[k]_{p,q}!}.\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}u(x,t)\right]_{t=0}$$

$= \frac{\partial_{p,q}}{\partial_{p,q}x} U_k(x).$

Theorem 5: If $w(x,t) = \frac{\partial_{p,q}^r}{\partial_{p,q}t^r} u(x,t)$, then

$$W_k(x) = [k+1]_{p,q}\, [k+2]_{p,q} \dots [k+r]_{p,q} U_{k+r}(x).$$

Proof: $W_k(x) = \frac{1}{[k]_{p,q}\,!} . \left[\frac{\partial_{p,q}^k}{\partial_{p,q}t^k}\left(\frac{\partial_{p,q}^r}{\partial_{p,q}t^r} u(x,t)\right)\right]_{t=0}$

$$= \frac{1}{[k]_{p,q}\,!} . \frac{[k+r]_{p,q}!}{[k+r]_{p,q}!} . \left[\frac{\partial_{p,q}^{k+r}}{\partial_{p,q}t^{k+r}} u(x,t)\right]_{t=0}$$

$$= \frac{[k+r]_{p,q}!}{[k]_{p,q}\,!} . \frac{1}{[k+r]_{p,q}!}\left[\frac{\partial_{p,q}^{k+r}}{\partial_{p,q}t^{k+r}} u(x,t)\right]_{t=0}$$

$$= [k+1]_{p,q}\, [k+2]_{p,q} \dots [k+r]_{p,q} U_{k+r}(x).$$

Example 2: Suppose we want to solve the (p,q)-diffusion equation

$$\frac{\partial_{p,q}}{\partial_{p,q}t} u(x,t) = \frac{\partial_{p,q}^2}{\partial_{p,q}x^2} u(x,t)$$

subject to the initial condition

$$u(x,0) = e_{p,q}^x$$

where $e_{p,q}^x$ is the (p,q)-exponential function given as [9]

$$e_{p,q}^x = \sum_{k=0}^{\infty} p^{\binom{k}{2}} \frac{x^k}{[k]_{p,q}\,!}$$

and have the (p,q)-derivative

$$\frac{\partial_{p,q}}{\partial_{p,q}x} e_{p,q}^x = e_{p,q}^{px}.$$

By using the t-dimensional (p,q)-differential transform of (p,q)-diffusion equation, we write the following recursion

$$[k+1]_{p,q}\, U_{k+1}(x) = \frac{\partial_{p,q}^2}{\partial_{p,q}x^2} U_k(x),\ k = 0,1,2,\dots \tag{4}$$

Then the initial data is written as

$$U_0(x) = u(x,0) = e_{p,q}^x. \tag{5}$$

Now, substituting (5) into (4), we obtain the following $U_k(x)$ values successively

$$U_1(x) = \frac{p}{[1]_{p,q}!} e_{p,q}^{p^2x}$$

$$U_2(x) = \frac{p^6}{[2]_{p,q}!} e_{p,q}^{p^4 x}$$

$$U_3(x) = \frac{p^{15}}{[3]_{p,q}!} e_{p,q}^{p^6 x}$$

$\vdots$

$$U_k(x) = \frac{p^{k(2k-1)}}{[k]_{p,q}!} e_{p,q}^{p^{2k} x}.$$

Hence we get the solution of (p,q)-diffusion equation as follows

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k = \sum_{k=0}^{\infty} \frac{p^{k(2k-1)}}{[k]_{p,q}!} e_{p,q}^{p^{2k} x} t^k.$$

For $p = 1$, this result is also the same in Example 4.1 of [12].

Example 3: Consider the following partial (p,q)- differential equation

$$\frac{\partial_{p,q}}{\partial_{p,q} t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + \frac{\partial}{\partial x}(x u(x,t)) \tag{6}$$

with the initial condition

$u(x,0) = 2x^2.$

By using the t-dimensional (p,q)-differential transform of equation (6), we have

$$[k+1]_{p,q} U_{k+1}(x) = \frac{\partial^2}{\partial x^2} U_k(x) + \frac{\partial}{\partial x}(x U_k(x)) \tag{7}$$

where

$$U_0(x) = 2x^2. \tag{8}$$

Solving (7) with the initial condition (8), we successively achieve the values $U_k(x)$ as follows:

$$U_1(x) = \frac{4 + 6x^2}{[1]_{p,q}} = \frac{2.(3^1 - 1) + 2.3^1 x^2}{[1]_{p,q}!}$$

$$U_2(x) = \frac{16 + 18x^2}{[1]_{p,q}[2]_{p,q}} = \frac{2.(3^2 - 1) + 2.3^2 x^2}{[2]_{p,q}!}$$

$$U_3(x) = \frac{52 + 54x^2}{[1]_{p,q}[2]_{p,q}[3]_{p,q}} = \frac{2.(3^3 - 1) + 2.3^3 x^2}{[3]_{p,q}!}$$

$\vdots$

$$U_k(x) = \frac{2.(3^k - 1) + 2.3^k x^2}{[k]_{p,q}!}$$

Substituting all $U_k(x)$ into (2), we have the required solution of the partial (p,q)-differential equation to be

$$u(x,t) = 2x^2 + (4+6x^2)\frac{t}{[1]_{p,q}!} + (16+18x^2)\frac{t^2}{[2]_{p,q}!} + \cdots$$
$$+ (2.(3^k-1) + 2.3^k\, x^2)\frac{t^k}{[k]_{p,q}!} + \cdots$$

$$= \sum_{n=0}^{\infty} (2.(3^n-1) + 2.3^n\, x^2)\frac{t^n}{[n]_{p,q}!}.$$

To prove the next theorem, we will use the following lemmas.

Lemma 1.

$$D_{p,q}^n(fg)(x,t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} D_{p,q,t}^k(f)(x,tp^{n-k}) D_{p,q,t}^{n-k}(g)(x,tq^k).$$

Proof. The lemma can easily be proved by applying the principle of mathematical induction as in [9, Lemma 3.2.1].

Lemma 2.

$$f(x,p^n t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} t^k p^{\binom{k}{2}} (p-q)^k D_{p,q,t}^k f(x,t).$$

Proof. The lemma can easily be proved by using divided differences as in [8, Theorem 4].

Theorem 6: If $w(x,t) = u(x,t)v(x,t)$, then

$$W_k(x) = \sum_{n=0}^{k} U_n(x) V_{k-n}(x).$$

Proof: $W_k(x) = \frac{1}{[k]_{p,q}!}\left[D_{p,q,t}^k(u(x,t)v(x,t))\right]_{t=0}$

$$= \frac{1}{[k]_{p,q}!}\left[\sum_{n=0}^{k} \begin{bmatrix} k \\ n \end{bmatrix}_{p,q} D_{p,q,t}^n u(x,tp^{k-n})\ D_{p,q,t}^{k-n} v(x,tq^n)\right]_{t=0}$$

$$= \sum_{n=0}^{k} \frac{1}{[k-n]_{p,q}!\,[n]_{p,q}!} \left[\begin{matrix} \displaystyle\sum_{r=0}^{k-n} \begin{bmatrix} k-n \\ r \end{bmatrix}_{p,q} t^r p^{\binom{r}{2}} (p-q)^r D_{p,q,t}^{n+r} u(x,t) \\ \displaystyle * \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_{p,q} t^s q^{\binom{s}{2}} (p-q)^s D_{p,q,t}^{k-n+s} v(x,t) \end{matrix}\right]_{t=0}$$

$$= \sum_{n=0}^{k} \frac{1}{[k-n]_{p,q}!\,[n]_{p,q}!} D_{p,q,t}^n u(x,t) D_{p,q,t}^{k-n} v(x,t)$$

$$= \sum_{n=0}^{k} \frac{1}{[n]_{p,q}!} D_{p,q,t}^{n} u(x,t) \frac{1}{[k-n]_{p,q}!} D_{p,q,t}^{k-n} v(x,t)$$

$$= \sum_{n=0}^{k} U_n(x) V_{k-n}(x).$$

Example 4: Consider the following nonlinear partial (p,q)- differential equation

$$\frac{\partial_{p,q}}{\partial_{p,q}t} u(x,t) = u^2(x,t) + \frac{\partial_{p,q}}{\partial_{p,q}x} u(x,t) \tag{9}$$

with the initial condition

$u(x,0) = 1 + 4x.$

By taking the t-dimensional (p,q)-differential transform of equation (9) and using Theorems 5 and 6, we have

$$[k+1]_{p,q} U_{k+1}(x) = \sum_{n=0}^{k} U_n(x) U_{k-n}(x) + \frac{\partial_{p,q}}{\partial_{p,q}x} U_k(x), \qquad k = 0,1,2,\dots$$

From the initial equation, we obtain

$U_0(x) = u(x,0) = 1 + 4x.$

Starting with $k = 0$, the values of $U_k(x)$ are successively computed as follows.

$$[1]_{p,q} U_1(x) = U_0(x) U_0(x) + \frac{\partial_{p,q}}{\partial_{p,q}x} U_0(x) = (1+4x)^2 + 4$$

$$\Longrightarrow U_1(x) = 16x^2 + 8x + 5,$$

$$[2]_{p,q} U_2(x) = 2U_0(x) U_1(x) + \frac{\partial_{p,q}}{\partial_{p,q}x} U_1(x)$$
$$= 2(1+4x)(16x^2 + 8x + 5) + (16(p+q)x + 8)$$

$$\Longrightarrow U_2(x) = \frac{128x^3 + 96x^2 + [56 + 16(p+q)]x + 18}{p+q},$$

⋮

Substituting all $U_k(x)$ in (2), we obtain the series solution as

$$u(x,t) = 1 + 4x + (16x^2 + 8x + 5)t + \frac{128x^3 + 96x^2 + [56 + 16(p+q)]x + 18}{p+q} t^2 + \cdots.$$

REFERENCES

1. Jackson, F. H., On q-functions and a certain difference operator, Trans. R. Soc. Edinb. **46**, 253-281, 1909.
2. Jackson, F. H., A q-form of Taylor's theorem. Messenger Math., **38**, 62-64, 1909.

3. Jackson, F. H., On q-definite integrals, Quarterly J. Pure Appl. Math., **41**, 193-203, 1910.
4. Ernst, T., The history of q-calculus and a new method, Licentiate Thesis, Report 2000:16, Department of Mathematics, Uppsala University, Uppsala, Accessed 2 March 2001.
5. Kac, V., Cheung, P., Quantum Calculus, New York, Springer, 2002.
6. El-Shahed, M., Gaber, M., Two-dimensional q-differential transformation and its application, App. Math. Comput., **217**, 9165-9172, 2011.
7. Akça, H., Benbourenane J., Eleuch, H., The q-derivative and differential equation, Journal of Physics: Conference Series, 1411:012002, 2019.
8. Aracı, S., Duran, U., Açıkgöz, M., Srivastava, H. M., A certain (p,q)-derivative operator and associated divided differences, J. Ineq. Appl., 2016:301, 2016.
9. Duran, U., Post quantum calculus, Master Thesis, Department of Mathematics, Gaziantep University, Gaziantep, Türkiye, 2016.
10. Sadjang, P. N., On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas, Results Math., **73**:39, 2018.
11. Corcino, R. B., On P,Q-binomial coefficients, Electron. J. Comb. Number Theory, **8**, Article ID A29, 2008.
12. Jafari, H., Haghbin, A., Hesam, S., Baleanu, D., Solving partial q-differential equations within reduced q-differential transform method, Rom. Journ. Phys., **59**(5-6), 399-407, 2014.