RC

▷ $f = O(g)$ iff $\exists\, n_0 \in \mathbb{N},\ c > 0 \in \mathbb{R}$ s.t. $\forall\, n \geq n_0$ $f(n) \leq c\, g(n)$

▷ $f = o(g)$ iff $\forall\, \delta > 0,\ \exists\, n_0 \in \mathbb{N}$ s.t. $f(n) \leq \delta\, g(n)$ $\forall\, n \geq n_0$

▷ $(1+x) \leq e^x \ \forall x$

▷ $\lim_{n\to\infty} \left(1 \pm \frac{x}{n}\right)^n = e^{\pm x}$; $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$

▷ $\sum_1^n i = \frac{n(n+1)}{2}$ $\quad \sum_1^n i^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad \sum i^3 = \frac{n^2(n+1)^2}{4}$

▷ $\sum_{i=0}^n c^i = \frac{c^{n+1}-1}{c-1}$; $\sum_{i=0}^{\infty} c^i = \frac{1}{1-c}$ ▷ $1+2+\dots+2^n = \frac{2^{n+1}-1}{2-1}$

▷ Jensen: if $f(x)$ convex: $\mathbb{E}[f(x)] \geq f(\mathbb{E}[x])$

▷ Binomial: $B(n,p)$ $\quad \mu = np$; $\sigma^2 = np(1-p)$

▷ Geometric: $P(X=j) = (1-p)^{j-1}p$; $\mu = 1/p$; $\sigma^2 = \frac{1-p}{p^2}$

▷ $\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[(X-\mathbb{E}[X])^2]$

▷ Markov: $X \geq 0$; $\forall\, a > 0$ $\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$

▷ Chebyshev: $\forall\, a > 0$ $\Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}$

▷ $\mathrm{Cov}[X,Y] = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$

▷ $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y]$

▷ Chernoff: $X = \sum X_i$, $\mu := \mathbb{E}[X]$; $X_i \to$ Bernoulli w/ p

▷ $\forall\, \delta > 0$

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

▷ $\forall\, 0 < \delta \leq 1$

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3}$$

▷ $R \geq 6\mu$; $\Pr[X \geq R] \leq 2^{-R}$

▷ $\Pr[X \leq (1-\delta)\mu] \leq \left(\dfrac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}$

▷ $\Pr[X \leq (1-\delta)\mu] \leq e^{-\mu\delta^2/2}$

▷ $\Pr[|X-\mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}$

▷ Unbiased X_i ; $\forall a > 0$

$\Pr[X \geq a] \leq e^{-a^2/2n}$

▷ Balls ; $M = \dfrac{3\ln(n)}{\ln\ln(n)}$; then for $n \geq e^{e^{e^4}}$,

$\Pr[\text{some bin} \geq M \text{ balls}] \leq \dfrac{1}{n}$

▷ Ramsey , $R(k,k)$: smallest n, s.t. any 2-coloring of the edges of K_n must contain a red or blue K_k

▷ $\forall n$, s.t. $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, $R(k,k) > n$

▷ Lovasz :

1. $\forall i$, $\Pr[E_i] \leq p$
2. Degree of $(E_1, \ldots, E_n) \leq d$
3. $4dp \leq 1$

▷ Then $\Pr\left[\bigcap_{j=1}^{n} \bar{E_j}\right] > 0$

▷ Markov Chains

$$\begin{bmatrix} M(a_1,a_1) & M(a_1,a_2) & \cdots & \\ M(a_2,a_1) & M(a_2,a_2) & M(a_2,a_3) & \cdots \end{bmatrix}$$

$$M[a_{t-1}, a_t] = Pr[X_t = a_t \mid X_{t-1} = a_{t-1}]$$

▷ $\bar{p}(0) = \langle 0, 0, \ldots, 1, 0, \ldots, 0 \rangle$

▷ $\bar{p}(t) = \bar{p}(0) \cdot M$

▷ Irreducible if all states $x, y \in \Omega$,

$\exists\ z^0 = x, z^1, \ldots, z^e = y$, s.t. $M[z^j, z^{j+1}] > 0\ \forall j$

▷ Recurrent $z \in \Omega$ if

$$\sum_{t=0}^{\infty} r_{z,z}^{t} = 1, \quad \text{o.w. transient}$$

$$r_{z,w}^{t} = Pr[X(t) = w \text{ and } \forall 1 \le s \le t-1,\ X(s) \neq w \mid X(0) = z]$$

▷ Expected time to travel:

$$h_{z,w} = \sum_{t=0}^{\infty} t \cdot r_{z,w}^{t}; \quad h_{z,z} < \infty \Rightarrow \text{+ve recurrent}$$

o.w. $\Rightarrow$ null recurrent

▷ Periodic:

$\exists\ k \ge 2$, s.t. $Pr[X(t+s) = z \mid X(t) = z] \neq 0$

iff s is divisible by k

▷ Ergodic = aperiodic & (+ve) recurrent (state)

▷ Ergodic chain = finite & irreducible & aperiodic

▷ Stationary distribution $\bar{\pi} = \bar{\pi} \cdot M$

- For any ergodic chain
  - Unique $\bar{\pi} = (\pi_1, \dots, \pi_{|\Omega|})$
  - $\forall\, x, y$; $\lim_{t \to \infty} M^t[x, y]$ exists & $\perp\!\!\!\perp x$
  - $\pi_y = \lim_{t \to \infty} M^t[x, y] = \frac{1}{h_{y,y}}$
- Any finite chain
  - At least 1 recurrent state
  - All recurrent are +ve recurrent
- General Chernoff

  $x_i \in [a_i, b_i]$ $\quad X = \sum_i X_i$

  - $P(X - \mathbb{E}[X] \geq t) \leq \exp\left(-\frac{2t^2}{\sum_i (b_i - a_i)^2}\right)$
  - $P(|X - \mathbb{E}[X]| \geq t) \leq 2\exp\left(-\frac{2t^2}{\sum_i (b_i - a_i)^2}\right)$
- Conditional expectation inequality

  $X = \sum_i X_i$; X_i is 0/1

  $\Rightarrow \Pr(X > 0) \geq \sum_{i=1}^{n} \frac{\Pr(X_i = 1)}{\mathbb{E}[X \mid X_i = 1]}$